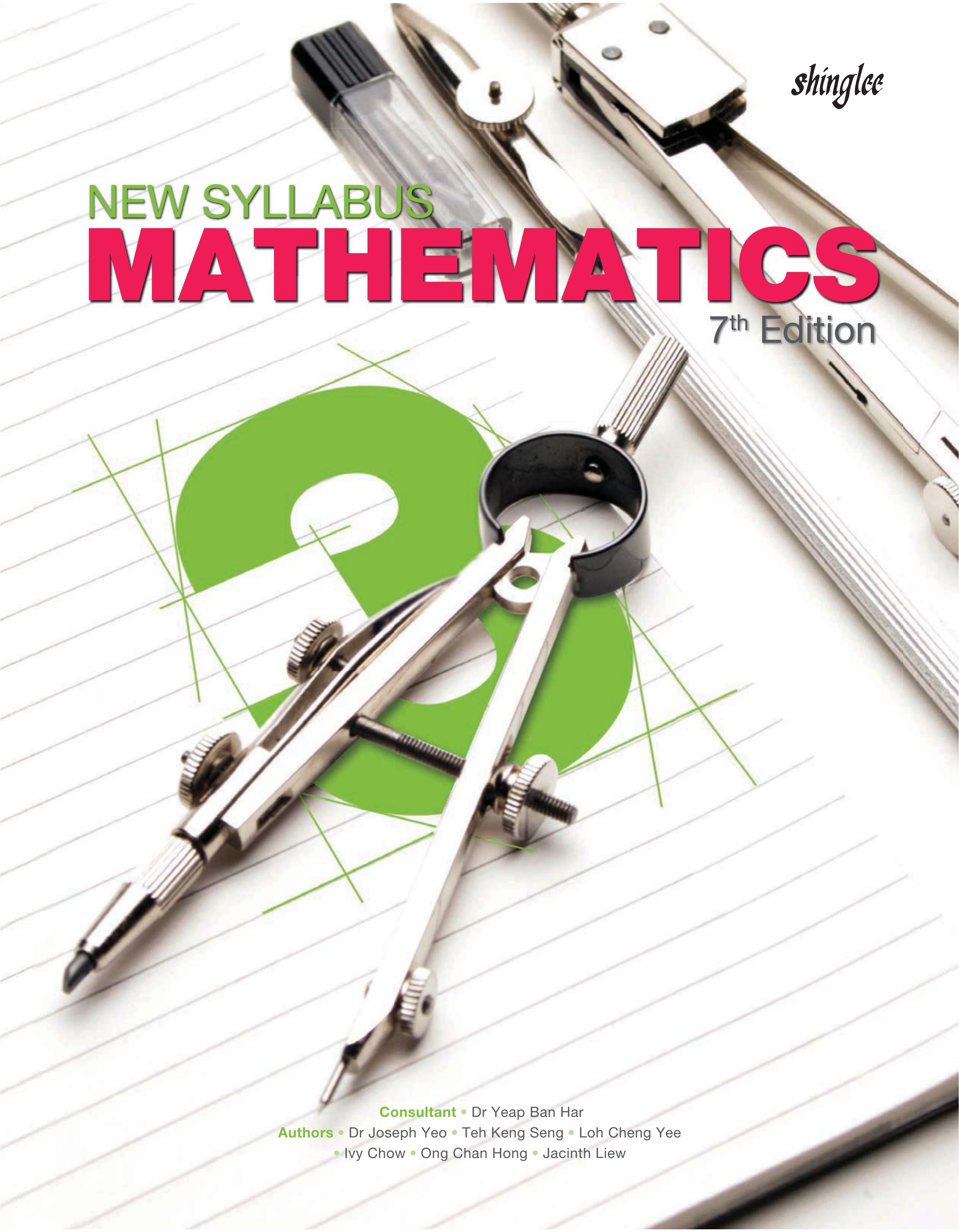
shinglee

NEW SYLLABUS MATHEMATICS

7th Edition

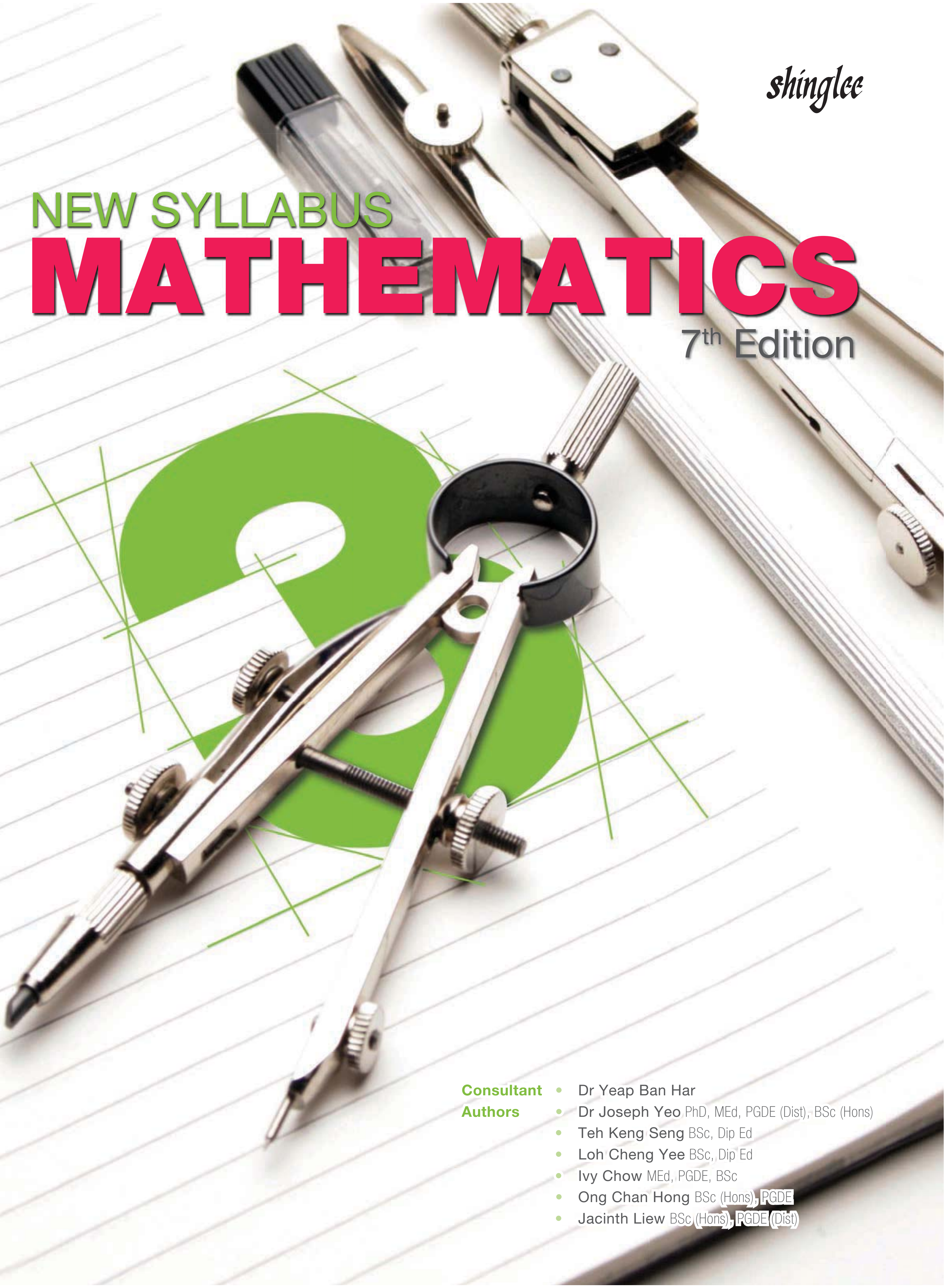


Consultant • Dr Yeap Ban Har

Authors • Dr Joseph Yeo • Teh Keng Seng • Loh Cheng Yee • Ivy Chow • Ong Chan Hong • Jacinth Liew

NEW SYLLABUS MATHEMATICS

7th Edition



shinglee

Consultant • Dr Yeap Ban Har

Authors • Dr Joseph Yeo PhD, MEd, PGDE (Dist), BSc (Hons)
• Teh Keng Seng BSc, Dip Ed
• Loh Cheng Yee BSc, Dip Ed
• Ivy Chow MEd, PGDE, BSc
• Ong Chan Hong BSc (Hons), PGDE
• Jacinth Liew BSc (Hons), PGDE (Dist)

SHINGLEE PUBLISHERS PTE LTD
120 Hillview Avenue #05-06/07
Kewalram Hillview Singapore 669594
Tel: 67601388 Fax: 67625684
email: info@shinglee.com.sg
http://www.shinglee.com.sg

First Published 1982
Reprinted 1983, 1984, 1985, 1986
Second Edition 1987
Reprinted 1987, 1988, 1989, 1990, 1991, 1992, 1993
Third Edition 1994
Reprinted 1994, 1995, 1996, 1997
Fourth Edition 1998
Reprinted 1999, 2000
Fifth Edition 2001
Reprinted 2002, 2003, 2004, 2005, 2006, 2007
Sixth Edition 2008
Reprinted 2009, 2010, 2011, 2012, 2013, 2014
Seventh Edition 2015

ISBN 978 981 237 932 0

Acknowledgements

All licensed images purchased under standard license agreement with www.shutterstock.com

APPROVED BY MINISTRY OF EDUCATION · for use from 2015-2019 ·

Printed in Singapore

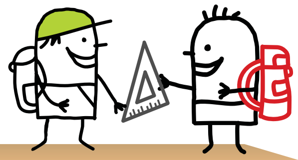
PREFACE

New Syllabus Mathematics (NSM)

is a series of textbooks specially designed to provide valuable learning experiences to engage the hearts and minds of students sitting for the GCE O-level examination in Mathematics. Included in the textbooks are **Investigation**, **Class Discussion**, **Thinking Time**, **Journal Writing**, **Performance Task** and **Problems in Real-World Contexts** to support the teaching and learning of Mathematics.

Every chapter begins with a chapter opener which motivates students in learning the topic. Interesting stories about Mathematicians, real-life examples and applications are used to arouse students' interest and curiosity so that they can appreciate the beauty of Mathematics in their surroundings.

The use of ICT helps students to visualise and manipulate mathematical objects more easily, thus making the learning of Mathematics more interactive. Ready-to-use interactive ICT templates are available at http://www.shinglee.com.sg/StudentResources/



KEY FEATURES

CHAPTER OPENER

Each chapter begins with a chapter opener to arouse students' interest and curiosity in learning the topic.

LEARNING OBJECTIVES

Learning objectives help students to be more aware of what they are about to study so that they can monitor their own progress.

RECAP

Relevant prerequisites will be revisited at the beginning of the chapter or at appropriate junctures so that students can build upon their prior knowledge, thus creating meaningful links to their existing schema.

WORKED EXAMPLE

This shows students how to apply what they have learnt to solve related problems and how to present their working clearly. A suitable heading is included in brackets to distinguish between the different Worked Examples.

PRACTISE NOW

At the end of each Worked Example, a similar question will be provided for immediate practice. Where appropriate, this includes further questions of progressive difficulty.

SIMILAR QUESTIONS

A list of similar questions in the Exercise is given here to help teachers choose questions that their students can do on their own.

EXERCISE

The questions are classified into three levels of difficulty – Basic, Intermediate and Advanced.

SUMMARY

At the end of each chapter, a succinct summary of the key concepts is provided to help students consolidate what they have learnt.

REVIEW EXERCISE

This is included at the end of each chapter for the consolidation of learning of concepts.

CHALLENGE YOURSELF

Optional problems are included at the end of each chapter to challenge and stretch high-ability students to their fullest potential.

REVISION EXERCISE

This is included after every few chapters to help students assess their learning.

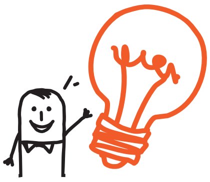
Learning experiences have been infused into Investigation, Class Discussion, Thinking Time, Journal Writing and Performance Task.

Investigation

Activities are included to guide students to investigate and discover important mathematical concepts so that they can construct their own knowledge meaningfully.

Class Discussion

Questions are provided for students to discuss in class, with the teacher acting as the facilitator. The questions will assist students to learn new knowledge, think mathematically, and enhance their reasoning and oral communication skills.

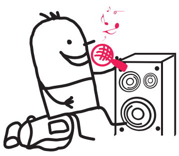


Thinking Time

Key questions are also included at appropriate junctures to check if students have grasped various concepts and to create opportunities for them to further develop their thinking.

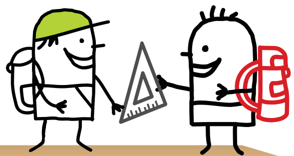
Journal Writing

Opportunities are provided for students to reflect on their learning and to communicate mathematically. It can also be used as a formative assessment to provide feedback to students to improve on their learning.



Performance Task

Mini projects are designed to develop research and presentation skills in the students.



MARGINAL NOTES

This contains important information that students should know.

This guides students on how to approach a problem.

This includes information that may be of interest to students.

This contains certain mathematical concepts or rules that students have learnt previously.

This contains puzzles, fascinating facts and interesting stories about Mathematics as enrichment for students.

This guides students to search on the Internet for valuable information or interesting online games for their independent and self-directed learning.

Contents

CHAPTER 1

Quadratic Equations and Functions **001**

1.1 Solving Quadratic Equations by Completing the Square 003
1.2 Solving Quadratic Equations by using Formula 010
1.3 Solving Quadratic Equations by Graphical Method 012
1.4 Solving Fractional Equations that can be reduced to Quadratic Equations 017
1.5 Applications of Quadratic Equations in Real-World Contexts 020
1.6 Graphs of Quadratic Functions 027
Summary 036
Review Exercise 1 038

CHAPTER 2

Linear Inequalities **041**

2.1 Inequalities 043
2.2 Problem Solving involving Inequalities 049
2.3 Solving Simultaneous Linear Inequalities 051
Summary 054
Review Exercise 2 055

CHAPTER 3

Indices and Standard Form **057**

3.1 Indices 059
3.2 Laws of Indices 060
3.3 Zero and Negative Indices 069
3.4 Rational Indices 075
3.5 Compound Interest 081
3.6 Standard Form 086
Summary 094
Review Exercise 3 095

Revision Exercise A **097**

CHAPTER 4

Coordinate Geometry 099

4.1 Gradient of a Straight Line 101

4.2 Length of a Line Segment 107

4.3 Equation of a Straight Line 113

Summary 119

Review Exercise 4 120

CHAPTER 5

Graphs of Functions and Graphical Solution 123

5.1 Graphs of Cubic Functions 125

5.2 Graphs of Reciprocal Functions 128

5.3 Graphs of Exponential Functions 135

5.4 Gradient of a Curve 139

5.5 Applications of Graphs in Real-World Contexts 144

Summary 162

Review Exercise 5 164

Revision Exercise B 167

CHAPTER 6

Further Trigonometry 171

6.1 Sine and Cosine of Obtuse Angles 173

6.2 Area of Triangle 181

6.3 Sine Rule 187

6.4 Cosine Rule 197

Summary 205

Review Exercise 6 207

CHAPTER 7

Applications of Trigonometry **211**

7.1 Angles of Elevation and Depression 213

7.2 Bearings 219

7.3 Three-Dimensional Problems 227

Summary 240

Review Exercise 7 241

CHAPTER 8

Arc Length, Area of Sector and Radian Measure **245**

8.1 Length of Arc 247

8.2 Area of Sector 256

8.3 Radian Measure 263

8.4 Arc Length and Area of Sector using Radian Measure 272

Summary 282

Review Exercise 8 283

Revision Exercise C **287**

CHAPTER 9

Congruence and Similarity Tests **291**

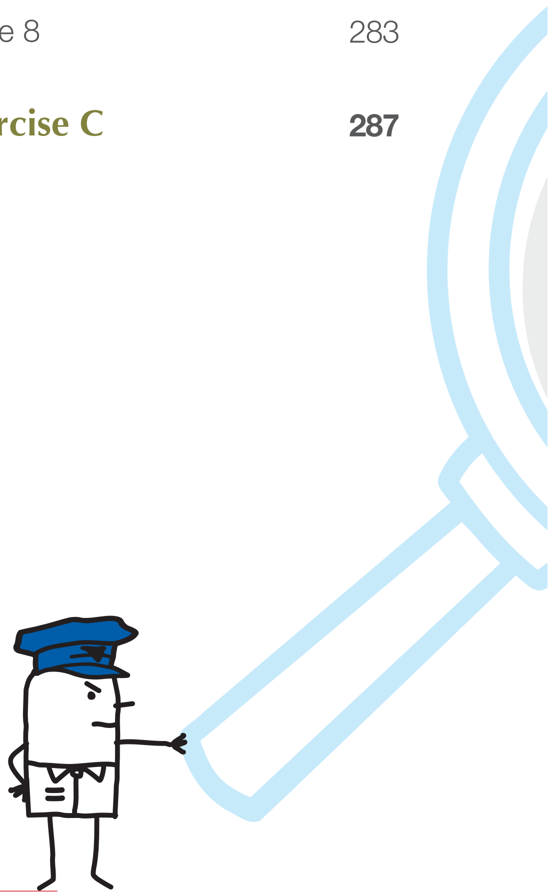
9.1 Congruence Tests 293

9.2 Similarity Tests 309

9.3 Applications of Congruent and Similar Triangles 324

Summary 328

Review Exercise 9 330



CHAPTER 10

Area and Volume of Similar Figures and Solids 335

10.1 Area of Similar Figures 337

10.2 Volume of Similar Solids 346

Summary 355

Review Exercise 10 355

CHAPTER 11

Geometrical Properties of Circles 359

11.1 Symmetric Properties of Circles 361

11.2 Angle Properties of Circles 380

Summary 401

Review Exercise 11 403

Revision Exercise D 407

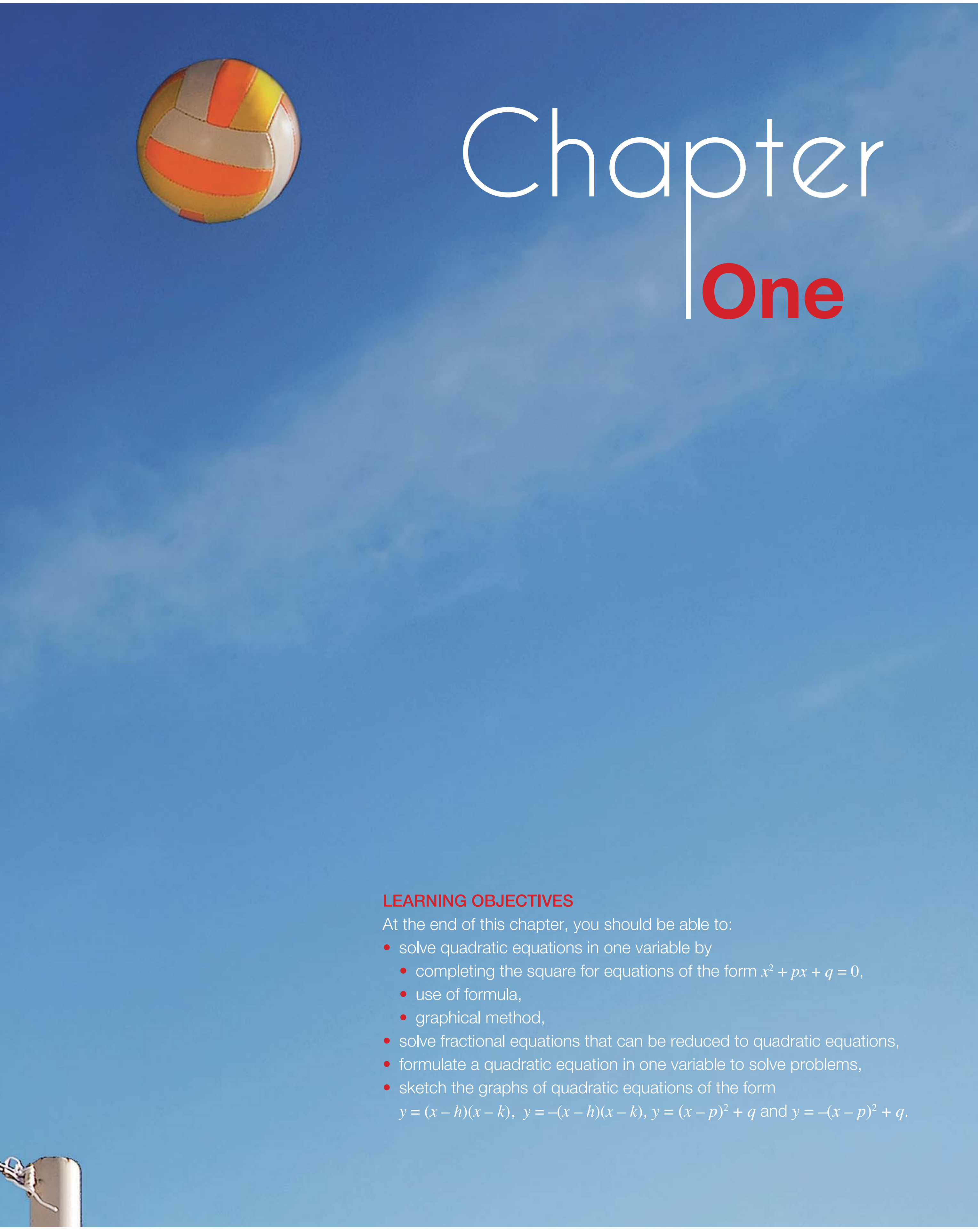
Problems in Real-World Contexts 411

Practise Now Answers 417

Answers 421

Quadratic Equations and Functions

A ball is thrown over the net. What do you notice about the path of the ball? The path of the ball can be described by a quadratic function. We can use the formula $h = ut + \frac{1}{2}at^2$ to find the height of the ball t seconds after leaving the hand. This idea is used in the study of mechanics.



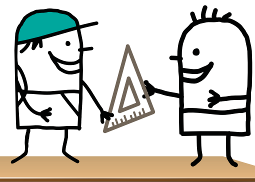
Chapter One

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- solve quadratic equations in one variable by
    - completing the square for equations of the form $x^2 + px + q = 0$,
    - use of formula,
    - graphical method,
- solve fractional equations that can be reduced to quadratic equations,
- formulate a quadratic equation in one variable to solve problems,
- sketch the graphs of quadratic equations of the form $y = (x - h)(x - k)$, $y = -(x - h)(x - k)$, $y = (x - p)^2 + q$ and $y = -(x - p)^2 + q$.

1.1 Solving Quadratic Equations by Completing the Square



Recap

In Book 2, we have learnt that a quadratic equation is of the form

$ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$.

Worked Example 1

(Solving a Quadratic Equation by Factorisation)

Solve the equation $x^2 - 5x - 6 = 0$.

Solution:

$x^2 - 5x - 6 = 0$

$(x - 6)(x + 1) = 0$ (factorise by using the multiplication frame)

$x - 6 = 0$ or $x + 1 = 0$

$x = 6$ $\quad$ $x = -1$

$\therefore x = 6$ or $x = -1$

RECALL

If two factors P and Q are such that $P \times Q = 0$, then either $P = 0$ or $Q = 0$ or both P and Q are equal to 0.

PRACTISE NOW 1

Solve each of the following equations.

(a) $x^2 + 7x - 8 = 0$

(b) $6y^2 + 7y - 20 = 0$

SIMILAR QUESTIONS

Exercise 1A Questions 1(a)-(f)

Solving Quadratic Equations of the Form $(x + a)^2 = b$

In Worked Example 1, we solved the equation by factorisation. However, the solutions of some quadratic equations cannot be obtained by factorisation. An example of this type of quadratic equation is $x^2 + 6x - 5 = 0$. If this equation can be written in the form $(x + a)^2 = b$, where a and b are real numbers, then it can be solved easily by taking the square roots on both sides of the equation to obtain the solutions.

Worked Example 2

(Solving a Quadratic Equation of the Form $(x + a)^2 = b$)

Solve the equation $(x + 3)^2 = 14$.

Solution:

$(x + 3)^2 = 14$

$x + 3 = \pm\sqrt{14}$ (take the square roots on both sides)

$x + 3 = \sqrt{14}$ or $x + 3 = -\sqrt{14}$

$x = \sqrt{14} - 3$ $\qquad$ $x = -\sqrt{14} - 3$

$= 0.742$ (to 3 s.f.) $\qquad$ $= -6.74$ (to 3 s.f.)

$\therefore x = 0.742$ or $x = -6.74$

PRACTISE NOW 2

Solve each of the following equations.

(a) $(x + 7)^2 = 100$ **(b)** $(y - 5)^2 = 11$

SIMILAR QUESTIONS

Exercise 1A Questions 2(a)-(h)

Completing the Square for a Quadratic Expression

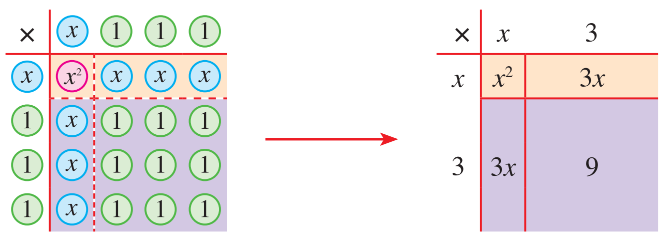
To express a quadratic equation of the form $x^2 + px + q = 0$ in the form $(x + a)^2 = b$, we first need to learn how to complete the square for a quadratic expression $x^2 + px$.

Let us consider the expansion of $(x + 3)^2$.

As shown in Book 2, we can use algebra discs to represent the expansion $(x + 3)^2 = x^2 + 6x + 9$ in the form of a square array (equal rows and columns of discs) as shown in Fig. 1.1**(a)** or a multiplication frame as shown in Fig. 1.1**(b)**.

We observe that in the square array,

- the x^2 disc is in the top left-hand corner,
- the nine 1 discs are arranged as a 3 by 3 square at the bottom right-hand corner,
- the six x discs are divided equally into 2 parts, i.e. $6x$ is divided into 2 parts of $3x$.



(a) (b)

Fig 1.1

Quadratic expressions of the form $(x + a)^2$ can be arranged into a multiplication frame similar to the example in Fig. 1.1.

However, not all quadratic expressions can be expressed in the form $(x + a)^2$. For example, the expression $x^2 + 6x$ can only be arranged as shown in Fig. 1.2.

×	x	3
x	x^2	$3x$
3	$3x$	

Fig. 1.2

Comparing Fig. 1.1 and Fig. 1.2, what number must be added to complete the square? We observe that 9 must be added to $x^2 + 6x$ to make it into $(x + 3)^2$.

However, $x^2 + 6x \neq (x + 3)^2$.

×	x	3
x	x^2	$3x$
3	$3x$	

$\neq$

×	x	3
x	x^2	$3x$
3	$3x$	9

Since we add 9 to $x^2 + 6x$, we must subtract 9 as follows:

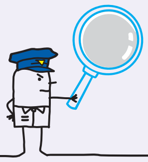
$$\begin{aligned} x^2 + 6x &= x^2 + 6x + 9 - 9 \\ &= (x + 3)^2 - 9 \end{aligned}$$

Pictorially, it looks like

×	x	3		
x	x^2	$3x$		
3	$3x$	9		-9

ATTENTION

Essentially, we add $9 - 9 = 0$ to $x^2 + 6x$ so that the equality will still hold.



Investigation

Completing the Square for Quadratic Expressions of the Form $x^2 + px$

To make a quadratic expression of the form $x^2 + px$ into a perfect square $(x + a)^2$, we have to add a number, b, to $x^2 + px$. In this investigation, we will find a relationship between b and p.

Copy and complete Table 1.1. The second one has been done for you.

	Quadratic Expression $x^2 + px$	**Number that must be added to complete the square,** b	$\frac{1}{2}$ × **coefficient of** x, $\frac{p}{2}$	**Quadratic expression of the form** $(x + a)^2 - b$
(a)	$x^2 + 4x$			
(b)	$x^2 + 6x$ $\begin{array}{c\|cc} \times & x & 3 \\ \hline x & x^2 & 3x \\ 3 & 3x & \end{array}$	$3^2 = 9$	$\frac{6}{2} = 3$	$x^2 + 6x$ $= x^2 + 6x + 3^2 - 3^2$ $= (x + 3)^2 - 9$ $\begin{array}{c\|cc} \times & x & 3 \\ \hline x & x^2 & 3x \\ 3 & 3x & 9 \end{array} \quad \boxed{-9}$
(c)	$x^2 + 8x$			
(d)	$x^2 + 10x$			

Table 1.1

1. What is the relationship between b and p?
2. To express $x^2 + px$ in the form $(x + a)^2 - b$, write down an expression of a and of b in terms of p.

From the investigation, on **completing the square**,

> if $x^2 + px = (x + a)^2 - b$, then $a = \frac{p}{2}$ and $b = \left(\frac{p}{2}\right)^2$,
>
> i.e. $x^2 + px = \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2$.

For a quadratic expression of the form $x^2 + px + q$, we can express it as follows:

$$\begin{aligned} x^2 + px + q &= (x^2 + px) + q \\ &= \left(x + \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 + q \end{aligned}$$

Worked Example 3

(Completing the Square for Quadratic Expressions)

Express each of the following expressions in the form $(x + a)^2 + b$.

(a) $x^2 + 10x$ **(b)** $x^2 - 5x$

(c) $x^2 + 2x + 3$

Solution:

(a) The coefficient of x is 10. Half of this is 5.

$$\begin{aligned} \therefore\ x^2 + 10x &= [x^2 + 10x + 5^2] - 5^2 \\ &= (x + 5)^2 - 25 \end{aligned}$$

(b) The coefficient of x is -5. Half of this is $-\frac{5}{2}$.

$$\begin{aligned} \therefore\ x^2 - 5x &= \left[x^2 - 5x + \left(-\frac{5}{2}\right)^2\right] - \left(-\frac{5}{2}\right)^2 \\ &= \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} \end{aligned}$$

(c) $x^2 + 2x + 3 = (x^2 + 2x) + 3$

The coefficient of x is 2. Half of this is 1.

$$\begin{aligned} x^2 + 2x + 3 &= [x^2 + 2x + 1^2] - 1^2 + 3 \\ &= (x + 1)^2 + 2 \end{aligned}$$

PRACTISE NOW 3

Express each of the following expressions in the form $(x + a)^2 + b$.

(a) $x^2 + 20x$ **(b)** $x^2 - 7x$

(c) $x^2 + \frac{1}{5}x$ **(d)** $x^2 + 6x - 9$

SIMILAR QUESTIONS

Exercise 1A Questions 3(a)-(h)

In Worked Example 4, we will show how to solve a quadratic equation by completing the square.

Worked Example 4

(Solving a Quadratic Equation by Completing the Square)

Solve the equation $x^2 + 4x - 3 = 0$, giving your answers correct to 2 decimal places.

Solution:

As $x^2 + 4x - 3$ cannot be easily factorised, we need to transform the equation $x^2 + 4x - 3 = 0$ into the form $(x + a)^2 = b$ as follows:

$$x^2 + 4x - 3 = 0$$

$$x^2 + 4x = 3$$ (rewrite the equation such that the constant term is on the RHS of the equation)

$$x^2 + 4x + \left(\frac{4}{2}\right)^2 = 3 + \left(\frac{4}{2}\right)^2$$ (add $\left(\frac{4}{2}\right)^2$ to both sides of the equation to complete the square for the LHS)

$$x^2 + 4x + 2^2 = 3 + 2^2$$

$$(x + 2)^2 = 7$$ (factorise the expression on the LHS and simplify the RHS)

$$x + 2 = \pm\sqrt{7}$$ (take the square roots on both sides)

$$x + 2 = \sqrt{7} \quad \text{or} \quad x + 2 = -\sqrt{7}$$

$$x = \sqrt{7} - 2 \qquad x = -\sqrt{7} - 2$$

$$= 0.65 \text{ (to 2 d.p.)} \qquad = -4.65 \text{ (to 2 d.p.)}$$

$\therefore\ x = 0.65$ or $x = -4.65$

ATTENTION

Alternatively, we can complete the square by:

$$x^2 + 4x - 3 = 0$$

$$\left[x^2 + 4x + \left(\frac{4}{2}\right)^2\right] - \left(\frac{4}{2}\right)^2 - 3 = 0$$

$$\left(x + \frac{4}{2}\right)^2 - 4 - 3 = 0$$

$$(x + 2)^2 = 7$$

In general, the steps taken to solve a quadratic equation $x^2 + px + q = 0$, where p and q are real numbers, by completing the square are as follows:

$$\boxed{x^2 + px + q = 0}$$

$\downarrow$

$$\boxed{x^2 + px = -q}$$ (add $-q$ to both sides of the equation)

$\downarrow$

$$\boxed{x^2 + px + \left(\frac{p}{2}\right)^2 = -q + \left(\frac{p}{2}\right)^2}$$ (add $\left(\frac{p}{2}\right)^2$ to both sides of the equation)

$\downarrow$

$$\boxed{\left(x + \frac{p}{2}\right)^2 = \frac{p^2}{4} - q}$$ (factorise the expression on the LHS and simplify the RHS)

$\downarrow$

$$\boxed{x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q}}$$ (take the square roots on both sides)

$\downarrow$

$$\boxed{x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}}$$ (add $-\frac{p}{2}$ to both sides of the equation)

PRACTISE NOW 4

SIMILAR QUESTIONS

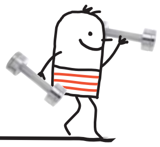
Exercise 1A Questions 4(a)-(h), 5(a)-(d), 6

1. Solve each of the following equations, giving your answers correct to 2 decimal places.

(a) $x^2 + 6x - 4 = 0$ **(b)** $x^2 + 7x + 5 = 0$

(c) $x^2 - x - 1 = 0$

2. Solve the equation $(x + 4)(x - 3) = 15$.



Exercise 1A

BASIC LEVEL

1. Solve each of the following equations.

(a) $2x^2 + 5x - 7 = 0$ **(b)** $4x^2 - 5x - 6 = 0$

(c) $7x + x^2 - 18 = 0$ **(d)** $4 - 3x - x^2 = 0$

(e) $x(3x - 1) = 2$ **(f)** $(7 - 3x)(x + 2) = 4$

2. Solve each of the following equations, giving your answers correct to 2 decimal places where necessary.

(a) $(x + 1)^2 = 9$ **(b)** $(2x + 1)^2 = 16$

(c) $(5x - 4)^2 = 81$ **(d)** $(7 - 3x)^2 = \frac{9}{16}$

(e) $(x + 3)^2 = 11$ **(f)** $(2x - 3)^2 = 23$

(g) $(5 - x)^2 = 7$ **(h)** $\left(\frac{1}{2} - x\right)^2 = 10$

INTERMEDIATE LEVEL

3. Express each of the following expressions in the form $(x + a)^2 + b$.

(a) $x^2 + 12x$ **(b)** $x^2 - 6x + 1$

(c) $x^2 + 3x - 2$ **(d)** $x^2 + 9x - 1$

(e) $x^2 + \frac{1}{2}x$ **(f)** $x^2 - \frac{2}{9}x$

(g) $x^2 + 0.2x$ **(h)** $x^2 - 1.4x$

4. Solve each of the following equations, giving your answers correct to 2 decimal places.

(a) $x^2 + 2x - 5 = 0$ **(b)** $x^2 + 17x - 30 = 0$

(c) $x^2 - 12x + 9 = 0$ **(d)** $x^2 - 5x - 5 = 0$

(e) $x^2 + \frac{1}{4}x - 3 = 0$ **(f)** $x^2 - \frac{6}{7}x + \frac{2}{49} = 0$

(g) $x^2 + 0.6x - 1 = 0$ **(h)** $x^2 - 4.8x + 2 = 0$

5. Solve each of the following equations.

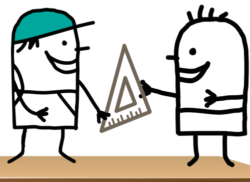
(a) $x(x - 3) = 5x + 1$ **(b)** $(x + 1)^2 = 7x$

(c) $(x + 2)(x - 5) = 4x$ **(d)** $x(x - 4) = 2(x + 7)$

ADVANCED LEVEL

6. Given the equation $y^2 - ay - 6 = 0$, where a is a constant, find the expressions for y in terms of a.

1.2 Solving Quadratic Equations by using Formula



The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$. Now, we shall use the method of completing the square to derive a formula for the solution to all quadratic equations.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{(divide throughout by } a\text{)}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{(rewrite the equation such that the constant term is on the RHS of the equation)}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{(add } \left(\frac{b}{2a}\right)^2 \text{ to both sides of the equation to make the LHS a perfect square)}$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{(factorise the expression on the LHS and simplify the RHS)}$$

$$= \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{(take the square roots on both sides)}$$

$$= \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

In general,

> if $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$, then
>
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The above formula for solving quadratic equations is usually used when the quadratic expression cannot be factorised easily.

Worked Example 5

(Solving a Quadratic Equation by using Formula)

Solve the equation $3x^2 + 4x - 5 = 0$.

Solution:

Comparing $3x^2 + 4x - 5 = 0$ with $ax^2 + bx + c = 0$, we have $a = 3$, $b = 4$ and $c = -5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(3)(-5)}}{2(3)}$$

$$= \frac{-4 \pm \sqrt{16 - (-60)}}{6}$$

$$= \frac{-4 \pm \sqrt{16 + 60}}{6}$$

$$= \frac{-4 \pm \sqrt{76}}{6}$$

$= 0.786$ (to 3 s.f.) or -2.12 (to 3 s.f.)

$\therefore$ $x = 0.786$ or $x = -2.12$

ATTENTION

Always ensure that the equation is in the form $ax^2 + bx + c = 0$ before substituting the values of a, b and c into the formula.

PRACTISE NOW 5

Solve each of the following equations.

(a) $2x^2 + 3x - 7 = 0$

(b) $5x^2 - 8x - 1 = 0$

(c) $(x - 1)^2 = 4x - 5$

(d) $(x + 3)(x - 1) = 8x - 7$

SIMILAR QUESTIONS

Exercise 1B Questions 1(a)-(f), 2(a)-(f), 3(a)-(f)

Class Discussion

Solutions to Quadratic Equations

Work in pairs.

Consider each of the following equations.

(a) $4x^2 - 12x + 9 = 0$

(b) $2x^2 + 5x + 8 = 0$

(c) $3x^2 + 5x - 4 = 0$

1. Find the value of $b^2 - 4ac$.

2. Use the quadratic formula to solve the equation. Are there any real solutions? Explain your answer.

3. What can you say about the sign of $b^2 - 4ac$ and the number of real solutions of a quadratic equation?

ATTENTION

The equation $x^2 - 6x + 9 = 0$ has only 1 real solution $x = 3$, i.e. 3 is the only real number that satisfies the equation.
The equation $x^2 + 9 = 0$ has no real solutions as there is no real number that satisfies the equation.

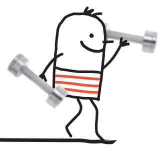
From the class discussion, we observe that

For a quadratic equation $ax^2 + bx + c = 0$,

- if $\boldsymbol{b^2 - 4ac > 0}$, the equation has *two real solutions*,
- if $\boldsymbol{b^2 - 4ac = 0}$, the equation has *one real solution*,
- if $\boldsymbol{b^2 - 4ac < 0}$, the equation has *no real solutions*.

SIMILAR QUESTIONS

Exercise 1B Questions 4(a)-(d)



Exercise 1B

BASIC LEVEL

1. Solve each of the following equations.

(a) $x^2 + 4x + 1 = 0$ **(b)** $3x^2 + 6x - 1 = 0$

(c) $2x^2 - 7x + 2 = 0$ **(d)** $3x^2 - 5x - 17 = 0$

(e) $-3x^2 - 7x + 9 = 0$ **(f)** $-5x^2 + 10x - 2 = 0$

2. Solve each of the following equations.

(a) $x^2 + 5x = 21$ **(b)** $10x^2 - 12x = 15$

(c) $8x^2 = 3x + 6$ **(d)** $4x^2 - 7 = 2x$

(e) $9 - 5x^2 = -3x$ **(f)** $16x - 61 = x^2$

INTERMEDIATE LEVEL

3. Solve each of the following equations.

(a) $x(x + 1) = 1$ **(b)** $3(x + 1)(x - 1) = 7x$

(c) $(x - 1)^2 - 2x = 0$ **(d)** $x(x - 5) = 7 - 2x$

(e) $(2x + 3)(x - 1) - x(x + 2) = 0$

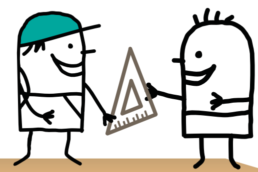
(f) $(4x - 3)^2 + (4x + 3)^2 = 25$

4. Solve each of the following equations if possible.

(a) $0.5(x^2 + 1) = x$ **(b)** $\frac{3}{4}x^2 + 2x - \frac{1}{2} = 0$

(c) $5x - 7 = x^2$ **(d)** $3x - 4 = (4x - 3)^2$

1.3 Solving Quadratic Equations by Graphical Method



In Sections 1.1 and 1.2, we have learnt how to solve quadratic equations by completing the square and by using the quadratic formula. Another method that can be used to find the solutions of the quadratic equation $ax^2 + bx + c = 0$ is by drawing the corresponding quadratic graph of $y = ax^2 + bx + c$ and to find the x-coordinates of the points of intersection of this graph with the x-axis ($y = 0$).

RECALL

When solving a pair of simultaneous equations

$y = ax^2 + bx + c$ --- (1)

and $y = 0$, --- (2)

we obtain the quadratic equation $ax^2 + bx + c = 0$.

Worked Example 6

(Solving a Quadratic Equation by Graphical Method)

The variables x and y are connected by the equation $y = 2x^2 - 5x - 6$.

(i) Complete the table for $y = 2x^2 - 5x - 6$.

x	-2	-1	0	1	2	3	4
y							

(ii) Draw the graph of $y = 2x^2 - 5x - 6$ for $-2 \leqslant x \leqslant 4$.

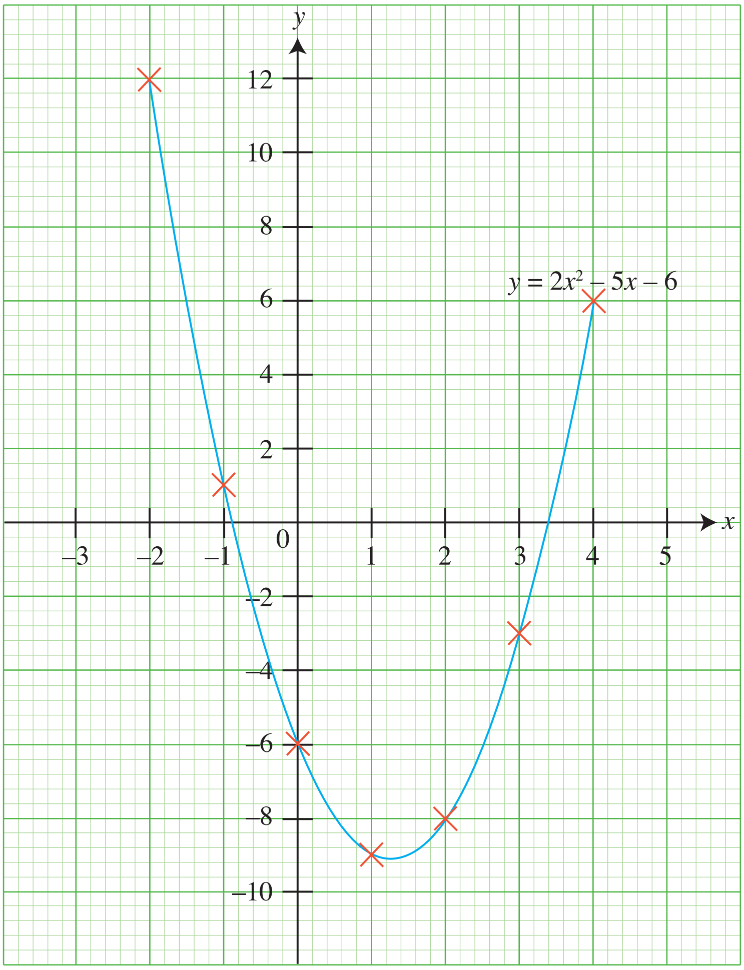
(iii) Hence, solve the equation $2x^2 - 5x - 6 = 0$.

Solution:

(i)

x	-2	-1	0	1	2	3	4
y	12	1	-6	-9	-8	-3	6

(ii)



(iii) From the graph, the x-coordinates of the points of intersection of $y = 2x^2 - 5x - 6$ and the x-axis (i.e. $y = 0$) are $x = -0.9$ and $x = 3.4$.

$\therefore$ The solutions of the equation $2x^2 - 5x - 6 = 0$ are $x = -0.9$ and $x = 3.4$.

RECALL

The solution of a pair of simultaneous linear equations is given by the coordinates of the *point of intersection* of the graphs of the two equations.

ATTENTION

For $2x^2 - 5x - 6 = 0$, the value of $b^2 - 4ac$ is $73 > 0$. Hence, there are two real solutions.

ATTENTION

The answers obtained by the graphical method can only be accurate up to half of a small square grid. In Worked Example 6, the solutions are accurate to the nearest 0.1.

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 1C Questions 1, 2, 4, 5, 8, 9

1. The variables x and y are connected by the equation $y = 2x^2 - 4x - 1$.
   (i) Complete the table for $y = 2x^2 - 4x - 1$.

x	-2	-1	0	1	2	3	4
y							

   (ii) Draw the graph of $y = 2x^2 - 4x - 1$ for $-2 \leqslant x \leqslant 4$.
   (iii) Hence, solve the equation $2x^2 - 4x - 1 = 0$.
2. By drawing the graph of $y = 7 - 4x - 3x^2$ for $-3 \leqslant x \leqslant 2$, solve the equation $7 - 4x - 3x^2 = 0$ graphically.

Worked Example 7

(Solving a Quadratic Equation by Graphical Method)

The variables x and y are connected by the equation $y = x^2 - 4x + 4$.

(i) Complete the table for $y = x^2 - 4x + 4$.

x	-1	0	1	2	3	4	5
y							

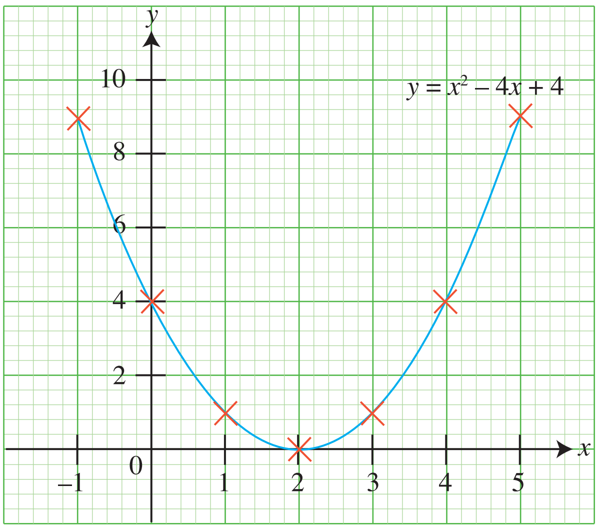
(ii) Draw the graph of $y = x^2 - 4x + 4$ for $-1 \leqslant x \leqslant 5$.
(iii) Hence, solve the equation $x^2 - 4x + 4 = 0$.

Solution:

(i)

x	-1	0	1	2	3	4	5
y	9	4	1	0	1	4	9

(ii)



ATTENTION

Equations such as $x^2 - 4x + 4 = 0$ are usually solved by factorisation as they can be easily factorised.

(iii) From the graph, the curve $y = x^2 - 4x + 4$ touches the x-axis (i.e. $y = 0$) at $x = 2$ only.

$\therefore$ The solution of the equation $x^2 - 4x + 4 = 0$ is $x = 2$.

ATTENTION

For $x^2 - 4x + 4 = 0$, the value of $b^2 - 4ac$ is 0. Hence, there is only one real solution.

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 1C Questions 3, 6, 7

1. The variables x and y are connected by the equation $y = x^2 - 6x + 9$.

(i) Complete the table for $y = x^2 - 6x + 9$.

x	-1	0	1	2	3	4	5	6
y								

(ii) Draw the graph of $y = x^2 - 6x + 9$ for $-1 \leqslant x \leqslant 6$.

(iii) Hence, solve the equation $x^2 - 6x + 9 = 0$.

2. By drawing the graph of $y = 8x - x^2 - 16$ for $0 \leqslant x \leqslant 8$, solve the equation $8x - x^2 - 16 = 0$ graphically.

Thinking Time

Draw the graph of $y = 2x^2 + 4x + 3$ for $-2 \leqslant x \leqslant 4$.

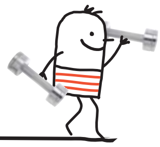
(i) State the number of points of intersection between the graph and the x-axis.
(ii) How many real solutions are there to the equation $2x^2 + 4x + 3 = 0$? Explain your answer.
(iii) Find the value of $b^2 - 4ac$ for the equation $2x^2 + 4x + 3 = 0$.
(iv) How does the value of $b^2 - 4ac$ obtained in **(iii)** relate to the number of points of intersection of the graph with the x-axis?

Journal Writing

To solve a quadratic equation, you have learnt the following 4 methods:

(i) Factorisation
(ii) Completing the square
(iii) Use of the quadratic formula
(iv) Graphical method

Write down the advantages and disadvantages of using each method. When solving a quadratic equation, how would you choose which method to use?



Exercise 1C

BASIC LEVEL

1. The variables x and y are connected by the equation $y = 2x^2 - 5x + 1$.

(i) Complete the table for $y = 2x^2 - 5x + 1$.

x	–1	0	1	2	3	4
y						

(ii) Draw the graph of $y = 2x^2 - 5x + 1$ for $-1 \leqslant x \leqslant 4$.

(iii) Hence, solve the equation $2x^2 - 5x + 1 = 0$.

2. The variables x and y are connected by the equation $y = 7 - 5x - 3x^2$.

(i) Complete the table for $y = 7 - 5x - 3x^2$.

x	–3	–2	–1	0	1	2
y						

(ii) Draw the graph of $y = 7 - 5x - 3x^2$ for $-3 \leqslant x \leqslant 2$.

(iii) Hence, solve the equation $7 - 5x - 3x^2 = 0$.

3. The variables x and y are connected by the equation $y = x^2 + 6x + 9$.

(i) Complete the table for $y = x^2 + 6x + 9$.

x	–5	–4	–3	–2	–1	0
y						

(ii) Draw the graph of $y = x^2 + 6x + 9$ for $-5 \leqslant x \leqslant 0$.

(iii) Hence, solve the equation $x^2 + 6x + 9 = 0$.

INTERMEDIATE LEVEL

4. **(i)** Draw the graph of $y = 3x^2 + 4x - 5$ for $-3 \leqslant x \leqslant 2$.

(ii) Hence, solve the equation $3x^2 + 4x - 5 = 0$ graphically.

5. By drawing the graph of $y = 5 - 2x - x^2$ for $-4 \leqslant x \leqslant 2$, solve the equation $5 - 2x - x^2 = 0$ graphically.

6. **(i)** Draw the graph of $y = 4x^2 + 12x + 9$ for $-4 \leqslant x \leqslant 2$.

(ii) Hence, solve the equation $4x^2 + 12x + 9 = 0$ graphically.

7. By drawing the graph of $y = 10x - 25 - x^2$ for $0 \leqslant x \leqslant 10$, solve the equation $10x - 25 - x^2 = 0$ graphically.

ADVANCED LEVEL

8. The profit, \$$P$ million, of a manufacturing company in its first 10 years of operation can be modelled by the equation $P = 2 - 0.1(x - 3)^2$, where x is the number of years of operation.

(a) Using a scale of 1 cm to represent 1 year, draw a horizontal x-axis for $0 \leqslant x \leqslant 10$. Using a scale of 2 cm to represent \$1 million, draw a vertical P-axis for $-4 \leqslant P \leqslant 3$. On your axes, plot the points given in the table and join them with a smooth curve.

(b) Use your graph to find the value of x when the profit of the company is zero.

9. During an annual carnival, participants are each expected to throw a balloon filled with water from the top of a platform onto a sandpit. Points are allocated based on the horizontal distance from the foot of the platform to where the balloon lands. Huixian throws a balloon. During the flight, its height above ground level, y cm, is represented by the equation $y = 200 + 7x - 6x^2$, where x is the horizontal distance, in metres, from the foot of the platform.

The table shows some values of x and the corresponding values of y.

x	0	1	2	3	4	5	6
y	200	201	190	167	132	85	26

(a) Using a scale of 1 cm to represent 1 m, draw a horizontal x-axis for $0 \leqslant x \leqslant 6$.

Using a scale of 2 cm to represent 50 cm, draw a vertical y-axis for $0 \leqslant y \leqslant 250$.

On your axes, plot the points given in the table and join them with a smooth curve.

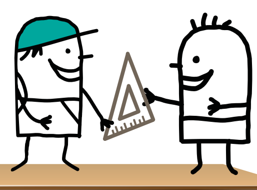
(b) Use your graph to find

(i) the positive solution of $200 + 7x - 6x^2 = 0$,

(ii) the horizontal distance from the foot of the platform when the balloon is 50 cm above the ground.

(c) Given that the flight of the balloon above ground level can only be modelled by the equation $y = 200 + 7x - 6x^2$ for $0 \leqslant x \leqslant t$, state the value of t. Explain your answer.

1.4 Solving Fractional Equations that can be reduced to Quadratic Equations



In Book 2, we have learnt that algebraic fractions are of the form $\frac{A}{B}$, where A and/or B are algebraic expressions, and $B \neq 0$. Equations that have one or more algebraic fractions are known as fractional equations. Examples of **fractional equations** are $\frac{2}{x+2} = 5x - 1$ and $\frac{3}{x+2} + \frac{x-1}{x-5} = 2$.

In this section, we will learn how to solve fractional equations that can be reduced to quadratic equations.

Worked Example 8

(Solving a Fractional Equation by reducing it to a Quadratic Equation)

Solve the equation $\frac{2}{x+2} = 5x - 1$.

Solution:

$$\frac{2}{x+2} = 5x - 1$$

$$\frac{2}{x+2} \times (x+2) = (5x-1) \times (x+2) \text{ (multiply both sides by } (x+2))$$

$$2 = (5x-1)(x+2)$$

$$2 = 5x^2 + 10x - x - 2$$

$$2 = 5x^2 + 9x - 2$$

$$0 = 5x^2 + 9x - 4$$

$$5x^2 + 9x - 4 = 0 \text{ (rewrite the equation in the form } ax^2 + bx + c = 0)$$

ATTENTION

In an equation, if $x = y$, then $y = x$. Hence, $0 = 5x^2 + 9x - 4$ is equivalent to $5x^2 + 9x - 4 = 0$.

Comparing $5x^2 + 9x - 4 = 0$ with $ax^2 + bx + c = 0$, we have $a = 5$, $b = 9$ and $c = -4$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-9 \pm \sqrt{9^2 - 4(5)(-4)}}{2(5)}$$

$$= \frac{-9 \pm \sqrt{81 - (-80)}}{10}$$

$$= \frac{-9 \pm \sqrt{81 + 80}}{10}$$

$$= \frac{-9 \pm \sqrt{161}}{10}$$

$= 0.369$ (to 3 s.f.) or -2.17 (to 3 s.f.)

$\therefore x = 0.369$ or $x = -2.17$

PRACTISE NOW 8

1. Solve each of the following equations.

(a) $\frac{6}{x+4} = x + 3$ **(b)** $\frac{3}{x+2} = 3x - 1$

2. Solve the equation $\frac{4}{x} = 2x - 3$.

SIMILAR QUESTIONS

Exercise 1D Questions 1(a)-(f), 3(a),(b), 4, 14(a)

Worked Example 9

(Solving a Fractional Equation by reducing it to a Quadratic Equation)

Solve the equation $\frac{3}{x+2} + \frac{x-1}{x-5} = 2$.

Solution:

$$\frac{3}{x+2} + \frac{x-1}{x-5} = 2$$

$$\left[\frac{3}{x+2} + \frac{x-1}{x-5}\right] \times (x+2)(x-5) = 2 \times (x+2)(x-5) \text{ (multiply both sides by } (x+2)(x-5))$$

$$\frac{3}{\cancel{x+2}} \times (\cancel{x+2})(x-5) + \frac{x-1}{\cancel{x-5}} \times (x+2)(\cancel{x-5}) = 2(x+2)(x-5)$$

$$3(x-5) + (x-1)(x+2) = 2(x+2)(x-5)$$

$$3x - 15 + x^2 + 2x - x - 2 = 2(x^2 - 5x + 2x - 10)$$

$$x^2 + 4x - 17 = 2(x^2 - 3x - 10)$$

$$x^2 + 4x - 17 = 2x^2 - 6x - 20$$

$$0 = x^2 - 10x - 3$$

$$x^2 - 10x - 3 = 0 \text{ (rewrite the equation in the form } ax^2 + bx + c = 0)$$

ATTENTION

The LCM of the denominators of $\frac{3}{x+2}$ and $\frac{x-1}{x-5}$ is $(x+2)(x-5)$.

Comparing $x^2 - 10x - 3 = 0$ with $ax^2 + bx + c = 0$, we have $a = 1$, $b = -10$ and $c = -3$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{100 - (-12)}}{2}$$

$$= \frac{10 \pm \sqrt{100 + 12}}{2}$$

$$= \frac{10 \pm \sqrt{112}}{2}$$

$= 10.3$ (to 3 s.f.) or -0.292 (to 3 s.f.)

$\therefore\ x = 10.3$ or $x = -0.292$

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 1D Questions 5(a)-(h), 14(b)-(d)

1. Solve each of the following equations.

(a) $\dfrac{1}{x-2}+\dfrac{2}{x-3}=5$ **(b)** $\dfrac{5}{x-3}-\dfrac{x-1}{x-2}=7$

2. Solve the equation $\dfrac{3}{x-2}-\dfrac{1}{(x-2)^2}=2$.

Thinking Time

Lixin is given the following fractional equation to solve:

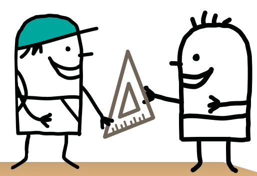
$$\frac{7}{x-3}-\frac{4}{x}=\frac{21}{x(x-3)}$$

Her working is as shown:

$$\begin{aligned}\frac{7}{x-3}\times x(x-3)-\frac{4}{x}\times x(x-3)&=\frac{21}{x(x-3)}\times x(x-3)\\ 7x-4(x-3)&=21\\ 7x-4x+12&=21\\ 3x&=9\\ x&=3\end{aligned}$$

Verify if the solution $x = 3$ is valid. Explain your answer.

1.5 Applications of Quadratic Equations in Real-World Contexts

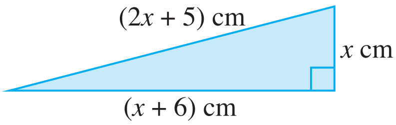


In order for quadratic equations to be applied to solve problems, we may have to formulate the quadratic equations first. Worked Examples 10 and 11 illustrate this.

Worked Example 10

(Application of Quadratic Equations in Real-World Contexts)

On a map, a piece of land is in the shape of a right-angled triangle with sides of length x cm, $(x + 6)$ cm and $(2x + 5)$ cm.



(i) From the information given, formulate an equation and show that it simplifies to $2x^2 + 8x - 11 = 0$.

(ii) Solve the equation $2x^2 + 8x - 11 = 0$, giving both answers correct to 3 decimal places.

(iii) Hence, find the perimeter of the triangle.

Solution:

(i) Using Pythagoras' Theorem,

$$x^2 + (x + 6)^2 = (2x + 5)^2$$
$$x^2 + x^2 + 12x + 36 = 4x^2 + 20x + 25$$
$$2x^2 + 8x - 11 = 0 \text{ (shown)}$$

(ii) Comparing $2x^2 + 8x - 11 = 0$ with $ax^2 + bx + c = 0$, we have $a = 2$, $b = 8$ and $c = -11$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-8 \pm \sqrt{8^2 - 4(2)(-11)}}{2(2)}$$
$$= \frac{-8 \pm \sqrt{64 - (-88)}}{4}$$
$$= \frac{-8 \pm \sqrt{64 + 88}}{4}$$
$$= \frac{-8 \pm \sqrt{152}}{4}$$
$$= 1.082 \text{ (to 3 d.p.) or } -5.082 \text{ (to 3 d.p.)}$$

(iii) Perimeter of the triangle $= x + (x + 6) + (2x + 5)$
$= x + x + 6 + 2x + 5$
$= 4x + 11$

Since the length of a triangle cannot be a negative value, $x = 1.082$.

$\therefore$ Perimeter of the triangle $= 4(1.082) + 11$
$= 15.3$ cm (to 3 s.f.)

Problem Solving Tip

Polya's 4-step Problem Solving Model

1. Understand the problem.
- A right-angled triangle with 3 sides is given in terms of x.

2. Devise a plan.
- Use Pythagoras' Theorem to formulate an equation in x.

3. Implement the plan.
- Simplify and solve the equation.

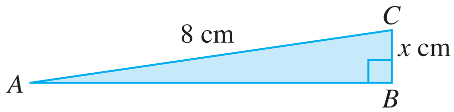
4. Check your answers.
- For answers in 3 significant figures, intermediate answers must be given to at least 4 significant figures.
- Since length is positive, the negative value of x is rejected.
- Use the value of x to compute the perimeter.

PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 1D Questions 6, 7

The figure shows a right-angled triangle ABC with dimensions as shown.



(i) If the perimeter of the triangle is 17 cm, write down an expression, in terms of x, for the length of AB.

(ii) Hence, formulate an equation in x and show that it simplifies to $2x^2 - 18x + 17 = 0$.

(iii) Solve the equation $2x^2 - 18x + 17 = 0$, giving both answers correct to 3 decimal places.

(iv) Hence, find the area of the triangle.

Worked Example 11

(Application of Quadratic Equations in Real-World Contexts)

A family decides to travel from Singapore to Kuala Lumpur, which are 315 km apart. The average speed of an aeroplane is 350 km/h more than the speed of a car. Let the average speed of the car be x km/h.

(i) Write down an expression, in terms of x, for the number of hours taken by the family if they choose to travel by aeroplane.

If they choose to travel by aeroplane instead of by car, they will be able to reach Kuala Lumpur 3 hours and 15 minutes earlier.

(ii) From the information given, formulate an equation in x and show that it reduces to

$$13x^2 + 4550x - 441\,000 = 0.$$

(iii) Solve the equation $13x^2 + 4550x - 441\,000 = 0$, giving both your answers correct to 2 decimal places.

(iv) Find the time taken by the family to travel by aeroplane, giving your answer correct to the nearest minute.

Problem Solving Tip

Polya's 4-step Problem Solving Model

1. Understand the problem.
- The distance and average speed of the aeroplane and the difference in travel time are given.
- What are the assumptions made? The distance travelled by the car and the aeroplane is exactly the same.

2. Devise a plan.
- Use the algebraic method and the relationship between speed, distance and time to formulate an equation in x.

3. Implement the plan.
- What is the time taken by the aeroplane? $\frac{315}{x+350}$ hours
- What is the time taken by the car? $\frac{315}{x}$ hours
- Which has a larger value?
- Hence, the difference is $\frac{315}{x} - \frac{315}{x+350} = 3\frac{1}{4}$, because 3 hours 15 minutes $= 3\frac{1}{4}$ hours.

4. Check your answers.
- For answers in 3 significant figures, intermediate answers must be given to at least 4 significant figures.
- Since speed is positive, the negative value of x is rejected.
- Use the value of x to compute the time taken to travel by aeroplane.

Solution:

(i) Number of hours taken by the family to travel by aeroplane = $\dfrac{315}{x+350}$

(ii)

$$\frac{315}{x} - \frac{315}{x+350} = 3\frac{15}{60}$$

$$\frac{315}{x} - \frac{315}{x+350} = 3\frac{1}{4}$$

$$\frac{315}{x} - \frac{315}{x+350} = \frac{13}{4}$$

$$\frac{315}{\cancel{x}} \times 4\cancel{x}(x+350) - \frac{315}{\cancel{x+350}} \times 4x\cancel{(x+350)} = \frac{13}{\cancel{4}} \times \cancel{4}x(x+350) \quad \text{(multiply both sides by } 4x(x+350)\text{)}$$

$$1260(x+350) - 1260x = 13x(x+350)$$

$$1260x + 441\,000 - 1260x = 13x^2 + 4550x$$

$$0 = 13x^2 + 4550x - 441\,000$$

$$13x^2 + 4550x - 441\,000 = 0 \text{ (shown)}$$

ATTENTION

Convert 3 hours 15 minutes into hours.

(iii) Comparing $13x^2 + 4550x - 441\,000 = 0$ with $ax^2 + bx + c = 0$, we have $a = 13$, $b = 4550$ and $c = -441\,000$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4550 \pm \sqrt{4550^2 - 4(13)(-441\,000)}}{2(13)}$$

$$= \frac{-4550 \pm \sqrt{43\,634\,500}}{26}$$

$= 79.06$ (to 2 d.p.) or -429.06 (to 2 d.p.)

$\therefore$ $x = 79.06$ or $x = -429.06$

(iv) Since the speed of the car cannot be a negative value, $x = 79.06$.

$\therefore$ Number of hours taken by the family to travel by aeroplane $= \dfrac{315}{79.06 + 350}$

$= 0.7342$ hours

$= 0.7342 \times 60$

$= 44$ minutes (to the nearest minute)

PRACTISE NOW 11

SIMILAR QUESTIONS

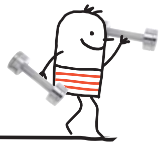
Exercise 1D Questions 2, 8-13, 15

Mr Lee drove from City P to City Q, which are 600 km apart. During his return journey, his average speed was increased by 7 km/h and the time taken was 15 minutes less.

(i) If he drove at an average speed of x km/h on his journey from City P to City Q, formulate an equation in x and show that it reduces to $x^2 + 7x - 16\,800 = 0$.

(ii) Solve the equation $x^2 + 7x - 16\,800 = 0$, giving both your answers correct to 2 decimal places.

(iii) Find the time taken for the return journey.



Exercise 1D

BASIC LEVEL

1. Solve each of the following equations.

(a) $\frac{8}{x} = 2x + 1$ **(b)** $3x - 1 = \frac{7}{x+4}$

(c) $\frac{x+1}{5-x} = x$ **(d)** $x + \frac{7}{x} = 9$

(e) $2x + 1 = \frac{x+1}{x-5}$ **(f)** $\frac{5x}{x+4} = 3x + 1$

2. The difference between two positive numbers, $\frac{12}{x+1}$ and $\frac{12}{x}$, is 1.

(i) Form an equation in x and show that it reduces to $x^2 + x - 12 = 0$.

(ii) Solve the equation $x^2 + x - 12 = 0$.

(iii) Hence, find the two numbers.

INTERMEDIATE LEVEL

3. Solve each of the following equations.

(a) $\frac{2}{x+1} = \frac{5x}{3-x}$

(b) $\frac{(x-2)(x-3)}{(x-1)(x+2)} = \frac{2}{3}$

4. Find the value(s) of x that satisfy the equation $\frac{x(x-3)}{(x+1)^2} = \frac{3}{5}$.

5. Solve each of the following equations.

(a) $\frac{x}{2} = \frac{4}{x} - 1$ **(b)** $\frac{2}{x+5} = 1 - \frac{x+1}{5}$

(c) $\frac{x-2}{5} + \frac{1}{2x-3} = 1$ **(d)** $\frac{3}{x} + \frac{2}{x+1} = 5$

(e) $\frac{1}{x+2} + \frac{1}{x-2} = \frac{3}{8}$ **(f)** $\frac{7}{x-1} - \frac{x+1}{x+3} = \frac{1}{2}$

(g) $\frac{5}{x-2} = 2 - \frac{4}{(x-2)^2}$ **(h)** $\frac{5}{x-1} + \frac{x}{(x-1)^2} = 1$

6. The perimeter of a rectangle is 112 cm and its breadth is x cm.

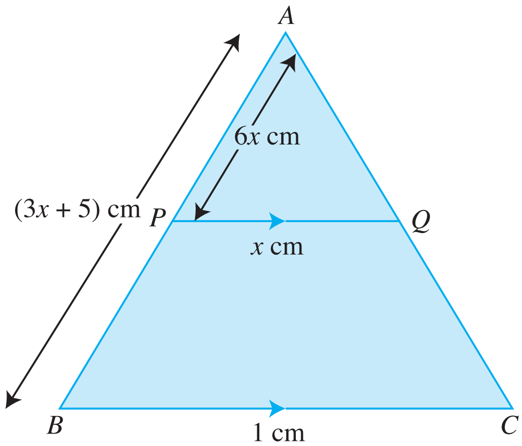
(i) Find, in terms of x, an expression for the length of the rectangle.

(ii) Given that the area of the rectangle is 597 cm^2, formulate an equation in x and show that it reduces to $x^2 - 56x + 597 = 0$.

(iii) Solve the equation $x^2 - 56x + 597 = 0$, giving both answers correct to 2 decimal places.

(iv) Hence, find the length of the diagonal of the rectangle.

7. The figure shows a triangle ABC in which $AP = 6x$ cm, $AB = (3x + 5)$ cm, $PQ = x$ cm and $BC = 1$ cm. P and Q are two points on the lines AB and AC respectively such that $\frac{AP}{AB} = \frac{PQ}{BC}$.



(i) Formulate an equation in x and show that it reduces to $3x^2 - x = 0$.

(ii) Solve the equation $3x^2 - x = 0$.

(iii) Find the length of PB.

8. There are 2 printers in a library. Printer A prints 60 pages every x minutes.

(i) Write down an expression, in terms of x, for the number of pages printed by Printer A in 1 minute.

(ii) Given that Printer B takes 2 minutes longer than Printer A to print 60 pages, write down an expression, in terms of x, for the number of pages printed by Printer B in 1 minute.

When both printers are in use, they are able to print a total of 144 pages in 1 minute.

(iii) Formulate an equation in x and show that it reduces to $6x^2 + 7x - 5 = 0$.

(iv) Solve the equation $6x^2 + 7x - 5 = 0$.

(v) Hence, find the time taken by Printer B to print 144 pages.

9. In January 2009, the price of rice in Singapore was \$$x$ per kilogram. A food catering company spent an average of \$350 on rice each month.

(i) Write down an expression, in terms of x, for the average amount of rice that this food catering company ordered in January 2009.

In January 2012, the price of each kilogram of rice had increased by 15 cents.

(ii) Given that the company continued to spend \$350 on rice each month, write down an expression, in terms of x, for the average amount of rice ordered in January 2012.

(iii) If the difference in the amount of rice ordered is 30 kg, formulate an equation in x and show that it reduces to $20x^2 + 3x - 35 = 0$.

(iv) Hence, find the price of each kilogram of rice in January 2012.

10. Rui Feng and Jun Wei represented their class in a 10 km race. They started running at the same speed of x km/h. After 2 km, Rui Feng increased his speed by 1 km/h and ran the remaining distance at a constant speed of $(x + 1)$ km/h. Jun Wei maintained his speed of x km/h throughout the race.

(i) Write down an expression, in terms of x, for the time taken by Rui Feng to complete the race.

(ii) Given that Rui Feng completed the race 40 minutes earlier than Jun Wei, formulate an equation in x and show that it reduces to $x^2 + x - 12 = 0$.

(iii) Solve the equation $x^2 + x - 12 = 0$. Explain why you reject one of the answers.

(iv) Hence, find the time taken by Rui Feng to complete the race, giving your answer in hours and minutes.

11. Amirah travels by coach from Singapore to Penang, which are 700 km apart, to visit her grandparents. She returns to Singapore by car at an average speed which is 30 km/h greater than that of a coach.

(i) If the average speed of the car is x km/h and the time taken for the whole journey is 20 hours, formulate an equation in x and show that it reduces to $x^2 - 100x + 1050 = 0$.

(ii) Solve the equation $x^2 - 100x + 1050 = 0$, giving both your answers correct to 2 decimal places.

(iii) Find the time taken for the return journey.

12. A tank, when full, can contain 1500 litres of water. Pump A can fill water into the tank at a rate of x litres per minute.

(i) Write down an expression, in terms of x, for the number of minutes taken by Pump A to fill the tank completely.

Pump B can fill water into the tank at a rate of $(x + 50)$ litres per minute.

(ii) Write down an expression, in terms of x, for the number of minutes taken by Pump B to fill the tank completely.

(iii) If Pump A takes 30 minutes longer than Pump B to fill the tank completely, formulate an equation in x to represent this information and show that it reduces to $x^2 + 50x - 150\,000 = 0$.

(iv) Solve the equation $x^2 + 50x - 150\,000 = 0$, giving both your answers correct to 2 decimal places.

(v) Find the time taken for Pump B to fill the tank completely, giving your answer in minutes and seconds, correct to the nearest second.

13. Two weeks before Nora went to New York for a holiday, she exchanged S\$2000 into US dollars (US\$) at Samy's Money Exchange at a rate of US\$1 = S\$$x$.

(i) Write down an expression, in terms of x, for the amount of US\$ she received from Samy's Money Exchange.

One week before her holiday, she exchanged another S\$1000 into US\$ at Chan's Money Exchange at a rate of US\$1 = S\$$(x + 0.05)$.

(ii) Write down an expression, in terms of x, for the amount of US\$ she received from Chan's Money Exchange.

(iii) If Nora received a total of US\$2370 from the two Money Exchanges, formulate an equation in x and show that it reduces to

$$237x^2 - 288.15x - 10 = 0.$$

(iv) Solve the equation $237x^2 - 288.15x - 10 = 0$, giving both your answers correct to 2 decimal places.

(v) Find the exchange rate between S\$ and US\$ offered by Chan's Money Exchange.

ADVANCED LEVEL

14. Solve each of the following equations.

(a) $\dfrac{4}{x-1} = \dfrac{x}{2x^2+3x-5}$

(b) $\dfrac{1}{x} + \dfrac{2}{x-1} + \dfrac{3}{x+1} = 0$

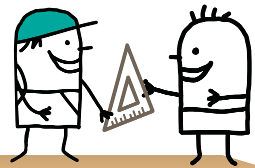
(c) $\dfrac{1}{x^2-9} - \dfrac{2}{3-x} = 1$

(d) $\dfrac{3}{x-3} + \dfrac{x+1}{x^2-5x+6} = 1$

15. During a test flight, an aircraft flies from Sandy Land to White City and back to Sandy Land. The distance between Sandy Land and White City is 450 km and the total time taken for the whole journey is 5 hours and 30 minutes. Given that there is a constant wind blowing from Sandy Land to White City and that the speed of the aircraft in still air is 165 km/h, find the speed of the wind. State the assumptions you have made to solve this problem.

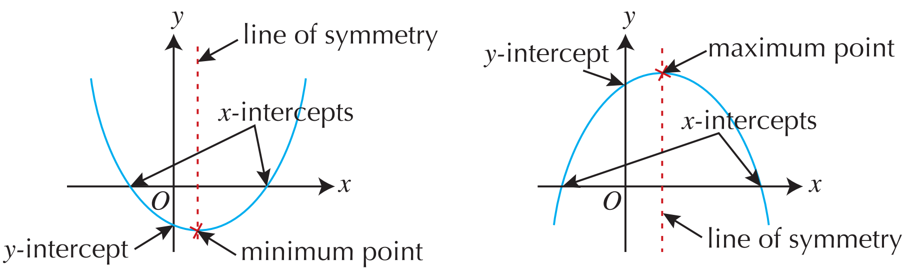
Hint: Let the speed of the wind be x km/h.

1.6 Graphs of Quadratic Functions



Recap

In Book 2, we have learnt how to draw the graphs of quadratic functions of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.



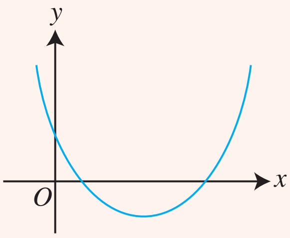
(a) $a > 0$ **(b)** $a < 0$

Fig. 1.3

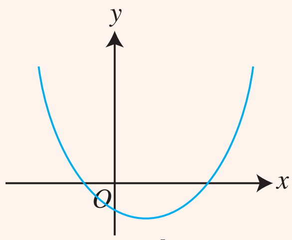
In this section, we will learn how to sketch the graphs of quadratic functions of the form $y = (x-h)(x-k)$ or $y = -(x-h)(x-k)$ and $y = (x-p)^2 + q$ or $y = -(x-p)^2 + q$.

Thinking Time

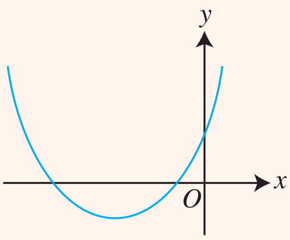
Determine which of the following sketches correspond to the quadratic functions $y = x^2 + 4x - 5$ and $y = -x^2 - 4x + 5$.



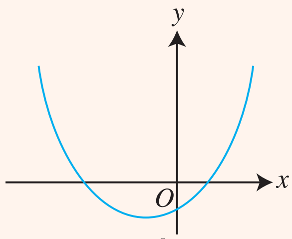
Graph 1



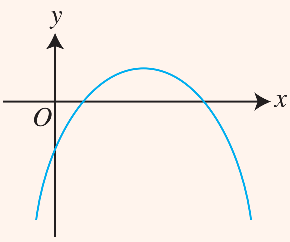
Graph 2



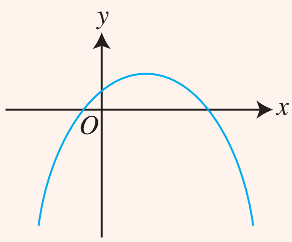
Graph 3



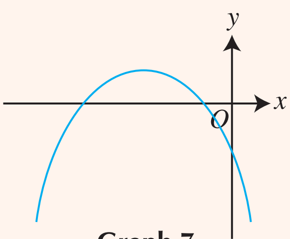
Graph 4



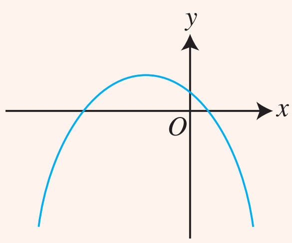
Graph 5



Graph 6

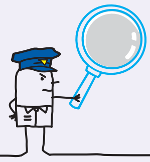


Graph 7



Graph 8

Graphs of the form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$



Investigation

Graphs of $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$

1. Use a graphing software to plot the graph of $y = (x - 3)(x - k)$ for $k = -2, -1, 0,$ 1 and 2.
2. Study the graphs and answer each of the following questions.
   (a) Does the graph open upwards or downwards?
   (b) Write down the coordinates of the point(s) where the graph cuts the x-axis, i.e. the x-intercepts.
   (c) Write down the coordinates of the point where the graph cuts the y-axis, i.e. the y-intercept.
   (d) What is the relationship between the x-intercepts and the line of symmetry?
   (e) State the equation of the line of symmetry of the graph.
   (f) Write down the coordinates of the maximum or the minimum point of the graph.
3. Repeat Steps 1 and 2 for $y = -(x - 3)(x - k)$, $y = (x - 5)(x - k)$ and $y = -(x - 5)(x - k)$.
4. By looking at the equation of each graph, how do you determine if it opens upwards or downwards?
5. By looking at the equation of each graph, how do you determine the coordinates of the points where the graph cuts the x-axis?
6. What can you say about the line of symmetry of each graph?

From the investigation, we observe that:

- For the equation $y = (x - h)(x - k)$, the graph opens upwards.
  The graph cuts the x-axis at $(h, 0)$ and $(k, 0)$. The graph is symmetrical about the vertical line that passes through the minimum point.
- For the equation $y = -(x - h)(x - k)$, the graph opens downwards.
  The graph cuts the x-axis at $(h, 0)$ and $(k, 0)$. The graph is symmetrical about the vertical line that passes through the maximum point.

Worked Example 12

(Sketching the Graph of $y = (x - h)(x - k)$)

Sketch the graph of $y = (x - 1)(x - 5)$.

Solution:

Since the coefficient of x^2 is 1, the graph opens upwards.

When $y = 0$,

$(x - 1)(x - 5) = 0$

$x - 1 = 0$ or $x - 5 = 0$

$x = 1$ $\quad\quad x = 5$

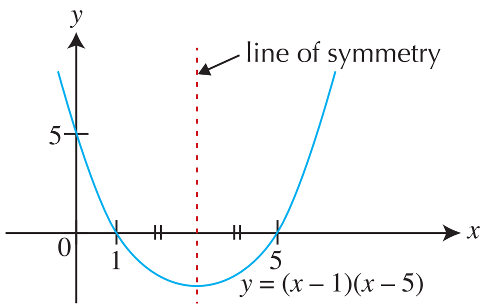
$\therefore$ The graph cuts the x-axis at (1, 0) and (5, 0).

When $x = 0$,

$y = (-1)(-5)$

$= 5$

$\therefore$ The graph cuts the y-axis at (0, 5).



Problem Solving Tip

Step 1: State the coefficient of x^2 to determine if the graph opens upwards or downwards.

Step 2: Obtain the x-intercepts by substituting $y = 0$ into the equation.

Step 3: Obtain the y-intercept by substituting $x = 0$ into the equation.

Step 4: Sketch the graph.

ATTENTION

The line of symmetry is halfway between the x-intercepts.

How do we find the x-coordinate of the midpoint? In Worked Example 12, it is $x = 3$.

With this information, how can we find the coordinates of the minimum point?

PRACTISE NOW 12

Sketch the graph of each of the following functions.

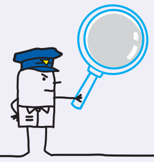
(a) $y = (x - 2)(x - 6)$ **(b)** $y = -(x - 3)(x + 1)$

(c) $y = (3 - x)(x + 5)$

SIMILAR QUESTIONS

Exercise 1E Questions 1(a)-(f), 3-6

Graphs of the form $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$



Investigation

Graphs of $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$

1. Use a graphing software to plot the graph of $y = (x - 2)^2 + q$ for $q = -4, -1, 0, 1$ and 4.
2. Study the graphs and answer each of the following questions.
   - **(a)** Does the graph open upwards or downwards?
   - **(b)** Write down the coordinates of the point(s) where the graph cuts the x-axis, i.e. the x-intercepts.
   - **(c)** Write down the coordinates of the point where the graph cuts the y-axis, i.e. the y-intercept.
   - **(d)** State the equation of the line of symmetry of the graph.
   - **(e)** Write down the coordinates of the maximum or the minimum point of the graph.
3. Repeat Steps 1 and 2 for $y = -(x - 2)^2 + q$, $y = (x + 3)^2 + q$ and $y = -(x + 3)^2 + q$.
4. By looking at the equation of each graph, how do you determine if it opens upwards or downwards?
5. By looking at the equation of each graph, how do you determine the coordinates of the maximum or the minimum point?
6. What can you say about the line of symmetry of each graph?

From the investigation, we observe that:

- For the equation $y = (x - p)^2 + q$, the graph opens upwards. The coordinates of the minimum point of the graph are (p, q) and the graph is symmetrical about the line $x = p$.
- For the equation $y = -(x - p)^2 + q$, the graph opens downwards. The coordinates of the maximum point of the graph are (p, q) and the graph is symmetrical about the line $x = p$.

INFORMATION

The graph of a quadratic function is a parabola. When it opens upwards, we say it is concave upwards.

Worked Example 13

(Sketching the Graph of $y = -(x - p)^2 + q$)

Given the quadratic function $y = -(x - 1)^2 + 4$,

(i) find the coordinates of the x- and y-intercepts,

(ii) write down the coordinates of the maximum point of the graph,

(iii) sketch the graph,

(iv) state the equation of the line of symmetry of the graph.

Solution:

(i) Since the coefficient of x^2 is -1, the graph opens downwards.

When $y = 0$,

$$-(x-1)^2 + 4 = 0$$
$$-(x-1)^2 = -4$$
$$(x-1)^2 = 4$$
$$x - 1 = 2 \quad \text{or} \quad x - 1 = -2$$
$$x = 3 \qquad\qquad x = -1$$

$\therefore$ The graph cuts the x-axis at (3, 0) and (–1, 0).

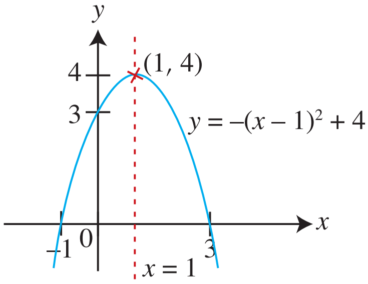
When $x = 0$,

$$y = -(-1)^2 + 4$$
$$= 3$$

$\therefore$ The graph cuts the y-axis at (0, 3).

(ii) The coordinates of the maximum point are (1, 4).

(iii)



(iv) The equation of the line of symmetry is $x = 1$.

Problem Solving Tip

Step 1: State the coefficient of x^2 to determine if the graph opens upwards or downwards.

Step 2: Since the equation is of the form $y = -(x - p)^2 + q$, the coordinates of the maximum point are (p, q).

Step 3: Obtain the x-intercepts by substituting $y = 0$ into the equation.

Step 4: Obtain the y-intercept by substituting $x = 0$ into the equation.

Step 5: Sketch the graph.

PRACTISE NOW 13

1. Given the quadratic function $y = -(x - 2)^2 + 9$,
- **(i)** find the coordinates of the x- and y-intercepts,
- **(ii)** write down the coordinates of the maximum point of the graph,
- **(iii)** sketch the graph,
- **(iv)** state the equation of the line of symmetry of the graph.

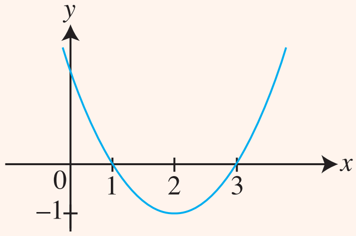
2. Given the quadratic function $y = (x + 1)^2 - 1$,
- **(i)** find the coordinates of the x- and y-intercepts,
- **(ii)** write down the coordinates of the minimum point of the graph,
- **(iii)** sketch the graph,
- **(iv)** state the equation of the line of symmetry of the graph.

SIMILAR QUESTIONS

Exercise 1E Questions 2(a)-(f), 9

Thinking Time

1. In Worked Example 13, we sketched the graph of $y = -(x - 1)^2 + 4$.
   Express $y = -(x - 1)^2 + 4$ in the factorised form $y = -(x - h)(x - k)$, and hence sketch the graph.
2. If we are given the following graph with x-intercepts at 1 and 3 and a minimum point at $(2, -1)$, can we express the equation of the curve in the form $y = (x - a)^2 + b$?



Worked Example 14

(Sketching the Graph of $y = ax^2 + bx + c$)

(i) Express $x^2 - 4x + 2$ in the form $(x - p)^2 + q$.

(ii) Write down the coordinates of the minimum point of the graph.

(iii) Hence, sketch the graph of $y = x^2 - 4x + 2$.

(iv) State the equation of the line of symmetry of the graph.

Solution:

(i) $x^2 - 4x + 2 = \left[x^2 - 4x + \left(-\frac{4}{2}\right)^2\right] - \left(-\frac{4}{2}\right)^2 + 2$

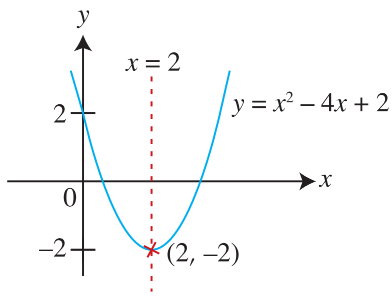
$= (x - 2)^2 - 2$

(ii) The coordinates of the minimum point are $(2, -2)$.

(iii) When $x = 0$,

$y = 2$

$\therefore$ The graph cuts the y-axis at $(0, 2)$.



(iv) The equation of the line of symmetry is $x = 2$.

Problem Solving Tip

Since the equation is of the form $y = (x - p)^2 + q$, the coordinates of the minimum point are (p, q).

PRACTISE NOW 14

SIMILAR QUESTIONS

Exercise 1E Questions 7, 8, 10

1. **(i)** Express $x^2 - 6x + 6$ in the form $(x - p)^2 + q$.
(ii) Write down the coordinates of the minimum point of the graph.
(iii) Hence, sketch the graph of $y = x^2 - 6x + 6$.
(iv) State the equation of the line of symmetry of the graph.

2. **(i)** Express $x^2 + x + 1$ in the form $(x + p)^2 + q$.
(ii) Write down the coordinates of the minimum point of the graph.
(iii) Hence, sketch the graph of $y = x^2 + x + 1$.
(iv) State the equation of the line of symmetry of the graph.

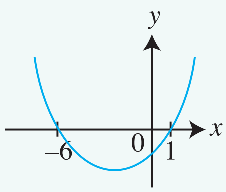
Class Discussion

Matching quadratic graphs with the corresponding functions

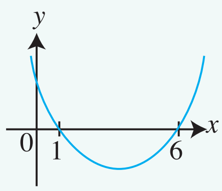
Work in pairs.

Match the graphs with their respective functions and justify your answers. If your classmate does not obtain the correct answer, explain to him what he has done wrong.

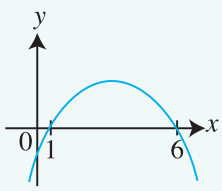
A: $y = -(x + 1)(x + 6)$	**B**: $y = (x + 1)(x + 6)$	**C**: $y = (x - 1)(x + 6)$	**D**: $y = -(x - 1)(x - 6)$
E: $y = x^2 - 7x + 6$	**F**: $y = -x^2 - 5x + 6$	**G**: $y = x^2 - 5x - 6$	**H**: $y = -x^2 + 5x + 6$



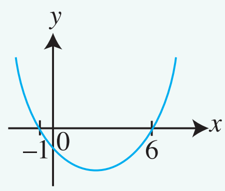
Graph 1



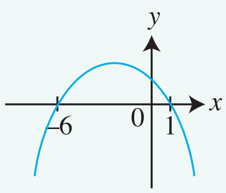
Graph 2



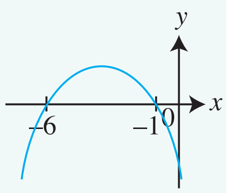
Graph 3



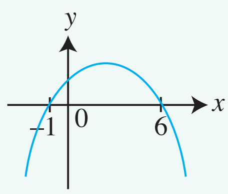
Graph 4



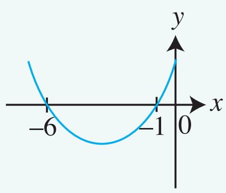
Graph 5



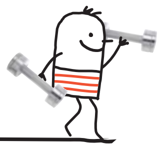
Graph 6



Graph 7



Graph 8



Exercise 1E

BASIC LEVEL

1. Sketch the graph of each of the following functions.

(a) $y = (x + 1)(x + 3)$ **(b)** $y = (x - 2)(x + 4)$

(c) $y = -(x + 1)(x - 5)$ **(d)** $y = -(x - 1)(x + 6)$

(e) $y = (3 - x)(x + 2)$ **(f)** $y = (2 - x)(4 - x)$

2. Sketch the graph of each of the following functions, stating the coordinates of the maximum or the minimum point and the equation of the line of symmetry.

(a) $y = x^2 + 2$ **(b)** $y = -x^2 - 6$

(c) $y = (x - 3)^2 + 1$ **(d)** $y = (x + 1)^2 - 3$

(e) $y = -(x + 2)^2 + 3$ **(f)** $y = -(x - 4)^2 - 1$

INTERMEDIATE LEVEL

3. **(i)** Factorise $x^2 + \frac{3}{4}x$.

(ii) Hence, sketch the graph of $y = x^2 + \frac{3}{4}x$.

4. Sketch the graph of $y = -(x^2 - x)$.

5. **(i)** Factorise $x^2 + x - 6$ completely.

(ii) Hence, sketch the graph of $y = x^2 + x - 6$.

6. Sketch the graph of $y = x^2 - 4x + 3$.

7. **(i)** Express $x^2 - 8x + 5$ in the form $(x - p)^2 + q$.

(ii) Hence, sketch the graph of $y = x^2 - 8x + 5$.

(iii) Write down the coordinates of the minimum point of the graph.

(iv) State the equation of the line of symmetry of the graph.

8. By first expressing $x^2 + 3x + 1$ in the form $(x + p)^2 + q$, sketch the graph of $y = x^2 + 3x + 1$. Write down the coordinates of the minimum point of the graph.

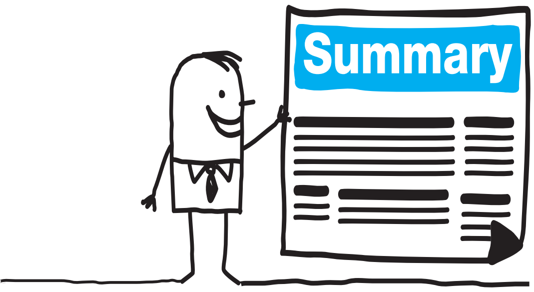
ADVANCED LEVEL

9. The graph of $y = (x - h)^2 + k$ has a minimum point at $\left(-\frac{1}{2}, \frac{3}{4}\right)$.

(i) State the value of h and of k.

(ii) Hence, sketch the graph of $y = (x - h)^2 + k$, indicating the coordinates of the point of intersection of the graph with the y-axis.

10. It is given that $-x^2 + 10x - 4$ can be expressed in the form $-(x - p)^2 + q$. By first finding the value of p and of q, sketch the graph of $y = -x^2 + 10x - 4$, indicating the coordinates of the maximum point of the graph.

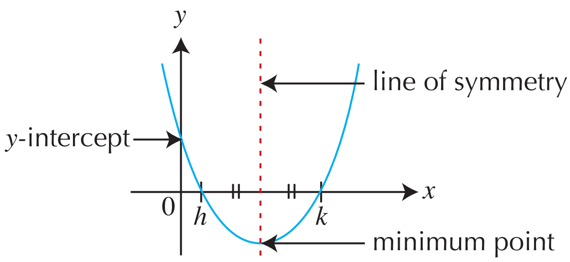


1. A quadratic equation in one variable can be solved by
   - completing the square,
   - using formula,
   - graphical method.

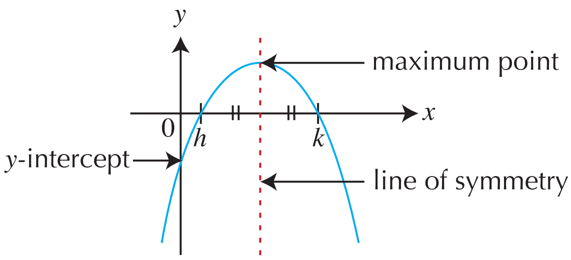
Completing the square	Using formula	Graphical method
To solve a quadratic equation in the form $x^2 + px + q = 0$ by **completing the square**: • Rewrite the equation such that the constant term is on the RHS of the equation, i.e. $x^2 + px = -q$. • Add $\left(\frac{p}{2}\right)^2$ to both sides of the equation to form $\left(x+\frac{p}{2}\right)^2 = \left(\frac{p}{2}\right)^2 - q$. • Take the square roots on both sides of the equation to solve for x.	The **formula** for solving a quadratic equation in the form $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Note that when $b^2 - 4ac < 0$, the equation has no real solutions.	To solve a quadratic equation in the form $ax^2 + bx + c = 0$ by **graphical method**: • Draw the graph of $y = ax^2 + bx + c$. • The solution(s) of the equation are given by the x-coordinate(s) of the point(s) of intersection of the graph with the x-axis.

2. Equations that have one or more algebraic fractions are known as **fractional equations**.
   To solve a fractional equation:
   - Multiply both sides of the equation by the LCM of the denominators.
   - Reduce it to a quadratic equation.
   - Solve the equation either by factorisation or by using the quadratic formula.

3. Graphs of quadratic functions of the form $y = (x - h)(x - k)$ or $y = -(x - h)(x - k)$



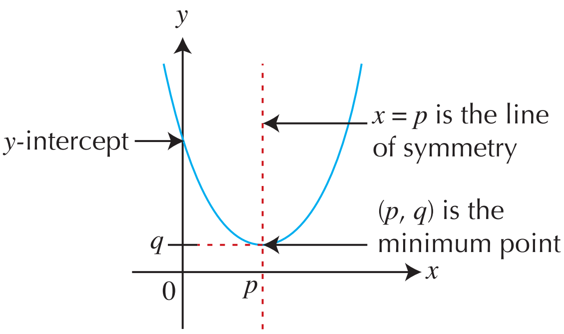
$y = (x - h)(x - k)$



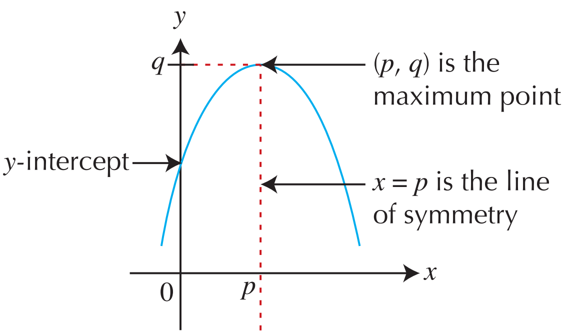
$y = -(x - h)(x - k)$

The line of symmetry passes through the midpoint of the x-intercepts.

4. Graphs of quadratic functions of the form $y = (x - p)^2 + q$ or $y = -(x - p)^2 + q$



$y = (x - p)^2 + q$



$y = -(x - p)^2 + q$

Review Exercise 1

1. Solve each of the following equations by completing the square.
   (a) $x^2 + 8x + 5 = 0$ (b) $x^2 + 7x - 3 = 0$
   (c) $x^2 - 11x - 7 = 0$ (d) $x^2 + 1.2x = 1$

2. By using the quadratic formula, solve each of the following equations.
   (a) $2x^2 + 6x + 1 = 0$ (b) $3x^2 - 7x - 2 = 0$
   (c) $-4x^2 + x + 5 = 0$ (d) $3x^2 = 5x + 1$

3. Solve each of the following equations.
   (a) $(x-3)^2 = \frac{4}{25}$ (b) $(4-x)^2 = 12$
   (c) $(x-1)(x+3) = 9$ (d) $x(x+4) = 17$

4. (i) Solve the equation $2x^2 - 7x + 4 = 0$, giving your answers correct to 2 decimal places.
   (ii) Hence, find the values of y that satisfy the equation $2(y-1)^2 - 7(y-1) + 4 = 0$.

5. Form a quadratic equation in the form $ax^2 + bx + c = 0$, where a, b and c are integers, given each of the following solutions.
   (a) $x = 2$, $x = \frac{6}{7}$ (b) $x = -\frac{1}{2}$, $x = -\frac{2}{3}$

6. Solve each of the following equations.
   (a) $x - 1 = \frac{5}{x+7}$ (b) $\frac{x-1}{x+4} = \frac{2x}{x-3}$
   (c) $\frac{1}{x} - 5x = 5$ (d) $\frac{5}{x} = 3 - \frac{x}{x-3}$
   (e) $\frac{2}{x+1} + \frac{1}{x-3} = 5$ (f) $\frac{x}{x+1} + \frac{1}{5} = \frac{3}{x-2}$
   (g) $\frac{5}{x-2} - \frac{3}{x^2-4} = \frac{2}{7}$
   (h) $\frac{1}{2x+1} + \frac{x+3}{2x^2-5x-3} = 2$

7. (i) Express $y = x^2 - 7x + 12$ in the form $y = (x-h)(x-k)$.
   (ii) Hence, sketch the graph of $y = x^2 - 7x + 12$.

8. (i) Express $y = -x^2 + 5x - 4$ in the form $y = -(x-p)^2 + q$.
   (ii) Hence, sketch the graph of $y = -x^2 + 5x - 4$.

9. The difference between the reciprocals of two consecutive positive integers is $\frac{1}{12}$. Find the two numbers.

10. In November 2013, the exchange rate between Australian dollars (A\$) and Singapore dollars (S\$) offered by a money changer was A\$100 = S\$$x$. In December 2013, the exchange rate offered was A\$100 = S\$(x – 5). Mr Neo found that, for every S\$650 he exchanged in December 2013, he would receive A\$20 more than if he exchanged in November 2013.
    (i) Formulate an equation in x.
    (ii) Hence, find the amount of Singapore dollars Mr Neo received if he exchanged A\$1250 in November 2013.

11. Farhan travelled by car from Town A to Town B, 40 km apart, at an average speed of x km/h.

(i) Write down an expression, in terms of x, for the time taken by Farhan to travel from Town A to Town B.

Khairul travelled by van from Town B to Town A at an average speed that was 30 km/h less than that of the car.

(ii) Write down an expression, in terms of x, for the time taken by Khairul to travel from Town B to Town A.

(iii) Given that Farhan took 10 minutes less than Khairul to complete the journey, form an equation in x and show that it reduces to $x^2 - 30x - 7200 = 0$.

(iv) Solve the equation $x^2 - 30x - 7200 = 0$, giving both your answers correct to 2 decimal places.

(v) Find the time taken by Khairul to travel from Town B to Town A.

12. In November 2013, the price of petrol was x cents per litre.

(i) Write down an expression, in terms of x, for the number of litres of petrol that could be bought with \$60 in November 2013.

In December 2013, the price had increased by 10 cents per litre.

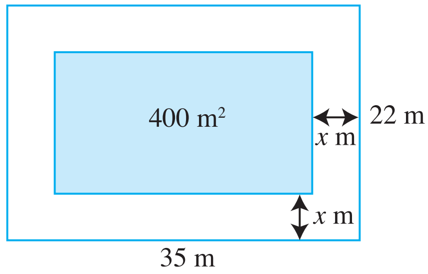
(ii) Write down an expression, in terms of x, for the number of litres of petrol that could be bought with \$60 in December 2013.

(iii) Given that an additional $1\frac{3}{7}$ litres of petrol could be bought in November 2013 than in December 2013, form an equation in x and show that it reduces to $x^2 + 10x - 42\,000 = 0$.

(iv) Solve the equation $x^2 + 10x - 42\,000 = 0$.

(v) Find the number of litres of petrol that could be bought with \$34 in December 2013.

13. A rectangular function room has dimensions 35 m by 22 m. Part of the floor is covered with ceramic tiles, as shown by the shaded rectangle in the figure.



(i) Given that the part of the floor which is not covered by the tiles has a uniform width of x m, write down an expression, in terms of x, for the length and the breadth of the floor covered by the tiles.

(ii) Given that the floor area covered by the tiles is 400 m^2, formulate an equation in x and show that it reduces to $2x^2 - 57x + 185 = 0$.

(iii) Solve the equation $2x^2 - 57x + 185 = 0$, giving both your answers correct to 2 decimal places.

(iv) State the width of the floor that is not covered by the tiles.

14. A stone was thrown from the top of a vertical tower into the sea. Its position during the flight is represented by the equation $y = 60 + 25x - x^2$, where y metres is the height of the stone above sea level and x metres is the horizontal distance from the foot of the tower.

(a) **(i)** Solve the equation $60 + 25x - x^2 = 0$, giving both your answers correct to 1 decimal place.

(ii) Explain briefly what the positive solution in **(a)(i)** represents.

(b) The table shows some values of x and the corresponding values of y.

x	0	2	4	6	8	10	12	14	16	18	20	22
y	60	106	144	174	196	210	216	214	204	186	160	126

Using a scale of 2 cm to represent 5 m, draw a horizontal x-axis for $0 \leqslant x \leqslant 22$.

Using a scale of 2 cm to represent 20 m, draw a vertical y-axis for $0 \leqslant y \leqslant 220$.

On your axes, plot the points given in the table and join them with a smooth curve.

(c) Use your graph to find

(i) the greatest height reached by the stone,

(ii) the horizontal distance from the foot of the tower when the stone is 180 m above sea level.

15. A water tank can be filled with water by two pipes in $11\frac{1}{9}$ minutes. If the smaller pipe takes 5 minutes longer than the larger pipe to fill the tank, find the time taken by each pipe to fill the tank.

16. A boat travels 12 km upstream and back in 1 hour and 30 minutes. Given that the speed of the current is 5 km/h, find the speed of the boat in still water.

Hint: Let the speed of the boat in still water be x km/h.

Challenge Yourself

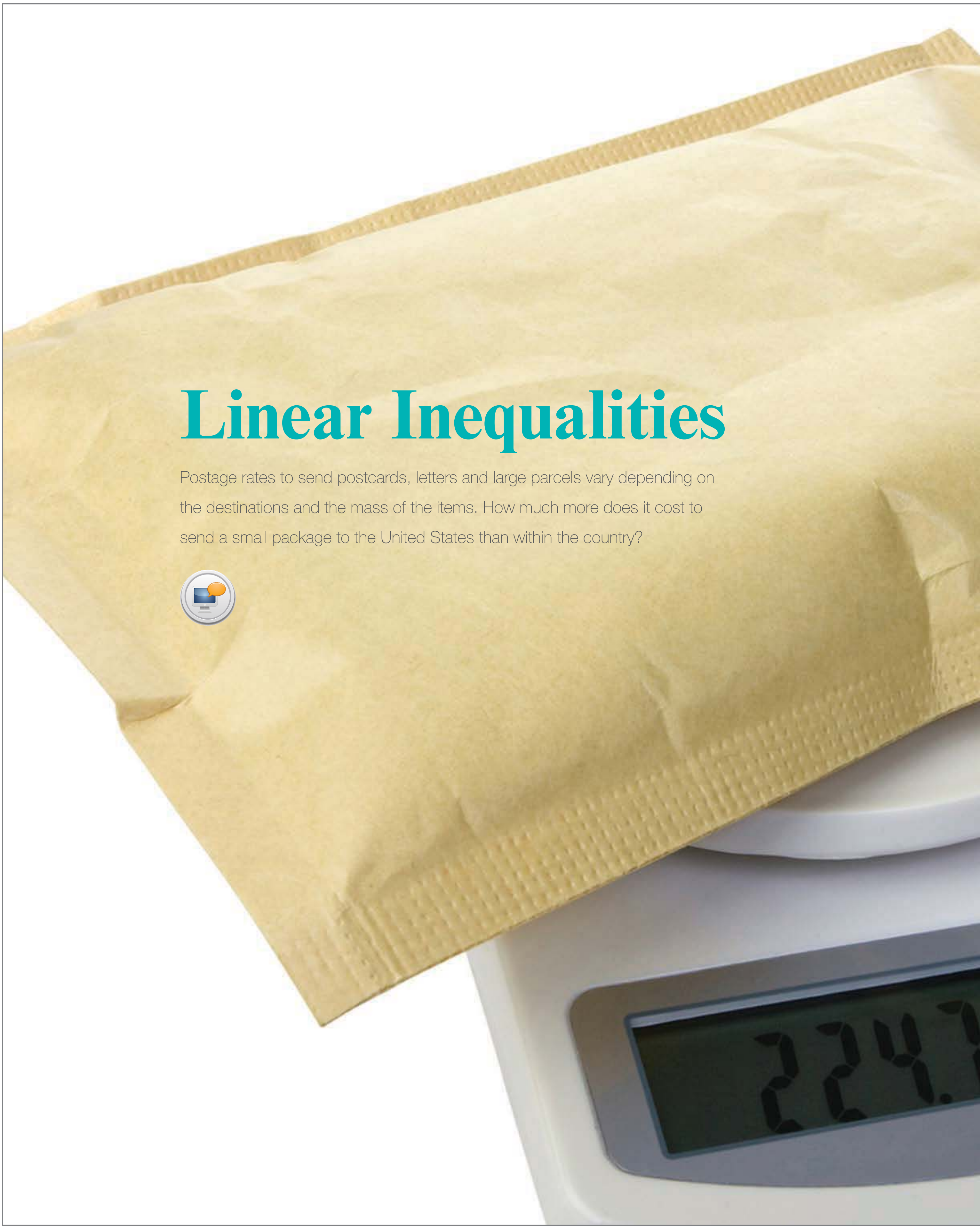
1. A two-digit number is such that the sum of its digits is 6 while the product of its digits is $\frac{1}{3}$ of the original number. Find the original number.

Hint: Let x be one of the digits.

2. If $x = h$ and $x = k$ are the real solutions of the quadratic equation $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$, we say that $x = h$ and $x = k$ are the roots of the equation. The sum of the roots, $h + k$, and the product of the roots, hk, can be expressed in terms of the coefficients a, b and c. Find an expression for $h + k$ and for hk in terms of a, b and/or c.

Linear Inequalities

Postage rates to send postcards, letters and large parcels vary depending on the destinations and the mass of the items. How much more does it cost to send a small package to the United States than within the country?



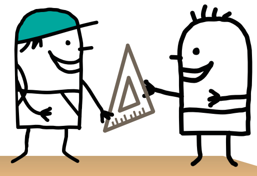
Chapter Two

LEARNING OBJECTIVES

At the end of this chapter,
you should be able to:

- solve linear inequalities in one variable and represent the solution on a number line,
- apply linear inequalities to solve word problems.

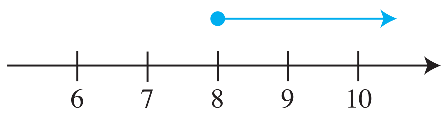
2.1 Inequalities



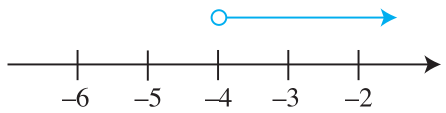
Recap

In Book 1, we have learnt how to solve a simple inequality in the form $ax > b$, $ax \geqslant b$, $ax < b$ and $ax \leqslant b$, where a and b are integers.
'To solve an inequality' means to find all the solutions that satisfy the inequality.

For example, if $4x \geqslant 32$,
then $\qquad x \geqslant 8$. (divide both sides by 4)



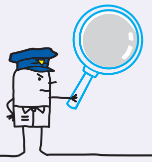
If $-5y < 20$,
then $5y > -20$, (multiply both sides by -1, change the inequality sign)
i.e. $y > -4$. (divide both sides by 5)



In general, to solve an inequality, we

- multiply or divide both sides by a *positive* number *without* having to reverse the inequality signs,
  i.e. if $x \geqslant y$ and $c > 0$, then $cx \geqslant cy$ and $\frac{x}{c} \geqslant \frac{y}{c}$.
- *reverse* the inequality sign if we multiply or divide both sides by a *negative* number,
  i.e. if $x \geqslant y$ and $d < 0$, then $dx \leqslant dy$ and $\frac{x}{d} \leqslant \frac{y}{d}$.

In this section, we will learn how to solve linear inequalities in one variable and represent the solution on a number line.



Investigation

Inequalities

By considering a number line, fill in each blank with '>' or '<'.

1. **(a)** **(i)** $6 < 12$ **(ii)** $6 + 2 \ \square\ 12 + 2$ **(iii)** $6 - 4 \ \square\ 12 - 4$

(b) If $6 < 12$ and a is a real number, then $6 + a \ \square\ 12 + a$ and $6 - a \ \square\ 12 - a$.

(c) If $12 > 6$ and a is a real number, then $12 + a \ \square\ 6 + a$ and $12 - a \ \square\ 6 - a$.

2. **(a)** **(i)** $-6 \ \square\ 12$ **(ii)** $-6 + 2 \ \square\ 12 + 2$ **(iii)** $-6 - 4 \ \square\ 12 - 4$

(b) If $-6 < 12$ and a is a real number, then $-6 + a \ \square\ 12 + a$ and $-6 - a \ \square\ 12 - a$.

(c) If $12 > -6$ and a is a real number, then $12 + a \ \square\ -6 + a$ and $12 - a \ \square\ -6 - a$.

3. **(a)** **(i)** $6 \ \square\ -12$ **(ii)** $6 + 2 \ \square\ -12 + 2$ **(iii)** $6 - 4 \ \square\ -12 - 4$

(b) What do you observe about your answers in Question 3**(a)**?

4. Do the conclusions which you have drawn in Questions 1 to 3 apply to $6 \leqslant 12$? What about $12 \geqslant 6$?

From the investigation, we observe that when we add or subtract a positive or a negative number from both sides of an inequality, the inequality sign does not change,

> i.e. if $x > y$, then $x + a > y + a$ and $x - a > y - a$.
> If $x \geqslant y$, then $x + a \geqslant y + a$ and $x - a \geqslant y - a$.
>
> Similarly, if $x < y$, then $x + a < y + a$ and $x - a < y - a$.
> If $x \leqslant y$, then $x + a \leqslant y + a$ and $x - a \leqslant y - a$.

SIMILAR QUESTIONS

Exercise 2A Questions 1(a),(b)

Journal Writing

One example of a real-life application of inequalities is the speed limit of vehicles travelling on expressways.
Can you think of other real-life applications of inequalities? Write down as many as you can.

Worked Example 1

(Solving Linear Inequalities)

Solve each of the following inequalities, illustrating each solution on a number line.

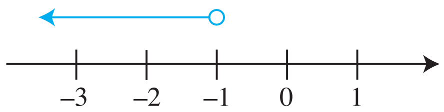
(a) $x + 4 < 3$ **(b)** $-4y - 5 \geqslant 11$

Solution:

(a) $x + 4 < 3$

$x + 4 - 4 < 3 - 4$ (subtract 4 from both sides)

$x < -1$



(b) $-4y - 5 \geqslant 11$

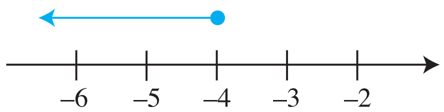
$-4y - 5 + 5 \geqslant 11 + 5$ (add 5 to both sides)

$-4y \geqslant 16$

$4y \leqslant -16$ (multiply both sides by –1 and change the inequality sign)

$\frac{4y}{4} \leqslant \frac{-16}{4}$ (divide by 4 on both sides)

$y \leqslant -4$



PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 2A Questions 2(a)-(f)

Solve each of the following inequalities, illustrating each solution on a number line.

(a) $x - 3 \geqslant 7$ **(b)** $-2y + 4 > 3$

Thinking Time

1. Given an equation in the form $ax + b = c$, where a, b and c are constants and $a > 0$, list the steps you would take to find the value of x. Would the steps change if $a < 0$?
2. Given an inequality in the form $ax + b > c$, where a, b and c are constants and $a > 0$, list the steps you would take to find the range of values of x. How would the steps change if $a < 0$?
3. Given an inequality in the form $ax + b \geqslant c$, where a, b and c are constants and $a > 0$, list the steps you would take to find the range of values of x. How would the steps change if $a < 0$?

For the inequalities $ax + b < c$, $ax + b \leqslant c$, $ax + b > c$ and $ax + b \geqslant c$, we say that the corresponding linear equation is $ax + b = c$.

4. How is the solution of each inequality related to that of the corresponding linear equation?

Worked Example 2

(Solving Linear Inequalities)

Solve the inequality $8 - x > 3$ and illustrate the solution on a number line.

(i) If x is a prime number, write down the largest possible value of x that satisfies the inequality.

(ii) Write down the positive integer values of x that satisfy the inequality.

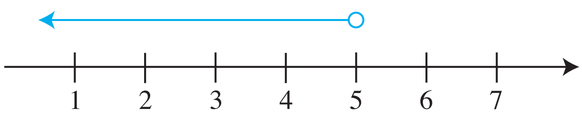
Solution:

$8 - x > 3$

$8 - 8 - x > 3 - 8$ (subtract 8 from both sides)

$-x > -5$

$x < 5$ (multiply both sides by -1 and change the inequality sign)



(i) Largest prime value of x is 3

(ii) Positive integer values of x are 1, 2, 3 and 4

PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 2A Question 3

Solve the inequality $5 - x < -9$ and illustrate the solution on a number line.

(i) If x is a prime number, write down the smallest possible value of x that satisfies the inequality.

(ii) Given that x is a perfect cube, find the smallest possible value of x.

Worked Example 3

(Solving Linear Inequalities)

Solve each of the following inequalities.

(a) $3x - 2 > 2(1 - x)$ **(b)** $\frac{y}{4} \leqslant \frac{y+1}{7}$

Solution:

(a)

$$3x - 2 > 2(1 - x)$$

$$3x - 2 > 2 - 2x \text{ (expand the RHS)}$$

$$3x - 2 + 2x > 2 - 2x + 2x \text{ (add } 2x \text{ to both sides)}$$

$$5x - 2 > 2$$

$$5x - 2 + 2 > 2 + 2 \text{ (add 2 to both sides)}$$

$$5x > 4$$

$$x > \frac{4}{5}$$

(b)

$$\frac{y}{4} \leqslant \frac{y+1}{7}$$

$$4 \times 7 \times \frac{y}{4} \leqslant 4 \times 7 \times \frac{y+1}{7} \text{ (multiply by } 4 \times 7 \text{ on both sides)}$$

$$7y \leqslant 4(y + 1)$$

$$7y \leqslant 4y + 4 \text{ (expand the RHS)}$$

$$7y - 4y \leqslant 4y + 4 - 4y \text{ (subtract } 4y \text{ from both sides)}$$

$$3y \leqslant 4$$

$$y \leqslant \frac{4}{3}$$

$$y \leqslant 1\frac{1}{3}$$

Problem Solving Tip

The LCM of 4 and 7 is 4×7.

PRACTISE NOW 3

SIMILAR QUESTIONS

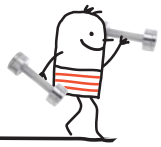
Exercise 2A Questions 2(g)-(l), 4, 5(a),(b), 6(a)-(h), 7, 8

1. Solve each of the following inequalities.

(a) $15x + 1 < 5(3 + x)$ **(b)** $\frac{16y}{3} \geqslant \frac{y+1}{2}$

(c) $\frac{1}{2}(z-4) \leqslant \frac{1}{3}(z+1)+2$

2. Given that p satisfies the inequality $\frac{3}{4}(p-2)+\frac{1}{2} > \frac{1}{2}(p-1)$, find the smallest possible value of p if p is a perfect square.



Exercise 2A

BASIC LEVEL

1. Fill in each box with '<', '>', '⩽' or '⩾'.

(a) $5 + h \ \square \ 7 + h$, where h is a real number

(b) $5 - k \ \square \ 7 - k$, where k is a real number

2. Solve each of the following inequalities, illustrating each solution on a number line.

(a) $a + 2 < 3$ **(b)** $b - 3 \geqslant 4$

(c) $-c + 3 > 5$ **(d)** $4 - d \leqslant 4$

(e) $-2e - 1 \leqslant 2$ **(f)** $2 + 5f < 0$

(g) $g - 7 \geqslant 1 - g$ **(h)** $5h > 4(h + 1)$

(i) $8j + 3 < 2(7 - j)$ **(j)** $4k + 5 \geqslant 2(-2k)$

(k) $2(m - 5) \leqslant 2 - m$ **(l)** $3(1 - 4n) > 8 - 7n$

3. Solve the inequality $7 + 2x \leqslant 16$ and illustrate the solution on a number line.

(i) If x is an integer, write down the largest possible value of x that satisfies the inequality.

(ii) Given that x is a perfect square, find the largest possible value of x.

4. Solve the inequality $3 - 4x > 3x - 18$ and illustrate the solution on a number line.

(i) If x is a prime number, write down the possible value(s) of x that satisfies the inequality.

(ii) Does $x = 0$ satisfy the inequality? Explain your answer.

INTERMEDIATE LEVEL

5. Solve each of the following inequalities, illustrating each solution on a number line.

(a) $4(p + 1) < -3(p - 4)$

(b) $6 - (1 - 2q) \geqslant 3(5q - 2)$

6. Solve each of the following inequalities.

(a) $\dfrac{4a}{3} \geqslant 2$

(b) $\dfrac{2b-1}{3} > \dfrac{3b}{5}$

(c) $\dfrac{c+4}{4} > \dfrac{c+1}{3}$

(d) $\dfrac{2-d}{2} < \dfrac{3-d}{4} + \dfrac{1}{2}$

(e) $\dfrac{1}{4}(e-2) + \dfrac{2}{3} < \dfrac{1}{6}(e-4)$

(f) $\dfrac{f+1}{2} + \dfrac{3f+1}{4} \leqslant \dfrac{3f-1}{4} + 2$

(g) $\dfrac{1}{5}(3g+4) - \dfrac{1}{3}(g+1) \geqslant 1 - \dfrac{1}{3}(g+5)$

(h) $4\left(\dfrac{h}{3} + \dfrac{3}{4}\right) < 3\left(\dfrac{h}{2} - 5\right)$

7. Given that p satisfies the inequality $\dfrac{1}{6}(2-p) - 3 \geqslant \dfrac{p}{10}$, find the largest possible value of p.

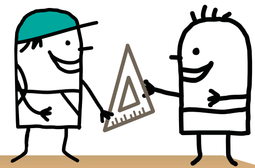
ADVANCED LEVEL

8. Given that $\dfrac{1}{3}(2x-7) \leqslant \dfrac{3x+2}{2}$,

(i) solve the inequality,

(ii) find the smallest possible value of x^2.

2.2 Problem Solving involving Inequalities



In this section, we shall take a look at how inequalities are used to solve problems.

Worked Example 4

(Problem Solving involving Inequalities)

Devi scored 66 marks for her first class test and 72 marks for her second class test. What is the minimum mark she must score for her third class test to meet her target of obtaining an average of 75 marks or more for the three tests?

Solution:

Let x be the marks scored by Devi in her third class test.

$$\begin{aligned}
\frac{66+72+x}{3} &\geqslant 75 \\
3\times\frac{66+72+x}{3} &\geqslant 3\times 75 \text{ (multiply by 3 on both sides)} \\
66+72+x &\geqslant 225 \\
138+x &\geqslant 225 \\
138+x-138 &\geqslant 225-138 \text{ (subtract 138 from both sides)} \\
x &\geqslant 87
\end{aligned}$$

$\therefore$ Devi must score at least 87 marks for her third class test.

PRACTISE NOW 4

The minimum mark to obtain a Grade A is 75. Priya managed to achieve an average of Grade A for three of her Science quizzes. What is the minimum mark she scored in her first quiz if she scored 76 and 89 marks in her second and third quiz respectively?

SIMILAR QUESTIONS

Exercise 2B Questions 1, 6, 7

Worked Example 5

(Problem Solving involving Inequalities)

An IQ test consists of 20 multiple choice questions. 3 points are awarded for a correct answer and 1 point is deducted for a wrong answer. No points are awarded or deducted for an unanswered question. Raj attempted a total of 19 questions and his total score for the IQ test was above 32. Find the minimum number of correct answers he obtained.

Solution:

Let x and y be the number of correct answers and incorrect answers respectively.

$x + y = 19$ ----------- (1)

$3 \times x + (-1) \times y > 32$

i.e. $3x - y > 32$ ----------- (2)

From (1),

$y = 19 - x$ ----------- (3)

Substitute (3) into (2):

$3x - (19 - x) > 32$

$3x - 19 + x > 32$

$4x - 19 + 19 > 32 + 19$ (add 19 to both sides)

$4x > 51$

$x > \frac{51}{4}$ (divide by 4 on both sides)

$x > 12.75$

$\therefore$ Raj obtained at least 13 correct answers.

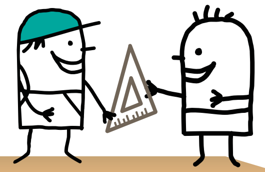
PRACTISE NOW 5

Vishal has 12 pieces of \$10 and \$5 notes in his wallet. If the total value of all the notes is less than \$95, what is the maximum number of \$10 notes that he has?

SIMILAR QUESTIONS

Exercise 2B Questions 2, 8, 9

2.3 Solving Simultaneous Linear Inequalities



Simultaneous Linear Inequalities

To solve linear inequalities simultaneously, we find the solution(s) to each inequality separately, then we consider only the common solutions of the inequalities.

For example, given that $x \geqslant 5$ and $x \leqslant 8$, then the range of values of x which satisfies both inequalities is $5 \leqslant x \leqslant 8$.

Does $x = 1$ satisfy *both* $3x \leqslant x + 6$ *and* $2x + 4 < 3x + 6$? Does $x = -3$ satisfy *both* $3x \leqslant x + 6$ *and* $2x + 4 < 3x + 6$?

ATTENTION

To check if $x = 1$ satisfies the inequality $3x \leqslant x + 6$, substitute $x = 1$ into the inequality and check if LHS $\leqslant$ RHS.

Worked Example 6

(Solving Simultaneous Linear Inequalities)

Find the range of values of x for which $3x \leqslant x + 6$ and $2x + 4 < 3x + 6$.

Solution:

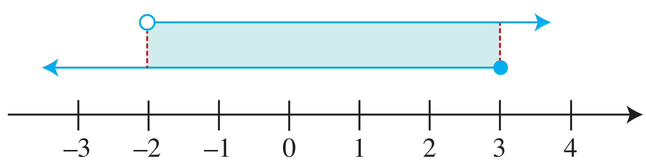
Solving the two linear inequalities separately,

$$\begin{aligned} 3x &\leqslant x + 6 \\ 3x - x &\leqslant x + 6 - x \\ 2x &\leqslant 6 \\ x &\leqslant \frac{6}{2} \\ x &\leqslant 3 \end{aligned}$$

and

$$\begin{aligned} 2x + 4 &< 3x + 6 \\ 2x + 4 - 3x &< 3x + 6 - 3x \\ -x + 4 &< 6 \\ -x + 4 - 4 &< 6 - 4 \\ -x &< 2 \\ x &> -2 \end{aligned}$$

Representing $x \leqslant 3$ and $x > -2$ on a number line,



ATTENTION

-2 is not a solution to the inequality.

$\therefore$ The solutions satisfying both inequalities lie in the overlapping shaded region, i.e. $-2 < x \leqslant 3$.

PRACTISE NOW 6

Find the range of values of x for which $2x - 3 \leqslant 7$ and $2x + 1 > -3x - 4$.

SIMILAR QUESTIONS

Exercise 2B Questions 3(a),(b), 4(a),(b), 10

Worked Example 7

(Solving Simultaneous Linear Inequalities)

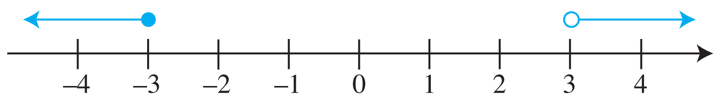
Solve the inequalities $4x + 14 \leqslant x + 5 < 3x - 1$.

Solution:

Solving the two linear inequalities separately,

$$\begin{array}{rclcrcl}
4x + 14 &\leqslant& x + 5 & \text{and} & x + 5 &<& 3x - 1 \\
4x + 14 - x &\leqslant& x + 5 - x & & x + 5 - 3x &<& 3x - 1 - 3x \\
3x + 14 &\leqslant& 5 & & -2x + 5 &<& -1 \\
3x + 14 - 14 &\leqslant& 5 - 14 & & -2x + 5 - 5 &<& -1 - 5 \\
3x &\leqslant& -9 & & -2x &<& -6 \\
x &\leqslant& -3 & & 2x &>& 6 \\
 & & & & x &>& 3
\end{array}$$

Representing $x \leqslant -3$ and $x > 3$ on a number line,



$\therefore$ The simultaneous linear inequalities have **no solution**.

ATTENTION

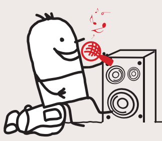
Since there is no overlapping region on the number line, there is no solution that satisfies both inequalities.

PRACTISE NOW 7

1. Solve the inequalities $8x + 13 \leqslant 4x - 3 < 5x - 11$.
2. Solve the inequalities $\frac{y-2}{3} < \frac{2y+1}{5} \leqslant 3$.

SIMILAR QUESTIONS

Exercise 2B Questions 5(a)-(d), 11, 12, 13(a)-(d), 14(a)-(d), 15(a)-(d), 16-18

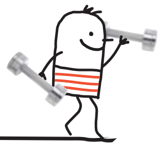


Performance Task

The table shows the postage rates for letters and small packages to Malaysia offered by a local company.

Mass (m g)	Postage (cents)
$0 < m \leqslant 20$	45
$20 < m \leqslant 50$	55
$50 < m \leqslant 100$	85
$100 < m \leqslant 200$	185
$200 < m \leqslant 300$	285

Search on the Internet for the postage rates for parcels to Thailand, New Zealand and the United Kingdom, displaying your findings in a table similar to the above.



Exercise 2B

BASIC LEVEL

1. On weekends, a movie ticket costs \$10.50. Form an inequality and solve it to find the maximum number of tickets Kate can buy with \$205.

2. The length and breadth of a rectangle are x cm and y cm respectively. If the rectangle has an area of 24 cm^2, state the possible pairs of integer values of x and y, where $x > y$.

3. Find the range of values of x which satisfy each of the following pairs of inequalities.

(a) $x - 4 \leqslant 3$ and $3x \geqslant -6$

(b) $2x + 5 < 15$ and $3x - 2 > -6$

4. Find the integer values of x which satisfy each of the following pairs of inequalities.

(a) $5x - 1 < 4$ and $3x + 5 \geqslant x + 1$

(b) $2x - 5 \geqslant 1$ and $3x - 1 < 26$

5. Solve each of the following pairs of inequalities, illustrating each solution on a number line.

(a) $-4 \leqslant 2x \leqslant 3x - 2$ **(b)** $1 - x < -2 \leqslant 3 - x$

(c) $3x - 3 < x - 9 < 2x$ **(d)** $2x \leqslant x + 6 < 3x + 5$

INTERMEDIATE LEVEL

6. Mr Chua's car consumes petrol at an average rate of 8 litres daily. Before Mr Chua begins his journey, he tops up the petrol in his car to 100 litres. Given that he will next top up the petrol in his car when there are 20 litres left, form an inequality and solve it to find the maximum number of days he can travel before he has to top up the petrol in his car.

7. If the sum of three consecutive integers is less than 75, find the cube of the largest possible integer.

8. In a Math Olympiad quiz, 5 points are awarded for a correct answer and 2 points are deducted for a wrong answer or if a question is left unanswered. Shirley attempted all 30 questions and her total score for the quiz was not more than 66. Find the maximum number of correct answers she obtained.

9. Ethan opened his piggy bank to find 50 pieces of \$5 and \$2 notes. If the total value of all the notes is more than \$132, find the minimum number of \$5 notes he has.

10. Given that x is a prime number, find the values of x for which $\frac{1}{2}x - 4 > \frac{1}{3}x$ and $\frac{1}{6}x + 1 < \frac{1}{8}x + 3$.

11. An integer x is such that $x + 2 < 5\sqrt{17} < x + 3$. State the value of x.

12. Given that x is a prime number, find the value of x for which $3x - 2 \geqslant 10 \geqslant x + 4$.

13. Solve each of the following pairs of inequalities.

(a) $3 - a \leqslant a - 4 \leqslant 9 - 2a$

(b) $1 - b < b - 1 < 11 - 2b$

(c) $3 - c < 2c - 1 < 5 + c$

(d) $3d - 5 < d + 1 \leqslant 2d + 1$

14. Solve each of the following pairs of inequalities.

(a) $\frac{a}{4} + 3 \leqslant 4 \leqslant \frac{a}{2} + 6$

(b) $\frac{b}{3} \geqslant \frac{b}{2} + 1 \geqslant b - 1$

(c) $2(1 - c) > c - 1 \geqslant \frac{c - 2}{7}$

(d) $d - 5 < \frac{2d}{5} \leqslant \frac{d}{2} + \frac{1}{5}$

15. Find the integer values of x which satisfy each of the following inequalities.

(a) $3x - 5 < 26 \leqslant 4x - 6$ (b) $3x + 2 < 19 < 5x - 4$

(c) $-4 \leqslant 7 - 3x \leqslant 2$ (d) $-10 < 7 - 2x \leqslant -1$

16. Given that $0 \leqslant x \leqslant 7$ and $1 \leqslant y \leqslant 5$, find

(a) the greatest possible value of $x + y$,

(b) the least possible value of $x - y$,

(c) the largest possible value of xy,

(d) the smallest possible value of $\frac{x}{y}$,

(e) the least and greatest possible values of x^2.

17. Given that $-4 \leqslant a \leqslant -1$ and $-6 \leqslant b \leqslant -2$, find

(a) the least possible value of $a + b$,

(b) the greatest possible value of $a - b$,

(c) the smallest possible value of ab,

(d) the largest possible value of $\frac{a}{b}$,

(e) the least and greatest possible values of a^2,

(f) the largest value of $b^2 - a$.

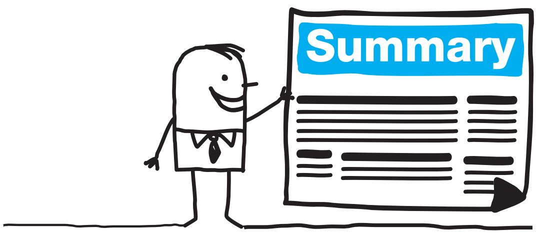
ADVANCED LEVEL

18. State whether each of the following statements is true or false. If your answer is 'false', offer an explanation to support your case.

(a) If $a > b$ and both a and b are negative, then $\frac{a}{b} > 1$.

(b) If $a > b$ and both a and b are negative, then $a^3 > b^3$.

(c) If $a > b$ and both a and b are negative, then $\frac{b}{a} - \frac{a}{b} > 0$.



1. Solving a Linear Inequality

Case	Adding a number to both sides of the inequality	Subtracting a number from both sides of the inequality
$x > y$	$x + a > y + a$	$x - a > y - a$
$x \geqslant y$	$x + a \geqslant y + a$	$x - a \geqslant y - a$
$x < y$	$x + a < y + a$	$x - a < y - a$
$x \leqslant y$	$x + a \leqslant y + a$	$x - a \leqslant y - a$

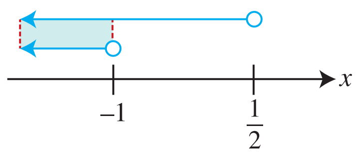
Case	Multiplying a positive number c to both sides of the inequality	Dividing a positive number c from both sides of the inequality	Multiplying a negative number d to both sides of the inequality	Dividing a positive number d from both sides of the inequality
$x > y$	$cx > cy$	$\frac{x}{c} > \frac{y}{c}$	$dx < dy$	$\frac{x}{d} < \frac{y}{d}$
$x \geqslant y$	$cx \geqslant cy$	$\frac{x}{c} \geqslant \frac{y}{c}$	$dx \leqslant dy$	$\frac{x}{d} \leqslant \frac{y}{d}$
$x < y$	$cx < cy$	$\frac{x}{c} < \frac{y}{c}$	$dx > dy$	$\frac{x}{d} > \frac{y}{d}$
$x \leqslant y$	$cx \leqslant cy$	$\frac{x}{c} \leqslant \frac{y}{c}$	$dx \geqslant dy$	$\frac{x}{d} \geqslant \frac{y}{d}$

When solving a linear inequality of the form $ax + b > cx + d$, apply the above guidelines to reduce the inequality to the form $x > k$ or $x < k$. This is called the **solution** of the inequality.

2. When solving a pair of simultaneous linear inequalities, we only consider the common solutions of the inequalities with the aid of a number line.

For example, to solve $2x + 1 < x < 1 - x$, we have

$$2x + 1 < x \qquad \text{and} \qquad x < 1 - x$$
$$x < -1 \qquad\qquad 2x < 1$$
$$x < \frac{1}{2}$$



The common solution is $x < -1$.

Review Exercise 2

1. Solve each of the following inequalities, illustrating each solution on a number line.

(a) $a - 2 \leqslant 3$
(b) $2b + 1 < 5 - 4b$
(c) $c \geqslant \frac{1}{2}c - 1$
(d) $\frac{1}{2}d > 1 + \frac{1}{3}d$
(e) $2(e - 3) \geqslant 1$
(f) $5(f - 4) \leqslant 2f$
(g) $-3 - g > 2g - 7$
(h) $18 - 3h < 5h - 4$

2. Solve each of the following inequalities.

(a) $3 + \frac{a}{4} > 5 + \frac{a}{3}$
(b) $\frac{4b}{9} - 5 < 3 - \frac{2b}{3}$
(c) $\frac{4c}{9} - \frac{3}{4} \geqslant c - \frac{1}{2}$
(d) $\frac{d-2}{3} < \frac{2d+3}{5} + \frac{5}{8}$
(e) $\frac{1}{3}(e+2) \geqslant \frac{2}{3} + \frac{1}{4}(e-1)$
(f) $5 - \frac{2f-5}{6} \leqslant \frac{f+3}{2} + \frac{2(f+1)}{3}$

3. Solve each of the following pairs of inequalities.

(a) $5 - a \leqslant a - 6 \leqslant 10 - 3a$

(b) $4 - b < 2b - 1 < 7 + b$

(c) $4c - 1 < \frac{1}{2} \leqslant 3c + 2$

(d) $2d + 1 \geqslant d > 3d - 20$

4. Given that $x \leqslant 14\frac{1}{2}$, state the largest possible value of x if x is

(a) an integer, (b) a prime number,

(c) a rational number.

5. Given that $27 - 2x \leqslant 8$, find

(a) the least value of x,

(b) the least integer value of x.

6. Find the integer values of x which satisfy each of the following inequalities.

(a) $5x > 69 - 2x$ and $27 - 2x \geqslant 4$

(b) $-10 \leqslant x < -4$ and $2 - 5x < 35$

7. Given that $-1 \leqslant x \leqslant 5$ and $2 \leqslant y \leqslant 6$, find the greatest and least values of

(a) $y - x$, (b) $\frac{x}{y}$,

(c) $\frac{x^2}{y}$.

8. Given that $-3 \leqslant x \leqslant 7$ and $4 \leqslant y \leqslant 10$, find

(a) the smallest possible value of $x - y$,

(b) the largest possible value of $\frac{x}{y}$,

(c) the largest possible value of $x^2 - y^2$,

(d) the smallest possible value of $x^3 + y^3$.

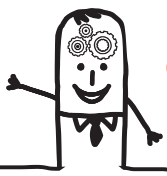
9. The perimeter of a square is at most 81 cm. What is the greatest possible area of the square? Give your answer correct to 4 significant figures.

10. The masses of a sheet of writing paper and an envelope are 3 g and 5 g respectively. It costs 60 cents to send a letter with a mass not exceeding 20 g. Michael has 60 cents worth of stamps. If x is the number of sheets of writing paper, form an inequality in x and find the maximum number of sheets of writing paper that he can use.

11. Lixin is 3 years younger than Rui Feng. If the sum of their ages is at most 50 years, find the maximum possible age of Lixin 5 years ago.

12. Jun Wei has 16 \$1 coins and some 20-cent coins in his pocket. Given that the total value of the coins in his pocket is at most \$22, find the maximum number of 20-cent coins that he has.

13. In a set of 20 True/False questions, 2 points are awarded for a correct answer and 1 point is deducted for a wrong answer. No points are awarded or deducted for an unanswered question. Amirah answered 15 questions and left the remaining questions unanswered. If her total score is greater than 24, find the maximum number of questions she answered wrongly.



Challenge Yourself

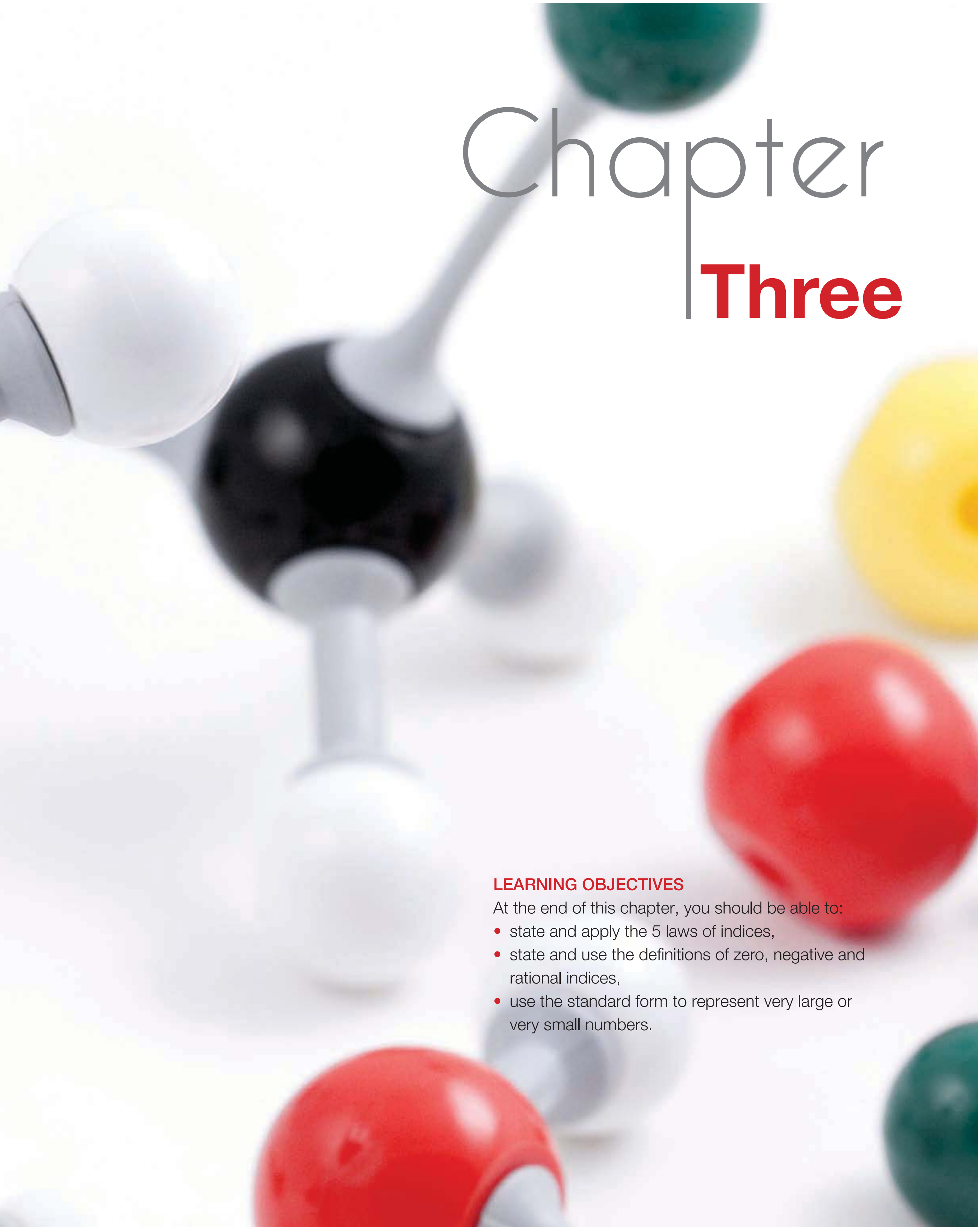
1. It is given that $6 \leqslant x \leqslant 8$ and $0.2 \leqslant y \leqslant 0.5$. If $z = \frac{x}{y}$, find the limits in which z must lie.

2. Find the range of values of x for which $\frac{3x - 5}{x^2 - 14x + 49} > 0$, $x \neq 7$. Explain your answer.

Indices and Standard Form

The world population is estimated to be about 7 000 000 000 in 2012.
The Bohr radius of a hydrogen atom is estimated to be about 0.000 000 000 053 m.
As it is troublesome to write very large or very small numbers in this manner, they can be represented using standard form which involves indices.

In this chapter, we will learn about indices and standard form.



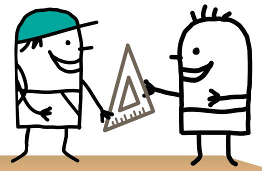
Chapter Three

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- state and apply the 5 laws of indices,
- state and use the definitions of zero, negative and rational indices,
- use the standard form to represent very large or very small numbers.

3.1 Indices



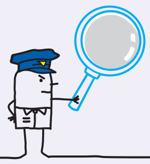
In Book 1, we have learnt how to represent $5 \times 5 \times 5 \times 5$ as 5^4 (read as '5 to the power of 4'). The digit 5 in 5^4 is known as the **base**, and the digit 4 in 5^4 is known as the **index** (plural: indices). 5^4 is called the index notation of $5 \times 5 \times 5 \times 5$.

$5^4 \leftarrow$ index

$\uparrow$

base

Write $3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$ in index notation: _____



Investigation

Indices

"Dad, could you please give me an allowance of \$10 on the first day of this month, \$20 on the second day and \$30 on the third day, increasing by \$10 each day until the 31st day of this month? Then I will promise not to ask for allowance ever again."

"How about giving me 1 cent on the first day of this month, 2 cents on the second day and 4 cents on the third day, doubling the amount each day until the 31st day of this month? Then I will promise not to ask for allowance ever again."

Calculate the amount of allowance the father has to give to his son on the 31st day of the month.

From the investigation, 2^{30} cents may seem to be a small amount, but it is actually equal to 1 073 741 824 cents or about \$10.7 million. This shows how 'powerful' the index 30 is!

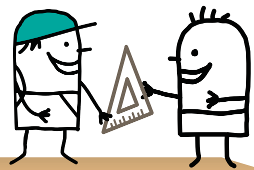
Class Discussion

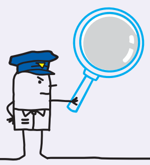
Comparing Numbers written in Index Form

Work in pairs.

1. Describe in your own words the meaning of 2^{10} and 10^2.
2. Without using a calculator, explain whether 2^{10} or 10^2 is greater.
   Hint: $2^{10} = (\quad)^2 \;\square\; 10^2$ (Convert 2^{10} to a number with a different base but with index 2)
3. Without using a calculator, explain whether 3^7 or 7^3 is greater.
   Hint: $3^7 = 3(3)^6 = 3(\quad)^3 \;\square\; 7^3$ (Convert 3^7 to a number with a different base but with index 3)
4. If a and b are positive integers such that $b > a$, when will
   (i) $a^b = b^a$, **(ii)** $a^b < b^a$?
   Hint: Use systematic listing, e.g. for $a = 1$, tabulate values for a^b and b^a for integer values of b. Repeat for values of $a = 2, 3, 4, \ldots$
5. In general, if a and b are positive integers such that $b > a$, then $a^b > b^a$ with some exceptions. Based on your work in Question 4, list the values of a and b for which the above statement is not true.

3.2 Laws of Indices



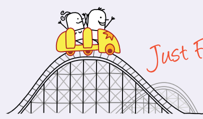


Investigation

Law 1 of Indices

Copy and complete the following.

1. $7^2 \times 7^3 = (7 \times 7) \times (\qquad)$
   $= 7 \times 7 \times \ldots \times 7$
   $= 7^{\square}$
   $= 7^{2+\square}$
2. $6^4 \times 6^5 = (\qquad) \times (\qquad)$
   $= 6 \times 6 \times \ldots \times 6$
   $= 6^{\square}$
   $= 6^{4+\square}$



Just For Fun

10^{100} is called a googol, while $10^{10^{100}}$, i.e. $10^{(10^{100})}$, is called a googolplex. Hence, a googolplex has one googol (or 10^{100}) zeros. If one page of newspaper can print 30 000 digits, you will need at least 10^{95} pages of newspaper. However, the whole universe is estimated to contain about 10^{78} to 10^{82} atoms, so it is not possible to print out all the zeros of a googolplex. If you think a googolplex is a very large number, search on the Internet for 'Graham's number', which is so much larger than the googolplex that a new notation has to be used to represent it.

3. $a^3 \times a^4 = (a \times a \times a) \times (\qquad\qquad)$
$= a \times a \times \ldots \times a$
$= a^{\square}$
$= a^{3+\square}$

4. $a^m \times a^n = (\underbrace{a \times a \times a \times \ldots \times a \times a}_{m \text{ times}}) \times (\underbrace{a \times a \times \ldots \times a}_{\square \text{ times}})$

$= \underbrace{a \times a \times \ldots \times a}_{m + \square \text{ times}}$

$= a^{m+\square}$

In general, if a is a real number, and m and n are positive integers, then

Law 1 of Indices: $a^m \times a^n = a^{m+n}$

Worked Example 1

(Law 1 of Indices)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $5^8 \times 5^9$ **(b)** $(-2)^3 \times (-2)$
(c) $6b^3c^5 \times 2bc^4$

Solution:

(a) $5^8 \times 5^9 = 5^{8+9}$ (Law 1)
$= 5^{17}$

(b) $(-2)^3 \times (-2) = (-2)^{3+1}$ (Law 1)
$= (-2)^4$

(c) $6b^3c^5 \times 2bc^4 = 12b^{3+1}c^{5+4}$ (Law 1)
$= 12b^4c^9$

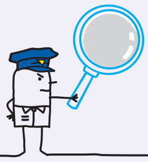
PRACTISE NOW 1

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $4^7 \times 4^5$ **(b)** $(-3)^6 \times (-3)$
(c) $a^{12} \times a^8$ **(d)** $2xy^4 \times 3x^5y^3$

SIMILAR QUESTIONS

Exercise 3A Questions 1(a)-(d), 5(a)-(c)



Investigation

Law 2 of Indices

Copy and complete the following. Assume $a \neq 0$.

1. $3^5 \div 3^2 = \dfrac{3 \times 3 \times 3 \times 3 \times 3}{\boxed{\qquad}}$

$= 3^{\boxed{\quad}}$

$= 3^{5-\boxed{\quad}}$

2. $\dfrac{10^6}{10^4} = \dfrac{\boxed{\qquad}}{\boxed{\qquad}}$

$= 10^{\boxed{\quad}}$

$= 10^{6-\boxed{\quad}}$

3. $a^7 \div a^3 = \dfrac{a \times a \times a \times a \times a \times a \times a}{\boxed{\qquad}}$

$= a^{\boxed{\quad}}$

$= a^{7-\boxed{\quad}}$

4. $a^m \div a^n = \dfrac{\overbrace{a \times a \times a \times \ldots \times a \times a}^{m \text{ times}}}{\underbrace{a \times a \times \ldots \times a}_{\boxed{\quad} \text{ times}}}$

$= \underbrace{a \times a \times \ldots \times a}_{m - \boxed{\quad} \text{ times}}$

$= a^{m-\boxed{\quad}}$

In general, if a is a real number, and m and n are positive integers such that $m > n$, then

Law 2 of Indices: $a^m \div a^n = a^{m-n}$, if $a \neq 0$

Worked Example 2

(Law 2 of Indices)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $7^8 \div 7^3$

(b) $(-5)^6 \div (-5)$

(c) $12b^6c^3 \div 6b^4c^2$

Solution:

(a) $7^8 \div 7^3 = 7^{8-3}$ (Law 2)

$= 7^5$

(b) $(-5)^6 \div (-5) = (-5)^{6-1}$ (Law 2)

$= (-5)^5$

(c) $12b^6c^3 \div 6b^4c^2 = \dfrac{12b^6c^3}{6b^4c^2}$ (Law 2)

$= 2b^{6-4}c^{3-2}$

$= 2b^2c$

PRACTISE NOW 2

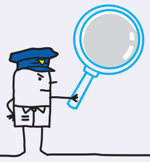
SIMILAR QUESTIONS

Exercise 3A Questions 2(a)-(d), 5(d)-(f)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $9^7 \div 9^3$ **(b)** $(-4)^8 \div (-4)$

(c) $a^{10} \div a^6$ **(d)** $27x^9y^4 \div 9x^6y^3$



Investigation

Law 3 of Indices

Copy and complete the following.

1. $(2^5)^2 = 2^5 \times 2^5$

$= 2^{5+\square}$ (using Law 1 of Indices)

$= 2^{5\times\square}$

2. $(10^4)^3 = 10^4 \times 10^4 \times 10^4$

$= 10^{4+\square+\square}$ (using Law 1 of Indices)

$= 10^{4\times\square}$

3. $(a^m)^n = (\underbrace{a^m \times a^m \times \ldots \times a^m}_{n \text{ times}})$

$= a^{\overbrace{m+m+\ldots+m}^{\square \text{ times}}}$

$= a^{m\times\square}$

In general, if a is a real number, and m and n are positive integers, then

Law 3 of Indices: $(a^m)^n = a^{mn}$

Worked Example 3

(Laws of Indices)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $(5^7)^3$ **(b)** $(h^8)^2$

(c) $(7^p)^4 \times (7^2)^p$

Solution:

(a) $(5^7)^3 = 5^{7\times 3}$ (Law 3)

$= 5^{21}$

(b) $(h^8)^2 = h^{8\times 2}$ (Law 3)

$= h^{16}$

(c) $(7^p)^4 \times (7^2)^p = 7^{p\times 4} \times 7^{2\times p}$ (Law 3)

$= 7^{4p} \times 7^{2p}$

$= 7^{4p+2p}$ (Law 1)

$= 7^{6p}$

PRACTISE NOW 3

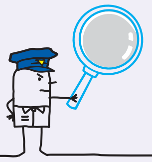
SIMILAR QUESTIONS

Exercise 3A Questions 3(a),(b), 6(a)-(c)

1. Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $(6^3)^4$ **(b)** $(k^5)^9$ **(c)** $(3^q)^6 \times (3^4)^q$

2. Given that $x^8 \times (x^3)^n \div (x^n)^2 = x^{10}$, find the value of n.



Investigation

Law 4 of Indices

Copy and complete the following.

1. $2^3 \times 7^3 = (2 \times 2 \times 2) \times (\underline{\qquad\qquad})$

$= (2 \times 7) \times (2 \times 7) \times (2 \times \underline{\quad})$

$= (2 \times \underline{\quad})^3$

2. $(-3)^2 \times (-4)^2 = (-3) \times (-3) \times (\underline{\quad}) \times (\underline{\quad})$

$= [(-3) \times (-4)] \times [(-3) \times (\underline{\quad})]$

$= [(-3) \times (\underline{\quad})]^{\underline{\quad}}$

3. $a^n \times b^n = \underbrace{(a \times a \times \ldots \times a)}_{n \text{ times}} \times \underbrace{(b \times b \times \ldots \times b)}_{\underline{\quad} \text{ times}}$

$= \underbrace{(a \times b) \times (a \times b) \times \ldots \times (a \times b)}_{\underline{\quad} \text{ times}}$

$= (a \times b)^{\underline{\quad}}$

In general, if a and b are real numbers, and n is a positive integer, then

Law 4 of Indices: $a^n \times b^n = (a \times b)^n$

Another useful version of Law 4 of Indices can be written as $(a \times b)^n = a^n \times b^n$ or $(ab)^n = a^nb^n$.

Worked Example 4

(Laws of Indices)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $2^4 \times 7^4$ **(b)** $(2h^2)^6$

(c) $(xy^2)^3 \times (-3x^2y)^4$ **(d)** $(4x^2y^3)^3 \div (xy^3)^2$

Solution:

(a) $2^4 \times 7^4 = (2 \times 7)^4$ (Law 4)

$= 14^4$

(b) $(2h^2)^6 = 2^6(h^2)^6$ (Law 4)

$= 64h^{2 \times 6}$ (Law 3)

$= 64h^{12}$

(c) $(xy^2)^3 \times (-3x^2y)^4 = x^3y^{2 \times 3} \times (-3)^4x^{2 \times 4}y^4$ (Law 4 and Law 3)

$= x^3y^6 \times 81x^8y^4$

$= 81x^{3+8}y^{6+4}$ (Law 1)

$= 81x^{11}y^{10}$

(d) $(4x^2y^3)^3 \div (xy^3)^2 = 4^3x^{2 \times 3}y^{3 \times 3} \div x^2y^{3 \times 2}$ (Law 4 and Law 3)

$= 64x^6y^9 \div x^2y^6$

$= 64x^{6-2}y^{9-6}$ (Law 2)

$= 64x^4y^3$

PRACTISE NOW 4

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $3^7 \times 8^7$ **(b)** $(5b^4)^3$

(c) $(-2c^2d^5)^5$ **(d)** $(m^2n)^4 \times (m^4n^3)^5$

(e) $(-p^7q^5)^2 \div (3p^3q^2)^3$

SIMILAR QUESTIONS

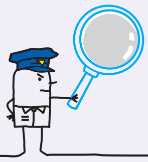
Exercise 3A Questions 3(c)-(f), 6(d),(e), 7(a)-(d), 9(a)-(d), 10

Class Discussion

Simplification using the Laws of Indices

Work in pairs.

There are two ways of simplifying $(xy^2)^4 \times (3x^2y)^4$. Explain these two ways using the laws of indices.



Investigation

Law 5 of Indices

Copy and complete the following.

1. $8^3 \div 5^3 = \dfrac{8^3}{5^3}$

$$= \frac{8 \times 8 \times 8}{\square \times \square \times \square}$$

$$= \frac{8}{\square} \times \frac{8}{\square} \times \frac{8}{\square}$$

$$= \left(\frac{8}{\square}\right)^3$$

2. $(-12)^4 \div (-7)^4 = \dfrac{(-12)^4}{(-7)^4}$

$$= \frac{(-12) \times (-12) \times (-12) \times (-12)}{\square \times \square \times \square \times \square}$$

$$= \frac{(-12)}{\square} \times \frac{(-12)}{\square} \times \frac{(-12)}{\square} \times \frac{(-12)}{\square}$$

$$= \left[\frac{(-12)}{\square}\right]^4$$

3. Consider $a^n \div b^n$, where $b \neq 0$.

$$a^n \div b^n = \frac{\overbrace{a \times a \times \ldots \times a}^{n \text{ times}}}{\underbrace{\square \times \square \times \ldots \times \square}_{n \text{ times}}}$$

$$= \underbrace{\frac{a}{\square} \times \frac{a}{\square} \times \ldots \times \frac{a}{\square}}_{\square \text{ times}}$$

$$= \left(\frac{a}{\square}\right)^n$$

In general, if a and b are real numbers, and n is a positive integer, then

Law 5 of Indices: $a^n \div b^n = \left(\dfrac{a}{b}\right)^n$, if $b \neq 0$

Another useful version of Law 5 of Indices can be written as $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ if $b \neq 0$.

Worked Example 5

(Laws of Indices)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $10^8 \div 2^8$ **(b)** $\left(\dfrac{2}{b}\right)^5$

(c) $\left(\dfrac{2p^2}{q^3}\right)^4 \div \dfrac{8p^3}{q^{16}}$

Solution:

(a) $10^8 \div 2^8 = \left(\dfrac{10}{2}\right)^8$ (Law 5)

$= 5^8$

(b) $\left(\dfrac{2}{b}\right)^5 = \dfrac{2^5}{b^5}$ (Law 5)

$= \dfrac{32}{b^5}$

(c) $\left(\dfrac{2p^2}{q^3}\right)^4 \div \dfrac{8p^3}{q^{16}} = \dfrac{2^4\,p^{2\times4}}{q^{3\times4}} \times \dfrac{q^{16}}{8p^3}$ (Law 5 and Law 3)

$= \dfrac{16p^8}{q^{12}} \times \dfrac{q^{16}}{8p^3}$

$= 2p^{8-3}q^{16-12}$ (Law 2)

$= 2p^5q^4$

PRACTISE NOW 5

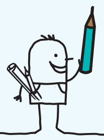
SIMILAR QUESTIONS

Exercise 3A Questions 4(a)-(f), 6(f), 8(a)-(d)

Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $21^3 \div 7^3$ **(b)** $(26^5)^3 \div 13^{15}$

(c) $\left(\dfrac{p^2}{q}\right)^3 \div \dfrac{q^7}{p^5}$ **(d)** $\left(\dfrac{3x^2}{x^3}\right)^3 \div \dfrac{27x^7}{x^{21}}$



Journal Writing

Nora and Farhan were asked to simplify $\left(\dfrac{2x^2}{y}\right)^3$.

Nora wrote this: $\left(\dfrac{2x^2}{y}\right)^3 = \dfrac{2x^6}{y^3}$

Farhan wrote this: $\left(\dfrac{2x^2}{y}\right)^3 = \dfrac{8x^5}{y}$

Both Nora and Farhan obtained the wrong solutions.

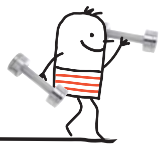
(i) Highlight the possible misconceptions that Nora and Farhan might have.

(ii) Provide the correct solution.

Class Discussion

Is $(a + b)^n = a^n + b^n$? Is $(a - b)^n = a^n - b^n$?

Work in pairs.
We have learnt that $(a \times b)^n = a^n \times b^n$ and $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.
Is it true that $(a + b)^n = a^n + b^n$?
How about $(a - b)^n = a^n - b^n$?
Discuss with your classmates. Provide some counterexamples if necessary.



Exercise 3A

BASIC LEVEL

1. Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $2^3 \times 2^7$ **(b)** $(-4)^6 \times (-4)^5$

(c) $x^8 \times x^3$ **(d)** $(3y^2) \times (8y^7)$

2. Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $5^8 \div 5^5$ **(b)** $(-7)^{11} \div (-7)^4$

(c) $6x^7 \div x^3$ **(d)** $(-15y^9) \div 5y^4$

3. Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $(9^2)^4$ **(b)** $(h^2)^5$

(c) $3^{14} \times (5^2)^7$ **(d)** $2^3 \times 9^3$

(e) $(2k^6)^3$ **(f)** $(-3x^6y^2)^4$

4. Simplify each of the following, leaving your answer in index notation where appropriate.

(a) $14^{13} \div 7^{13}$ **(b)** $(9^5)^4 \div 3^{20}$

(c) $\left(\frac{m}{2}\right)^5$ **(d)** $\left(\frac{3}{n^2}\right)^3$

(e) $\left(\frac{p^4}{q}\right)^6$ **(f)** $\left(-\frac{x}{y^2}\right)^4$

INTERMEDIATE LEVEL

5. Simplify each of the following.

(a) $h^2k \times h^{11}k^9$ **(b)** $(-m^7n^3) \times 4m^{11}n^9$

(c) $11p^6q^7 \times 2p^3q^{10}$ **(d)** $h^9k^6 \div h^5k^4$

(e) $15m^8n^7 \div 3m^2n$ **(f)** $(-10x^5y^6) \div (-2xy^3)$

6. Simplify each of the following.

(a) $(a^2)^3 \times a^5$ **(b)** $(b^3)^7 \times (b^4)^5$

(c) $(c^6)^5 \div (-c^2)$ **(d)** $(-3d^3)^2 \div (2d)^3$

(e) $(e^3)^5 \div (-e^2)^4$ **(f)** $(4f^6)^3 \div (-2f^3)^3$

7. Simplify each of the following.

(a) $(ab^2)^3 \times (2a^2b)^3$ **(b)** $c^2d^2 \times (-5c^3d^3)^2$

(c) $(8e^5f^3)^2 \div (e^3f)^3$ **(d)** $16g^8h^7 \div (-2g^3h^2)^3$

8. Simplify each of the following.

(a) $\left(\frac{2a^2}{b}\right) \times \left(\frac{a}{b^2}\right)^2$ **(b)** $\left(\frac{c}{d^2}\right)^3 \times \left(\frac{c^3}{2d}\right)^2$

(c) $\left(\frac{3e^3}{f^2}\right)^4 \div \left(\frac{27e^9}{f^{11}}\right)$ **(d)** $\left(\frac{g^2}{h^3}\right)^6 \div \left(\frac{-3g^5}{2h^2}\right)^3$

ADVANCED LEVEL

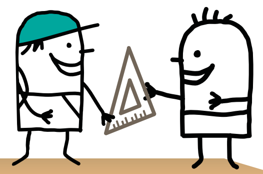
9. Simplify each of the following.

(a) $\dfrac{(2x^2y)^3}{(10xy^3)^2} \times \dfrac{(5xy^4)^3}{4xy}$ **(b)** $\dfrac{8x^8y^4}{(2xy^2)^2} \times \dfrac{(4x^2y^2)^2}{(3xy)^2}$

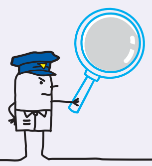
(c) $\dfrac{(2xy^2)^5}{(4x^2y)^2(xy^3)}$ **(d)** $\dfrac{4x^2y^4 \times 8x^4y^2}{(4x^2y^2)^2}$

10. Given that $\dfrac{(2p^3q^4)^4}{(-3q^5)^2} \div \dfrac{(4p^2q)^2}{9} = \dfrac{p^{a+b}}{q^{a-b}}$, form a pair of simultaneous equations in a and b and hence find the value of a and of b.

3.3 Zero and Negative Indices



In Section 3.2, all the 5 laws of indices apply when the indices are positive integers. In this section, we will consider what happens if the indices are zero or negative integers.



Investigation

Zero Index

3^4 means 3 multiplied by itself 4 times, i.e. $3 \times 3 \times 3 \times 3 = 81$. What does 3^0 mean?

1. Fill in the values of 3^2 and 3^1 in Table 3.1.

Index Form	Value	
3^4	81	$\div 3$
3^3	27	$\div 3$
3^2		$\div 3$
3^1		$\div 3$
3^0		

Table 3.1

2. What number must you divide 81 (i.e. 3^4) to get 27 (i.e. 3^3)?

3. What number must you divide 27 (i.e. 3^3) to get the value of 3^2?

4. What number must you divide 3^2 to get the value of 3^1?

5. **(a)** By continuing this pattern, what number must you divide 3^1 to get the value of 3^0?

(b) Complete the last row in Table 3.1.

6. Complete Table 3.2.

Index Form	Value
$(-2)^4$	16
$(-2)^3$	-8
$(-2)^2$	
$(-2)^1$	
$(-2)^0$	

$\div(-2)$ $\div(-2)$ $\div(-2)$ $\div(-2)$

Table 3.2

7. Does the pattern work for 0^4, 0^3, 0^2, 0^1 and 0^0? Explain your answer.

Since $5^3 \div 5^3 = \dfrac{5 \times 5 \times 5}{5 \times 5 \times 5}$

$= 1,$

and using the second law of indices,

$$5^3 \div 5^3 = 5^{3-3}$$
$$= 5^0.$$

$\therefore\ 5^0 = 1$

Since $a^4 \div a^4 = \dfrac{a \times a \times a \times a}{a \times a \times a \times a}$

$= 1,$

and using the second law of indices,

$$a^4 \div a^4 = a^{4-4}$$
$$= a^0.$$

$\therefore\ a^0 = 1$

$\therefore$ For the second law of indices to apply for $n = 0$, we define $a^0 = 1$, where a is a real number.

Definition 1: $a^0 = 1$, if $a \neq 0$

Worked Example 6

(Zero Indices)

Evaluate each of the following.

(a) 121^0 **(b)** $2x^0$

(c) $(2x)^0$

Solution:

(a) $121^0 = 1$

(b) $2x^0 = 2 \times 1$
$= 2$

(c) $(2x)^0 = 1$

PRACTISE NOW 6

1. Evaluate each of the following.

(a) 2015^0 **(b)** $(-7)^0$

(c) $3y^0$ **(d)** $(3y)^0$

2. Find the value of each of the following.

(a) $3^0 \times 3^3 \div 3^2$ **(b)** $3^0 + 3^2$

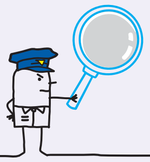
SIMILAR QUESTIONS

Exercise 3B Questions 1(a)-(f), 2(a)-(d)

Thinking Time

Is $-5^0 = (-5)^0$? Explain your answer.

Negative Indices



Investigation

Negative Indices

Now that we know $3^0 = 1$, we want to find out what 3^{-1} and 3^{-2} are equal to.

1. Continuing the same pattern in the previous investigation, complete Table 3.3.

Index Form	Value	
3^2	9	
3^1	3	$\div 3$
3^0		$\div 3$
3^{-1}		$\div 3$
3^{-2}		$\div 3$

Table 3.3

Problem Solving Tip

$$\frac{1}{3} \div 3 = \frac{1}{3} \times \frac{1}{3}$$
$$= \frac{1}{3 \times 3}$$
$$= \frac{1}{3^2}$$

2. Continuing the same pattern in the previous investigation, complete Table 3.4.

Index Form	Value	
$(-2)^2$	4	
$(-2)^1$	-2	$\div(-2)$
$(-2)^0$		$\div(-2)$
$(-2)^{-1}$		$\div(-2)$
$(-2)^{-2}$		$\div(-2)$

Table 3.4

3. What do you think 0^{-2} is equal to? Explain your answer.

Since $a^4 \div a^7 = \dfrac{a \times a \times a \times a}{a \times a \times a \times a \times a \times a \times a}$

$$= \frac{1}{a^3},$$

and using the second law of indices,

$$a^4 \div a^7 = a^{4-7}$$
$$= a^{-3}.$$

$$\therefore a^{-3} = \frac{1}{a^3}$$

$\therefore$ For the second law of indices to apply for negative indices, we define a^{-n} as $\dfrac{1}{a^n}$, where a is a real number and $a \neq 0$.

Definition 2: $a^{-n} = \dfrac{1}{a^n}$, if $a \neq 0$

Worked Example 7

(Negative Indices)

Evaluate each of the following.

(a) 4^{-3} **(b)** $(-7)^{-1}$

(c) $\left(\frac{3}{5}\right)^{-2}$

Solution:

(a) $4^{-3} = \frac{1}{4^3}$

$= \frac{1}{64}$

(b) $(-7)^{-1} = \frac{1}{(-7)^1}$

$= -\frac{1}{7}$

(c) $\left(\frac{3}{5}\right)^{-2} = \frac{1}{\left(\frac{3}{5}\right)^2}$

$= 1 \div \frac{3^2}{5^2}$ (Law 3 and Law 5)

$= 1 \times \frac{5^2}{3^2}$

$= \frac{25}{9}$

$= 2\frac{7}{9}$

PRACTISE NOW 7

Evaluate each of the following.

(a) 6^{-2} **(b)** $(-8)^{-1}$

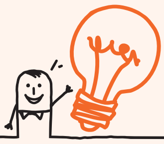
(c) $\left(\frac{4}{5}\right)^{-3}$ **(d)** $\left(\frac{1}{9}\right)^{-1}$

SIMILAR QUESTIONS

Exercise 3B Questions 3(a)-(d)

Extension of Laws of Indices to Zero and Negative Indices

In Section 3.2, we have learnt that all the 5 laws of indices apply when the indices are positive integers. With the definitions of zero and negative indices, we can now extend all the 5 laws of indices to include *all integer indices*.



Thinking Time

Copy and complete the following.

1. If a and b are real numbers, and m and n are **integers**, then

Law 1 of Indices: $a^m \times a^n =$ ________ if $a \neq 0$

Law 2 of Indices: $a^m \div a^n =$ ________ if $a \neq 0$

Law 3 of Indices: $(a^m)^n =$ ________ if ________

Law 4 of Indices: $a^n \times b^n =$ ________ if $a, b \neq 0$

Law 5 of Indices: $a^n \div b^n =$ ________ if ________

2. Notice that some conditions on the bases a and b are now different.

(i) Why is it necessary for $a \neq 0$ in Law 1?

(ii) Why is it necessary for $a, b \neq 0$ in Law 4?

3. **(i)** What happens if $m = n$ in Law 2?

(ii) What happens if $m = 0$ in Law 2?

ATTENTION

Notice that it is no longer necessary for $m > n$ for Law 2 of Indices. Why is this so?

Worked Example 8

(Applications of Laws of Indices, Zero and Negative Indices)

Simplify each of the following, leaving your answer in positive index form.

(a) $a^{-7} \times a^4 \div a^{-3}$

(b) $\dfrac{8b^{-6}c^3}{(2b^2c)^3}$

(c) $(2d)^0 \div (d^2e^{-4})^{-1}$

(d) $3a \div a^{-2} + a^2 \times a - \dfrac{6a^{-1}}{2a^{-4}}$

Solution:

(a)
$$\begin{aligned} a^{-7} \times a^4 \div a^{-3} &= a^{-7+4-(-3)} \\ &= a^{-3-(-3)} \\ &= a^0 \\ &= 1 \end{aligned}$$

(b)
$$\begin{aligned} \frac{8b^{-6}c^3}{(2b^2c)^3} &= \frac{8b^{-6}c^3}{8b^6c^3} \\ &= b^{-6-6}c^{3-3} \\ &= b^{-12}c^0 \\ &= \frac{1}{b^{12}} \quad \left(\text{since } b^{-12} = \frac{1}{b^{12}} \text{ and } c^0 = 1\right) \end{aligned}$$

(c) $(2d)^0 \div (d^2e^{-4})^{-1} = 1 \div \dfrac{1}{d^2e^{-4}}$

$= 1 \times d^2e^{-4}$

$= \dfrac{d^2}{e^4}$

(d) $3a \div a^{-2} + a^2 \times a - \dfrac{6a^{-1}}{2a^{-4}}$

$= 3a^{1-(-2)} + a^{2+1} - \dfrac{6a^{-1-(-4)}}{2}$

$= 3a^3 + a^3 - 3a^3$

$= a^3$

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 3B Questions 4(a)-(d), 10(a)-(h)

Simplify each of the following, leaving your answer in positive index form.

(a) $a^{-1} \times a^3 \div a^{-2}$

(b) $(b^{-5}c^2)^{-3}$

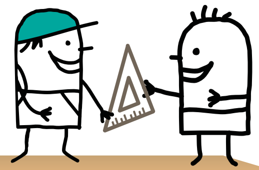
(c) $\dfrac{16d^{-2}e}{(2d^{-1}e)^3}$

(d) $5f^0 \div 3(f^{-2})^2$

(e) $18g^{-6} \div 3(g^{-2})^2$

(f) $6h^2 \div 2h^{-2} - h \times h^3 - \dfrac{4}{h^{-4}}$

3.4 Rational Indices



Positive n^{th} Root

In Book 1, we have learnt about the square root and the cube root of a number, e.g.

$\sqrt{9} = 3$ since $3^2 = 3 \times 3 = 9$

and $\sqrt[3]{27} = 3$ since $3^3 = 3 \times 3 \times 3 = 27$.

Since $3^4 = 3 \times 3 \times 3 \times 3 = 81$, then we can define the positive 4th root of 81 to be

$$\sqrt[4]{81} = 3.$$

In general,

> if a is a positive number such that $a = b^n$ for some positive number b, then b is the **positive n^{th} root** of a, and we write $b = \sqrt[n]{a}$.

An expression that involves the **radical sign** $\sqrt[n]{}$ is called a radical expression.

Worked Example 9

(Finding the Positive n^{th} Root)

Evaluate each of the following without the use of a calculator.

(a) $\sqrt[4]{625}$ **(b)** $\sqrt[5]{243}$

Solution:

(a) By prime factorisation, $625 = 5 \times 5 \times 5 \times 5 = 5^4$.

$$\therefore \sqrt[4]{625} = \sqrt[4]{5\times5\times5\times5}$$
$$= 5$$

(b) By prime factorisation, $243 = 3 \times 3 \times 3 \times 3 \times 3 = 3^5$.

$$\therefore \sqrt[5]{243} = \sqrt[5]{3\times3\times3\times3\times3}$$
$$= 3$$

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 3B Questions 5(a)-(d)

Evaluate each of the following without the use of a calculator.

(a) $\sqrt[4]{256}$ **(b)** $\sqrt[5]{1024}$

(c) $\sqrt[3]{\frac{8}{27}}$

Rational Indices

So far, all the indices that we have discussed are integers. What happens if the indices are non-integer rational numbers?

For example, what is $3^{\frac{1}{2}}$ equal to?

Let $p = 3^{\frac{1}{2}}$. Then $p^2 = \left(3^{\frac{1}{2}}\right)^2$

$$= 3^{\frac{1}{2}\times 2} \quad \text{(using Law 3 of Indices)}$$
$$= 3^1$$
$$= 3$$
$$p = \pm\sqrt{3}$$

However, we define $3^{\frac{1}{2}}$ to be equal to the **positive square root** of 3 (and not the positive and negative square roots of 3) because we want $y = a^{\frac{1}{x}}$ to be a function, i.e. for every value of x, there should be exactly one value of y.

Hence, $3^{\frac{1}{2}} = \sqrt{3}$.

Class Discussion

Rational Indices

Work in pairs.

Copy and complete the following.

What is $5^{\frac{1}{3}}$ equal to?

Let $p = 5^{\frac{1}{3}}$. Then $p^3 = (\quad)^3$

$= 5^{\quad \times 3}$ (using Law 3 of Indices)

$= 5^1$

$= 5$

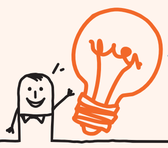
$\therefore p =$ ________

In this case, there is only one possible value of p.

Hence, $5^{\frac{1}{3}} =$ ________

In general, if n is a **positive integer**, we define

$$\boxed{\textbf{Definition 3: } a^{\frac{1}{n}} = \sqrt[n]{a}\text{, if } a > 0}$$



Thinking Time

Consider $a^{\frac{1}{n}} = \sqrt[n]{a}$.

1. What happens if $a < 0$?

2. What happens if $a = 0$?

Worked Example 10

(Rational Indices)

Rewrite each of the following in the radical form and hence evaluate the result without the use of a calculator.

(a) $16^{\frac{1}{2}}$ **(b)** $27^{-\frac{1}{3}}$

Solution:

(a) $16^{\frac{1}{2}} = \sqrt{16}$

$= 4$

(b) $27^{-\frac{1}{3}} = \dfrac{1}{27^{\frac{1}{3}}}$

$= \dfrac{1}{\sqrt[3]{27}}$

$= \dfrac{1}{3}$

ATTENTION

If $a = 0$, $a^{\frac{1}{n}} = \sqrt[n]{a}$ is not defined if n is negative, as $0^{-\frac{1}{3}} = \dfrac{1}{\sqrt[3]{0}}$.

PRACTISE NOW 10

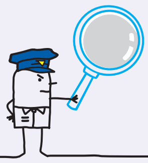
Rewrite each of the following in the radical form and hence evaluate the result without the use of a calculator.

(a) $36^{\frac{1}{2}}$ **(b)** $8^{-\frac{1}{3}}$

(c) $(-125)^{-\frac{1}{3}}$

SIMILAR QUESTIONS

Exercise 3B Questions 6(a)-(c)



Investigation

Rational Indices

Copy and complete the following.

What is $5^{\frac{2}{3}}$ equal to?

(a) $5^{\frac{2}{3}} = 5^{2 \times \frac{1}{\square}}$

$= (5^2)^{\frac{1}{\square}}$

$= \sqrt[\square]{5^2}$

(b) $5^{\frac{2}{3}} = 5^{\frac{1}{\square} \times 2}$

$= \left(5^{\frac{1}{\square}}\right)^2$

$= \left(\sqrt[\square]{5}\right)^2$

In general, if m and n are **positive integers**,

$$\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} \text{ or } \left(\sqrt[n]{a}\right)^m, \text{ if } a > 0}$$

Worked Example 11

(Rational Indices)

(a) Evaluate $125^{\frac{2}{3}}$ without the use of a calculator.

(b) Simplify $\frac{1}{\sqrt{x^n}}$, expressing your answer in index form.

Solution:

(a) $125^{\frac{2}{3}} = (\sqrt[3]{125})^2$

$= 5^2$

$= 25$

(b) $\frac{1}{\sqrt{x^n}} = \frac{1}{x^{\frac{n}{2}}}$

$= x^{-\frac{n}{2}}$

Problem Solving Tip

$125^{\frac{2}{3}} \leftarrow$ root

The denominator of the index is always the root.
You can think of it as 'below the ground'.

PRACTISE NOW 11

SIMILAR QUESTIONS

Exercise 3B Questions 6(d)-(f), 7(a)-(f)

1. Evaluate each of the following without the use of a calculator.

(a) $64^{\frac{2}{3}}$ **(b)** $32^{-\frac{3}{5}}$

(c) $100^{1.5}$

2. Simplify each of the following, expressing your answer in index form.

(a) $\sqrt[3]{a^n}$ **(b)** $\frac{1}{\sqrt[5]{x^2}}$

Extension of Laws of Indices to Rational Indices

Earlier in this section, we have extended all the 5 laws of indices to include all integer indices. With the definition of rational indices, we can now extend all the 5 laws of indices to include *all rational indices*.

Thinking Time

Copy and complete the following.

1. If a and b are real numbers, and m and n are **rational numbers**, then

Law 1 of Indices: $a^m \times a^n =$ ________ if $a > 0$

Law 2 of Indices: $a^m \div a^n =$ ________ if $a > 0$

Law 3 of Indices: $(a^m)^n =$ ________ if ________

Law 4 of Indices: $a^n \times b^n =$ ________ if $a, b > 0$

Law 5 of Indices: $a^n \div b^n =$ ________ if ________

2. Notice that some conditions on the bases a and b are now different.
   (i) Why is it necessary for $a > 0$ in Law 1?
   (ii) Why is it necessary for $a, b > 0$ in Law 4?
3. What happens if you do not take care of the conditions?
   The following shows a ridiculous proof that conclude that $1 = -1$.
   Explain what is wrong with the proof.
   $$1 = \sqrt{1} = \sqrt{(-1)\times(-1)} = \sqrt{-1}\times\sqrt{-1} = \left(\sqrt{-1}\right)^2 = (-1)^{\frac{1}{2}\times 2} = (-1)^1 = -1$$

Worked Example 12

(Rational Indices)

Simplify each of the following, expressing your answers in positive index form.

(a) $\sqrt[3]{m}\times\sqrt[4]{m^3}$ **(b)** $\left(m^{\frac{1}{3}}n^{-2}\right)^{\frac{3}{5}}$

(c) $(mn)^{\frac{2}{3}}\div\left(m^{\frac{3}{4}}n^{\frac{1}{3}}\right)^2$

Solution:

(a)
$$\begin{aligned}\sqrt[3]{m}\times\sqrt[4]{m^3} &= m^{\frac{1}{3}}\times m^{\frac{3}{4}}\\ &= m^{\frac{1}{3}+\frac{3}{4}}\\ &= m^{\frac{13}{12}}\end{aligned}$$

(b)
$$\begin{aligned}\left(m^{\frac{1}{3}}n^{-2}\right)^{\frac{3}{5}} &= m^{\frac{1}{3}\times\frac{3}{5}}n^{\left(-2\times\frac{3}{5}\right)}\\ &= m^{\frac{1}{5}}n^{-\frac{6}{5}}\\ &= \frac{m^{\frac{1}{5}}}{n^{\frac{6}{5}}}\end{aligned}$$

(c)
$$\begin{aligned}(mn)^{\frac{2}{3}}\div\left(m^{\frac{3}{4}}n^{\frac{1}{3}}\right)^2 &= m^{\frac{2}{3}}n^{\frac{2}{3}}\div m^{\left(\frac{3}{4}\times 2\right)}n^{\left(\frac{1}{3}\times 2\right)}\\ &= m^{\frac{2}{3}}n^{\frac{2}{3}}\div m^{\frac{3}{2}}n^{\frac{2}{3}}\\ &= m^{\frac{2}{3}-\frac{3}{2}}n^{\frac{2}{3}-\frac{2}{3}}\\ &= m^{\frac{2}{3}-\frac{3}{2}}n^0\\ &= m^{-\frac{5}{6}}(1)\\ &= \frac{1}{m^{\frac{5}{6}}}\end{aligned}$$

PRACTISE NOW 12

SIMILAR QUESTIONS

Exercise 3B Questions 11(a)-(f), 12(a)-(f), 16(a)-(d)

Simplify each of the following, expressing your answers in positive index form.

(a) $(m^2)^{\frac{5}{6}}\times m^{\frac{1}{3}}$ **(b)** $\sqrt[5]{m}\div\sqrt[3]{m^2}$ **(c)** $(m^{-3}n^5)^{-\frac{1}{3}}$

(d) $\dfrac{m^{-\frac{1}{3}}n^{-\frac{1}{4}}}{\left(m^2n^{-\frac{1}{3}}\right)^{-2}}$ **(e)** $(25m^2n^{-4})^{\frac{1}{2}}\left(m^3n^{-\frac{2}{5}}\right)^2$ **(f)** $\left(m^2n^{-\frac{1}{7}}\right)\times\sqrt[5]{(m^5n^{-5})}$

Equations involving Indices

To solve an equation such as $x^2 = 100$, we take the square root on both sides to obtain $x = \pm 10$. Similarly, to solve the equation $y^3 = 64$, we take the cube root on both sides to obtain $y = 4$.

If we are given the equation $2^x = 32$, how do we find the value of x?

Worked Example 13

(Equations involving Indices)

Solve each of the following equations.

(a) $2^x = 32$ **(b)** $3^y = \frac{1}{9}$

(c) $9^z = 27$

Solution:

(a)
$$\begin{aligned} 2^x &= 32 \\ &= 2^5 \\ x &= 5 \end{aligned}$$

(b)
$$\begin{aligned} 3^y &= \frac{1}{9} \\ &= \frac{1}{3^2} \\ &= 3^{-2} \\ y &= -2 \end{aligned}$$

(c)
$$\begin{aligned} 9^z &= 27 \\ (3^2)^z &= 3^3 \\ 3^{2z} &= 3^3 \\ 2z &= 3 \\ z &= \frac{3}{2} \\ &= 1\frac{1}{2} \end{aligned}$$

PRACTISE NOW 13

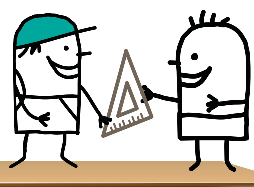
SIMILAR QUESTIONS

Exercise 3B Questions 8(a)-(d)

Solve each of the following equations.

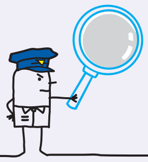
(a) $5^x = 125$ **(b)** $7^y = \frac{1}{49}$ **(c)** $8^z = 16$

3.5 Compound Interest



In Book 1, we have learnt that the amount of simple interest one has to pay depends on the amount of money borrowed, i.e. the principal, the interest rate per annum and the loan period in years.

For simple interest, the interest earned every year is the same because it is calculated based on the original principal. Now what happens if the interest is **compounded** yearly? This means that after each year, the interest earned will be counted in the new principal to earn more interest.



Investigation

Simple Interest and Compound Interest

Mr Wong wants to place \$1000 in a bank as a fixed deposit for 3 years.

Bank *A* offers a *simple* interest rate of 2% per annum.

1. Calculate the interest earned and the total amount of money he will have after 3 years.

Bank *B* offers an interest rate of 2% per annum *compounded* yearly.

2. Copy and complete the following to find the interest earned and the total amount of money he will have at the end of each year.

1st year: Principal $P_1 = \$1000$,

Interest $I_1 = \$1000 \times 2\%$

$= \$\_\_\_\_\_\_\_\_$

Total amount at the end of the 1st year, $A_1 = P_1 + I_1$

$= \$1000 + \$\_\_\_\_\_\_\_\_$

$= \$1020$

2nd year: Principal $P_2 = A_1 = \$1020$,

Interest $I_2 = \$\_\_\_\_\_\_\_\_ \times 2\%$

$= \$\_\_\_\_\_\_\_\_$

Total amount at the end of the 2nd year, $A_2 = P_2 + I_2$

$= \$1020 + \$\_\_\_\_\_\_\_\_$

$= \$\_\_\_\_\_\_\_\_$

3rd year: Principal $P_3 = A_2 = \$\_\_\_\_\_\_\_\_$,

Interest $I_3 = \$\_\_\_\_\_\_\_\_ \times 2\%$

$= \$\_\_\_\_\_\_\_\_$

Total amount at the end of the 3rd year, $A_3 = P_3 + I_3$

$= \$1040.40 + \$\_\_\_\_\_\_\_\_$

$= \$\_\_\_\_\_\_\_\_$ (to the nearest cent)

3. Which bank offers a higher interest and by how much?

From the investigation, finding the compound interest for each year is very tedious. What happens if Mr Wong puts \$1000 in Bank B for 10 years? How do we calculate the compound interest at the end of 10 years?

Hence, there is a need to find a formula to calculate compound interest easily.

From the investigation, we observe the following:

Total amount at the end of the 1st year, $A_1 = P_1 + I_1$

$$= \$1000 + \$1000 \times \frac{2}{100}$$

$$= \$1000\left(1+\frac{2}{100}\right)$$

Total amount at the end of the 2nd year,

$A_2 = P_2 + I_2$

$$= \$1000\left(1+\frac{2}{100}\right) + \$1000\left(1+\frac{2}{100}\right) \times \frac{2}{100}$$

$$= \$1000\left(1+\frac{2}{100}\right)\left(1+\frac{2}{100}\right) \quad \text{(Extract common factor } 1000\left(1+\frac{2}{100}\right)\text{)}$$

$$= \$1000\left(1+\frac{2}{100}\right)^2$$

Total amount at the end of the 3rd year,

$A_3 = P_3 + I_3$

$$= \$1000\left(1+\frac{2}{100}\right)^2 + \$1000\left(1+\frac{2}{100}\right)^2 \times \frac{2}{100}$$

$$= \$1000\left(1+\frac{2}{100}\right)^2\left(1+\frac{2}{100}\right) \quad \text{(Extract common factor } 1000\left(1+\frac{2}{100}\right)^2\text{)}$$

$$= \$1000\left(1+\frac{2}{100}\right)^3$$

By looking at the pattern above, what do you think the total amount of money at the end of 4th year will be? \$________________

In general, the formula for finding **compound interest** is

$$\boxed{A = P\left(1+\frac{R}{100}\right)^n,}$$

where $\boldsymbol{A}$ is the **total amount**, $\boldsymbol{P}$ is the principal, $\boldsymbol{R\%}$ is the **interest rate per annum** (or year) and $\boldsymbol{n}$ is the **number of years**.

Note that if the interest is **compounded monthly**, then $R\%$ is the **interest rate per month** and n is the **number of months**.

INFORMATION

What happens if the interest is compounded daily instead of monthly or yearly?

What happens if the interest is compounded every hour? Every minute? Every second?

That is, what happens if interest is compounded continuously?

Then you will get this formula:

$$A = Pe^{\frac{Rn}{100}},$$

where $e \approx 2.718$ is a mathematical constant.

You may be tempted to think that interest compounded continuously will earn you a lot more interest.

Try using this formula in Worked Example 14 and you will observe that the interest earned is only a bit more. Why is this so?

Worked Example 14

(Compound Interest)

Find the compound interest on \$5000 for 7 years at 3% per annum, compounded annually.

Solution:

$P = \$5000$, $R = 3$, $n = 7$

At the end of 7 years, total amount accumulated is

$$A = P\left(1+\frac{R}{100}\right)^n$$

$$= \$5000\left(1+\frac{3}{100}\right)^7$$

$$= \$6149.37 \text{ (to the nearest cent)}$$

$$\therefore \text{Compound interest } I = A - P$$

$$= \$6149.37 - \$5000$$

$$= \$1149.37$$

Problem Solving Tip

Unlike the simple interest formula where the interest I can be found directly, the compound interest formula does not allow you to find I directly.

PRACTISE NOW 14

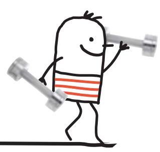
1. Find the compound interest on \$3000 for 4 years at 5% per annum, compounded annually.
2. Find the compound interest on \$1500 for 2 years at 2% per annum, compounded **(a)** annually, **(b)** monthly.
3. Mrs Lee places \$4000 in a bank as a fixed deposit for 2 years. The bank offers an interest compounded yearly. At the end of 2 years, she receives a total of \$4243.60. Find the interest rate.

SIMILAR QUESTIONS

Exercise 3B Questions 9, 13-15, 17

Problem Solving Tip

For Question 2, find the interest rate per month first.



Exercise 3B

BASIC LEVEL

1. Evaluate each of the following.

   (a) 17^0 **(b)** $\left(-\frac{2}{7}\right)^0$

   (c) $4a^0$ **(d)** $-8b^0$

   (e) $(72cd^2)^0$ **(f)** $7(e^8)^0$

2. Find the value of each of the following.

   (a) $2^0 \times 2^4$ **(b)** $7^2 \times 7^0 \div 7$

   (c) $8^0 - 8^2$ **(d)** $6^3 + 6^0 - 6$

3. Evaluate each of the following.

   (a) 7^{-3} **(b)** $(-5)^{-1}$

   (c) $\left(\frac{3}{4}\right)^{-2}$ **(d)** $\left(\frac{5}{3}\right)^{-1}$

4. Evaluate each of the following.

   (a) $(7^2)^{-2} \div 7^{-4}$ **(b)** $5^0 - 5^{-2}$

   (c) $(2^{15})^0 + \left(\frac{3}{5}\right)^{-1}$ **(d)** $\left(\frac{3}{4}\right)^{-2} \times 3^2 \times 2015^0$

5. Evaluate each of the following without the use of a calculator.

(a) $\sqrt{196}$ (b) $\sqrt[3]{125}$

(c) $\sqrt[5]{\dfrac{1}{32}}$ (d) $\sqrt[4]{\dfrac{16}{81}}$

6. Rewrite each of the following in the radical form and hence evaluate the result without the use of a calculator.

(a) $81^{\frac{1}{2}}$ (b) $(-27)^{\frac{1}{3}}$

(c) $16^{-\frac{1}{4}}$ (d) $4^{1.5}$

(e) $8^{-\frac{5}{3}}$ (f) $(-1000)^{\frac{2}{3}}$

7. Simplify each of the following, expressing your answer in index form.

(a) $\sqrt[4]{a}$ (b) $\sqrt[3]{b^2}$

(c) $(\sqrt[5]{c})^4$ (d) $\dfrac{1}{\sqrt[6]{d}}$

(e) $\dfrac{1}{\sqrt[8]{e^4}}$ (f) $\dfrac{1}{(\sqrt[3]{f})^5}$

8. Solve each of the following equations.

(a) $11^a = 1331$ (b) $2^b = \dfrac{1}{128}$

(c) $9^c = 243$ (d) $10^d = 0.01$

9. Kate places \$5000 in her bank account. The bank offers an interest of 8% per annum compounded yearly. Find the total interest in her account at the end of 3 years.

INTERMEDIATE LEVEL

10. Simplify each of the following, expressing your answers in positive index form.

(a) $5a^4 \times 3a^2 \div a^{-3}$ (b) $-24b^{-6} \div (3b^{-3})^2$

(c) $(3c)^0 \div (c^{-3}d^5)^{-2}$ (d) $\dfrac{(4e^{-6}f^3)^2}{8e^{12}f^6}$

(e) $(3g^{-3}h^{-1})^2 \times (-4g^3h^{-2})^2$ (f) $(j^2k^{-1})^{-3} \times \left(\dfrac{j^2}{k^3}\right)^{-3}$

(g) $\dfrac{(m^5n^3) \times (m^2)^{-2}}{(m^{-1}n)^2}$ (h) $(5p)^3 - 10p \times 7p^2 + \dfrac{6}{p^{-3}}$

11. Simplify each of the following, expressing your answer in positive index form.

(a) $\sqrt{a} \times \sqrt[3]{a}$ (b) $\sqrt[3]{b^2} \div \sqrt[6]{b}$

(c) $c^{\frac{4}{5}} \times c^{\frac{1}{2}} \div c^{-\frac{2}{5}}$ (d) $d^{\frac{1}{10}} \div d^{-\frac{1}{5}} \times d^{-\frac{3}{2}}$

(e) $(e^{-3}f^4)^{-\frac{1}{2}}$ (f) $\left(g^{\frac{2}{3}}h^{-\frac{4}{5}}\right)^{\frac{3}{2}}$

12. Simplify each of the following, expressing your answer in positive index form.

(a) $(a^{-2}b^3)^{\frac{1}{3}} \times (a^4b^{-5})^{\frac{1}{2}}$ (b) $\left(c^{-3}d^{\frac{3}{5}}\right)^{-2} \times \left(c^{\frac{4}{5}}d^{-\frac{2}{5}}\right)^5$

(c) $\dfrac{e^{-\frac{1}{3}}f^{-\frac{1}{4}}}{\left(e^2f^{-\frac{1}{3}}\right)^{-2}}$ (d) $\left(\dfrac{g^{-2}h^2}{25}\right)^{-\frac{1}{2}}$

(e) $(4j^4k)^{\frac{1}{2}} \div 2h^3k^{-\frac{1}{2}}$ (f) $\left(m^3n^{-\frac{1}{4}}\right)^4 \div \sqrt[5]{32m^4n^{-8}}$

13. Rui Feng deposited \$15 000 in an account that pays 5.68% compound interest per year. Find the total amount in the account after 6 years if the interest is compounded

(a) monthly, (b) half-yearly.

14. Mr Tan invested \$5000 in an endowment fund for 5 years. The fund pays an interest compounded yearly. At the end of 5 years, he received a total of \$5800. Find the interest rate.

15. Mr Chua borrows a sum of money from the bank which charges a compound interest of 4.2% per annum, compounded quarterly. Given that Mr Chua had to pay \$96.60 in interest payments at the end of the first year, find the original sum of money borrowed, giving your answer correct to the nearest cent.

ADVANCED LEVEL

16. Simplify each of the following, expressing your answers in positive index form.

(a) $\left(\dfrac{x^{-4}y^{7}z^{-6}}{x^{3}y^{-1}z^{3}}\right)^{3} \times \left(\dfrac{x^{5}y^{2}z^{-6}}{x^{-3}y^{-5}z^{4}}\right)^{-4}$

(b) $\left(\dfrac{x^{3}y^{-4}z^{7}}{x^{-5}y^{2}}\right)^{3} \div \left(\dfrac{x^{-4}yz^{-5}}{x^{7}y^{-3}}\right)^{-2}$

(c) $\dfrac{ab^{n}}{bc} \times \dfrac{c^{n}d}{cd} \div \dfrac{b^{n+2}}{c^{n+3}}$

(d) $\dfrac{(a+b)^{n}}{bc^{2}} \div \dfrac{(a+b)^{n+3}}{abc}$

17. Kate has \$8000 to invest in either Company *A* or Company *B*.

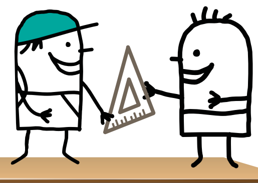
> Company *A*: 4.9% per annum simple interest
> Company *B*: 4.8% per annum compound interest, compounded half-yearly

Kate wishes to invest the money for a period of 4 years.

(i) Which company should she invest in? Explain your answer.

(ii) Calculate the difference in interest earned after 4 years.

3.6 Standard Form



Class Discussion

Standard Form

Work in pairs.

Table 3.5 shows some examples of measurements which involve very large or very small numbers.

		Ordinary Notation	Standard Form
(i)	Singapore's population in 2013	5 300 000	5.3×10^{6}
(ii)	Distance between Earth and the sun	149 600 000 km	1.496×10^{8} km
(iii)	Mass of a dust particle	0.000 000 000 753 kg	7.53×10^{-10} kg
(iv)	Mass of an oxygen atom	0.000 000 000 000 000 000 000 000 002 656 kg	2.656×10^{-27} kg
(v)	Number of grains of sand in a bag	29 000	$2.9 \times 10^{\square}$
(vi)	Speed of light	300 000 000 m/s	$3 \times 10^{\square}$ m/s
(vii)	Wavelength of violet light	0.000 038 cm	$3.8 \times 10^{\square}$ cm
(viii)	Mass of a water molecule	0.000 000 000 000 000 000 000 0299	$2.99 \times 10^{\square}$ g

Table 3.5

1. The examples in **(i)-(ii)** involve very large numbers. What do you observe about the powers of 10 in each standard form?
2. The examples in **(iii)-(iv)** involve very small numbers. What do you observe about the powers of 10 in each standard form?
3. Complete the last column for **(v)-(viii)** in Table 3.5.

Table 3.6 shows numbers expressed in standard form and numbers not expressed in standard form.

	Standard Form	**Not Standard Form**
(i)	4.5×10^4	45×10^3
(ii)	2.06×10^8	0.206×10^5
(iii)	3.71×10^{21}	$3.71 \times 10^{21.2}$
(iv)	8.00×10^{-3}	$8.00 \times 10^{-3\frac{1}{2}}$
(v)	9.25×10^{-10}	$92.5 \times 10^{-1.01}$
(vi)	1.0×10^{-16}	10×10^{-17}

Table 3.6

4. For a number in the form $A \times 10^n$ to be considered as standard form, what can you say about A and n? Explain your answer.

The world population, estimated to be about 7 000 000 000 in 2012, can be written as 7.0×10^9. The Bohr radius of a hydrogen atom is 0.000 000 000 053 m, which can be written as 5.3×10^{-11} m.

Both 7.0×10^9 and 5.3×10^{-11} are examples of numbers expressed in standard form.

In general, a number is said to be expressed in **standard form**, or **scientific notation**, when it is written as

$\boldsymbol{A \times 10^n}$, where $\boldsymbol{1 \leqslant A < 10}$ and $\boldsymbol{n}$ is an **integer**.

Worked Example 15

(Standard Form)

Express each of the following numbers in standard form.

(a) 149 600 000 **(b)** 0.000 038

Solution:

(a) $149\,600\,000 = 1.496 \times 10^8$ (move the decimal point 8 places to the left)

(b) $0.000\,038 = 3.8 \times 10^{-5}$ (move the decimal point 5 places to the right)

PRACTISE NOW 15

SIMILAR QUESTIONS

Exercise 3C Questions 1(a)-(d), 2(a)-(d)

1. Express each of the following numbers in standard form.
   (a) 5 300 000 **(b)** 600 000 000
   (c) 0.000 048 **(d)** 0.000 000 000 21
2. Express each of the following as an integer or a decimal.
   (a) 1.325×10^6 **(b)** 4.4×10^{-3}

Common Prefixes

Have you used an external hard disk with a capacity of 512 *giga*bytes? Have you used a micrometer in the laboratory?

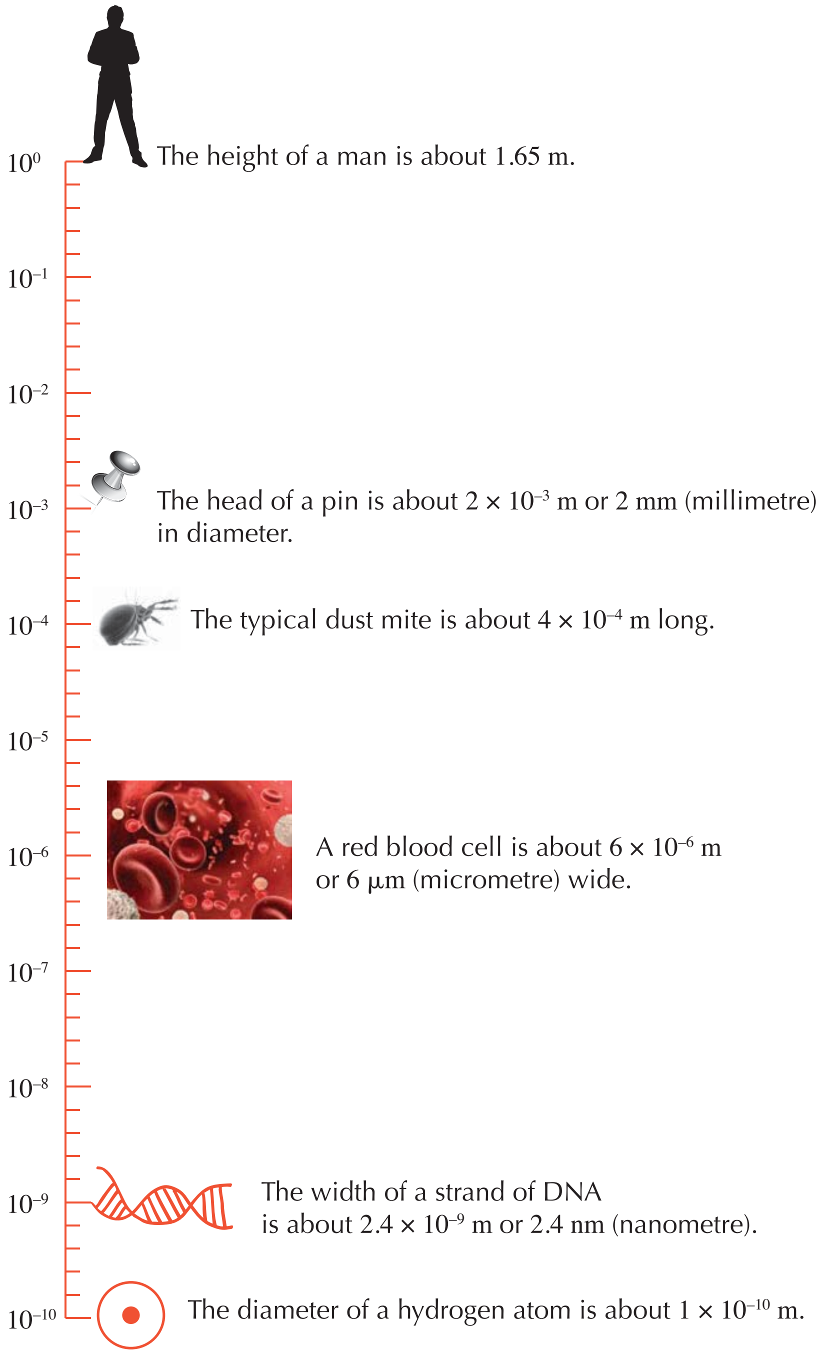
Prefixes are commonly used in our daily lives to denote certain powers of 10. They are related to the special names in the SI system of units. Table 3.7 lists some of the common prefixes and their symbols used for very large and very small numbers.

Power of 10	Name	SI Prefix	Symbol	Numerical Value
10^{12}	trillion	tera-	T	1 000 000 000 000
10^9	billion	giga-	G	1 000 000 000
10^6	million	mega-	M	1 000 000
10^3	thousand	kilo-	k	1000
10^{-3}	thousandth	milli-	m	$0.001 = \frac{1}{1000}$
10^{-6}	millionth	micro-	μ	$0.000\,001 = \frac{1}{1\,000\,000}$
10^{-9}	billionth	nano-	n	$0.000\,000\,001 = \frac{1}{1\,000\,000\,000}$
10^{-12}	trillionth	pico-	p	$0.000\,000\,000\,001 = \frac{1}{1\,000\,000\,000\,000}$

Table 3.7

Hence, 8.2×10^9 is read as 8.2 billion. How do you read 3.7×10^{12}?
5.6×10^{-6} is read as 5.6 millionth. How do you read 4.9×10^{-9}?

The figure below shows a range of prefixes used in our daily lives.



INFORMATION

Do you know that one billion is not always 10^9 or one thousand million? In the 'long scale' used by most European countries (including the UK) in most of the 19th and 20th centuries, one billion means 10^{12} or one million million. In 1974, the UK switched to the 'short scale', which is used by the USA. In the 'short scale', one billion is 10^9 or one thousand million. Search on the Internet for 'long and short scales' to find out more.

Internet Resources

Search on the Internet for 'powers of 10' to view a video starting from a picnic in Chicago to our galaxy 10^{24} metres away and then back to the picnic to within a proton at 10^{-16} metres.

Worked Example 16

(Common Prefixes in our Daily Lives)

For each of the following, give your answer in standard form.

(a) A male African elephant can weigh as heavy as 7000 kilograms. Express this weight in grams.

(b) The average lifespan of a certain molecule is 0.5 nanoseconds.
Given that 1 nanosecond = 10^{-9} seconds, express this time in seconds.

(c) A steam power plant in Singapore has a capacity of 250 megawatts.
Given that 1 megawatt = 10^{6} watts, express this capacity in watts.

Solution:

(a) 7000 kilograms
$= 7000 \times 10^3$ grams
$= 7 \times 10^3 \times 10^3$ grams
$= 7 \times 10^{3+3}$ grams
$= 7 \times 10^6$ grams

(b) 0.5 nanoseconds
$= 0.5 \times 10^{-9}$ seconds
$= 5 \times 10^{-1} \times 10^{-9}$ seconds
$= 5 \times 10^{-1+(-9)}$ seconds
$= 5 \times 10^{-10}$ seconds

(c) 250 megawatts
$= 250 \times 10^6$ watts
$= 2.5 \times 10^2 \times 10^6$ watts
$= 2.5 \times 10^{2+6}$ watts
$= 2.5 \times 10^8$ watts

PRACTISE NOW 16

For each of the following, give your answer in standard form.

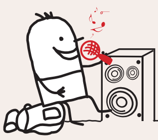
(a) The diameter of a human hair is 25.4 micrometres.
Given that 1 micrometre = 10^{-6} metres, express this diameter in metres.

(b) A rain gauge measures rainfall over a period of time. The average annual rainfall in Singapore is 2340 mm. Express this measurement in centimetres.

(c) An external hard drive has a capacity of 4.0 terabytes .
Given that 1 terabyte = 10^{12} bytes, express this capacity in bytes.

SIMILAR QUESTIONS

Exercise 3C Questions 3-5



Performance Task

If you are buying 1 thumbdrive with a capacity of 1 gigabyte (1 GB), are you getting exactly 1 billion bytes of computer space? Search on the Internet to find out how many bytes 1 GB is actually equal to. Why is it not possible for the manufacturer to produce a thumbdrive with exactly 1 billion bytes?

Hint: How does the number 2 in 2^{30} play a part here?

Search on the Internet to find out why computer storage systems always come in the form of 128 MB, 256 MB, 512 MB and so on. Present your findings to your class.

Worked Example 17

(Applying the Orders of Operation in Standard Form)

With the use of a calculator, find the value of each of the following, giving your answer in standard form, correct to 3 significant figures.

(a) $(7.2 \times 10^3) \times (2.05 \times 10^5)$ **(b)** $\dfrac{4.5\times10^{-3}}{9\times10^{-1}}$

(c) $1.35 \times 10^3 + 2.37 \times 10^4$ **(d)** $\dfrac{2.58\times10^{-3}-4.19\times10^{-4}}{3.17\times10^{2}}$

Solution:

(a) $(7.2 \times 10^3) \times (2.05 \times 10^5)$

$= \boxed{(}\,\boxed{7}\,\boxed{.}\,\boxed{2}\,\boxed{\times10^x}\,\boxed{3}\,\boxed{)}\,\boxed{\times}\,\boxed{(}\,\boxed{2}\,\boxed{.}\,\boxed{0}\,\boxed{5}\,\boxed{\times10^x}\,\boxed{5}\,\boxed{)}\,\boxed{=}$

$= 1\ 476\ 000\ 000$

$= 1.48 \times 10^9$ (to 3 s.f.)

(b) $\dfrac{4.5\times10^{-3}}{9\times10^{-1}}$

$= \boxed{(}\,\boxed{4}\,\boxed{.}\,\boxed{5}\,\boxed{\times10^x}\,\boxed{-}\,\boxed{3}\,\boxed{)}\,\boxed{\div}\,\boxed{(}\,\boxed{9}\,\boxed{\times10^x}\,\boxed{-}\,\boxed{1}\,\boxed{)}\,\boxed{=}$

$= 5 \times 10^{-3}$

(c) $1.35 \times 10^3 + 2.37 \times 10^4$

$= \boxed{1}\,\boxed{.}\,\boxed{3}\,\boxed{5}\,\boxed{\times10^x}\,\boxed{3}\,\boxed{+}\,\boxed{2}\,\boxed{.}\,\boxed{3}\,\boxed{7}\,\boxed{\times10^x}\,\boxed{4}\,\boxed{=}$

$= 25\ 050$

$= 2.5050 \times 10^4$

$= 2.51 \times 10^4$ (to 3 s.f.)

(d) $\dfrac{2.58\times10^{-3}-4.19\times10^{-4}}{3.17\times10^{2}}$

$= \boxed{(}\,\boxed{2}\,\boxed{.}\,\boxed{5}\,\boxed{8}\,\boxed{\times10^x}\,\boxed{-}\,\boxed{3}\,\boxed{-}\,\boxed{4}\,\boxed{.}\,\boxed{1}\,\boxed{9}\,\boxed{\times10^x}\,\boxed{-}\,\boxed{4}\,\boxed{)}$

$\boxed{\div}\,\boxed{3}\,\boxed{.}\,\boxed{1}\,\boxed{7}\,\boxed{\times10^x}\,\boxed{2}\,\boxed{=}$

$= 6.817 \times 10^{-6}$

$= 6.82 \times 10^{-6}$ (to 3 s.f.)

ATTENTION

The buttons on calculators vary with each model. Refer to the manual of your calculator.

PRACTISE NOW 17

With the use of a calculator, find the value of each of the following, giving your answer in standard form, correct to 3 significant figures.

(a) $(1.14 \times 10^5) \times (4.56 \times 10^4)$

(b) $(4.2 \times 10^{-4}) \times (2.6 \times 10^2)$

(c) $(2.4 \times 10^8) \div (6 \times 10^4)$

(d) $\dfrac{3.5\times10^{-5}}{1.4\times10^{8}}$

(e) $1.14 \times 10^5 + 4.56 \times 10^4$

(f) $4 \times 10^4 - 2.6 \times 10^6$

(g) $\dfrac{2.37\times10^{-3}+3.25\times10^{-4}}{4.1\times10^{5}}$

(h) $\dfrac{6.3\times10^{6}}{1.5\times10^{2}-3\times10^{-1}}$

SIMILAR QUESTIONS

Exercise 3C Questions 6(a)-(h), 7(a)-(f), 8-11

Worked Example 18

(Applications of Standard Form)

The approximate mass of the moon is 7×10^{19} tonnes while the mass of the earth is approximately 6×10^{24} tonnes. Calculate the number of times that the earth is as heavy as the moon, giving your answer correct to the nearest 1000.

Solution:

Number of times the earth is as heavy as the moon $= \dfrac{6 \times 10^{24}}{7 \times 10^{19}}$

$$= \frac{6}{7} \times 10^{24-19}$$

$$= \frac{6}{7} \times 10^{5}$$

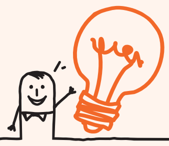
$= 86\ 000$ (to the nearest 1000)

PRACTISE NOW 18

SIMILAR QUESTIONS

Exercise 3C Questions 12-14

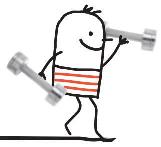
A Secure Digital (SD) memory card has a capacity of 512 megabytes. Each photograph has a size of 640 kilobytes. Assuming that 1 MB = 10^6 bytes and 1 kB = 10^3 bytes , how many photographs can this memory card store?



Thinking Time

Fill in the blanks to convert between ordinary notation and standard form.

1. The distance of the planet Mercury from the sun is about 57 910 000 km (______ × 10^7 km) while the distance of Pluto from the sun is about 5 945 900 000 km (______ × $10^{\square}$ km).
2. The moon travels around the earth at a speed of about 3 683 000 m/h (______ × $10^{\square}$ km/h).
3. If the diameter of an air molecule is about 0.000 000 0004 m (______ × $10^{\square}$ m) and there are 500 000 000 000 000 000 000 000 000 (______ × $10^{\square}$) molecules in a room, then the total volume of the air molecules in the room is ______ × $10^{\square}$ m^3.
4. There are about 100 trillion (______ × $10^{\square}$) cells in the human body. The human heart contains about ______ million (2×10^9) cells and beats about 42 000 000 (______ × $10^{\square}$) times each year.



Exercise 3C

BASIC LEVEL

1. Express each of the following numbers in standard form.

(a) 85 300 **(b)** 52 700 000

(c) 0.000 23 **(d)** 0.000 000 094

2. Express each of the following as an integer or a decimal.

(a) 9.6×10^3 **(b)** 4×10^5

(c) 2.8×10^{-4} **(d)** 1×10^{-6}

3. Microwaves are a form of electromagnetic radiation with frequencies between 300 000 000 Hz and 300 GHz. Giving your answer in standard form, express

(i) 300 000 000 Hz in MHz,

(ii) 300 GHz in MHz.

(1 MHz = 10^6 Hz and 1 GHz = 10^9 Hz)

INTERMEDIATE LEVEL

4. **(i)** A nitrogen atom has an atomic radius of a picometres (pm), where $a = 70$ and 1 pm $= 10^{-12}$ m. Express this radius in metres. Give your answer in standard form.

(ii) An oxygen atom has an atomic radius of b nanometres (nm), where $b = 0.074$ and 1 nm $= 10^{-9}$ m. Express this radius in metres. Give your answer in standard form.

(iii) Express $a : b$ as a ratio of two integers in its simplest form.

5. The mean distance from the earth to the sun is c megametres (Mm), where $c = 1500$ and 1 Mm $= 10^6$ m. The mean distance from Pluto to the sun is d terametres (Tm), where $d = 5.91$ and 1 Tm $= 10^{12}$ m. Express d as a percentage of c. Give your answer in standard form.

6. With the use of a calculator, find the value of each of the following, giving your answer in standard form, correct to 3 significant figures.

(a) $(2.34 \times 10^5) \times (7.12 \times 10^{-4})$

(b) $(5.1 \times 10^{-7}) \times (2.76 \times 10^{-3})$

(c) $(13.4 \times 10^4) \div (4 \times 10^5)$

(d) $\dfrac{3 \times 10^{-4}}{9 \times 10^{-8}}$

(e) $2.54 \times 10^3 + 3.11 \times 10^4$

(f) $6 \times 10^5 - 3.1 \times 10^7$

(g) $\dfrac{4.37 \times 10^{-4} + 2.16 \times 10^{-5}}{3 \times 10^{-3}}$

(h) $\dfrac{2.4 \times 10^{-10}}{7.2 \times 10^{-6} - 3.5 \times 10^{-8}}$

7. With the use of a calculator, find the value of each of the following, giving your answer in standard form, correct to 3 significant figures.

(a) $(1.35 \times 10^{-4})^3$ **(b)** $6(3.4 \times 10^3)^2$

(c) $\sqrt{1.21 \times 10^8}$ **(d)** $\sqrt[3]{9.261 \times 10^6}$

(e) $\dfrac{2.3 \times 10^{-2} \times 4.7 \times 10^3}{2 \times 10^3}$ **(f)** $\dfrac{8 \times 10^2 + 2.5 \times 10^3}{2 \times 10^{-2} - 3.4 \times 10^{-3}}$

8. Given that $P = 7.5 \times 10^3$ and $Q = 5.25 \times 10^4$, express each of the following in standard form.

(a) $2P \times 4Q$ **(b)** $Q - P$

9. Given that $x = 2 \times 10^{-3}$ and $y = 7 \times 10^{-4}$, evaluate $x + 8y$, giving your answer in standard form.

10. Given that $M = 3.2 \times 10^6$ and $N = 5.0 \times 10^7$, find the value of each of the following, giving your answer in standard form.

(a) MN **(b)** $\dfrac{M}{N}$

11. Given that $R = \dfrac{M}{EI}$, find the value of R when $M = 6 \times 10^4$, $E = 4.5 \times 10^8$ and $I = 4 \times 10^2$. Give your answer in standard form.

12. Light travels at a speed of 300 000 000 m/s.

(i) Express this speed in standard form.

(ii) Given that the mean distance from the sun to Jupiter is 778.5 million kilometres, find the time taken, in minutes and seconds, for light to travel from the sun to Jupiter.

13. On a journey from Planet P to Venus, a rocket is travelling at a constant speed. During this journey, the rocket travels past the moon in 4 days. The distance from Planet P to the moon is 4.8×10^5 km.

(i) Find the distance travelled by the rocket in 12 days. Give your answer in standard form.

(ii) Given that the distance between Planet P and Venus is 4.8×10^7 km, find the time taken, in days, for the journey.

ADVANCED LEVEL

14. The table shows the approximate population of the world in the past centuries.

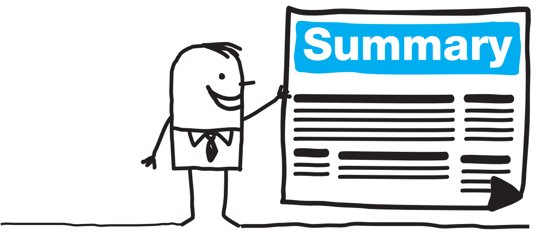
Year	World population
1549	4.20×10^8
1649	5.45×10^8
1749	7.28×10^8
1849	1.17×10^9

Find

(i) the increase in population from 1549 to 1649,

(ii) the number of times that the population in 1849 is as large as that in 1649,

(iii) the number of times that the population of China is as large as that of the world in 1749, given that the population of China in year 2000 is approximately 1.23 billion, where 1 billion = 10^9.



1. In index form, $a^n = \underbrace{a \times a \times \ldots \times a \times a}_{n \text{ times}}$, where n is a positive integer.

2. **Laws of Indices**

Assume that a and b are real numbers and m and n are positive integers.

Law 1 of Indices: $\boldsymbol{a^m \times a^n = a^{m+n}}$

Law 2 of Indices: $\boldsymbol{a^m \div a^n = a^{m-n}}$, if $a \neq 0$

Law 3 of Indices: $\boldsymbol{(a^m)^n = a^{mn}}$

Law 4 of Indices: $\boldsymbol{a^n \times b^n = (a \times b)^n}$

Law 5 of Indices: $\boldsymbol{a^n \div b^n = \left(\frac{a}{b}\right)^n}$, if $b \neq 0$

If m and n are integers, the conditions for the above 5 laws of indices are different (see page 74).

If m and n are rational numbers, the conditions for the above 5 laws of indices are again different (see pages 79–80).

3. **Zero Indices**

If a is a real number, we define $\boxed{a^0 = 1 \text{ if } a \neq 0}$.

4. **Negative Indices**

If a is a real number, we define $\boxed{a^{-n} = \frac{1}{a^n}, \text{ if } a \neq 0}$.

5. **Rational Indices**

If n is a positive integer, we define $\boxed{a^{\frac{1}{n}} = \sqrt[n]{a} \text{ if } a > 0}$.

If m and n are positive integers, $\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m, \text{ if } a > 0}$.

6. **Compound Interest Formula**

$$\boxed{A = P\left(1+\frac{R}{100}\right)^n,}$$

where A is the total amount, P is the principal, $R\%$ is the interest rate per annum (or year) and n is the number of years

7. A number is said to be expressed in **standard form**, or **scientific notation**, when it is written as $\boxed{A \times 10^n, \text{ where } 1 \leqslant A < 10 \text{ and } n \text{ is an } \textbf{integer}}$.

Review Exercise 3

1. Simplify each of the following.
   (a) $(a^3b) \times (a^4b^3)$
   (b) $(6a^5b^4) \div (2a^3b^2)$
   (c) $(-3a^3b^5)^3$
   (d) $\left(\frac{2a^2b}{b^3}\right)^3 \div \left(\frac{16a^5}{ab^7}\right)$

2. Express each of the following as a power of 5.
   (a) $5^{24} \div 5^8$
   (b) $\frac{1}{125}$
   (c) $\sqrt[5]{5}$

3. Evaluate each of the following without the use of a calculator.
   (a) $5^2 \div 5^{-1} \times 5^0$
   (b) $2^{-2} - 3^{-2}$
   (c) $3^{-2} + \left(\frac{1}{3}\right)^{-1} - (-3)^0$
   (d) $\left(\frac{2}{5}\right)^3 \div \left(\frac{9}{2}\right)^{-2}$

4. Evaluate each of the following without the use of a calculator.
   (a) $\sqrt[4]{81}$
   (b) $\sqrt[3]{\frac{27}{125}}$
   (c) $16^{1.5}$
   (d) $1024^{-\frac{3}{5}}$

5. Simplify each of the following, expressing your answer in positive index form.
   (a) $\left(\frac{3}{x}\right)^{-4}$
   (b) $3 \div x^{-3}$

6. Simplify each of the following, leaving your answer in positive index form.
   (a) $(x^3y^{-2}) \times (x^{-3}y^5)$
   (b) $(5x^2y^3)^0 \div (-2x^{-3}y^5)^{-2}$
   (c) $\left(\frac{x^2}{y^{-3}}\right)^4 \div \left(\frac{x^5}{y^7}\right)^3$
   (d) $\frac{\left(3x^{-2}y^5\right)^2 \times \left(-2x^3y^{-2}\right)^2}{9x^4y^6}$

7. Simplify each of the following, expressing the answers in positive index form.
   (a) $\sqrt[5]{p^3} \times \sqrt[3]{8p}$
   (b) $\left(p^{-3}q^{\frac{3}{5}}\right)^{-\frac{2}{3}} \times \left(p^{\frac{4}{5}}q^{-\frac{2}{3}}\right)^3$
   (c) $\frac{p^{\frac{2}{3}}q^{-\frac{2}{5}}}{\left(p^2q^{-\frac{1}{5}}\right)^{-3}}$
   (d) $\left(p^{-\frac{1}{3}}q^2\right)^5 \times \sqrt[3]{27\left(p^{-3}q^2\right)}$

8. (a) Given that $4^{-6} \times 4^x = 1$, find the value of x.
   (b) Find the value of x^3 for which $x^{-3} = 7$.
   (c) If $5^{12} \times 5^{-2} \div 5^x = 25$, find the value of x.

9. Solve each of the following equations.
   (a) $16^a = 8$
   (b) $2015^b = 1$
   (c) $\frac{10^c}{10} = 0.01$
   (d) $\frac{2^{d-6}}{2} = 2^9$

10. Nora deposited \$15 000 in an account that pays 4.12% per annum compound interest. Find the total amount in the account after 3 years if the interest is compounded
(a) monthly, **(b)** half-yearly.

11. A bank pays investors 4% per annum compound interest, compounded half-yearly. Find the original amount Rui Feng invested if he received \$5800 as interest at the end of 3 years. Leave your answer correct to the nearest dollar.

12. With the use of a calculator, evaluate each of the following, giving your answer in standard form, correct to 3 significant figures.
(a) $(6.4 \times 10^6) \times (5.1 \times 10^{-3})$
(b) $(2.17 \times 10^{-5}) \div (7 \times 10^4)$
(c) $(3.17 \times 10^4) + (2.26 \times 10^5)$
(d) $(4.15 \times 10^{-3}) - (5.12 \times 10^{-4})$
(e) $\dfrac{5.1\times10^{-6} - 2.34\times10^{5}}{4.87\times10^{-3} + 9\times10^{-2}}$
(f) $\dfrac{8.43\times10^{7} + 6.8\times10^{8}}{(1.01\times10^{4})^{3}}$

13. Given that $a = 110\ 000\ 000$, $b = 12\ 100\ 000$ and $c = 0.000\ 007$, find the value of each of the following, giving your answer in standard form.
(a) $a - b$ **(b)** $\sqrt[3]{ab}$
(c) $6c^2$ **(d)** $\dfrac{ac}{b}$

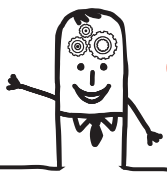
14. The diameter of a circular microorganism is 7 nanometres (nm). Find the
(i) circumference in metres,
(ii) area in square metres,
of the microorganism. Give your answer in standard form. (Take $\pi = 3.142$ and 1 nm = 10^{-9} m.)

15. An astronomical unit (abbreviated as AU, au, a.u., or ua) is a unit of length defined as 149 597 870 700 metres, which is roughly or exactly the mean distance between the earth and the sun. One astronomical unit (au) is estimated to be 150 gigametres (Gm).
(i) Express this astronomical unit in metres, correct to 4 significant figures. Give your answer in standard form.
(ii) The speed of light is 3×10^8 m/s. How long does light take to travel a distance of 1 au? (Take 1 Gm = 10^9 m.)

16. The distance from 2 planets is 240 megametres (1 Mm = 10^6 m).
(i) Express this distance in metres. Give your answer in standard form.

A rocket travels a distance of one metre in 8000 nanoseconds (ns) and 1 ns = 10^{-9} s.
(ii) How long does the rocket take to travel from one planet to the other? Give your answer in seconds.

17. The mass of a hydrogen atom is approximately 1.66×10^{-24} g and that of an oxygen atom is approximately 2.66×10^{-23} g.
(i) Find the mass of a water molecule which consists of two hydrogen and one oxygen atoms. Give your answer in standard form correct to 3 significant figures.
(ii) Find the approximate number of water molecules in a cup of water which has a mass of 280 g. Give your answer in standard form correct to 3 significant figures.



Challenge Yourself

1. Use the numerals 2, 3 and 4 to form a number as large as possible. Show that it is the largest.

2. What is the last digit of 3^{2015}?

3. Find the exact value of $\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}$.

A1 Revision Exercise

1. Evaluate each of the following, without the use of a calculator.

   (a) $\left(\frac{1}{2}\right)^{-3}$ (b) $0.04^{-1.5}$

   (c) $\left(1\frac{9}{16}\right)^{-\frac{1}{2}}$ (d) $9^{2.5} \div 27^{1\frac{1}{3}}$

   (e) $9^{\frac{1}{2}} - 0.36^{-\frac{1}{2}}$

2. Simplify each of the following, expressing your answer in positive index form.

   (a) $a^5 \div a^{-2}$ (b) $b^4 \div \sqrt{b} \times b^{-7}$

   (c) $\left(\frac{c^{-3}d}{c^2 d^{-2}}\right)^{-5}$

3. Solve each of the following equations.

   (a) $7^x = 2^4 \div 4^2$ (b) $(2y+3)^{\frac{1}{2}} = 5$

4. (i) Solve the inequality $3 - 5p > 17$.

   (ii) Hence, write down the greatest integer value of p which satisfies $17 < 3 - 5p$.

5. When 5 is subtracted from twice a number, the resulting number is less than 12. When 1 is subtracted from thrice the number, the resulting number is greater than 12. List the possible integer values of the number.

6. The diameter of a circular microorganism is 8.8 nanometres. Find the

   (i) circumference in m,

   (ii) area in m^2,

   of the microorganism.

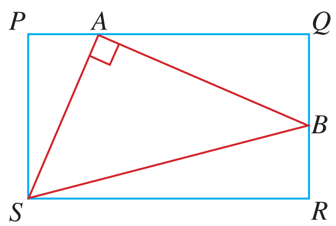
   Give your answers in standard form.

   (Take $\pi = 3.142$, 1 nanometre = 10^{-9} metres.)

7. The perimeter of a square exceeds that of another by 100 cm and the area of the larger square exceeds three times that of the smaller square by 325 cm^2. Find the length of a side of each of the squares.

8. The distance between P and Q is 330 km. A train, A, travelling from P to Q at an average speed of x km/h takes half an hour less than another train, B, travelling from Q to P at an average speed of $(x - 5)$ km/h. Form an equation in x and find the time taken for each train to travel between P and Q.

9. In the figure, $PQRS$ is a rectangle. The point A lies on PQ and the point B lies on QR such that $\angle SAB = 90°$.

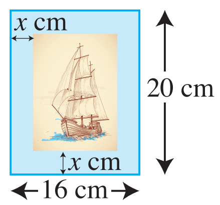


   (i) Given that $PQ = 8$ cm, $QR = 4$ cm, $QB = 3$ cm and $PA = y$ cm, write down expressions in terms of y for AS^2 and AB^2.

   (ii) Form an equation in y and show that it reduces to $y^2 - 8y + 12 = 0$.

   (iii) Solve the equation $y^2 - 8y + 12 = 0$ and hence find the two possible values of the area of ΔABS.

10. In the figure, the width of the border of the picture is x cm. The picture has an area of 160 cm^2.



   (i) Form an equation in x and show that it reduces to $x^2 - 18x + 40 = 0$.

   (ii) Solve the equation $x^2 - 18x + 40 = 0$, giving both your answers correct to 2 significant figures.

   (iii) Write down the width of the border, giving your answer correct to 2 significant figures.

A2 Revision Exercise

1. Simplify each of the following, expressing your answer in positive index form.
   (a) $(2ab^2)^3$ (b) $c^3 \times c^{-2} \div c^0$
   (c) $\left(\frac{2}{d}\right)^{-2}$ (d) $2 \div 4e^{-3}$

2. Given that $5^{-6} \div 5^p = 125^0$, find the value of $\frac{1}{2^p}$.

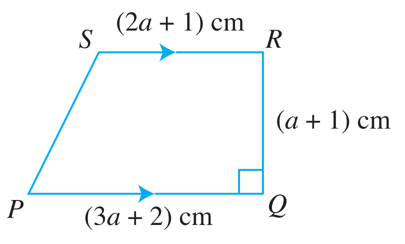
3. Solve each of the following equations.
   (a) $10^{2x+3} = 0.001$ (b) $y^{-2} = \frac{1}{81}$
   (c) $(2z-1)^{\frac{1}{3}} = 2$

4. Given that $1 \leqslant x \leqslant 3$ and $3 \leqslant y \leqslant 6$, find
   (a) the largest possible value of $x^2 - y$,
   (b) the smallest possible value of $\frac{x}{y}$.

5. Solve the inequality $-3 \leqslant 2q + 7 < 23$, illustrating the solution on a number line.

6. Given that $\frac{4x}{5} - \frac{3}{10} \leqslant x - 2\frac{1}{4}$, find the smallest possible value of x such that
   (a) x is a prime number,
   (b) x is an integer,
   (c) x is a rational number.

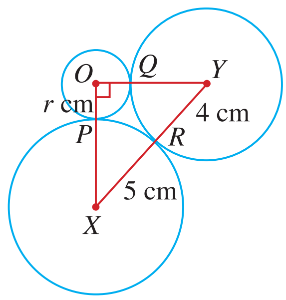
7. The figure shows a trapezium $PQRS$ in which SR is parallel to PQ, $\angle PQR = 90°$, $PQ = (3a + 2)$ cm, $QR = (a + 1)$ cm and $RS = (2a + 1)$ cm.



   (i) Find an expression, in terms of a, for the area of the trapezium $PQRS$.
   (ii) Given that the area of the trapezium is 9 cm², form an equation in a and show that it reduces to $5a^2 + 8a - 15 = 0$.
   (iii) Solve the equation $5a^2 + 8a - 15 = 0$, giving both your answers correct to 2 decimal places.
   (iv) Hence, find the length of PQ.

8. A water tank can be filled with water by two pipes in 1 hour and 20 minutes. If the smaller pipe takes 2 hours longer than the larger pipe to fill the tank, find the time taken by each pipe to fill the tank.

9. The figure shows three circles with centres at O, X and Y. P, Q and R are the points of contact.

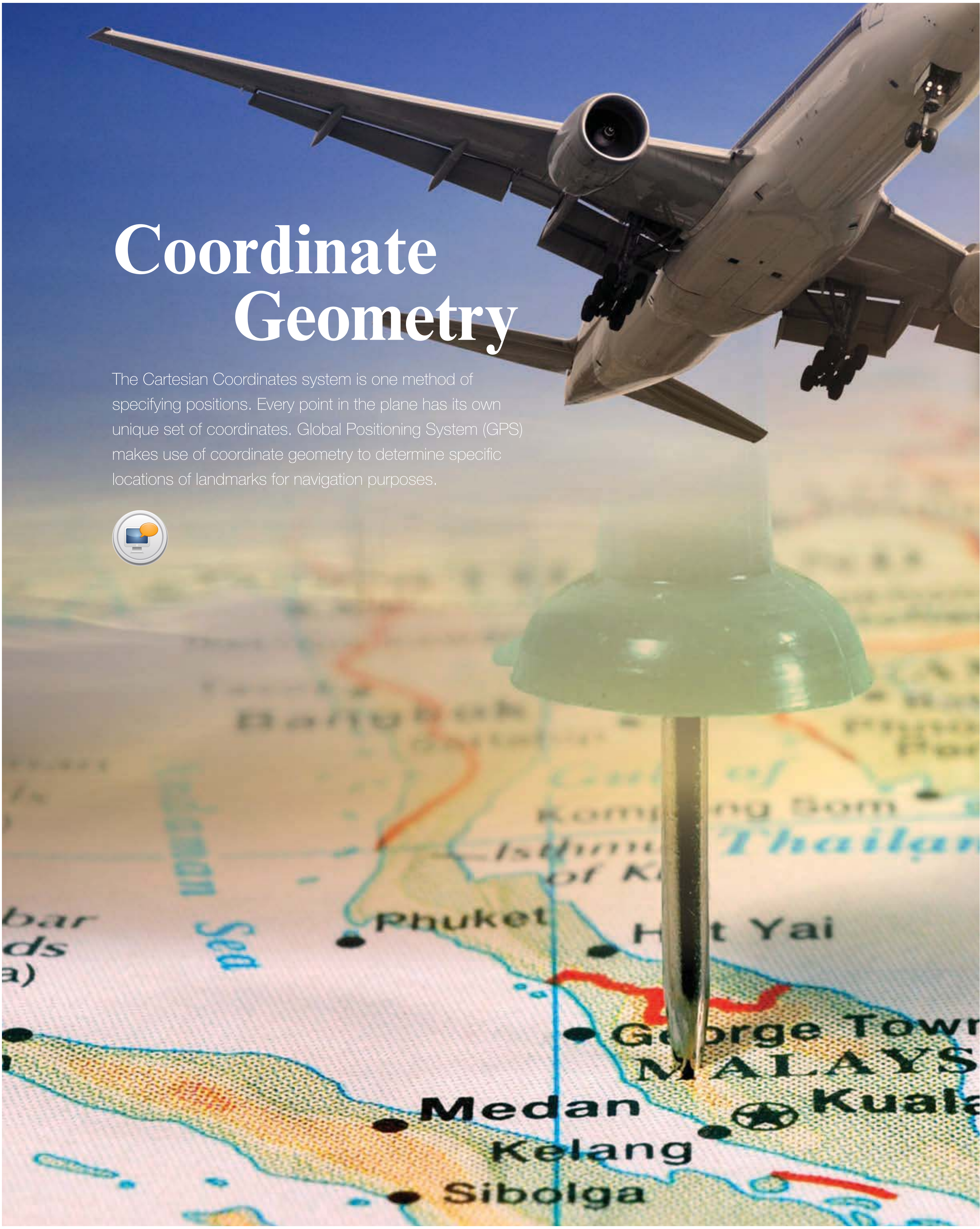


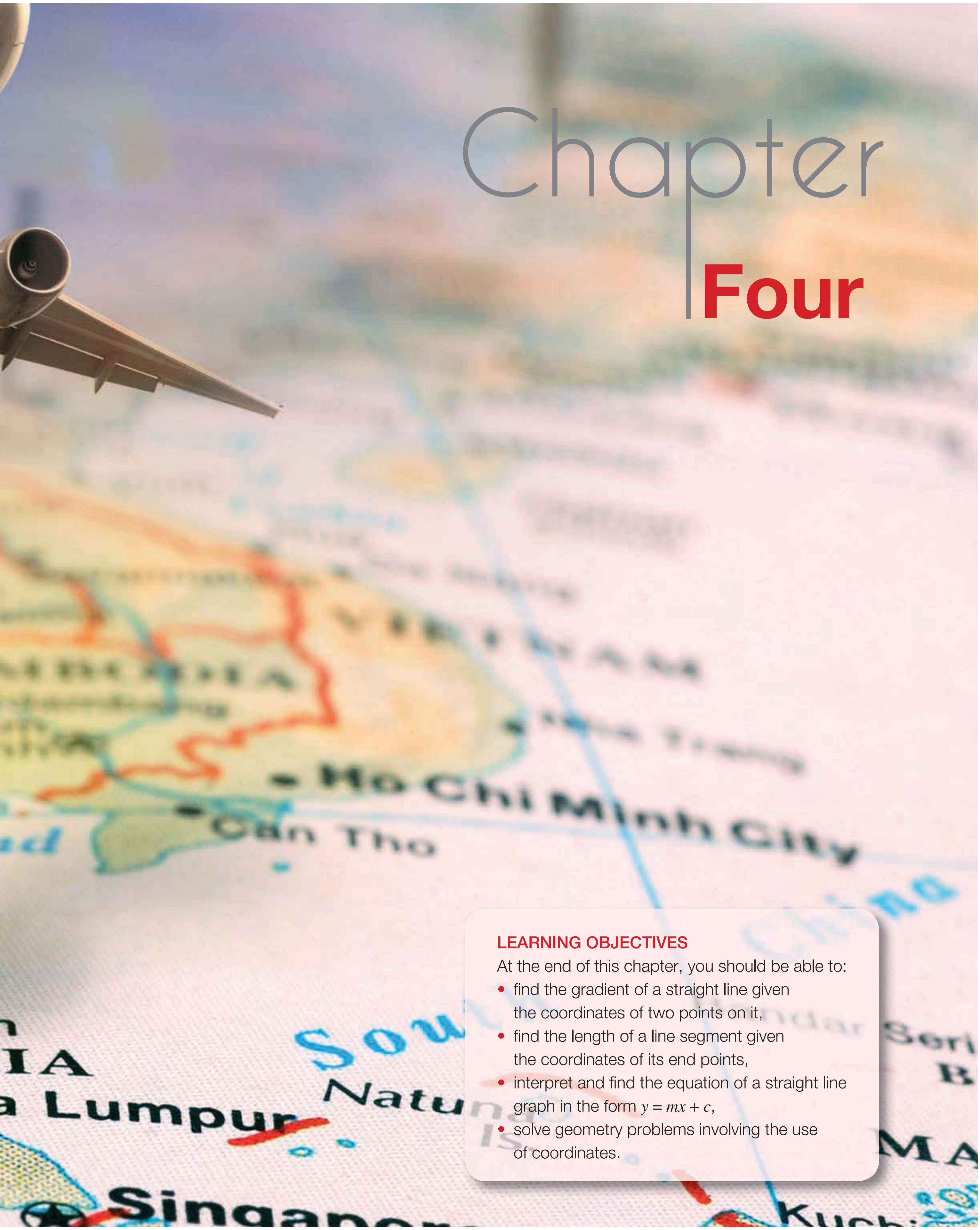
   Given that $XR = 5$ cm, $YR = 4$ cm, $OP = r$ cm and $\angle XOY = 90°$,
   (i) write down an equation in r and show that it reduces to $r^2 + 9r - 20 = 0$,
   (ii) find the length of OX and of OY.

10. Michael cycles from A to B, covering a total distance of 50 km. For the first 40 km of his journey, his average speed is x km/h but for the last 10 km, his average speed is 5 km/h less. The total journey takes 2 hours and 40 minutes. Form an equation in x and solve it to find his average speed for the first 40 km of the journey.

Coordinate Geometry

The Cartesian Coordinates system is one method of specifying positions. Every point in the plane has its own unique set of coordinates. Global Positioning System (GPS) makes use of coordinate geometry to determine specific locations of landmarks for navigation purposes.





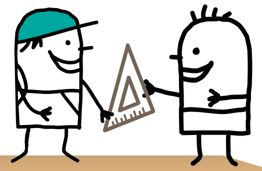
Chapter Four

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- find the gradient of a straight line given the coordinates of two points on it,
- find the length of a line segment given the coordinates of its end points,
- interpret and find the equation of a straight line graph in the form $y = mx + c$,
- solve geometry problems involving the use of coordinates.

4.1 Gradient of a Straight Line



Recap

In Book 1, we have learnt that a rectangular or **Cartesian** plane consists of two number lines intersecting at right angles at the point O, known as the **origin**. The horizontal and vertical axes are called the ***x*-axis** and the ***y*-axis** respectively.

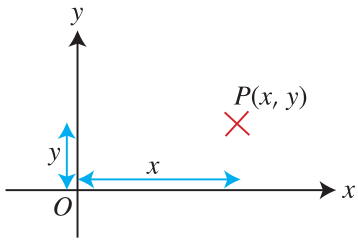


Fig. 4.1

Each point P in the plane is located by an ordered pair (x, y). We say that P has coordinates (x, y).

Rene Descartes, a French philosopher in the early 17th century invented the coordinate system. His use of (x, y) as ordered pairs enhanced the inter-relationship between geometrical curves and algebraic equations. He was also the first person to declare the words "I think, therefore I am."

Gradient of a Straight Line

In Book 1, we have also learnt that the gradient of a straight line is the ratio of the vertical change to the horizontal change.

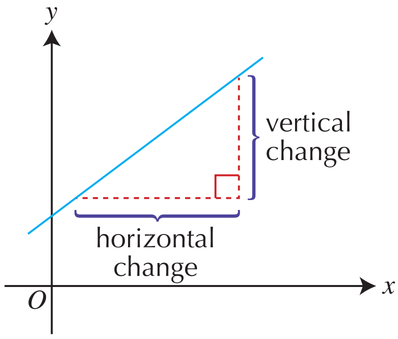
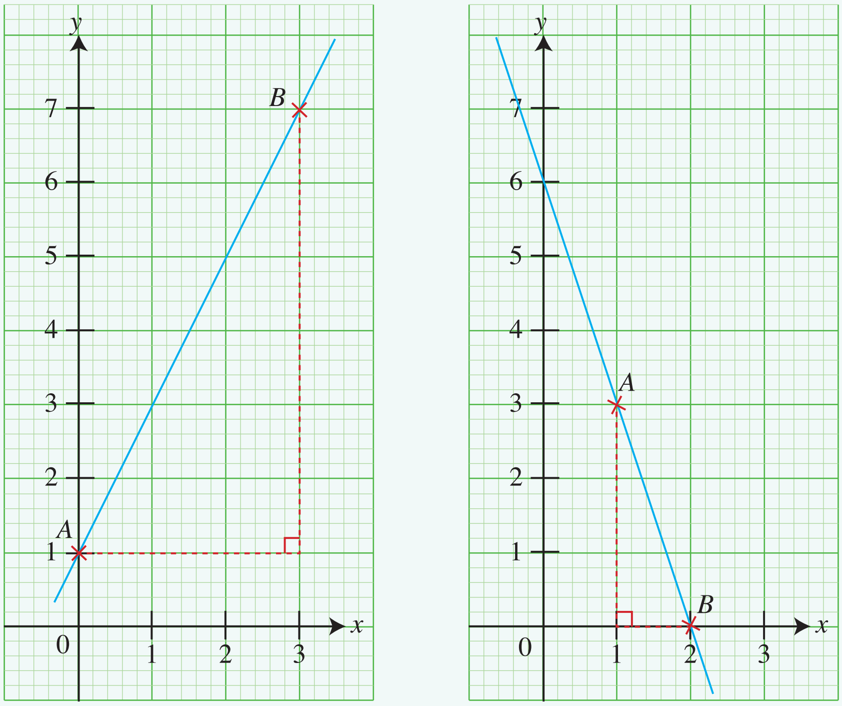


Fig. 4.2

Class Discussion

Finding the Gradient of a Straight Line

1. In Fig. 4.3**(a)** and **(b)**, A and B are two points on the line.



(a) **(b)**

Fig. 4.3

(i) Find the gradient of each line segment.

(ii) Choose two other points that lie on each of the line segments and calculate the gradient of each line segment. Compare your answers with those obtained in **(i)**. What do you notice? Explain your answer.

2. Given any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, how would you find the gradient of the line passing through A and B?

3. Using your answer in Question 2, find the gradient of the line passing through each of the following pairs of points.

(a) $(-1, 4)$ and $(3, 7)$ **(b)** $(-4, -3)$ and $(2, -11)$

(c) $(6, 3)$ and $(-4, 3)$ **(d)** $(2, -1)$ and $(2, 8)$

Compare your answers with those obtained by your classmates.

From the class discussion, we observe that if $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a line, then

$$\boxed{\text{gradient of } AB = \frac{y_2 - y_1}{x_2 - x_1}.}$$

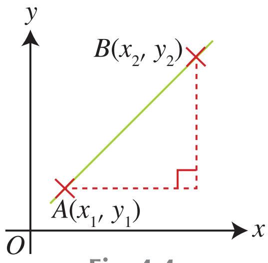
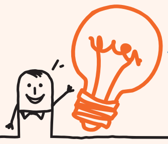


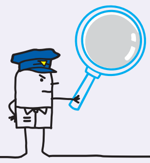
Fig. 4.4



Thinking Time

Instead of writing the gradient of AB as $\frac{y_2 - y_1}{x_2 - x_1}$, we can also write it as $\frac{y_1 - y_2}{x_1 - x_2}$.

Is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$? Explain your answer.



Investigation

Gradient of a Straight Line

1. Using a suitable geometry software, draw a line segment with the end-points as $A(-2, 1)$ and $B(0, 5)$.

2. Find the gradient of the line segment AB and record it in Table 4.1. Describe the gradient of the line segment AB using one of the following terms: positive, negative, zero or undefined.

3. Write down the value of $y_2 - y_1$ and of $x_2 - x_1$ in Table 4.1.

4. Repeat Steps 1–3 for each of the following pairs of points.

(a) $C(7, 5)$ and $D(4, 8)$ **(b)** $E(-2, 6)$ and $F(-4, 3)$

(c) $G(1, 1)$ and $H(3, 1)$ **(d)** $I(-4, 3)$ and $J(-4, 6)$

	Coordinates of End-points	Gradient of Line Segment	Sign of Gradient	$y_2 - y_1$	$x_2 - x_1$
(a)	$A(-2, 1)$ and $B(0, 5)$		positive	$5 - 1 =$	$0 - (-2) =$
(b)	$C(7, 5)$ and $D(4, 8)$				
(c)	$E(-2, 6)$ and $F(-4, 3)$				
(d)	$G(1, 1)$ and $H(3, 1)$				
(e)	$I(-4, 3)$ and $J(-4, 6)$				

Table 4.1

5. **(a)** When $y_2 - y_1 > 0$ and $x_2 - x_1 < 0$, what do you notice about the sign of the gradient?

(b) When $y_2 - y_1 < 0$ and $x_2 - x_1 > 0$, what do you notice about the sign of the gradient?

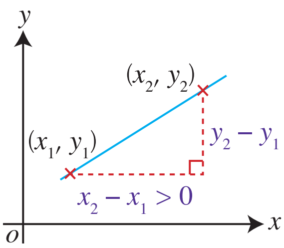
(c) When the signs of $y_2 - y_1$ and $x_2 - x_1$ are the same, what do you notice about the sign of the gradient?

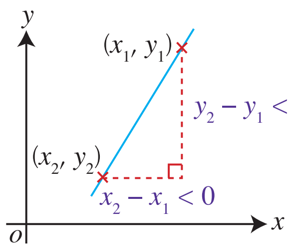
(d) When $y_2 - y_1 = 0$, what do you notice about the gradient of the line?

(e) When $x_2 - x_1 = 0$, what do you notice about the gradient of the line?

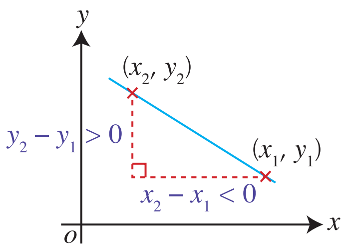
From the investigation, we observe that

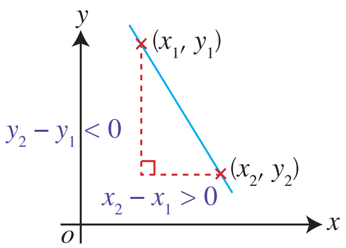
- the gradient of a straight line can be positive, negative, zero or undefined,
- if $y_2 - y_1$ and $x_2 - x_1$ have the same signs, the gradient of the straight line is positive,



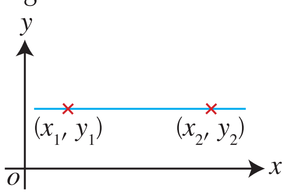


- if $y_2 - y_1$ and $x_2 - x_1$ have opposite signs, the gradient of the straight line is negative,

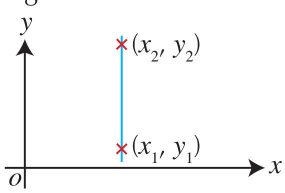




- if $y_2 - y_1 = 0$ or $y_2 = y_1$, the gradient of a horizontal line is zero,

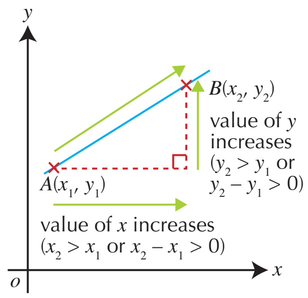


- if $x_2 - x_1 = 0$ or $x_2 = x_1$, the gradient of a vertical line is undefined.



ATTENTION

When the gradient of the line is positive, as the value of x increases (from point A to point B), the value of y also increases.



Worked Example 1

(Finding Gradient given Two Points)

Find the gradient of the line passing through each of the following pairs of points.

(a) $A(2, 3)$ and $B(7, 5)$ **(b)** $P(-2, 8)$ and $Q(1, -1)$

Solution:

(a) Gradient of $AB = \dfrac{y_2 - y_1}{x_2 - x_1}$ (Let $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (7, 5)$)

$$= \frac{5-3}{7-2}$$

$$= \frac{2}{5}$$

Alternatively,

Gradient of $AB = \dfrac{y_1 - y_2}{x_1 - x_2}$

$$= \frac{3-5}{2-7}$$

$$= \frac{-2}{-5}$$

$$= \frac{2}{5}$$

(b) Gradient of $PQ = \dfrac{y_2 - y_1}{x_2 - x_1}$ (Let $(x_1, y_1) = (-2, 8)$ and $(x_2, y_2) = (1, -1)$)

$$= \frac{-1-8}{1-(-2)}$$

$$= \frac{-9}{3}$$

$$= -3$$

PRACTISE NOW 1

Find the gradient of the line passing through each of the following pairs of points.

(a) $C(3, 1)$ and $D(6, 3)$ **(b)** $H(5, -7)$ and $K(0, -2)$

(c) $M(-4, 1)$ and $N(16, 1)$

SIMILAR QUESTIONS

Exercise 4A Questions 1(a)-(f), 2, 9

Worked Example 2

(Using the Gradient to Determine the Coordinates of a Point on the Line)

If the gradient of the line joining the points (k, 5) and (2, k) is –2, find the value of k.

Solution:

Gradient of line, $\frac{y_2 - y_1}{x_2 - x_1} = -2$

$$\frac{k-5}{2-k} = -2 \quad \text{(Let } (x_1, y_1) = (k, 5) \text{ and } (x_2, y_2) = (2, k)\text{)}$$

$$k - 5 = -2(2 - k)$$

$$k - 5 = -4 + 2k$$

$$-1 = k$$

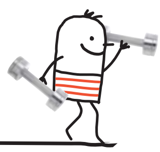
$\therefore k = -1$

PRACTISE NOW 2

SIMILAR QUESTIONS

If the gradient of the line joining the points (4, –9) and (–3, h) is –3, find the value of h.

Exercise 4A Questions 3-8



Exercise 4A

BASIC LEVEL

1. Find the gradient of the line passing through each of the following pairs of points.

(a) $A(0, 0)$ and $B(-2, 1)$

(b) $C(2, -3)$ and $D(1, 7)$

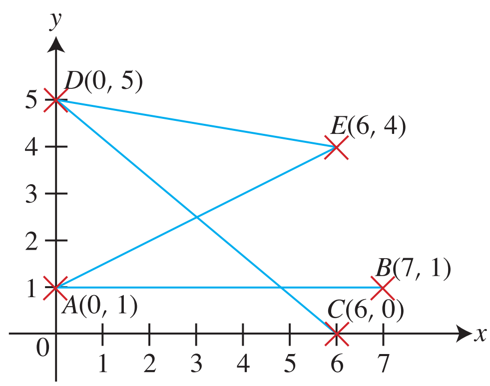
(c) $E(-2, 4)$ and $F(-5, 8)$

(d) $G(-4, 7)$ and $H(1, -8)$

(e) $I(-2, -5)$ and $J(2, 6)$

(f) $K(-7, 9)$ and $L(6, 9)$

2. The points $A(0, 1)$, $B(7, 1)$, $C(6, 0)$, $D(0, 5)$ and $E(6, 4)$ are shown in the diagram.



Find the gradient of each of the line segments AB, AE, DC and DE.

3. If the gradient of the line joining the points (–3, –7) and (4, p) is $\frac{3}{5}$, find the value of p.

4. The coordinates of A and B are (3k, 8) and (k, –3) respectively. Given that the gradient of the line segment AB is 3, find the value of k.

INTERMEDIATE LEVEL

5. The gradient of the line joining the points (9, a) and (2a, 1) is $\frac{2}{a}$, where $a \neq 0$. Find the possible values of a.

6. The points P, Q and R have coordinates (6, –11), (k, –9) and (2k, –3) respectively. If the gradient of PQ is equal to the gradient of PR, find the value of k.

7. The points $P(2, -3)$, $Q(3, -2)$ and $R(8, z)$ are collinear, i.e. they lie on a straight line. Find the value of z.

8. The line joining the points $A(2, t)$ and $B(7, 2t^2 + 7)$ has a gradient of 2. Find the possible values of t.

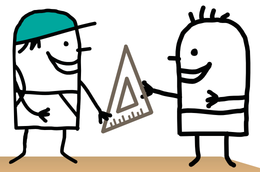
ADVANCED LEVEL

9. The coordinates of the vertices of a square $ABCD$ are $A(0, 6)$, $B(2, 1)$, $C(7, 3)$ and $D(5, 8)$.

(i) Find the gradient of all 4 sides of $ABCD$.

(ii) What do you observe about the gradients of the opposite sides of a square?

4.2 Length of a Line Segment



Consider the points $A(1, 1)$ and $B(7, 9)$ as shown in Fig. 4.5.

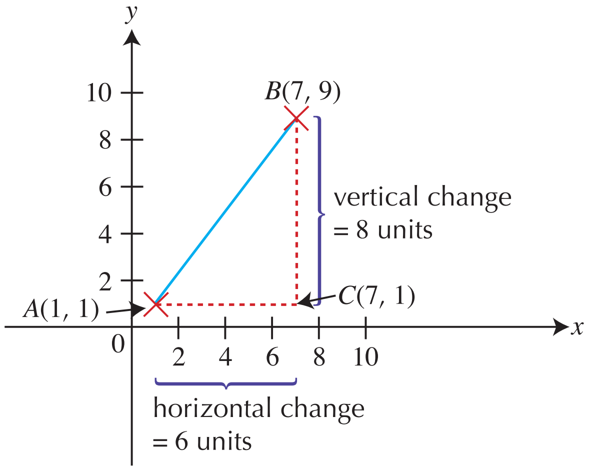


Fig. 4.5

ΔABC is formed by drawing AC parallel to the x-axis and BC parallel to the y-axis. The coordinates of the point C are given by (7, 1).

ATTENTION

Recall that a line segment is part of a line with two end-points. A line has no end-points so it does not have a length. However, a line segment has two end-points and so it has length.

ATTENTION

BC is vertical, i.e.
x-coordinate of C = x-coordinate of B.
AC is horizontal, i.e.
y-coordinate of C = y-coordinate of A.

Hence, $AC = 7 - 1 = 6$
and $\quad BC = 9 - 1 = 8$.

Using Pythagoras' Theorem,
$AB^2 = AC^2 + BC^2$
$\quad = 6^2 + 8^2$
$\quad = 100$
$AB = \sqrt{100}$
$\quad = 10$ units

Consider any two points P and Q with coordinates (x_1, y_1) and (x_2, y_2) respectively. By completing the right-angled ΔPQR, we have the coordinates of R as (x_2, y_1).

Hence, $PR = x_2 - x_1$
and $\quad QR = y_2 - y_1$.

Using Pythagoras' Theorem,
$PQ^2 = PR^2 + QR^2$
$\quad = (x_2 - x_1)^2 + (y_2 - y_1)^2$

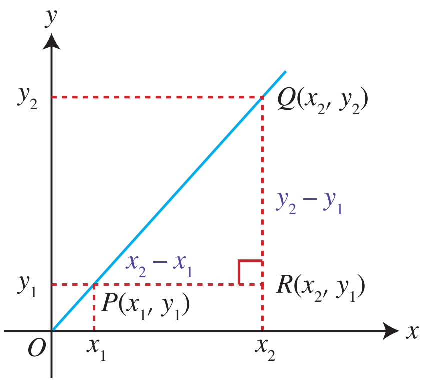


Fig. 4.6

In general, the length of any line segment PQ, where the coordinates of the points P and Q are (x_1, y_1) and (x_2, y_2) respectively, is

$$\boxed{PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.}$$

Thinking Time

Instead of writing the length of line segment PQ as $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$,

we can also write it as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$? Explain your answer.

Worked Example 3

(Finding the Length of a Line Segment)

Given that the coordinates of the points A and B are (–4, 1) and (6, –5) respectively, find the length of the line segment AB.

Solution:

Length of line segment $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{[6-(-4)]^2 + (-5-1)^2}$

$= \sqrt{10^2 + (-6)^2}$

$= \sqrt{100+36}$

$= \sqrt{136}$

= 11.7 units (to 3 s.f.)

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 4B Questions 1(a)-(d)

Find the length of the line segment joining each of the following pairs of points.

(a) $C(6, 2)$ and $D(3, -2)$

(b) $M(-1, 5)$ and $N(6, -4)$

(c) $P(2, 7)$ and $Q(8, 7)$

Worked Example 4

(Using the Length to Determine the Coordinates of a Point on the Line)

Given that the coordinates of the points A and B are (–3, 2) and (1, –6) respectively, find the coordinates of the point C that lies on the y-axis such that $AC = BC$.

Hence, find the area of ΔACO, where O is the origin.

Solution:

Let the coordinates of C be $(0, k)$.

$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{[0-(-3)]^2 + (k-2)^2}$

$= \sqrt{(0+3)^2 + (k-2)^2}$

$= \sqrt{9 + (k-2)^2}$

Problem Solving Tip

Since C lies on the y-axis, its x-coordinate is 0.

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(0-1)^2 + [k-(-6)]^2}$$
$$= \sqrt{(-1)^2 + (k+6)^2}$$
$$= \sqrt{1+(k+6)^2}$$

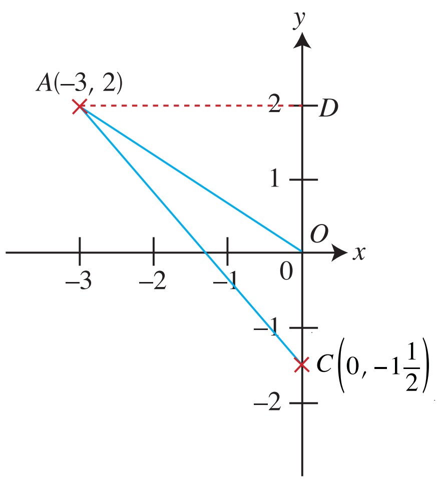
Since $AC = BC$,

$$\sqrt{9+(k-2)^2} = \sqrt{1+(k+6)^2}$$
$$\left(\sqrt{9+(k-2)^2}\right)^2 = \left(\sqrt{1+(k+6)^2}\right)^2 \quad \text{(square both sides of the equation)}$$
$$9 + (k-2)^2 = 1 + (k+6)^2$$
$$9 + k^2 - 4k + 4 = 1 + k^2 + 12k + 36$$
$$k^2 - 4k + 13 = k^2 + 12k + 37$$
$$-16k = 24$$
$$16k = -24$$
$$k = -\frac{24}{16}$$
$$= -\frac{3}{2}$$
$$= -1\frac{1}{2}$$

$\therefore$ Coordinates of C are $\left(0, -1\frac{1}{2}\right)$

Area of $\Delta ACO = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times OC \times AD$$
$$= \frac{1}{2} \times 1\frac{1}{2} \times 3$$
$$= \frac{9}{4}$$
$$= 2\frac{1}{4} \text{ units}^2$$



PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 4B Questions 2-8

Given that the coordinates of the points C and D are $(4, -1)$ and $(-2, 7)$ respectively, find

(a) the coordinates of the point E that lies on the y-axis such that $CE = DE$,

(b) the coordinates of the point F that lies on the x-axis such that $CF = DF$.

Hence, find the area of ΔOEF, where O is the origin.

Worked Example 5

(Using the Length to show that a Triangle is Right-angled)

A triangle has vertices $A(0, -5)$, $B(-2, 1)$ and $C(10, 5)$. Show that ΔABC is a right-angled triangle and identify the right angle.

Solution:

$$\begin{aligned} AB^2 &= (-2-0)^2 + [1-(-5)]^2 \\ &= (-2)^2 + 6^2 \\ &= 4 + 36 \\ &= 40 \end{aligned}$$

$$\begin{aligned} BC^2 &= [10-(-2)]^2 + (5-1)^2 \\ &= 12^2 + 4^2 \\ &= 144 + 16 \\ &= 160 \end{aligned}$$

$$\begin{aligned} AC^2 &= (10-0)^2 + [5-(-5)]^2 \\ &= 10^2 + 10^2 \\ &= 100 + 100 \\ &= 200 \end{aligned}$$

Since $AB^2 + BC^2 = 40 + 160$
$= 200$
$= AC^2$,

the triangle is a right-angled triangle with $\angle ABC = 90°$. (Converse of Pythagoras' Theorem)

Problem Solving Tip

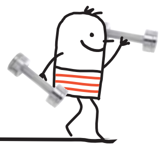
Check that the sum of squares of the two shorter sides is equal to the square of the longest side.

PRACTISE NOW 5

1. A triangle has vertices $D(6, 1)$, $E(2, 3)$ and $F(-1, -3)$. Show that ΔDEF is a right-angled triangle and identify the right angle.
2. A triangle has vertices $P(-3, 1)$, $Q(6, 3)$ and $R(1, 8)$. Determine if ΔPQR is a right-angled triangle.

SIMILAR QUESTIONS

Exercise 4B Questions 9-11



Exercise 4B

BASIC LEVEL

1. Find the length of the line segment joining each of the following pairs of points.

(a) $A(2, 3)$ and $B(9, 7)$

(b) $C(3, 6)$ and $D(-5, 9)$

(c) $E(-1, 4)$ and $F(8, -3)$

(d) $G(-10, 2)$ and $H(-4, -7)$

2. If the distance between the points $A(p, 0)$ and $B(0, p)$ is 10, find the possible values of p.

INTERMEDIATE LEVEL

3. Given that the coordinates of the points P and Q are $(-2, 6)$ and $(9, 3)$ respectively, find

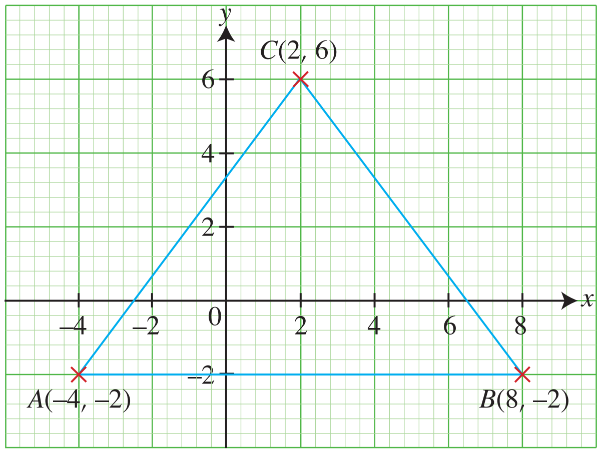
(a) the coordinates of the point R that lies on the y-axis such that $PR = QR$,

(b) the coordinates of the point S that lies on the x-axis such that $PS = QS$.

4. A line segment has two end-points $M(3, 7)$ and $N(11, -6)$. Find the coordinates of the point W that lies on the y-axis such that W is equidistant from M and from N.

Hint: The term 'equidistant' means 'same distance'.

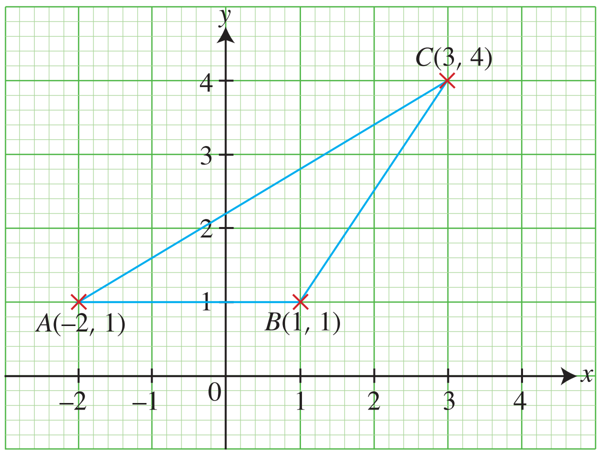
5. The vertices of ΔABC are $A(-4, -2)$, $B(8, -2)$ and $C(2, 6)$.



(i) Find the perimeter and the area of ΔABC.

(ii) Hence, find the length of the perpendicular from A to BC.

6. The diagram shows ΔABC with vertices $A(-2, 1)$, $B(1, 1)$ and $C(3, 4)$.



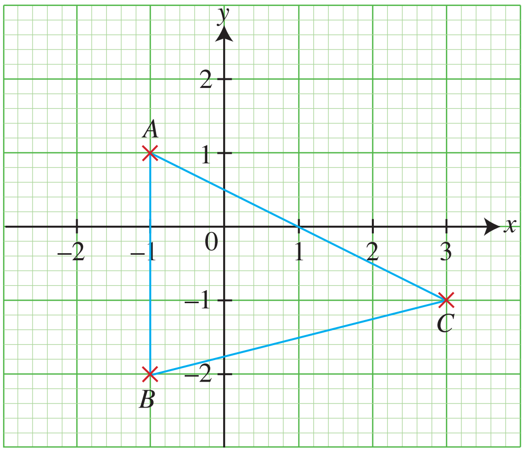
(i) Find the area of ΔABC.

(ii) Find the length of AC, giving your answer correct to 2 decimal places.

(iii) Given that $ABCD$ is a parallelogram, find the coordinates of D.

(iv) Given that K is the point $(t, 4)$ and the area of ΔBCK is 12 units^2, find the possible values of t.

7. The diagram shows ΔABC with vertices $A(-1, 1)$, $B(-1, -2)$ and $C(3, -1)$.



(i) Find the lengths of AB, BC and AC.

(ii) Find the area of ΔABC.

The coordinates of a point E are $(3, k)$ and the area of ΔBCE is 14 units^2.

(iii) Find the possible values of k.

8. The distance between the points $(1, 2t)$ and $(1 - t, 1)$ is $\sqrt{11 - 9t}$. Find the possible values of t.

9. **(i)** Show that the points $A(-1, 2)$, $B(5, 2)$ and $C(2, 5)$ are the vertices of an isosceles triangle.

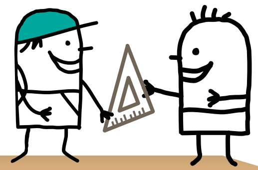
(ii) Find the area of ΔABC.

10. By showing that the points $P(3, 4)$, $Q(3, 1)$ and $R(8, 4)$ are the vertices of a right-angled triangle, find the length of the perpendicular from P to QR.

ADVANCED LEVEL

11. The vertices of ΔPQR are $P(1, 3)$, $Q(5, 4)$ and $R(5, 15)$. Find the length of the perpendicular from Q to PR.

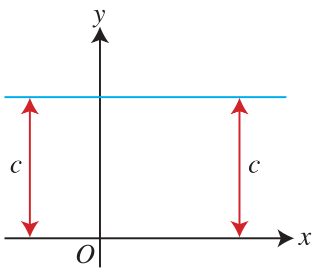
4.3 Equation of a Straight Line

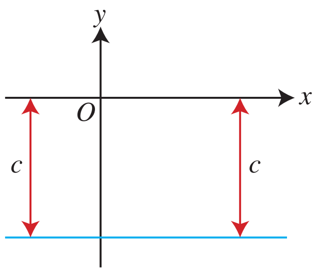


Recap

Horizontal Lines

If a line is parallel to the x-axis and its distance from the x-axis is c, then every point on the line has the same y-coordinate.



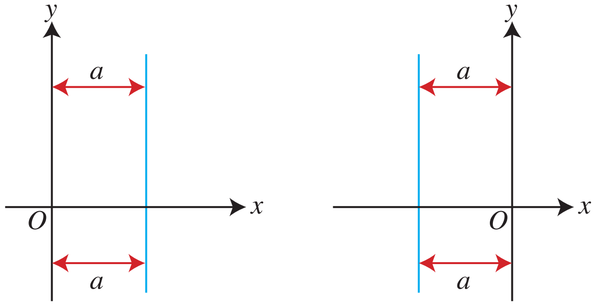


The gradient of a horizontal line is 0.

The equation of a horizontal line is $y = c$ or $y = -c$.

Vertical Lines

If a line is parallel to the y-axis and its distance from the y-axis is a, then every point on the line has the same x-coordinate.



The gradient of a vertical line is undefined.

The equation of a vertical line is $x = a$ or $x = -a$.

Equation of a Straight Line $y = mx + c$

In Book 1, we have learnt that the equation of a straight line is in the form $\boldsymbol{y = mx + c}$, where the constant $\boldsymbol{m}$ is the gradient of the line and the constant $\boldsymbol{c}$ is the **y-intercept**.

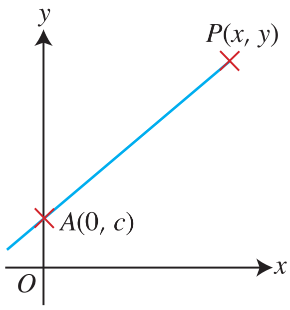


Fig. 4.7

In Fig. 4.7, the straight line passes through the points $A(0, c)$ and $P(x, y)$. If the gradient of the line is m, then

Gradient of line $= m$

$$\frac{y-c}{x-0} = m$$

$$y - c = mx$$

$$\therefore\ y = mx + c$$

In general,

> for a straight line passing through the point $(0, c)$ and with gradient m, the equation is
>
> $$\boldsymbol{y = mx + c}.$$

The equation $\boldsymbol{y = mx + c}$ is known as the **gradient-intercept form** of the equation of a straight line. In this equation, $\boldsymbol{m}$ gives the gradient of the straight line, $\boldsymbol{c}$ gives the intercept on the y-axis and $(0, \boldsymbol{c})$ is the point where the line cuts the y-axis.

Worked Example 6

(Finding the y-intercept given the Gradient and Coordinates of a Point)

Given that $y = 3x + c$ passes through the point (3, 1), find the value of c.

Solution:

Since (3, 1) lies on the line $y = 3x + c$, the coordinates (3, 1) must satisfy the equation,

i.e. $1 = 3(3) + c$ (substitute $x = 3$ and $y = 1$)

$= 9 + c$

$\therefore c = -8$

PRACTISE NOW 6

1. Given that $y = 5x + a$ passes through the point (–1, 2), find the value of a.
2. The point (6, 8) lies on the line $y = -4x + b$. Find the value of b.

SIMILAR QUESTIONS

Exercise 4C Questions 1, 2

Worked Example 7

(Finding the Equation of a Straight Line given the Coordinates of 2 Points)

Find the equation of the straight line passing through each of the following pairs of points.

(a) $A(1, 2)$ and $B(3, 7)$ **(b)** $C(2, 3)$ and $D(7, 3)$

(c) $E(5, 1)$ and $F(5, 6)$

Solution:

(a) Gradient of $AB = \frac{7-2}{3-1}$

$= \frac{5}{2}$

Equation of AB is in the form $y = \frac{5}{2}x + c$

Since (1, 2) lies on the line,

$2 = \frac{5}{2}(1) + c$

$c = -\frac{1}{2}$

$\therefore$ Equation of AB is $y = \frac{5}{2}x - \frac{1}{2}$

ATTENTION

We can also substitute (3, 7) into the equation of AB to find the value of c.

(b) $C(2, 3)$ and $D(7, 3)$ have the same y-coordinate of value 3.

$\therefore$ CD is a horizontal line with equation $y = 3$.

(c) $E(5, 1)$ and $F(5, 6)$ have the same x-coordinate of value 5.

$\therefore$ EF is a vertical line with equation $x = 5$.

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 4C Questions 3(a)-(h), 4(a)-(f), 5-17

Find the equation of the straight line passing through each of the following pairs of points.

(a) $A(-2, 1)$ and $B(5, 3)$

(b) $C(6, 4)$ and $D(-4, 4)$

(c) $E(-3, 5)$ and $F(-3, 8)$

Journal Writing

When we want to find the equation of a straight line, what information do we need?

Consider each of the cases below.

Case 1: Given the gradient m and the y-intercept c
Case 2: Given the gradient m and the coordinates of a point (a, b)
Case 3: Given the coordinates of two points (a, b) and (c, d)

For each case, describe how you would find the equation of the straight line.

Exercise 4C

BASIC LEVEL

1. Given that $y = -x + c$ passes through the point $(1, 2)$, find the value of c.

2. The point $(-3, 3)$ lies on the line $y = 4x + k$. Find the value of k.

3. Find the equation of the straight line passing through each of the following pairs of points.

(a) $A(0, 0)$ and $B(1, -1)$
(b) $C(1, 3)$ and $D(2, 5)$
(c) $E(2, 4)$ and $F(-2, 3)$
(d) $G(-6, -5)$ and $H(4, 4)$
(e) $I(-2, -4)$ and $J(1, -7)$
(f) $K(-7, -5)$ and $L(-1, -1)$
(g) $M(8, 0)$ and $N(-9, 0)$
(h) $O(0, 0)$ and $P(0, 7)$

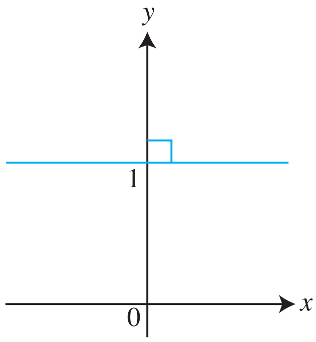
4. Find the equation of each of the straight lines, given the gradient and the coordinates of a point that lies on it.

(a) $\frac{1}{3}$, $(0, 0)$
(b) 3, $(1, 1)$
(c) -3, $(2, -5)$
(d) $-\frac{1}{2}$, $(5, 7)$
(e) 0, $(5, 4)$
(f) a, $(0, a)$

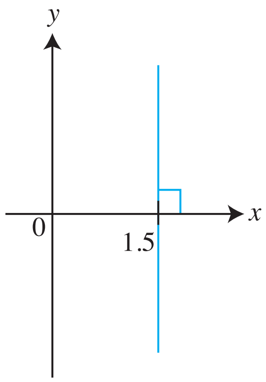
5. Write down the equation of the straight line which passes through the origin and with gradient 2.

6. In each of the following diagrams, find the gradient and the y-intercept of the line where possible. State the equation of each line.

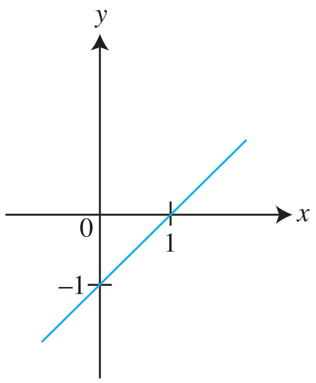
(a)



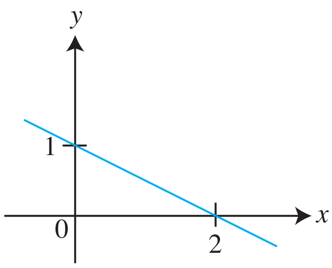
(b)



(c)

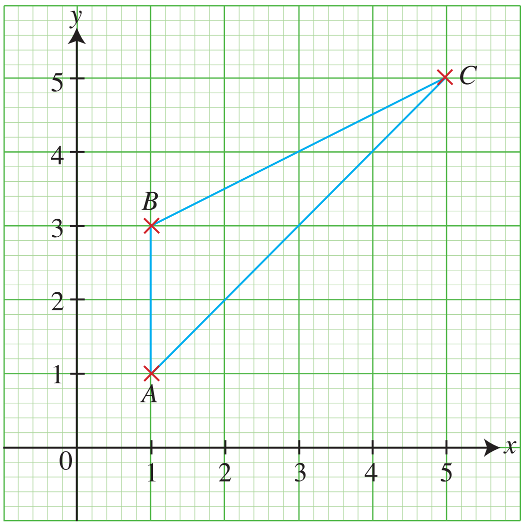


(d)



INTERMEDIATE LEVEL

7. The diagram shows ΔABC with vertices $A(1, 1)$, $B(1, 3)$ and $C(5, 5)$.



(i) Find the area of ΔABC.

(ii) Find the gradient of the line passing through B and C.

(iii) Find the equation of the line passing through A and C.

8. The lines $2x – 5 = ky$ and $(k + 1)x = 6y – 3$ have the same gradient. Find the possible values of k.

9. Given the line $\frac{x}{3}+\frac{y}{2} = 1$,

(i) make y the subject of the formula $\frac{x}{3}+\frac{y}{2} = 1$,

(ii) find the gradient of the line,

(iii) find the coordinates of the point at which the line cuts the x-axis.

10. (i) Find the equation of the straight line which passes through the point $(–3, 5)$ and with gradient $-\frac{2}{3}$.

(ii) Given that the line in **(i)** also passes through the point $(p, 3)$, find the value of p.

11. Find the equation of the straight line passing through the point (3, –2) and having the same gradient as the line $2y = 5x + 7$.

12. **(i)** Find the equation of the straight line which passes through the point (3, 1) and with gradient 3.

(ii) Hence, find the coordinates of the point of intersection of the line in **(i)** with the line $y = x$.

13. The line l has equation $5x + 6y + 30 = 0$. Given that P is the point (3, –1), find

(i) the coordinates of the point where l crosses the x-axis,

(ii) the coordinates of the point of intersection of l with the line $x = 2$,

(iii) the equation of the line passing through P and having the same gradient as l,

(iv) the equation of the line passing through P and having a gradient of 0.

14. A straight line l passes through the points $A(0, 3)$ and $B(3, 12)$.

(a) Find

(i) the gradient of the line l,

(ii) the equation of the line l.

(b) The line $x = 3$ is the line of symmetry of ΔABC. Find the coordinates of C.

ADVANCED LEVEL

15. If the line $mx = ny + 2$ has the same gradient as the x-axis, find the value of m. State the condition for the line to be parallel to the y-axis instead.

16. The line l has equation $3x + 4y = 24$. It crosses the x-axis at the point A and the y-axis at the point B. Find

(i) the coordinates of A and of B,

(ii) the length of the line segment AB,

(iii) the coordinates of the point C that lies on the line l such that C is equidistant from the coordinate axes,

(iv) the equation of the line OC, where O is the origin.

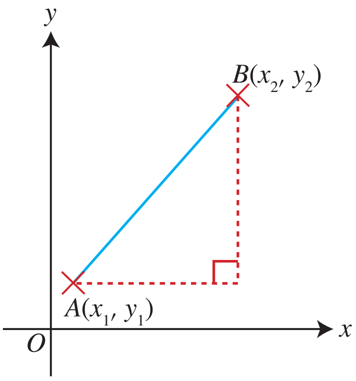
17. The coordinates of the points P and Q are (2, 3) and (9, 5) respectively.

(i) Find the coordinates of the point where the line passing through P and Q intersects the x-axis.

(ii) Given that $y = 5$ is the line of symmetry of ΔPQR, find the coordinates of R.

(iii) Calculate the length of PQ.

(iv) Hence, find the length of the perpendicular from R to PQ.

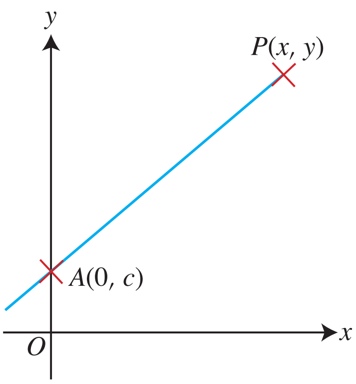
<table>
<tr>
<td>1.</td>
<td>
</td>
<td>

Gradient

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a line, then gradient of AB is $\frac{y_2 - y_1}{x_2 - x_1}$.

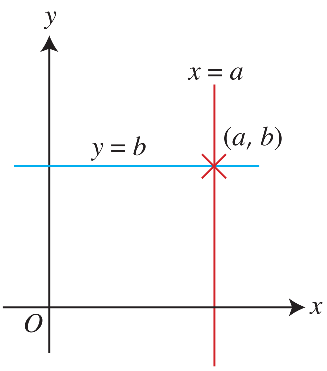
Length

The length of any line segment AB, where the coordinates of the points A and B are (x_1, y_1) and (x_2, y_2) respectively is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

</td>
</tr>
<tr>
<td>2.</td>
<td>
</td>
<td>

Equation of an oblique line

The equation of a straight line passing through the point $(0, c)$ and with gradient m is $\boldsymbol{y = mx + c}$.

</td>
</tr>
<tr>
<td>3.</td>
<td>
</td>
<td>

Equation of a horizontal line

The equation of a straight line that is parallel to the x-axis and which passes through the point (a, b) is $\boldsymbol{y = b}$. It has a gradient of **0**.

Equation of a vertical line

The equation of a straight line that is parallel to the y-axis and which passes through the point (a, b) is $\boldsymbol{x = a}$. Its gradient is **undefined**.

</td>
</tr>
</table>

Review Exercise 4

1. A straight line has a gradient of 2 and passes through the point (0, –3).

(i) Write down the equation of the straight line.

(ii) Given that the line also passes through the point (4, k), find the value of k.

2. The equation of a straight line is $6x + 2y = 7$.

(i) Find the gradient of the line.

Another line with equation $y = mx + c$ has the same gradient as $6x + 2y = 7$ and passes through the point (3, 5).

(ii) Find the value of c.

3. The coordinates of the points A and B are (1, 5) and (2, –3) respectively. Find the equation of the line passing through the origin and having the same gradient as AB.

4. The line l has equation $3x - 4y = 24$. It intersects the x-axis at A and the y-axis at B. Given that M is the point (4, –3), find

(i) the gradient of l,

(ii) the length of AB,

(iii) the equation of the line passing through B and having the same gradient as OM, where O is the origin.

5. The coordinates of the points A and B are (0, 6) and (8, 0) respectively.

(i) Find the equation of the line passing through A and B.

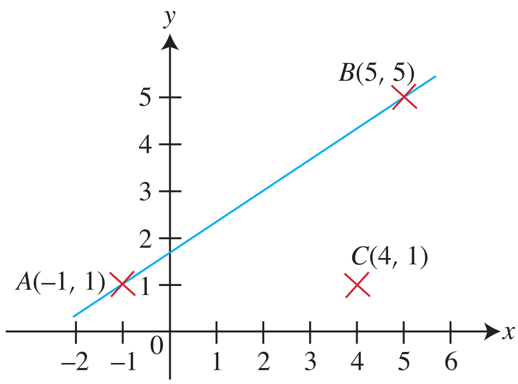
Given that the line $y = x + 1$ cuts the line AB at the point M, find

(ii) the coordinates of M,

(iii) the equation of the line which passes through M and is parallel to the x-axis,

(iv) the equation of the line which passes through M and is parallel to the y-axis.

6. The diagram shows the line l passing through the points $A(–1, 1)$ and $B(5, 5)$.



Given that C is the point (4, 1), find

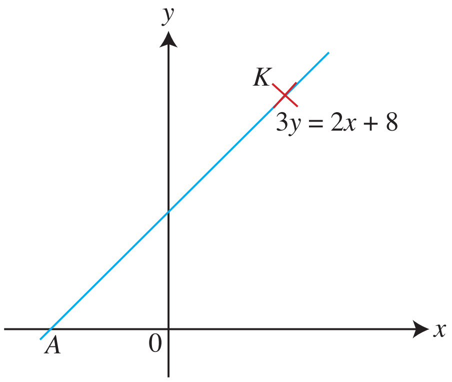
(i) the gradient of l,

(ii) the equation of l,

(iii) the area of ΔABC,

(iv) the length of BC, giving your answer correct to 2 decimal places.

7. The diagram shows the graph of $3y = 2x + 8$. The line cuts the x-axis at A and the point K lies on the line such that its distance from the x-axis is twice that from the y-axis.



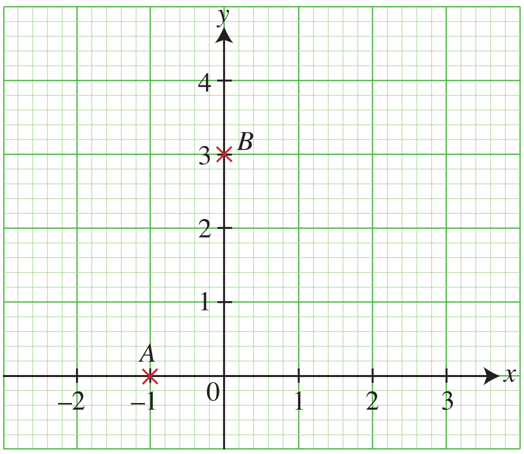
Find

(i) the coordinates of the point A,

(ii) the coordinates of the point K,

(iii) the area of ΔAKH, where H is the point $(4, 0)$.

8. The diagram shows the points $A(-1, 0)$ and $B(0, 3)$.



(i) Find the equation of the line passing through A and B.

(ii) Given that the length of AB is $\sqrt{h}$ units, find the value of h.

The point $(-5, k)$ lies on BA produced.

(iii) Find the value of k.

(iv) Given that $y = x + 1$ is the line of symmetry of ΔABC, find the coordinates of C.

9. The equation of a straight line l_1 is $5y + 12x - 60 = 0$. It intersects the x-axis at P and the y-axis at Q.

(i) Write down the coordinates of P and of Q.

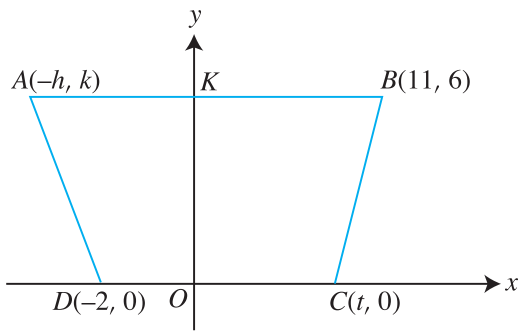
(ii) Find the length of PQ.

Another line l_2 has the same gradient as l_1 and passes through the point $(0, -2)$.

(iii) Find the equation of the line l_2.

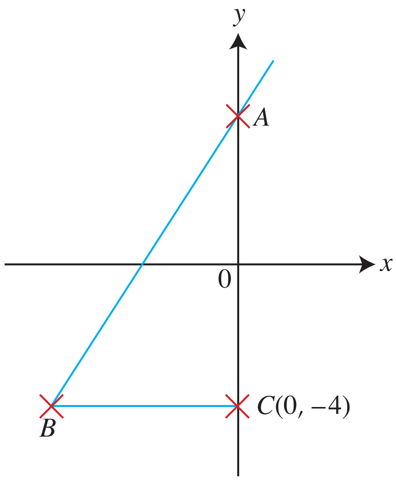
(iv) Given that the y-axis is the line of symmetry of ΔPQR, find the coordinates of R.

10. The diagram shows a trapezium $ABCD$ in which AB is parallel to DC and the area of $ABCD$ is 84 units^2.



Given that the vertices are $A(-h, k)$, $B(11, 6)$, $C(t, 0)$ and $D(-2, 0)$, find an expression for t in terms of h.

11. In the diagram, C is the point $(0, -4)$ and A is a point on the y-axis. The line AB meets the horizontal line through C at B.



(i) Write down the equation of the line passing through B and C.

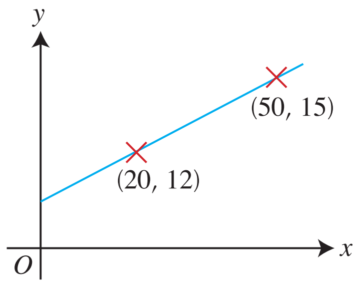
(ii) Given that the equation of the line passing through A and B is $2y - 5x = 4$, find the area of ΔABC.

(iii) Given that the length of AB is $\sqrt{l}$ units, find the value of l.

(iv) Find the length of the perpendicular from C to AB.

(v) Given that the coordinates of D are $(2.4, 2)$, show that $ABCD$ is a parallelogram.

12. A spring is suspended freely. When a mass of 20 g is attached to the spring, it has a length of 12 cm. When a mass of 50 g is attached to the spring, it has a length of 15 cm. The graph below shows how the length, y cm, of the spring, varies with the mass, x g, attached to it.

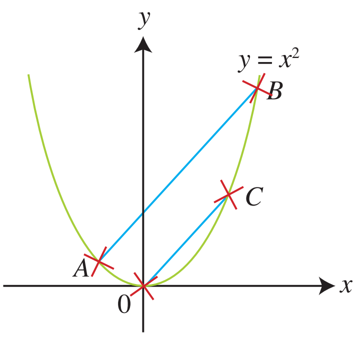


(i) Find an expression for y in terms of x.

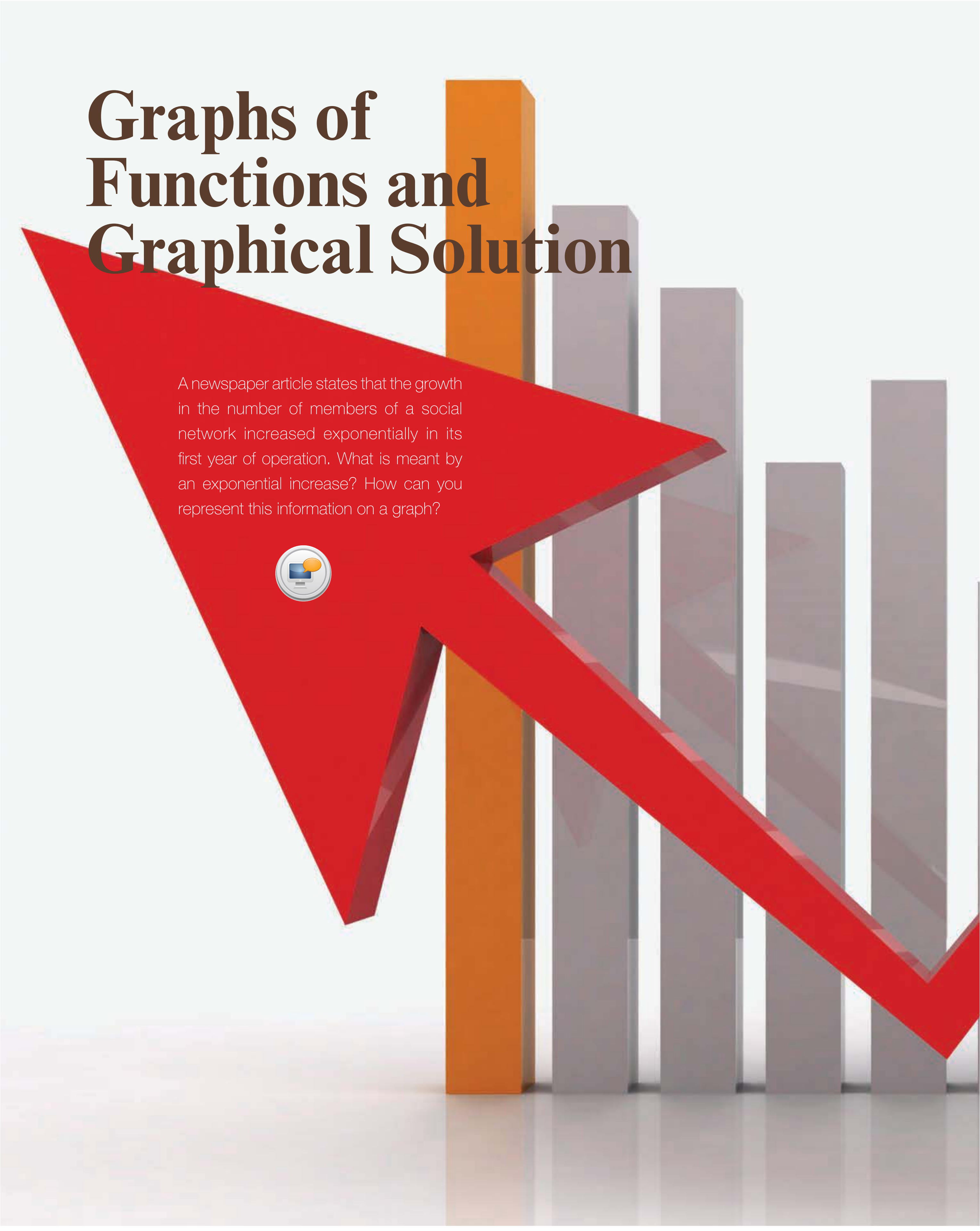
(ii) State what the value of the y-intercept represents.

Challenge Yourself

The diagram shows two points A and B on the graph of $y = x^2$. A line, drawn from the origin O and parallel to AB, intersects the graph at C. The coordinates of A, B and C are (a, a^2), (b, b^2) and (c, c^2) respectively. Show that $a + b = c$.



Graphs of Functions and Graphical Solution



A newspaper article states that the growth in the number of members of a social network increased exponentially in its first year of operation. What is meant by an exponential increase? How can you represent this information on a graph?

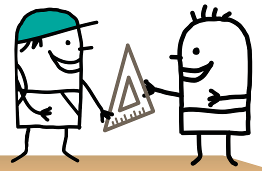
Chapter Five

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- draw the graphs of simple sums of power functions $y = ax^n$, where $n = 3, 2, 1, 0, -1$ and -2,
- draw the graphs of exponential functions $y = ka^x$, where a is a positive integer,
- estimate the gradient of a curve by drawing a tangent,
- interpret and analyse data from tables and graphs, including distance-time and speed-time graphs.

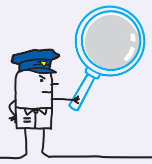
5.1 Graphs of Cubic Functions



In Book 2, we have learnt how to draw the graphs of $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$. In Chapter 1 of this book, we have learnt how to sketch graphs of the form $y = (x - h)(x - k)$ and $y = -(x - h)(x - k)$, where h and k are constants, and $y = (x - p)^2 + q$ and $y = -(x - p)^2 + q$, where p and q are constants.

In this section, we will learn how to draw the graphs of cubic functions.

In general, cubic functions are of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c and d are constants and $a \neq 0$.



Investigation

Graphs of Cubic Functions

1. Using a graphing software, draw each of the following graphs.

(a) $y = x^3$ **(b)** $y = 2x^3$

(c) $y = 5x^3$ **(d)** $y = -x^3$

(e) $y = -2x^3$ **(f)** $y = -5x^3$

2. For the graph of $y = ax^3$, where a is a constant, how does the value of a affect the shape of the graph?

3. Using a graphing software, draw each of the following graphs.

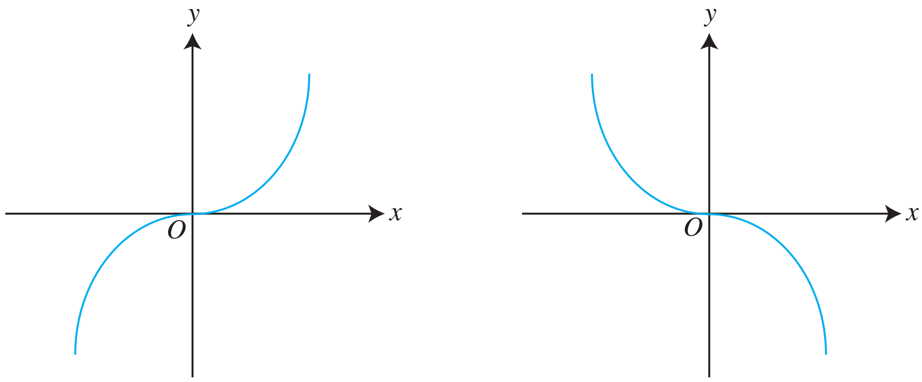
(a) $y = x^3 - x^2 + 1$ **(b)** $y = x^3 + 4x^2 - 3$

(c) $y = x^3 + x$ **(d)** $y = -x^3 + x^2 - 2$

(e) $y = -x^3 + 2x^2 + 1$ **(f)** $y = -x^3 - 0.5x - 1$

4. For each of the graphs in Question 3, how does the coefficient of x^3 affect the shape of the graph?

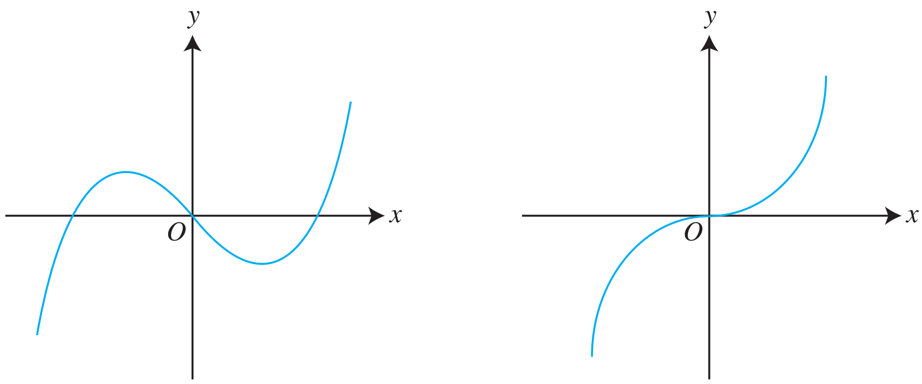
From the investigation, we observe that for the graph of a cubic function of the form $y = ax^3$, the graph takes the shape in Fig. 5.1 **(a)** and **(b)** for positive and negative values of the coefficient of x^3.



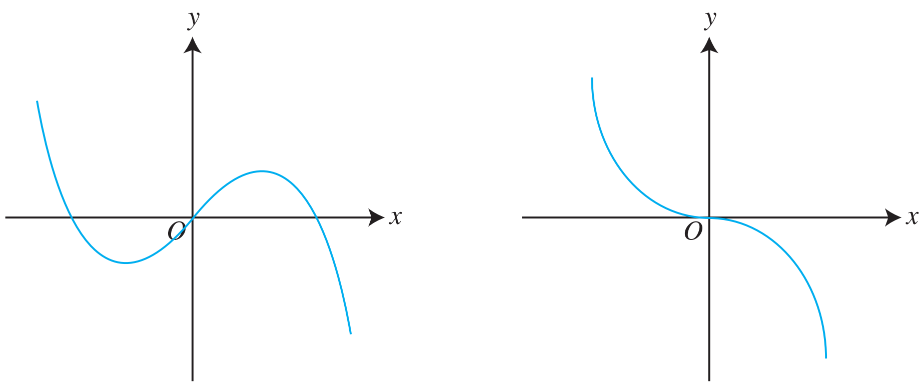
(a) $y = ax^3$, where $a > 0$ **(b)** $y = ax^3$, where $a < 0$

Fig. 5.1

In general, the graph of a cubic function, i.e. a function of the form $y = ax^3 + bx^2 + cx + d$ would take the shape in Fig. 5.2 (but may not pass through the origin if $d \neq 0$).



(a) $y = ax^3 + bx^2 + cx + d$, where $a > 0$



(b) $y = ax^3 + bx^2 + cx + d$, where $a < 0$

Fig. 5.2

Worked Example 1

(Drawing the Graph of a Cubic Function)

Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 10 units on the y-axis, draw the graphs of $y = x^3 + 3$ and $y = -x^3 - 3$ for $-3 \leqslant x \leqslant 3$.

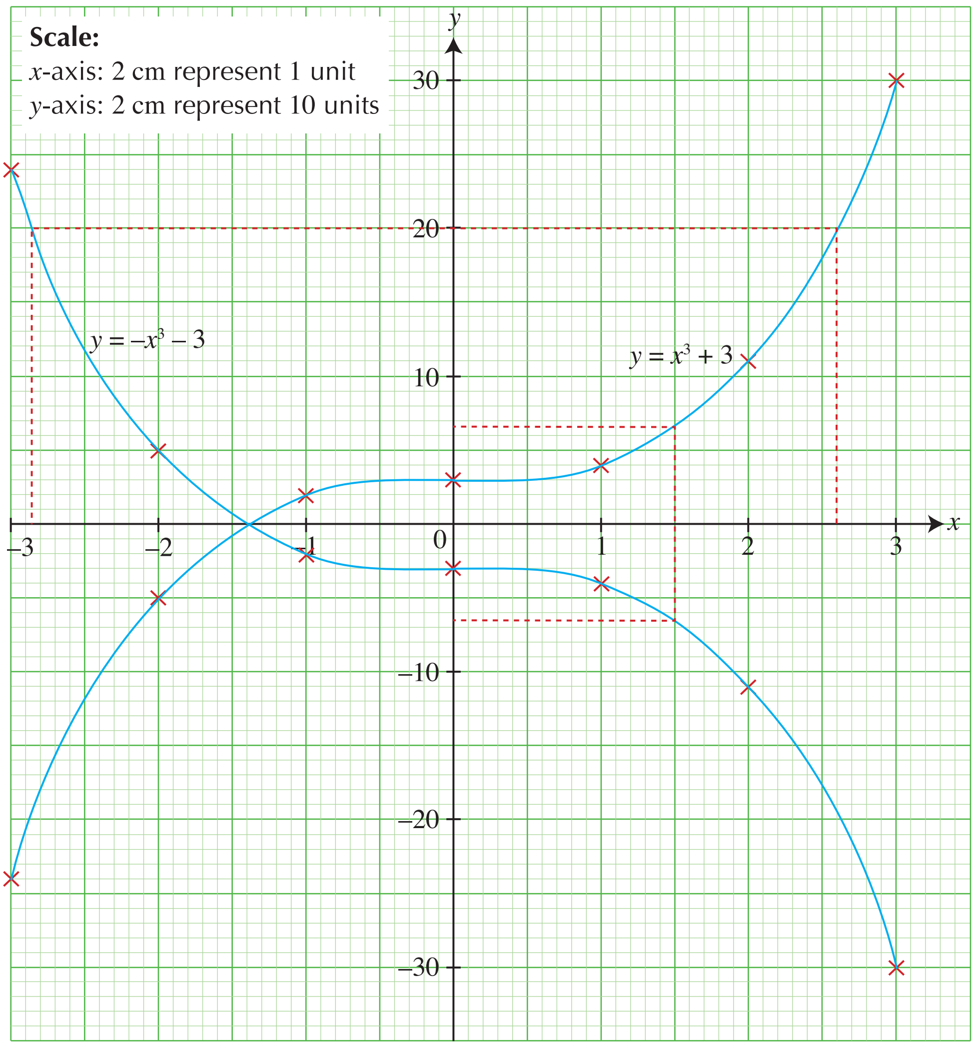
For each graph, find

(i) the value of y when $x = 1.5$,

(ii) the value of x when $y = 20$.

Solution:

x	–3	–2	–1	0	1	2	3
$y = x^3 + 3$	–24	–5	2	3	4	11	30
$y = -x^3 - 3$	24	5	–2	–3	–4	–11	–30



(i) Consider $y = x^3 + 3$.

From the graph, when $x = 1.5$, $y = 6.5$.

Consider $y = -x^3 - 3$.

From the graph, when $x = 1.5$, $y = -6.5$.

(ii) Consider $y = x^3 + 3$.

From the graph, when $y = 20$, $x = 2.6$.

Consider $y = -x^3 - 3$.

From the graph, when $y = 20$, $x = -2.85$.

PRACTISE NOW 1

SIMILAR QUESTIONS

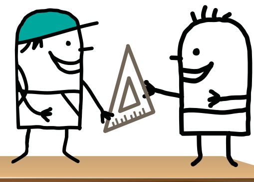
Exercise 5A Questions 1, 2, 5, 6, 11

Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 10 units on the y-axis, draw the graphs of $y = x^3 + 2$ and $y = -x^3 - 2$ for $-3 \leqslant x \leqslant 3$.

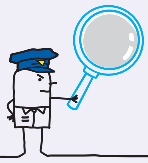
For each graph, find

(i) the value of y when $x = 2.5$, **(ii)** the value of x when $y = 15$.

5.2 Graphs of Reciprocal Functions



Graph of $y = \dfrac{a}{x}$



Investigation

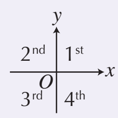
Graphs of $y = \dfrac{a}{x}$

Using a graphing software, draw the graph of $y = \dfrac{a}{x}$ for $a = 1$, $a = 5$, $a = -1$ and $a = -3$.

1. **(i)** For $a > 0$, which quadrants do the graphs lie in?
   (ii) For $a < 0$, which quadrants do the graphs lie in?
2. What can you say about the rotational symmetry of each of the graphs?
3. Do the graphs intersect the x-axis and the y-axis? Explain your answer.

INFORMATION

The four quadrants on the Cartesian plane are labelled 1st, 2nd, 3rd and 4th as follows:

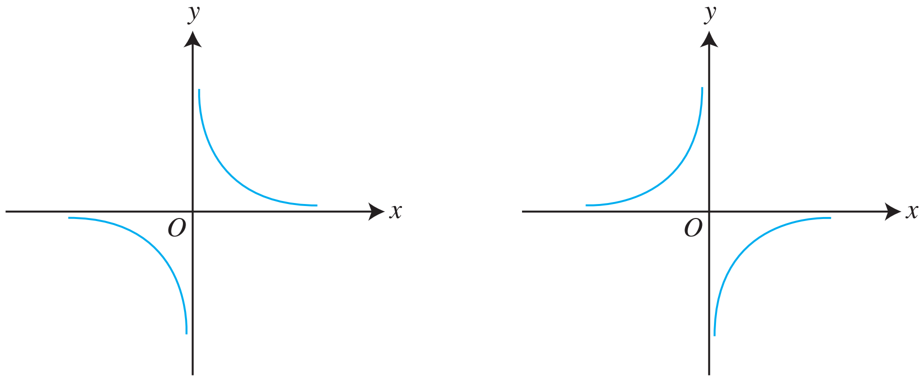


RECALL

The order of rotational symmetry about a particular point is the number of distinct ways in which a figure can map onto itself by rotation in 360°.

From the investigation, we observe that for the graph of $y = \frac{a}{x}$,

- when $x = 0$, the function $y = \frac{a}{x}$ is not defined, i.e. there is a break when $x = 0$,
- there is rotational symmetry of order 2 about the origin, i.e. it maps onto itself twice by rotation in 360°,
- if $a > 0$, the graph would take the shape in Fig. 5.3**(i)**,
  if $a < 0$, the graph would take the shape in Fig. 5.3**(ii)**.



(i) $y = \frac{a}{x}$, where $a > 0$ **(ii)** $y = \frac{a}{x}$, where $a < 0$

Fig. 5.3

Consider the graph in Fig. 5.3**(i)** $y = \frac{a}{x}$, where $a > 0$. The graph consists of two parts that lie in the 1st and 3rd quadrants. In the 1st quadrant, we observe that:

- as x increases, y decreases;
- as x approaches zero, y becomes very large;
  e.g. for $a = 1$, $y = \frac{1}{x}$, if $x = 0.000\,001$, $y = \frac{1}{0.000\,001} = 1\,000\,000$;
- as x becomes very large, y approaches zero;
  e.g. for $a = 1$, $y = \frac{1}{x}$, if $x = 1\,000\,000$, $y = \frac{1}{1\,000\,000} = 0.000\,001$;
- the curve gets very close to the x-axis and y-axis but never touches them.

INFORMATION

As the positive value of x decreases, the value of y increases rapidly and it gets very close to the y-axis. We say that the value of y approaches **infinity**, written as $y \to \infty$.

Can you describe the part of the graph that is in the 3rd quadrant?

Can you describe the graph of $y = \frac{a}{x}$, where $a < 0$?

Thinking Time

What are the equations of the lines of symmetry of the graph $y = \frac{a}{x}$ when

(a) $a > 0$? **(b)** $a < 0$?

Worked Example 2

(Drawing the Graph of $y = \frac{a}{x}$)

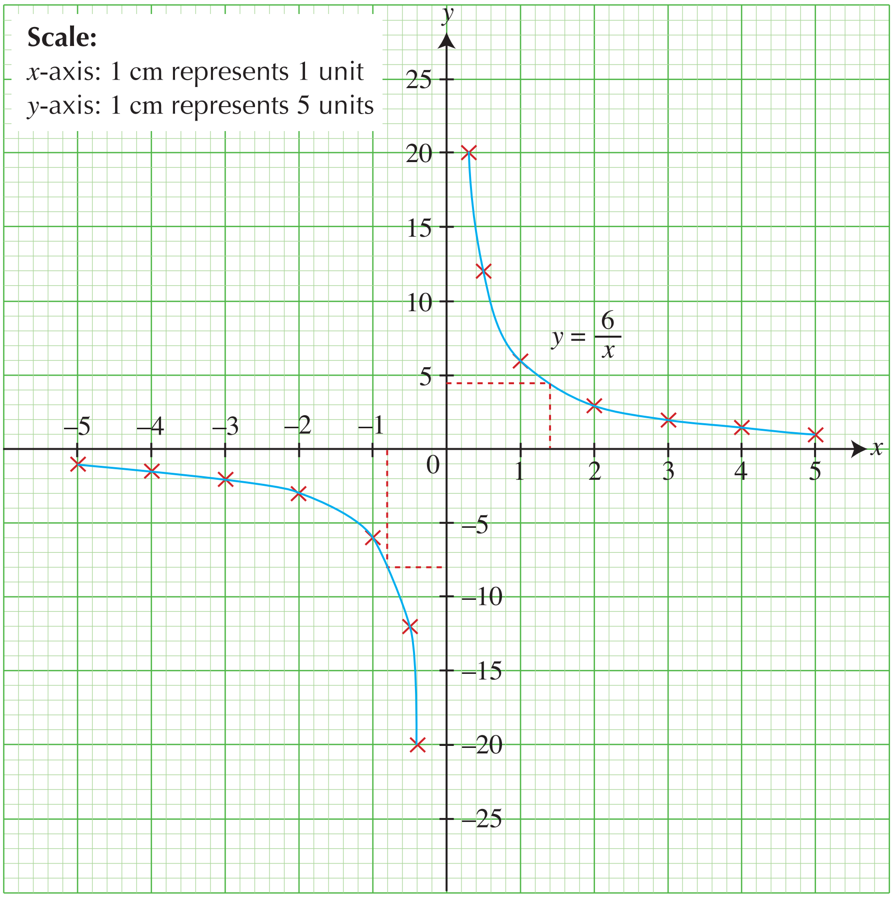
Using a scale of 1 cm to represent 1 unit on the x-axis and 1 cm to represent 5 units on the y-axis, draw the graph of $y = \frac{6}{x}$ for $-5 \leqslant x \leqslant 5$, $x \neq 0$. Find

(i) the value of y when $x = 1.4$,

(ii) the value of x when $y = -8$.

Solution:

x	–5	–4	–3	–2	–1	–0.5	–0.3	0.3	0.5	1	2	3	4	5
y	–1.2	–1.5	–2	–3	–6	–12	–20	20	12	6	3	2	1.5	1.2



(i) From the graph, when $x = 1.4$, $y = 4.5$.

(ii) From the graph, when $y = -8$, $x = -0.8$.

ATTENTION

For **(i)**, although the answer is 4.2857… by calculation, the answer obtained from the graph can only be accurate up to half of a small square grid, which is 0.5. Similarly, for **(ii)**, although $x = -0.75$ by calculation, the answer obtained from the graph is accurate to half of a small square grid, i.e. 0.1.

PRACTISE NOW 2

Using a scale of 1 cm to represent 1 unit on both axes, draw the graph of $y = \frac{3}{x}$ for $-5 \leqslant x \leqslant 5$, $x \neq 0$. Find

(i) the value of y when $x = 2.5$, **(ii)** the value of x when $y = -1.2$.

SIMILAR QUESTIONS

Exercise 5A Questions 3, 7, 8, 12

Graph of $y = \frac{a}{x^2}$

Investigation

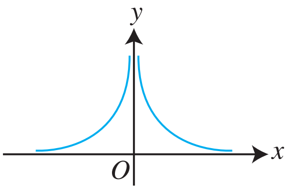
Graphs of $y = \frac{a}{x^2}$

Using a graphing software, draw the graph of $y = \frac{a}{x^2}$ for $a = 2$, $a = 4$, $a = -1$ and $a = -3$.

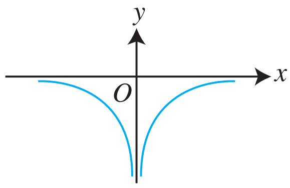
1. **(i)** For $a > 0$, which quadrants do the graphs lie in?
   (ii) For $a < 0$, which quadrants do the graphs lie in?
2. What can you say about the line symmetry of each of the graphs?
3. Do the graphs intersect the x-axis and the y-axis? Explain your answer.

From the investigation, we observe that for the graph of $y = \frac{a}{x^2}$,

- when $x = 0$, the function $y = \frac{a}{x^2}$ is not defined, i.e. there is a break when $x = 0$,
- if $a > 0$, the values of y are always positive, i.e. the graph lies entirely above the x-axis;
  if $a < 0$, the values of y are always negative, i.e. the graph lies entirely below the x-axis.
- the graph is symmetrical about the y-axis, i.e. the y-axis is the line of symmetry,
- if $a > 0$, the graph would take the shape in Fig. 5.4**(i)**,
  if $a < 0$, the graph would take the shape in Fig. 5.4**(ii)**.



(i) $y = \frac{a}{x^2}$, where $a > 0$



(ii) $y = \frac{a}{x^2}$, where $a < 0$

Fig. 5.4

Consider the graph in Fig. 5.4**(i)** $y = \frac{a}{x^2}$, where $a > 0$. The graph consists of two parts that lie in the 1st and 2nd quadrants. In the 1st quadrant, we observe that:

- as x increases, y decreases;
- as x approaches zero, y becomes very large;
  e.g. for $a = 1$, $y = \frac{1}{x^2}$, if $x = 0.001$, $y = \frac{1}{0.001^2} = 1\,000\,000$;
- as x becomes very large, y approaches zero;
  e.g. for $a = 1$, $y = \frac{1}{x^2}$, if $x = 1000$, $y = \frac{1}{1000^2} = 0.000\,001$;
- the curve gets very close to the x-axis and y-axis but never touches them.

Can you describe the part of the graph that is in the 2nd quadrant?

Can you describe the graph of $y = \frac{a}{x^2}$, where $a < 0$?

Worked Example 3

(Drawing the Graph of $y = \frac{a}{x^2}$)

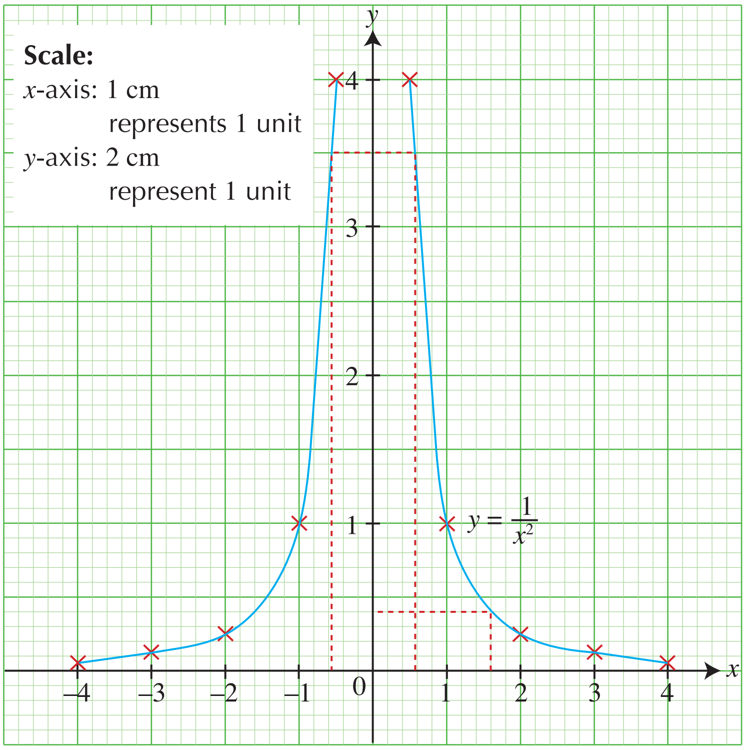
Using 1 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = \frac{1}{x^2}$ for $-4 \leqslant x \leqslant 4$, $x \neq 0$. Find

(i) the value of y when $x = 1.6$,

(ii) the values of x when $y = 3.5$.

Solution:

x	–4	–3	–2	–1	–0.5	0.5	1	2	3	4
y	0.06	0.11	0.25	1	4	4	1	0.25	0.11	0.06



(i) From the graph, when $x = 1.6$, $y = 0.4$.

(ii) From the graph, when $y = 3.5$, $x = 0.6$ or $x = -0.6$.

ATTENTION

The accuracy of the answer can only be accurate up to half of a small square grid.

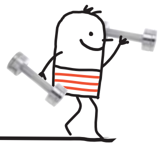
PRACTISE NOW 3

Using 1 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = -\frac{2}{x^2}$ for $-4 \leqslant x \leqslant 4$, $x \neq 0$. Find

(i) the value of y when $x = 1.5$, **(ii)** the values of x when $y = -3.2$.

SIMILAR QUESTIONS

Exercise 5A Questions 4, 9, 10, 13



Exercise 5A

BASIC LEVEL

1. The table below shows some values of x and the corresponding values of y, where $y = x^3$.

x	–3	–2	–1	0	1	2	3
y		–8			1		27

(a) Copy and complete the table.

(b) Using a scale of 2 cm to represent 1 unit, draw a horizontal x-axis for $-3 \leqslant x \leqslant 3$.

Using a scale of 2 cm to represent 5 units, draw a vertical y-axis for $-27 \leqslant y \leqslant 27$.

On your axes, plot the points given in the table and join them with a smooth curve.

(c) Use your graph to find

(i) the value of y when $x = 1.5$,

(ii) the value of x when $y = 12$.

2. The table below shows some values of x and the corresponding values of y, correct to 1 decimal place, where $y = 2x^3 + 3$.

x	–2	–1.5	–1	–0.5	0	0.5	1	1.5	2
y	–13	–3.8	1	p	3	3.3	5	9.8	19

(a) Find the value of p.

(b) Using a scale of 4 cm to represent 1 unit, draw a horizontal x-axis for $-2 \leqslant x \leqslant 2$.

Using a scale of 1 cm to represent 5 units, draw a vertical y-axis for $-15 \leqslant y \leqslant 20$.

On your axes, plot the points given in the table and join them with a smooth curve.

(c) Use your graph to find

(i) the value of y when $x = -1.2$,

(ii) the value of x when $y = 14$.

3. The table below shows some values of x and the corresponding values of y, where $y = \frac{4}{x}$.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4	5
y	16			2		1	

(a) Copy and complete the table.

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = \frac{4}{x}$ for $\frac{1}{4} \leqslant x \leqslant 5$.

(c) Use your graph to find

(i) the value of y when $x = 3.6$,

(ii) the value of x when $y = 1.5$.

4. The table below shows some values of x and the corresponding values of y, correct to 1 decimal place, where $y = \frac{10}{x^2}$.

x	1	2	3	4	5
y	10	2.5	a	0.6	b

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = \frac{10}{x^2}$ for $1 \leqslant x \leqslant 5$.

(c) Use your graph to find

(i) the value of y when $x = 2.8$,

(ii) the value of x when $y = 4.4$.

INTERMEDIATE LEVEL

5. Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = 3x - x^3$ for $-3 \leqslant x \leqslant 3$. Use your graph to find

(i) the value of y when $x = 1.4$,

(ii) the values of x when $y = -6.6$.

6. Using a suitable scale, draw the graph of $y = x^3 - 6x^2 + 13x$ for $0 \leqslant x \leqslant 5$. Use your graph to find
   (a) the value(s) of y when
      (i) $x = 1.5$, **(ii)** $x = 3.5$, **(iii)** $x = 4.45$.
   (b) the value of x when
      (i) $y = 7$, **(ii)** $y = 15$, **(iii)** $y = 22$.

7. Using a scale of 4 cm to represent 1 unit on both axes, draw the graph of $y = -\frac{2}{x} - 1$ for $\frac{1}{2} \leqslant x \leqslant 4$. Use your graph to find
   (i) the value of y when $x = 2.5$,
   (ii) the value of x when $y = -1.6$.

8. The table below shows some values of x and the corresponding values of y, correct to 1 decimal place, where $y = x - \frac{3}{x}$.

x	0.5	1	2	3	4	5	6
y	–5.5	–2	0.5	h	3.3	4.4	k

   (a) Find the value of h and of k.
   (b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = x - \frac{3}{x}$ for $0.5 \leqslant x \leqslant 6$.
   (c) Use your graph to find
      (i) the value of y when $x = 1.6$,
      (ii) the value of x when $y = -2.5$.

9. Using a scale of 2 cm to represent 1 unit on the x-axis and 4 cm to represent 1 unit on the y-axis, draw the graph of $y = 2 - \frac{3}{x^2}$ for $1 \leqslant x \leqslant 6$. Use your graph to find
   (i) the value of y when $x = 1.5$,
   (ii) the value of x when $y = 1.5$.

10. Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = x + \frac{2}{x^2}$ for $1 \leqslant x \leqslant 6$. Use your graph to find
   (i) the value of y when $x = 5.4$,
   (ii) the values of x when $y = 3$.

ADVANCED LEVEL

11. Using a suitable scale, draw the graph of $y = x^3 - 2x - 1$ for $-3 \leqslant x \leqslant 3$.
   (a) Use your graph to find the x-coordinates of the points of intersection of the curve with the x-axis.
   (b) On the same axes, draw the straight line $y = x$ for $-3 \leqslant x \leqslant 3$.
      (i) Write down the x-coordinates of the points at which the line $y = x$ meets the curve $y = x^3 - 2x - 1$.
      (ii) Hence, state the solutions of the equation $x^3 - 2x - 1 = x$. Explain your answer.

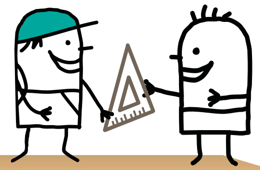
12. The variables x and y are connected by the equation $y = x + \frac{1}{2x} - 1$.
   The table below shows some values of x and the corresponding values of y, correct to 1 decimal place.

x	0.1	0.5	0.8	1	1.5	2	2.5	3	3.5	4
y	4.1	0.5	0.4	0.5	0.8	1.3	p	2.2	2.6	3.1

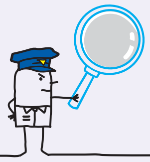
   (a) Calculate the value of p.
   (b) Using a scale of 4 cm to represent 1 unit on both axes, draw the graph of $y = x + \frac{1}{2x} - 1$ for $0.1 \leqslant x \leqslant 4$.
   (c) Use your graph to find the values of x in the range $0.1 \leqslant x \leqslant 4$ for which $x + \frac{1}{2x} = 1$.

13. Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph $y = \frac{1}{4}x^2 + \frac{8}{x} - 9$ for $0.5 \leqslant x \leqslant 7$.
   (a) Use your graph to find the minimum value of y in the given range.
   (b) By drawing suitable straight lines to the graph, solve each of the following equations, giving your answers correct to 1 decimal place.
      (i) $\frac{1}{4}x^2 + \frac{8}{x} = 6$
      (ii) $\frac{1}{4}x^2 + \frac{8}{x} = x + 4$
      (iii) $\frac{1}{4}x^2 + 2x = 15 - \frac{8}{x}$

5.3 Graphs of Exponential Functions



Graphs of $y = a^x$ and $y = ka^x$



Investigation

Graphs of $y = a^x$ and $y = ka^x$

1. Using a graphing software, draw each of the following graphs.

(a) $y = 2^x$ **(b)** $y = 3^x$

(c) $y = 4^x$ **(d)** $y = 5^x$

2. For each of the graphs in Question 1, answer the following questions.

(a) Write down the coordinates of the point where the graph intersects the y-axis.

(b) As x increases, what happens to the value of y?

(c) Does the graph intersect the x-axis?

3. How does the value of a affect the shape of the graph of $y = a^x$?

4. Using a graphing software, draw each of the following graphs.

(a) $y = 2^x$ **(b)** $y = 3(2^x)$

(c) $y = 5(2^x)$ **(d)** $y = -2^x$

(e) $y = -4(2^x)$

5. For each of the graphs in Question 4, answer the following questions.

(a) Write down the coordinates of the point where the graph intersects the y-axis.

(b) As x increases, what happens to the value of y?

(c) Does the graph intersect the x-axis?

6. How does the value of k affect the shape of the graph of $y = ka^x$?

From the investigation, we observe that for the graph of $y = a^x$,

- the values of y are always positive, i.e. the graph lies entirely above the x-axis,
- the graph intersects the y-axis at (0, 1).

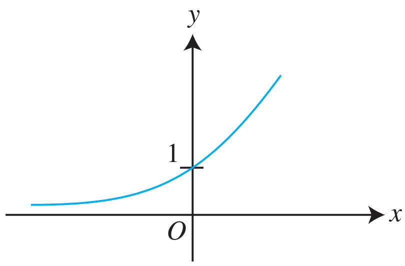


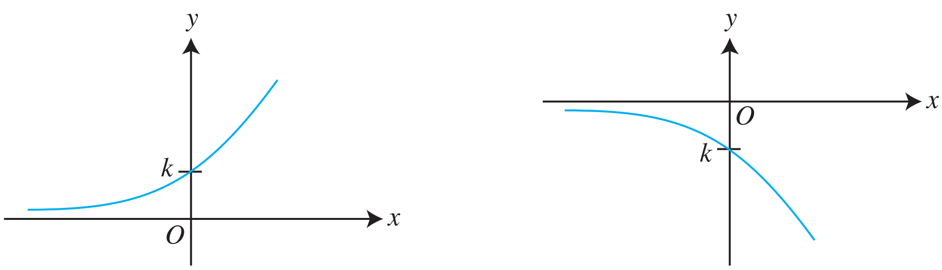
Fig. 5.5

As the positive value of x increases and tends to the right of the graph, the value of y increases very rapidly and approaches infinity. When x is negative and tends to the left of the graph, y becomes smaller as x becomes smaller. The curve gets very close to the x-axis but never touches it.

For the graph of $y = ka^x$,

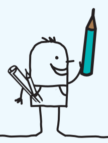
- if $k > 0$, the values of y are always positive, i.e. the graph lies entirely above the x-axis (see Fig. 5.6**(i)**),

  if $k < 0$, the values of y are always negative, i.e. the graph lies entirely below the x-axis (see Fig. 5.6**(ii)**),
- the graph intersects the y-axis at (0, k).



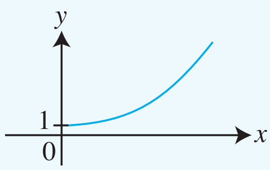
(i) $y = ka^x$, where $k > 0$ **(ii)** $y = ka^x$, where $k < 0$

Fig. 5.6



Journal Writing

A newspaper article states that the growth in the number of members of a social network increased **exponentially** in its first year of operation and can be represented by the equation $y = 28^x$, where x is the number of months and y is the number of members.



(i) Describe how the number of members of the social network changes with time.

(ii) Search on the Internet for more real-life applications of exponential graphs.

Worked Example 4

(Graph of $y = a^x$)

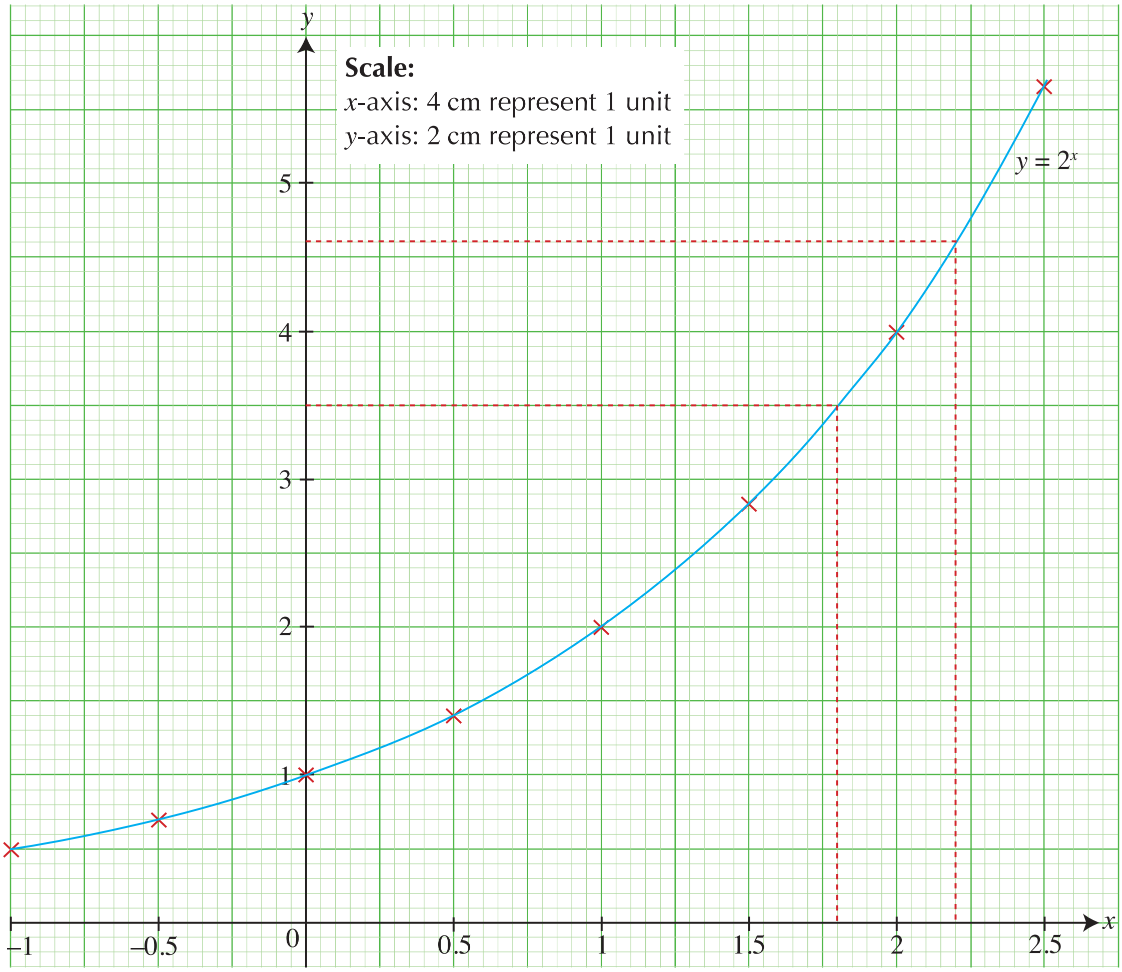
Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 2^x$ for $-1 \leqslant x \leqslant 2.5$. Use your graph to find

(i) the value of y when $x = 1.8$,

(ii) the value of x when $y = 4.6$.

Solution:

x	–1	–0.5	0	0.5	1	1.5	2	2.5
y	0.5	0.71	1	1.41	2	2.83	4	5.66



(i) From the graph, when $x = 1.8$, $y = 3.5$.

(ii) From the graph, when $y = 4.6$, $x = 2.2$.

PRACTISE NOW 4

Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 3^x$ for $-2 \leqslant x \leqslant 2$. Use your graph to find

(i) the value of y when $x = -1$, **(ii)** the value of x when $y = 0.7$.

SIMILAR QUESTIONS

Exercise 5B Questions 1-5

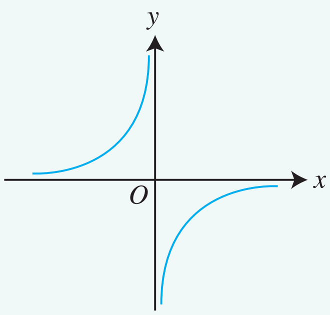
Class Discussion

Matching Graphs of Power Functions with the Corresponding Functions

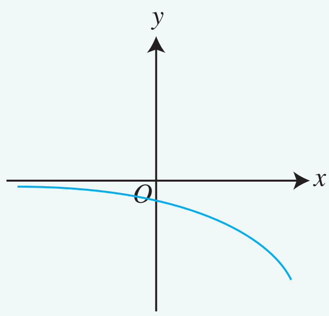
Work in pairs.

Match the graphs with their respective functions and justify your answers. If your classmate does not obtain the correct answer, explain to him what he has done wrong.

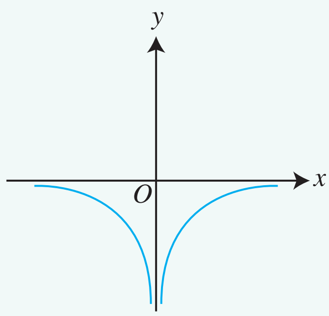
A: $y = 2x^3$	**B**: $y = -\dfrac{6}{x}$	**C**: $y = \dfrac{5}{2x^2}$	**D**: $y = 5^x$
E: $y = -\dfrac{3}{x^2}$	**F**: $y = -2(6^x)$	**G**: $y = \dfrac{1}{2x}$	**H**: $y = -3x^3$



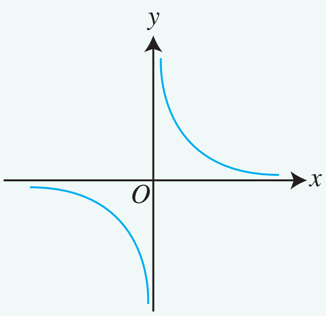
Graph 1



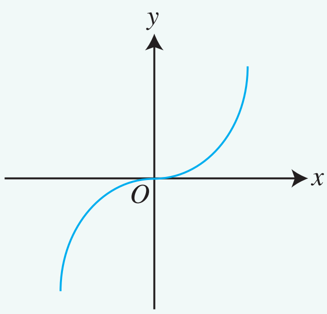
Graph 2



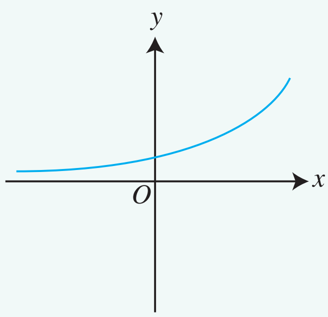
Graph 3



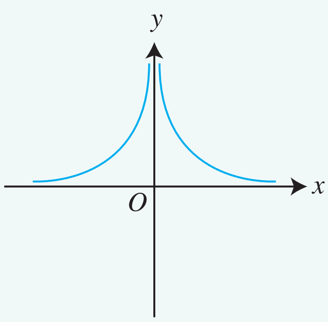
Graph 4



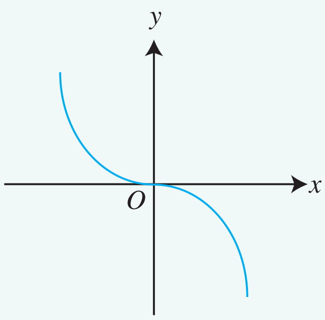
Graph 5



Graph 6



Graph 7

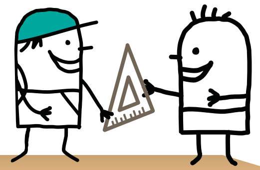


Graph 8

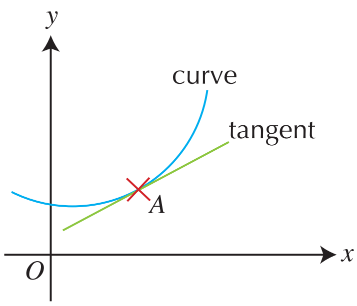
SIMILAR QUESTIONS

Exercise 5B Question 6

5.4 Gradient of a Curve



When a straight line touches a curve at a single point A, the line is called the **tangent** to the curve at the point A.



When a line l_1 touches the curve at P, l_1 is called the tangent to the curve at P. Similarly, when a line l_2 touches the curve at Q, l_2 is called the tangent to the curve at Q.

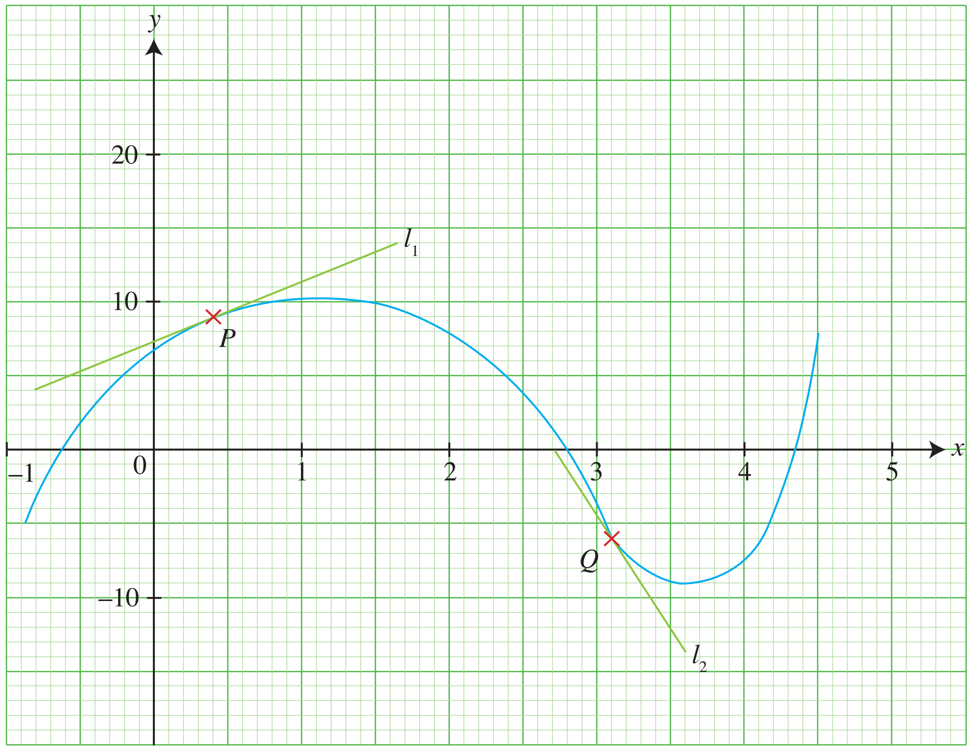


Fig. 5.7

The **gradient of the curve at a point** is defined as the **gradient of the tangent to the curve at that point**. Hence, the gradient of the curve at P in Fig. 5.7 is equal to the gradient of the line l_1 and the gradient of the curve at Q is equal to the gradient of the line l_2.

Worked Example 5

(Gradient of a Curve)

The variables x and y are connected by the equation $y = \frac{1}{2}(5x - x^2)$.

The table below shows some values of x and the corresponding values of y.

x	$-\frac{1}{2}$	0	1	2	$2\frac{1}{2}$	3	4	5
y	a	0	2	3	b	3	2	0

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = \frac{1}{2}(5x - x^2)$ for $-\frac{1}{2} \leqslant x \leqslant 5$.

(c) By drawing a tangent, find the gradient of the curve at the point (1, 2).

(d) The gradient of the curve at the point (h, k) is zero.

  (i) Draw the tangent at the point (h, k).

  (ii) Hence, find the value of h and of k.

Solution:

(a) When $x = -\frac{1}{2}$,

$$y = \frac{1}{2}\left[5\left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)^2\right]$$
$$= -1\frac{3}{8}$$
$$\therefore a = -1\frac{3}{8}$$
$$= -1.375$$

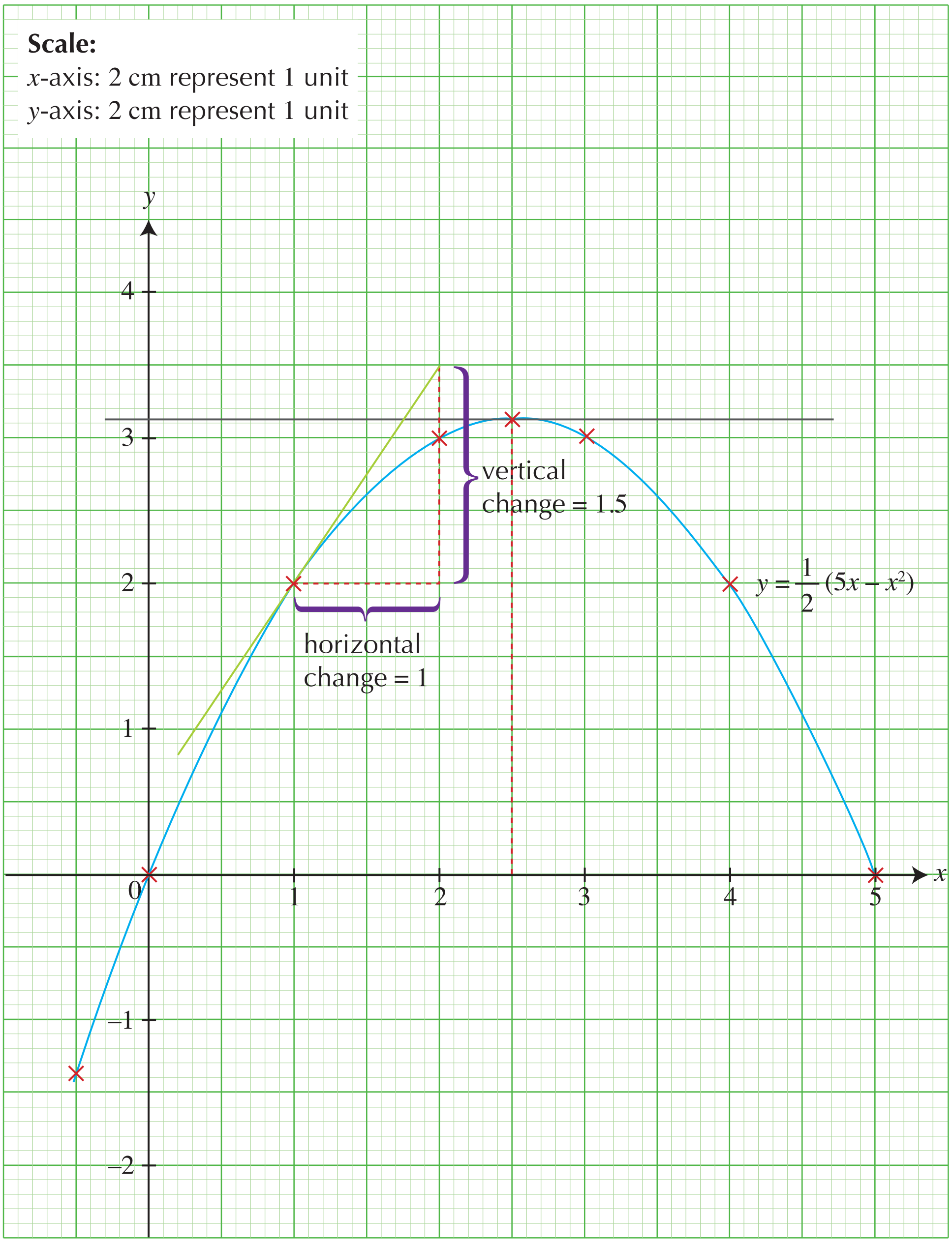
When $x = 2\frac{1}{2}$,

$$y = \frac{1}{2}\left[5\left(2\frac{1}{2}\right) - \left(2\frac{1}{2}\right)^2\right]$$
$$= 3\frac{1}{8}$$
$$\therefore b = 3\frac{1}{8}$$
$$= 3.125$$

Problem Solving Tip

Give the value of a and of b in decimals for easy plotting of points.

(b)



(c) A tangent is drawn to the curve at the point (1, 2).

From the graph,

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}}$$
$$= \frac{1.5}{1}$$
$$= 1.5$$

(d) A line parallel to the x-axis at the maximum point of the curve has a gradient equal to zero. From the graph and table, $h = 2.5$, $k = 3\frac{1}{8}$.

RECALL

A line parallel to the x-axis has a gradient equal to zero.

PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 5B Questions 7, 8, 10

The variables x and y are connected by the equation $y = x^2 - 4x$.

The table below shows some values of x and the corresponding values of y.

x	–1	0	1	2	3	4	5
y	a	0	–3	–4	b	0	5

(a) Find the value of a and of b.

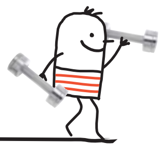
(b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = x^2 - 4x$ for $-1 \leqslant x \leqslant 5$.

(c) By drawing a tangent, find the gradient of the curve at the point where $x = 2.8$.

(d) The gradient of the curve at the point (h, k) is zero.

(i) Draw the tangent at the point (h, k).

(ii) Hence, find the value of h and of k.



Exercise 5B

BASIC LEVEL

1. The table below shows some values of x and the corresponding values of y, where $y = 4^x$.

x	–1	–0.5	0	0.5	1	1.5	2	2.5
y	0.25		1	2	4			

(a) Copy and complete the table.

(b) Using a scale of 4 cm to represent 1 unit, draw a horizontal x-axis for $-1 \leqslant x \leqslant 2.5$.

Using a scale of 1 cm to represent 2 units, draw a vertical y-axis for $0 \leqslant y \leqslant 32$.

On your axes, plot the points given in the table and join them with a smooth curve.

(c) Use your graph to find

(i) the value of y when $x = 1.8$,

(ii) the value of x when $y = 0.4$.

2. The variables x and y are connected by the equation $y = 3(2^x)$.

The table below shows some values of x and the corresponding values of y correct to 1 decimal place.

x	–1	–0.5	0	0.5	1	1.5	2	2.5
y	1.5	2.1			6		12	17.0

(a) Copy and complete the table.

(b) Using a scale of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 3(2^x)$ for $-1 \leqslant x \leqslant 2.5$.

(c) Use your graph to find

(i) the values of y when $x = 0.7$ and $x = 2.3$,

(ii) the values of x when $y = 2.5$ and $y = 7.4$.

3. Using a scale of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = -2(3^x)$ for $-2 \leqslant x \leqslant 2$. Use your graph to find

(i) the value of y when $x = 1.2$,

(ii) the value of x when $y = -6.7$.

INTERMEDIATE LEVEL

4. The table below shows some values of x and the corresponding values of y, correct to 1 decimal place, where $y = 2 + 2^x$.

x	–1	–0.5	0	1	1.5	2	2.5	3
y	a	2.7	3	4	4.8	6	b	10

(a) Find the value of a and of b.

(b) Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 2 + 2^x$ for $-1 \leqslant x \leqslant 3$.

(c) Use your graph to find

(i) the values of y when $x = -0.7$ and $x = 2.7$,

(ii) the values of x when $y = 5.3$ and $y = 7.5$.

5. Using a scale of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graph of $y = 3^x$ for $-2 \leqslant x \leqslant 2$.

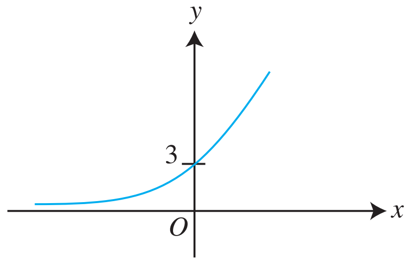
(a) Use your graph to find the value of x when $y = 5.8$.

(b) On the same axes, draw the graph of $y = \frac{1}{2}x - \frac{1}{x}$, $x \neq 0$.

(i) Write down the coordinates of the point at which the graph of $y = \frac{1}{2}x - \frac{1}{x}$ meets the curve $y = 3^x$.

(ii) Hence, state the solution of the equation $3^x + \frac{1}{x} - \frac{1}{2}x = 0$.

6. The sketch represents the graph of $y = ka^x$, where $a > 0$.



Write down the value of k.

7. The table below shows some values of x and the corresponding values of y, where $y = (x + 2)(4 - x)$.

x	–2	–1	0	1	2	3	4
y	0		8	9			0

(a) Copy and complete the table.

(b) Using a scale of 2 cm to represent 1 unit on both axes, draw the graph of $y = (x + 2)(4 - x)$ for $-2 \leqslant x \leqslant 4$.

(c) By drawing a tangent, find the gradient of the curve at the point where $x = -1$.

(d) The gradient of the curve at the point (h, k) is zero.

(i) Draw the tangent at the point (h, k).

(ii) Hence, find the value of h and of k.

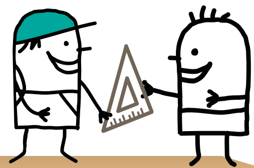
8. (a) Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = 12 + 10x - 3x^2$ for $-2 \leqslant x \leqslant 5$.
   (b) Find the gradient of the curve when $x = 4$.
   (c) Find the gradient of the curve at the point where the curve intersects the y-axis.

ADVANCED LEVEL

9. Using a scale of 4 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis, draw the graph of $y = 2^x + \frac{1}{x^2}$ for $-2 \leqslant x \leqslant 3$.
   (a) (i) On the same axes, draw the line $y = 1 - x$.
   (ii) Hence, solve the equation $2^x + \frac{1}{x^2} - 1 + x = 0$.
   (b) Explain why the graph of $y = 2^x + \frac{1}{x^2}$ will not lie below the x-axis for all real values of x.

10. (a) Using suitable scale, draw the graph of $y = 1 + \frac{1}{x}$ for $0.5 \leqslant x \leqslant 3$.
   (b) On the same axes, draw the line $y = -x$.
   (c) Hence, find the coordinates on the graph of $y = 1 + \frac{1}{x}$ at which the gradient of the curve is -1.

5.5 Applications of Graphs in Real-World Contexts



In this section, we will apply our knowledge of coordinate geometry and graphs to analyse and interpret graphs in various real-world contexts, including distance-time and speed-time graphs. Important features of the graphs such as intercepts, gradients, variables and scale of the x- and y-axes will provide information to help us analyse the graphs.

For example, the gradient of a line segment in a distance-time graph gives the speed. If the graph is a curve, the gradient of the curve at a point will give the speed at that instant.

Class Discussion

Linear Distance-Time Graphs

Work in pairs.

Fig. 5.8 shows the graph of a cyclist's journey between 0800 and 1200. The graph can be divided into 4 sections – 0800 to 0900, 0900 to 0930, 0930 to 1030 and 1030 to 1200.

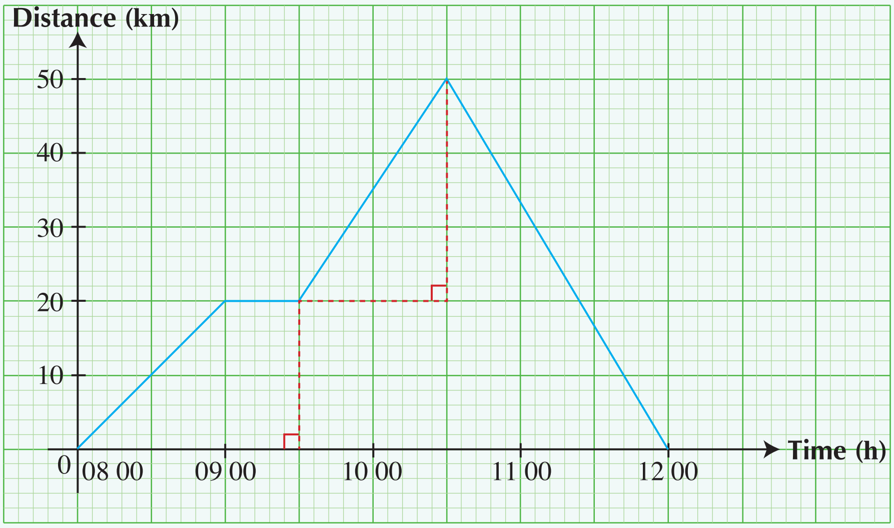


Fig. 5.8

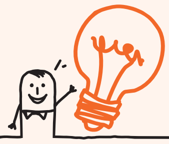
Since the gradient of the graph from 0800 to 0900 $= \dfrac{20 \text{ km}}{1 \text{ h}}$

$= 20$ km/h,

the cyclist travels at a constant speed of 20 km/h in the first hour.

1. Consider the section of the graph from 0900 to 0930. Since the graph is a horizontal line, what is its gradient? State clearly what this gradient represents.
2. Find the gradient of the section of the graph from 0930 to 1030. What does this gradient tell you about the motion of the cyclist?
3. Find the gradient of the section of the graph from 1030 to 1200. What does the negative gradient represent? Describe briefly the motion of the cyclist.
4. Explain why the average speed of the cyclist cannot be calculated by using $\dfrac{20+0+30+50}{4}$ km/h. Hence, find the **average speed** of the cyclist for the whole journey.

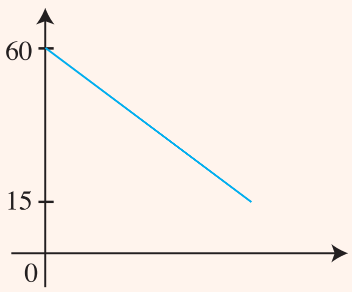
The average speed of an object is defined as the total distance travelled by the object per unit time.



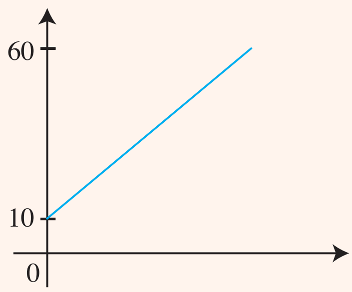
Thinking Time

Match the scenarios with their respective graphs and justify your answers.

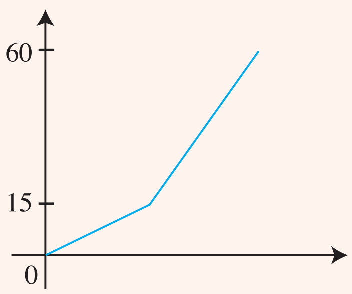
A: A few years ago, the exchange rate between Singapore dollars and Hong Kong dollars was S\$1 = HK\$6.	**B:** The height of water in a uniform cylindrical container increased at a constant rate from 10 cm to 60 cm.	**C:** Mr Neo was driving at a constant speed of 60 km/h when he suddenly applied the brakes and came to a stop.
D: The battery level in a smartphone decreased non-uniformly from 60% to 15%.	**E:** The temperature of a substance in a freezer decreased uniformly from 60 °C to 15 °C in 20 minutes.	**F:** A plant grew slowly at a constant rate to a height of 15 cm when it was kept indoors for 4 weeks, then grew more quickly at a constant rate to a height of 60 cm when it was placed outdoors for the next 4 weeks.



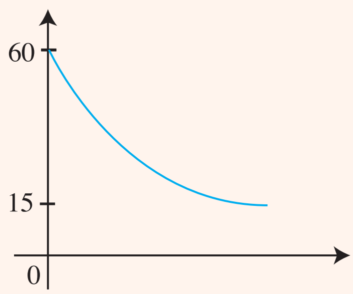
Graph 1



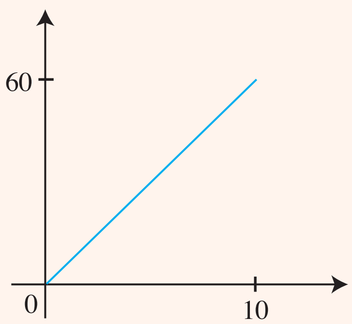
Graph 2



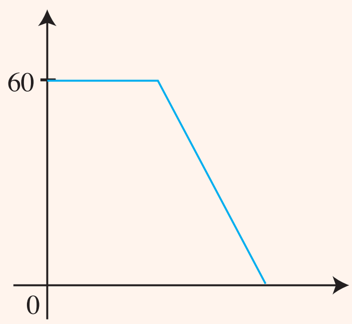
Graph 3



Graph 4



Graph 5



Graph 6

Worked Example 6

(Distance-Time Curve)

A train started from station A and travelled to station B 8 km from A.

The table below shows the readings of the time, in minutes, since leaving station A and the corresponding distance, in km, from A.

Time (in minutes)	1	2	3	4	5	6	7	8
Distance (in km)	0.3	1.1	2.3	4.8	6.8	7.4	7.8	8.0

(a) Using a scale of 2 cm to represent 2 minutes on the horizontal axis and 2 cm to represent 2 km on the vertical axis, plot the points given in the table and join them with a smooth curve.

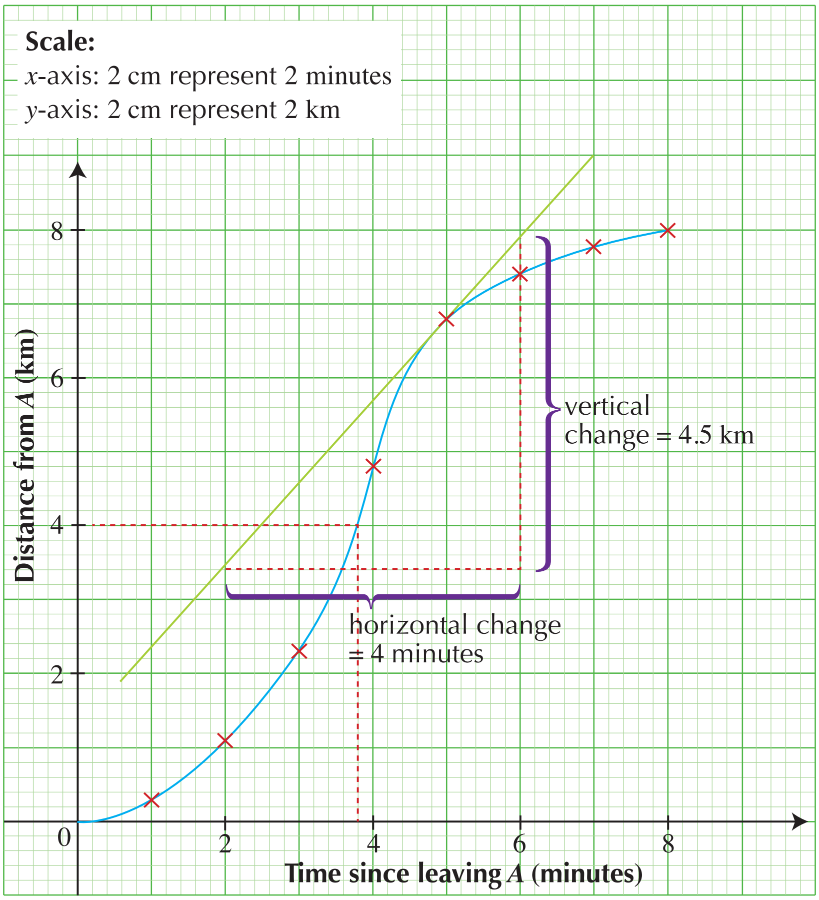
(b) Use your graph to estimate the time taken to travel the first 4 km of the journey.

(c) By drawing a tangent, find the approximate speed of the train 5 minutes after it has left station A.

(d) By considering the gradient of the graph, compare and describe briefly the motion of the train during the first 4 minutes and the last 4 minutes of the journey.

Solution:

(a)



(b) From the graph, the train takes approximately 3.8 minutes to travel the first 4 km.

(c) The gradient of the tangent at the point 5 minutes after it left station A gives the speed at that particular point. It is called the instantaneous speed. A tangent is drawn to the curve at the point 5 minutes after it has left station A.

From the graph,

$$\begin{aligned} \text{Gradient} &= \frac{\text{vertical change}}{\text{horizontal change}} \\ &= \frac{4.5 \text{ km}}{4 \text{ minutes}} \\ &= \frac{4.5 \text{ km}}{\frac{4}{60} \text{ h}} \\ &= 67.5 \text{ km/h} \end{aligned}$$

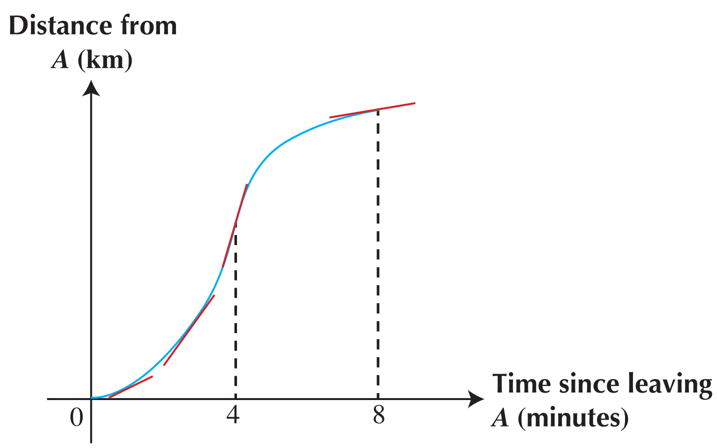
$\therefore$ The speed of the train 5 minutes after it has left station A is approximately 67.5 km/h.

ATTENTION

The graphical method of finding the gradient of a curve yields only approximate results. A slight change in the drawing may give very different results.

(d) During the first 4 minutes, the speed of the train increases as the gradient of the curve increases.

During the last 4 minutes, the speed of the train decreases as the gradient of the curve decreases.



PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 5C Questions 1-3, 6-9

A train started from station P and travelled to station Q, 7.6 km from P.

The table below shows the readings of the time, in minutes, since leaving station P and the corresponding distance, in km, from P.

Time (in minutes)	1	2	3	4	5	6	7	8
Distance (in km)	0.2	0.8	2.6	5.0	6.5	7.2	7.5	7.6

(a) Using a scale of 2 cm to represent 2 minutes on the horizontal axis and 2 cm to represent 2 km on the vertical axis, plot the points given in the table and join them with a smooth curve.

(b) Use your graph to estimate the time taken to travel the first 4 km of the journey.

(c) By drawing a tangent, find the approximate speed of the train 6 minutes after it has left station P.

(d) By considering the gradient of the graph, compare and describe briefly the motion of the train during the first 4 minutes and the last 4 minutes of the journey.

Thinking Time

Fig. 5.9 shows three containers, each of which is being filled with liquid at a constant rate from a tap to a height of a cm. The containers are initially empty.

The graph of Fig. 5.9**(a)** shows the height (h cm) of the liquid as the container is being filled in t seconds.

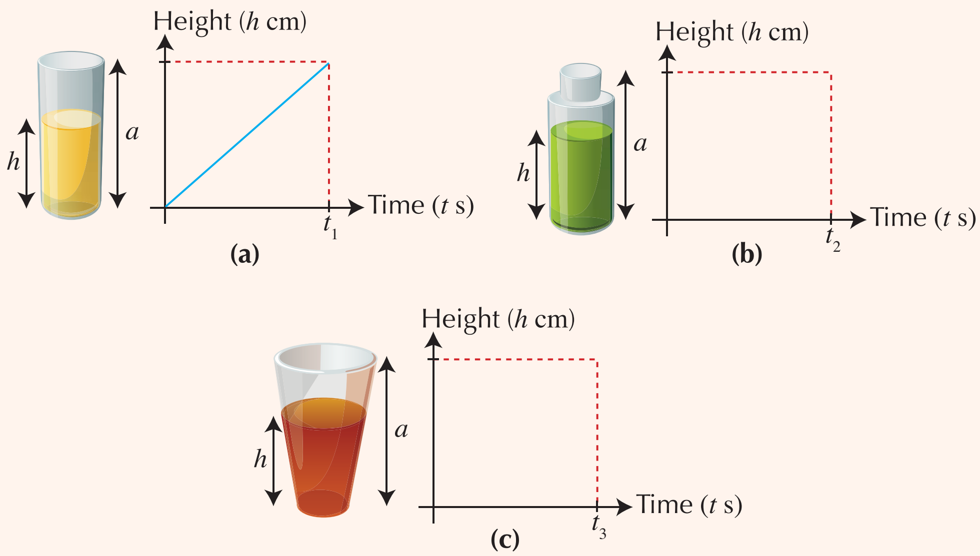


Fig. 5.9

1. Explain clearly why the graph of Fig. 5.9**(a)** is a straight line.

2. Complete the graphs in Fig. 5.9**(b)** and **(c)**. Explain the shape of each graph obtained.

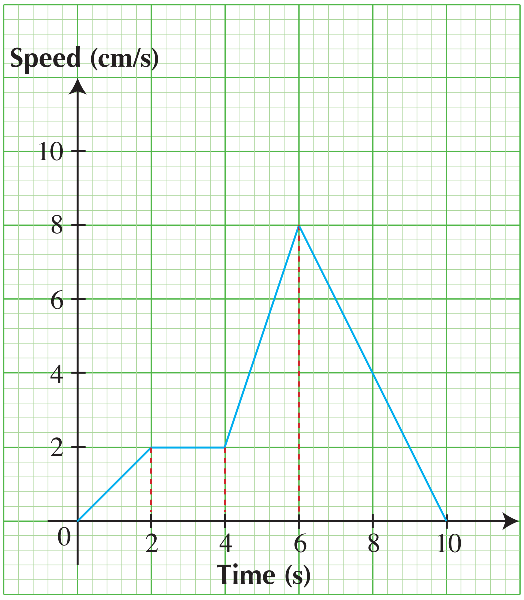
SIMILAR QUESTIONS

Exercise 5C Questions 10, 19

Worked Example 7

(Speed-Time Graph)

The graph shows the speed of an object over a period of 10 seconds.



(i) Find the acceleration in the first 2 seconds,

(ii) Given that the distance travelled is given by the area under the speed-time graph, find the average speed during the whole journey.

(iii) Find the deceleration in the last 2 seconds.

Solution:

(i) Acceleration $= \dfrac{2 \text{ cm/s}}{2 \text{ s}}$

$= 1 \text{ cm/s}^2$

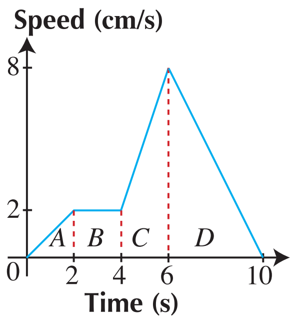
(ii) Total distance = area under graph

$= \text{Area of } (A + B + C + D)$

$= \left(\frac{1}{2}\times2\times2\right)+(2\times2)+\frac{1}{2}(2+8)\times2+\left(\frac{1}{2}\times8\times4\right)$

$= 2 + 4 + 10 + 16$

$= 32 \text{ cm}$



Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$= \dfrac{32}{10}$

$= 3.2 \text{ cm/s}$

(iii) Deceleration in the last 2 seconds = deceleration in the last 4 seconds

$= \dfrac{8-0}{10-6}$

$= 2 \text{ cm/s}^2$

ATTENTION

For **(i)**, the gradient can be found by $\dfrac{\text{vertical change}}{\text{horizontal change}}$ or by using the formula $\dfrac{y_2 - y_1}{x_2 - x_1}$.

ATTENTION

Acceleration of an object
= rate of change of speed
= gradient of a speed-time graph

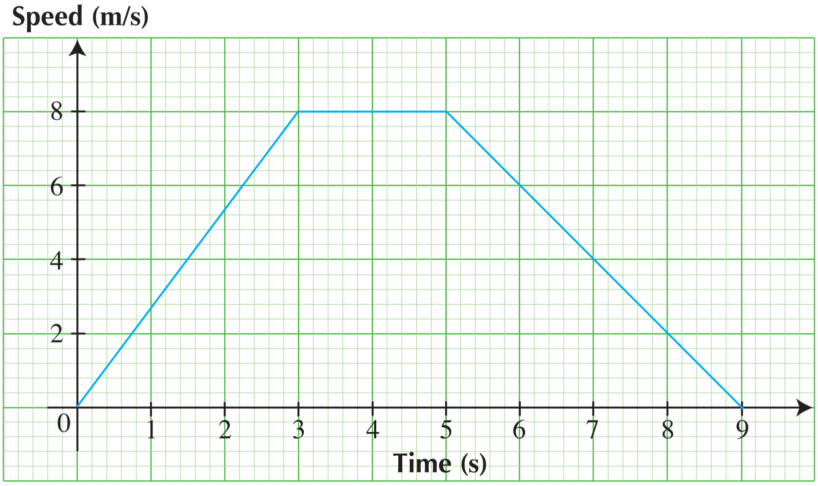
If the gradient is negative, we say that the object is decelerating. The unit for acceleration is always of speed per unit time, i.e. if the unit of speed is cm/s, then the unit of acceleration is cm/s².

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 5C Questions 4, 5, 11, 12

The graph shows the speed of an object over a period of 9 seconds.



(i) Find the acceleration in the first 3 seconds.

(ii) Given that the distance travelled is given by the area under the speed-time graph, find the average speed during the whole journey.

(iii) Find the deceleration in the last 3 seconds.

Worked Example 8

(Speed-Time Graph)

A particle moves along a straight line from A to B so that, t seconds after leaving A, its speed, v m/s, is given by $v = 3t^2 - 15t + 20$.

The table below shows some values of t and the corresponding values of v.

t	0	1	1.5	2	2.5	3	4	5
v	20	8	a	2	b	2	8	20

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 second on the horizontal axis and 2 cm to represent 5 m/s on the vertical axis, draw the graph of $v = 3t^2 - 15t + 20$ for $0 \leqslant t \leqslant 5$.

(c) Use your graph to estimate

(i) the value of t when the speed is 10 m/s,

(ii) the time at which the acceleration is zero,

(iii) the gradient at $t = 4$, and explain what this value represents,

(iv) the time interval when the speed is less than 15 m/s.

Solution:

(a) When $t = 1.5$,

$$v = 3(1.5)^2 - 15(1.5) + 20$$
$$= 4.25$$

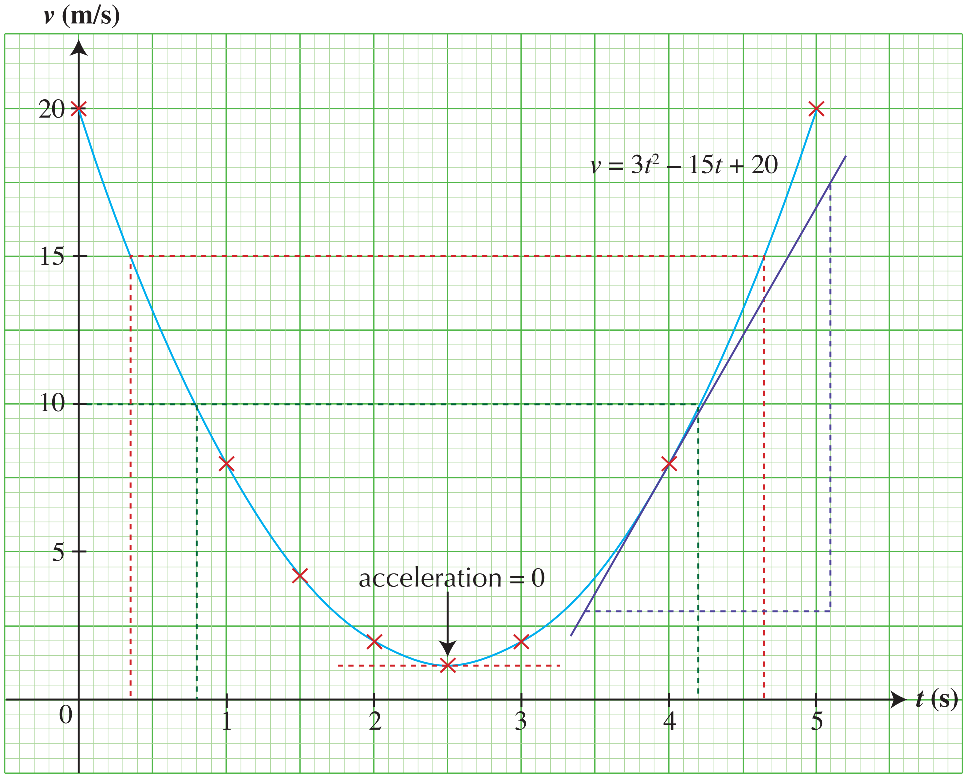
$\therefore a = 4.25$

When $t = 2.5$,

$$v = 3(2.5)^2 - 15(2.5) + 20$$
$$= 1.25$$

$\therefore b = 1.25$

(b)



(c) **(i)** From the graph, when $v = 10$,

$t = 0.8$ or $t = 4.2$.

(ii) The acceleration is zero when the gradient of the curve is zero.

From the graph, the acceleration is zero at $t = 2.5$.

(iii) A tangent is drawn to the curve at the point $t = 4$.

From the graph,

$$\text{Gradient} = \frac{\text{vertical change}}{\text{horizontal change}}$$
$$= \frac{14.5}{1.65}$$
$$\approx 9$$

$\therefore$ The acceleration of the particle at $t = 4$ is approximately 9 m/s^2.

(iv) From the graph, when $v < 15$, $0.35 < t < 4.65$.

ATTENTION

In **(iii)**, the unit for acceleration is $\frac{m/s}{s}$, i.e. m/s^2.

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 5C Questions 13-16, 20-22

A particle moves along a straight line from P to Q so that, t seconds after leaving P, its speed, v m/s, is given by $v = 2t^2 - 9t + 12$.

The table below shows some values of t and the corresponding values of v.

t	0	1	2	3	4	5
v	12	5	a	3	8	b

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 second on the horizontal axis and 2 cm to represent 5 m/s on the vertical axis, draw the graph of $v = 2t^2 - 9t + 12$ for $0 \leqslant t \leqslant 5$.

(c) Use your graph to estimate

(i) the values of t when the speed is 7 m/s,

(ii) the time at which the acceleration is zero,

(iii) the gradient at $t = 4.5$, and explain what this value represents,

(iv) the time interval when the speed is less than 10 m/s.

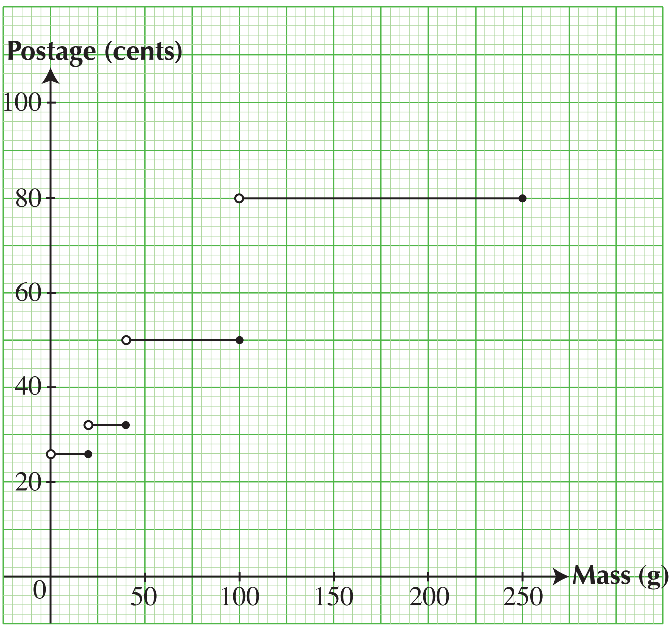
Other Graphs

Graphs can also be used in other real-world contexts such as in the calculation of postage rates, parking charges and labour costs.

Worked Example 9

(Graphs involving Rates)

The step-function graph below shows the local postage rates for letters, postcards and small packages offered by Company A.



Problem Solving Tip

When the mass m is such that $0 < m \leqslant 20$, the postage is 26 cents. Similarly, for $20 < m \leqslant 40$, the postage is 32 cents.

ATTENTION

An empty node ○ indicates that the point is excluded from the graph and a shaded node • indicates that the point is included.
For example, when the mass is 100 g, the postage is 50 cents, not 80 cents.

(a) Write down the postage to mail a letter with a mass of 80 g.

Company B offers the following postage rates: 60 cents for the first 50 g and 0.2 cents for each subsequent gram.

(b) Given that Huixian wishes to post a letter with a mass of 150 g, insert the graph corresponding to the rates offered by Company B and use your graph to determine which company offers a lower postage.

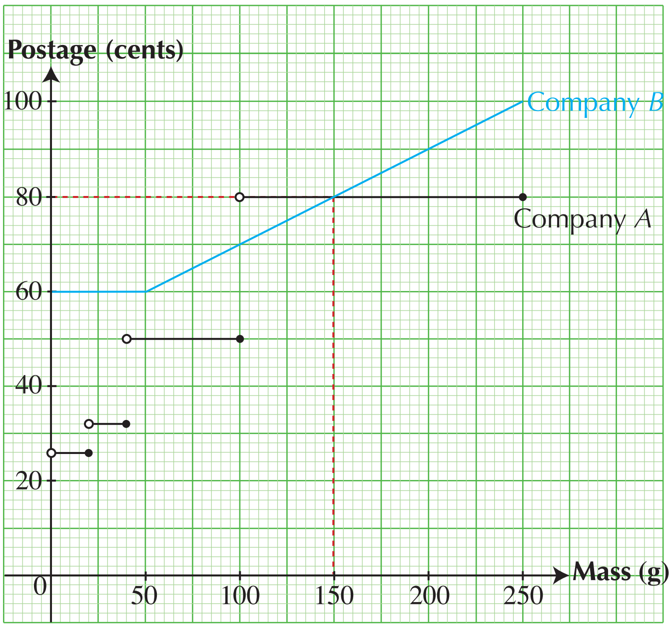
Solution:

(a) From the graph, the postage is 50 cents.

(b) For the first 50 g, the postage is 60 cents.

When the mass is 100 g, the postage is $60 + 0.2 \times (100 - 50) = 70$ cents.

When the mass is 250 g, the postage is $60 + 0.2 \times (250 - 50) = 100$ cents.



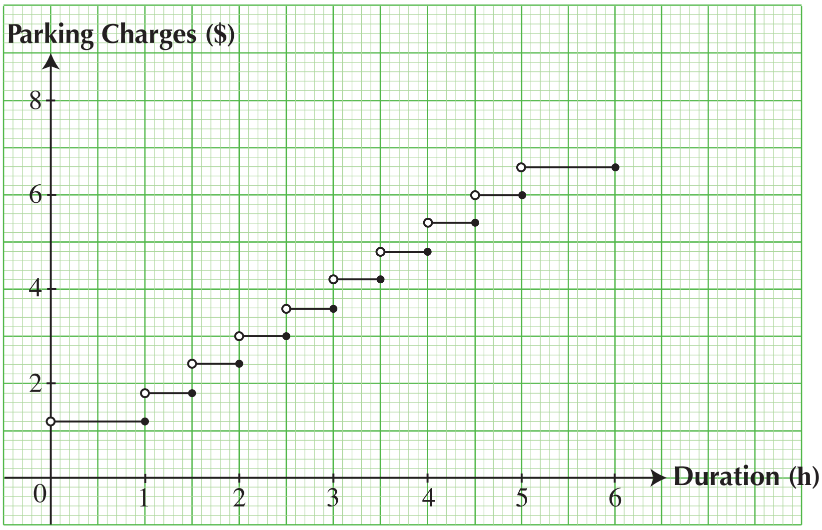
From the graph, for a mass of 150 g, both companies charge the same postage of 80 cents.

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 5C Question 17

The step-function graph below shows the parking charges for the first 6 hours at Carpark *X*.



Carpark *Y* has the following charges:

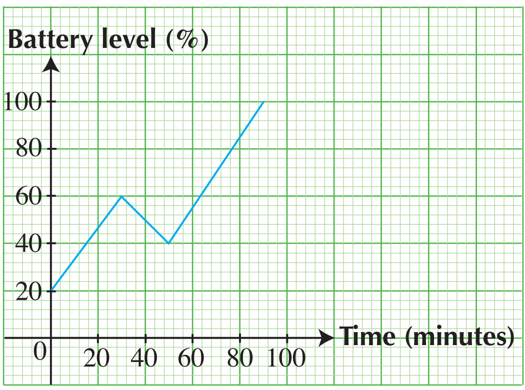
Free for the first 12 minutes 2.5 cents per minute thereafter

Insert the graph corresponding to the rates offered by Carpark *Y* and use your graph to determine which carpark Mr Wong should choose if he has to park for 2 hours.

Worked Example 10

(Other Types of Graphs)

The graph below shows the battery level of a smartphone. It had an initial level of 20%, increasing to 60% in half an hour while connected to the power supply. Farhan then removed the smartphone from the power supply to watch a 20-minute long video clip, before connecting the smartphone to the power supply again.



(a) Find the battery level of the smartphone when Farhan was exactly halfway through the video clip.

(b) Find the rate of increase in the battery level of the smartphone when it was connected to the power supply again.

Solution:

(a) From the graph, the battery level was 50%.

(b) To find the rate of increase in the battery level, we need to calculate the gradient of the line from the 50^{th} minute to the 90^{th} minute.

$$\text{Gradient} = \frac{100-40}{90-50}$$
$$= 1.5\%/\text{minute}$$

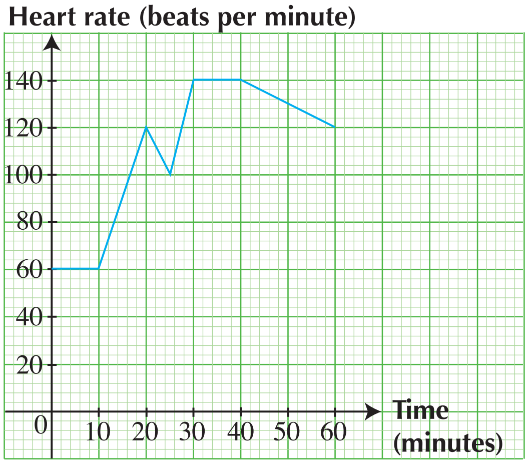
$\therefore$ The rate of increase in the battery level is 1.5%/minute.

PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 5C Question 18

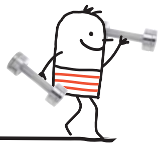
The graph below shows the heart rate, in beats per minute, of an adult who is at the park. He rests at the bench for the first 10 minutes, after which he begins to brisk walk for 10 minutes. He then slows down for 5 minutes, before brisk walking again for a further 5 minutes. He then jogs at a constant speed for 10 minutes, before gradually slowing down.



(a) Write down his resting heart rate.

(b) Find the rate of increase in his heart rate as he brisk walks for the first time.

(c) Find the rate of decrease in his heart rate as he slows down in the last 20 minutes.



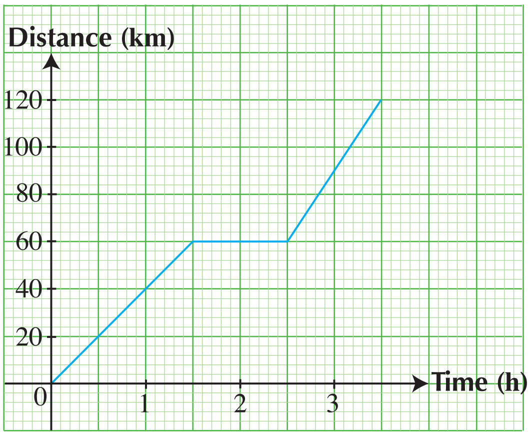
Exercise 5C

BASIC LEVEL

1. A cyclist set out at 0900 for a destination 40 km away. He cycled at a constant speed of 15 km/h until 1030. Then he rested for half an hour before completing his journey at a constant speed of 20 km/h.
   (i) Draw the distance-time graph to represent the journey.
   (ii) Hence, find the time at which the cyclist reached his destination, giving your answer to the nearest minute.

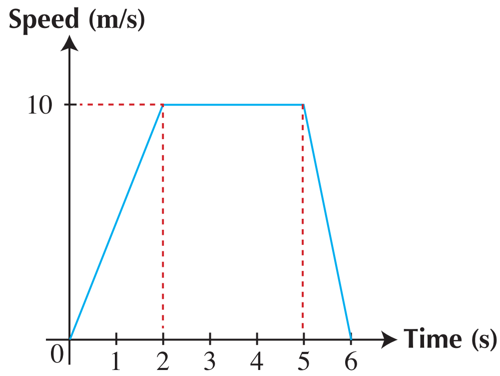
2. Raj starts a 30-km journey at 0900. He maintains a constant speed of 20 km/h for the first 45 minutes and then stops for a rest. He then continues his journey at a constant speed of 30 km/h, finally arriving at his destination at 1120.
   (i) Find the distance travelled in the first 45 minutes.
   (ii) Draw the distance-time graph to represent the journey.
   (iii) Hence, state the duration of his stop, giving your answer in minutes.

3. The figure shows the distance-time graph of a car.



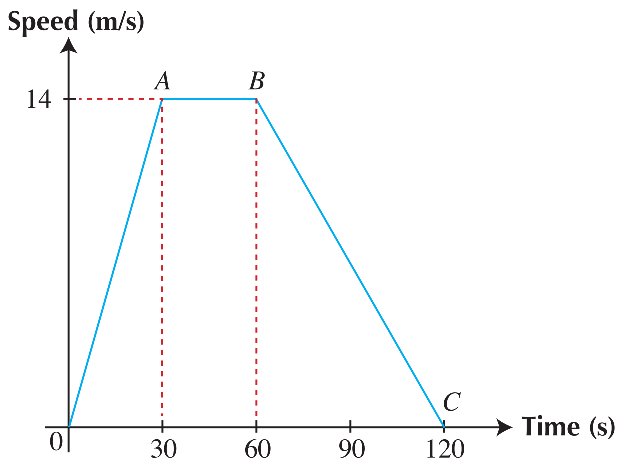
   (i) Find the duration during which the car is not moving.
   (ii) Find the average speed of the car in the first 2 hours of the journey.
   (iii) Find the average speed of the car for the whole journey.
   (iv) Draw the speed-time graph of the car for the whole journey.

4. The graph shows the speed-time graph of a car.



   (i) Find the acceleration in the first 2 seconds.
   (ii) Given that the distance travelled is given by the area under the speed-time graph, find the average speed during the whole journey.

5. The graph shows the speed, v m/s, of a car after t seconds.



   (i) State what the gradient of OA represents.
   (ii) Find the speed of the car when $t = 15$.

INTERMEDIATE LEVEL

6. A lift moves from ground level to a height of 60 metres in 10 seconds, stops for 10 seconds and then descends to the ground in 10 seconds. The table shows the height, h m, of the lift on the upward and downward journeys, t seconds after leaving ground level.

t (in seconds)	0	2	4	6	8	10
h (in m)	0	3	16	44	57	60

t (in seconds)	20	22	24	26	28	30
h (in m)	60	57	44	16	3	0

(i) Using a scale of 2 cm to represent 5 seconds, draw a horizontal t-axis for $0 \leqslant t \leqslant 30$.

Using a scale of 1 cm to represent 5 metres, draw a vertical h-axis for $0 \leqslant h \leqslant 60$.

On your axes, plot the points given in the table and join them with a smooth curve.

(ii) Find the gradient of the graph at $t = 8$ and explain briefly what this gradient represents.

A construction worker, waiting at the 40-metre level, starts to walk down at $t = 5$.

(iii) Assuming that he descends at a steady speed of 0.8 m/s, use your graph to find the time when the worker and the lift are at the same height.

7. A company which manufactures automated vehicles is putting them on a test run. One of the vehicles starts from a point X and travels to a point Y, 3 km away. The table shows the distance, d km, of the vehicle from X, t minutes after leaving X.

Time (in minutes)	0	1	2	3	4	5	6
Distance (in km)	0	0.2	0.7	1.8	2.5	2.9	3.0

(a) Using a scale of 2 cm to represent 1 minute on the horizontal axis and 4 cm to represent 1 km on the vertical axis, plot the points given in the table and join them with a smooth curve.

(b) Use your graph to find

(i) the approximate time taken to travel the first 1 km,

(ii) the gradient of the graph when $t = 1\frac{1}{2}$ and explain briefly what this value represents,

(iii) the time taken to travel the last 1 km.

8. Ethan and Michael start moving towards each other at the same time. The initial distance between them is 32 km.

(a) Given that Ethan is cycling at a constant speed of 20 km/h and Michael is walking at a constant speed of 7 km/h, draw a distance-time graph to illustrate this information.

(b) Use your graph to find

(i) how long it will take for them to pass each other,

(ii) the times when they will be 5 km apart.

9. At 0900, Shirley travels to meet Kate, who stays 20 km away. Shirley travels at a uniform speed of 18 km/h for half an hour. She rests for 20 minutes and then continues her journey at a uniform speed of 8 km/h.

At 0900, Kate sets off from home on the same road to meet Shirley and travels at a uniform speed of 7 km/h.

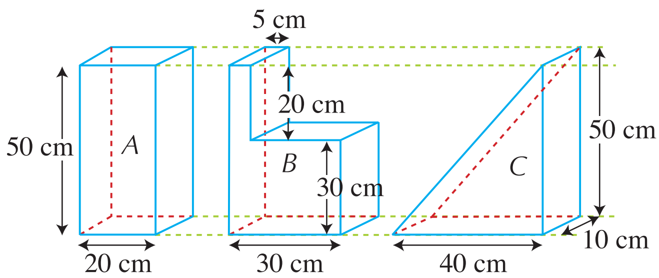
(a) Draw the distance-time graph for the above information.

(b) Use your graph to find

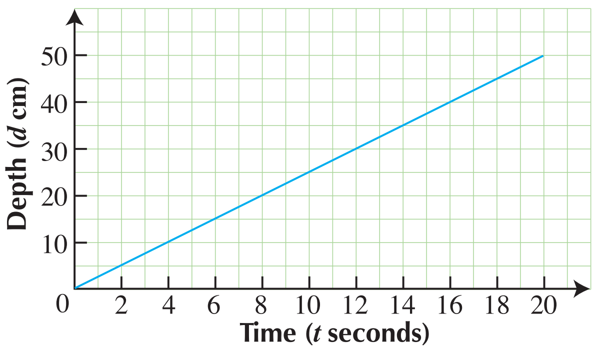
(i) the time at which Shirley and Kate meet,

(ii) the distance away from Kate's home when they meet.

10. The figure shows three containers, each with a height of 50 cm and a width of 10 cm. The other dimensions are as shown. The containers are initially empty and it takes 20 seconds to fill each container at a constant rate.

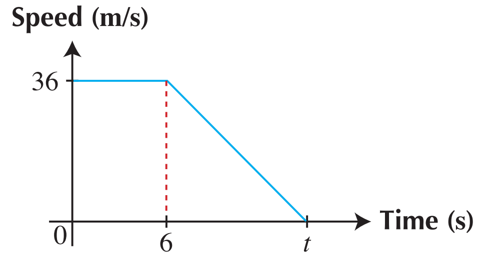


The diagram below shows the relationship between the depth, d cm, of the liquid and the time, t seconds, taken to fill container A.



On the same diagram, sketch the graph of the depth of the liquid against time for container B and container C.

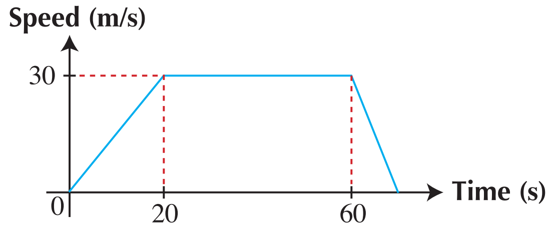
11. The diagram shows the speed-time graph of an object which travels at a constant speed of 36 m/s and then slows down at a rate of 12 m/s^2, coming to rest at time t seconds.



(i) Find the value of t.

(ii) Given that the distance travelled when the object is slowing down is 54 m, find the average speed for the whole journey.

12. The diagram shows the speed-time graph of a train.



(i) Find the acceleration of the train during the first 20 seconds.

(ii) Given that the train decelerates at a rate of 0.75 m/s^2, find the time taken for the whole journey.

13. A particle moves along a straight line from A to B so that, t minutes after leaving A, its speed, v m/min, is given by $v = t^2 - 7t + 16$.

t (minutes)	0	1	2	3	4	5	6
v (m/min)	16	10	6	a	4	6	b

(a) Find the value of a and of b.

(b) Using a scale of 2 cm to represent 1 minute on the horizontal axis and 1 cm to represent 1 m/min on the vertical axis, draw the graph of $v = t^2 - 7t + 16$ for $0 \leqslant t \leqslant 6$.

(c) Use your graph to estimate

(i) the value(s) of t when the speed is 7 m/min,

(ii) the time at which the speed is a minimum,

(iii) the gradient at $t = 2$, and explain what this value represents,

(iv) the time interval when the speed is not more than 5 m/min.

14. The speed of a body, v m/s, after time t seconds is given in the table.

t (s)	0	2	4	6	8	10	12
v (m/s)	0	2	7	12	19	28	42

(i) Using a scale of 1 cm to represent 1 second on the horizontal axis and 1 cm to represent 5 m/s on the vertical axis, plot the graph of v against t for $0 \leqslant t \leqslant 12$.

(ii) Use your graph to estimate the speed of the body when $t = 5$ and when $t = 11$.

(iii) By drawing two tangents, find the acceleration of the body when $t = 4$ and when $t = 10$.

15. Object P moves along a straight line from A to B so that, t hours after leaving A, its speed, v km/h, is given by $v = 3t^2 - 17t + 30$.

t (h)	0	1	2	3	4	5
v (km/h)	30	16	h	6	k	20

(a) Find the value of h and of k.

(b) Using a scale of 2 cm to represent 1 hour on the horizontal axis and 2 cm to represent 5 km/h on the vertical axis, draw the graph of $v = 3t^2 - 17t + 30$ for $0 \leqslant t \leqslant 5$.

(c) Use your graph to estimate

(i) the time at which the speed is a minimum,

(ii) the gradient at $t = 4.5$, and explain what this value represents,

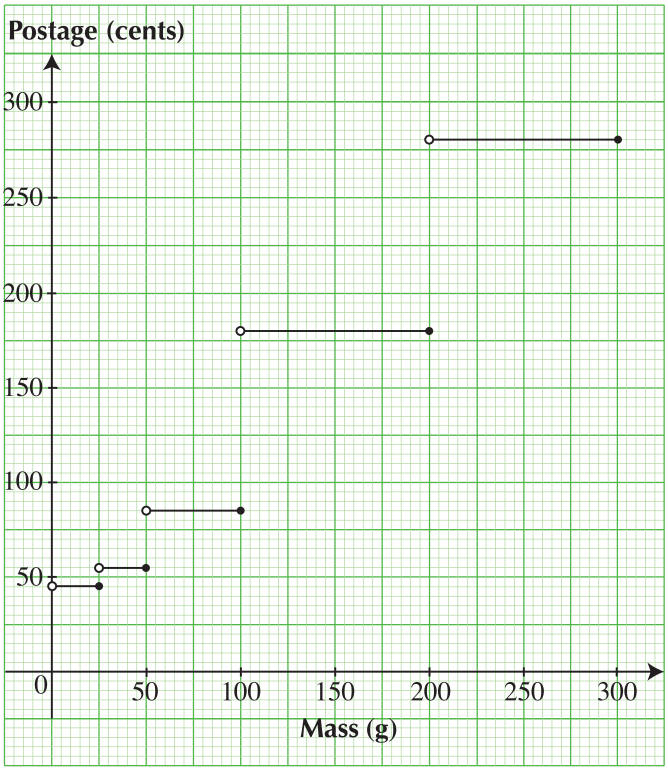
(iii) the time interval when the speed does not exceed 10 km/h.

Object Q moves along a straight line from A to B with a constant speed of 24 km/h.

(d) Use your graph to determine the value of t at which both objects have the same speed.

16. A taxi starts from rest and accelerates at a uniform rate for 45 seconds to reach a speed of 30 m/s. It then travels at this constant speed. Sketch the speed-time graph and use it to find the speed after 10 seconds.

17. The step-function graph below shows the postage rates to Malaysia for letters and small packages offered by Company A.



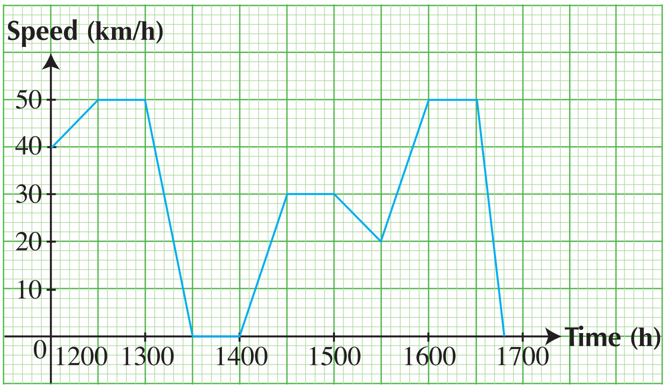
(a) Write down the postage to mail a letter with a mass of 50 g to Malaysia.

Company B offers the following postage rates to Malaysia: \$1 for the first 80 g and 1 cent for each subsequent gram.

(b) Given that Devi wishes to post a small package with a mass of 220 g to Malaysia, determine which company offers a lower postage. Show your working to support your answer.

INTERMEDIATE LEVEL

18. The graph below shows the speed of a coach as it ferried passengers from Blue Town to Summer City via the highway one afternoon. The coach made only one stop during the journey.



(a) Write down the duration of the stop.

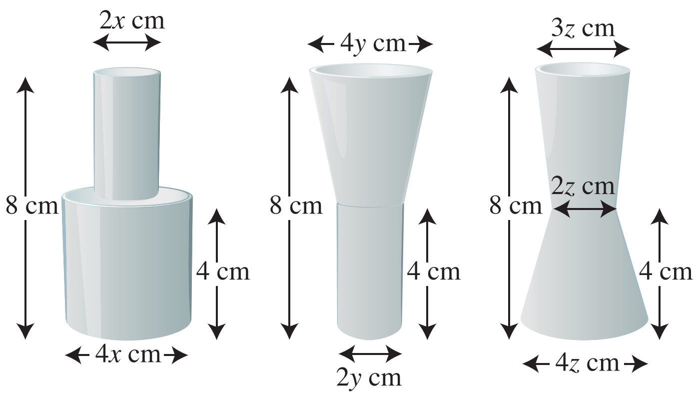
(b) Find the initial acceleration of the coach.

(c) Given that the distance travelled is given by the area under the speed-time graph, explain why the distance between Blue Town and Summer City is less than 250 km.

(d) Determine the time when the coach reached Summer City.

ADVANCED LEVEL

19. The diagram below shows three containers of a fixed volume and varying cross-sectional areas, each with a height of 8 cm. The containers are initially empty. A tap is used to fill each of the containers at a constant rate.



Given that it takes 60 seconds to fill each of the containers, sketch the graph of the height (h cm) of the water level against time (t s) for each of the containers.

20. A toy car starts from a point A and moves towards a point B, which it reaches after 7 seconds. The speed, v cm/s, after t seconds, is given in the table.

t (s)	0	1	2	3	4	5	6	7
v (cm/s)	0	4.5	8	10.5	12	12.5	12	10.5

(a) Using a scale of 2 cm to represent 1 second on the horizontal axis and 2 cm to represent 2 cm/s on the vertical axis, plot the graph of v against t for $0 \leqslant t \leqslant 7$.

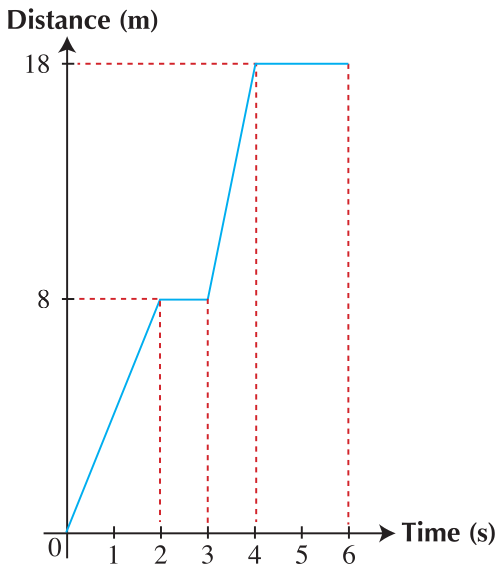
(b) Use your graph to estimate

(i) the acceleration of the body when $t = 2$ and when $t = 6$,

(ii) the time interval when the speed is greater than 11 m/s.

(c) Given that this motion can be modelled by the equation $v = at^2 + bt + c$, where a, b and c are constants, find the values of a, b and c.

21. The graph shows the distance-time graph of a body during a period of 6 seconds.



Sketch the speed-time graph for the same journey.

22. The speed of an object, v m/s, at time t seconds, is given by $v = 6 + 2t$.

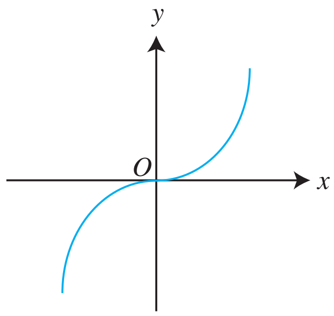
(a) Sketch the speed-time graph for the motion.

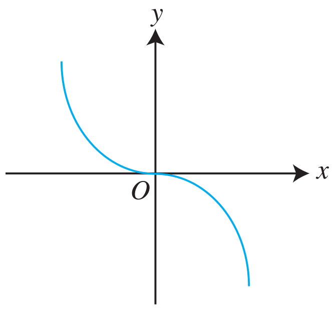
(b) Find the speed when $t = 3$.

(c) Sketch the acceleration-time graph for the motion.

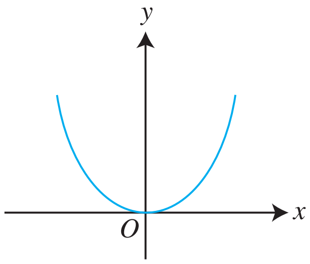
1. **Graphs of Power Functions $y = ax^n$**

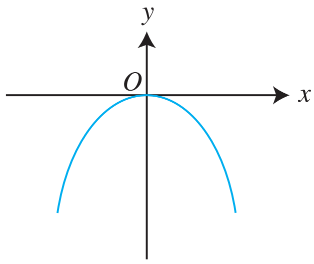
(a) $n = 3,\ a > 0,\ y = ax^3$ $n = 3,\ a < 0,\ y = ax^3$



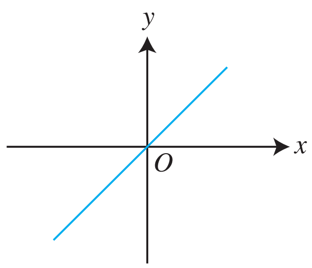


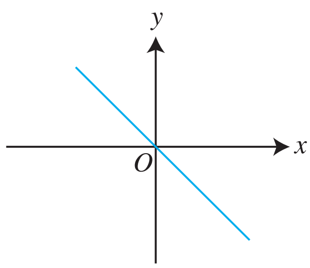
(b) $n = 2,\ a > 0,\ y = ax^2$ $n = 2,\ a < 0,\ y = ax^2$



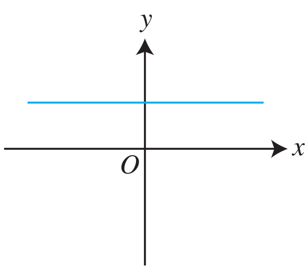


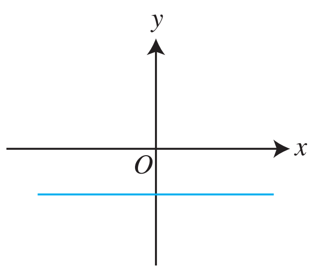
(c) $n = 1,\ a > 0,\ y = ax$ $n = 1,\ a < 0,\ y = ax$



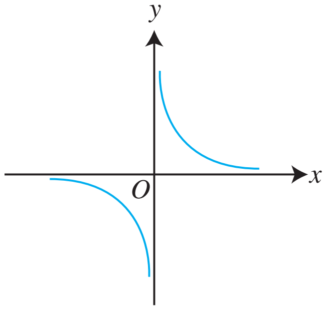


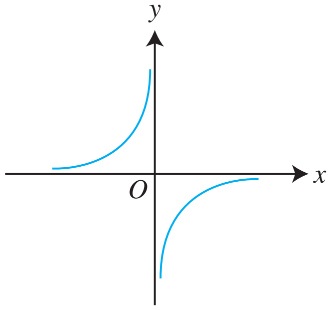
(d) $n = 0,\ a > 0,\ y = a$ $n = 0,\ a < 0,\ y = a$



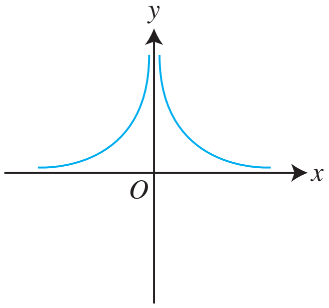


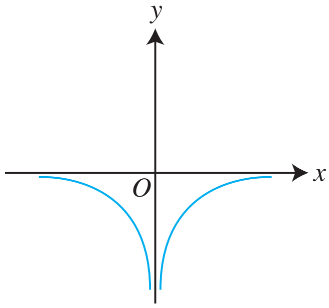
(e) $n = -1,\ a > 0,\ y = \dfrac{a}{x}$ $\qquad$ $n = -1,\ a < 0,\ y = \dfrac{a}{x}$





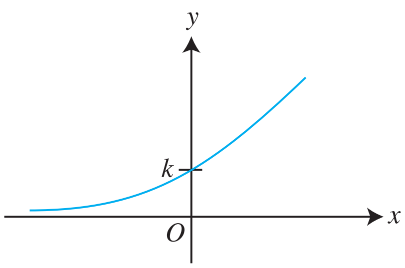
(f) $n = -2,\ a > 0,\ y = \dfrac{a}{x^2}$ $\qquad$ $n = -2,\ a < 0,\ y = \dfrac{a}{x^2}$

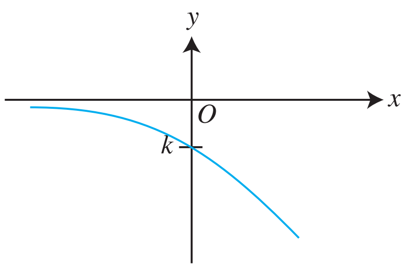




2. Graph of $y = ka^x$, where $a > 0$

$k > 0,\ y = ka^x$ $\qquad$ $k < 0,\ y = ka^x$





3. Gradient of a Curve

The gradient of a curve at a point can be obtained by drawing a tangent to the curve at that point and finding the gradient of the tangent.

Review Exercise 5

1. The table below shows some values of x and the corresponding values of y, where $y = x^3 - 3x - 10$.

x	–3	–2	–1	0	1	2	3	4
y	–28			–10			8	42

   (a) Copy and complete the table.

   (b) Using a scale of 1 cm to represent 1 unit, draw a horizontal x-axis for $-3 \leqslant x \leqslant 4$.

   Using a scale of 1 cm to represent 5 units, draw a vertical y-axis for $-28 \leqslant y \leqslant 42$.

   On your axes, plot the points given in the table and join them with a smooth curve.

   (c) Use your graph to find
   - **(i)** the value of y when $x = 1.8$,
   - **(ii)** the value of x when $y = 10$.

2. The variables x and y are connected by the equation $y = x(x - 2)(x + 2)$.

   The table below shows some values of x and the corresponding values of y.

x	–3	–2	–1	0	1	2	3
y	–15	0		0	–3		

   (a) Copy and complete the table.

   (b) Using a scale of 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, draw the graph of $y = x(x - 2)(x + 2)$ for $-3 \leqslant x \leqslant 3$.

   (c) Use your graph to find
   - **(i)** the value of y when $x = 1.4$,
   - **(ii)** the value of x when $y = 4.5$,
   - **(iii)** the solutions to the equation $x(x - 2)(x + 2) = 0$.

3. Using a scale of 4 cm to represent 1 unit on the x-axis and 2 cm to represent 1 unit on the y-axis, draw the graph of $y = 1 - 2x - \frac{1}{x}$ for $-4 \leqslant x \leqslant -0.25$.

   (a) Use your graph to find
   - **(i)** the value of y when $x = -0.75$,
   - **(ii)** the values of x when $y = 4.5$.

   (b) Write down the coordinates of the point on the curve where the tangent to the curve is a horizontal line.

4. Using a suitable scale, draw the graph of $y = 3^x - 2$ for $-1.5 \leqslant x \leqslant 2$. Using your graph,
   - **(i)** solve the equation $3^x = 2$,
   - **(ii)** find the coordinates of the point on the graph of $y = 3^x - 2$ where the gradient of the tangent is 2.

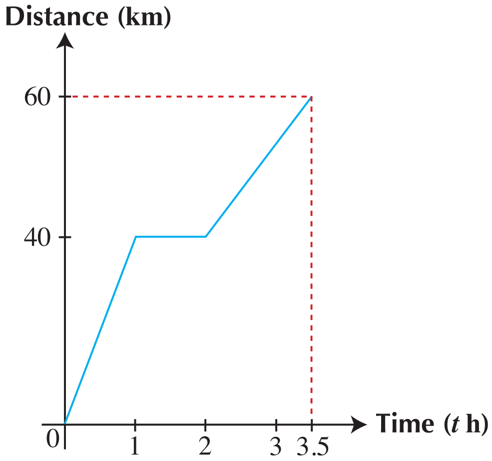
5. Using a scale of 4 cm to represent 1 unit on both axes, draw the graph of $y = x - 2 + \frac{3}{x}$ for $0.5 \leqslant x \leqslant 4$.
   - **(i)** State the minimum value of y and the corresponding value of x.
   - **(ii)** Find the range of values of x for which $y < 2.2$.
   - **(iii)** By drawing a tangent, find the gradient of the graph at the point where $x = 3$.
   - **(iv)** Using your graph, find the value of x for which $2x + \frac{3}{x} = 8$.

6. A coach travelled from the airport to the hotel in 20 minutes at a constant speed of 45 km/h. After stopping for half an hour, it travelled back to the airport at a constant speed of 60 km/h.

   (i) Draw the distance-time graph to represent the journey.

   (ii) Hence, find the average speed of the coach for the whole journey.

7. The graph shows the distance-time graph of a heavy goods vehicle.



   (a) Find

   (i) the time interval during which the vehicle stopped to unload goods,

   (ii) the speed when $t = 3$,

   (iii) the maximum speed during the journey,

   (iv) the average speed for the whole journey.

   (b) Sketch the speed-time graph for the motion.

8. The intensity of illumination, I units, at a point on a screen a distance of D cm from the light source is modelled by the equation $I = \frac{k}{D^2}$, where k is a constant.

   (i) Using the data in the table, find the value of k.

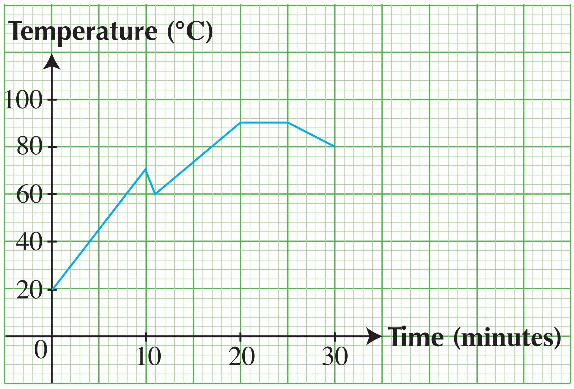
Distance (D cm)	10		20	25		
Intensity (I units)		1250	800		294	200

   (ii) Complete the table and use a suitable scale to draw the graph of I against D.

   (iii) From the graph, find the intensity of illumination when the light source is 30 cm from the screen.

   (iv) What can you say about the relationship between I and D^2?

9. The graph below shows the temperature of soup as it was being heated in an electric cooker. Ten minutes later, Mrs Wong added some fresh vegetables from the refrigerator, continued to heat the soup until it reached a temperature of 90 °C and switched off the cooker after five minutes.



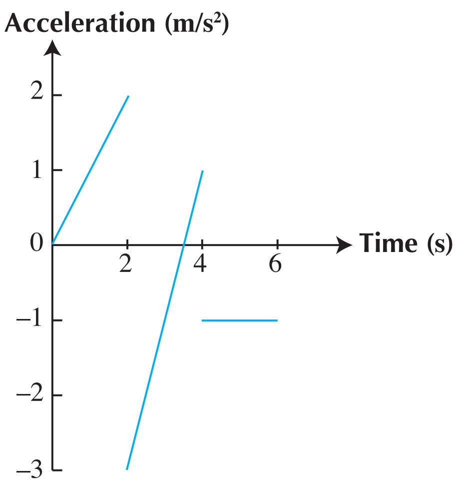
   (a) Find the rate of increase in the temperature of the soup in the first ten minutes.

   (b) Suggest why there is a drop in the temperature between the 10th and the 11th minute.

   (c) Find the rate of decrease in the temperature of the soup after the cooker was switched off.

Challenge Yourself

The graph shows an acceleration-time graph of an object travelling in a straight line.

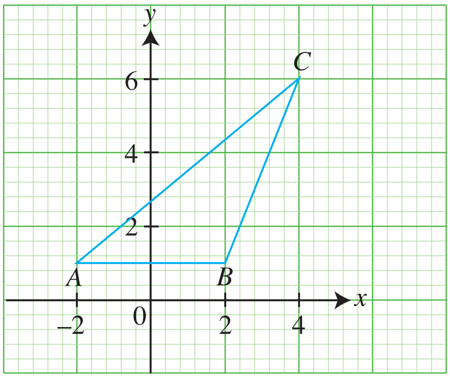


Sketch a possible speed-time graph for the motion of the object. Explain your answer.

B1 Revision Exercise

1. Find the equation of the line joining the points $A(5, 7)$ and $B(8, 12)$.

2. The figure shows the points $A(–2, 1)$, $B(2, 1)$ and $C(4, 6)$.



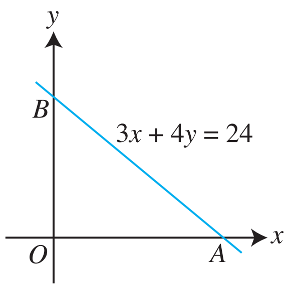
Find

(i) the area of ΔABC,

(ii) the coordinates of the point D such that $ABCD$ is a parallelogram,

(iii) the coordinates of the point P such that $ACPB$ is a parallelogram.

3. The line $3x + 4y = 24$ cuts the x-axis and the y-axis at A and B respectively.



Find

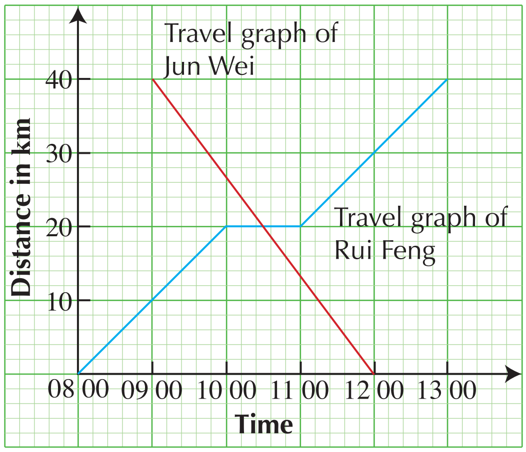
(i) the coordinates of A and of B,

(ii) the area of ΔABC, where C is the point $(–5, 0)$,

(iii) the gradient of BC,

(iv) the length of the perpendicular from C to AB.

4. The travel graphs below show the journeys of Rui Feng and Jun Wei. Rui Feng starts from the train station at 0800 and travels towards the airport, 40 km away. Jun Wei starts from the airport at 0900 and travels towards the train station.



From the graph, find

(i) Jun Wei's average speed for the whole journey,

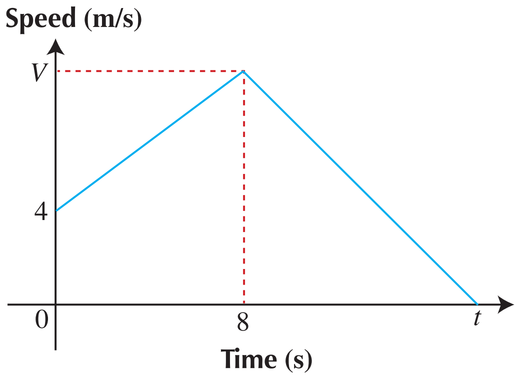
(ii) the time when Rui Feng and Jun Wei meet and how far they are from the airport when they meet,

(iii) the time interval during which Rui Feng took a rest,

(iv) the distance between Rui Feng from the airport when Jun Wei reaches the train station.

B1 Revision Exercise

5. The diagram shows the speed-time graph of a particle over a period of t seconds.



(i) Given that the acceleration of the particle during the first 8 seconds of its motion is 1.5 m/s^2, find the value of V.

(ii) Given that the retardation of the particle is 1 m/s^2, find the value of t.

6. The variables x and y are connected by the equation $y = \frac{12}{x} + x - 6$.

The table below shows some values of x and the corresponding values of y.

x	1	1.5	2	3	4	5	6	7	8
y	7	h	2	1	1	1.4	2	k	3.5

(a) Find the value of h and of k.

(b) Using a scale of 2 cm to represent 1 unit on each axis, draw the graph of $y = \frac{12}{x} + x - 6$ for $1 \leqslant x \leqslant 8$.

(c) Use your graph to find

(i) the value of y when $x = 2.3$,

(ii) the minimum value of y.

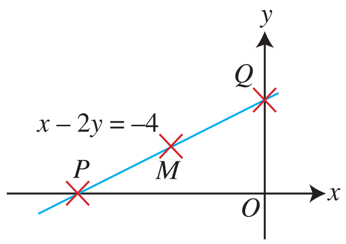
(d) By drawing a tangent, find the gradient of the curve at the point where $x = 5$.

(e) Use your graph to obtain one solution of the equation $x^2 + 12 = 10x$.

B2 Revision Exercise

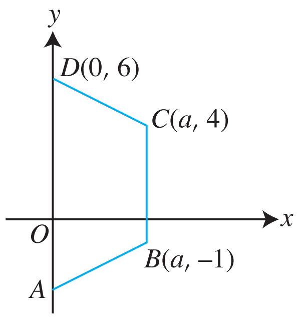
1. Find the equation of the line passing through the point (2, –5) and parallel to the line $5x + 7y = 46$.

2. The line $x - 2y = -4$ cuts the x-axis and the y-axis at P and Q respectively. M is a point on PQ such that it is equidistant from the coordinate axes.



Find
(i) the coordinates of P and of Q,
(ii) the coordinates of M,
(iii) the area of ΔPMO.

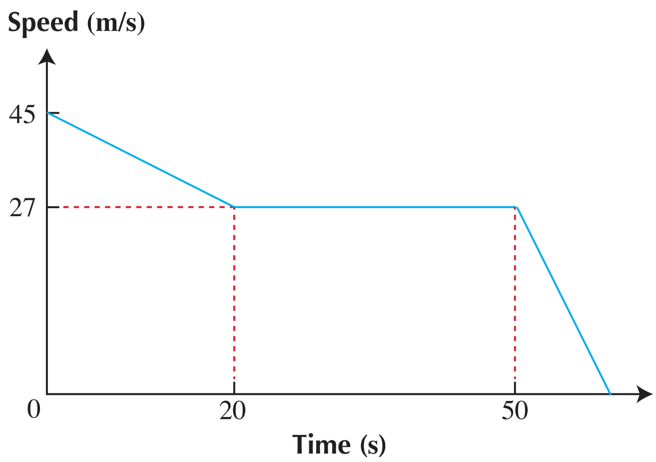
3. $ABCD$ is a trapezium in which $DC = \sqrt{13}$ units. The coordinates of B, C and D are $(a, -1)$, $(a, 4)$ and $(0, 6)$ respectively and the gradient of AB is $\frac{2}{3}$.



Find
(i) the coordinates of the point C,
(ii) the coordinates of the point A,
(iii) the area of the trapezium $ABCD$,
(iv) the length of AB,
(v) the equation of AB.

4. A coach leaves Watertown for Sandcity 120 km away at 1100 and travels at a uniform speed of 50 km/h. An hour later, a car travelling at a uniform speed of 80 km/h leaves Sandcity for Watertown by the same route.
(a) Draw the distance-time graph to represent the journey.
(b) Use your graph to find
(i) the time when the car meets the coach and the distance from Watertown at this instant,
(ii) the distance between the coach and the car at 1300.

5. The diagram shows the speed-time graph of a car which decelerates uniformly from 45 m/s to 27 m/s in 30 seconds. It then travels at a constant speed of 27 m/s for 30 seconds.



(i) Given that the car begins to decelerate uniformly at 0.6 m/s² until it comes to rest, find the total time taken for the journey. Give your answer in minutes and seconds.
(ii) Sketch the acceleration-time graph for the motion.

6. A string of length 15 cm is used to form a rectangle. Given that the area of the rectangle is y cm^2 and that one side of the rectangle is x cm long, show that $y = \frac{1}{2}x(15 - 2x)$.

The table below shows some values of x and the corresponding values of y.

x	0.5	1	2	3	4	5	6	6.5
y	3.5	6.5	11	13.5	14	12.5	9	p

(i) Find the value of p.

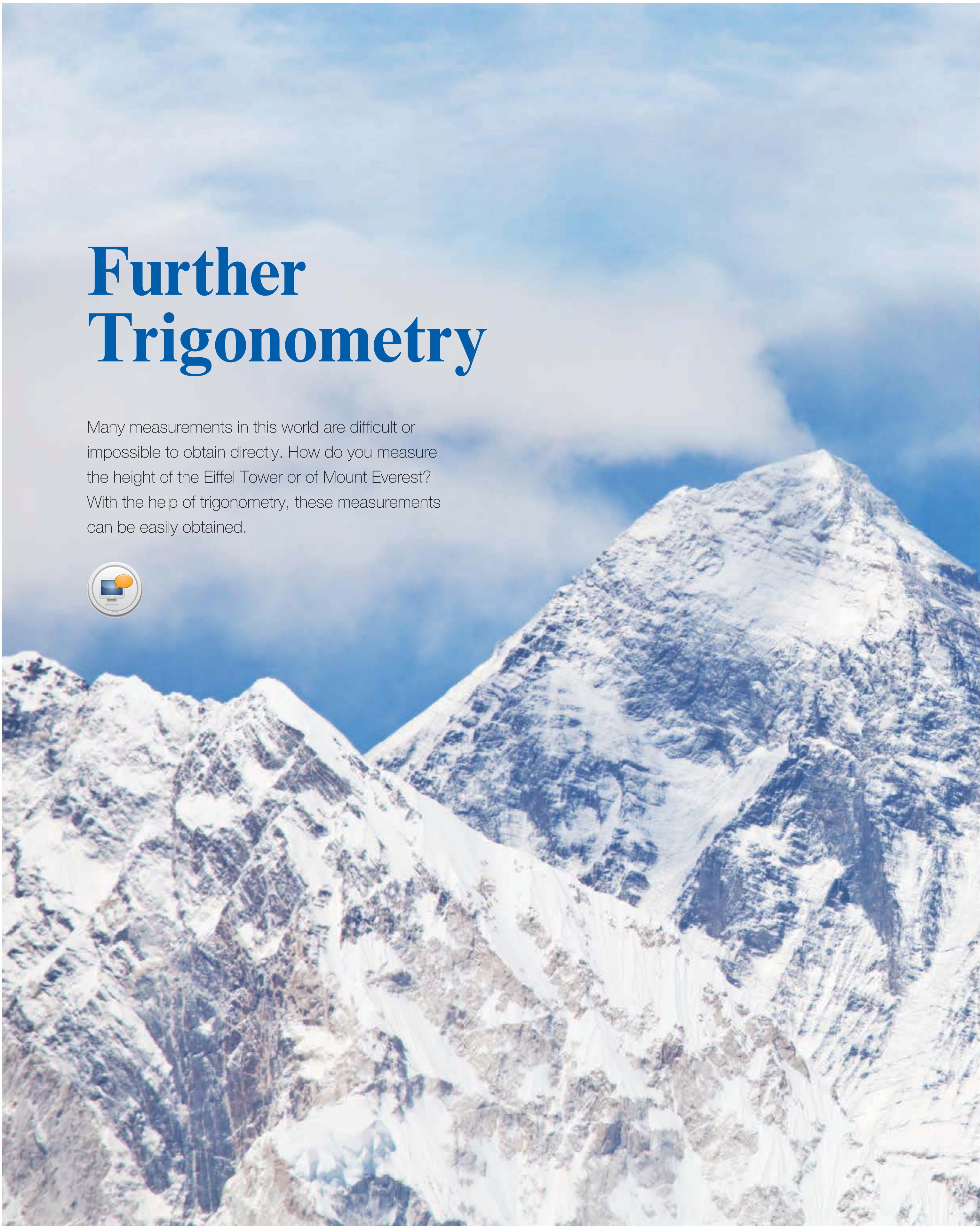
(ii) Using a scale of 2 cm to represent 1 unit, draw a horizontal x-axis for $0 \leqslant x \leqslant 7$.
Using a scale of 1 cm to represent 1 unit, draw a vertical y-axis for $0 \leqslant y \leqslant 16$.
On your axes, plot the points given in the table and join them with a smooth curve.

(iii) Use your graph to find the solutions of the equation $x(15 - 2x) = 23$.

(iv) By drawing a tangent, find the gradient of the curve at the point (5, 12.5).

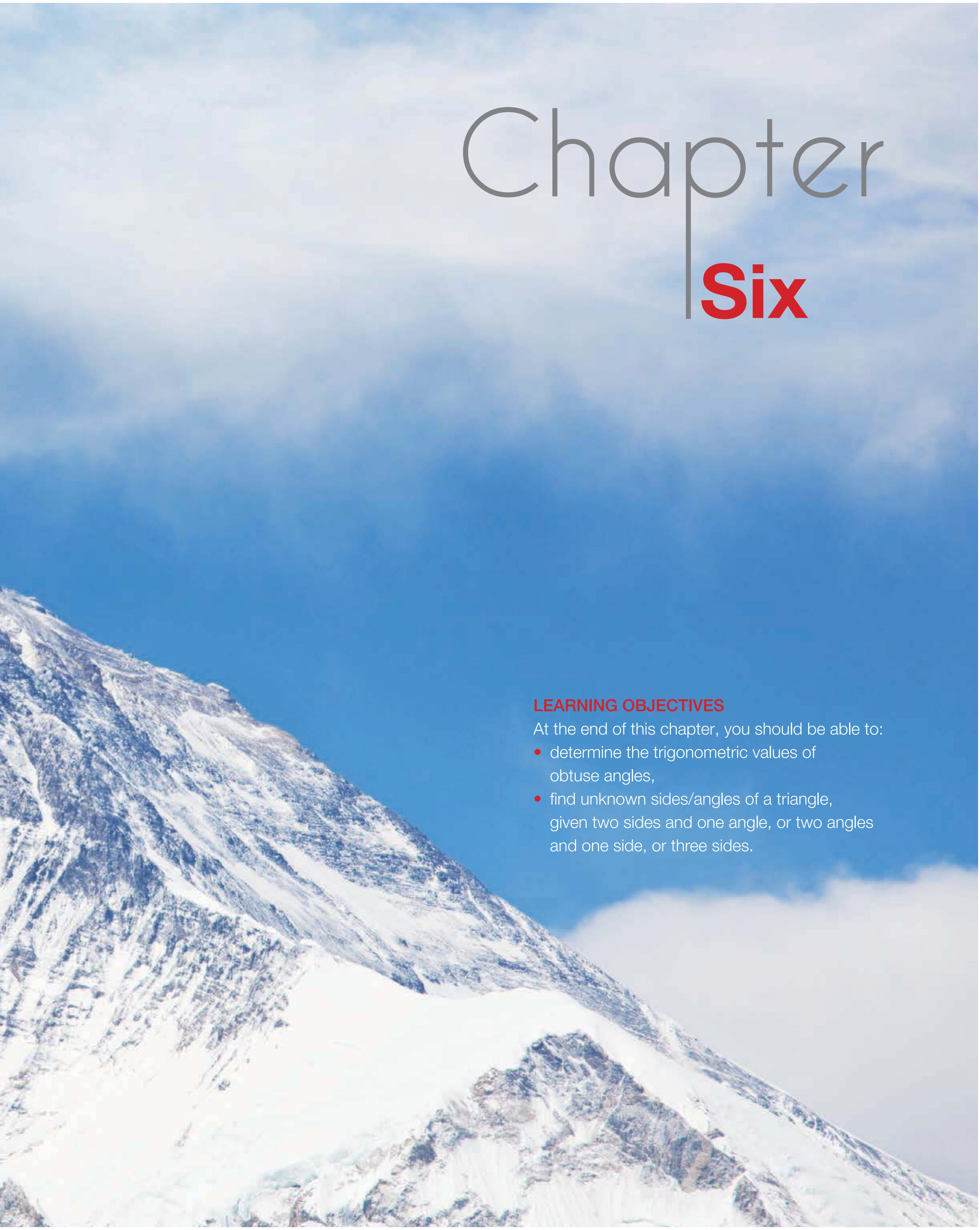
(v) Use your graph to find the maximum value of y and the corresponding value of x.

(vi) Hence, write down the dimensions of the rectangle when the area is a maximum. What can you say about your answer?



Further Trigonometry

Many measurements in this world are difficult or impossible to obtain directly. How do you measure the height of the Eiffel Tower or of Mount Everest? With the help of trigonometry, these measurements can be easily obtained.



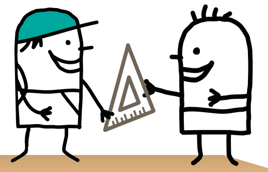
Chapter Six

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- determine the trigonometric values of obtuse angles,
- find unknown sides/angles of a triangle, given two sides and one angle, or two angles and one side, or three sides.

6.1 Sine and Cosine of Obtuse Angles



Recap

In Book 2, we have learnt that the trigonometric ratios of an acute angle A are defined based on a right-angled triangle as follows:

$$\sin A = \frac{\text{opp}}{\text{hyp}}$$

$$\cos A = \frac{\text{adj}}{\text{hyp}}$$

$$\tan A = \frac{\text{opp}}{\text{adj}}$$

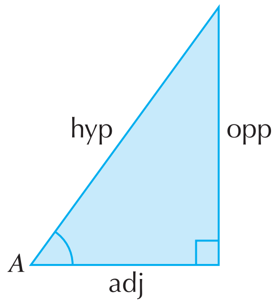


Fig. 6.1

What happens if A is an **obtuse** angle as shown below?

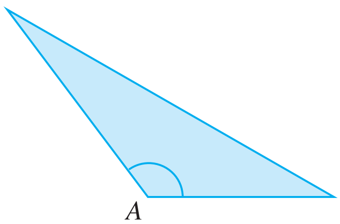


Fig. 6.2

In order to find the sides and angles of an obtuse-angled triangle, we will need to extend the definitions of trigonometric ratios. In this chapter, we will learn the trigonometric ratios of the sine and cosine of obtuse angles.

Sine and Cosine of Obtuse Angles

Fig. 6.3**(a)** shows a circle with centre O and radius r units.

$P(x, y)$ is a point on the circle and ΔOPQ is a right-angled triangle. $\angle A$ is an acute angle.

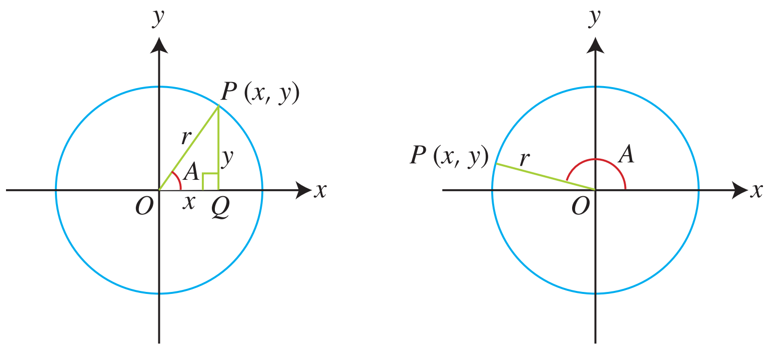
$$\therefore \sin A = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} \text{ and } \cos A = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$$

In other words, we have extended the definition of the sine and cosine of an angle A in terms of the coordinates of a point $P(x, y)$:

$$\boxed{\sin A = \frac{y}{r} \text{ and } \cos A = \frac{x}{r}}$$

If A is an acute angle, then x, y and r are positive.

In other words, sin A and cos A are positive if A is acute.



(a) $\angle A$ is acute **(b)** $\angle A$ is obtuse

Fig. 6.3

Fig. 6.3**(b)** shows a circle with centre O and radius r units.

$P(x, y)$ is a point on the circle and $\angle A$ is an **obtuse** angle.

Using the extended definitions,

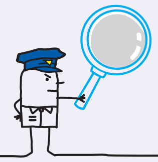
$$\sin A = \frac{y}{r} \text{ and } \cos A = \frac{x}{r}.$$

However, x is now negative, but y and r are still positive.

In other words, if A is an obtuse angle, then sin A is still positive but cos A is negative.

Use your calculator to find the value of sin 150° and of cos 150°.

Which trigonometric ratio is positive and which one is negative?



Investigation

Relationship between Trigonometric Ratios of Acute and Obtuse Angles

Use your calculator to evaluate the sine and cosine of each of the following pairs of angles, leaving your answers correct to 3 significant figures where necessary.

	A	$180° - A$	$\sin A$	$\sin (180° - A)$	$\cos A$	$\cos (180° - A)$
(a)	30°	150°				
(b)	76°	104°				
(c)	111°	69°				
(d)	167°	13°				

Table 6.1

1. What do you notice about $\sin A$ and $\sin (180° - A)$?

2. What do you notice about $\cos A$ and $\cos (180° - A)$?

In general, if A is acute or obtuse,

$$\boxed{\begin{aligned}\sin A &= \sin (180° - A)\\ \cos A &= -\cos (180° - A)\end{aligned}}$$

To prove the above relationship, consider Fig. 6.4 which shows a circle with centre O and radius r units. $P(x, y)$ is a point on the circle where $x < 0$ and $y > 0$.

$\angle A$ is an obtuse angle.

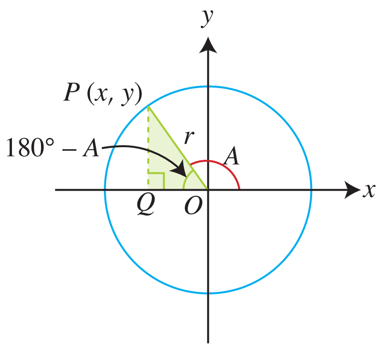


Fig. 6.4

Using the extended definitions,

$$\sin A = \frac{y}{r} \text{ and } \cos A = \frac{x}{r}. \text{ --- (1)}$$

In the right-angled ΔOPQ,

$OQ = -x$ (so that OQ will be positive since $x < 0$)

and $PQ = y$ (since $y > 0$).

Since 180° – A is an acute angle, we can use the definitions for acute angles:

$$\sin (180° - A) = \frac{\text{opp}}{\text{hyp}} = \frac{PQ}{OP} = \frac{y}{r}$$

and $\cos (180° - A) = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{OQ}{OP} = \dfrac{-x}{r}$ --- (2)

Comparing (1) and (2),

$$\sin A = \frac{y}{r} = \sin (180° - A)$$

but $\cos A = \dfrac{x}{r} = -\cos (180° - A)$

Worked Example 1

(Relationship between Trigonometric Ratios of Acute and Obtuse Angles)

Given that sin 55° = 0.819 and cos 136° = –0.719 when corrected to 3 significant figures, find the value of each of the following without the use of a calculator.

(a) sin 125° **(b)** cos 44°

Solution:

(a) $\sin 125° = \sin (180° - 125°)$
$= \sin 55°$
$= 0.819$

(b) $\cos 44° = -\cos (180° - 44°)$
$= -\cos 136°$
$= -(-0.719)$
$= 0.719$

PRACTISE NOW 1

1. Given that sin 84° = 0.995 and cos 129° = –0.629 when corrected to 3 significant figures, find the value of each of the following without the use of a calculator.

(a) sin 96° **(b)** cos 51°

2. Given that sin 172° = 0.139 and cos 40° = 0.766 when corrected to 3 significant figures, find the value of sin 8° – cos 140° without the use of a calculator.

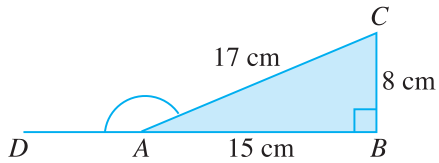
SIMILAR QUESTIONS

Exercise 6A Questions 1(a)-(f), 2(a),(b), 3(a)-(c)

Worked Example 2

(Relationship between Trigonometric Ratios of Acute and Obtuse Angles)

In the figure, DAB is a straight line, $\angle ABC = 90°$, $AB = 15$ cm, $BC = 8$ cm and $AC = 17$ cm.



Find the value of each of the following.

(a) $\sin \angle DAC$ **(b)** $\cos \angle DAC$

(c) $\tan \angle ACB$

Solution:

(a)
$$\begin{aligned}\sin \angle DAC &= \sin (180° - \angle DAC)\\ &= \sin \angle BAC\\ &= \frac{\text{opp}}{\text{hyp}} \quad (\text{since } \sin A = \sin (180° - A))\\ &= \frac{BC}{AC}\\ &= \frac{8}{17}\end{aligned}$$

(b)
$$\begin{aligned}\cos \angle DAC &= -\cos (180° - \angle DAC)\\ &= -\cos \angle BAC\\ &= -\frac{\text{adj}}{\text{hyp}} \quad (\text{since } \cos A = -\cos (180° - A))\\ &= -\frac{AB}{AC}\\ &= -\frac{15}{17}\end{aligned}$$

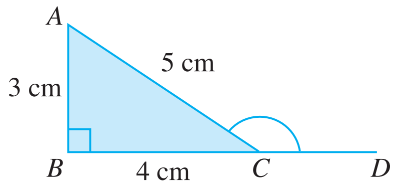
(c)
$$\begin{aligned}\tan \angle ACB &= \frac{\text{opp}}{\text{adj}}\\ &= \frac{AB}{BC}\\ &= \frac{15}{8}\\ &= 1\frac{7}{8}\end{aligned}$$

PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 6A Questions 4, 5, 10-12

1. In the figure, BCD is a straight line, $\angle ABC = 90°$, $AB = 3$ cm, $BC = 4$ cm and $AC = 5$ cm.

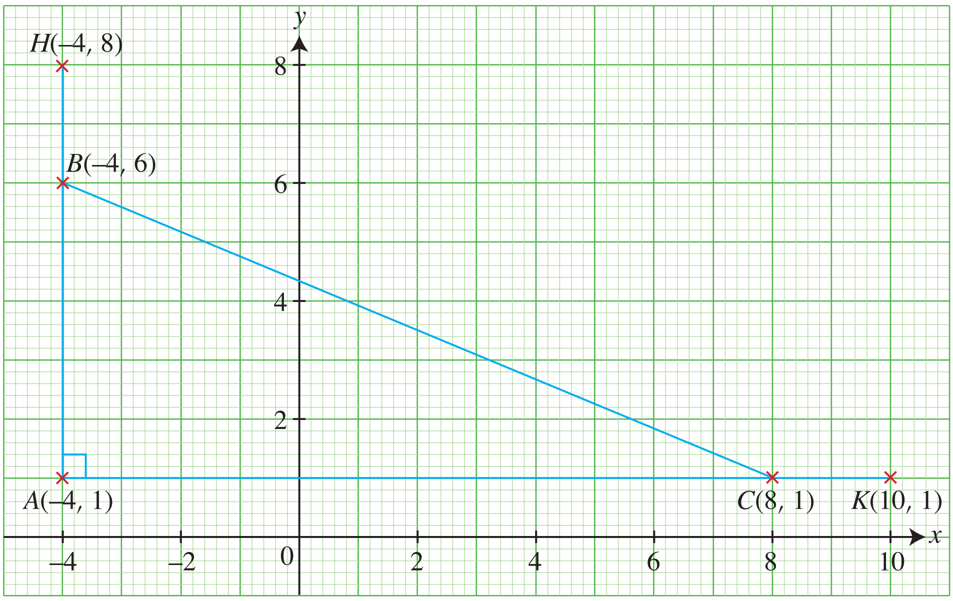


Find the value of each of the following.

(a) $\sin \angle ACD$ **(b)** $\cos \angle ACD$

(c) $\tan \angle BAC$

2. The figure shows ΔABC with vertices $A(-4, 1)$, $B(-4, 6)$ and $C(8, 1)$. $H(-4, 8)$ and $K(10, 1)$ lie on AB produced and AC produced respectively.



ATTENTION

The scale for the x-axis and the y-axis must be the same for the trigonometric ratios of the angles to be correct. If the scale used is different, the lengths of the sides of the triangle would not match the distance between two points represented by coordinates.

(a) Find the length of BC.

(b) State the value of each of the following.

(i) $\sin \angle HBC$ **(ii)** $\cos \angle BCK$

(iii) $\tan \angle ABC$

Worked Example 3

(Solving Simple Trigonometric Equations)

Given that $0° \leqslant x \leqslant 180°$, find the possible values of x for each of the following equations.

(a) $\sin x = 0.45$ **(b)** $\cos x = -0.834$

Solution:

(a) Since $\sin x$ is positive, x can either be an acute angle or an obtuse angle.

$\sin x = 0.45$

$x = \sin^{-1} 0.45 = 26.7°$ (to 1 d.p.)

or $180° - 26.7° = 153.3°$ (to 1 d.p.)

$\therefore x = 26.7°$ or $153.3°$

ATTENTION

Always leave your answer in degrees correct to 1 decimal place unless otherwise stated.

(b) Since $\cos x$ is negative, x is an obtuse angle.

$\cos x = -0.834$

$x = \cos^{-1} (-0.834)$

$= 146.5°$ (to 1 d.p.)

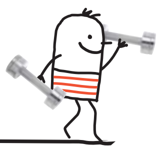
PRACTISE NOW 3

Given that $0° \leqslant x \leqslant 180°$, find the possible values of x for each of the following equations.

(a) $\sin x = 0.415$ **(b)** $\cos x = -0.234$ **(c)** $\cos x = 0.104$

SIMILAR QUESTIONS

Exercise 6A Questions 6(a)-(d), 7(a)-(d), 8(a)-(d), 9(a)-(f), 13, 14(a), (b)



Exercise 6A

BASIC LEVEL

1. Express each of the following as a trigonometric ratio of the acute angle.

(a) $\sin 110°$ **(b)** $\sin 176°$

(c) $\sin 98°$ **(d)** $\cos 99°$

(e) $\cos 107°$ **(f)** $\cos 175°$

2. Given that $\sin 32° = 0.530$ and $\cos 145° = -0.819$ when corrected to 3 significant figures, find the value of each of the following without the use of a calculator.

(a) $\sin 148°$ **(b)** $\cos 35°$

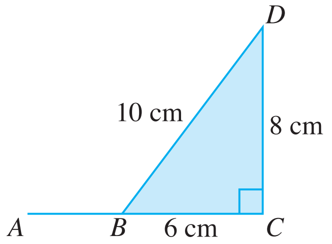
3. Given that $\sin 45° = \cos 45° = 0.707$ when corrected to 3 significant figures, find the value of each of the following without the use of a calculator.

(a) $2 \cos 45° + 3 \sin 135°$

(b) $3 \cos 135° + 4 \sin 135°$

(c) $\cos 135° - 2 \sin 45°$

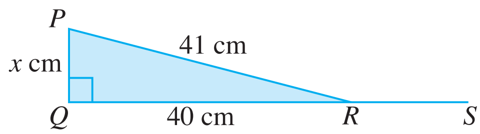
4. In the figure, ABC is a straight line, $\angle BCD = 90°$, $BC = 6$ cm, $CD = 8$ cm and $BD = 10$ cm.



Find the value of each of the following.

(a) $\sin \angle ABD$ **(b)** $\cos \angle DBA$
(c) $\tan \angle CBD$

5. In the figure, QRS is a straight line, $\angle PQR = 90°$, $PQ = x$ cm, $QR = 40$ cm and $PR = 41$ cm.



(a) Find the value of x.
(b) Find the value of each of the following.
  (i) $\sin \angle PRS$ **(ii)** $\cos \angle PRS$
  (iii) $\tan \angle PRQ$

6. Find an acute angle whose sine is
(a) 0.52, **(b)** 0.75,
(c) 0.875, **(d)** 0.3456.

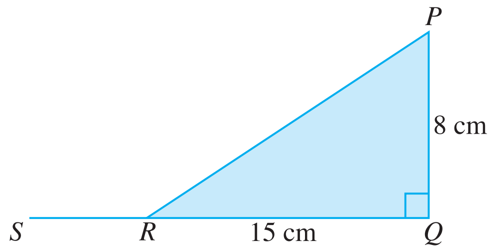
7. Find an obtuse angle whose sine is
(a) 0.52, **(b)** 0.75,
(c) 0.875, **(d)** 0.3456.

8. Find an acute angle whose cosine is
(a) 0.67, **(b)** 0.756,
(c) 0.5, **(d)** 0.985.

9. Given that $0° \leqslant x \leqslant 180°$, find the possible values of x for each of the following equations.
(a) $\sin x = 0.753$ **(b)** $\sin x = 0.952$
(c) $\sin x = 0.4714$ **(d)** $\cos x = -0.238$
(e) $\cos x = -0.783$ **(f)** $\cos x = 0.524$

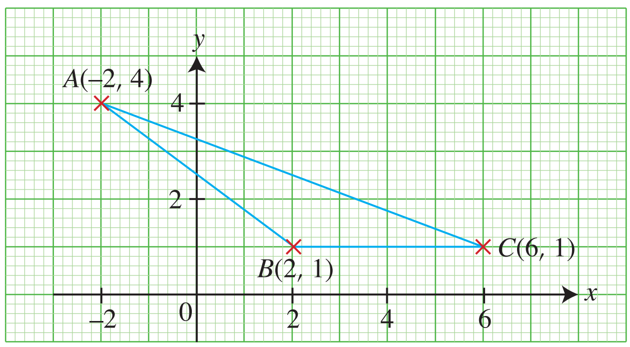
INTERMEDIATE LEVEL

10. In the figure, SRQ is a straight line, $\angle PQR = 90°$, $PQ = 8$ cm and $QR = 15$ cm.



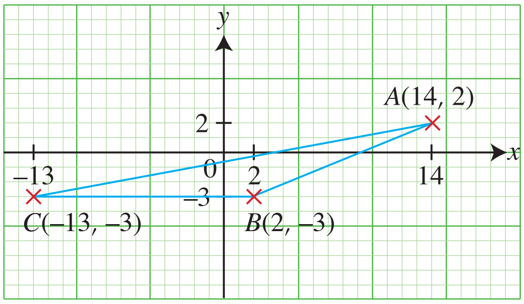
Find the value of each of the following.
(a) $\sin \angle PRS$ **(b)** $\cos \angle SRP$
(c) $\tan \angle PRQ$

11. The figure shows ΔABC with vertices $A(-2, 4)$, $B(2, 1)$ and $C(6, 1)$.



Find the value of each of the following.
(a) $\sin \angle ABC$ **(b)** $\cos \angle ABC$
(c) $\tan \angle ACB$

12. The figure shows ΔABC with vertices $A(14, 2)$, $B(2, -3)$ and $C(-13, -3)$.



Find the value of each of the following.

(a) $\sin \angle ABC$ **(b)** $\cos \angle ABC$

(c) $\tan \angle ACB$

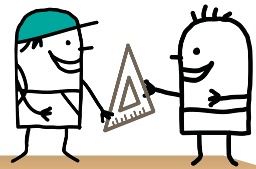
13. Given that $\sin x° = \sin 27°$, where $0° < x < 180°$, write down the possible values of x.

ADVANCED LEVEL

14. Given that $0° < x < 180°$, find the possible values of x for each of the following equations.

(a) $\sin (x + 10°) = 0.47$ **(b)** $\cos (x - 10°) = -0.56$

6.2 Area of Triangle



In primary school, we have learnt that the area of a triangle is given by the formula:

$$\text{Area of triangle} = \frac{1}{2} \times \text{base} \times \text{height or } \frac{1}{2}bh$$

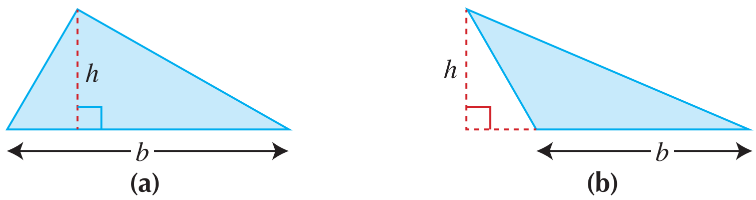
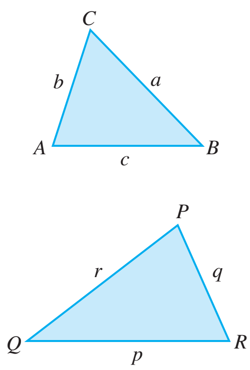


Fig. 6.5

INFORMATION

In this chapter, we shall use small letters to denote the lengths of the sides facing the angles, which are correspondingly denoted by capital letters.
Hence, we label two triangles as follows:



What happens if the height of a triangle is not given?

In real life, if a farmer has a triangular field with a given base, it is not easy to find the height of the triangle.

For example, if he is to start measuring from the point B in Fig. 6.6**(a)**, when he reaches D on AC, $\angle BDC$ may not be a right angle so BD may not be the height.

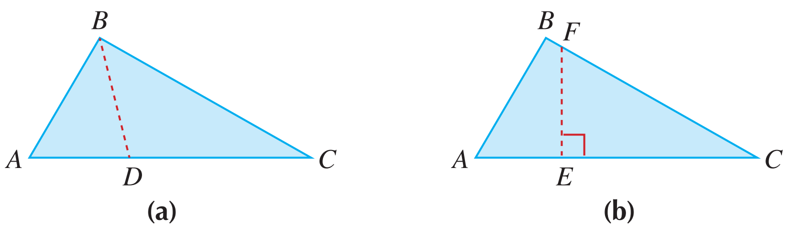


Fig. 6.6

If the farmer is to choose a point E on AC in Fig. 6.6**(b)** and he walks in the direction perpendicular to AC, he may not end up at B.

In other words, there is a need to find another formula for the area of a triangle.

Fig. 6.7 shows two triangles.
In Fig. 6.7**(a)**, $\angle C$ is acute, while in Fig. 6.7**(b)**, $\angle C$ is obtuse.

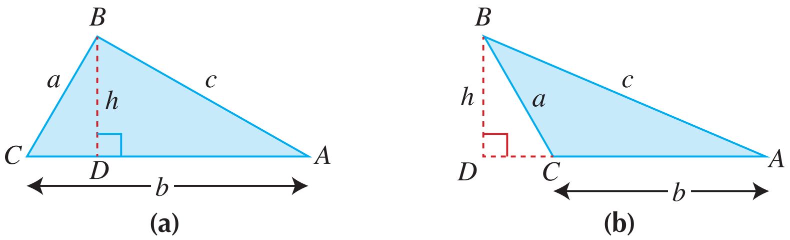


Fig. 6.7

In Fig. 6.7**(a)**, consider ΔBCD.

Then $\sin C = \dfrac{\text{opp}}{\text{hyp}}$

$$= \frac{h}{a}$$

$$h = a \sin C$$

$\therefore$ Area of $\Delta ABC = \dfrac{1}{2}bh$

$$= \frac{1}{2}b(a \sin C)$$

$$= \frac{1}{2}ab \sin C$$

In Fig. 6.7**(b)**, consider ΔBCD.

Then $\sin C = \dfrac{\text{opp}}{\text{hyp}}$

$$= \frac{h}{a}$$

$\therefore \sin \angle ACB = \sin (180° - \angle ACB)$

$$= \sin \angle BCD$$

$$= \frac{h}{a}$$

i.e. $h = a \sin \angle ACB$

$$= a \sin C$$

$\therefore$ Area of $\Delta ABC = \dfrac{1}{2}bh$

$$= \frac{1}{2}ab \sin C$$

By considering $\sin A$ and $\sin B$ in a similar way, we can show that:

Area of $\Delta ABC = \dfrac{1}{2}bc \sin A$ and $\dfrac{1}{2}ac \sin B$ respectively.

In general,

$$\boxed{\textbf{Area of } \Delta ABC = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B}$$

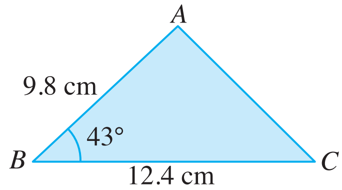
ATTENTION

In the formula $\dfrac{1}{2}ab \sin C$, notice that the angle C is in between the two sides a and b, i.e. C is called the **included** angle.

Worked Example 4

(Finding the Area of a Triangle)

Find the area of ΔABC, given that $AB = 9.8$ cm, $BC = 12.4$ cm and $\angle ABC = 43°$.



Solution:

We have $a = 12.4$, $c = 9.8$ and $B = 43°$.

$$\begin{aligned} \text{Area of } \Delta ABC &= \frac{1}{2}ac \sin B \\ &= \frac{1}{2} \times 12.4 \times 9.8 \times \sin 43° \\ &= 41.4 \text{ cm}^2 \end{aligned}$$

PRACTISE NOW 4

Find the area of ΔABC, given that $BC = 31.8$ m, $AC = 24.8$ m and $\angle ACB = 49°$.

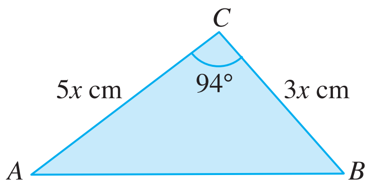
SIMILAR QUESTIONS

Exercise 6B Questions 1(a)-(f), 2-6, 8, 9

Worked Example 5

(Problem involving Area of a Triangle)

In ΔABC, $CA = 5x$ cm, $CB = 3x$ cm and $\angle ACB = 94°$.



Given that the area of ΔABC is 145 cm², find the value of x.

Solution:

We have $a = 3x$, $b = 5x$ and $C = 94°$.

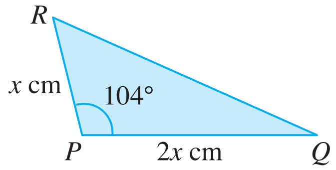
$$\begin{aligned} \text{Area of } \Delta ABC &= \frac{1}{2}ab \sin C \\ 145 &= \frac{1}{2} \times 3x \times 5x \times \sin 94° \\ &= 7.5x^2 \sin 94° \\ x^2 &= \frac{145}{7.5 \sin 94°} \\ x &= \sqrt{\frac{145}{7.5 \sin 94°}} \quad \text{(since } x \text{ is positive)} \\ &= 4.40 \text{ (to 3 s.f.)} \end{aligned}$$

PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 6B Questions 7, 10-12

1. In ΔPQR, $PQ = 2x$ cm, $PR = x$ cm and $\angle QPR = 104°$.



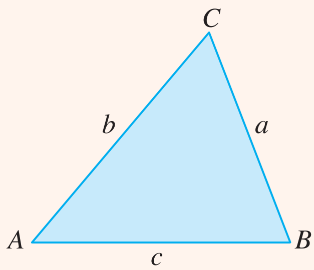
Given that the area of ΔPQR is 12.5 cm^2, find the value of x.

2. In ΔXYZ, $XY = 5$ cm, $YZ = 6$ cm and the area of ΔXYZ is 12 cm^2. Find $\angle XYZ$.

Thinking Time

In real life, a 'triangular' field is not always exactly a triangle. Therefore, a small error in measuring the included angle may result in a large error in the area of the triangle if we use the formula $\frac{1}{2}ab\sin C$. However, the error is usually not so large if we can use the lengths of the three sides of the triangle to find its area.

Heron of Alexandria (around AD 75) established a formula for finding the area of a triangle using the lengths of its sides only. The area of ΔABC is given by $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$ is half of the perimeter.



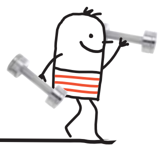
Verify that the above formula is correct for each of the following cases:

(a) $a = 6$ cm, $b = 8$ cm and $c = 10$ cm

(b) $a = 8$ cm, $b = 9$ cm and $c = 10$ cm

(c) $a = 5$ cm, $b = 3$ cm and $c = 7$ cm

Can you find a proof for this formula?

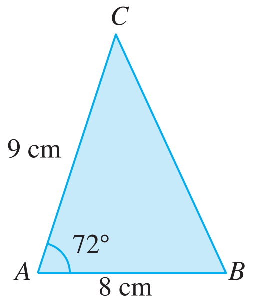


Exercise 6B

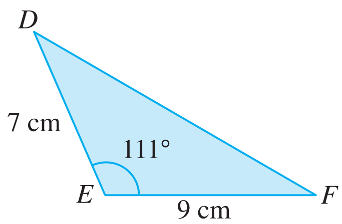
BASIC LEVEL

1. Find the area of each of the following figures.

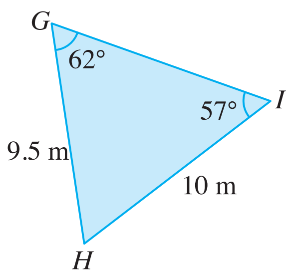
(a)



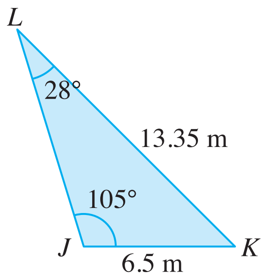
(b)



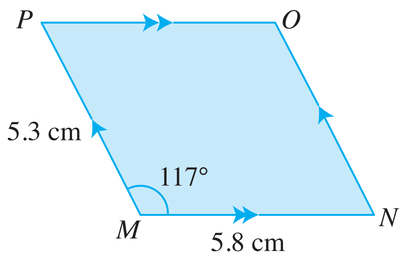
(c)



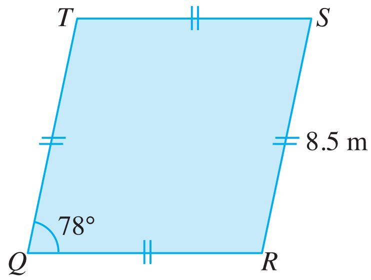
(d)



(e)



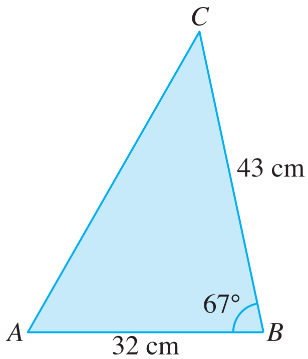
(f)



2. Find the area of ΔABC, given that $AB = 22$ cm, $AC = 15$ cm and $\angle BAC = 45°$.

3. In ΔPQR, $\angle P = 72°$, $q = 152$ cm and $r = 125$ cm. Find the area of ΔPQR.

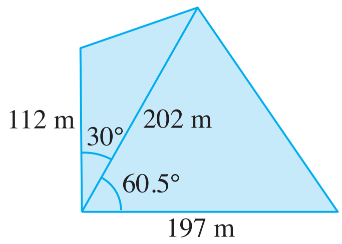
4. In ΔABC, $AB = 32$ cm, $BC = 43$ cm and $\angle ABC = 67°$.



(i) Find the area of ΔABC.

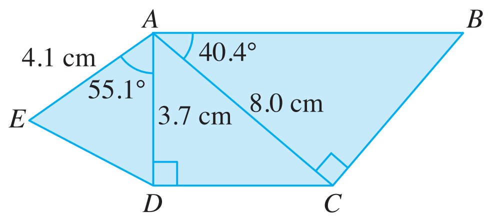
(ii) Hence, find the perpendicular distance from A to BC.

5. The figure shows the plan of two neighbouring estates in the form of 2 triangles.



Calculate the area of the two estates.

6. In the figure, $\angle ADC = \angle ACB = 90°$, $\angle EAD = 55.1°$, $\angle CAB = 40.4°$, $AE = 4.1$ cm, $AD = 3.7$ cm and $AC = 8.0$ cm.

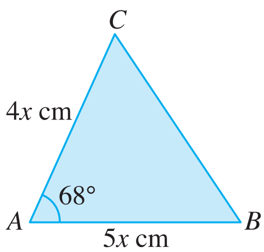


Find

(i) $\angle ACD$, **(ii)** the length of AB,

(iii) the area of ΔAED.

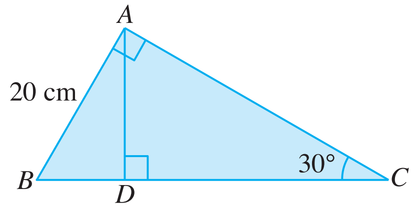
7. In ΔABC, $AB = 5x$ cm, $AC = 4x$ cm and $\angle BAC = 68°$.



Given that the area of ΔABC is 97 cm^2, find the value of x.

INTERMEDIATE LEVEL

8. In the figure, $AB = 20$ cm, $\angle BAC = 90°$, $\angle ACB = 30°$ and AD is perpendicular to BC.



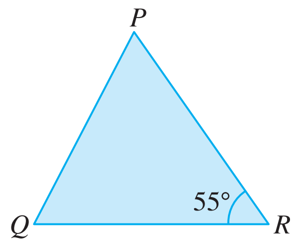
Find

(i) $\angle BAD$, **(ii)** BD,

(iii) the area of ΔABC.

9. The diagonals of a parallelogram have lengths 15.6 cm and 17.2 cm. They intersect at an angle of 120°. Find the area of the parallelogram.

10. In ΔPQR, $\angle PRQ = 55°$, $3QR = 4PR$ and the area of ΔPQR is 158 cm^2.



Find the length of QR.

11. Given that the area of a rhombus is 40 cm^2 and that each side has a length of 15 cm, find the angles of the rhombus.

ADVANCED LEVEL

12. In quadrilateral $ABCD$, $AB = 3.2$ cm, $BC = 5.1$ cm, $\angle CBD = 34.4°$ and the length of the diagonal BD is 7.5 cm. Given further that the area of ΔABD is 11.62 cm^2 and $\angle ABD$ is obtuse, find

(i) the area of ΔBCD,

(ii) $\angle ABD$.

6.3 Sine Rule

Investigation

Sine Rule

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template 'Sine Rule'.

Fig. 6.8 shows a triangle *ABC* and a table of values.

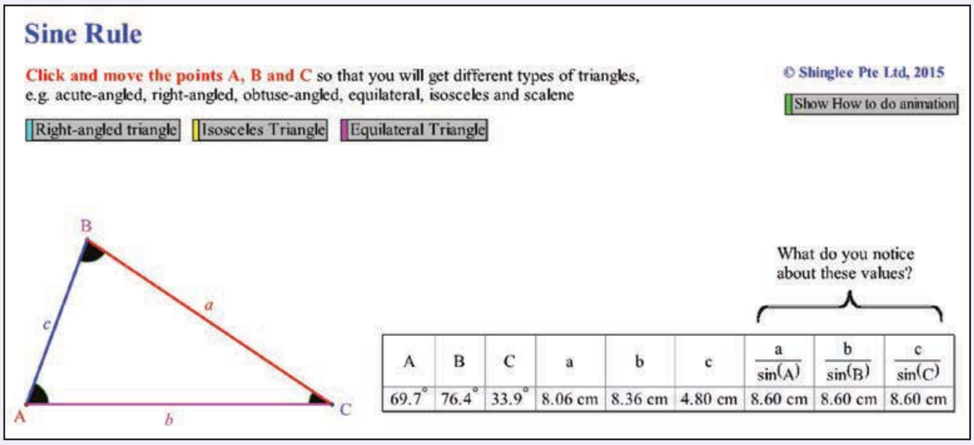


Fig. 6.8

1. The labelling of the sides of the triangle with reference to the vertices is important. Copy and complete the following.
   (a) The length of the side of the triangle opposite vertex *A* is labelled *a*.
   (b) The length of the side of the triangle opposite vertex *B* is labelled _____.
   (c) The length of the side of the triangle opposite vertex *C* is labelled _____.

2. Click and drag each of the vertices A, B and C to get different types of triangles. To obtain special triangles such as a right-angled triangle, an isosceles triangle and an equilateral triangle, click on the respective buttons in the template. For each triangle, copy and complete Table 6.2.

No.	$\angle A$	$\angle B$	$\angle C$	a	b	c	$\frac{a}{\sin A}$	$\frac{b}{\sin B}$	$\frac{c}{\sin C}$
1.									
2.									
3.									
4.									
5.									
6.									

Table 6.2

3. What do you notice about the last 3 columns in Table 6.2?
4. Click on the button 'Show how to do animation' in the template and it will show you how to add 10 more entries to the table as the points A, B and C move automatically. What do you notice about the last 3 columns of the table in the template?
5. Hence, write down a formula relating the quantities in the last 3 columns of the table. This is called the **Sine Rule**. Notice that for each fraction, *the side must be opposite the angle*.
6. Do you think the relationship $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ is also true? Explain your answer.
7. Copy and complete the following:

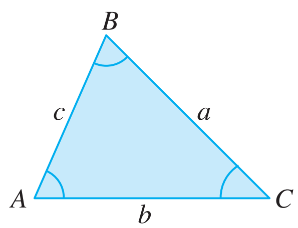
   The lengths of the sides of a triangle are p__________ to the sine of the angles opposite the sides.

RECALL

In a triangle,
- the largest angle is opposite the longest side,
- the smallest angle is opposite the shortest side.

From the investigation, we can conclude that:

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ or } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},}$$



where A, B and C are the three interior angles of ΔABC opposite the sides whose lengths are a, b and c respectively.

This is called the **Sine Rule**.

We can prove the sine rule using the formula for the area of a triangle obtained from the previous section, as shown on the next page.

For any triangle ABC,

$$\frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C.$$

Dividing each side by $\frac{1}{2}abc$,

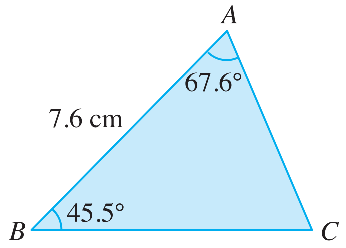
$$\frac{\frac{1}{2}bc\sin A}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac\sin B}{\frac{1}{2}abc} = \frac{\frac{1}{2}ab\sin C}{\frac{1}{2}abc}.$$

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Worked Example 6

(Using Sine Rule when given 2 angles and 1 side)

In ΔABC, $\angle A = 67.6°$, $\angle B = 45.5°$ and $AB = 7.6$ cm. Find



(i) $\angle C$,

(ii) the length of BC,

(iii) the length of AC.

Solution:

(i) $\angle C = 180° - 67.6° - 45.5°$ ($\angle$ sum of a Δ)

$= 66.9°$

(ii) Using sine rule,

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 67.6°} = \frac{7.6}{\sin 66.9°}$$

$$a = \frac{7.6\sin 67.6°}{\sin 66.9°}$$

$= 7.64$ cm (to 3 s.f.)

$\therefore BC = 7.64$ cm

(iii) Using sine rule,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 45.5°} = \frac{7.6}{\sin 66.9°}$$

$$b = \frac{7.6\sin 45.5°}{\sin 66.9°}$$

$= 5.89$ cm (to 3 s.f.)

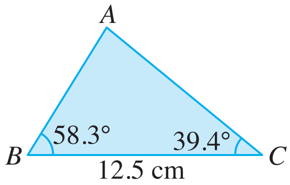
$\therefore AC = 5.89$ cm

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 6C Questions 1(a)-(c), 2, 3, 7-11

In ΔABC, $\angle B = 58.3°$, $\angle C = 39.4°$ and $BC = 12.5$ cm. Find

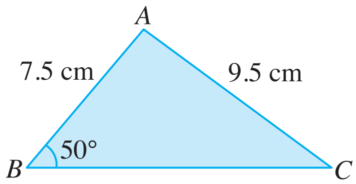


(i) $\angle A$, **(ii)** the length of AB, **(iii)** the length of AC.

Worked Example 7

(Using Sine Rule when given 2 sides and 1 non-included angle)

In ΔABC, $\angle B = 50°$, $AB = 7.5$ cm and $AC = 9.5$ cm. Find



(i) $\angle C$, **(ii)** $\angle A$,
(iii) the length of BC.

Solution:

(i) Using sine rule,

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin C}{7.5} = \frac{\sin 50°}{9.5}$$

$$\sin C = \frac{7.5 \sin 50°}{9.5}$$

$$= 0.6048 \text{ (to 4 s.f.)}$$

$\angle C = \sin^{-1} 0.6048 = 37.21°$ (to 2 d.p.)

or $180° - 37.21° = 142.78°$ (to 2 d.p.)

Since $c < b$, then $\angle C < \angle B$, hence $\angle C$ cannot be 142.78°.

$\therefore \angle C = 37.2°$ (to 1 d.p.)

(ii) $\angle A = 180° - 50° - 37.21°$ ($\angle$ sum of a Δ)
$= 92.8°$ (to 1 d.p.)

(iii) Using sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 92.79°} = \frac{9.5}{\sin 50°}$$

$$a = \frac{9.5 \sin 92.79°}{\sin 50°}$$

$$= 12.4 \text{ cm (to 3 s.f.)}$$

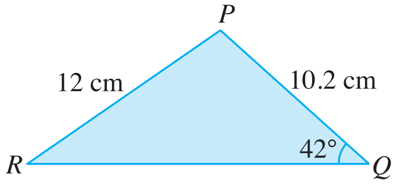
$\therefore BC = 12.4$ cm

PRACTISE NOW 7

SIMILAR QUESTIONS

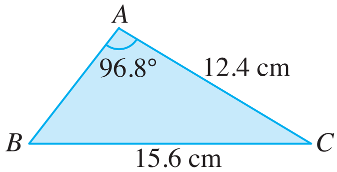
Exercise 6C Questions 4(a)-(c), 5, 6, 12, 16

1. In ΔPQR, $\angle Q = 42°$, $PR = 12$ cm and $PQ = 10.2$ cm. Find



(i) $\angle R$, **(ii)** $\angle P$, **(iii)** the length of QR.

2. In ΔABC, $\angle BAC = 96.8°$, $AC = 12.4$ cm and $BC = 15.6$ cm. Find



(i) $\angle ABC$, **(ii)** $\angle BCA$, **(iii)** the length of AB.

Worked Example 8

(Ambiguous Case of Sine Rule)

In ΔABC, $\angle ABC = 55°$, $AB = 16.3$ cm and $AC = 14.3$ cm. Find $\angle ACB$, $\angle BAC$ and the length of BC.

Solution:

Using sine rule,

$$\frac{\sin \angle ACB}{16.3} = \frac{\sin 55°}{14.3}$$

$$\sin \angle ACB = \frac{16.3 \sin 55°}{14.3}$$

$$= 0.9337 \text{ (to 4 s.f.)}$$

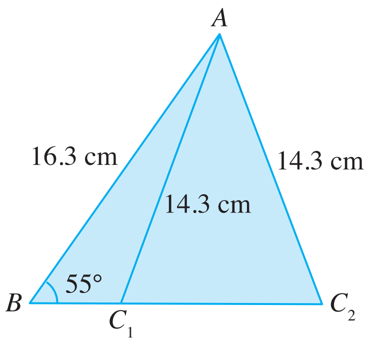
$\angle ACB = \sin^{-1} 0.9337 = 69.02°$ (to 2 d.p.)

or $180° - 69.02° = 110.98°$ (to 2 d.p.) (Since $c > b$, then $\angle C > \angle B$, i.e. $\angle ACB > 55°$, hence both answers are possible.)

When $\angle ACB = 69.02°$, $\angle BAC = 180° - 55° - 69.02°$
$= 55.98°$

When $\angle ACB = 110.98°$, $\angle BAC = 180° - 55° - 110.98°$
$= 14.02°$

Notice that it is possible to construct two different triangles from the information above.



A set of information that will give two sets of solutions is said to be *ambiguous*. Hence, for ambiguous cases, two sets of solutions will be obtained.

Case 1: when $\angle ACB = 69.02°$ and $\angle BAC = 55.98°$

Using sine rule,

$$\frac{a}{\sin 55.98°} = \frac{16.3}{\sin 69.02°}$$

$$a = \frac{16.3 \sin 55.98°}{\sin 69.02°}$$

$$= 14.5 \text{ cm (to 3 s.f.)}$$

$\therefore$ $\angle ACB = 69.0°$, $\angle BAC = 56.0°$ and $BC = 14.5$ cm

Case 2: when $\angle ACB = 110.98°$ and $\angle BAC = 14.02°$

Using sine rule,

$$\frac{a}{\sin 14.02°} = \frac{16.3}{\sin 110.98°}$$

$$a = \frac{16.3 \sin 14.02°}{\sin 110.98°}$$

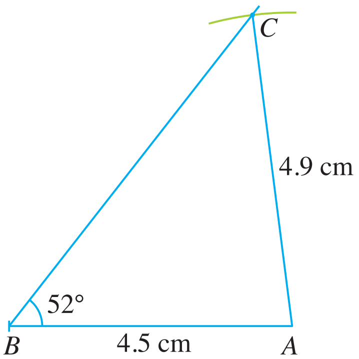
$$= 4.23 \text{ cm (to 3 s.f.)}$$

$\therefore\ \angle ACB = 111.0°$, $\angle BAC = 14.0°$ and $BC = 4.23$ cm

If sides b, c and $\angle B$ (i.e. 2 sides and a non-included angle) are given, take note of each of the following cases.

Case 1: $b > c$

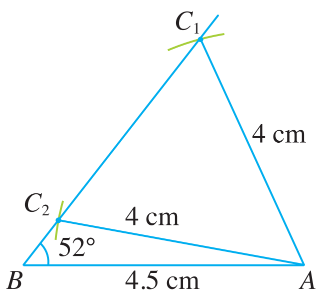
For example, construct ΔABC such that $\angle ABC = 52°$, $AC = 4.9$ cm and $AB = 4.5$ cm.



When $b > c$, one triangle can be constructed.
Therefore, there is only one value of $\angle ACB$.

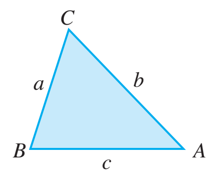
Case 2: $b < c$

For example, construct ΔABC such that $\angle ABC = 52°$, $AC = 4$ cm and $AB = 4.5$ cm.



When $b < c$, two triangles can be constructed.
Therefore, there are two possible values of $\angle ACB$.

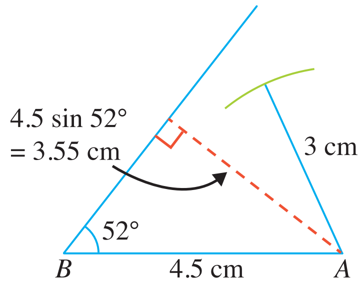
RECALL



In ΔABC, a is the side opposite to $\angle A$, b is the side opposite to $\angle B$ and c is the side opposite to $\angle C$.

Case 3: $b < c \sin B$, where $c \sin B$ is the perpendicular distance from A to the line through B

For example, construct ΔABC such that $\angle ABC = 52°$, $AB = 4.5$ cm and $AC = 3$ cm.



When $b < c \sin B$, no triangle can be constructed.

PRACTISE NOW 8

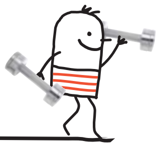
In ΔABC, $\angle ABC = 46°$, $AB = 9.8$ cm and $AC = 7.1$ cm. Find $\angle ACB$, $\angle BAC$ and the length of BC.

SIMILAR QUESTIONS

Exercise 6C Questions 13(a)-(f), 14(a)-(d), 15, 17

Journal Writing

To solve a triangle means to find all the unknown sides and/or angles. From the worked examples, state the given conditions when sine rule can be used to solve a triangle.

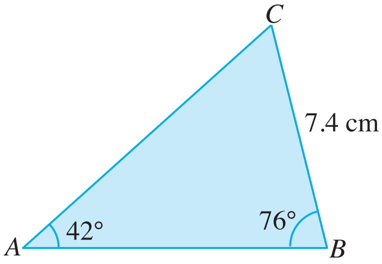


Exercise 6C

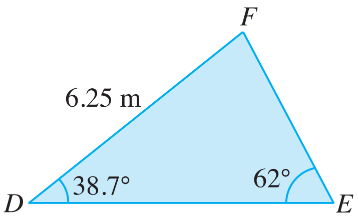
BASIC LEVEL

1. For each of the following triangles, find the unknown angles and sides.

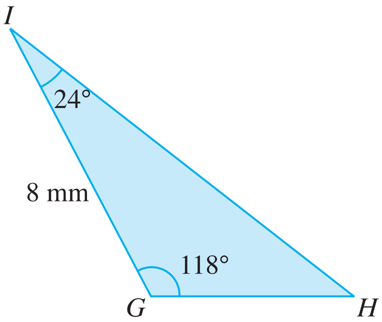
(a)



(b)



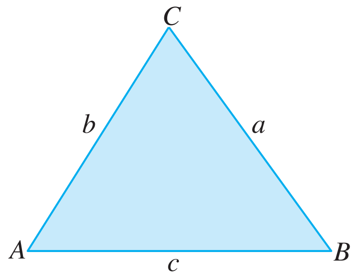
(c)



2. In ΔPQR, $QR = 7$ cm, $\angle PQR = 47°$ and $\angle PRQ = 97°$. Find the length of PQ.

3. In ΔPQR, $\angle P = 75°$, $\angle Q = 60°$ and $q = 14$ cm. Find the length of the longest side.

4. For each of the following triangles ABC, find the unknown angles and sides.

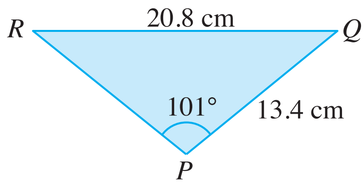


(a) $\angle A = 92.0°$, $b = 6.93$ cm and $a = 15.3$ cm

(b) $\angle B = 98.0°$, $a = 14.5$ m and $b = 17.4$ m

(c) $\angle C = 35.0°$, $b = 8.7$ cm and $c = 9.5$ cm

5. In ΔPQR, $\angle P = 101°$, $PQ = 13.4$ cm and $QR = 20.8$ cm. Find



(i) $\angle R$, **(ii)** $\angle Q$,

(iii) the length of PR.

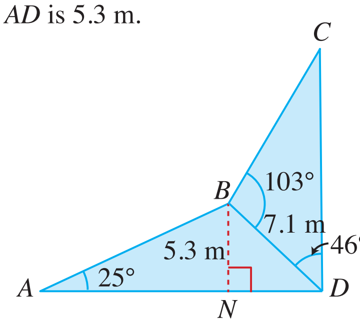
6. In ΔABC, $\angle ABC = 91°$, $BC = 7.4$ cm and $AC = 11.6$ cm. Find

(i) $\angle BAC$, **(ii)** $\angle ACB$,

(iii) the length of AB.

INTERMEDIATE LEVEL

7. The figure shows a metal framework in which AD is horizontal with $BD = 7.1$ m, $\angle BAD = 25°$, $\angle BDC = 46°$, $\angle DBC = 103°$ and the height of B above AD is 5.3 m.



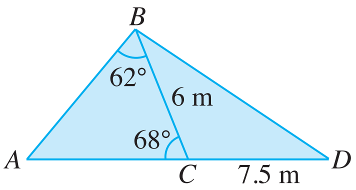
Find

(i) the length of the metal bar AB,

(ii) the angle that BD makes with BN,

(iii) the length of the metal bar CD.

8. In the figure, A, C and D are three points along a straight road where $\angle ABC = 62°$, $\angle ACB = 68°$, $BC = 6$ m and $CD = 7.5$ m.

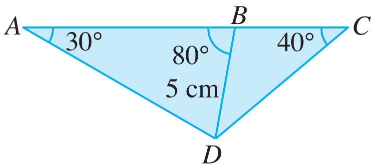


Find

(i) the distance AC,

(ii) the area of the region enclosed by AB, BD and DA.

9. An experiment is carried out to determine the extension of springs. Springs are attached to a horizontal bar at A, B and C and are joined to a mass D.



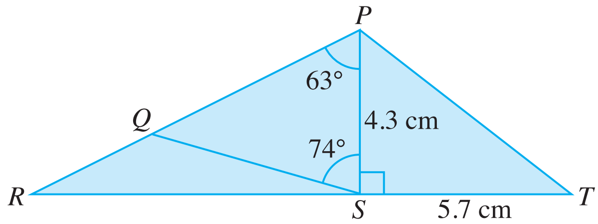
Given that $\angle ACD = 40°$, $\angle CAD = 30°$, $\angle ABD = 80°$ and $BD = 5$ cm, find

(i) the distance between A and B,

(ii) the distance between B and C,

(iii) the vertical distance between the mass and the horizontal bar.

10. In the figure, RST is a straight line, $\angle PST = 90°$, $\angle SPR = 63°$, $\angle PSQ = 74°$, $PS = 4.3$ cm and $ST = 5.7$ cm.

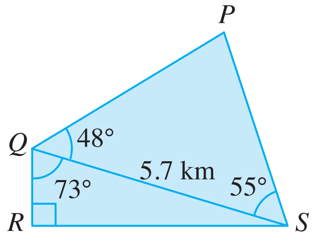


(i) Determine if QS is parallel to PT.

(ii) Find the length of PR.

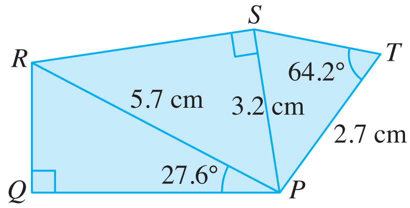
(iii) Find the length of QS.

11. In the figure, $PQRS$ is a nature reserve. A 5.7 km long walkway connects Q to S. It is given that $\angle QRS = 90°$, $\angle SQR = 73°$, $\angle PQS = 48°$ and $\angle PSQ = 55°$.



Find the area of the nature reserve.

12. In the figure, $\angle PQR = \angle PSR = 90°$, $\angle QPR = 27.6°$, $\angle PTS = 64.2°$, $PR = 5.7$ cm, $PS = 3.2$ cm and $PT = 2.7$ cm.



Find

(i) the length of QR,

(ii) $\angle SPR$,

(iii) $\angle PST$.

13. For the data of each of the following triangles, determine whether it is an ambiguous case. Explain your answer.

(a) ΔABC, $\angle A = 92°$, $b = 7.5$ cm, $a = 8.5$ cm

(b) ΔDEF, $\angle D = 47°$, $d = 75$ m, $e = 80$ m

(c) ΔGHI, $g = 37$ mm, $h = 37$ mm, $\angle G = 58°$

(d) ΔJKL, $j = 19$ cm, $k = 15$ cm, $\angle K = 39°$

(e) ΔMNO, $n = 80$ m, $o = 67$ m, $\angle O = 43°$

(f) ΔPQR, $p = 19$ mm, $q = 25$ mm, $\angle Q = 52°$

14. Determine whether it is possible to construct each of the following triangles with the given conditions.

(a) ΔABC, $AB = 6$ cm, $BC = 8$ cm, $\angle ABC = 90°$ and $\angle ACB = 35°$

(b) ΔPQR, $PQ = 6$ cm, $PR = 5$ cm, $\angle PQR = 30°$ and $\angle PRQ = 36.9°$

(c) ΔLMN, $LM = 6.9$ cm and $LN = 7.8$ cm, $\angle LMN = 42°$ and $\angle LNM = 57°$

(d) ΔGHK, $GH = 6.4$ cm and $GK = 12.8$ cm, $\angle GHK = 90°$ and $\angle HGK = 60°$

15. In ΔABC, $\angle BAC = 58°$, $BC = 14.0$ cm and $AC = 15.4$ cm. Find $\angle ABC$, $\angle ACB$ and the length of AB.

ADVANCED LEVEL

16. On a map whose scale is 8 cm to 1 km, an undeveloped plot of land is shown as a quadrilateral $ABCD$. The length of the diagonal AC is 7 cm, $\angle BAC = 55°$, $\angle BCA = 77°$, $\angle DAC = 90°$ and $\angle DCA = 40°$. Find

(i) the length, in cm, of the side AB on the map,

(ii) the length, in km, which is represented by AD,

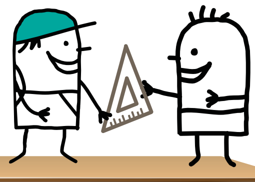
(iii) the area, in km², which is represented by ΔADC.

17. In ΔABC, $\angle A = 35°$, $BC = 5$ cm and $\sin B = \frac{4}{3}\sin A$.

(i) Calculate two possible values of $\angle B$.

(ii) Find the length of AC.

6.4 Cosine Rule



From the journal writing in Section 6.3, we have observed that the Sine Rule can be used to solve a triangle if the following are given:

(1) Two angles and the length of one side (see Worked Example 6); or

(2) The lengths of two sides and one non-included angle (see Worked Example 7).

What happens if the lengths of two sides and an included angle are given (see Fig. 6.9)? Can you try to use sine rule to solve the triangle?

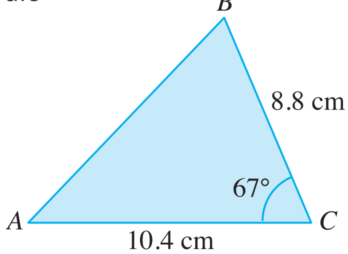


Fig. 6.9

Investigation

Cosine Rule

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template 'Cosine Rule'.

Fig 6.10 shows a triangle *ABC* and a table of values.

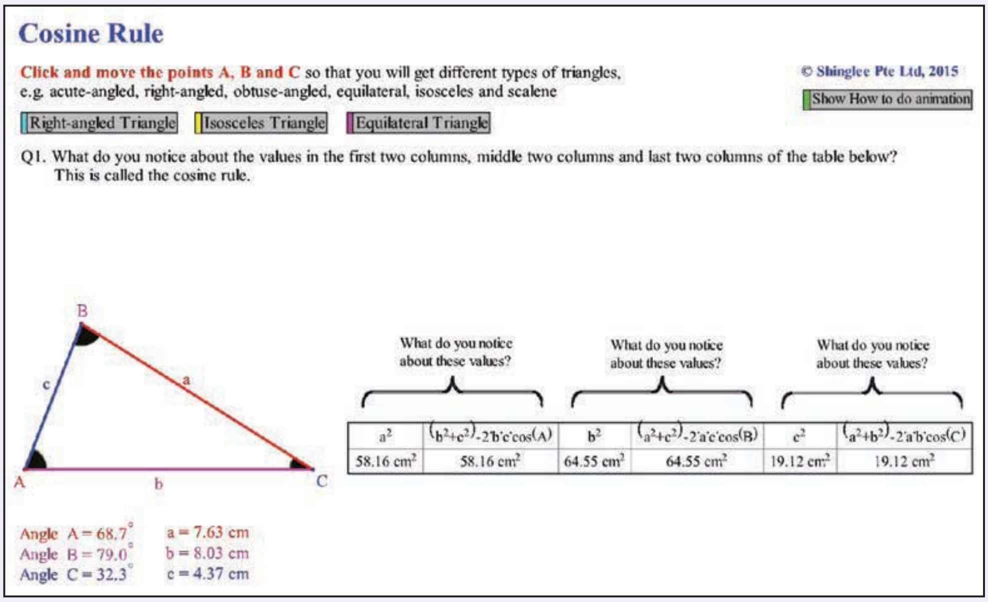


Fig 6.10

1. The labelling of the sides of the triangle with reference to the vertices is important. Copy and complete the following.

(a) The length of the side of the triangle opposite vertex *A* is labelled *a*.

(b) The length of the side of the triangle opposite vertex *B* is labelled _____.

(c) The length of the side of the triangle opposite vertex *C* is labelled _____.

2. Click and drag each of the vertices A, B and C to get different types of triangles. To obtain special triangles such as a right-angled triangle, an isosceles triangle and an equilateral triangle, click on the respective buttons in the template. For each triangle, copy and complete Table 6.3.

No.	$\angle A$	$\angle B$	$\angle C$	a^2	$b^2 + c^2 - 2bc \cos A$	b^2	$a^2 + c^2 - 2ac \cos B$	c^2	$a^2 + b^2 - 2ab \cos C$
1.									
2.									
3.									
4.									
5.									
6.									

Table 6.3

3. What do you notice about the last 6 columns in Table 6.3?
4. Click on the button 'Show how to do animation' in the template and it will show you how to add 10 more entries to the table as the points A, B and C move automatically. What do you notice about the last 6 columns of the table in the template?
5. Hence, write down a formula relating the quantities in the last 6 columns of the table. This is called the **Cosine Rule**. Notice that for each fraction, the *side by itself must be opposite the angle*.
6. For each of the formulae in Question 5, make the angle the subject of the formula.

From the investigation, we can conclude that:

$$a^2 = b^2 + c^2 - 2bc \cos A \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 = a^2 + c^2 - 2ac \cos B \qquad \text{or} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab \cos C \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

where A, B and C are the three interior angles of ΔABC opposite the sides whose lengths are a, b and c respectively.

This is called the **Cosine Rule**.

We can prove the cosine rule as follows. Without loss of generality, we will just prove $a^2 = b^2 + c^2 - 2bc \cos A$.

There are three cases: $\angle A$ is an acute angle, $\angle A$ is an obtuse angle and $\angle A$ is a right angle. We will show the case when A is acute.

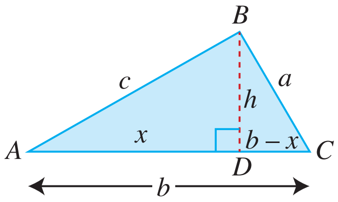


Fig. 6.11

In ΔBCD,

$$a^2 = h^2 + (b - x)^2 \text{ (Pythagoras' Theorem)}$$
$$= h^2 + b^2 - 2bx + x^2$$
$$= b^2 + (h^2 + x^2) - 2bx \text{ --- (1)}$$

In ΔBAD,

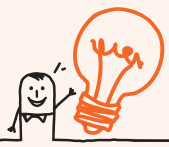
$$c^2 = h^2 + x^2 \text{ (Pythagoras' Theorem) --- (2)}$$

and $\cos A = \frac{x}{c}$,

i.e. $x = c \cos A$ --- (3)

Substituting (2) and (3) into (1),

$$a^2 = b^2 + (h^2 + x^2) - 2bx$$
$$= b^2 + c^2 - 2bc \cos A \text{ (proven)}$$



Thinking Time

1. Prove the Cosine Rule for Case 2 where $\angle A$ is an obtuse angle.

2. Let us consider Case 3 where $\angle A$ is a right angle.

(a) What happens to the formula for the Cosine Rule $a^2 = b^2 + c^2 - 2bc \cos A$ if $A = 90°$?

(b) Is this formula always true if $A = 90°$? Explain your answer.

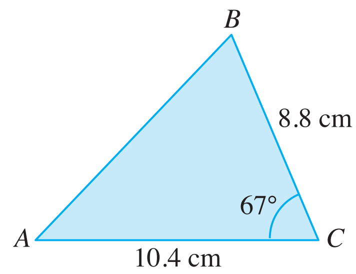
3. Copy and complete the following:

_______________ Theorem is a special case of the Cosine Rule.

Worked Example 9

(Using Cosine Rule when given 2 sides and 1 included angle)

In ΔABC, $BC = 8.8$ cm, $AC = 10.4$ cm and $\angle ACB = 67°$. Find



(i) the length of AB, **(ii)** $\angle ABC$,
(iii) $\angle BAC$.

Solution:

(i) Using cosine rule,

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$AB^2 = 8.8^2 + 10.4^2 - 2 \times 8.8 \times 10.4 \times \cos 67°$$
$$= 114.1 \text{ (to 4 s.f.)}$$
$$\therefore AB = \sqrt{114.1}$$
$$= 10.7 \text{ cm (to 3 s.f.)}$$

(ii) Using sine rule,

$$\frac{\sin \angle ABC}{AC} = \frac{\sin \angle ACB}{AB}$$
$$\frac{\sin \angle ABC}{10.4} = \frac{\sin 67°}{10.68}$$
$$\sin \angle ABC = \frac{10.4 \sin 67°}{10.68}$$
$$= 0.8963 \text{ (to 4 s.f.)}$$
$$\angle ABC = \sin^{-1} 0.8963 = 63.68° \text{ (to 2 d.p.)}$$
$$\text{or } 180° - 63.68° = 116.32° \text{ (to 2 d.p.)}$$

Since $AC < AB$, then $\angle B < \angle C$, hence $\angle B$ cannot be 116.32°.

$\therefore \angle ABC = 63.7°$ (to 1 d.p.)

(iii) $\angle BAC = 180° - 63.68° - 67°$ ($\angle$ sum of a Δ)
$= 49.3°$ (to 1 d.p.)

ATTENTION

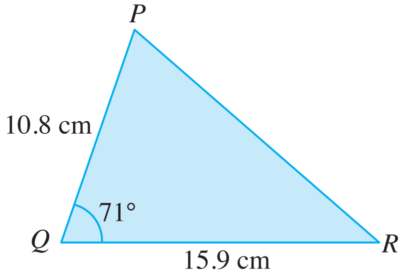
In order for the final answer to be accurate to three significant figures, any intermediate working must be correct to four significant figures, i.e. $AB \approx 10.68$ cm.

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 6D Questions 1-3, 7-10

In ΔPQR, $PQ = 10.8$ cm, $QR = 15.9$ cm and $\angle PQR = 71°$. Find



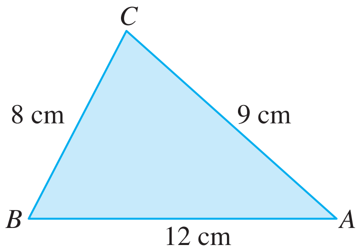
(i) the length of PR, **(ii)** $\angle QPR$, **(iii)** $\angle PRQ$.

Worked Example 10

(Using Cosine Rule when given 3 sides)

In ΔABC, $AB = 12$ cm, $BC = 8$ cm and $AC = 9$ cm.

Find the size of the smallest angle.



Solution:

The smallest angle is the angle opposite the shortest side, i.e. $\angle BAC$.

Using cosine rule,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{9^2 + 12^2 - 8^2}{2 \times 9 \times 12}$$

$$= \frac{161}{216}$$

$$\angle A = \cos^{-1}\frac{161}{216} = 41.8° \text{ (to 1 d.p.)}$$

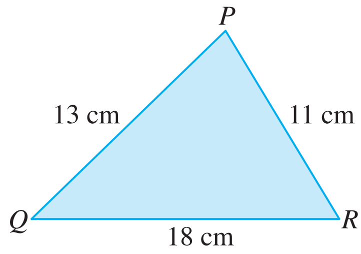
$\therefore$ The smallest angle is 41.8°.

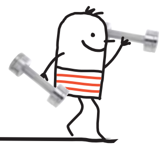
PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 6D Questions 4-6, 11-17

In ΔPQR, $PQ = 13$ cm, $QR = 18$ cm and $PR = 11$ cm. Find the size of the largest angle.





Exercise 6D

BASIC LEVEL

1. In ΔABC, $a = 5$ cm, $b = 7$ cm and $\angle C = 60°$. Find c.

2. In ΔGHI, $g = 9$ cm, $i = 7$ cm and $\angle H = 30°$. Find h.

3. In ΔMNO, $m = 4.2$ cm, $n = 5.8$ cm and $\angle O = 141.4°$. Find o.

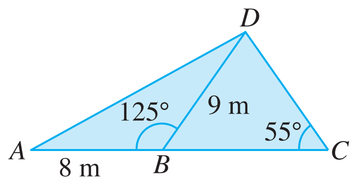
4. In ΔXYZ, $x = 7$ m, $y = 8$ m and $z = 9$ m. Find the unknown angles.

5. In ΔABC, $AB = 6.7$ cm, $BC = 3.8$ cm and $AC = 5.3$ cm. Find the size of the smallest angle.

6. In ΔPQR, $PQ = 7.8$ cm, $QR = 9.1$ cm and $PR = 4.9$ cm. Find the size of the largest angle.

INTERMEDIATE LEVEL

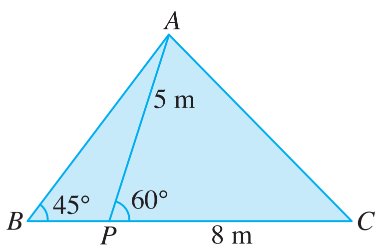
7. In the figure, the point B lies on AC such that $AB = 8$ m, $BD = 9$ m, $\angle ABD = 125°$ and $\angle BCD = 55°$.



Find

(i) the length of CD, **(ii)** the length of AD.

8. The figure shows the cross section of the roof of an old cottage. It is given that $AP = 5$ m, $PC = 8$ m, $\angle APC = 60°$ and $\angle ABC = 45°$.



Find

(i) the length of AB, **(ii)** the length of AC.

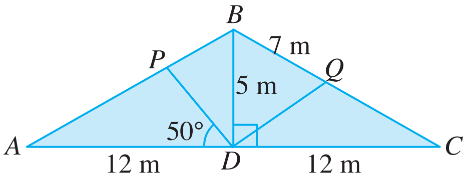
9. In ΔABC, $BC = 4$ cm. M is the midpoint of BC such that $AM = 4$ cm and $\angle AMB = 120°$. Find

(i) the length of AC,

(ii) the length of AB,

(iii) $\angle ACB$.

10. The figure shows the supports of the roof of a building in which $BD = 5$ m, $AD = CD = 12$ m, $BQ = 7$ m and $\angle PDA = 50°$.



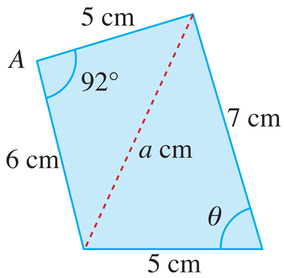
Find

(i) $\angle BAD$,

(ii) the length of the support PD,

(iii) the length of the support DQ.

11. The figure shows a quadrilateral with the dimensions as shown.

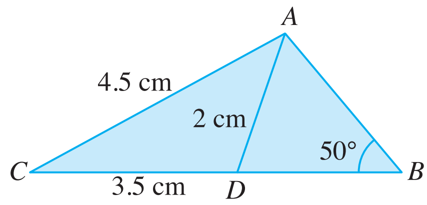


Find

(i) the value of a,

(ii) θ.

12. In the figure, D is a point on CB such that $AD = 2$ cm, $AC = 4.5$ cm, $CD = 3.5$ cm and $\angle ABD = 50°$.



Find

(i) $\angle ADB$,

(ii) the shortest distance from A to CB,

(iii) the length of BD.

13. On a map whose scale is 2 cm to 5 km, a farm is shown as a triangle XYZ. Given that $XY = 9$ cm, $YZ = 12$ cm and $XZ = 8$ cm, find

(i) the length, in km, which is represented by XZ,

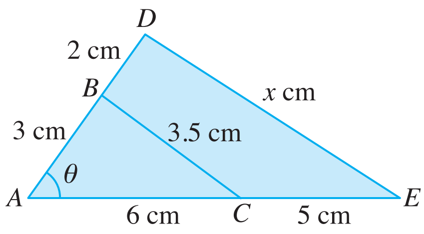
(ii) $\angle YXZ$,

(iii) the area, in km^2, which is represented by ΔXYZ.

14. In a trapezium $ABCD$, AB is parallel to DC, $AB = 4.5$ cm, $BC = 5$ cm, $CD = 7.5$ cm and $AD = 6$ cm. The point X lies on CD such that BX is parallel to AD. Find $\angle BCX$ and the length of BD.

ADVANCED LEVEL

15. The figure shows two triangles ABC and ADE.



(i) Determine if ΔADE is an enlargement of ΔABC.

(ii) Find the value of $\cos\theta$.

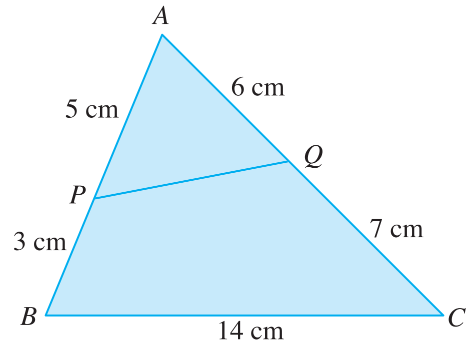
(iii) Hence, find the value of x.

16. In ΔABC, $AB = 8$ cm, $BC = 5$ cm and $CA = 6$ cm. BC is produced to R so that $CR = 3$ cm.

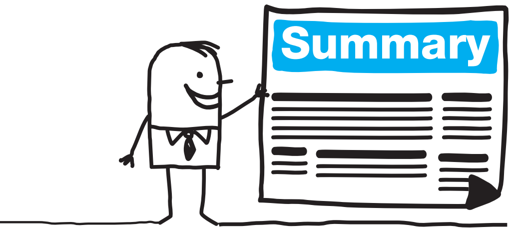
(i) Express $\cos\angle BCA$ in the form $\frac{p}{q}$, where p and q are integers.

(ii) Hence, find the length of AR.

17. In the figure, the point P lies on AB such that $AP = 5$ cm and $PB = 3$ cm. The point Q lies on AC such that $AQ = 6$ cm and $QC = 7$ cm.



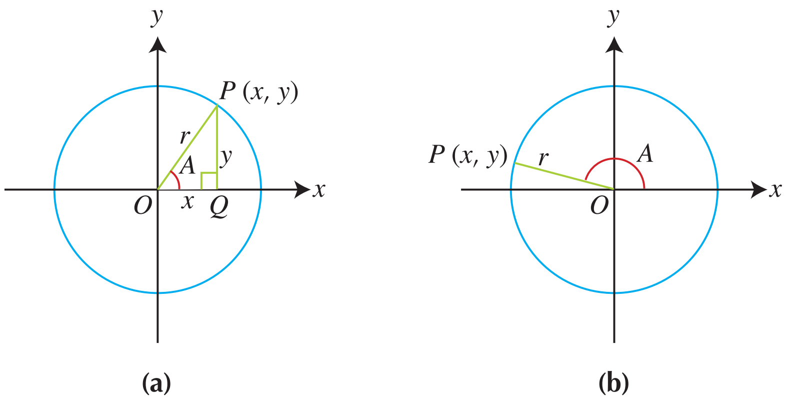
Find the length of PQ.



1. For any angle A, the sine and cosine of an angle A are defined as follows:

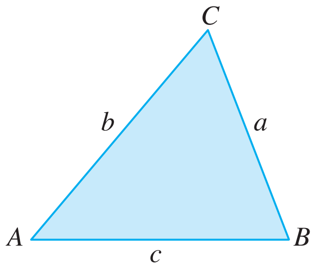
$$\sin A = \frac{y}{r} \text{ and } \cos A = \frac{x}{r},$$

where (x, y) are the coordinates of a point P on a circle with centre O and radius r as shown.



2. $\sin A = \sin (180° - A)$

$\cos A = -\cos (180° - A)$



3. Area of $\Delta ABC = \frac{1}{2} ab \sin C$

4. **Sine Rule**

In any ΔABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

The Sine Rule can be used to solve a triangle (i.e. find the unknown sides and angles) if the following are given:

- two angles and the length of one side; or
- the lengths of two sides and one **non-included** angle

5. **Cosine Rule**

In any ΔABC,

$$a^2 = b^2 + c^2 - 2bc \cos A \qquad \qquad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 = a^2 + c^2 - 2ac \cos B \qquad \text{or} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

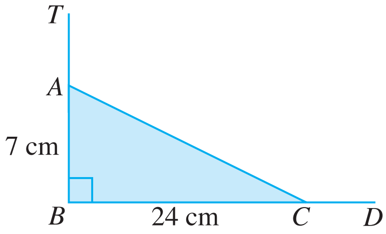
$$c^2 = a^2 + b^2 - 2ab \cos C \qquad \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

The Cosine Rule can be used to solve a triangle if the following are given:

- the lengths of all three sides; or
- the lengths of two sides and an **included** angle

Review Exercise 6

1. In the figure, A is a point on TB such that $AB = 7$ cm and C is a point on BD such that $BC = 24$ cm.



Given that $\angle TBD = 90°$, find

(a) AC,

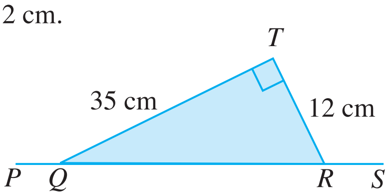
(b) the value of each of the following.

(i) $\tan \angle ACB$

(ii) $\cos \angle ACD$

(iii) $\sin \angle TAC$

2. In the figure, $PQRS$ is a straight line and $\angle QTR = 90°$. It is also given that $QT = 35$ cm and $TR = 12$ cm.



(a) Find the length of QR.

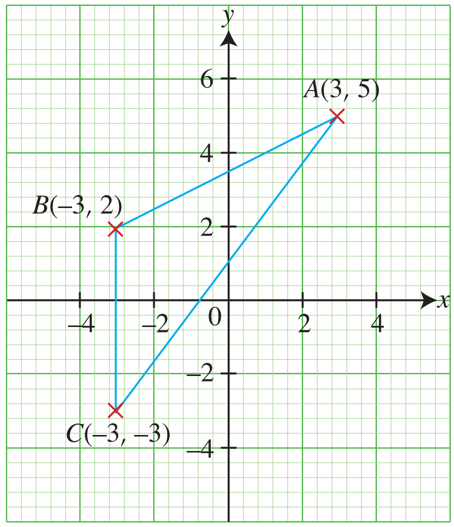
(b) Express each of the following as a fraction in its simplest form.

(i) $\sin \angle PQT$

(ii) $\cos \angle PQT$

(iii) $\tan \angle TQR + \tan \angle TRQ$

3. The figure shows ΔABC with vertices $A(3, 5)$, $B(-3, 2)$ and $C(-3, -3)$.



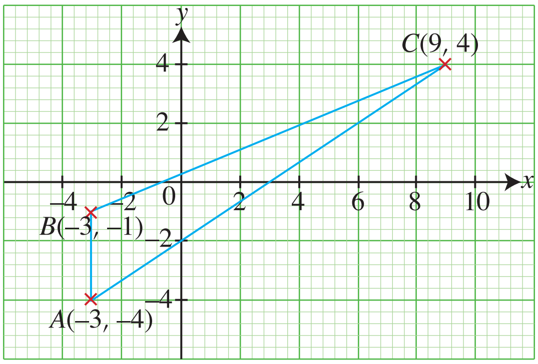
Find the value of each of the following.

(i) $\cos \angle ABC$

(ii) $\sin \angle ABC$

(iii) $\tan \angle ACB$

4. The figure shows the points $A(-3, -4)$, $B(-3, -1)$ and $C(9, 4)$.



Find

(i) the length of BC and of AC,

(ii) the value of $\sin \angle ABC$ and of $\cos \angle ABC$,

(iii) the area of ΔABC,

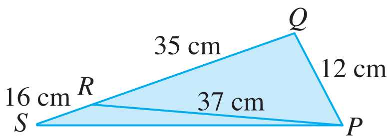
(iv) the length of the perpendicular from B to AC.

5. Given that $0° < x < 180°$, find the possible values of x for each of the following equations.

(a) $\sin x = 0.419$ **(b)** $\cos x = 0.932$

(c) $\tan x = 0.503$ **(d)** $\cos x = -0.318$

6. In ΔPQR, $PQ = 12$ cm, $QR = 35$ cm and $PR = 37$ cm.



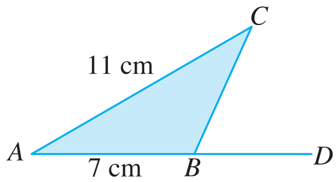
(a) Explain why $\angle PQR$ is a right angle.

(b) QR is produced to S such that $RS = 16$ cm. Find

(i) the value of $\cos \angle PRS$ and of $\sin \angle PRS$,

(ii) $\angle RPS$.

7. In the figure, ABD is a straight line, $AB = 7$ cm, $AC = 11$ cm and $\sin \angle CBD = \frac{3}{4}$.



Find

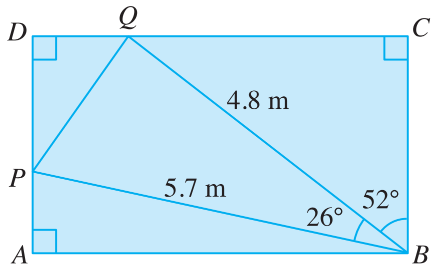
(i) $\sin \angle ACB$, giving your answer as a fraction in its simplest form,

(ii) $\angle BAC$,

(iii) the area of ΔABC,

(iv) the length of BC.

8. In the figure, $ABCD$ is a rectangular hoarding and PQ, BP and BQ are three pieces of wood nailed at the back to support the hoarding.



Given that $BP = 5.7$ m, $BQ = 4.8$ m, $\angle PBQ = 26°$ and $\angle CBQ = 52°$, find

(i) the width of the hoarding, BC,

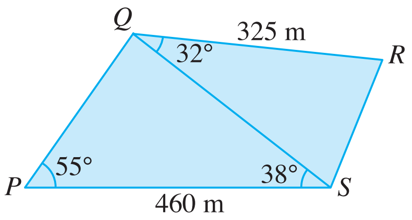
(ii) the length of AP,

(iii) the area enclosed by the three pieces of wood, ΔPBQ,

(iv) the length of PQ,

(v) $\angle BPQ$.

9. The figure shows a park $PQRS$ where $PS = 460$ m, $QR = 325$ m, $\angle PSQ = 38°$, $\angle QPS = 55°$ and $\angle RQS = 32°$.



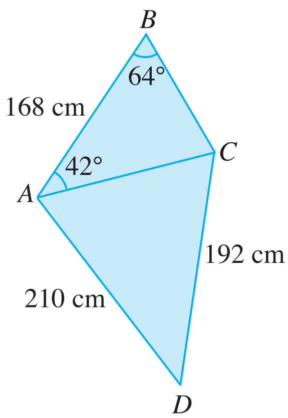
(i) Find the length of QS.

(ii) Find the length of RS.

(iii) What is the shortest distance between Q and PS?

(iv) Find the area occupied by the park.

10. Four children are standing in the field at the points A, B, C and D, playing a game of 'Catch Me'. It is given that $AB = 168$ cm, $AD = 210$ cm, $CD = 192$ cm, $\angle BAC = 42°$ and $\angle ABC = 64°$.

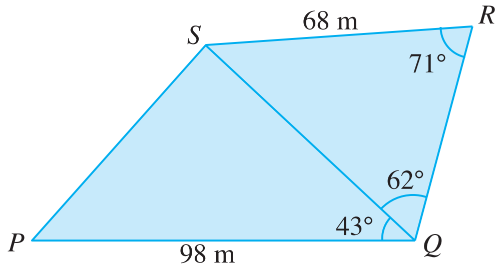


(i) Given that Huixian, who is standing at A, runs towards Lixin who is standing at C, find the distance AC that Huixian has to run to reach Lixin.

(ii) Find $\angle ADC$.

(iii) Calculate the area of $ABCD$, giving your answer in m^2.

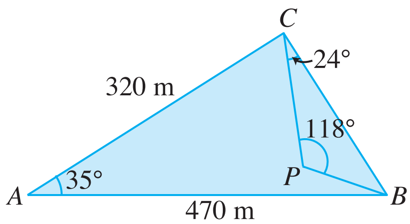
11. The figure shows a quadrilateral $PQRS$. Given that $PQ = 98$ m, $RS = 68$ m, $\angle SQR = 62°$, $\angle PQS = 43°$ and $\angle SRQ = 71°$, find



(i) the area of ΔSQR,

(ii) the length of PS.

12. The figure shows a triangular park ABC where $AB = 470$ m, $AC = 320$ m and $\angle BAC = 35°$. P is a lamp post inside the park such that $\angle BPC = 118°$ and $\angle PCB = 24°$.

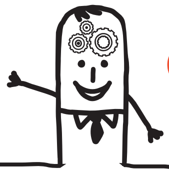


Find

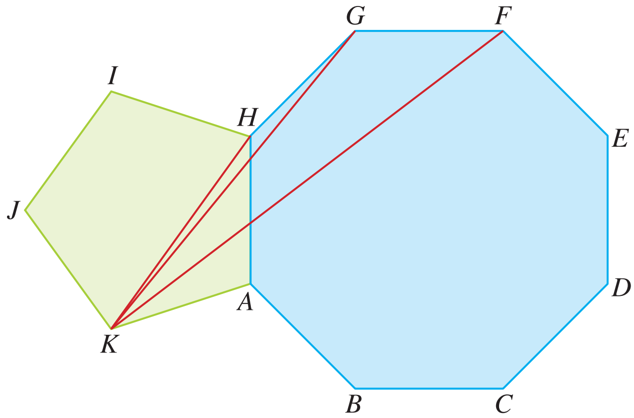
(i) the area of the park,

(ii) the distance between B and C,

(iii) the distance between P and C.



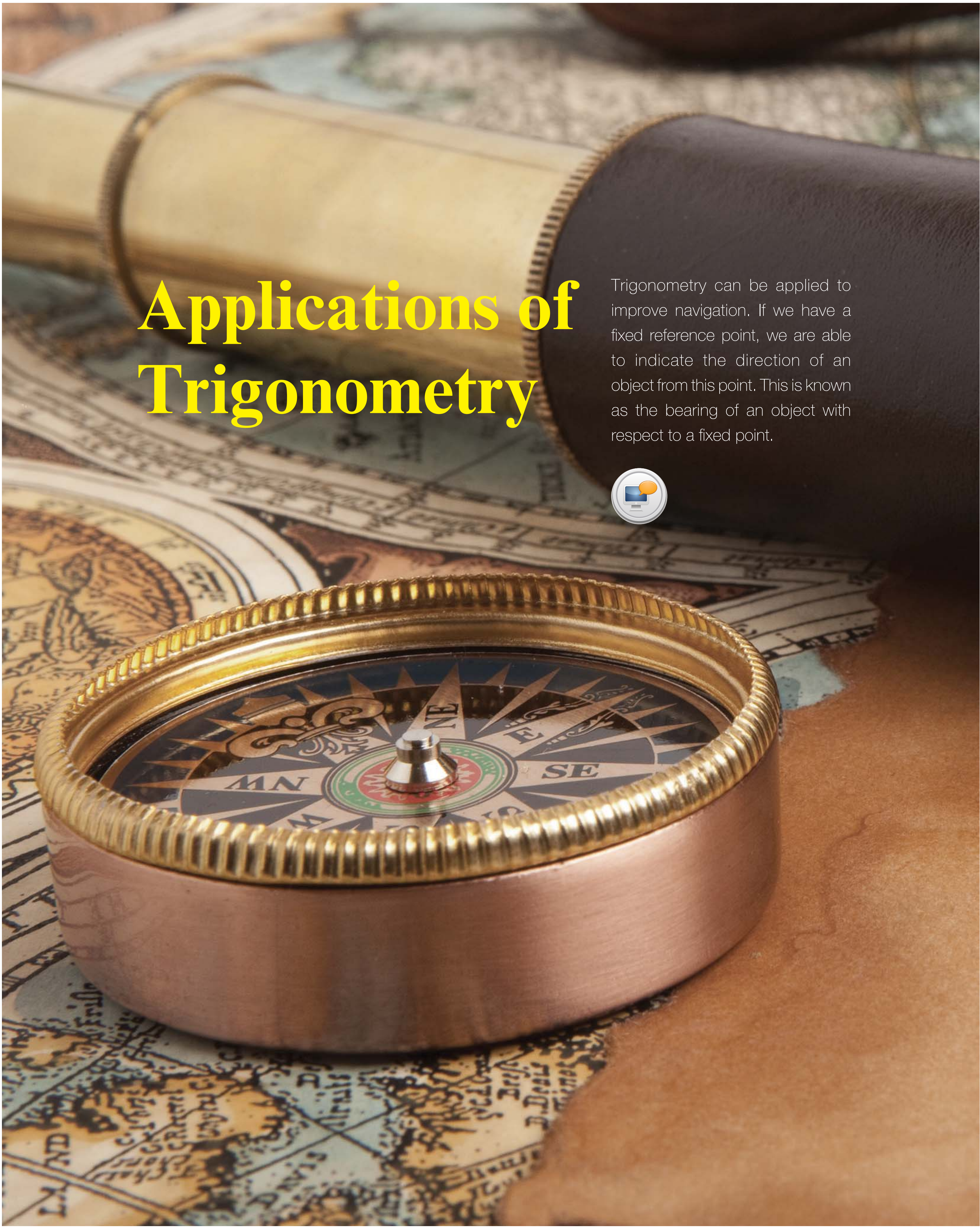
Challenge Yourself



In the figure, *ABCDEFGH* is a regular octagon with sides 8 cm and *AHIJK* is a regular pentagon. Find

(a) the length of *HK* and of *GK*,

(b) the area of ΔFGK.



Applications of Trigonometry

Trigonometry can be applied to improve navigation. If we have a fixed reference point, we are able to indicate the direction of an object from this point. This is known as the bearing of an object with respect to a fixed point.

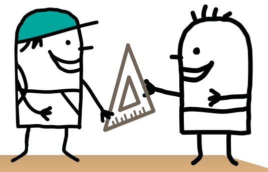
Chapter Seven

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- solve simple practical problems in two and three dimensions including those involving angles of elevation and depression and bearings.

7.1 Angles of Elevation and Depression



Recap

In Book 2, we have learnt that trigonometry can be used to find the heights of buildings and mountains. We have also learnt that a clinometer may be used to obtain the angle of elevation of the top of an object, as shown in Fig. 7.1.

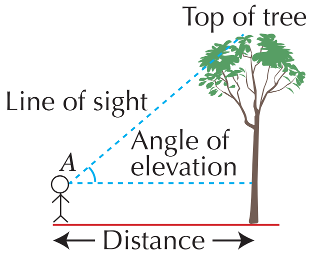


Fig. 7.1

In this section, we will learn about angles of elevation and depression and how they may be used to solve simple problems.

Angles of Elevation and Depression

Fig. 7.2 shows Khairul standing in front of a vertical wall BC. A is the point where his eyes are and AD is an imaginary horizontal line from his eyes to the wall.

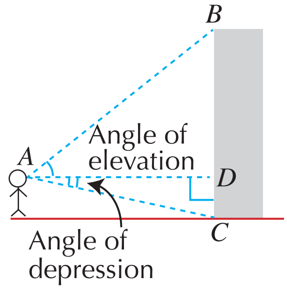


Fig. 7.2

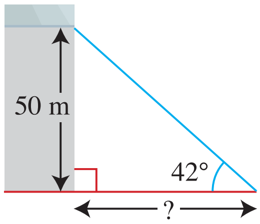
When Khairul looks at the top of the wall, B, the angle between the horizontal AD and the line of sight AB, i.e. $\angle BAD$, is called the **angle of elevation**.

When Khairul looks at the bottom of the wall, C, the angle between the horizontal AD and the line of sight AC, i.e. $\angle CAD$, is called the **angle of depression**.

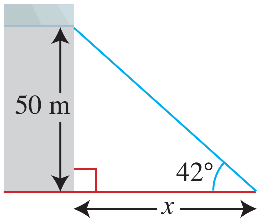
Worked Example 1

(Problem involving Angle of Elevation)

A window of a building is 50 m above the ground. Given that the angle of elevation of the window from a point on the ground is 42°, find the distance of the point on the ground from the foot of the building.



Solution:



Let x m be the distance of the point on the ground from the foot of the building.

$$\tan 42° = \frac{50}{x}$$

$$x \tan 42° = 50$$

$$x = \frac{50}{\tan 42°}$$

$$= 55.5 \text{ (to 3 s.f.)}$$

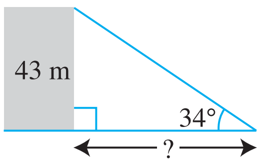
∴ The distance of the point on the ground from the foot of the building is 55.5 m.

PRACTISE NOW 1

SIMILAR QUESTIONS

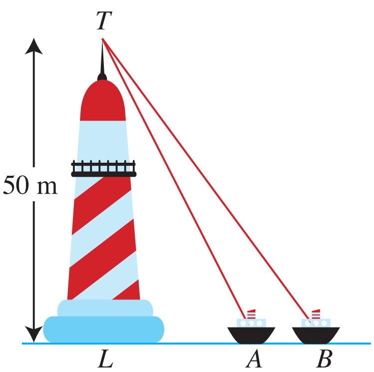
Exercise 7A Questions 1-3, 7-10, 13

1. The angle of elevation of the top of an office tower of height 43 m from a point on level ground is 34°. Find the distance of the point on the ground from the foot of the tower.



2. A lighthouse TL has a height of 50 m. The angles of elevation of the top of the lighthouse T from boat A and boat B are 48° and 38° respectively.

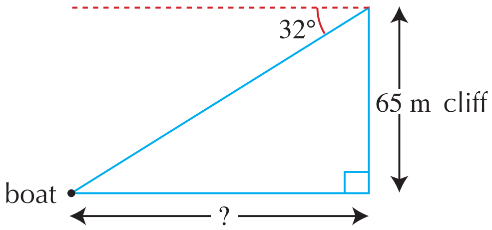
Find the distance between boats A and B.



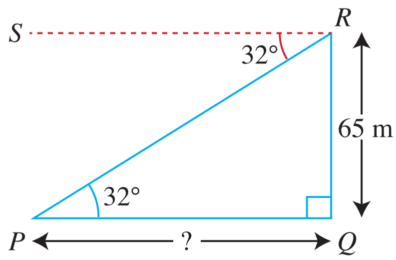
Worked Example 2

(Problem involving Angle of Depression)

A cliff is 65 m high. Given that the angle of depression of a boat from the top of the cliff is 32°, find the distance between the boat and the base of the cliff.



Solution:



Method 1:

$\angle RPQ = 32°$ (alt. ∠s, SR // PQ)

$\tan 32° = \dfrac{65}{PQ}$

$\therefore PQ = \dfrac{65}{\tan 32°}$

$= 104$ m (to 3 s.f.)

Method 2:

$\angle PRQ + 32° = 90°$ ($\angle QRS$ is a right angle.)

$\therefore \angle PRQ = 90° - 32°$

$= 58°$

$\tan 58° = \dfrac{PQ}{65}$

$\therefore PQ = 65 \tan 58°$

$= 104$ m (to 3 s.f.)

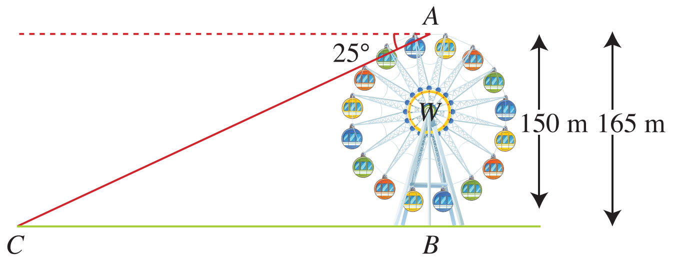
∴ The distance between the boat and the base of the cliff is 104 m.

PRACTISE NOW 2

SIMILAR QUESTIONS

Exercise 7A Questions 4-6, 11, 12, 14

1. The Singapore Flyer is an iconic giant observation wheel built on top of a terminal building. The diameter of the wheel is approximately 150 m and the highest point of the wheel is about 165 m above the ground. From the point A at the top of wheel, Nora observes that the angle of depression of a sports car C on the ground is 25°.

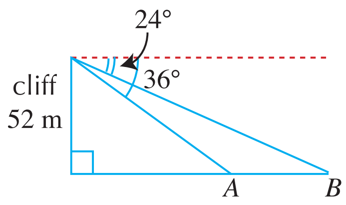


Find

(i) the distance of the sports car from a point B which is on ground level directly below A,

(ii) the angle of depression of the sports car from the centre of the wheel, W.

2. From the top of a cliff 52 m high, the angles of depression of two ships A and B due east of it are 36° and 24° respectively.

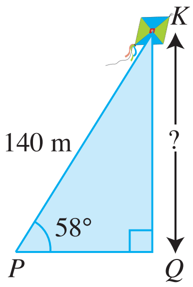


Calculate the distance between the two ships.

Exercise 7A

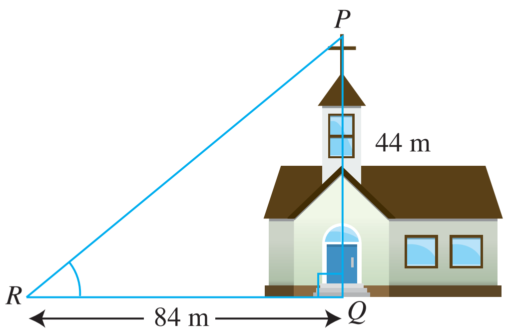
BASIC LEVEL

1. Lixin, standing at P, is flying a kite attached to a string of length 140 m. The angle of elevation of the kite K from her hand is 58°. Assuming that the string is taut, find the height of the kite above her hand.



2. Two buildings on level ground are 120 m and 85 m tall respectively. Given that the angle of elevation of the top of the taller building from the top of the shorter building is 33.9°, find the distance between the two buildings.

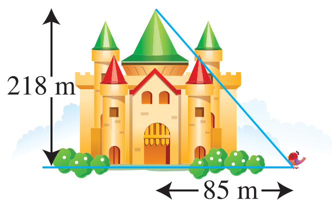
3. At a certain time in a day, a church spire PQ, 44 m high, casts a shadow RQ, 84 m long. Find the angle of elevation of the top of the spire from the point R.



4. A building is 41 m high. Given that the angle of depression of a fire hydrant from the top of the building is 33°, find the distance between the fire hydrant and the foot of the building.

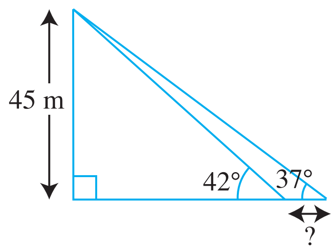
5. A boat is 65.7 m away from the base of the cliff. Given that the angle of depression of the boat from the top of the cliff is 28.9°, find the height of the cliff.

6. A castle has a height of 218 m. Given that an unusual bird is 85 m away from the foot of the castle, find the angle of depression of the bird from the top of the castle.



INTERMEDIATE LEVEL

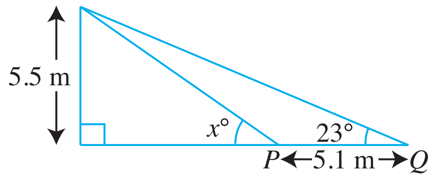
7. A clock tower has a height of 45 m. The angles of elevation of the top of the clock tower from two points on the ground are 42° and 37° respectively.



Find the distance between the two points.

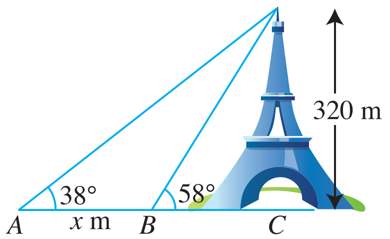
8. A castle stands on top of a mountain. At a point on level ground which is 55 m away from the foot of the mountain, the angles of elevation of the top of the castle and the top of the mountain are 60° and 45° respectively. Find the height of the castle.

9. An overhead bridge has a height of 5.5 m. The angles of elevation of the top of the bridge from two points P and Q on the ground are $x°$ and 23° respectively.



Given that the distance between P and Q is 5.1 m, find the value of x.

10. The Eiffel Tower in Paris has a height of 320 m. When Kate stands at the point A, the angle of elevation of the top of the tower is 38°. Kate walks x metres to a point B and observes that the angle of elevation of the top of the tower is now 58°.



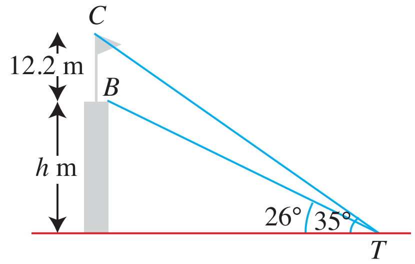
Find the value of x.

11. From the top of a cliff 88 m high, the angles of depression of two boats due west of it are 23° and 18° respectively. Calculate the distance between the two boats.

12. A satellite dish stands at the top of a cliff. From the top of the satellite dish, the angle of depression of a ship which is 80 m away from the base of the cliff is 37°. From the foot of the satellite dish, the angle of depression of the same ship is 32°. Find the height of the satellite dish.

ADVANCED LEVEL

13. A flagpole of height 12.2 m is placed on top of a building of height h metres. From a point T on level ground, the angle of elevation of the base of the flagpole B is 26° and the angle of elevation of the top of the flagpole C is 35°.



Find the value of h.

14. A tower with a height of 27 m stands at the top of a cliff. From the top of the tower, the angle of depression of a guard house is 56°. From the foot of the tower, the angle of depression of the same guard house is 49°. Find

(i) the distance between the base of the cliff and the guard house,

(ii) the height of the cliff.

7.2 Bearings

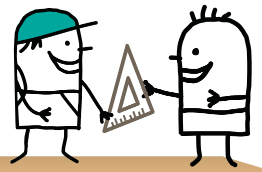


Fig. 7.3 shows the positions of four points A, B, C and D relative to an origin O. N, E, S and W represent the directions north, east, south and west from O respectively.

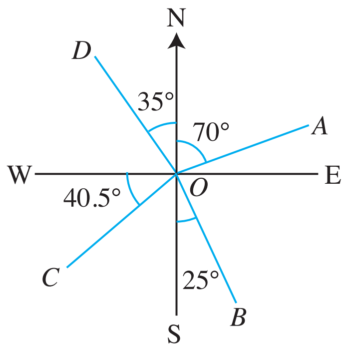


Fig. 7.3

The bearing of A from O is an angle measured from the north, at O, in a clockwise direction and is always written as a three-digit number. Hence, the bearing of A from O is 070°.

The bearing of B from O is equal to 180° – 25°. Hence, the bearing of B from O is 155°.

The bearing of C from O is equal to 270° – 40.5°. Hence, the bearing of C from O is 229.5°.

What is the bearing of D from O?

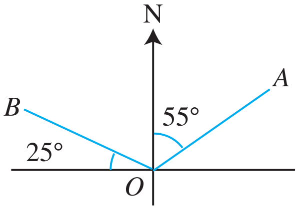
The bearing of N is taken as 000° or 360°. The bearing of E from O is 090°. Similarly, the bearing of S from O is 180°. What is the bearing of W from O?

When reading compass bearings, directions are usually measured from either the north or the south. For example, 070° is written as N70°E and 210° is written as S30°W. When reading true bearings, directions are given in terms of the angles measured clockwise from the north.

Worked Example 3

(Finding the Bearing)

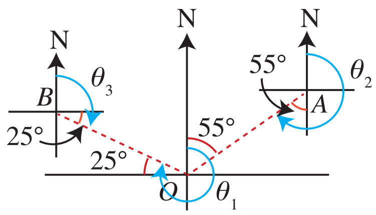
Find the bearing of



(a) A from O, **(b)** B from O,
(c) O from A, **(d)** O from B.

Solution:

(a) Bearing of A from O is 055°



(b) The bearing of B from O is given by the reflex angle θ_1, which is (270° + 25°).

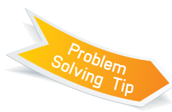
$\therefore$ Bearing of B from O is 295°

(c) The bearing of O from A is given by the reflex angle θ_2, which is (180° + 55°).

$\therefore$ Bearing of O from A is 235°

(d) The bearing of O from B is given by the obtuse angle θ_3, which is (90° + 25°).

$\therefore$ Bearing of O from B is 115°



For **(c)**, follow the steps as shown.

Step 1: Draw the north line from A.

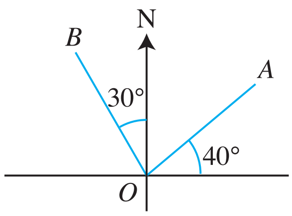
Step 2: Draw the angle clockwise from the north line to OA.

Step 3: Find the angle θ_2.

PRACTISE NOW 3

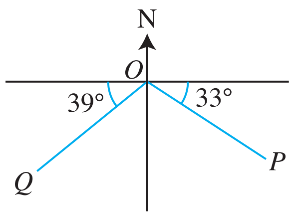
1. Find the bearing of

(a) A from O, **(b)** B from O,
(c) O from A, **(d)** O from B.



2. Find the bearing of

(a) P from O, **(b)** Q from O,
(c) O from P, **(d)** O from Q.



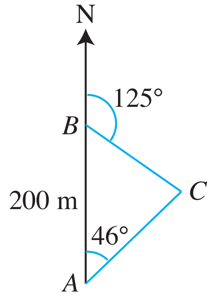
SIMILAR QUESTIONS

Exercise 7B Questions 1-3, 6-8

Worked Example 4

(Problem involving Bearings)

Three points A, B and C are on level ground such that B is due north of A, the bearing of C from A is 046° and the bearing of C from B is 125°. Given that the distance between A and B is 200 m, find the distance of C from A.



Solution:

Since the bearing of C from B is 125°,

$\angle ABC = 180° - 125° = 55°$

i.e. $\angle ACB = 180° - 46° - 55°$ ($\angle$ sum of a Δ)

$= 79°$

Using sine rule,

$$\frac{AC}{\sin 55°} = \frac{200}{\sin 79°}$$

$$AC = \frac{200 \sin 55°}{\sin 79°}$$

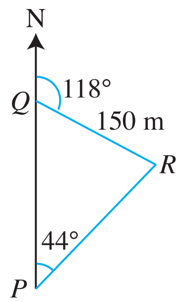
$= 167$ m (to 3 s.f.)

PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 7B Questions 4, 5, 9, 14

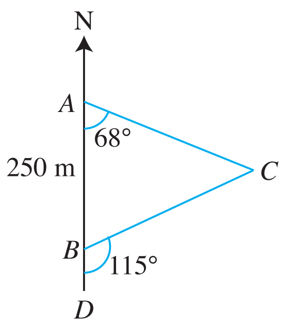
1. Three points P, Q and R are on level ground such that P is due south of Q, the bearing of R from Q is 118° and the bearing of R from P is 044°. Given that the distance between Q and R is 150 m, find the distance of P from Q.



2. In the figure, A, B and C are points of an amusement park on level ground, with A due north of B. Given that $\angle BAC = 68°$, $AB = 250$ m and $\angle CBD = 115°$, find

(i) the bearing of B from C,

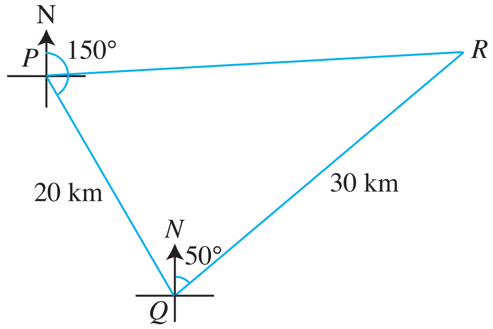
(ii) the length of AC and of BC.



Worked Example 5

(Problem involving Bearings)

A boat sailed 20 km from a point P to an island Q, on a bearing of 150°. It then sailed another 30 km on a bearing of 50° to a lighthouse R. Find the distance and the bearing of the lighthouse from P.



Solution:

$\angle PQR = 30° + 50° = 80°$

Using cosine rule,

$$PR^2 = 20^2 + 30^2 - 2 \times 20 \times 30 \times \cos 80°$$
$$= 1092 \text{ (to 4 s.f.)}$$
$$PR = \sqrt{1092}$$
$$= 33.0 \text{ km (to 3 s.f.)}$$

∴ The lighthouse is 33.0 km away from P.

Problem Solving Tip

To find distance PR using cosine rule, we need to find $\angle PQR$ first.

Using sine rule,

$$\frac{\sin \angle QPR}{30} = \frac{\sin 80°}{33.05}$$
$$\sin \angle QPR = \frac{30 \sin 80°}{33.05}$$
$$= 0.8940 \text{ (to 4 s.f.)}$$
$$\angle QPR = \sin^{-1} 0.8940$$
$$= 63.39° \text{ (to 2 d.p.)}$$

$150° - 63.39° = 86.61°$ (to 2 d.p.)

∴ Bearing of the lighthouse from P is 086.6°

Problem Solving Tip

To find the bearing of R from P, we need to find $\angle QPR$ first.

PRACTISE NOW 5

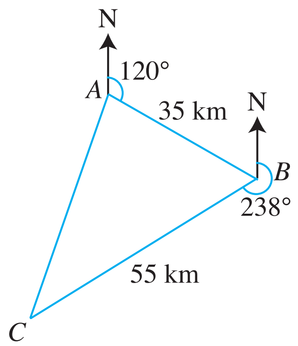
SIMILAR QUESTIONS

Exercise 7B Questions 10, 11

The figure shows three towns on level ground. Given that the bearing of B from A is 120°, the bearing of C from B is 238°, AB = 35 km and BC = 55 km, find

(i) the distance between towns A and C,

(ii) the bearing of town C from town A.



Worked Example 6

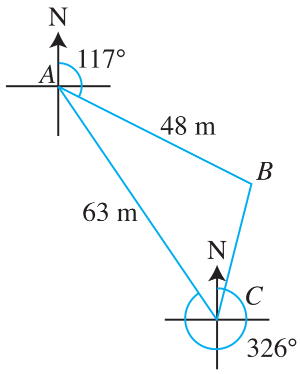
(Problem involving Bearings)

The figure shows three points on level ground. The bearing of B from A is 117°, the bearing of A from C is 326°, AB = 48 m and AC = 63 m. Calculate

(i) the length of BC,

(ii) the bearing of B from C,

(iii) the shortest distance from B to AC.



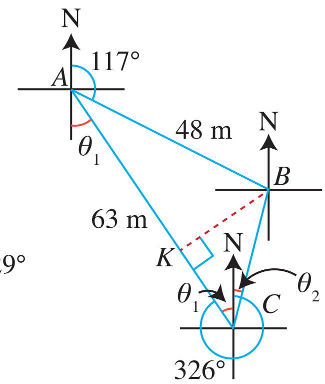
Solution:

(i) $\theta_1 = 360° - 326°$ (∠s at a point)

$= 34°$

i.e. $\angle BAC = 180° - 117° - 34°$ (adj. ∠s on a str. line)

$= 29°$



Using cosine rule,

$$BC^2 = 48^2 + 63^2 - 2 \times 48 \times 63 \times \cos 29°$$

$$= 983.3 \text{ (to 4 s.f.)}$$

$$BC = \sqrt{983.3}$$

$$= 31.4 \text{ km (to 3 s.f.)}$$

Problem Solving Tip

Since we are given two sides of the triangle (AB and AC), in order to find the third side (BC), we first have to find $\angle BAC$.
From the bearings given in the question, how can we find $\angle BAC$?

(ii) Using sine rule,

$$\frac{\sin \angle ACB}{48} = \frac{\sin 29°}{31.36}$$

$$\sin \angle ACB = \frac{48 \sin 29°}{31.36}$$

$$= 0.7421 \text{ (to 4 s.f.)}$$

$$\angle ACB = \sin^{-1} 0.7421$$

$$= 47.91° \text{ (to 2 d.p.)}$$

$\theta_2 = 47.91° - \theta_1$

$= 13.9°$ (to 1 d.p.)

$\therefore$ Bearing of B from C is 013.9°

Problem Solving Tip

To find the bearing of B from C, i.e. θ_2, which angles do we need to find?

(iii) The shortest distance from B to AC is BK, where BK is perpendicular to AC. In ΔABK,

$$\sin \angle BAC = \frac{BK}{48}$$

$$BK = 48 \sin 29°$$

$$= 23.3 \text{ m (to 3 s.f.)}$$

Problem Solving Tip

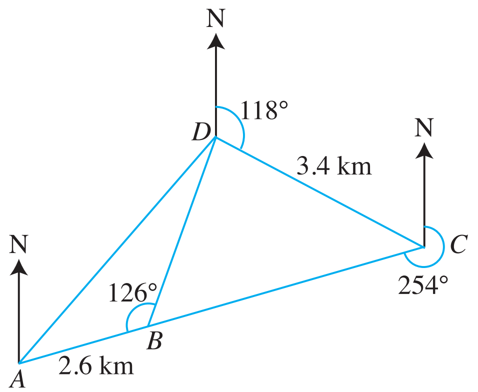
What is the shortest distance from a point to a line?

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 7B Questions 12, 13, 15

A, B, C and D are four points on level ground. The bearing of C from D is 118° and the bearing of A from C is 254°.



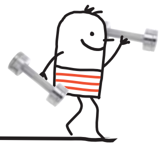
Given that $AB = 2.6$ km, $CD = 3.4$ km, $\angle ABD = 126°$ and ABC is a straight line, find

(i) the bearing of B from D,

(ii) the distance between B and D,

(iii) the distance between A and D,

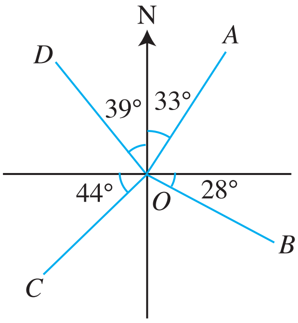
(iv) the shortest distance from B to AD.



Exercise 7B

BASIC LEVEL

1. The figure shows the positions of O, A, B, C and D.

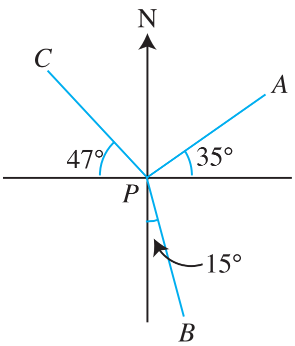


Find the bearing of

(a) A from O, **(b)** B from O,

(c) C from O, **(d)** D from O.

2. The figure shows the positions of P, A, B and C.



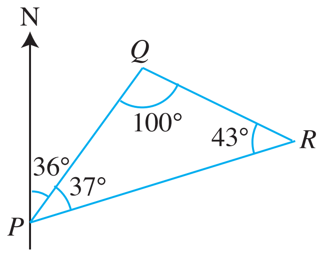
Find the bearing of

(a) A from P, **(b)** B from P,

(c) C from P, **(d)** P from A,

(e) P from B, **(f)** P from C.

3. The figure shows the positions of P, Q and R.



Find the bearing of

(a) Q from P, **(b)** P from Q,

(c) R from P, **(d)** P from R,

(e) Q from R, **(f)** R from Q.

4. A point Q is 24 km from P and on a bearing of 072° from P. From Q, Vishal walks at a bearing of 320° to a point R, located directly north of P. Find

(a) the distance between P and R,

(b) the distance between Q and R.

5. A petrol kiosk P is 12 km due north of another petrol kiosk Q. The bearing of a police station R from P is 135° and that from Q is 120°. Find the distance between P and R.

INTERMEDIATE LEVEL

6. A, B, C and D are the four corners of a rectangular plot marked out on level ground. Given that the bearing of B from A is 040° and that the bearing of C from A is 090°, find the bearing of

(a) B from C, **(b)** A from C,

(c) D from C.

7. P, Q and R are three points on level ground. Given that the bearing of R from P is 135°, $\angle PQR = 55°$ and $\angle PRQ = 48°$, find the bearing of

(a) P from R, **(b)** Q from R,

(c) P from Q.

8. A, B and C are three points on level ground. Given that the bearing of B from A is 122°, $\angle CAB = 32°$ and $\angle ABC = 86°$, find the possible bearing(s) of C from B.

9. A bus stop is 280 m due north of a taxi stand. Nora walks from the taxi stand in the direction 050°. Calculate how far she has to walk before she is

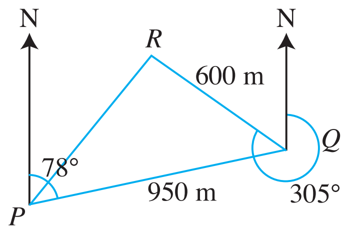
(a) equidistant from the bus stop and the taxi stand,

(b) as close as possible to the bus stop,

(c) due east of the bus stop.

10. A helicopter flies 30 km from a point P to another point Q on a bearing of 128°. It then flies another 25 km to a point R on a bearing of 295°. Find the distance between P and R.

11. In the figure, P, Q and R are points on level ground in a park. R is 600 m from Q and on a bearing of 305° from Q. Q is 950 m from P and on a bearing of 078° from P.

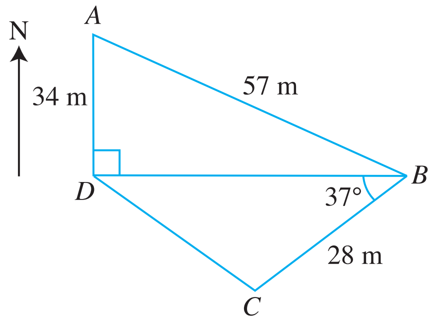


Find

(i) the distance between P and R,

(ii) the bearing of R from P.

12. The figure shows four points on level ground. A is due north of D, B is due east of D and $\angle DBC = 37°$.



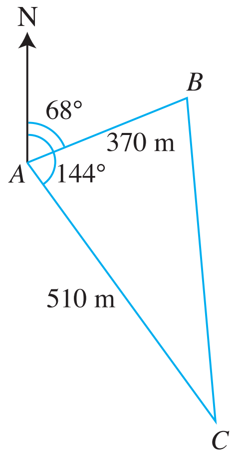
Given that $AD = 34$ m, $AB = 57$ m and $BC = 28$ m, find

(i) the bearing of B from A,

(ii) the shortest distance from C to BD,

(iii) the bearing of D from C.

13. A, B and C are three points on level ground. The bearing of B from A is 068° and the bearing of C from A is 144°.



Given that $AB = 370$ m and $AC = 510$ m, find

(i) the distance between B and C,

(ii) $\angle ACB$,

(iii) the bearing of C from B,

(iv) the shortest distance from A to BC.

ADVANCED LEVEL

14. Two cruise ships P and Q leave the port at the same time. P sails at 10 km/h on a bearing of 030° and Q sails at 12 km/h on a bearing of 300°. Find their distance apart and the bearing of P from Q after 2 hours.

15. P, Q and R represent three ports. Q is 35 km from P and on a bearing of 032° from P. R is 65 km from P and on a bearing of 108° from P.

(a) Find

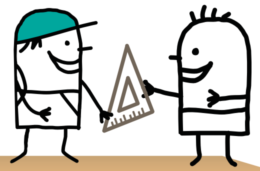
(i) the distance between Q and R,

(ii) the bearing of R from Q.

A ship sets sail at 0930 from P directly to R at an average speed of 30 km/h and reaches a point S due south of Q.

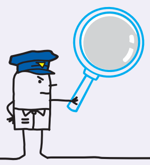
(b) Find the time when it reaches S.

7.3 Three-Dimensional Problems



A plane is a flat surface like the floor or the surface of a whiteboard. It has two dimensions (2D) – length and breadth.

A solid has three dimensions (3D) – length, breadth and height/thickness/depth.



Investigation

Visualising 3D Solids

1. Look at your school desk or table. It has a rectangular top (ignore the rounded corners, if any). Fig. 7.4 shows a photo of a school desk viewed from the top. Measure the angles of the two corners of the rectangular top, $\angle ABC$ and $\angle BCD$. Do you get 90° for both angles?

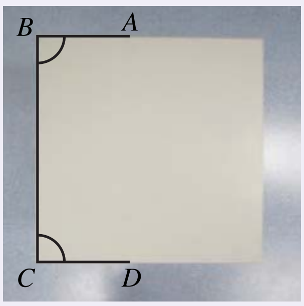
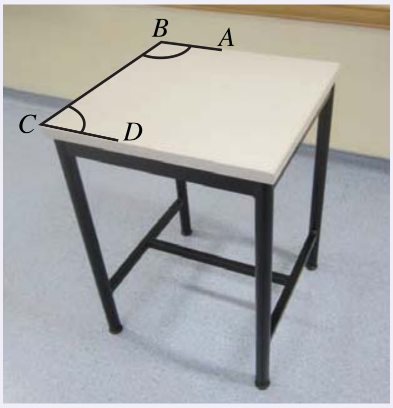
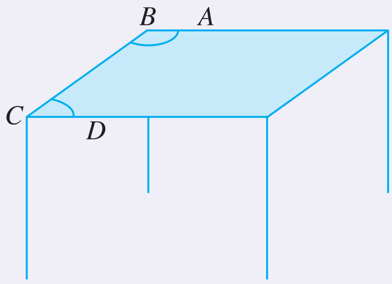


Fig. 7.4

2. Fig. 7.5**(a)** shows a photo of the same desk viewed from the side. Measure $\angle ABC$ and $\angle BCD$ again. Do you get 90°, smaller than 90°, or larger than 90°?



(a)



(b)

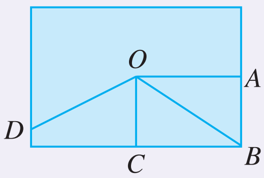
Fig. 7.5

In other words, drawing a 3D solid or object on a flat surface **may make a right angle look smaller or larger than 90°**.

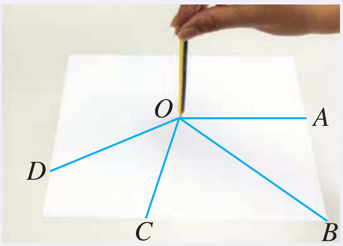
3. Fig. 7.6**(a)** shows a plane with a few lines drawn on it.

Place a pencil perpendicular to the plane in Fig. 7.6**(a)**, as shown in Fig. 7.6**(b)**.

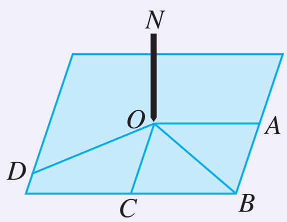
Use a set square to check whether the pencil is perpendicular to every line on the plane in Fig. 7.6**(a)**.



(a)



(b)



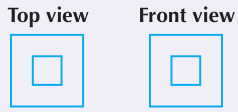
(c)

Fig. 7.6

Your pencil is called a **normal** to the plane since it is perpendicular to every line on the plane.

4. In Fig. 7.6**(c)**, $\angle NOA$ looks like it is a 90° angle, but $\angle NOB$ does not look like a 90° angle. Is $\angle NOB = 90°$? Explain your answer.

The following figures show the top view and the front view of a structure.



Notice that there are no hidden lines. Draw the side view of the structure.

5. Fig. 7.7 shows a cuboid. Dotted lines represent lines that are hidden, i.e. you cannot see them from the front.

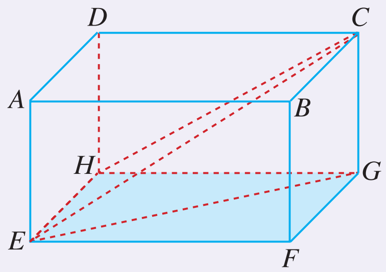


Fig. 7.7

There are two methods to determine whether a triangle in the above cuboid is a right-angled triangle.

Method 1: Find a rectangle

To determine whether ΔEFG and ΔCGH are right-angled triangles:

(a) ΔEFG lies on the plane $EFGH$. Is the plane $EFGH$ a rectangle? Explain your answer.

(b) Thus, is $\angle EFG = 90°$ and ΔEFG a right-angled triangle? Explain your answer.

(c) Using the same method as above, determine whether ΔCGH is a right-angled triangle by identifying the appropriate rectangle and the right angle of the triangle.

Method 2: Find a normal to a plane

To determine whether ΔCGE and ΔCHE are right-angled triangles:

(a) $EFGH$ is a horizontal plane. Is the vertical line CG a normal to the plane $EFGH$? Explain your answer.

(b) Is the line GE a line on the plane $EFGH$?

(c) Thus, is $\angle CGE = 90°$ and ΔCGE a right-angled triangle? Explain your answer.

(d) Using the same method as above, determine whether ΔCHE is a right-angled triangle by shading the appropriate plane, and identifying the corresponding normal and the right angle of ΔCHE.

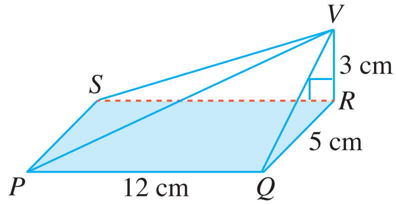
Problem Solving Tip

A normal to a plane is perpendicular to every line on the plane.

Worked Example 7

(Three-dimensional Problem)

The figure shows a pyramid with a rectangular base $PQRS$ and vertex V vertically above R.



Given that $PQ = 12$ cm, $QR = 5$ cm and $VR = 3$ cm, find

(i) $\angle VQR$, **(ii)** $\angle VPR$.

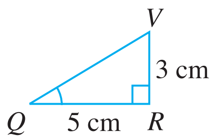
Solution:

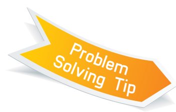
(i) In ΔVQR,

$$\tan \angle VQR = \frac{3}{5}$$

$$\angle VQR = \tan^{-1}\frac{3}{5}$$

$$= 31.0° \text{ (to 1 d.p.)}$$





The basic technique used in solving a three-dimensional problem is to reduce it to a problem in a plane.

(ii) In ΔPQR, $\angle PQR = 90°$.

Using Pythagoras' Theorem,

$$PR^2 = PQ^2 + QR^2$$

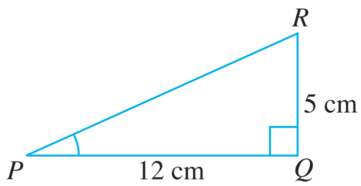
$$= 12^2 + 5^2$$

$$= 144 + 25$$

$$= 169$$

$$PR = \sqrt{169}$$

$$= 13 \text{ cm}$$

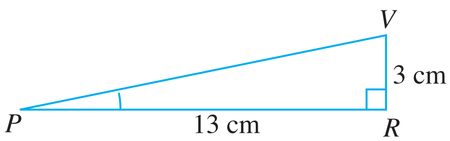


In ΔVRP, $\angle VRP = 90°$ (VR is the normal to the plane $PQRS$).

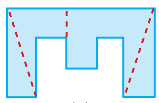
$$\tan \angle VPR = \frac{3}{13}$$

$$\angle VPR = \tan^{-1}\frac{3}{13}$$

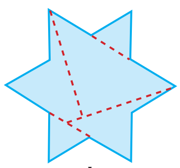
$$= 13.0° \text{ (to 1 d.p.)}$$



Two shapes are as shown.



(a)



(b)

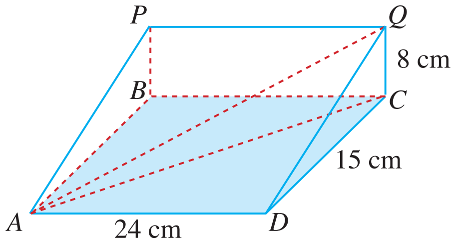
Cut each shape along the dotted lines and use the pieces to form two squares.

PRACTISE NOW 7

SIMILAR QUESTIONS

Exercise 7C Questions 1, 6, 7, 14

1. The figure shows a wedge with a horizontal base $ABCD$ and a vertical face $PQCB$. $APQD$ is a rectangular sloping surface and ΔABP and ΔDCQ are right-angled triangles in the vertical plane.



Given that $CQ = BP = 8$ cm, $DC = AB = 15$ cm and $AD = BC = PQ = 24$ cm, find

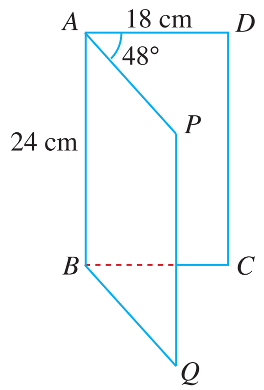
(i) $\angle BAC$, **(ii)** $\angle AQC$, **(iii)** $\angle CDQ$.

2. The figure shows a photo frame which can be opened about AB. $ABCD$ and $ABQP$ are rectangles. The frame is opened through 48° as shown.

Given that $AB = 24$ cm and $AP = AD = 18$ cm, find

(i) the length of the straight line CQ,

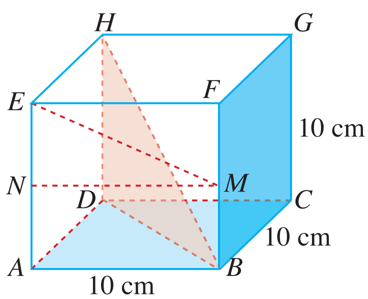
(ii) $\angle CAQ$.



Worked Example 8

(Three-dimensional Problem)

The figure shows a cube of length 10 cm. M and N are the midpoints of BF and AE respectively.



Find

(i) $\angle MEN$, **(ii)** $\angle EMN$,

(iii) $\angle MEF$, **(iv)** $\angle HBD$.

Solution:

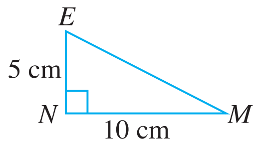
(i) In ΔMEN,

$\tan \angle MEN = \dfrac{10}{5}$

$= 2$

$\angle MEN = \tan^{-1} 2$

$= 63.4°$ (to 1 d.p.)



(ii) In ΔMEN,

$\tan \angle EMN = \dfrac{5}{10}$

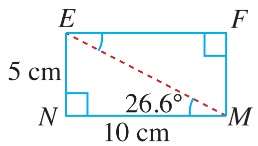
$= \dfrac{1}{2}$

$\angle EMN = \tan^{-1} \dfrac{1}{2}$

$= 26.6°$ (to 1 d.p.)

(iii) $\angle MEF = \angle EMN$ (alt. $\angle$s, EF // NM)

$\therefore\ \angle MEF = 26.6°$



(iv) In ΔBCD, $\angle BCD = 90°$.

Using Pythagoras' Theorem,

$DB^2 = BC^2 + DC^2$

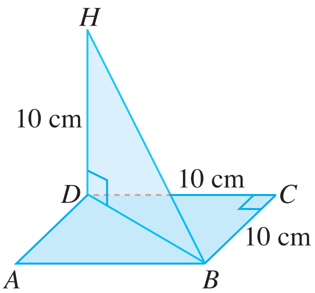
$= 10^2 + 10^2$

$= 100 + 100$

$= 200$

$DB = \sqrt{200}$

$= 14.14$ cm (to 4 s.f.)



In ΔHBD, $\angle HDB = 90°$.

$\tan \angle HBD = \dfrac{10}{14.14}$

$= 0.7072$ (to 4 s.f.)

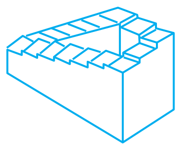
$\angle HBD = \tan^{-1} 0.7072$

$= 35.3°$ (to 1 d.p.)

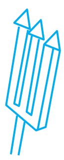
Problem Solving Tip

In **(ii)**, an alternative method to solve for $\angle EMN$ is to use the sum of angles in a triangle, i.e. $\angle EMN = 180° - \angle ENM - \angle MEN$.

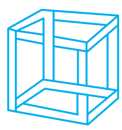
Is it possible to construct each of the following three-dimensional objects?



(a)



(b)



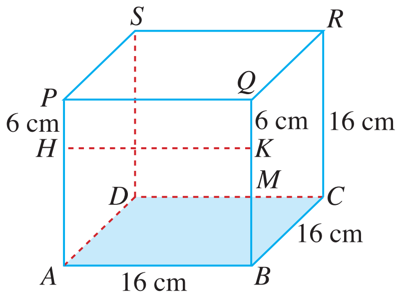
(c)

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 7C Questions 2, 3, 8, 9, 15

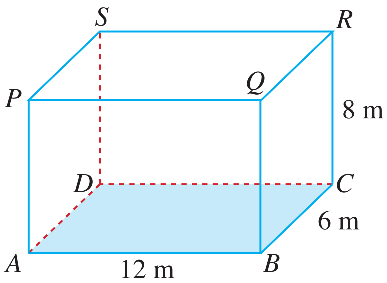
1. The figure shows a cube of length 16 cm.



Given that $PH = QK = 6$ cm, find

(i) $\angle BCK$, **(ii)** $\angle SBD$, **(iii)** $\angle BDK$.

2. The figure shows a cuboid where $AB = 12$ m, $BC = 6$ m and $CR = 8$ m.



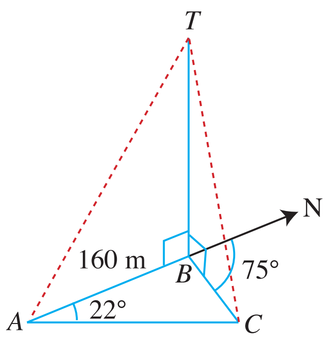
Find

(i) $\angle ABP$, **(ii)** $\angle BCQ$, **(iii)** $\angle CAR$.

Worked Example 9

(Angle of Elevation in a Three-dimensional Problem)

Three points A, B and C are on level ground. B is due north of A, the bearing of C from A is 022° and the bearing of C from B is 075°.



(i) Given that A and B are 160 m apart, find the distance between B and C.

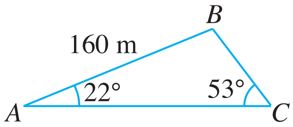
A vertical mast BT stands at B such that $\tan \angle TAB = \frac{3}{16}$.

(ii) Find the angle of elevation of T from C.

Solution:

(i) $\angle BCA = 75° - 22°$ (ext. $\angle$ of a Δ = sum of int. opp. $\angle$s)

$= 53°$



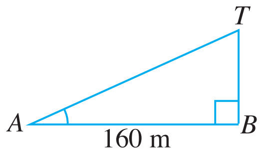
Using sine rule,

$$\frac{BC}{\sin 22°} = \frac{160}{\sin 53°}$$

$$BC = \frac{160 \sin 22°}{\sin 53°}$$

$$= 75.0 \text{ m (to 3 s.f.)}$$

(ii) In ΔTAB,



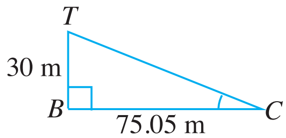
$$\tan \angle TAB = \frac{3}{16}$$

$$\frac{BT}{160} = \frac{3}{16}$$

$$BT = \frac{3}{16} \times 160$$

$$= 30 \text{ m}$$

In ΔTCB,



$$\tan \angle TCB = \frac{30}{75.05}$$

$$\angle TCB = \tan^{-1} \frac{30}{75.05}$$

$$= 21.8° \text{ (to 1 d.p.)}$$

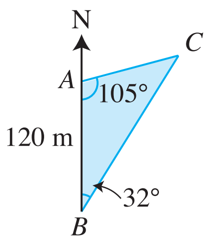
$\therefore$ The angle of elevation of T from C is 21.8°.

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 7C Questions 4, 5, 10, 11, 16

Three points A, B and C are on level ground. B is due south of A, the bearing of C from B is 032° and $\angle CAB = 105°$.



(i) Given that A and B are 120 m apart, find the distance between B and C.

A vertical mast CT of height 25 m stands at C.

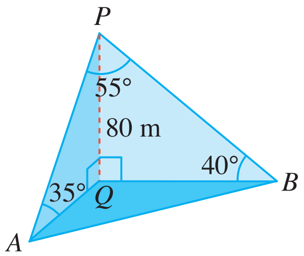
(ii) Find the angle of elevation of T from B.

Worked Example 10

(Angle of Depression in a Three-dimensional Problem)

Priya is in a cable car P at a height of 80 m above the ground. She observes a statue at A and fountain at B. Given that the angles of depression of the statue and the fountain are 35° and 40° respectively and that $\angle APB = 55°$, find the distance between A and B.

Solution:



In ΔAPQ, $\angle AQP = 90°$.

$$\sin 35° = \frac{80}{AP}$$

$$AP = \frac{80}{\sin 35°}$$

$$= 139.5 \text{ m (to 4 s.f.)}$$

In ΔBPQ, $\angle BQP = 90°$.

$$\sin 40° = \frac{80}{BP}$$

$$BP = \frac{80}{\sin 40°}$$

$$= 124.5 \text{ m (to 4 s.f.)}$$

Using cosine rule,

$$AB^2 = AP^2 + BP^2 - 2 \times AP \times BP \cos 55°$$

$$= 139.5^2 + 124.5^2 - 2 \times 139.5 \times 124.5 \cos 55°$$

$$= 15\,040 \text{ (to 4 s.f.)}$$

$$AB = \sqrt{15\,040}$$

$$= 123 \text{ m (to 3 s.f.)}$$

Problem Solving Tip

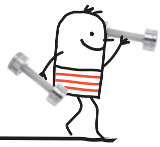
In ΔABP, in order to find AB, we first have to find AP and BP. Since ΔAPQ and ΔBPQ are right-angled triangles, how can we find AP and BP?

PRACTISE NOW 10

Rui Feng is on the top T of an observation tower OT. The height of the tower is 54 m. He observes a car that has broken down at A, causing a traffic jam to B. Given that the angles of depression of A and B are 42° and 38° respectively and that $\angle ATB = 48°$, find the distance between A and B.

SIMILAR QUESTIONS

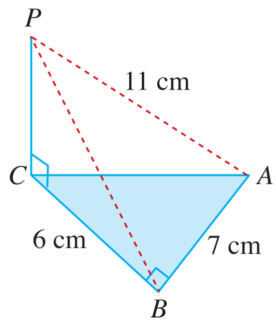
Exercise 7C Questions 12, 13, 17



Exercise 7C

BASIC LEVEL

1. The figure shows ΔABC, right-angled at B and lying in a horizontal plane. P is a point vertically above C.



Given that $AB = 7$ cm, $BC = 6$ cm and $AP = 11$ cm, find

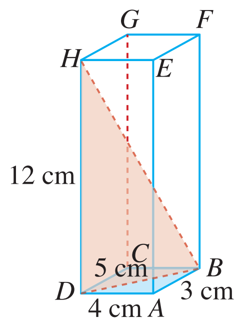
(i) AC,

(ii) PC,

(iii) $\angle PAC$,

(iv) the angle of elevation of P from B.

2. The figure shows a rectangular box in which $AB = 3$ cm, $AD = 4$ cm, $BD = 5$ cm and $DH = 12$ cm.

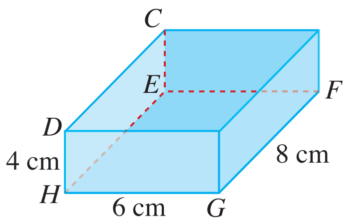


Find

(i) the length of BH, **(ii)** $\angle BDC$,

(iii) $\angle HBD$.

3. A rectangular block of sugar has a horizontal base $EFGH$. The corners C and D are vertically above E and H respectively. It is given that $DH = 4$ cm, $GH = 6$ cm and $FG = 8$ cm.



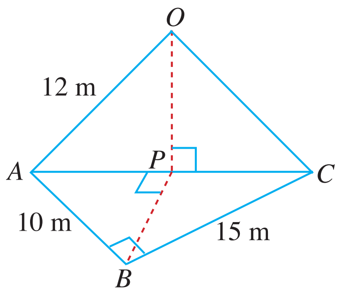
Find

(i) $\angle DGH$,

(ii) HF,

(iii) the angle of elevation of D from F.

4. The figure shows three points A, B and C on horizontal ground where $\angle ABC$ is a right angle. AOC represents a vertical triangular wall with P as the foot of the perpendicular from O to AC.



Given that $\angle APB = 90°$, $AB = 10$ m, $BC = 15$ m and $OA = 12$ m, find

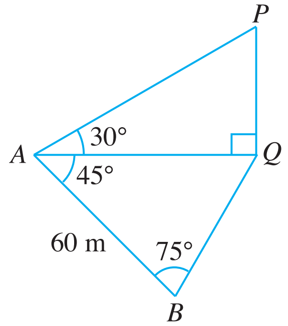
(i) $\angle BAC$,

(ii) the length of AP,

(iii) the length of OP,

(iv) the angle of elevation of O from B.

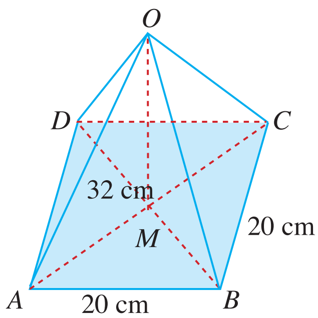
5. In the figure, the angle of elevation of the top of a vertical tower PQ from a point A is 30°.



Given that Q, the foot of the tower, is on the same horizontal plane as A and B, and that $AB = 60$ m, $\angle BAQ = 45°$ and $\angle ABQ = 75°$, find the height of the tower.

INTERMEDIATE LEVEL

6. $OABCD$ is a pyramid. The square base $ABCD$ has sides of length 20 cm and lies in a horizontal plane. M is the point of intersection of the diagonals of the base and O is vertically above M.



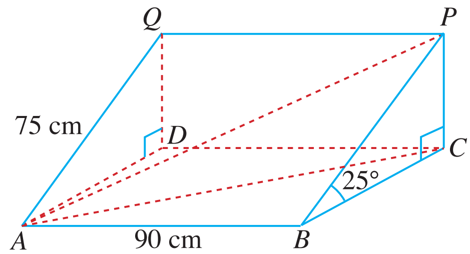
Given that $OA = 32$ cm, find

(i) the length of AM,

(ii) the height of the pyramid,

(iii) $\angle OAM$.

7. In the figure, $ABPQ$ is the rectangular sloping surface of a desk with $ABCD$ lying in a horizontal plane. Q and P lie vertically above D and C.

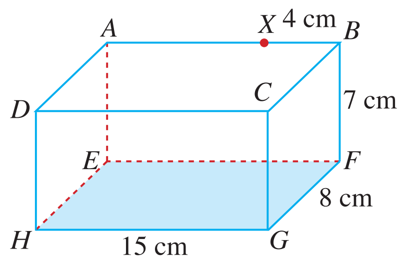


Given that $AB = PQ = 90$ cm, $AQ = BP = 75$ cm and $\angle PBC = \angle QAD = 25°$, find

(i) AC, **(ii)** $\angle PAC$,

(iii) $\angle CAB$.

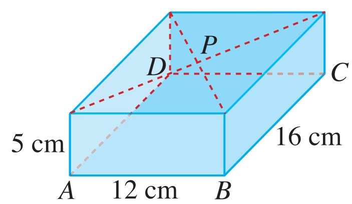
8. In the figure, a cuboid has a horizontal base $EFGH$ where $HG = 15$ cm, $GF = 8$ cm and $BF = 7$ cm. X is a point on AB such that $XB = 4$ cm.



Find

(i) $\angle CEG$, **(ii)** $\angle GXF$.

9. P is the centre of the upper face of the rectangular block with $ABCD$ as its base.



Find

(i) $\angle PAC$, **(ii)** $\angle PAB$.

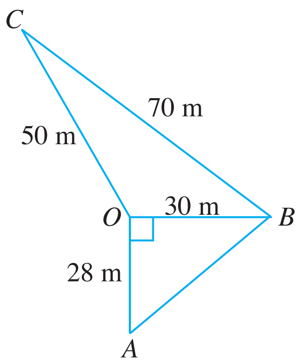
10. P, Q and R are three points on level ground with Q due east of P and R due south of P. A vertical mast PT stands at P and the angle of elevation of the top T from Q is 3.5°. Given that $PQ = 1000$ m and $PR = 750$ m, find

(i) the bearing of Q from R,

(ii) the height of the mast,

(iii) the angle of elevation of T from R.

11. The figure shows four points O, A, B and C which lie on level ground in a campsite. O is due north of A and B is due east of O.



(a) Given that $OA = 28$ m, $OB = 30$ m, $OC = 50$ m and $BC = 70$ m, find

(i) the bearing of A from B,

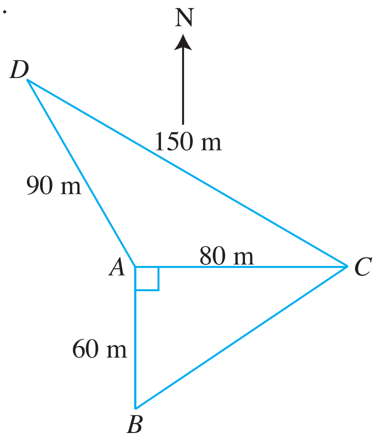
(ii) $\angle COB$,

(iii) the bearing of C from O.

A vertical flag pole stands at the point B such that the angles of elevation from O, A and C are measured.

(b) Given that the greatest of these 3 angles of elevation is 29°, calculate the height of the flag pole.

12. A, B, C and D are four points on horizontal ground. B is due south of A and the bearing of C from A is 090°.



(a) Given that $AB = 60$ m, $AC = 80$ m and $CD = 150$ m, find

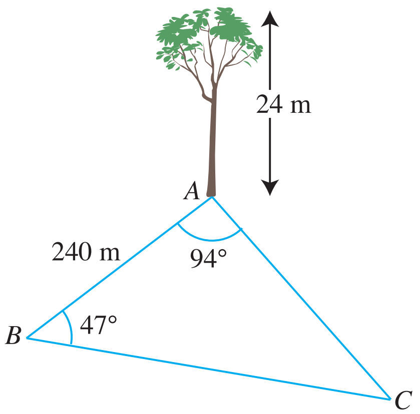
(i) the bearing of C from B,

(ii) the bearing of D from A.

A vertical mast stands at A and the angle of depression of C from the top of the mast is 8.6°.

(b) Find the height of the mast and the angle of depression of D from the top of the mast.

13. A tree of height 24 m stands vertically at A on the ground of an island. Two boats are at B and C such that $\angle BAC = 94°$, $\angle ABC = 47°$ and $AB = 240$ m.



(a) Find

(i) the distance between B and C,

(ii) the area of ΔABC,

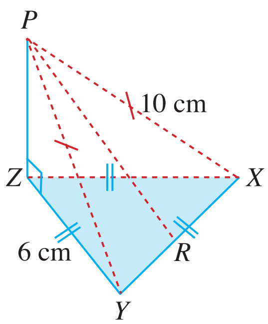
(iii) the shortest distance from A to BC.

The boat at B sails in a straight line towards C.

(b) Find the greatest angle of depression of the boat from the top of the tree.

ADVANCED LEVEL

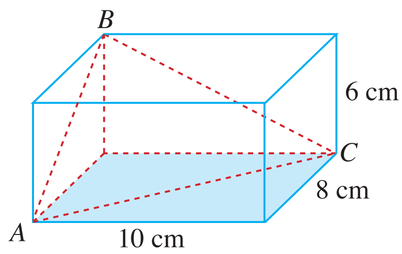
14. In the figure, XYZ is an equilateral triangle with sides of length of 6 cm lying in a horizontal plane. P lies vertically above Z, R is the midpoint of XY and $PX = PY = 10$ cm.



Find

(i) $\angle PYZ$, **(ii)** $\angle PRZ$.

15. The figure shows a block of wood in the shape of a cuboid with dimensions 10 cm by 8 cm by 6 cm. Huixian cuts the block into two pieces such that the cutting tool passes through the points A, B and C as shown.

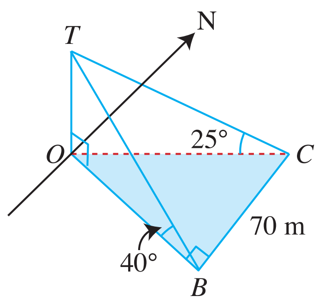


Given that the triangular surface ACB on one piece of the block is to be coated with varnish, find

(i) $\angle ABC$,

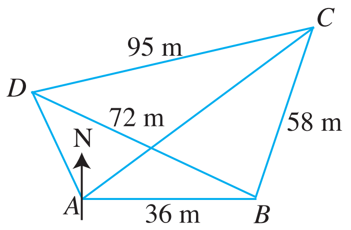
(ii) the area of the surface that is to be coated with varnish.

16. Amirah stands at a point B, due east of a vertical tower OT, and observes that the angle of elevation of the top of the tower T is 40°. She walks 70 m due north and finds that the angle of elevation of T from her new position at C is 25°.



Find the height of the tower and hence distance OB.

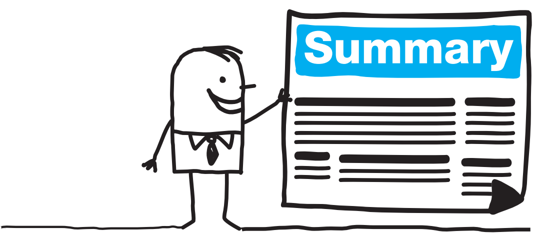
17. The figure shows a plot of land $ABCD$ such that B is due east of A and the bearing of C from A is 048°.



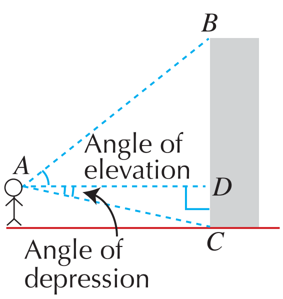
(i) Given that $AB = 36$ m, $BC = 58$ m, $BD = 72$ m and $CD = 95$ m, find the bearing of C from B.

A vertical control tower of height 35 m stands at B. Shirley cycles from C to D and reaches a point P where the angle of depression of P from the top of the tower is the greatest.

(ii) Find the angle of depression of P from the top of the tower.



1. Angles of Elevation and Depression



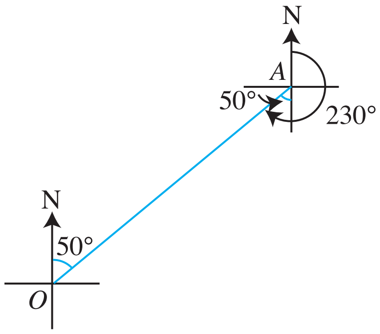
Angle of Elevation

When a person looks at the top of the wall, B, the angle between the horizontal AD and the line of sight AB, i.e. $\angle BAD$, is called the angle of elevation.

Angle of Depression

When a person looks at the bottom of the wall, C, the angle between the horizontal AD and the line of sight AC, i.e. $\angle CAD$, is called the angle of depression.

2. **Bearings**



The bearing of a point A from another point O is an angle measured from the north, at O, in a clockwise direction and it always written as a three-digit number.

The bearing of A from O is 050°.

The bearing of O from A is 230°.

3. The basic technique used in solving a three-dimensional problem is to reduce it to a problem in a plane.

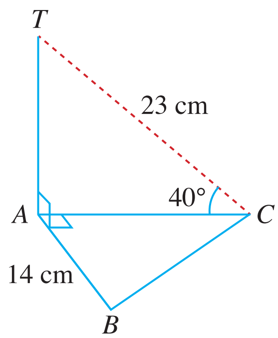
Review Exercise 7

1. Two points A and B, 35 m apart on level ground, are due east of the foot of a castle. The angles of elevation of the top of the castle from A and B are 47° and 29° respectively. Find the height of the castle.

2. The angle of depression of a rabbit from the top of a cliff is 24°. After the rabbit hops a distance of 80 m horizontally towards the base of the cliff, the angle of depression of the rabbit from the top of the cliff becomes 32°. Find the height of the cliff.

3. The captain of a ship observes that the angle of elevation of a lighthouse is 12°. When he sails a further distance of 200 m away from the lighthouse, the angle of elevation becomes 10°. Find the height of the lighthouse.

4. ABC is a triangle lying on a horizontal plane with $\angle BAC = 90°$ and $AB = 14$ cm. T is a point vertically above A, $TC = 23$ cm and the angle of elevation of T from C is 40°.



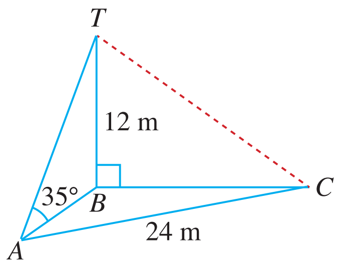
Find

(i) the height of AT,

(ii) the angle of elevation of T from B,

(iii) the length of BC.

5. $ABCT$ is a triangular pyramid with ΔABC as its base and BT as its height. It is given that $AC = 24$ m, $BT = 12$ m, $\angle TAB = 35°$ and $\angle ABC = 90°$, find

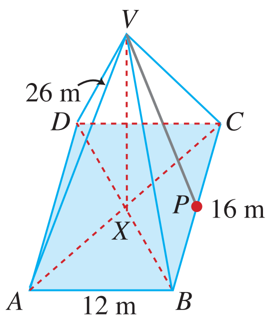


(i) the length of AB,

(ii) the length of BC,

(iii) the angle of depression of C from T.

6. The figure shows a roof in the shape of a right pyramid on a horizontal rectangular base $ABCD$, where $AB = 12$ m, $BC = 16$ m and $VA = 26$ m.



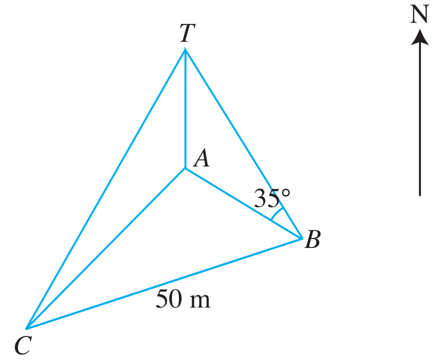
(i) Given that X is the midpoint of AC, find the height, VX, of the roof.

(ii) Find $\angle AVC$.

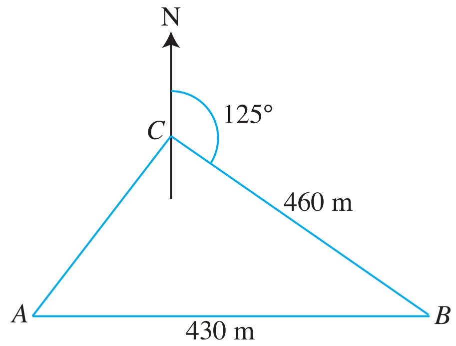
A passer-by notices a crack along VP, where P is the midpoint of BC.

(iii) Find the length of the crack.

7. Three points A, B and C lie on a horizontal ground. T is the top of a vertical tower standing on A. The bearings of B and C from A are 135° and 225° respectively and the bearing of C from B is 250°. If the distance between B and C is 50 m and the angle of elevation of T from B is 35°, calculate the height of the tower and the angle of elevation of T from C.



8. In the figure, A, B and C are three points on a horizontal field. A is due west of B, the bearing of B from C is 125°, $AB = 430$ m and $BC = 460$ m.



(a) Find

(i) the distance between A and C,

(ii) $\angle ACB$,

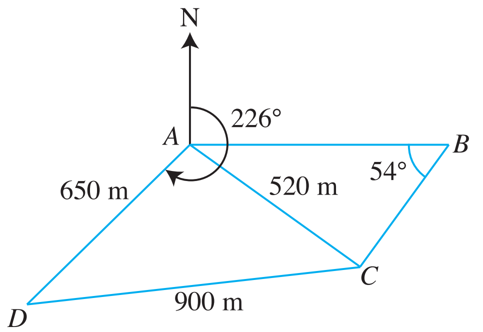
(iii) the bearing of C from A,

(iv) the area of ΔABC.

At a certain instant, a hot air balloon is at a point which is directly above C.

(b) Given that the angle of elevation of the hot air balloon from B is 5.2°, find the angle of elevation of the hot air balloon from A.

9. The figure shows the positions of four points A, B, C and D on level ground. B is due east of A, the bearing of D from A is 226°, $\angle ABC = 54°$, $AC = 520$ m, $AD = 650$ m and $CD = 900$ m.



(a) Find

(i) $\angle CAD$,

(ii) the bearing of C from A,

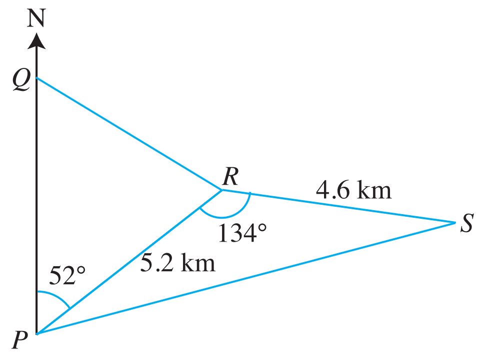
(iii) the distance between B and C,

(iv) the shortest distance from A to CD.

A vertical tower of height 80 m stands at A. Devi walks from C to D and reaches a point P where the angle of elevation of the top of the tower from P is the greatest.

(b) Find the angle of elevation of the top of the tower from P.

10. The figure shows the position of a post office P and three train stations Q, R and S. Q is due north of P, the bearing of R from P is 052°, $\angle PRS = 134°$, $PR = 5.2$ km and $RS = 4.6$ km.



(i) Calculate the distance between P and S.

(ii) Find the bearing of S from R.

(iii) Given that the bearing of R from Q is 122°, find the distance between P and Q.

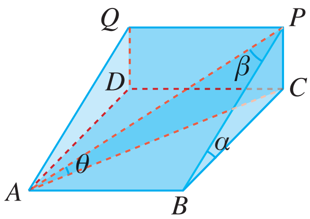
An office building of height 75 m stands at R.

(iv) Given that Huixian walks in a straight line along PS, calculate the greatest angle of elevation of the top of the building as she walks along PS.

A policeman is standing at a point due north of P such that he is equidistant from both P and R.

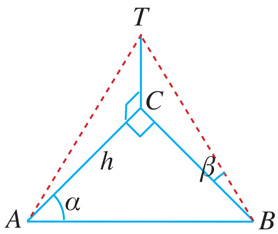
(v) Find the distance between the policeman and P.

1. In the figure, $ABCD$ is horizontal and $ABPQ$ is a rectangular plane.

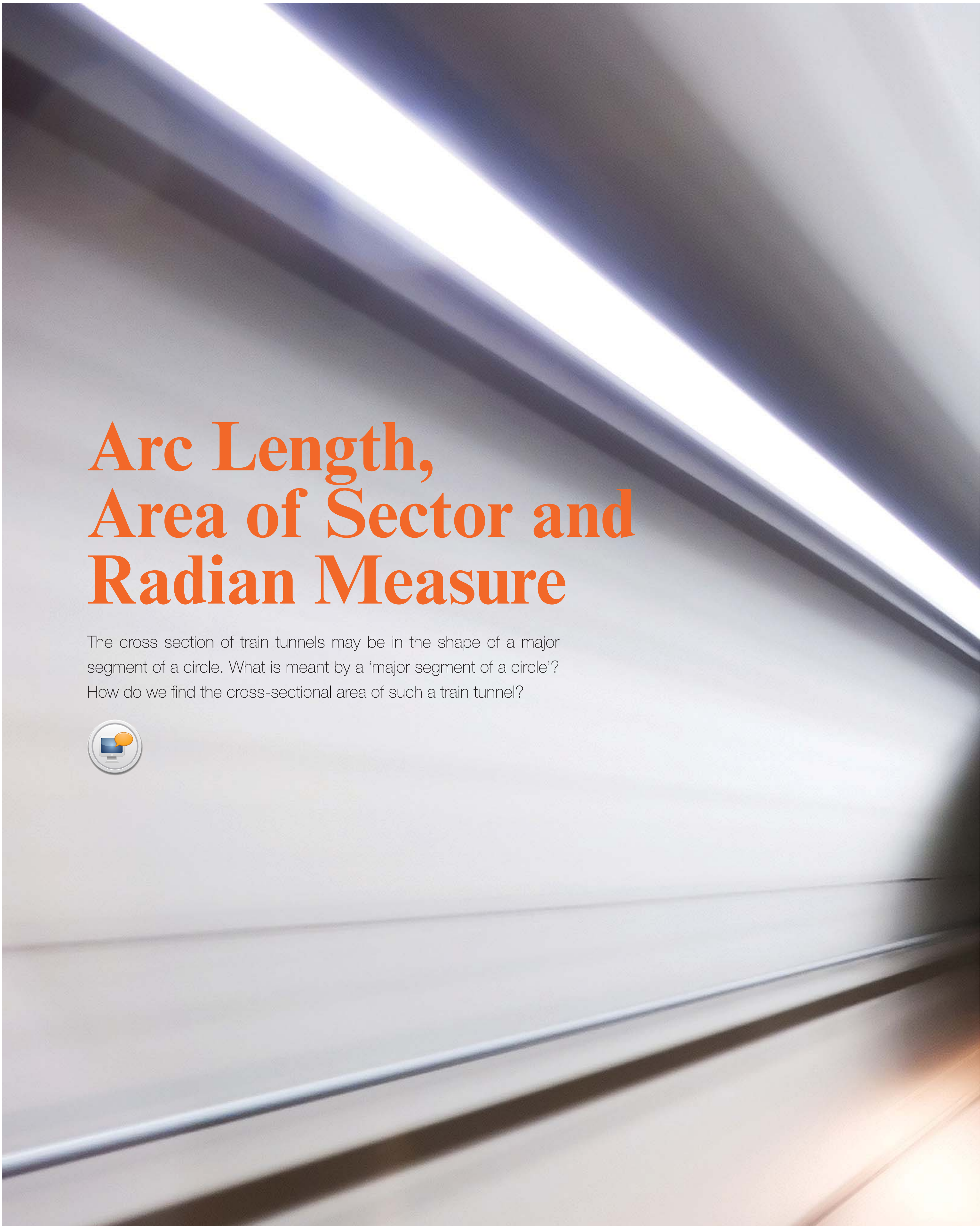


Given that $\angle PCA = \angle PCB = \angle QDC = 90°$, $\angle PBC = \alpha$, $\angle APB = \beta$ and $\angle PAC = \theta$, express $\sin \theta$ in terms of the trigonometric ratios of α and β only.

2. In the figure, ΔABC is on horizontal ground and CT is vertical.

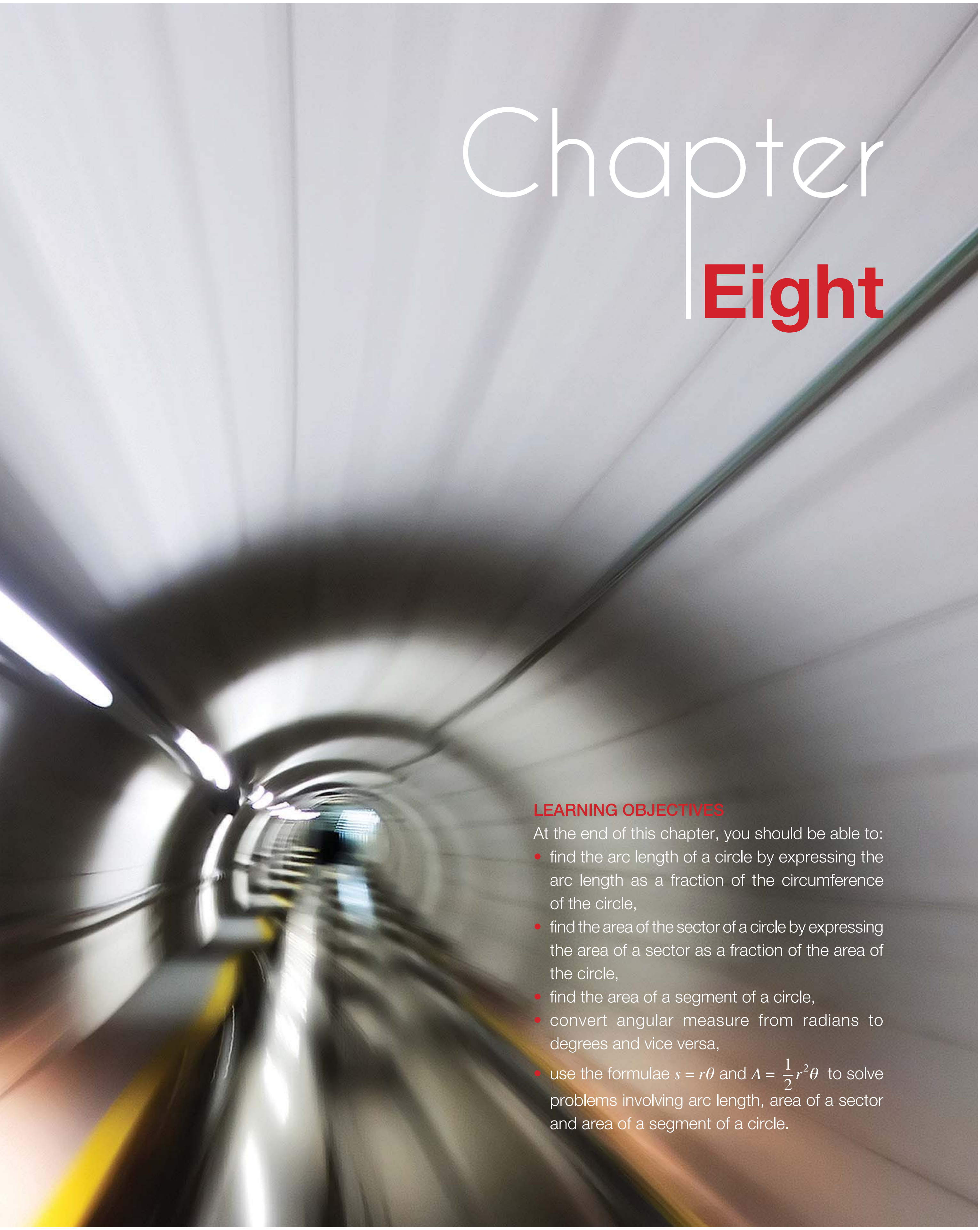


Given that $\angle ACB = 90°$, $\angle BAC = \alpha$, $\angle TBC = \beta$ and $AC = h$ metres, express the length of TB in terms of h and the trigonometric ratios of α and β.



Arc Length, Area of Sector and Radian Measure

The cross section of train tunnels may be in the shape of a major segment of a circle. What is meant by a 'major segment of a circle'? How do we find the cross-sectional area of such a train tunnel?



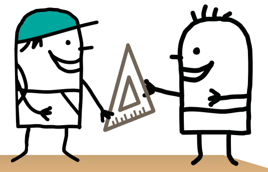
Chapter Eight

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- find the arc length of a circle by expressing the arc length as a fraction of the circumference of the circle,
- find the area of the sector of a circle by expressing the area of a sector as a fraction of the area of the circle,
- find the area of a segment of a circle,
- convert angular measure from radians to degrees and vice versa,
- use the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ to solve problems involving arc length, area of a sector and area of a segment of a circle.

8.1 Length of Arc



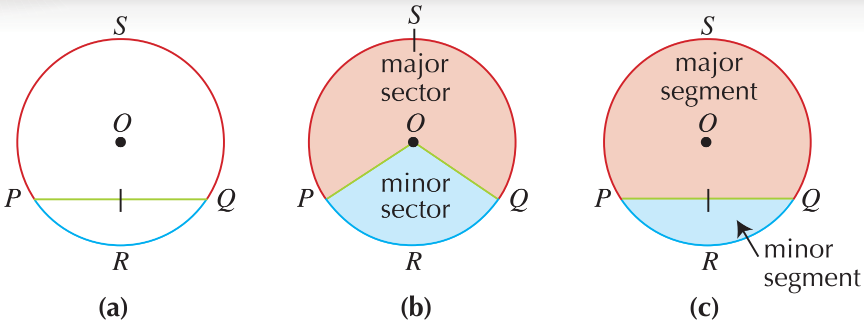
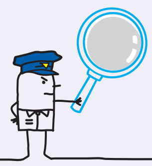


Fig. 8.1

Fig. 8.1**(a)** shows a circle with centre O. The line PQ is called a **chord**. PRQ is part of the circumference which is called an **arc**. The arc PRQ is called the **minor arc** and the arc PSQ is called the **major arc**.

The part of a circle enclosed by any two radii of a circle and an arc is called a **sector**. In Fig. 8.1**(b)**, the region enclosed by the radii OP, OQ and the minor arc PRQ is called a **minor sector** of the circle. The region enclosed by the radii OP, OQ and the major arc PSQ is called a **major sector** of the circle.

In Fig. 8.1**(c)**, the chord PQ divides the circle into two **segments**. The region enclosed by the chord PQ and the *minor arc* PRQ is called a **minor segment**. The region enclosed by the chord PQ and the *major arc* PSQ is called the **major segment**.



Investigation

Arc Length

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Arc Length as shown.

1. The template shows a circle with $\angle AOB$ subtended by the (blue) arc AB at the centre O.

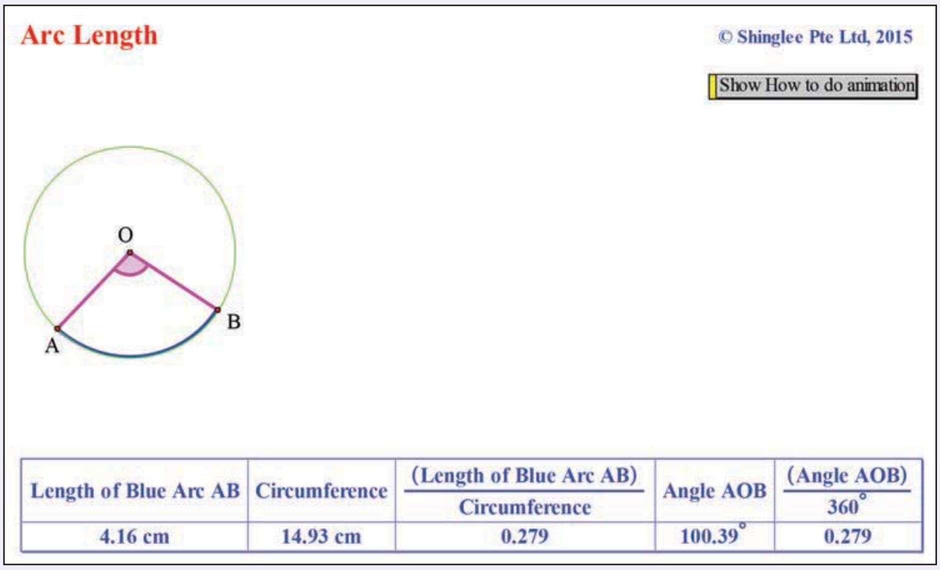


Fig. 8.2

2. Click and drag point B to change the size of the (blue) arc AB. To change the radius of the circle, move point A. Copy and complete Table 8.1.

No.	Length of Blue Arc AB	Circumference of Circle	$\dfrac{\text{Length of Blue Arc } AB}{\text{Circumference of Circle}}$	$\angle AOB$	$\dfrac{\angle AOB}{360°}$
(a)					
(b)					
(c)					
(d)					
(e)					

Table 8.1

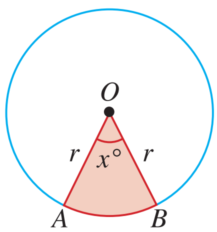
3. What do you notice about the third last column and the last column of Table 8.1?

4. Click on the button 'Show how to do animation' in the template and it will show you how to add 10 more entries to the table as the points A and B move automatically. What do you notice about the third last column and the last column of the table in the template?

5. Hence, write down a formula for finding the length of an arc of a circle.

From the investigation, we observe that

$$\begin{aligned}\text{arc length} &= \frac{x°}{360°} \times \text{circumference} \\ &= \frac{x°}{360°} \times 2\pi r,\end{aligned}$$

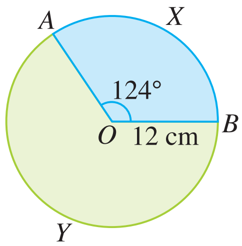


where $x°$ is the angle subtended by the arc at the centre of the circle of radius r.

Worked Example 1

(Finding the Arc Length)

In the figure, O is the centre of a circle of radius 12 cm and $\angle AOB = 124°$.



Find

(i) the length of the minor arc AXB,

(ii) the perimeter of the major sector $OAYB$.

Solution:

(i) Length of minor arc $AXC = \frac{124°}{360°} \times 2\pi \times 12$

$= 26.0$ cm (to 3 s.f.)

(ii) Perimeter of major sector = length of arc $AYB + OA + OB$

$= \frac{360° - 124°}{360°} \times 2\pi \times 12 + 12 + 12$

$= 73.4$ cm (to 3 s.f.)

Problem Solving Tip

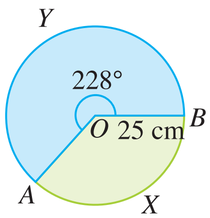
Alternatively,
Length of arc AYB
$= 2\pi r$ – length of arc AXB.

PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 8A Questions 1(a)-(d), 2(a)-(c), 3(a), (b), 4(a)-(d), 5-9

1. In the figure, O is the centre of a circle of radius 25 cm and reflex $\angle AOB = 228°$.

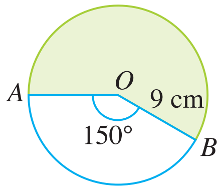


Find

(i) the length of the major arc AYB,

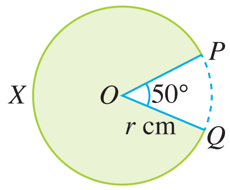
(ii) the perimeter of the minor sector $OAXB$.

2. In the figure, O is the centre of a circle of radius 9 cm and $\angle AOB = 150°$.



Find the perimeter of the shaded region, giving your answer in the form $a + b\pi$, where a and b are rational numbers.

3. The figure shows the design of a logo in which a sector has been removed from the circle, centre O and radius r cm.

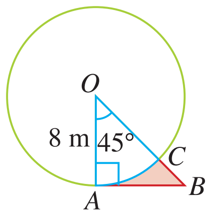


Given that the length of the major arc PXQ is 36 cm and $\angle POQ = 50°$, find the value of r.

Worked Example 2

(Finding the Arc Length)

In the figure, O is the centre of a circle of radius 8 m. The points A and C lie on the circumference of the circle and OCB is a straight line.



Given that $\angle AOB = 45°$ and OA is perpendicular to AB, find the perimeter of the shaded region ABC.

Solution:

$\angle ABC = 180° - 90° - 45°$

$= 45°$

i.e. ΔAOB is isosceles and $AB = 8$ m.

Length of arc $AC = \frac{45°}{360°} \times 2\pi \times 8$

$= 2\pi$ m

Using Pythagoras' Theorem,

$OB^2 = 8^2 + 8^2$

$OB = \sqrt{8^2 + 8^2}$

$= \sqrt{128}$

$= 11.31$ m (to 4 s.f.)

$BC = OB - OC$

$= 11.31 - 8$

$= 3.31$ m

$\therefore$ Perimeter of shaded region $ABC = AB + BC +$ length of arc AC

$= 8 + 3.31 + 2\pi$

$= 17.6$ m (to 3 s.f.)

ATTENTION

For accuracy, we should work in terms of π in the intermediate steps.

ATTENTION

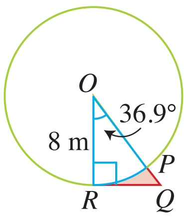
In order for the final answer to be accurate to three significant figures, any intermediate working must be correct to four significant figures.

PRACTISE NOW 2

SIMILAR QUESTIONS

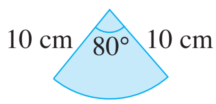
Exercise 8A Questions 10-12, 14

1. In the figure, O is the centre of a circle of radius 8 m. The points P and R lie on the circumference of the circle and OPQ is a straight line.



Given that $\angle QOR = 36.9°$ and OR is perpendicular to QR, find the perimeter of the shaded region PQR.

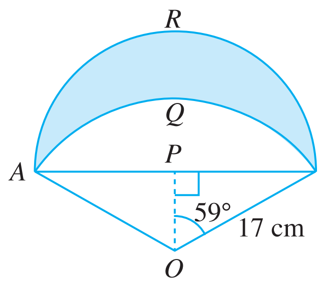
2. The figure shows a sector of a circle of radius 10 cm. Given that the angle at the centre of the circle is 80°, find the perimeter of the sector.



Worked Example 3

(Finding the Arc Length)

In the figure, AQB is the minor arc of a circle with centre O and radius 17 cm. ARB is a semicircle with AB as its diameter and P as its centre. ΔOPB is a right-angled triangle with $\angle OPB = 90°$ and $\angle POB = 59°$.



Find

(i) the length of AB,

(ii) the perimeter of the shaded region.

Solution:

(i) In ΔOPB,

$$\sin \angle POB = \frac{PB}{OB}$$

$$\sin 59° = \frac{PB}{17}$$

$$PB = 17 \sin 59°$$

$$= 14.57 \text{ cm (to 4 s.f.)}$$

$$\therefore AB = 2PB$$

$$= 2(14.57)$$

$$= 29.1 \text{ cm (to 3 s.f.)}$$

(ii) $\angle AOB = 2\angle POB$

$$= 2(59°)$$

$$= 118°$$

Length of arc $AQB = \dfrac{118°}{360°} \times 2\pi \times 17$

$$= 35.01 \text{ cm (to 4 s.f.)}$$

Length of arc $ARB = \dfrac{1}{2} \times 2\pi \times 14.57$

$$= 45.77 \text{ cm (to 4 s.f.)}$$

$\therefore$ Perimeter of shaded region $= 35.01 + 45.77$

$$= 80.8 \text{ cm (to 3 s.f.)}$$

ATTENTION

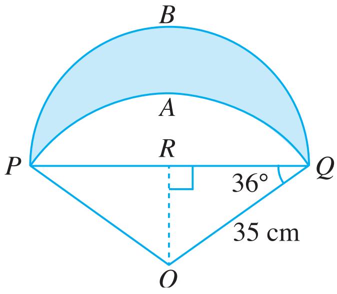
In ΔOPB, PB is the side opposite $\angle POB$ and OB is the hypotenuse.

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 8A Questions 13, 15

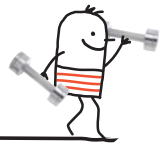
In the figure, PAQ is the minor arc of a circle with centre O and radius 35 cm. PBQ is a semicircle with PQ as its diameter and R as its centre. ΔORQ is a right-angled triangle with $\angle ORQ = 90°$ and $\angle RQO = 36°$.



Find

(i) the length of PQ,

(ii) the perimeter of the shaded region.

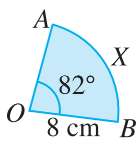


Exercise 8A

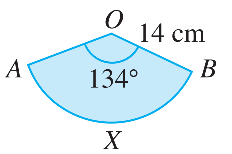
BASIC LEVEL

1. Find the length of each of the following arcs AXB.

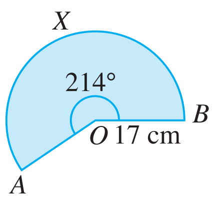
(a)



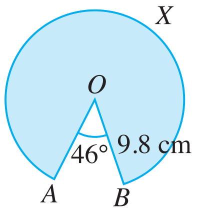
(b)



(c)



(d)

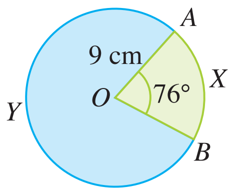


2. For each of the following circles, find

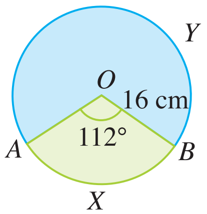
(i) the length of the minor arc AXB,

(ii) the perimeter of the major sector $OAYB$.

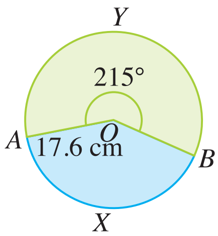
(a)



(b)

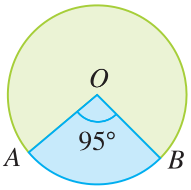


(c)



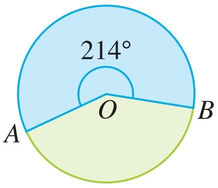
3. Find the radius of each of the following circles.

(a)



Length of minor arc = 26.53 cm

(b)



Length of major arc = 104.6 cm

4. The radius of a circle is 14 m. Find the angle at the centre of the circle subtended by an arc of length

(a) 12 m, **(b)** 19.5 m,

(c) 64.2 m, **(d)** 84.6 m,

giving your answers correct to the nearest degree.

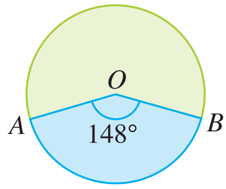
5. The hour hand of a large clock mounted on a clock tower travels through an angle of 45°. If the hour hand is 1.5 m long, how far does the tip of the hour hand travel?

INTERMEDIATE LEVEL

6. A piece of wire 32 cm long is bent to form a sector of a circle of radius 6 cm. Find the angle subtended by the wire at the centre of the circle.

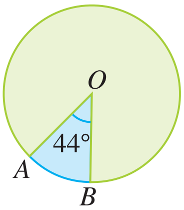
7. Find the radius of each of the following circles.

(a)



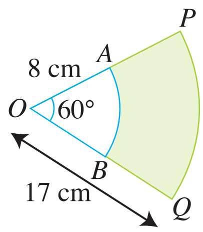
Perimeter of minor sector = 77.91 cm

(b)



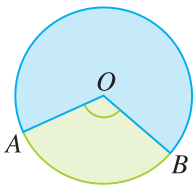
Perimeter of major sector = 278.1 cm

8. The figure shows two sectors OAB and OPQ with O as the common centre. The lengths of OA and OQ are 8 cm and 17 cm respectively.



Given that $\angle AOB = \angle POQ = 60°$, find the perimeter of the shaded region, giving your answer in the form $a + b\pi$, where a and b are rational numbers.

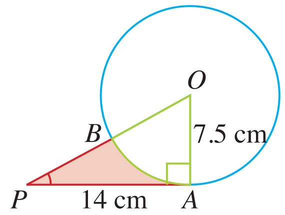
9. In the figure, the length of the minor arc is $\frac{7}{24}$ of the circumference of the circle.



(i) Find $\angle AOB$.

(ii) Given that the diameter of the circle is 14 cm, find the length of the minor arc.

10. In the figure, O is the centre of a circle of radius 7.5 cm. The points A and B lie on the circumference of the circle and OBP is a straight line.

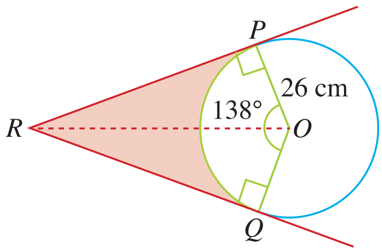


Given that PA = 14 cm and OA is perpendicular to AP, find

(i) $\angle POA$,

(ii) the perimeter of the shaded region PBA.

11. The figure shows a circle with centre O and radius 26 cm. Triangles OPR and OQR are congruent and $\angle OPR = \angle OQR = 90°$.

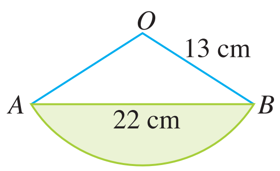


Given that $\angle POQ = 138°$, find

(i) the length of QR,

(ii) the perimeter of the shaded region.

12. The figure shows a sector of a circle with centre O and radius 13 cm.

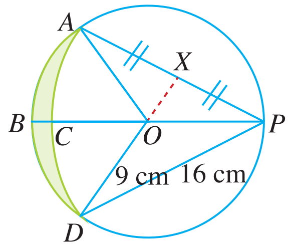


Given that the length of the chord AB = 22 cm,

(i) show that $\angle AOB$ is approximately 115.6°,

(ii) find the perimeter of the shaded region.

13. In the figure, ABD is the minor arc of a circle with centre O and radius 9 cm. ACD is the minor arc of a circle with centre P and radius 16 cm. The point X lies on AP such that $AX = XP$.



Find

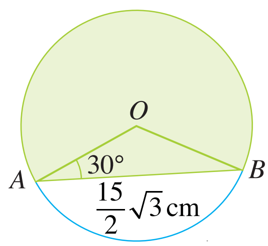
(i) $\angle APB$,

(ii) $\angle AOB$,

(iii) the perimeter of the shaded region.

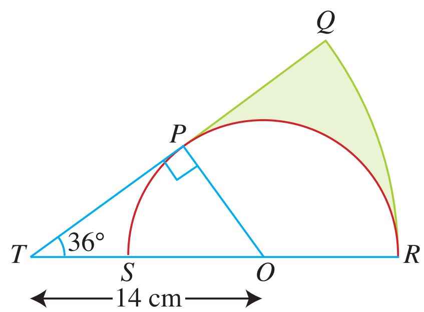
ADVANCED LEVEL

14. In the figure, O is the centre of a circle. The points A and B lie on the circumference of the circle such that $AB = \frac{15}{2}\sqrt{3}$ cm and $\angle OAB = 30°$.



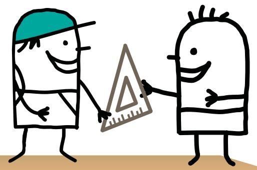
Find the perimeter of the shaded region.

15. The figure shows a semicircle with centre O and diameter SR. QR is an arc of another circle with centre T and T lies on RS produced.



Given that TO = 14 cm, $\angle OTP = 36°$ and OP is perpendicular to PT, find the perimeter of the shaded region.

8.2 Area of Sector



Investigation

Area of Sector

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Sector Area as shown below.

1. The template shows a circle with centre O and a (yellow) sector OAB.

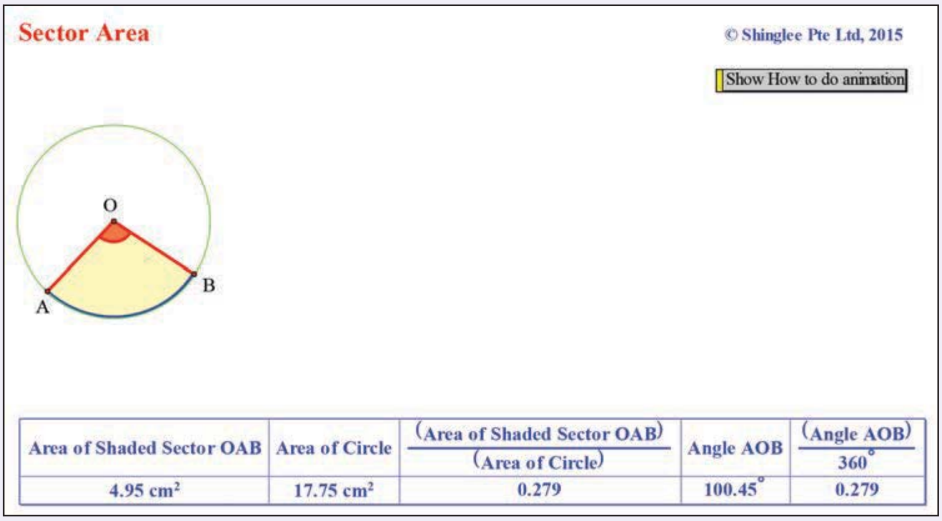


Fig. 8.3

2. Click and drag point B to change the size of the (yellow) sector OAB. To change the radius of the circle, move point A. Copy and complete Table 8.2.

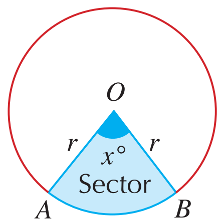
No.	Area of Shaded Sector OAB	Area of Circle	$\frac{\text{Area of Sector } OAB}{\text{Area of Circle}}$	$\angle AOB$	$\frac{\angle AOB}{360^\circ}$
(a)					
(b)					
(c)					
(d)					
(e)					

Table 8.2

3. What do you notice about the third last column and the last column of Table 8.2?
4. Click on the button 'Show how to do animation' in the template and it will show you how to add 10 more entries to the table as the points A and B move automatically. What do you notice about the third last column and the last column of the table in the template?
5. Hence, write down a formula for finding the area of a sector of a circle.

From the investigation, we observe that

$$\text{area of a sector} = \frac{x^\circ}{360^\circ} \times \text{area of the circle}$$
$$= \frac{x^\circ}{360^\circ} \times \pi r^2,$$

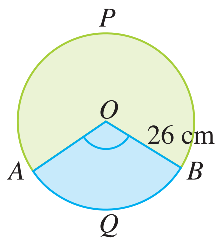


where x° is the angle of the sector subtended at the centre of the circle of radius r.

Worked Example 4

(Finding the Area of a Sector)

In the figure, O is the centre of a circle of radius 26 cm. The length of minor arc AQB is 52 cm.



(i) Show that $\angle AOB$ is approximately 114.6°.

(ii) Hence, find the area of the major sector $OAPB$.

Solution:

(i) Since the length of minor arc AQB is 52 cm,

$$\frac{\angle AOB}{360^\circ} \times 2\pi \times 26 = 52 \quad \left(\text{use arc length} = \frac{x^\circ}{360^\circ} \times 2\pi r\right)$$

$$\frac{\angle AOB}{360^\circ} = \frac{52}{52\pi}$$

$$= \frac{1}{\pi}$$

$$\angle AOB = \frac{1}{\pi} \times 360^\circ$$

$$= 114.6^\circ \text{ (to 1 d.p.)}$$

(ii) Reflex $\angle AOB = 360° - 114.59°$ ($\angle$s at a point)

$= 245.4°$

Area of major sector $OAPB = \frac{245.4°}{360°} \times \pi \times 26^2$

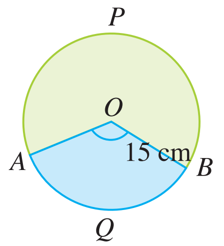
$= 1450 \text{ cm}^2$ (to 3 s.f.)

PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 8B Questions 1(a)-(f), 2(a)-(c), 3(a)-(d), 4(a)-(d), 5(a),(b), 6(a)-(d), 7-9

In the figure, O is the centre of a circle of radius 15 cm. The length of minor arc AQB is 33 cm.



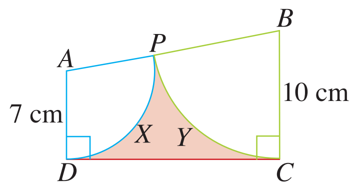
(i) Show that $\angle AOB$ is approximately 126.1°.

(ii) Hence, find the area of the major sector $OAPB$.

Worked Example 5

(Finding the Area of a Sector)

In the figure, $ABCD$ is a trapezium in which AD is parallel to BC, $AD = 7$ cm, $BC = 10$ cm and $\angle ADC = \angle BCD = 90°$. PXD is an arc of a circle centre A and PYC is an arc of a circle centre B.

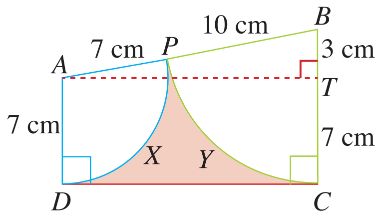


(i) Show that $\angle BAD$ is approximately 100.2°.

(ii) Hence, find the area of the shaded region.

Solution:

(i) Draw a line AT such that T lies on BC and AT is perpendicular to BC.



$$\begin{aligned} BT &= BC - TC \\ &= 10 - 7 \\ &= 3 \text{ cm} \end{aligned}$$

$$\begin{aligned} AB &= 7 + 10 \ (AB = AP + PB) \\ &= 17 \text{ cm} \end{aligned}$$

In ΔATB,

$$\sin \angle BAT = \frac{3}{17}$$

$$\begin{aligned} \angle BAT &= \sin^{-1} \frac{3}{17} \\ &= 10.16° \text{ (to 2 d.p.)} \end{aligned}$$

ATTENTION

In ΔATB, BT is the side opposite $\angle BAT$ and AB is the hypotenuse.

$$\begin{aligned} \angle BAD &= 90° + 10.16° \\ &= 100.16° \\ &= 100.2° \text{ (to 1 d.p.)} \end{aligned}$$

(ii) Using Pythagoras' Theorem,

$$\begin{aligned} AT^2 &= 17^2 - 3^2 \\ AT &= \sqrt{17^2 - 3^2} \\ &= \sqrt{280} \\ &= 16.73 \text{ cm (to 4 s.f.)} \end{aligned}$$

$$\begin{aligned} \angle ABT &= 180° - 90° - 10.16° \\ &= 79.84° \text{ (to 2 d.p.)} \end{aligned}$$

Area of shaded region

= Area of trapezium $ABCD$ – area of sector APD – area of sector BCP

$$= \frac{1}{2}(7+10)(16.73) - \frac{100.2°}{360°} \times \pi \times 7^2 - \frac{79.84°}{360°} \times \pi \times 10^2$$

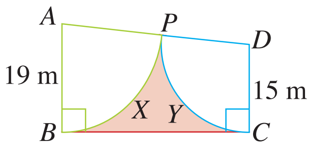
= 29.7 cm^2 (to 3 s.f.)

PRACTISE NOW 5

SIMILAR QUESTIONS

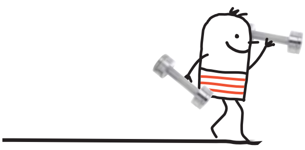
Exercise 6B Questions 10-13

In the figure, $ABCD$ is a trapezium in which AB is parallel to DC, AB = 19 m, DC = 15 m and $\angle ABC = \angle DCB = 90°$. PXB is an arc of a circle centre A and PYC is an arc of a circle centre D.



(i) Show that $\angle ADC$ is approximately 96.8°.

(ii) Hence, find the area of the shaded region.



Exercise 8B

BASIC LEVEL

1. Copy and complete the table for sectors of a circle.

	Radius	**Angle at centre**	**Arc length**	**Area**	**Perimeter**
(a)	7 cm	72°			
(b)	35 mm				136 mm
(c)		270°		1848 mm^2	
(d)		150°	220 cm		
(e)	14 m		55 m		
(f)		75°		154 cm^2	

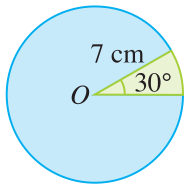
2. For each of the following circles, find

(i) the perimeter,

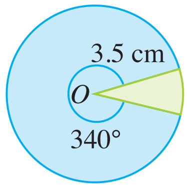
(ii) the area,

of the minor sector.

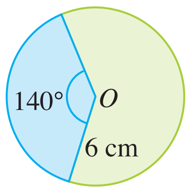
(a)



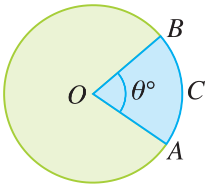
(b)



(c)



3. The figure shows a circle with centre O and $\angle AOB = \theta°$. The circumference of the circle is 88 cm.

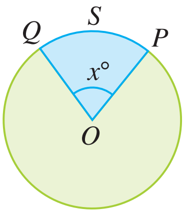


Find the length of arc ACB and the area of sector $OACB$ for each of the following values of θ.

(a) 60 **(b)** 99

(c) 126 **(d)** 216

4. The figure shows a circle with centre O and $\angle POQ = x°$. The area of the circle is 3850 cm^2.



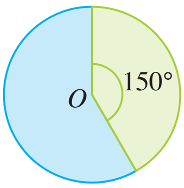
Find the area of sector $OPSQ$ and the length of arc PSQ for each of the following values of x.

(a) 36 **(b)** 84

(c) 108 **(d)** 198

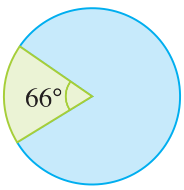
5. Find the radius of each of the following circles.

(a)



Area of minor sector = 114 cm^2

(b)



Area of major sector = 369 cm^2

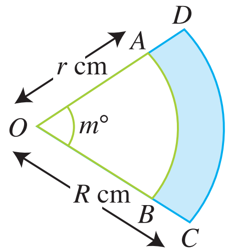
6. The diameter of a circle is 18 cm. Find the angle subtended by the arc of a sector with each of the following areas.

(a) 42.6 cm^2 **(b)** 117.2 cm^2

(c) 214.5 cm^2 **(d)** 18.9 cm^2

INTERMEDIATE LEVEL

7. The figure shows two sectors OAB and ODC with O as the common centre.



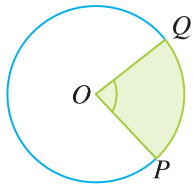
Given that $OA = r$ cm, $OC = R$ cm and $\angle AOB = m°$, find the perimeter and the area of the shaded region $ABCD$ for each of the following cases.

(i) $r = 10$, $R = 20$, $m = 45$

(ii) $r = 5$, $R = 8$, $m = 120$

(iii) $r = 35$, $R = 49$, $m = 160$

8. In the figure, the area of the shaded sector POQ is $\frac{5}{18}$ of the area of the whole circle.

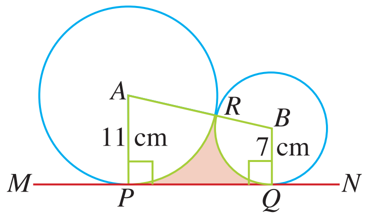


(i) Find $\angle POQ$.

(ii) Given that the area of the shaded sector is 385 cm^2, find the diameter of the circle.

9. During an Art lesson, students are required to make a shape in the form of a sector of a circle of radius 12 cm. If the perimeter of the shape is 38 cm, find the area of the paper used in making the shape.

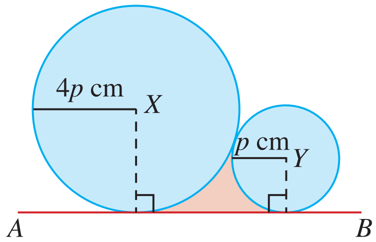
10. The figure shows two circular discs of radii 11 cm and 7 cm touching each other at R and lying on a straight line $MPQN$.



(i) Show that $\angle PAB$ is approximately 77.2°.

(ii) Hence, find the area of the shaded region.

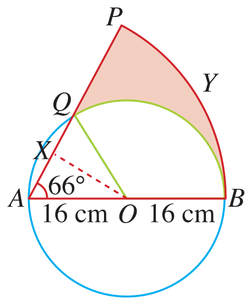
11. Two circular discs of radii $4p$ cm and p cm touch each other externally and lie on a straight line AB as shown.



Find an expression, in terms of p, for the area enclosed by the two discs and the line AB.

ADVANCED LEVEL

12. In the figure, O is the centre of a circle with radius 16 cm. PYB is the minor arc of a circle centre A and radius 32 cm. OX divides ΔOAQ into two congruent triangles.



Given that $\angle OAQ = 66°$, find

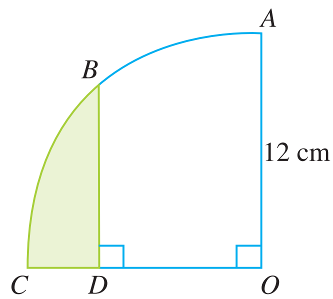
(i) $\angle BOQ$,

(ii) the length of AQ,

(iii) the perimeter of the shaded region,

(iv) the area of the shaded region.

13. The figure shows a quadrant of a circle of radius 12 cm.



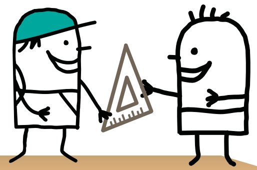
Given that B is the midpoint of the arc AC, find

(i) the length of BD,

(ii) the perimeter of the shaded region,

(iii) the area of the shaded region.

8.3 Radian Measure



Conversion Between Degrees and Radians

So far we have been using the measurement of 360° to denote the angle for one complete revolution. However, this value is arbitrary and in some branches of mathematics, angular measurement cannot be conveniently done in degrees. Hence, a new unit, called the **radian**, is introduced to describe the magnitude of an angle.

INFORMATION

This system of angular measurement involving radians, is applied especially to those branches of mathematics which involve the differentiation and integration of trigonometric ratios.

Consider a circle with centre O and radius r units (see Fig. 8.4**(a)**).

Suppose the arc AB has a length of r units.

Then the measure of the angle θ, subtended by the arc AB at the centre O, is defined to be one **radian**.

In other words, a radian is the size of the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

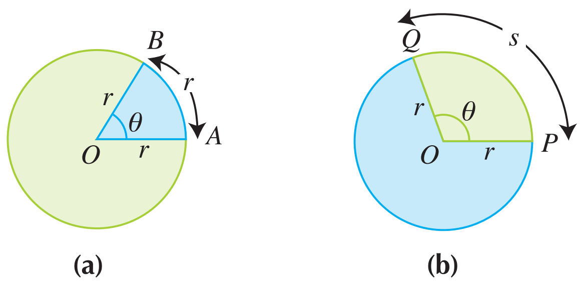


Fig. 8.4

In general, the angle θ, in radians, subtended at the centre of a circle with radius r units by an arc PQ of length s units, is equal to the ratio of the arc length to the radius:

$$\boxed{\theta \text{ (in radians)} = \frac{s}{r}}$$

Since $\frac{s}{r}$ is a ratio, θ has *no units*, i.e. radian is *not* a unit.

The abbreviation of 'radian' is 'rad'.

For example, if the radius of the circle is 2 cm and arc PQ = 2 cm, then θ = 1 rad; if the radius of the circle is 2 cm and arc PQ = 4 cm, then θ = 2 rad.

Investigation

Visualise the Size of an Angle of 1 radian

For this activity, you will require a sheet of paper, a pair of compasses and a piece of string (5 cm long).

1. On a sheet of paper, use a rule and a pair of compasses to draw a circle, centre O and with radius 5 cm.
2. Mark a point A on the circumference, as shown in Fig. 8.5.
3. Using a piece of string of length 5 cm, mark a point B on the circumference such that the length of arc AB is 5 cm.
4. Consider the minor sector AOB. Notice that the radius is equal to the arc length. What is the value of θ in radians?

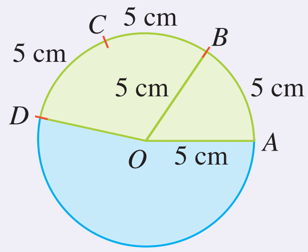


Fig. 8.5

5. Repeat Step 3 to mark out arcs BC, CD, DE, … of length 5 cm each around the circle.
6. **(a)** Estimate the size of the angle, in radians, subtended at the centre by
   (i) a semicircle, **(ii)** a circle.
   (b) Using this estimate, what is the approximate size of $\angle AOB$ (1 radian) in degrees?
   (c) Hence, what is an angle of 1 radian approximately equal to in degrees?
7. **(a)** Using the formula for the circumference of a circle, find the exact value of the angle at the centre of the circle in radians.
   (b) Hence, find the exact value of an angle of 1 radian in degrees.

RECALL

θ (in radians) $= \frac{s}{r}$

In general, for a circle of radius r units,

if arc $PQ = r$, then $\theta = 1$ rad;
if arc $PQ = 2r$, then $\theta = 2$ rad;
if arc $PQ = 2\pi r$, then $\theta = 2\pi$ rad.

When arc $PQ = 2\pi r$, it means that the arm OP has made one complete revolution, i.e. OM has moved through an angle of 360°, i.e. 2π rad = 360°.

$$\boxed{\pi \text{ rad} = 180°}$$

$$1 \text{ rad} = \frac{180°}{\pi} \approx 57.3°$$

$$1° = \frac{\pi}{180°} \approx 0.017\,46 \text{ rad}$$

INFORMATION

The table gives a conversion table for some special angles.

Angle (°)	Angle (rad)
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$

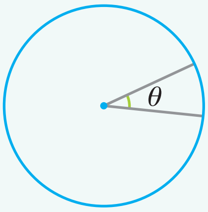
Class Discussion

Estimate the Size of Angles in Radians

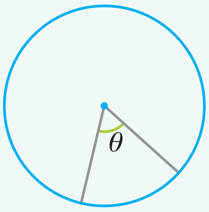
Work in pairs.

Look at the figures below.

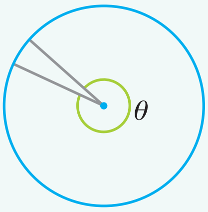
(a)



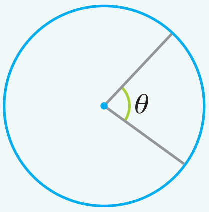
(b)



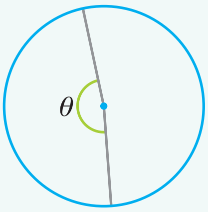
(c)



(d)



(e)



(f)

Identify which figure shows an angle θ of

(i) 1 radian, **(ii)** 2 radians, **(iii)** 3 radians,

(iv) 6 radians, **(v)** 0.5 radians, **(vi)** 1.5 radians.

Worked Example 6

(Converting from Radians to Degrees)

Convert each of the following angles from radians to degrees.

(a) $\frac{\pi}{12}$ **(b)** 2.8

Solution:

(a) Since π radians = 180°,

$$\frac{\pi}{12}\text{ rad} = \frac{180°}{12}$$
$$= 15°$$

(b) Since π radians = 180°,

$$1\text{ radian} = \frac{180°}{\pi}.$$
$$2.8\text{ radians} = 2.8 \times \frac{180°}{\pi}$$
$$= 160.4° \text{ (to 1 d.p.)}$$

Gottfried Wilhelm von Leibnitz (1646-1716) was a German genius who won recognition in law, philosophy, religion, literature, metaphysics and mathematics. He developed a method of calculating π without reference to a circle. He proved that $\frac{\pi}{4}$ could be determined to any desired degree of accuracy by the formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

PRACTISE NOW 6

Convert each of the following angles from radians to degrees.

(a) $\frac{\pi}{15}$ **(b)** $\frac{3\pi}{2}$

(c) 3.04 **(d)** 8

SIMILAR QUESTIONS

Exercise 8C Questions 1(a)-(d)

Worked Example 7

(Converting from Degrees to Radians)

Convert each of the following angles from degrees to radians.

(a) 27° **(b)** 315°

Solution:

(a) Since 180° = π radians,

$$27° = \frac{\pi}{180°} \times 27°$$
$$= 0.471 \text{ radians (to 3 s.f.)}$$

(b) Since 180° = π radians,

$$315° = \frac{\pi}{180°} \times 315°$$
$$= \frac{7\pi}{4}$$
$$= 5.50 \text{ radians (to 3 s.f.)}$$

PRACTISE NOW 7

Convert each of the following angles from degrees to radians.

(a) 36° **(b)** 288° **(c)** 197.5° **(d)** 400°

SIMILAR QUESTIONS

Exercise 8C Questions 2(a)-(d), 3(a)-(d)

Use of Calculators

Worked Example 8

(Use of a Calculator)

Find the value of each of the following.

(a) sin 1.2 **(b)** cos 0.879

(c) tan 1.012

INFORMATION

Most scientific calculators enable us to find the value of a trigonometric function in both radians and degrees by setting the mode to either 'radian' or 'degree'.

Solution:

(a) To find the value of sin 1.2, first set the calculator to the 'radian' mode.

Press $\boxed{\sin}\boxed{1}\boxed{.}\boxed{2}\boxed{=}$ and the display shows 0.932 039 086,

i.e. sin 1.2 = 0.932 (to 3 s.f.)

(b) To find the value of cos 0.879, press $\boxed{\cos}\boxed{0}\boxed{.}\boxed{8}\boxed{7}\boxed{9}\boxed{=}$ and the display shows 0.637 921 564,

i.e. cos 0.879 = 0.638 (to 3 s.f.)

(c) Press $\boxed{\tan}\boxed{1}\boxed{.}\boxed{0}\boxed{1}\boxed{2}\boxed{=}$ and the display shows 1.599 298 86,

i.e. tan 1.012 = 1.60 (to 3 s.f.)

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 8C Questions 4(a)-(f)

Find the value of each of the following.

(a) sin 0.65 **(b)** cos 0.235 **(c)** tan 1.23

Worked Example 9

(Use of a Calculator)

For each of the following, find the value of x in the range $0 < x < \frac{\pi}{2}$.

(a) $\sin x = 0.45$ **(b)** $\cos x = 0.605$

(c) $\tan x = 2.4$

Problem Solving Tip

Since $\frac{\pi}{2}$ radians = 90°,

$0 < x < \frac{\pi}{2}$ means $0° < x < 90°$,

i.e. x is an acute angle.

Solution:

(a) For $\sin x = 0.45$, press $\boxed{\sin^{-1}}\boxed{0}\boxed{.}\boxed{4}\boxed{5}\boxed{=}$ to get 0.466 765 339,

i.e. when $\sin x = 0.45$,

$x = 0.467$ radians (to 3 s.f.)

(b) For $\cos x = 0.605$, press $\boxed{\cos^{-1}}\boxed{0}\boxed{.}\boxed{6}\boxed{0}\boxed{5}\boxed{=}$ to get 0.921 030 459,

i.e. when $\cos x = 0.605$,

$x = 0.921$ radians (to 3 s.f.)

(c) For $\tan x = 2.4$, press $\boxed{\tan^{-1}}\boxed{2}\boxed{.}\boxed{4}\boxed{=}$ to get 1.176 005 207,

i.e. when $\tan x = 2.4$,

$x = 1.18$ radians (to 3 s.f.)

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 8C Questions 5(a)-(f)

For each of the following, find the value of x in the range $0 < x < \frac{\pi}{2}$.

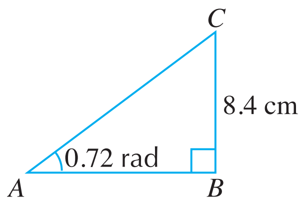
(a) $\sin x = 0.87$ **(b)** $\cos x = 0.347$ **(c)** $\tan x = 0.88$

Worked Example 10

(Solving a Triangle in Radian Mode)

In ΔABC, $\angle ABC = \frac{\pi}{2}$. Given that $BC = 8.4$ cm and $\angle BAC = 0.72$ radians, calculate the length of

(i) AB, **(ii)** AC.



Solution:

(a) $\tan \angle CAB = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AB}$

$\tan 0.72 = \frac{8.4}{AB}$

$\therefore AB = \frac{8.4}{\tan 0.72}$

$= 9.58$ cm (to 3 s.f.)

(b) $\sin \angle CAB = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AC}$

$\sin 0.72 = \frac{8.4}{AC}$

$\therefore AC = \frac{8.4}{\sin 0.72}$

$= 12.7$ cm (to 3 s.f.)

Problem Solving Tip

$\angle ABC = \frac{\pi}{2} = \frac{180°}{2} = 90°$

i.e. ΔABC is a right-angled triangle.

ATTENTION

We can also use Pythagoras' Theorem to find the length of AC, i.e. $AC^2 = AB^2 + BC^2$

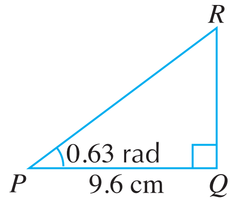
PRACTISE NOW 10

SIMILAR QUESTIONS

Exercise 8C Questions 6-8

In ΔPQR, $\angle PQR = \frac{\pi}{2}$. Given that $PQ = 9.6$ cm and $\angle QPR = 0.63$ radians, calculate the length of

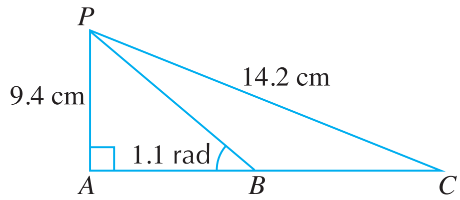
(i) QR, **(ii)** PR.



Worked Example 11

(Solving a Triangle in Radian Mode)

In ΔPAC, $PA = 9.4$ cm, $PC = 14.2$ cm and $\angle PAC = \frac{\pi}{2}$. B lies on AC such that $\angle PBA = 1.1$ radians.



Find

(i) $\angle ACP$ in radians,

(ii) the length of AB,

(iii) the length of BC.

Solution:

(i) In ΔPAC,

$$\sin \angle ACP = \frac{9.4}{14.2}$$

$$\therefore \angle ACP = \sin^{-1}\left(\frac{9.4}{14.2}\right)$$

$$= 0.723 \text{ radians (to 3 s.f.)}$$

(ii) In ΔPAB,

$$\tan 1.1 = \frac{9.4}{AB}$$

$$AB = \frac{9.4}{\tan 1.1}$$

$$= 4.78 \text{ cm (to 3 s.f.)}$$

(iii) Using Pythagoras' Theorem,

$PC^2 = PA^2 + AC^2$

$14.2^2 = 9.4^2 + AC^2$

$AC^2 = 14.2^2 - 9.4^2$

$= 113.28$

$AC = \sqrt{113.28}$

$= 10.64$ cm (to 4 s.f.)

$\therefore BC = AC - AB$

$= 10.64 - 4.78$

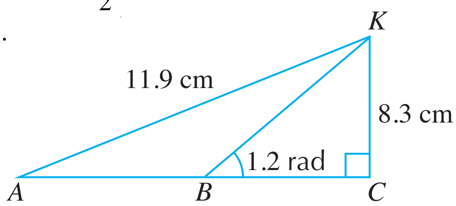
$= 5.86$ cm (to 3 s.f.)

PRACTISE NOW 11

SIMILAR QUESTIONS

Exercise 8C Questions 9, 10

In ΔACK, $KC = 8.3$ cm, $AK = 11.9$ cm and $\angle ACK = \frac{\pi}{2}$.
B lies on AC such that $\angle KBC = 1.2$ radians.

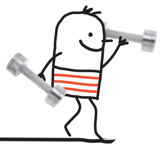


Find

(i) $\angle KAC$ in radians,

(ii) the length of BC,

(iii) the length of AB.



Exercise 8C

BASIC LEVEL

1. Convert each of the following angles from radians to degrees, giving your answer correct to 1 decimal place.

(a) $\frac{5\pi}{6}$ **(b)** $\frac{\pi}{7}$

(c) 3.2 **(d)** 2.56

2. Convert each of the following angles from degrees to radians.

(a) 37.4° **(b)** 78.9°

(c) 142° **(d)** 308°

3. Convert each of the following angles from degrees to radians, leaving your answer in terms of π.

(a) 15° **(b)** 18°

(c) 75° **(d)** 225°

4. Find the value of each of the following, where all the angles are in radians.

(a) $\sin 0.8$ **(b)** $\cos 0.543$

(c) $\tan 1.5$ **(d)** $\sin \frac{\pi}{8}$

(e) $\cos 0.45\pi$ **(f)** $\tan \frac{2\pi}{5}$

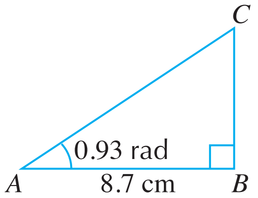
5. For each of the following, find the value of x in the range $0 < x < \frac{\pi}{2}$.

(a) $\sin x = 0.74$ **(b)** $\cos x = 0.17$

(c) $\tan x = 0.48$ **(d)** $\sin x = 0.147$

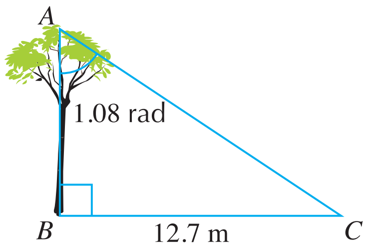
(e) $\cos x = 0.769$ **(f)** $\tan x = 1.256$

6. In ΔABC, $\angle ABC = \frac{\pi}{2}$. Given that $AB = 8.7$ cm and $\angle CAB = 0.93$ radians, calculate the length of



(i) BC, **(ii)** AC.

7. A point C on level ground is 12.7 m away from the foot B of a tree AB.



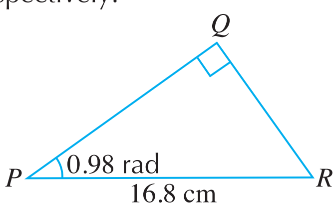
Given that $\angle BAC = 1.08$ radians, find

(i) the height of the tree,

(ii) the distance between A and C.

INTERMEDIATE LEVEL

8. The figure shows the cross section of a piece of cake PQR, in which $PR = 16.8$ cm, $\angle QPR = 0.98$ radians and $\angle PQR = \frac{\pi}{2}$. The edges PQ and QR are to be sprinkled with chocolate rice and icing sugar respectively.

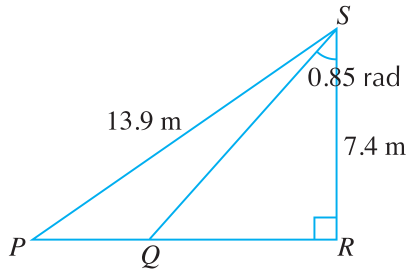


Find the length of the cake which is

(i) to be sprinkled with chocolate rice,

(ii) to be sprinkled with icing sugar.

9. At Large Splash, there is a swimming pool with 2 water slides SP and SQ.



Given that the points P, Q and R lie on the surface of the swimming pool such that $SP = 13.9$ m, $SR = 7.4$ m and $\angle QSR = 0.85$ radians, find

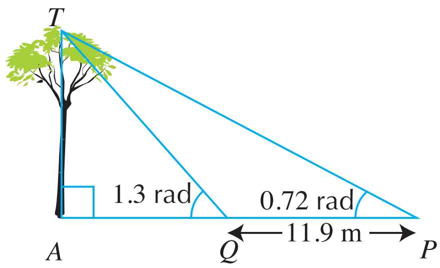
(i) $\angle PSQ$, the angle between the slides,

(ii) the length of slide SQ,

(iii) PQ, the distance between the bottom of both slides.

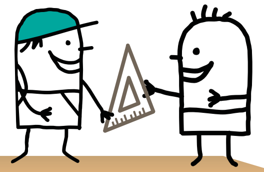
ADVANCED LEVEL

10. Two points P and Q, 11.9 m apart on level ground, are due east of the foot A of a tree TA.



Given that $\angle TPA = 0.72$ radians and $\angle TQA = 1.3$ radians, find the height of the tree.

8.4 Arc Length and Area of Sector using Radian Measure



From the investigation in Section 8.1 on the formula for arc length, we have discovered that

$$\frac{\text{Length of arc } AB}{\text{Circumference}} = \frac{\angle AOB}{360°}$$

$$\text{i.e. Length of arc } AB = \frac{x°}{360°} \times \text{Circumference}$$

However, if $\angle AOB = \theta$ is measured in radians, then

$$\frac{\text{Length of arc } AB}{\text{Circumference}} = \frac{\theta}{2\pi}$$

$$\begin{aligned}\text{i.e. Length of arc } AB &= \frac{\theta}{2\pi} \times \text{Circumference} \\ &= \frac{\theta}{2\pi} \times 2\pi r \\ &= r\theta\end{aligned}$$

In general,

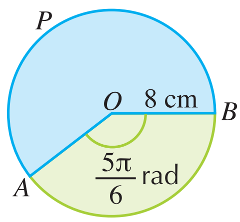
$$\boxed{s = r\theta,}$$

where s is the length of an arc of a circle of radius r units and θ, in radians, is the angle subtended by the arc at the centre of the circle.

Worked Example 12

(Finding Arc Length using Radian Measure)

In the figure, O is the centre of a circle of radius 8 cm and $\angle AOB = \frac{5\pi}{6}$ radians.



Find the length of the major arc APB.

Solution:

Reflex $\angle AOB = 2\pi - \frac{5\pi}{6}$ $(360° = 2\pi)$

$= \frac{7\pi}{6}$ radians

Length of major arc $APB = 8 \times \frac{7\pi}{6}$ (using $s = r\theta$, θ in radians)

$= \frac{28\pi}{3}$

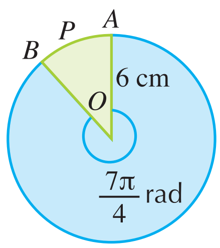
$= 29.3$ cm (to 3 s.f.)

PRACTISE NOW 12

SIMILAR QUESTIONS

Exercise 8D Questions 1(a)-(d)

In the figure, O is the centre of a circle of radius 6 cm and reflex $\angle AOB = \frac{7\pi}{4}$ radians. Find the length of the minor arc APB.



From the investigation in Section 8.2 on the formula for area of a sector, we have discovered that

$$\frac{\text{Area of sector } OAB}{\text{Area of circle}} = \frac{\angle AOB}{360°}$$

$$\text{i.e. Area of sector } OAB = \frac{x°}{360°} \times \text{Area of circle}$$

However, if $\angle AOB = \theta$ is measured in radians, then

$$\frac{\text{Area of sector } OAB}{\text{Area of circle}} = \frac{\theta}{2\pi}$$

$$\begin{aligned}\text{i.e. Area of sector } OAB &= \frac{\theta}{2\pi} \times \text{Area of circle} \\ &= \frac{\theta}{2\pi} \times \pi r^2 \\ &= \frac{1}{2} r^2 \theta\end{aligned}$$

In general,

$$\boxed{A = \frac{1}{2} r^2 \theta,}$$

where A is the area of a sector of a circle of radius r units and θ, in radians, is the angle subtended by the arc at the centre of the circle.

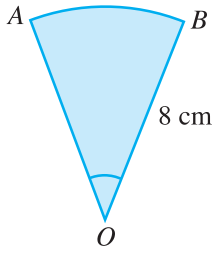
Worked Example 13

(Finding Area of a Sector using Radian Measure)

The figure shows a sector OAB of radius 8 cm and a perimeter of 23 cm. Find

(i) $\angle AOB$ in radians,

(ii) the area of the sector OAB.



Solution:

(i) Given that the perimeter = 23 cm,

$$\begin{aligned} r + r + s &= 23 \\ 8 + 8 + s &= 23 \\ 16 + s &= 23 \\ s &= 23 - 16 \\ &= 7 \text{ cm} \end{aligned}$$

Using $s = r\theta$,

$$7 = 8\theta$$

$$\theta = \frac{7}{8} \text{ radians}$$

$$\therefore \angle AOB = \frac{7}{8} \text{ radians}$$

(ii) Area of sector OAB

$$= \frac{1}{2} \times 8^2 \times \frac{7}{8}$$

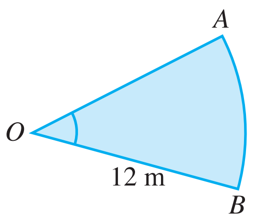
$$= 28 \text{ cm}^2$$

PRACTISE NOW 13

SIMILAR QUESTIONS

Exercise 8D Questions 2(a)-(d), 3, 4(a)-(f), 5-9, 20

1. The figure shows a sector OAB of radius 12 m and a perimeter of 33 m.

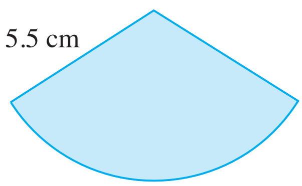


Find

(i) $\angle AOB$ in radians,

(ii) the area of the sector OAB.

2. A circle of radius 5.5 cm has a sector with an area of 30.25 cm^2.

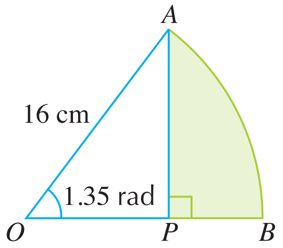


Calculate the perimeter of this sector.

Worked Example 14

(Finding Arc Length and Area of Sector using Radian Measure)

In the figure, AB is an arc of a circle centre O and radius 16 cm. P is a point on OB such that AP is perpendicular to OB and $\angle AOB = 1.35$ radians.



(i) Show that OP is approximately 3.50 cm.
(ii) Find the length of AP.
(iii) Find the area of the shaded region.

Solution:

(i) In ΔAOP,

$$\cos 1.35 = \frac{OP}{16}$$
$$\therefore OP = 16 \cos 1.35$$
$$= 3.50 \text{ cm (to 3 s.f.)}$$

(ii) In ΔAOP,

$$\sin 1.35 = \frac{AP}{16}$$
$$\therefore AP = 16 \sin 1.35$$
$$= 15.6 \text{ cm (to 3 s.f.)}$$

(iii) Area of shaded region = area of sector OAB – area of ΔAOP

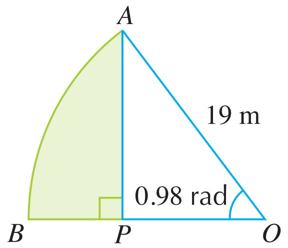
$$= \frac{1}{2} \times 16^2 \times 1.35 - \frac{1}{2} \times 3.504 \times 15.61$$
$$= 145 \text{ cm}^2$$

PRACTISE NOW 14

SIMILAR QUESTIONS

Exercise 8D Questions 10-12, 21-23

In the figure, AB is an arc of a circle centre O and radius 19 m. P is a point on OB such that AP is perpendicular to OB and $\angle AOB = 0.98$ radians.

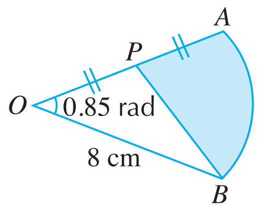


(i) Show that OP is approximately 10.6 m.
(ii) Find the length of AP.
(iii) Find the area of the shaded region.

Worked Example 15

(Problem involving Area of a Triangle)

In the figure, OAB is a sector of a circle with centre O and radius 8 cm.



P is the midpoint of OA and $\angle AOB = 0.85$ radians. Find

(i) the length of the arc AB,

(ii) the area of the shaded region.

Solution:

(i) Length of arc $AB = r\theta$

$= 0.85 \times 8$

$= 6.8$ cm

(ii) Area of sector $OAB = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 8^2 \times 0.85$

$= 27.2 \text{ cm}^2$

Area of $\Delta OPB = \frac{1}{2} \times OP \times OB \times \sin \angle POB$ (using formula Area $= \frac{1}{2}ab\sin C$)

$= \frac{1}{2} \times 4 \times 8 \times \sin 0.85$ ($OP = \frac{1}{2}OB$)

$= 12.02 \text{ cm}^2$ (to 4 s.f.)

$\therefore$ Area of shaded region = Area of sector OAB – area of ΔOPB

$= 27.2 - 12.02$

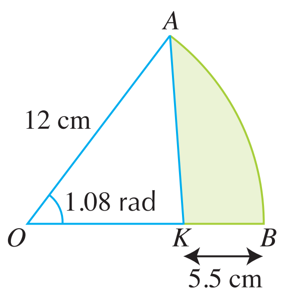
$= 15.2 \text{ cm}^2$ (to 3 s.f.)

PRACTISE NOW 15

SIMILAR QUESTIONS

Exercise 8D Questions 13-16, 24

In the figure, OAB is a sector of a circle with centre O and radius 12 cm. K is a point on OB such that $KB = 5.5$ cm and $\angle AOB = 1.08$ radians. Find



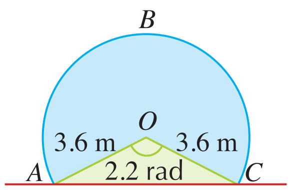
(i) the length of the arc AB,

(ii) the area of the shaded region.

Worked Example 16

(Finding Area of Segment using Radian Measure)

The figure shows the major segment ABC representing the cross section of a train tunnel, centre O and radius 3.6 m, such that $\angle AOC = 2.2$ radians.



(i) Show that AC is approximately 6.42 m.

(ii) Find the length of the major arc ABC, giving your answer in terms of π.

(iii) Find the area of the cross section of the tunnel, i.e. area of major segment ABC.

Solution:

(i) Using cosine rule,

$$AC^2 = OA^2 + OC^2 - 2(OA)(OC)\cos\angle AOC$$
$$= 3.6^2 + 3.6^2 - 2(3.6)(3.6)\cos 2.2$$
$$= 41.17 \text{ (to 4 s.f.)}$$
$$AC = 6.42 \text{ m (to 3 s.f.)}$$

(ii) Reflex $\angle AOC = (2\pi - 2.2)$ radians

$\therefore$ Length of major arc $ABC = 3.6 \times (2\pi - 2.2)$

$= (7.2\pi - 7.92)$ m

(iii) Area of major segment ABC = Area of major sector $OABC$ + area of ΔOAC

$$= \frac{1}{2} \times 3.6^2 \times (2\pi - 2.2) + \frac{1}{2} \times 3.6^2 \times \sin 2.2$$
$$= 31.7 \text{ m}^2 \text{ (to 3 s.f.)}$$

$\therefore$ Area of the cross section of the tunnel = 31.7 m^2

RECALL

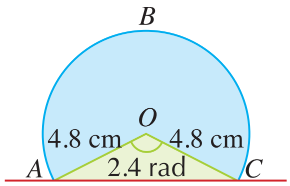
Area of triangle $= \frac{1}{2}ab\sin C$

PRACTISE NOW 16

SIMILAR QUESTIONS

Exercise 8D Questions 17-19, 25

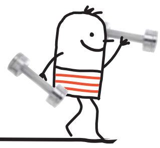
The figure shows the major segment *ABC* representing the cross section of a tunnel in a children's toy set, centre *O* and radius 4.8 cm, such that $\angle AOC = 2.4$ radians.



(i) Show that *AC* is approximately 8.95 cm.

(ii) Find the length of the major arc *ABC*, giving your answer in terms of π.

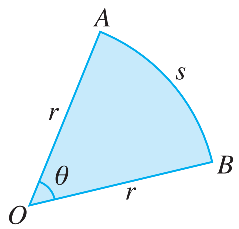
(iii) Find the area of the cross section of the tunnel, i.e. area of major segment *ABC*.



Exercise 8D

BASIC LEVEL

1. For each of the following sectors, find the arc length.



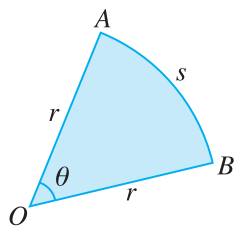
(a) $r = 6$ cm, $\theta = 1.6$ rad

(b) $r = 14$ cm, $\theta = 0.25$ rad

(c) $r = 25$ m, $\theta = 1.75$ rad

(d) $r = 12$ mm, $\theta = \frac{3}{4}$ rad

2. For each of the following sectors, find the area.



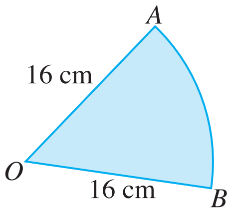
(a) $r = 8$ cm, $\theta = 2.2$ rad

(b) $r = 17$ cm, $\theta = 0.46$ rad

(c) $r = 33$ m, $\theta = \frac{1}{5}$ rad

(d) $r = 94$ mm, $\theta = 0.6$ rad

3. The figure shows a sector OAB of radius 16 cm and a perimeter of 50 cm.



Find

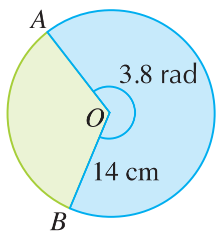
(i) $\angle AOB$ in radians,

(ii) the area of the sector OAB.

4. Copy and complete the table for sectors of a circle.

	Radius	Angle at centre	Arc length	Area
(a)	4 cm	1.25 rad		
(b)	6 cm		9 cm	
(c)		0.8 rad	9.6 m	
(d)		1.2 rad		60 m^2
(e)	8 mm			64 mm^2
(f)			6 mm	27 mm^2

5. OAB is a major sector of a circle centre O and radius 14 cm.



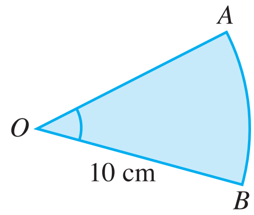
Given that reflex $\angle AOB = 3.8$ radians, find

(i) the perimeter of major sector OAB,

(ii) the area of the minor sector OAB.

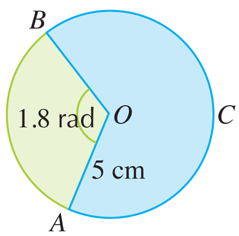
INTERMEDIATE LEVEL

6. Given that a sector OAB of radius 10 cm has an area of 60 cm^2, find the perimeter of the sector.



7. Given that a sector of radius 18 m has an area of 729 m^2, find the perimeter of the sector.

8. A, B and C are points on a circle with centre O and radius 5 cm, with obtuse $\angle AOB = 1.8$ radians.



(i) Find the area of minor sector AOB.

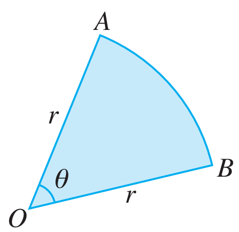
(ii) Write down an expression, in terms of π, for the size of reflex $\angle AOB$.

(iii) Find the length of major arc ACB, giving your answer in the form $a + b\pi$, where a and b are rational numbers.

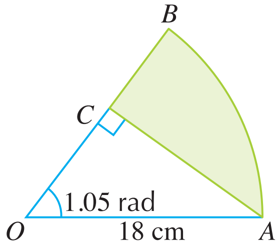
9. Given that the perimeter of the sector AOB is 18 cm and that the area is 8 cm^2,

(i) form a pair of simultaneous equations involving r and θ,

(ii) find the value of r and of θ.

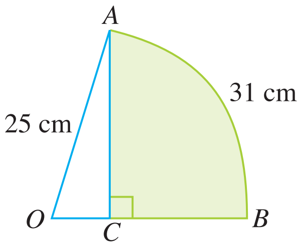


10. In the figure, AB is an arc of a circle centre O and radius 18 cm. C is a point on OB such that AC is perpendicular to OB and $\angle AOB = 1.05$ radians.



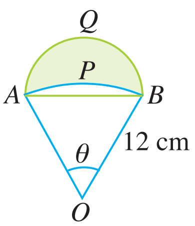
Find
(i) the length of AC,
(ii) the area of the shaded region.

11. In the figure, OAB is a sector of a circle centre O and radius 25 cm. AC is perpendicular to OB and the length of minor arc AB is 31 cm.



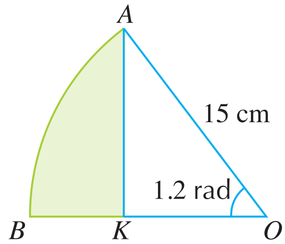
Find the area of the shaded region.

12. In the figure, APB is an arc of a circle centre O and radius 12 cm. ABQ is a semicircle with AB as diameter. ΔAOB is an equilateral triangle and $\angle AOB = \theta$ radians.



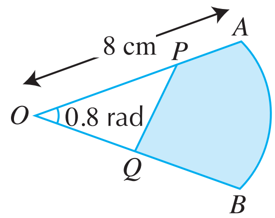
(i) Write down the value of θ.
(ii) Find the length of the arc APB.
(iii) Find the area of the shaded region.

13. In the figure, OAB is a sector of a circle with centre O and radius 15 cm. K is a point on OB such that $2OK = 3BK$ and $\angle AOB = 1.2$ radians.



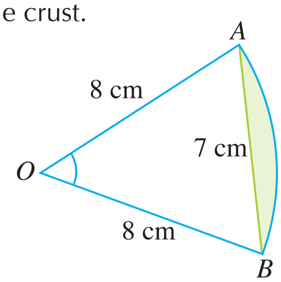
Find
(i) the length of the arc AB,
(ii) the area of the shaded region.

14. The figure shows a sector of a circle OAB, with centre O and radius 8 cm. The point P lies on OA such that $\frac{OP}{PA} = \frac{3}{2}$ and the point Q lies on OB such that $OQ : QB = 3 : 4$.



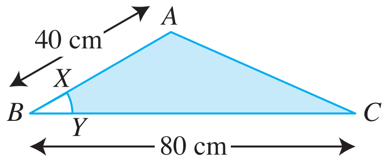
Given that $\angle AOB = 0.8$ radians, find
(i) the length of the arc AB,
(ii) the area of the shaded region.

15. The figure shows a slice of an apple pie in the shape of a sector of a circle with centre O and radius 8 cm. A cut is made along AB to remove some of the crust.



Given that $AB = 7$ cm, find
(i) the maximum number of slices that can be obtained from one full pie,
(ii) the area of the shaded segment that has been removed.

16. In ΔABC, $AB = 40$ cm, $BC = 80$ cm and $\angle ABC = \frac{\pi}{6}$. XY is an arc of a circle centre B and radius 10 cm.

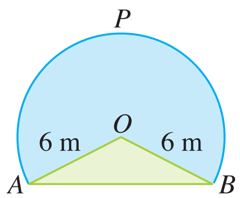


Find

(i) the area of ΔABC,

(ii) the area of the shaded region.

17. The figure shows the major segment APB centre O and radius 6 m.



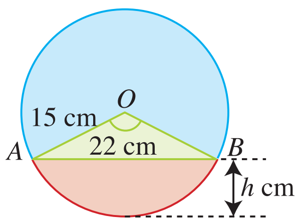
Given that the length of the major arc APB is 24 m, find

(i) reflex $\angle AOB$ in radians,

(ii) the length of AB,

(iii) the area of the major segment APB.

18. The figure shows the cross section of a water pipe of radius 15 cm. Water flowing in the pipe has a height of h cm and a horizontal width of 22 cm.

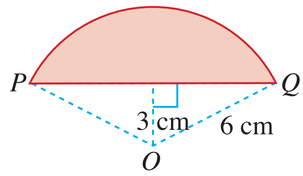


Find

(i) the value of h,

(ii) the area of the cross section which contains water.

19. The figure shows a minor segment of a circle centre O and radius 6 cm. PQ is a chord of the circle such that its distance from O is 3 cm.



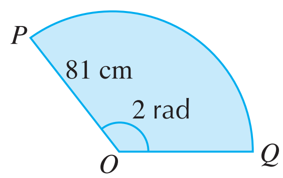
Find

(i) obtuse $\angle POQ$ in radians,

(ii) the area of the shaded segment.

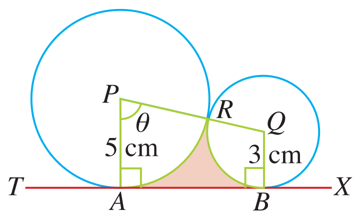
ADVANCED LEVEL

20. An arc PQ of a circle, centre O, subtends an angle of 2 radians at the centre.



This sector is folded to form a right circular cone so that the arc PQ becomes the circumference of the base. Find the height of the cone.

21. Two circles with centres P and Q touch each other externally at R. $TABX$ is a common tangent to the two circles where $\angle BAP = \angle ABQ = \frac{\pi}{2}$ radians.

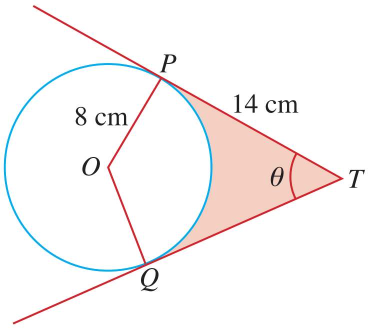


Given that the radii of the circles are 5 cm and 3 cm respectively and $\angle APQ = \theta$ radians, find

(i) the value of θ,

(ii) the area of the shaded region.

22. In the figure, O is the centre of the circle whose radius is 8 cm. TP and TQ are tangents to the circle such that $\angle OPT = \angle OQT = \frac{\pi}{2}$ radians.

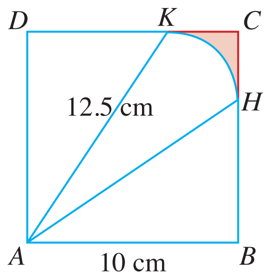


Given that $TP = 14$ cm and $\angle PTQ = \theta$ radians, find

(i) the value of θ,

(ii) the area of the shaded region.

23. In the figure, $ABCD$ is a square of side 10 cm.

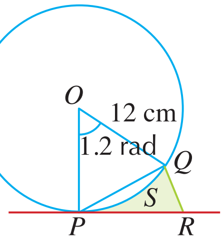


Given that HK is an arc of a circle centre A and radius AK where $AK = 12.5$ cm, find

(i) $\angle HAK$ in radians,

(ii) the area of the shaded region HCK.

24. The figure shows a circle centre O and radius 12 cm.



Given that $\angle POQ = 1.2$ radians and PR is a tangent to the circle where $\angle OPR = \frac{\pi}{2}$ radians and the length of PR is equal to that of the arc PSQ, find

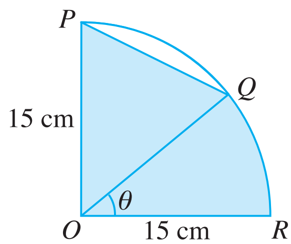
(i) the length of the arc PSQ,

(ii) the area of the segment PSQ,

(iii) the length of the chord PQ,

(iv) the area of the shaded region $PSQR$.

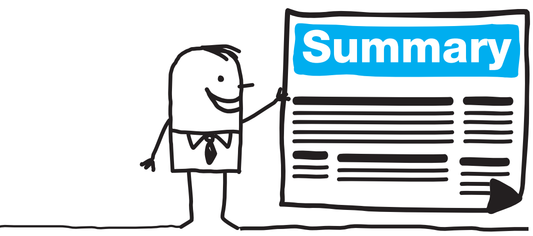
25. The figure shows a quadrant of a circle centre O and radius 15 cm. Q is a point on the arc such that the lengths of arcs PQ and QR are in the ratio 4 : 3.



Given that $\angle QOR = \theta$ radians, find

(i) the value of θ in terms of π,

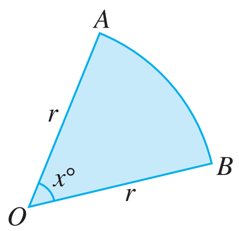
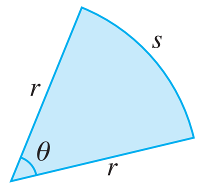
(ii) the area of the shaded region.



1. A radian is the size of the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.

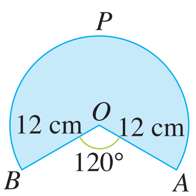
$$\pi \text{ radians} = 180°$$
$$1 \text{ radian} = \frac{180°}{\pi}$$
$$1° = \frac{\pi}{180°} \text{ radians}$$

2. To find the arc length and area of a sector:

For a sector subtending an angle $x°$ at the centre of a circle of radius r:	For a sector of a circle of radius r enclosed by two radii that subtend an angle of θ radians at the centre:
	
Arc length $AB = \frac{x°}{360°} \times 2\pi r$ Area of sector $OAB = \frac{x°}{360°} \times \pi r^2$	Arc length, $s = r\theta$ Area of sector, $A = \frac{1}{2}r^2\theta$

Review Exercise 8

1. In the figure, BPA is the major arc of a circle centre O and radius 12 cm.

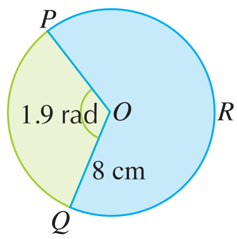


Given that $\angle BOA = 120°$, find

(i) the length of arc BPA,

(ii) the area of the sector $OBPA$.

2. The figure shows a circle with centre O, radius 8 cm and $\angle POQ = 1.9$ radians.



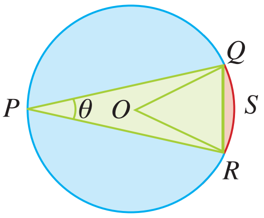
Find

(i) the area of the minor sector POQ,

(ii) reflex $\angle POQ$ in terms of π,

(iii) the length of the major arc PRQ in terms of π.

3. In the figure, O is the centre of a circle of radius 8 cm and ΔPQR is an isosceles triangle.

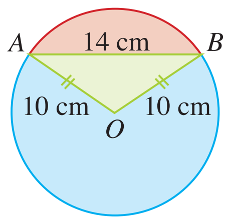


Given that $\angle QPR = \theta$ radians and $QR = 7$ cm, find

(i) the value of θ,

(ii) the area of the minor segment QSR.

4. A log cake baked for a Christmas party has a circular cross section with centre O and radius 10 cm. The top section of the cake is sliced off along AB and is to be replaced with cake toppings.

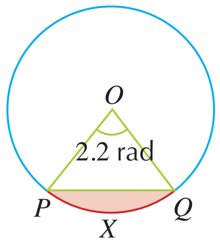


Given that $AB = 14$ cm, find

(i) $\angle AOB$ in radians,

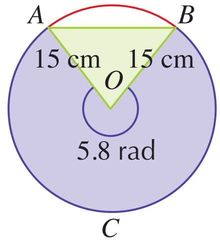
(ii) the cross-sectional area of the cake that is sliced off.

5. The figure shows the cross section of a drainage pipe. The shaded minor segment PQX of the circle centre O, represents the part of the pipe which contains mud.



Given that $\angle POQ = 2.2$ radians, express the area of the segment as a percentage of the area of the circle, giving your answer correct to the nearest integer.

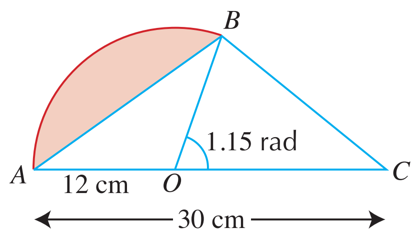
6. A dart board is in the shape of a circle with centre O and radius 15 cm. ΔOAB is shaded green such that $OA = OB = 15$ cm. The major segment $OACB$ is shaded purple such that reflex $\angle AOB = 5.8$ radians.



(i) Show that the area of ΔOAB is approximately 52.3 cm^2.

(ii) Given that the dart lands on the board, find the probability that it lands on the unshaded area, giving your answer in standard form.

7. In the figure, OAB is a sector of a circle with centre O and radius 12 cm. AOC is a straight line such that $AC = 30$ cm and $\angle BOC = 1.15$ radians.



Calculate

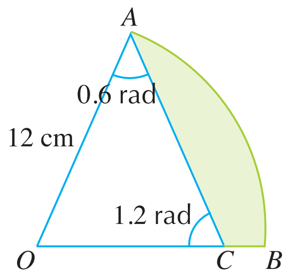
(i) the length of the arc AB,

(ii) the area of the shaded segment,

(iii) the area of ΔBOC,

(iv) the length of BC.

8. In the figure, OAB is a sector of a circle with centre O and radius 12 cm. C is a point on radius OB such that $\angle OAC$ = 0.6 radians and $\angle OCA$ = 1.2 radians.

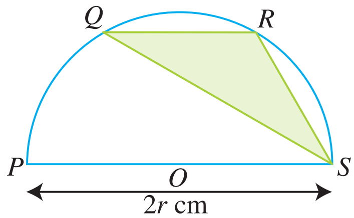


Find

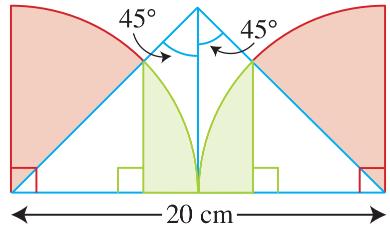
(i) the length of BC and of AC,

(ii) the area of the shaded region.

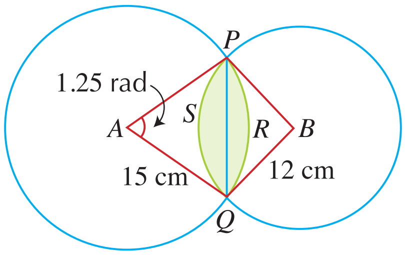
9. In the figure, $PQRS$ is a semicircle with centre at O. Given that $PS = 2r$ cm and the lengths of arcs PQ, QR and RS are equal, calculate the area of the shaded region, giving your answer in terms of r.



10. Find the total area of the shaded region in the figure below, giving your answer in terms of π.



11. The points P, R and Q lie on the circumference of the circle with centre A and radius 15 cm such that $\angle PAQ$ = 1.25 radians.



(a) Find

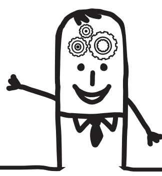
(i) the length of the chord PQ,

(ii) the area of the minor sector APQ,

(iii) the area of ΔPAQ.

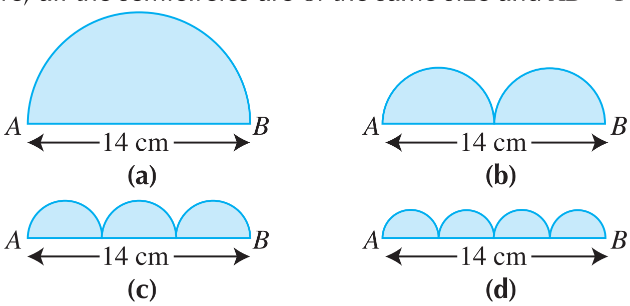
A second circle with centre B and radius 12 cm passes through the points P, S and Q.

(b) Calculate the area of the shaded region.



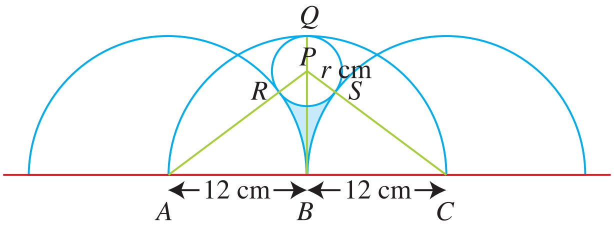
Challenge Yourself

1. The figures below show various semicircles.
In each figure, all the semicircles are of the same size and AB = 14 cm.



Find the perimeter of each figure, leaving your answer in terms of π. What do you observe? Can you generalise your observations?

2. The figure shows three semicircles each of radius 12 cm with centres at A, B and C in a straight line. A fourth circle with a centre at P and radius r cm is drawn to touch the other three semicircles.



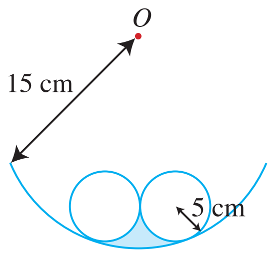
Given that BPQ is a straight line, find

(i) the value of r,

(ii) $\angle PAC$ in radians,

(iii) the area of the shaded region.

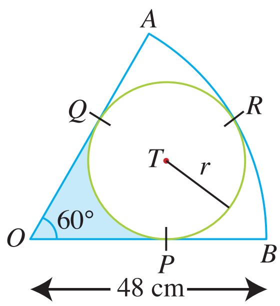
3. The figure shows two identical circles of radius 5 cm touching each other externally. The two circles also touch a larger circle, centre O and radius 15 cm internally.



(i) Show that the perimeter of the shaded region is $\frac{35\pi}{3}$ cm.

(ii) Find the area of the shaded region.

4. In the figure, the circle PQR with centre T and with radius r is enclosed in the sector OAB with centre at O and radius 48 cm.



Given that $\angle AOB = 60°$ and $\angle OPT = \angle OQT = \frac{\pi}{2}$ radians, find

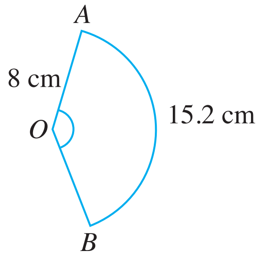
(i) the value of r,

(ii) the area of the shaded region.

C1 Revision Exercise

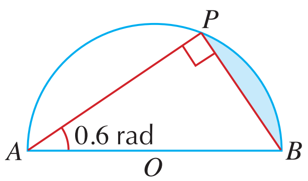
1. Given that $0 \leqslant x \leqslant 2\pi$, find the values of x for which $\sin x = 0.345$.

2. In the figure, OAB is a sector of a circle of radius 8 cm and the length of arc $AB = 15.2$ cm.



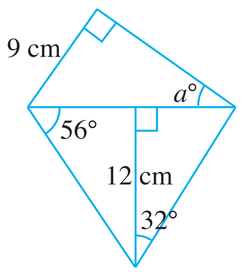
Find
(i) $\angle AOB$ in radians,
(ii) the area of the sector.

3. The figure shows a semicircle $AOBP$ where O is the centre and ΔAPB is right-angled at P.

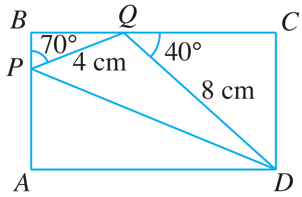


Given that $AB = 16$ cm and $\angle PAB = 0.6$ radians, find
(i) the length of AP,
(ii) the area of ΔPAB,
(iii) the area of the shaded region.

4. Find the value of a in the figure.

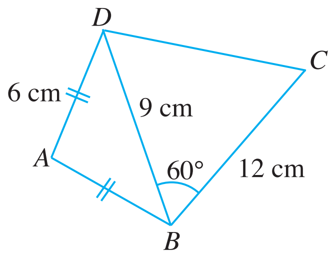


5. In the figure, $ABCD$ is a rectangle. The points P and Q lie on AB and BC respectively such that $PQ = 4$ cm, $QD = 8$ cm, $\angle BPQ = 70°$ and $\angle CQD = 40°$.



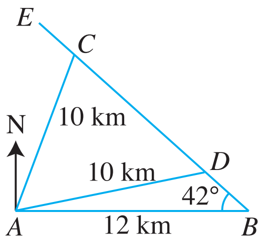
Find
(i) the length of CD,
(ii) $\angle APD$,
(iii) the length of AD.

6. The figure shows a quadrilateral $ABCD$ in which $AB = AD = 6$ cm, $BD = 9$ cm, $BC = 12$ cm and $\angle CBD = 60°$.



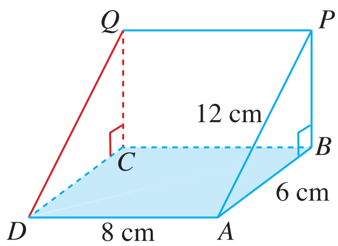
Find
(i) $\angle BAD$,
(ii) the area of ΔABD,
(iii) the area of the quadrilateral $ABCD$,
(iv) $\angle ABC$,
(v) the length of DC.

7. The figure shows a point B which lies 12 km due east of A. A straight road BE makes an angle of 42° with AB. C and D are two points on the road such that $AD = AC = 10$ km.



Find
(i) the bearing of C from A,
(ii) the bearing of D from A,
(iii) the distance between C and D.

8. The diagram shows a right triangular prism with $\angle ABP = 90°$ and $ABCD$ lying on a horizontal table.



Given that $AB = 6$ cm, $AD = 8$ cm and $AP = 12$ cm, find
(i) $\angle PAB$,
(ii) the length of PB,
(iii) $\angle PDB$.

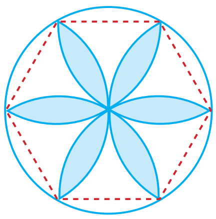
9. From the top T of an observation tower, which is 20 m high, the angle of depression of a ship, A, which is due east of T, is 25.4°. Another ship, B, which is due west of T, finds the angle of elevation of T to be 54.7°. Calculate the distance between A and B.

C2 Revision Exercise

1. Given that $0 \leqslant x \leqslant 3$, find the value of x for which $\cos x = -0.5$.

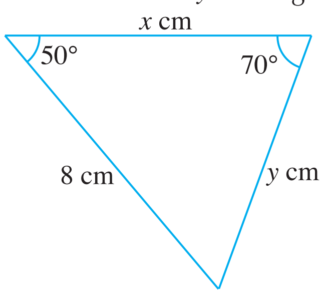
2. Given that $\sin 40° = 0.643$ and $\cos 15° = 0.966$ when corrected to 3 significant figures, find the value of each of the following without the use of a calculator.
(a) $\sin 140°$ **(b)** $\cos 165°$

3. In the figure, the radius of the circle and of each arc is 4 cm.

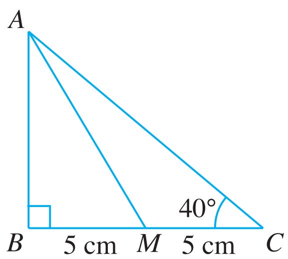


Find the area of the shaded region.

4. Find the value of x and of y in the given triangle.

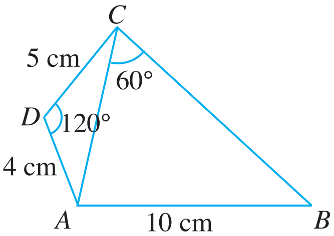


5. In the figure, ΔABC is right-angled at B and M is the midpoint of BC.



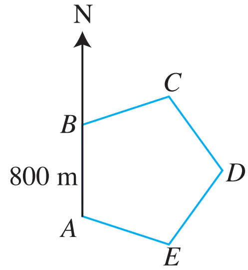
Given that $BM = MC = 5$ cm and $\angle ACB = 40°$, find
(i) the length of AM,
(ii) the length of AC,
(iii) $\angle AMB$,
(iv) $\angle CAM$.

6. The figure shows a quadrilateral $ABCD$ in which $AD = 4$ cm, $DC = 5$ cm, $AB = 10$ cm, $\angle ADC = 120°$ and $\angle ACB = 60°$.



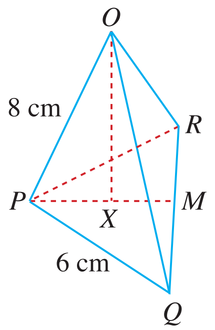
Find
(i) the length of AC,
(ii) $\angle DAC$,
(iii) $\angle ABC$
(iv) the length of BD,
(v) the area of the quadrilateral $ABCD$.

7. During a junior sailing competition, the course of the race is in the form of a regular pentagon $ABCDE$, where B is due north of A and $AB = 800$ m.



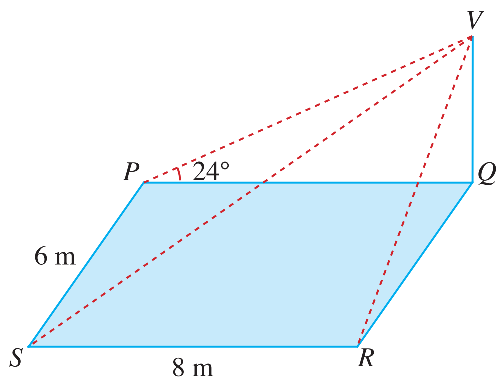
(a) Find the bearing of
  (i) C from B,
  (ii) E from A,
  (iii) D from E.
(b) Calculate the area of $ABCDE$, giving your answer in standard form correct to 4 significant figures.

8. *OPQR* is a triangular pyramid. *M* is the midpoint of *QR*, $PQ = QR = PR = 6$ cm and $OP = OQ = OR = 8$ cm.



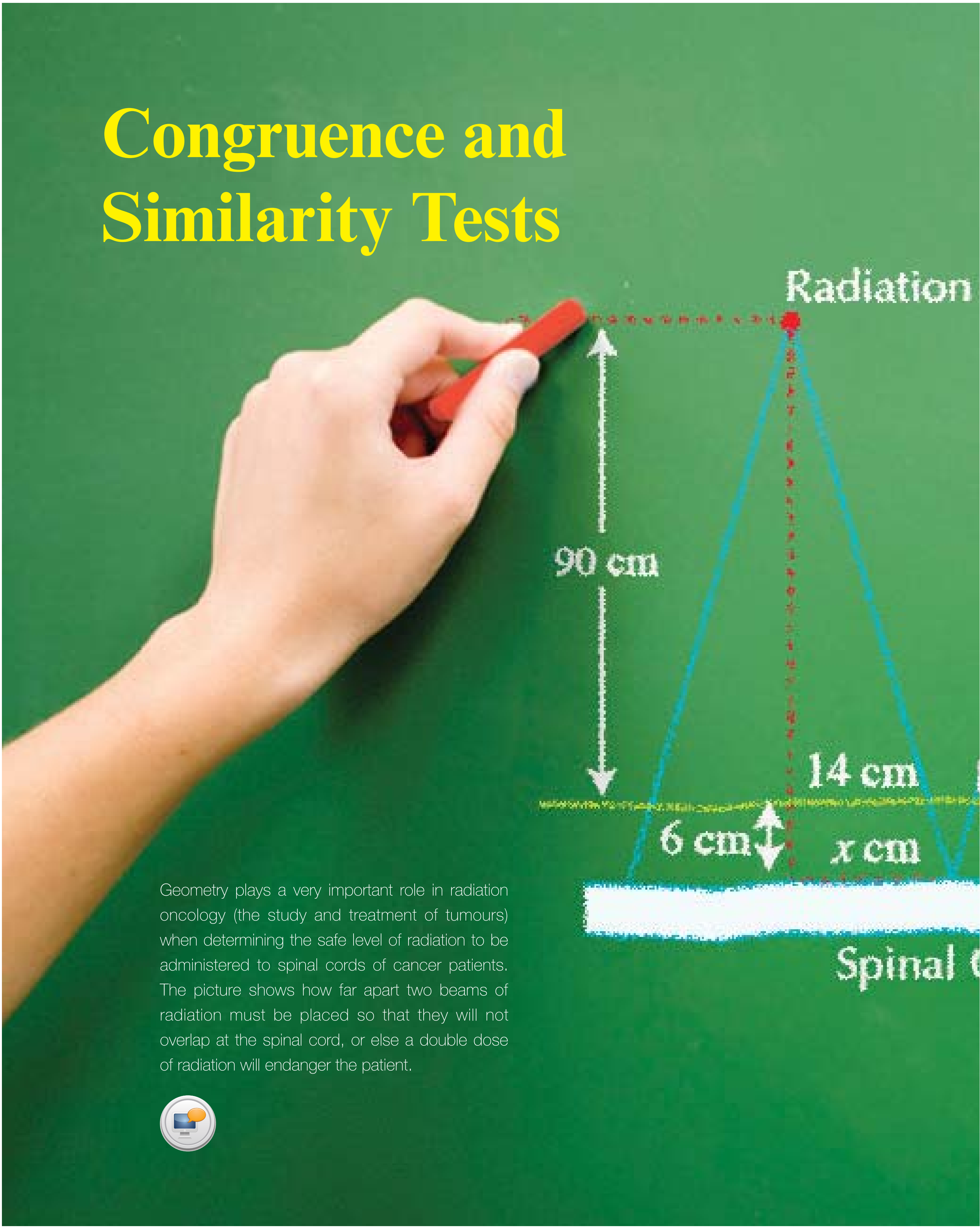
(a) Find
  (i) the height *OX* of the pyramid,
  (ii) *OPX*.
(b) Given that $PM : XM = 2 : 1$, find $\angle OMX$.

9. A vertical pole *VQ* stands at one corner of a horizontal rectangular field.



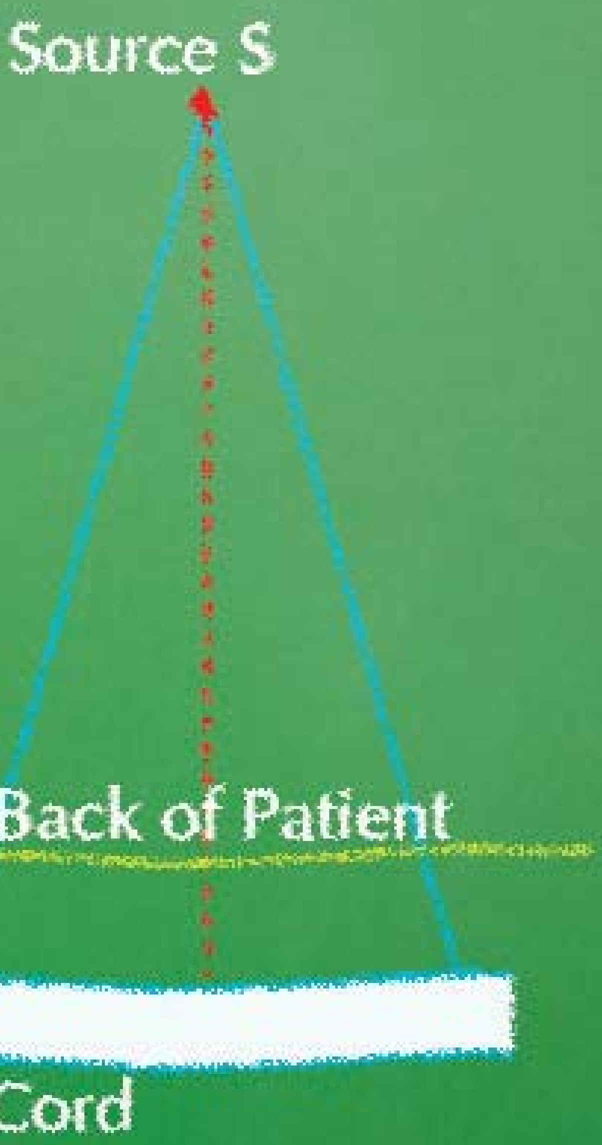
Given that $SR = 8$ m, $PS = 6$ m and the angle of elevation of *V* from *P* is 24°, find
(i) the distance between *V* and *Q*,
(ii) the distance between *V* and *S*,
(iii) the angle between *VS* and *VQ*,
(iv) the angle of elevation of *V* from *R*.

Congruence and Similarity Tests



Geometry plays a very important role in radiation oncology (the study and treatment of tumours) when determining the safe level of radiation to be administered to spinal cords of cancer patients. The picture shows how far apart two beams of radiation must be placed so that they will not overlap at the spinal cord, or else a double dose of radiation will endanger the patient.

Chapter Nine

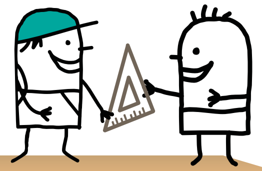


LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- apply the four congruence tests to determine whether two or more triangles are congruent,
- apply the three similarity tests to determine whether two or more triangles are similar,
- solve problems involving congruent and/or similar triangles.

9.1 Congruence Tests



Recap

In Book 2, we have learnt that congruent figures have exactly the same shape and size; and they can be mapped onto one another under translation, rotation and reflection.

For congruent triangles, this would mean that all the corresponding lengths are equal and all the corresponding angles are equal.

For example, ΔABC is congruent to ΔXYZ (and we write $\Delta ABC \cong \Delta XYZ$) if and only if

$AB = XY$
$BC = YZ$
$AC = XZ$
$B\hat{A}C = Y\hat{X}Z$
$A\hat{B}C = X\hat{Y}Z$
$A\hat{C}B = X\hat{Z}Y$

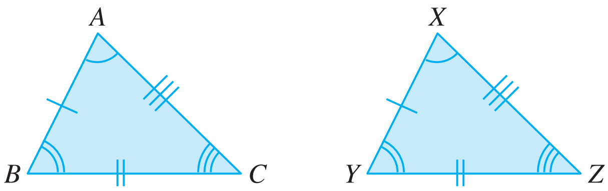


Fig. 9.1

ATTENTION

The vertices of the 2 triangles must match:

$A \leftrightarrow X$
$B \leftrightarrow Y$
$C \leftrightarrow Z$

INFORMATION

$A\hat{B}C$ can also be written as $\angle ABC$. Similarly, $X\hat{Y}Z$ can be written as $\angle XYZ$.

Consider ΔEFG, where

$EF = AB$
$FG = BC$
$EG = AC$
$F\hat{E}G = B\hat{A}C$
$E\hat{F}G = A\hat{B}C$
$E\hat{G}F = A\hat{C}B$

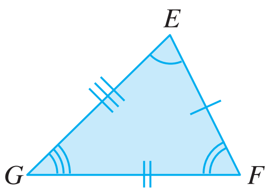


Fig. 9.2

ΔEFG is still congruent to ΔABC even though ΔEFG is laterally inverted, because ΔEFG can be mapped onto ΔABC by a reflection (and a translation if necessary).

In this section, we will investigate whether we need all the 6 conditions (i.e. 3 pairs of corresponding sides are equal and 3 pairs of corresponding angles are equal) to prove that two triangles are congruent.

SSS Congruence Test

Investigation

SSS Congruence Test

1. Cut 3 satay sticks so that the lengths of the sticks are 5 cm, 8 cm and 10 cm.
2. Try to form a triangle using the 3 satay sticks in as many ways as possible, such that the lengths of the triangle correspond to the lengths of the 3 sticks.

   (If you do not have satay sticks, you can try to construct the triangle using the three given lengths in as many ways as possible.)
3. Do you get the following triangle?

   If you get a different triangle, flip it over and see if it fits onto this triangle.

   Is it possible to get other triangles?

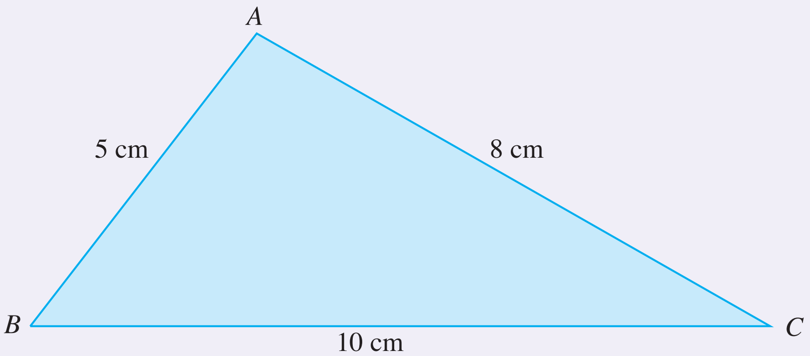


Fig. 9.3

4. Try using 3 other satay sticks of different lengths, such that the sum of the lengths of the two shorter sticks is greater than the length of the longest stick, and see if you always get a unique triangle (regardless of reflection) no matter how you try to form a triangle.
5. What can you conclude from this investigation?

ATTENTION

If the sum of the lengths of the two shorter sticks is less than or equal to the length of the longest stick, then you cannot form a triangle.

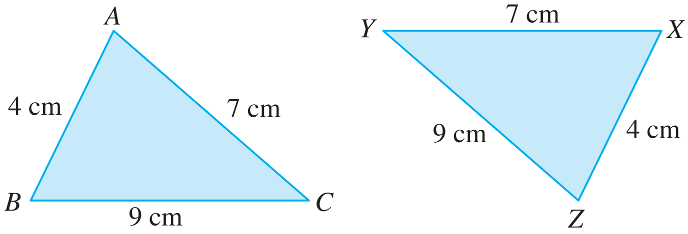
From the investigation, we observe the following:

> **SSS Congruence Test**: If the 3 sides of a triangle are equal to the 3 corresponding sides of another triangle, then the two triangles are congruent.

Worked Example 1

(Proving that Two Triangles are Congruent using the SSS Congruence Test)

Prove that the following two triangles are congruent.



Problem Solving Tip

How to match the vertices correctly:

Step 1: Match the vertex opposite the longest side for both triangles, i.e. $A \leftrightarrow X$.

Step 2: Match the vertex opposite the shortest side for both triangles, i.e. $C \leftrightarrow Y$.

Step 3: Match the last vertex, i.e. $B \leftrightarrow Z$.

In writing the proof, **all the vertices must match**.

Solution:

$A \leftrightarrow X$

$B \leftrightarrow Z$

$C \leftrightarrow Y$

$AB = XZ = 4$ cm

$BC = ZY = 9$ cm

$AC = XY = 7$ cm

$\therefore$ ΔABC • ΔXZY (SSS)

ATTENTION

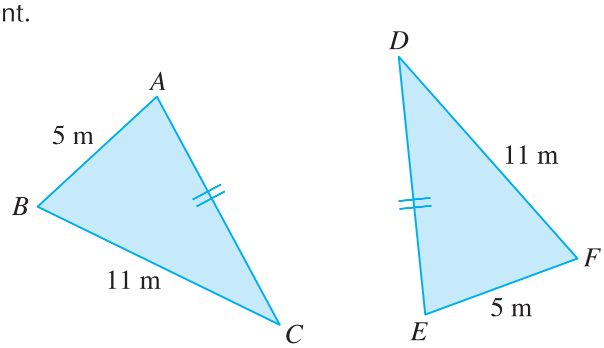
When ΔABC is congruent to ΔAZY, we can denote the relationship as follows: $\Delta ABC \equiv \Delta XZY$.

PRACTISE NOW 1

SIMILAR QUESTIONS

Exercise 9A Questions 1(a), 2(a), 3(a), 4(a),(b)

1. Copy and complete the proof to show that the following two triangles are congruent.



Solution:

$A \leftrightarrow$ ___

$B \leftrightarrow$ ___

$C \leftrightarrow$ ___

$AB =$ _____ $= 5$ m

$BC =$ _____ $=$ ___ m

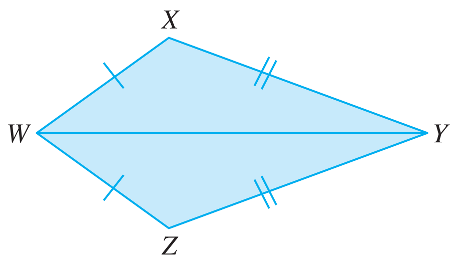
$AC =$ _____ (given)

$\therefore$ ΔABC • Δ_____ (_____)

ATTENTION

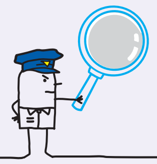
Even if two triangles are not congruent, we can still match the corresponding vertices because we must compare the longest side of one triangle with the longest side of the other triangle, etc.

2. The diagram shows a kite $WXYZ$.



Identify the two congruent triangles in the kite and prove that they are congruent.

SAS Congruence Test



Investigation

SAS Congruence Test

Part 1

1. Try to construct ΔXYZ such that $XY = 3$ cm, $YZ = 6$ cm and $X\hat{Y}Z = 50°$ in as many ways as possible.
Compare the triangle you have drawn with those drawn by your classmates.

2. Do you get the following triangles (not drawn to scale)?
Both triangles are actually congruent to each other: you can map both triangles together by a reflection.
Is it possible to get other triangles?

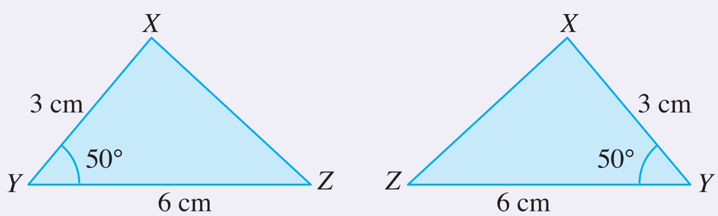


Fig. 9.4

3. Try to construct ΔXYZ for other dimensions, where XY and YZ have a fixed length and $X\hat{Y}Z$ is a fixed angle, in as many ways as possible and see if you always get a unique triangle (regardless of reflection).

4. Notice that the given $X\hat{Y}Z$ is between the two given sides XY and YZ: $X\hat{Y}Z$ is called the **included angle**.

5. What can you conclude from part 1 of this investigation?

Part 2

6. Try to construct ΔABC such that $AB = 5$ cm, $AC = 3$ cm and $A\hat{B}C = 30°$ in as many ways as possible.
7. Fig. 9.5 shows one possible ΔABC that satisfies the given dimensions.

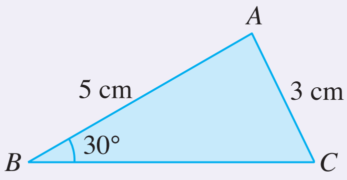


Fig. 9.5

Is it possible to construct a different ΔABC (excluding a laterally inverted triangle)?

8. Notice that the given $A\hat{B}C$ is not in between the two given sides AB and AC, i.e. $A\hat{B}C$ is *not the included angle*.
9. What can you conclude from part 2 of this investigation?

From the investigation, we observe the following:

> **SAS Congruence Test:** If 2 sides and the included angle of a triangle are equal to the 2 corresponding sides and the corresponding included angle of another triangle, then the two triangles are congruent.

If the given angle is not the included angle, then *SSA may **not** be a congruence test*.

For example, in the above investigation, there are two ways to construct ΔABC as shown in Fig. 9.6.

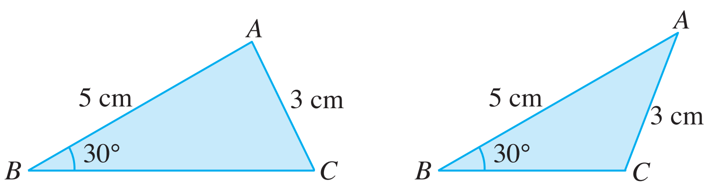


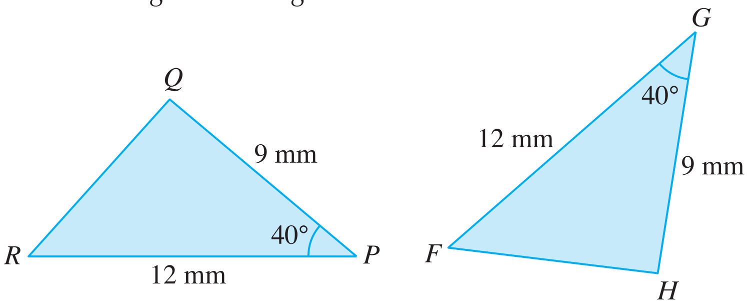
Fig. 9.6

Since the two triangles are not congruent, then SSA is not a congruence test in general, although there are exceptions (see RHS Congruence Test later in this section).

Worked Example 2

(Proving that Two Triangles are Congruent using the SAS Congruence Test)

Copy and complete the proof to show that the following two triangles are congruent.



Solution:

$P \leftrightarrow$ ___

$Q \leftrightarrow$ ___

$R \leftrightarrow$ ___

PQ = _____ = 9 mm

$Q\hat{P}R$ = _____ = ___°

PR = _____ = ___ mm

$\therefore \Delta PQR$ • Δ_____ (SAS)

Problem Solving Tip

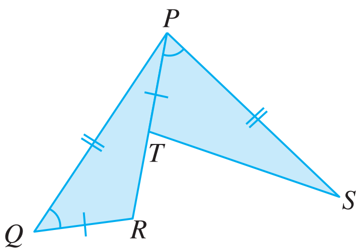
It is easier to match the vertices with the given angle first. Then match the vertex opposite either the 9 mm or 12 mm side for both triangles.

PRACTISE NOW 2

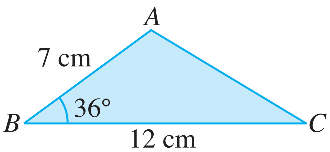
SIMILAR QUESTIONS

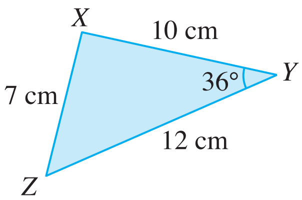
Exercise 9A Questions 1(b), 2(b), 3(b), 4(c),(d)

1. Prove that the following two triangles, where $P\hat{Q}R = S\hat{P}T$, are congruent.



2. Determine whether the following triangles are congruent.

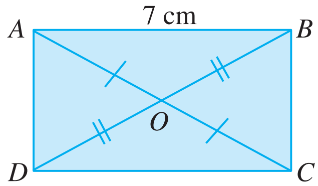




Worked Example 3

(Proving that Two Triangles are Congruent)

In the diagram, AOC and BOD are straight lines, $OA = OC$, $OB = OD$ and $AB = 7$ cm.



(i) Prove that ΔAOB is congruent to ΔCOD.

(ii) Find the length of CD.

Solution:

(i) $A \leftrightarrow C$

$O \leftrightarrow O$

$B \leftrightarrow D$

Problem Solving Tip

Match O first. Then from $OA = OC$, match A and C.

$OA = OC$ (given)

$A\hat{O}B = C\hat{O}D$ (vert. opp. $\angle$s)

$OB = OD$ (given)

$\therefore\ \Delta AOB \equiv \Delta COD$ (SAS)

(ii) Since $\Delta AOB \equiv \Delta COD$, then all the corresponding sides are equal.

$\therefore\ CD = AB$

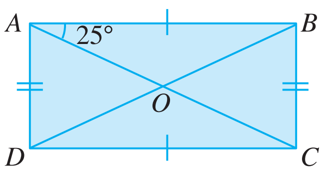
$= 7$ cm

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 9A Questions 5, 6

1. In the diagram, $ABCD$ is a rectangle, the two diagonals AC and BD intersect at O and $C\hat{A}B = 25°$.



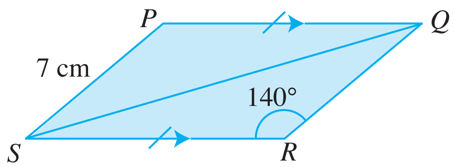
(i) Prove that ΔAOB is congruent to ΔCOD.

(ii) Find $B\hat{D}C$.

Problem Solving Tip

The diagonals of a rectangle bisect each other.

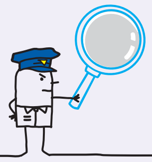
2. In the diagram, PQ is equal and parallel to SR, $PS = 7$ cm and $Q\hat{R}S = 140°$.



(i) Identify two congruent triangles and prove that they are congruent.

(ii) Find the length of QR and $Q\hat{P}S$.

AAS Congruence Test



Investigation

AAS Congruence Test

Part 1

1. Try to construct ΔPQR such that $PQ = 7$ cm, $Q\hat{P}R = 40°$ and $P\hat{Q}R = 60°$ in as many ways as possible.

Compare the triangles you have drawn with those drawn by your classmates.

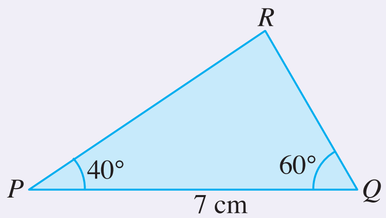


Fig. 9.7

2. Do you always get a unique triangle (regardless of reflection) as shown above (not drawn to scale)?

3. What can you conclude from part 1 of this investigation?

Part 2

4. Try to construct ΔABC such that $AB = 6$ cm, $B\hat{A}C = 50°$ and $A\hat{C}B = 70°$ in as many ways as possible.

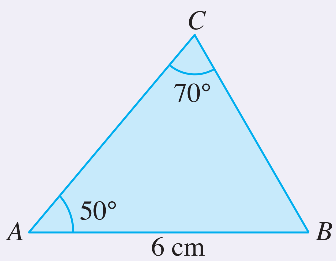


Fig. 9.8

Problem Solving Tip

An easier way to construct the vertex C is to find $A\hat{B}C$ first, and then construct $A\hat{B}C$.

5. Do you always get a unique triangle (regardless of reflection)?
6. What can you conclude from part 2 of this investigation?
7. Does it matter whether the given side of the triangle is between the 2 given angles (as in part 1) or if it is not between the 2 given angles (as in part 2)? Explain your answer.

   Hint: See Problem Solving Tip for Question 4 in this investigation.

From the investigation, we observe the following:

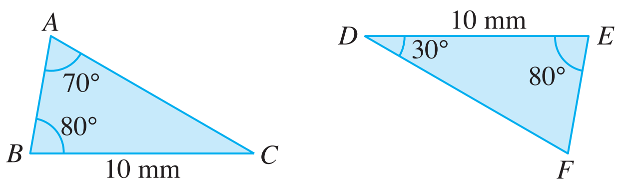
> **AAS Congruence Test:** If 2 angles and 1 side of a triangle are equal to the 2 corresponding angles and the corresponding side of another triangle, then the two triangles are congruent.

Since it does not matter whether or not the given side is between the two given angles, it does not matter whether we write AAS Congruence Test or ASA Congruence Test (unlike SAS Congruence Test where the given angle must be the included angle).

Worked Example 4

(Proving that Two Triangles are Congruent using the AAS Congruence Test)

Copy and complete the proof to show that the following two triangles are congruent.



Solution:

In ΔDEF, $E\hat{F}D = 180° - 80° - 30°$ ($\angle$ sum of a Δ)

$= 70°$

$A \leftrightarrow$ ___

$B \leftrightarrow$ ___

$C \leftrightarrow$ ___

$A\hat{B}C =$ _____ $= 80°$

$B\hat{A}C =$ _____ $=$ ___°

$BC =$ _____ $=$ ___ mm

$\therefore \Delta ABC$ • Δ_____ (AAS)

Problem Solving Tip

Sometimes there is a need to find the other unknown angle in either triangle.

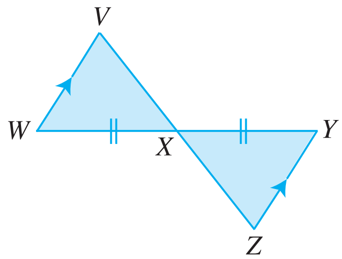
PRACTISE NOW 4

SIMILAR QUESTIONS

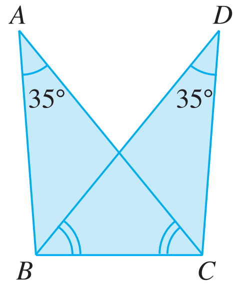
Exercise 9A Questions 1(c), 2(c), 3(c), 4(e),(f)

In each part, prove that the following two triangles are congruent.

(a)



(b)



RHS Congruence Test

Investigation

RHS Congruence Test

1. Try to construct ΔDEF such that $D\hat{E}F = 90°$, $DE = 3$ cm and $DF = 5$ cm in as many ways as possible.
   Compare the triangles you have drawn with those drawn by your classmates.

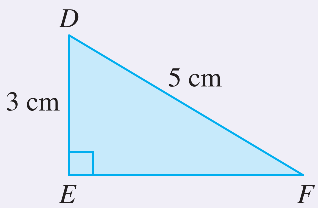


Fig. 9.9

2. Do you always get a unique triangle (regardless of reflection) as shown above (not drawn to scale)?
3. What can you conclude from the above investigation?
4. Do you notice that the above congruence test is a special case of SSA where the given angle is a right angle?

From the investigation, we observe the following:

> **RHS Congruence Test:** If the hypotenuse (H) and 1 side (S) of a right-angled (R) triangle are equal to the hypotenuse and 1 side of another right-angled triangle, then the two right-angled triangles are congruent.

INFORMATION

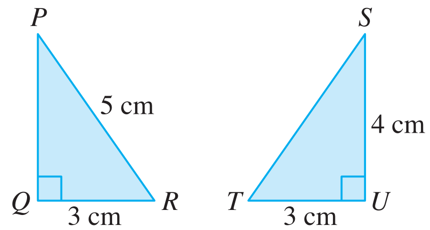
The 4 congruence tests covered so far are not the only congruence tests. There are more which are not included in the syllabus. An example is SSA, which is a congruence test if the given angle is obtuse.

In general, SSA is not a congruence test, but there are exceptions, one of which is the RHS Congruence Test.

Worked Example 5

(Proving that Two Triangles are Congruent using the RHS Congruence Test)

Copy and complete the proof to show that the following two triangles are congruent.



Solution:

By Pythagoras' Theorem,

$$\begin{aligned} ST &= \sqrt{TU^2 + SU^2} \\ &= \sqrt{3^2 + 4^2} \\ &= \sqrt{25} \\ &= 5 \text{ cm} \end{aligned}$$

$P \leftrightarrow$ ___

$Q \leftrightarrow$ ___

$R \leftrightarrow$ ___

$P\hat{Q}R$ = _____ = ___°

PR = _____ = ___ cm

QR = _____ = ___ cm

∴ ΔPQR • Δ_____ (RHS)

Problem Solving Tip

Sometimes there is a need to find the other unknown side in either triangle.

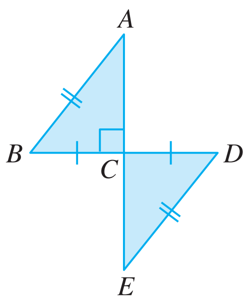
PRACTISE NOW 5

SIMILAR QUESTIONS

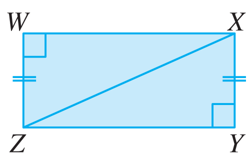
Exercise 9A Questions 1(d), 2(d), 3(d), 4(g),(h)

In each part, prove that the following two triangles are congruent.

(a)



(b)



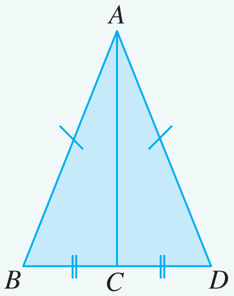
Class Discussion

Consolidation for Congruence Tests

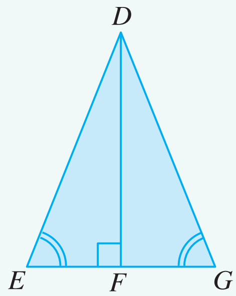
Work in pairs.

In each diagram, identify a pair of congruent triangles and prove that they are congruent.

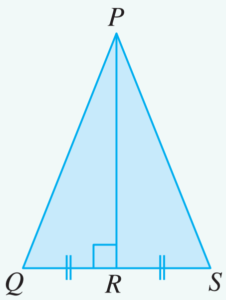
(a)



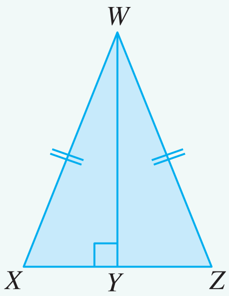
(b)



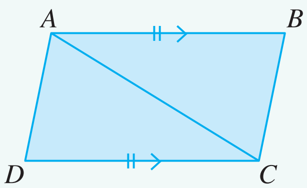
(c)



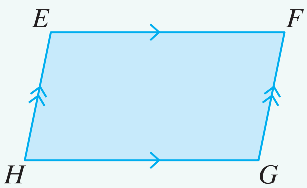
(d)



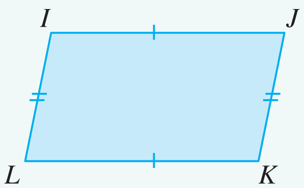
(e)



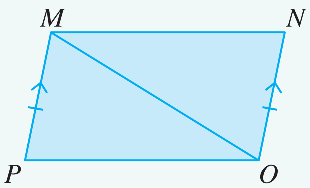
(f)



(g)

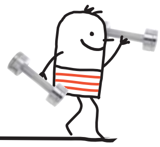


(h)



SIMILAR QUESTIONS

Exercise 9A Questions 7, 8



Exercise 9A

BASIC LEVEL

1. Identify a pair of congruent triangles from the following triangles (not drawn to scale), based on each of the following congruence tests:

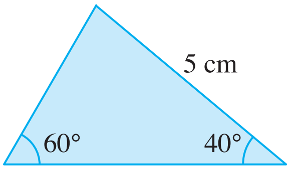
(a) SSS Congruence Test,

(b) SAS Congruence Test,

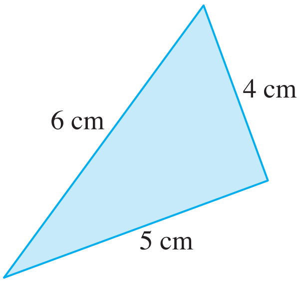
(c) AAS Congruence Test,

(d) RHS Congruence Test.

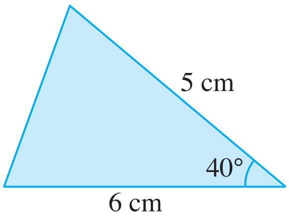
(i)



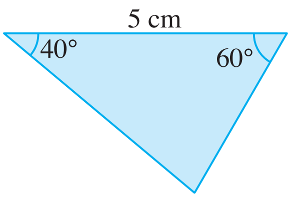
(ii)



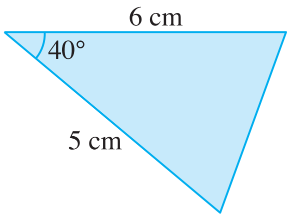
(iii)



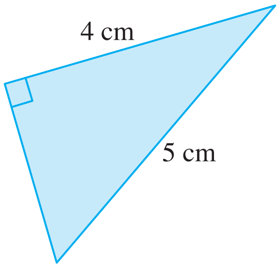
(iv)



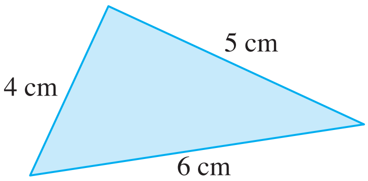
(v)



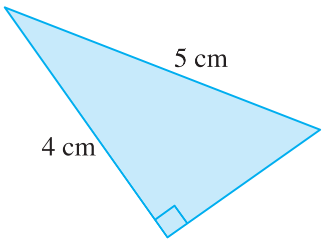
(vi)



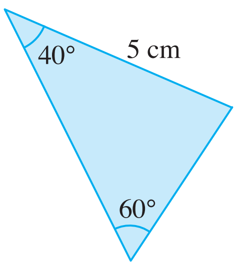
(vii)



(viii)

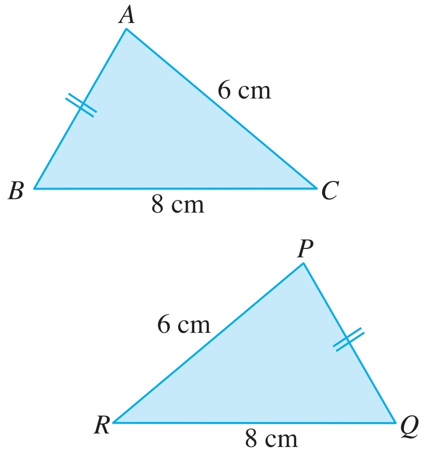


(ix)



2. Copy and complete the proof to show that each of the following pairs of triangles are congruent.

(a)



$A \leftrightarrow$ ___

$B \leftrightarrow$ ___

$C \leftrightarrow$ ___

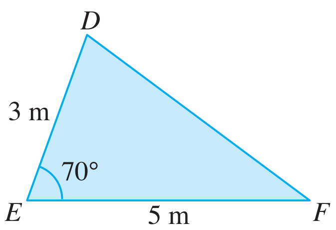
AB = _____ (given)

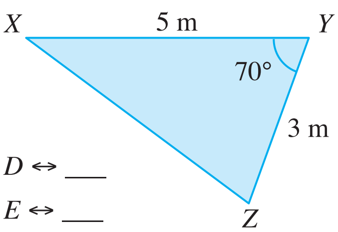
BC = _____ = 8 cm

AC = _____ = ___ cm

$\therefore$ ΔABC • Δ_____ (_____)

(b)





$D \leftrightarrow$ ___

$E \leftrightarrow$ ___

$F \leftrightarrow$ ___

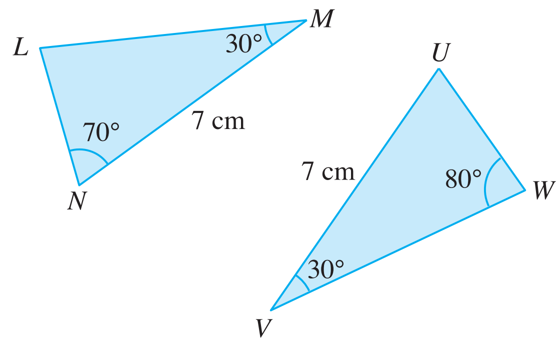
DE = _____ = 3 m

$D\hat{E}F$ = _____ = 70°

EF = _____ = ___ m

$\therefore$ ΔDEF • Δ_____ (_____)

(c)



$L \leftrightarrow$ ___

$M \leftrightarrow$ ___

$N \leftrightarrow$ ___

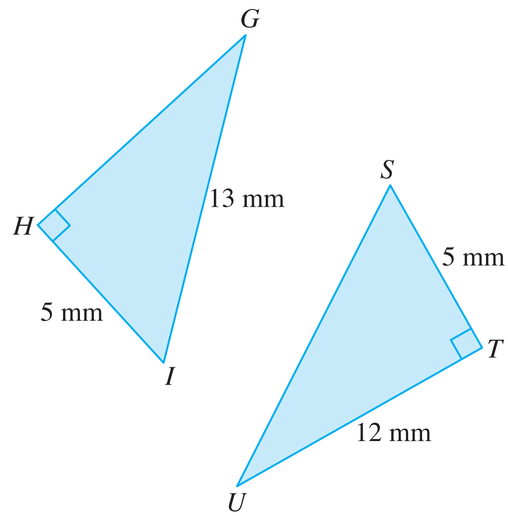
$L\hat{M}N$ = _____ = 30°

$L\hat{N}M$ = _____ = ___°

MN = _____ = ___ cm

$\therefore$ Δ_____ • ΔUVW (_____)

(d)



$G \leftrightarrow$ ___

$H \leftrightarrow$ ___

$I \leftrightarrow$ ___

$G\hat{H}I$ = _____ = ___°

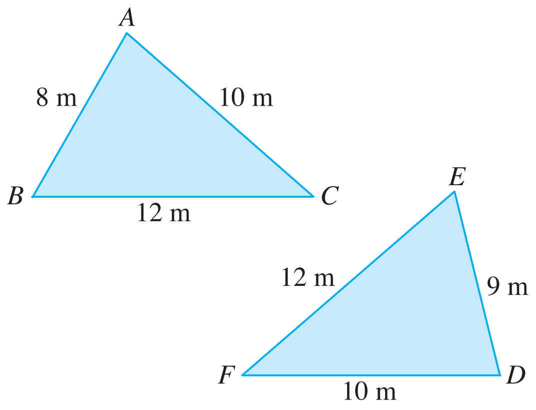
GI = _____ = 13 mm

HI = _____ = ___ mm

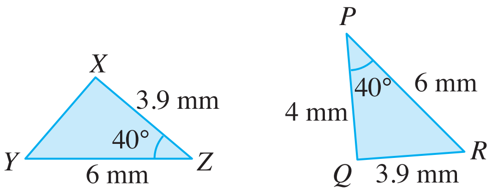
$\therefore$ Δ_____ • ΔSTU (_____)

3. Determine whether each of the following pairs of triangles are congruent.

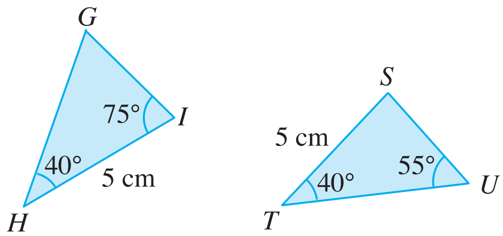
(a)



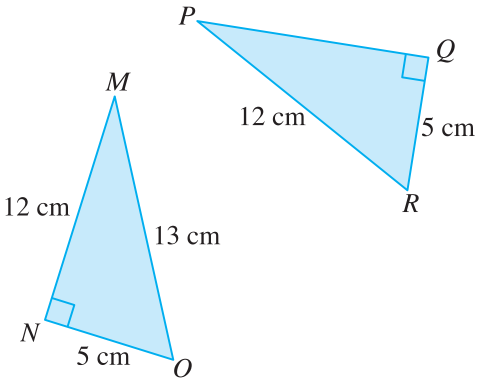
(b)



(c)



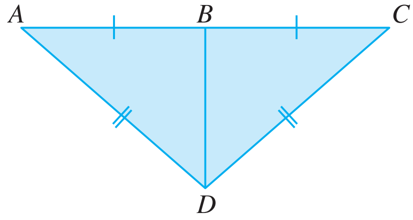
(d)



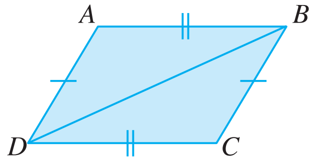
INTERMEDIATE LEVEL

4. In each diagram, identify a pair of congruent triangles and prove that they are congruent.

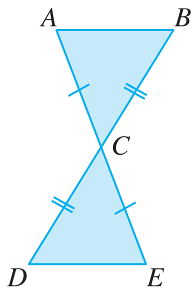
(a)



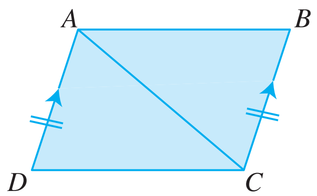
(b)



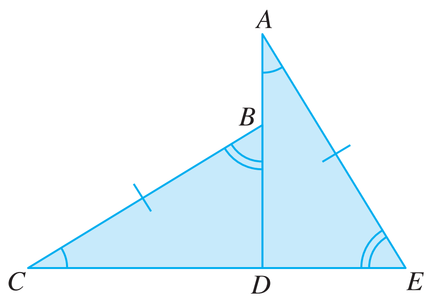
(c)



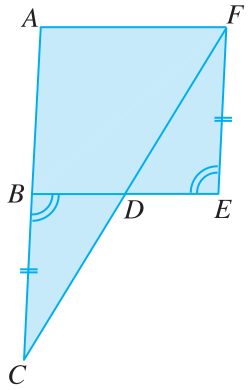
(d)



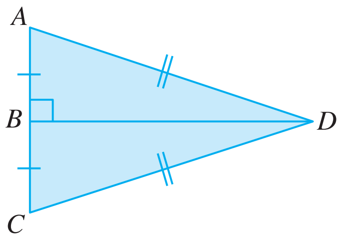
(e)



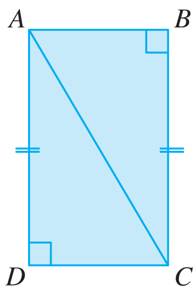
(f)



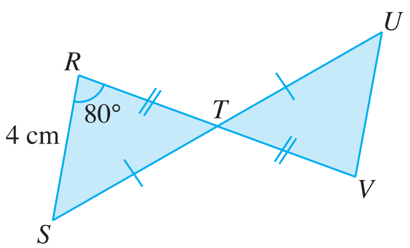
(g)



(h)



5. In the diagram, RTV and STU are straight lines, $RT = VT$, $ST = UT$, $RS = 4$ cm and $S\hat{R}T = 80°$.



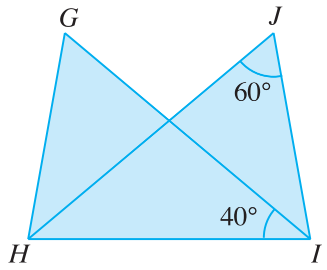
(i) Prove that ΔRST is congruent to ΔVUT.

(ii) Find the length of UV.

(iii) Find $U\hat{V}T$.

(iv) Hence, other than $RS = UV$, what can you conclude about the lines RS and UV?

6. In the diagram, $GH = JI$ and $GI = JH$.



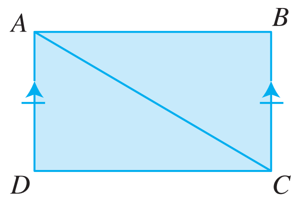
(i) Identify a triangle that is congruent to ΔGHI and prove that they are congruent.

(ii) If $H\hat{J}I = 60°$ and $G\hat{I}H = 40°$, find $G\hat{H}I$.

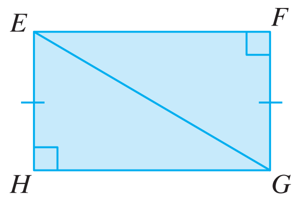
ADVANCED LEVEL

7. In each diagram, identify a pair of congruent triangles and prove that they are congruent.

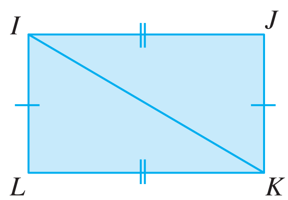
(a)



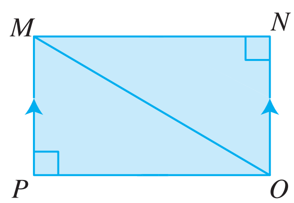
(b)



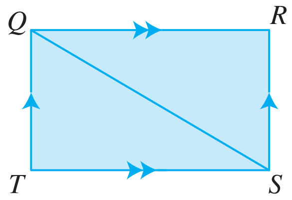
(c)



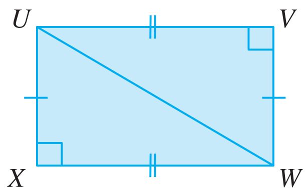
(d)



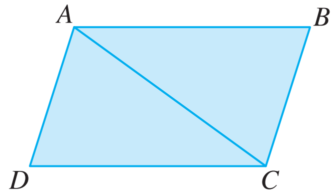
(e)



(f)

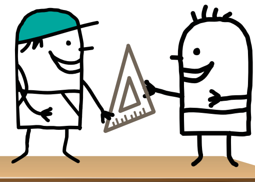


8. The diagram shows a parallelogram *ABCD*.



Use three different ways to prove that ΔABC and ΔCDA are congruent.

9.2 Similarity Tests



Recap

In Book 2, we have learnt that similar figures have exactly the same shape but *not* necessarily the same size (i.e. congruence is a special case of similarity); and they can be mapped onto one another under enlargement.

For similar triangles, this would mean that all the corresponding angles are equal, and all the corresponding sides are proportional (i.e. all the ratios of the corresponding sides are equal).

For example, ΔABC is similar to ΔXYZ if and only if

$B\hat{A}C = Y\hat{X}Z$

$A\hat{B}C = X\hat{Y}Z$

$A\hat{C}B = X\hat{Z}Y$

$\dfrac{AB}{XY} = \dfrac{BC}{YZ} = \dfrac{AC}{XZ}$

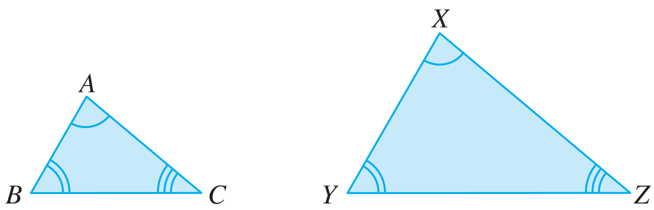


Fig. 9.10

ATTENTION

The vertices of the 2 triangles must match:

$A \leftrightarrow X$
$B \leftrightarrow Y$
$C \leftrightarrow Z$

In this section, we will investigate whether we need all the 6 conditions (i.e. 3 pairs of corresponding sides are equal and 3 ratios of corresponding sides are equal) to prove that two triangles are similar.

AA Similarity Test

Investigation

AA Similarity Test

1. Construct ΔABC and ΔXYZ of different sizes such that $B\hat{A}C = Y\hat{X}Z = 30°$ and $A\hat{B}C = X\hat{Y}Z = 50°$.

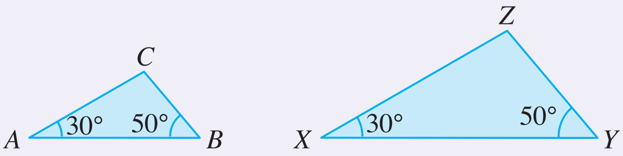


Fig. 9.11

2. Find $A\hat{C}B$ and $X\hat{Z}Y$ by using angle sum of triangle = 180°.
   Is $A\hat{C}B = X\hat{Z}Y$?
3. Measure the lengths of all the sides of the two triangles that you have constructed (not the ones shown in Fig. 9.11) and calculate the ratios $\frac{AB}{XY}$, $\frac{BC}{YZ}$ and $\frac{AC}{XZ}$ correct to 2 significant figures. Is $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$?
4. From what you have learnt in Book 2 which you have recapped at the beginning of Section 9.2, since all the corresponding angles of ΔABC and ΔXYZ are equal and all the 3 ratios of their corresponding sides are equal, are the two triangles similar?
5. It is very important to note that the above **given conditions** are **2 pairs of corresponding angles equal** (i.e. $B\hat{A}C = Y\hat{X}Z$ and $A\hat{B}C = X\hat{Y}Z$). Are these given conditions enough to prove that the two triangles are similar?

From the investigation, we observe the following:

> **AA Similarity Test:** If 2 angles of a triangle are equal to the 2 corresponding angles of another triangle, then the two triangles are similar.

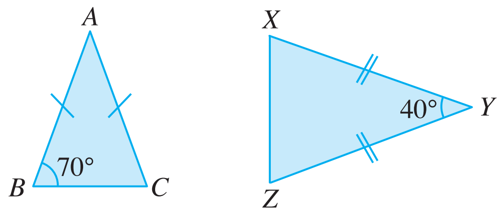
Thinking Time

1. Why is the AAA Similarity Test not necessary?
2. Do two congruent triangles satisfy the AA Similarity Test? Is congruence a special case of similarity?

Worked Example 6

(Proving that Two Triangles are Similar using the AA Similarity Test)

Copy and complete the proof to show that the following two triangles are similar.



Solution:

$A\hat{C}B = A\hat{B}C$ (base $\angle$s of isos. Δ)

$= 70°$

$Y\hat{X}Z = Y\hat{Z}X$

$= \dfrac{180° - 40°}{2}$ (base $\angle$s of isos. Δ)

$= 70°$

$A \Leftrightarrow Y$

$B \Leftrightarrow$ ___

$C \Leftrightarrow$ ___

$A\hat{B}C =$ _____ $= 70°$

$A\hat{C}B =$ _____ $=$ ___°

$\therefore$ ΔABC is similar to Δ_____ (2 pairs of corr. $\angle$s equal).

Problem Solving Tip

Sometimes there is a need to find the other unknown angles in both triangles.

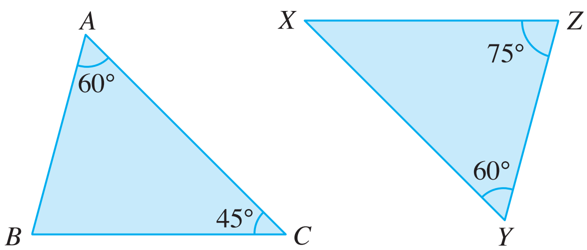
PRACTISE NOW 6

SIMILAR QUESTIONS

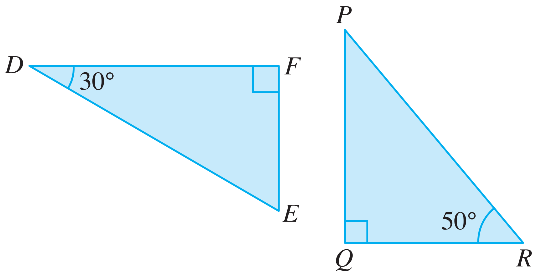
Exercise 9B Questions 1(a), 2(a), 3(a), 4(a), (b), 5(a)-(d)

1. In each part, determine whether the following two triangles are similar. Explain or prove your answers.

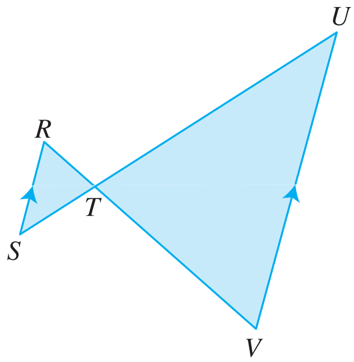
(a)



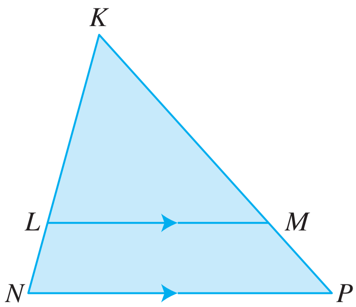
(b)



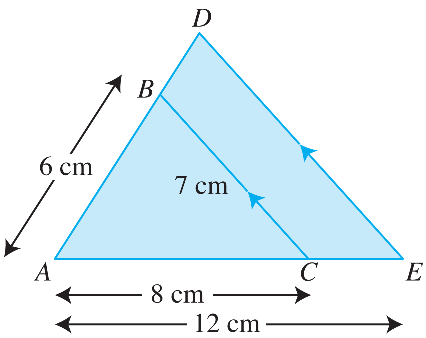
(c)



(d)



2. In the figure, BC is parallel to DE, $AB = 6$ cm, $AC = 8$ cm, $BC = 7$ cm and $AE = 12$ cm.

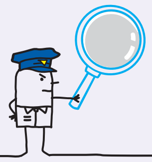


(i) Show that ΔABC is similar to ΔADE.

(ii) Find the length of DE and of BD.

(iii) What can you conclude about the ratios $\dfrac{AB}{BD}$ and $\dfrac{AC}{CE}$ for such a figure?

SSS Similarity Test



Investigation

SSS Similarity Test

1. Construct ΔDEF such that $DE = 2$ cm, $EF = 3$ cm and $DF = 4$ cm.
2. Construct ΔPQR such that $PQ = 4$ cm, $QR = 6$ cm and $PR = 8$ cm.

   Fig. 9.12 shows the two triangles (not drawn to scale).

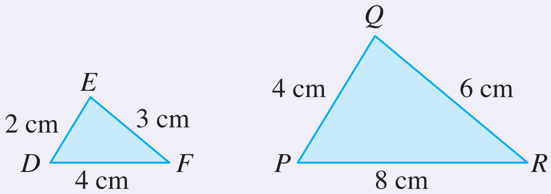


Fig. 9.12

3. Calculate the ratios $\frac{DE}{PQ}$, $\frac{EF}{QR}$ and $\frac{DF}{PR}$.

   Is $\frac{DE}{PQ} = \frac{EF}{QR} = \frac{DF}{PR}$?
4. Measure all the angles of the two triangles which you have constructed (not the ones shown in Fig. 9.12).

   Are $E\hat{D}F = Q\hat{P}R$, $D\hat{E}F = P\hat{Q}R$ and $D\hat{F}E = P\hat{R}Q$?
5. From what you have learnt in Book 2 which you have recapped at the beginning of Section 9.2, since all the corresponding angles of ΔDEF and ΔPQR are equal and all the 3 ratios of their corresponding sides are equal, are the two triangles similar?
6. It is very important to note that the above **given conditions** are **3 ratios of corresponding sides equal** (i.e. $\frac{DE}{PQ} = \frac{EF}{QR} = \frac{DF}{PR}$). Are these given conditions enough to prove that the two triangles are similar?

From the investigation, we observe the following:

> **SSS Similarity Test:** If the 3 ratios of the corresponding sides of two triangles are equal, then the two triangles are similar.

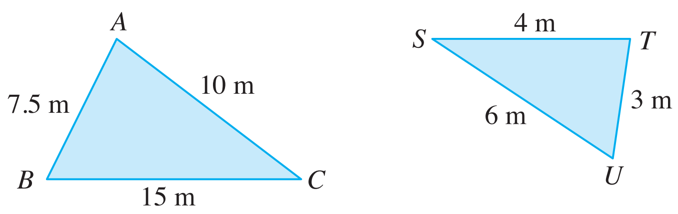
Thinking Time

What are the similarities and the differences between the SSS Congruence Test and the SSS Similarity Test?

Worked Example 7

(Proving that Two Triangles are Similar using the SSS Similarity Test)

Copy and complete the proof to show that the following two triangles are similar.



Solution:

$A \leftrightarrow T$

$B \leftrightarrow$ ___

$C \leftrightarrow$ ___

$$\frac{AB}{TU} = \frac{7.5}{3} = 2.5$$

$$\frac{AC}{TS} = \frac{}{} = \_\_\_$$

$$\frac{BC}{} = \frac{}{6} = \_\_\_$$

$\therefore$ ΔABC is similar to Δ_____ (3 ratios of corr. sides equal).

Problem Solving Tip

Match the vertex opposite the longest side for both triangles first, i.e. $A \leftrightarrow T$. Then do the same for the vertex opposite the shortest side and so on.

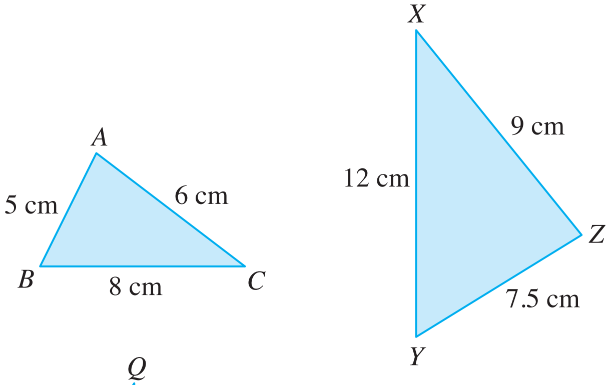
PRACTISE NOW 7

SIMILAR QUESTIONS

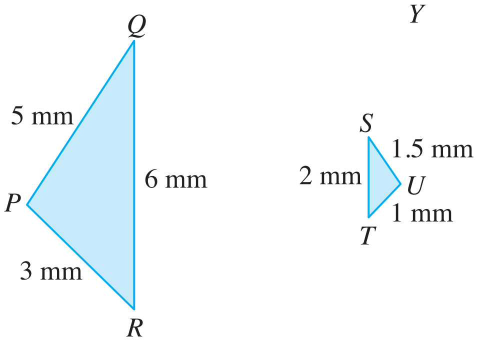
Exercise 9B Questions 1(b), 2(b), 3(b), 6

In each part, determine whether the following two triangles are similar. Explain or prove your answers.

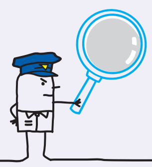
(a)



(b)



SAS Similarity Test



Investigation

SAS Similarity Test

1. Construct ΔABC such that $A\hat{B}C = 70°$, $AB = 3$ cm and $BC = 5$ cm.

2. Construct ΔPQR such that $P\hat{Q}R = 70°$, $PQ = 4.5$ cm and $QR = 7.5$ cm.

Fig. 9.13 shows the two triangles (not drawn to scale).

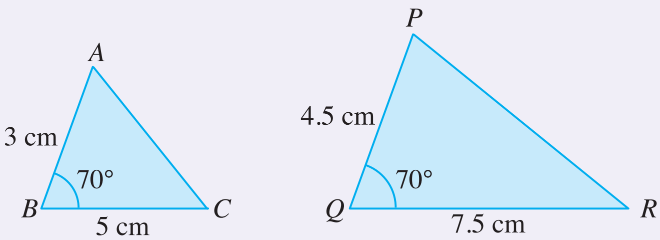


Fig. 9.13

3. Calculate the ratios $\frac{PQ}{AB}$ and $\frac{QR}{BC}$.

   Is $\frac{PQ}{AB} = \frac{QR}{BC}$?

4. Measure the length of AC and of PR in the two triangles which you have constructed (not the ones shown in Fig. 9.13). Then calculate $\frac{PR}{AC}$.

   Is $\frac{PR}{AC} = \frac{PQ}{AB} = \frac{QR}{BC}$?

5. Measure all the other unknown angles in the two triangles which you have constructed.

   Are $B\hat{A}C = Q\hat{P}R$ and $A\hat{C}B = P\hat{R}Q$?

6. From what you have learnt in Book 2 which you have recapped at the beginning of Section 9.2, since all the corresponding angles of ΔABC and ΔPQR are equal and all the 3 ratios of their corresponding sides are equal, are the two triangles similar?

7. It is very important to note that the above **given conditions** are **2 ratios of corresponding sides equal** and **the pair of *included* angles equal** (i.e. $\frac{PQ}{AB} = \frac{QR}{BC}$ and $P\hat{Q}R = A\hat{B}C$). Are these given conditions enough to prove that the two triangles are similar?

From the investigation, we observe the following:

SAS Similarity Test: If 2 ratios of the corresponding sides of two triangles are equal and the pair of included angles are also equal, then the two triangles are similar.

Thinking Time

1. What are the similarities and the differences between the SAS Congruence Test and the SAS Similarity Test?
2. Unlike the need for the AAS Congruence Test, why is the AAS Similarity Test not necessary?
3. Is there a RHS Similarity Test? Investigate and explain your answer.

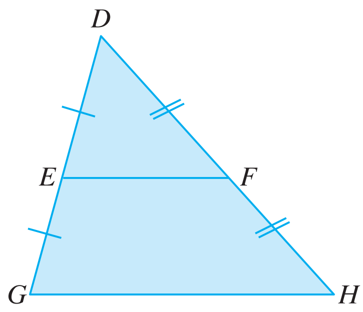
INFORMATION

The 3 similarity tests covered so far are not the only similarity tests. There are more, such as the RHS Similarity Test, which are not included in the syllabus.

Worked Example 8

(Proving that Two Triangles are Similar using the SAS Similarity Test)

Copy and complete the proof to show that the following two triangles are similar.



Solution:

$D \leftrightarrow$ ___

$E \leftrightarrow$ ___

$F \leftrightarrow$ ___

$E\hat{D}F$ = _____ (common angle)

$$\frac{DE}{DG} = \frac{1}{2}$$

$$\frac{DF}{DH} = \text{——}$$

$$\therefore \frac{DE}{DG} = \frac{DF}{DH}$$

$\therefore$ ΔDEF is similar to Δ_____ (2 ratios of corr. sides and included $\angle$ equal).

ATTENTION

The common angle is the included angle between the given corresponding sides.

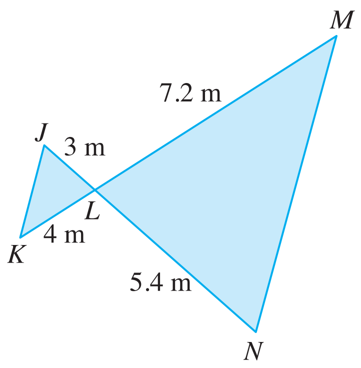
PRACTISE NOW 8

SIMILAR QUESTIONS

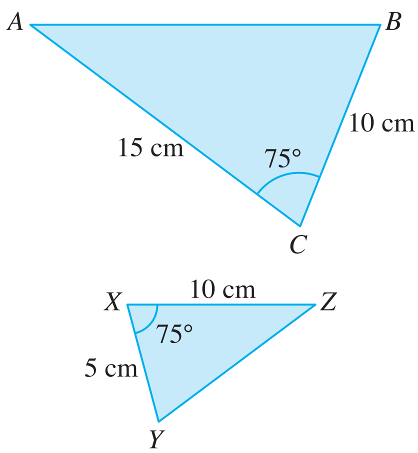
Exercise 9B Questions 1(c), 2(c), 3(c), 4(c),(d)

In each part, determine whether the following two triangles are similar. Explain or prove your answers.

(a)



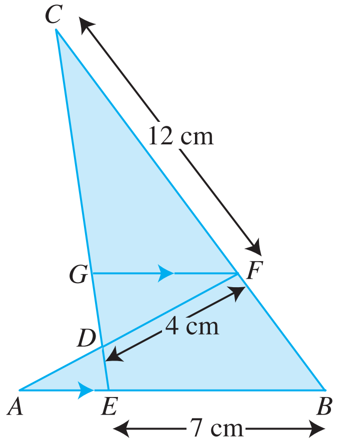
(b)



Worked Example 9

(Using Congruence Tests and Similarity Tests)

In the diagram, GF is parallel to AB, $AB = CF = 12$ cm, $BF = FG$, $DF = 4$ cm and $EB = 7$ cm.



(i) Show that ΔABF is congruent to ΔCFG.

(ii) Show that ΔCDF is similar to ΔADE.

(iii) Find the length of DE.

Solution:

(i) $A \leftrightarrow C$

$B \leftrightarrow F$

$F \leftrightarrow G$

$AB = CF$ (given)

$A\hat{B}F = C\hat{F}G$ (corr. $\angle$s, $AB \,//\, GF$)

$BF = FG$ (given)

$\therefore$ ΔABF • ΔCFG (SAS)

Problem Solving Tip

In **(i)**, since $BF = FG$ (given), then the vertices opposite these sides must match, i.e. $A \leftrightarrow C$; similarly for $AB = CF$. Since only 2 pairs of equal sides are given, we may consider using the SAS Congruence Test.

(ii) $C \leftrightarrow A$

$D \leftrightarrow D$

$F \leftrightarrow E$

$C\hat{D}F = A\hat{D}E$ (vert. opp. $\angle$s)

Since ΔCFG and ΔABF are congruent,

then $F\hat{C}G = B\hat{A}F$,

i.e. $F\hat{C}D = E\hat{A}D$.

$\therefore$ ΔCDF is similar to ΔADE (2 pairs of corr. $\angle$s equal).

Problem Solving Tip

In **(ii)**, since the lengths of ΔADE are not given, we may consider using the AA Similarity Test.

(iii) Since $AB = CF = 12$ cm, then $AE = 12 - 7 = 5$ cm.

Since ΔADE and ΔCDF are similar, then

$$\frac{DE}{DF} = \frac{AE}{CF}$$

i.e. $$\frac{DE}{4} = \frac{5}{12}$$

$$\therefore \quad DE = \frac{5}{12} \times 4$$

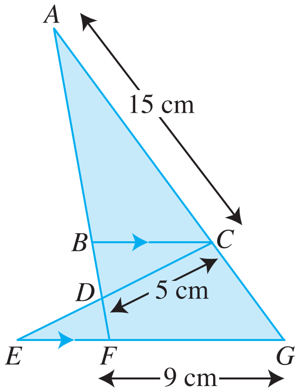
$$= 1\frac{2}{3} \text{ cm}$$

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 9B Questions 7-12

1. In the diagram, BC is parallel to EG, $AC = EG = 15$ cm, $BC = CG$, $CD = 5$ cm and $FG = 9$ cm.

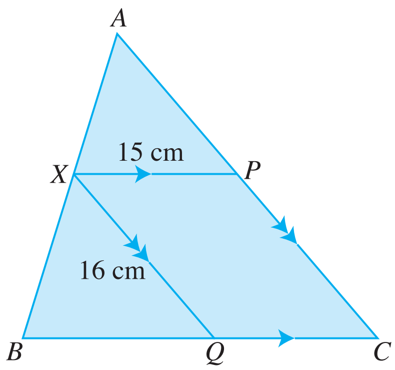


(i) Show that ΔABC and ΔECG are congruent.

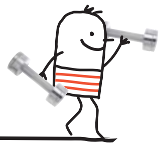
(ii) Show that ΔACD and ΔEFD are similar.

(iii) Find the length of DF.

2. The side AB of ΔABC is divided at X in the ratio 3 : 4. P and Q are points on CA and CB such that XP and XQ are parallel to BC and AC respectively.



Given that $XP = 15$ cm and $XQ = 16$ cm, find the length of BQ and of AC.



Exercise 9B

BASIC LEVEL

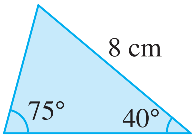
1. Identify a pair of similar triangles from the following triangles (not drawn to scale), based on in each of the following similarity tests:

(a) AA Similarity Test,

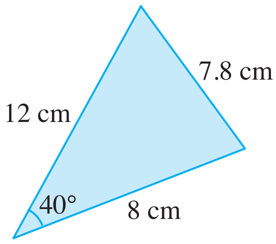
(b) SSS Similarity Test,

(c) SAS Similarity Test.

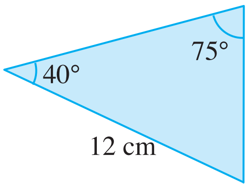
(i)



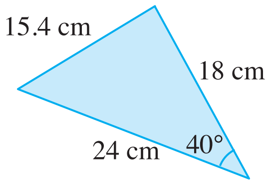
(ii)



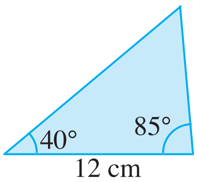
(iii)



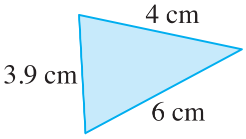
(iv)



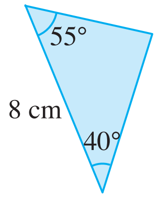
(v)



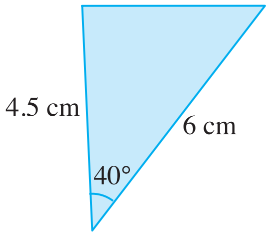
(vi)



(vii)

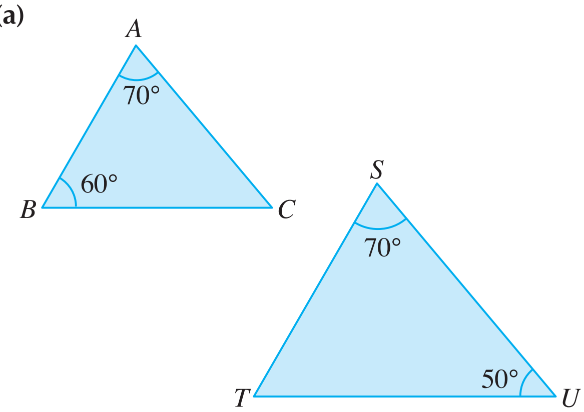


(viii)



2. Copy and complete the proof to show that each of the following pairs of triangles are similar.

(a)



$S\hat{T}U = 180° - 70° - 50°$ (∠ sum of a Δ)

$= \_\_\_°$

$A \leftrightarrow \_\_\_$

$B \leftrightarrow \_\_\_$

$C \leftrightarrow \_\_\_$

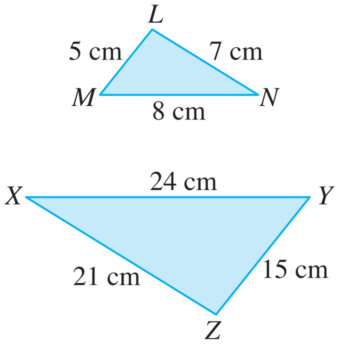
$B\hat{A}C = \_\_\_\_\_ = 70°$

$A\hat{B}C = \_\_\_\_\_ = \_\_\_°$

∴ Δ*ABC* is similar to Δ_____

(_____ pairs of corr. _____ equal).

(b)



$X \leftrightarrow$ ___

$Y \leftrightarrow$ ___

$Z \leftrightarrow$ ___

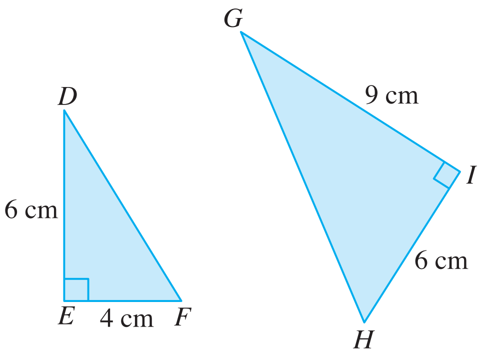
$\frac{XY}{NM} = \frac{24}{\quad} =$ ___

$\frac{XZ}{\quad} = \frac{\quad}{\quad} =$ ___

$\frac{YZ}{\quad} = \frac{\quad}{\quad} =$ ___

$\therefore$ ΔXYZ and Δ_____ are similar
(_____ ratios of corr. _____ equal).

(c)



$D \leftrightarrow$ ___

$E \leftrightarrow$ ___

$F \leftrightarrow$ ___

$D\hat{E}F =$ ___ $=$ ___°

$\frac{DE}{GI} = \frac{\quad}{\quad} =$ ___

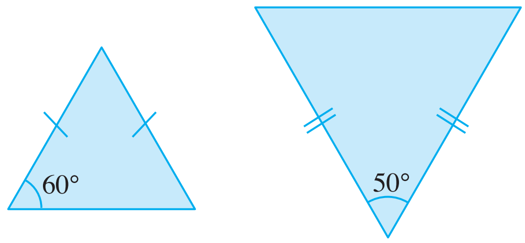
$\frac{EF}{\quad} = \frac{\quad}{\quad} =$ ___

$\therefore \frac{DE}{GI} = \frac{EF}{\quad}$

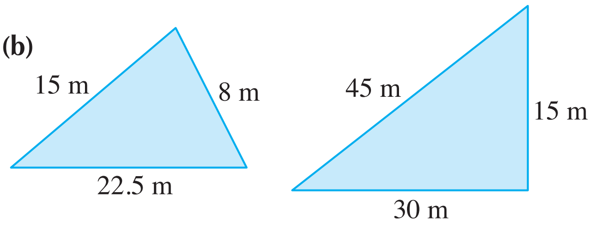
$\therefore$ ΔDEF is similar to Δ_____
(_____ ratios of corr. _____ and _____ angle equal).

3. Determine whether each of the following pairs of triangles are similar.

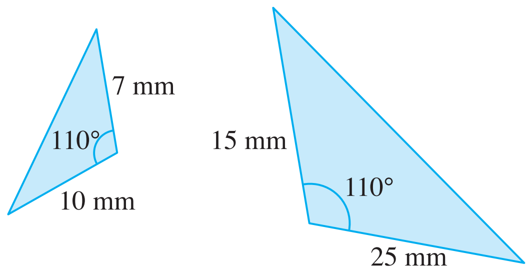
(a)



(b)



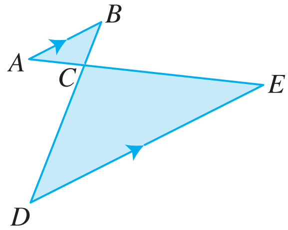
(c)



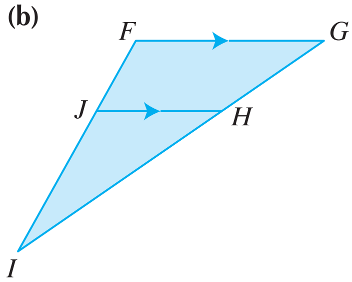
INTERMEDIATE LEVEL

4. In each diagram, identify a pair of similar triangles and prove that they are similar.

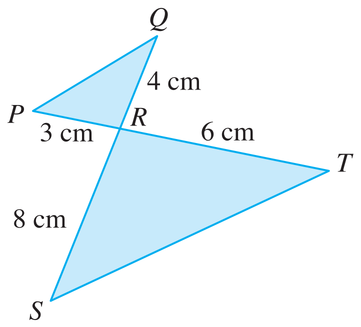
(a)



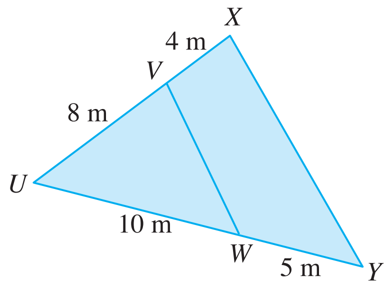
(b)



(c)

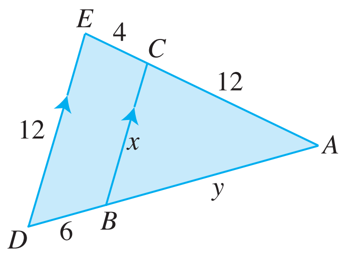


(d)

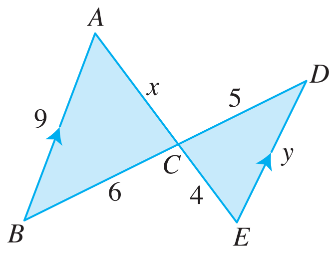


5. In each of the following figures, identify the similar triangles and find the value of x and of y. All lengths are given in cm.

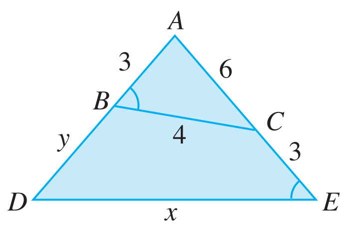
(a)



(b)

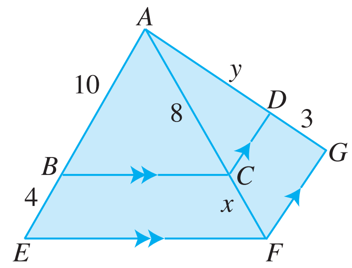


(c)

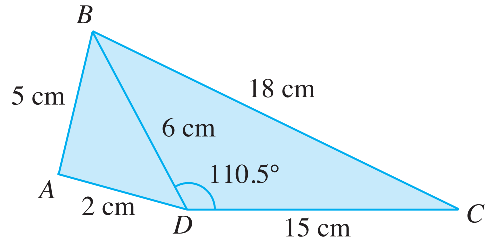


$A\hat{B}C = A\hat{E}D$

(d)



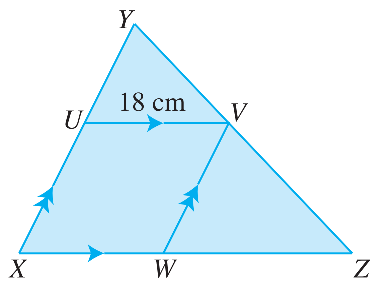
6. In the diagram, $AB = 5$ cm, $AD = 2$ cm, $BD = 6$ cm, $CD = 15$ cm, $BC = 18$ cm and $B\hat{D}C = 110.5°$.



(i) Prove that ΔABD and ΔDCB are similar.

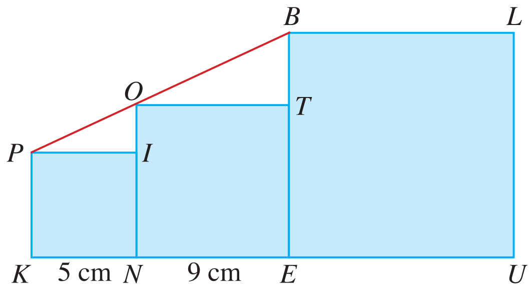
(ii) Find $D\hat{A}B$.

7. In the diagram, U divides YX in the ratio $2 : 3$. $UVWX$ is a rhombus in which $UV = 18$ cm.

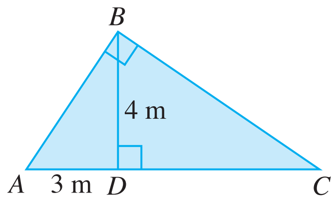


Find the length of XY and of WZ.

8. In the diagram, the lengths of the sides of the squares *PINK* and *NOTE* are 5 cm and 9 cm respectively. Given that *POB* is a straight line, find the length of the side of the square *BLUE*.



9. In the diagram, $A\hat{B}C = B\hat{D}C = 90°$, $AD = 3$ m and $BD = 4$ m.

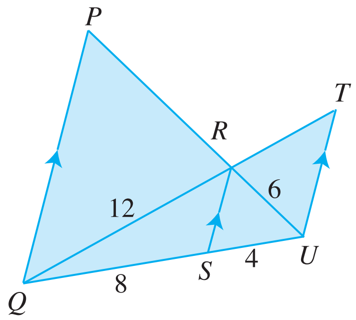


(i) Identify 3 similar triangles and prove that they are similar.

(ii) Find the length of *BC* and of *CD*.

ADVANCED LEVEL

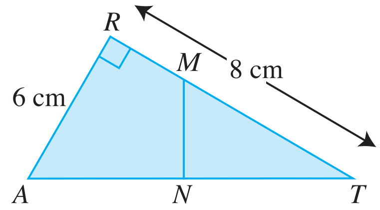
10. In the diagram, *PQ*, *RS* and *TU* are parallel lines, $QS = 8$ cm, $SU = 4$ cm, $QR = 12$ cm and $RU = 6$ cm.



(i) Find the length of *RT* and of *PR*.

(ii) Calculate the ratio *PQ* : *TU*.

11. *RAT* is a right-angled triangle with $A\hat{R}T = 90°$, $RA = 6$ cm and $RT = 8$ cm. If the triangle is folded along the line *MN*, vertex *A* coincides with the vertex *T*.

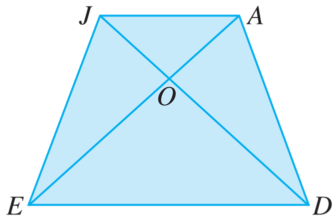


(i) Prove that *MN* is perpendicular to *AT*.

(ii) Name a pair of similar triangles and prove that they are similar.

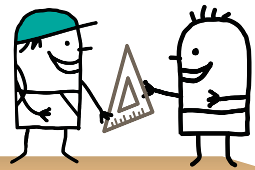
(iii) Hence, find the length of *MN*.

12. In the quadrilateral *JADE*, $AO = JO$ and $EO = DO$.



Explain why ΔAJD is congruent to ΔJAE.

9.3 Applications of Congruent and Similar Triangles

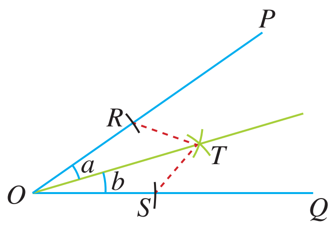


In this section, we will apply the concepts of congruent and similar triangles to solve problems in mathematics and in real life.

Worked Example 10

(Application of Congruent Triangles)

In Book 1, we have learnt how to construct the bisector of a given angle as shown in the diagram. Prove that OT is the angle bisector of $P\hat{O}Q$.



RECALL

Draw arcs OR and OS, where $OR = OS$.

Draw arcs RT and ST, where $RT = ST$.

Prove that OT is the angle bisector of $P\hat{O}Q$, i.e. $\angle a = \angle b$.

Solution:

$R \leftrightarrow S$

$O \leftrightarrow O$

$T \leftrightarrow T$

$OR = OS$

$RT = ST$

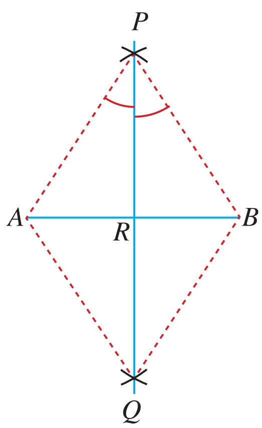
$OT = OT$ (common side)

$\therefore$ ΔROT and ΔSOT are congruent (SSS Congruence Test).

$\therefore$ $a = b$, i.e. OT is the angle bisector of $P\hat{O}Q$.

PRACTISE NOW 10

In Book 1, we have learnt how to construct the perpendicular bisector of a given line segment as shown in the diagram. Prove that PQ is the perpendicular bisector of AB.



SIMILAR QUESTIONS

Exercise 9C Questions 1, 2, 4, 5, 7

RECALL

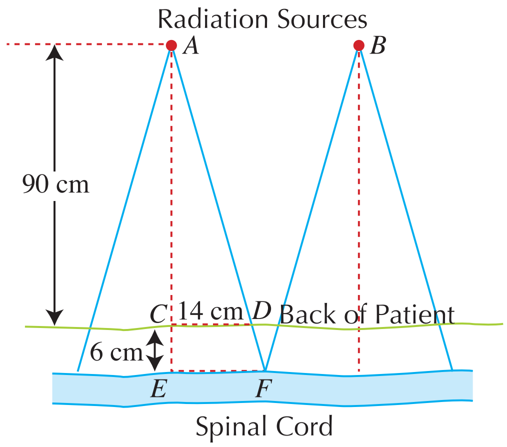
Draw arcs AP, AQ, BP and BQ, where $AP = AQ = BP = BQ$.

Prove that PQ is the perpendicular bisector of AB.

Worked Example 11

(Application of Similar Triangles)

The chapter opener mentions the importance of geometry in radiation oncology (the study and treatment of tumours). The diagram below shows how far apart two beams of radiation must be placed so that they will not overlap at the spinal cord, or else a double dose of radiation will endanger the patient.



Find the distance between the two radiation sources A and B.

Solution:

$A \leftrightarrow A$

$C \leftrightarrow E$

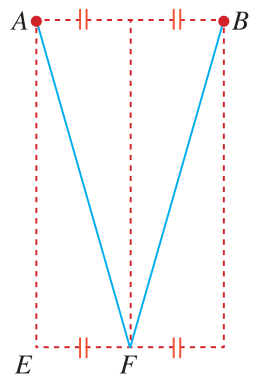
$D \leftrightarrow F$

$$\frac{EF}{CD} = \frac{AE}{AC}$$

$$\frac{EF}{14} = \frac{90+6}{90}$$

$$EF = \frac{96}{90} \times 14$$

$$= 14.93 \text{ cm (to 4 s.f.)}$$



$\therefore AB = 2 \times EF$

$= 2 \times 14.93$

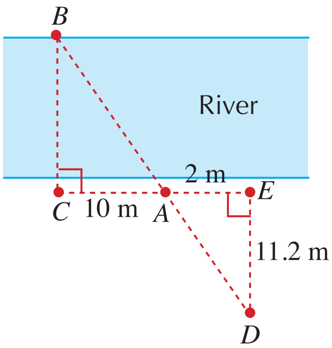
$= 29.9$ cm (to 3 s.f.)

PRACTISE NOW 11

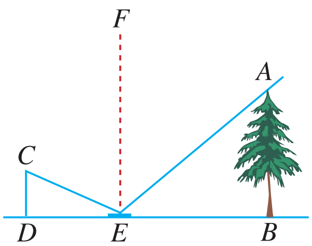
SIMILAR QUESTIONS

Exercise 9C Questions 3, 6

1. To estimate the width of a river, a man selects an object B at the far bank and stands at C such that BC is perpendicular to the river bank (see diagram). He then walks perpendicular to BC for 10 metres until he reaches A, where he plants a vertical pole. He continues to walk another 2 metres until point E, before he walks along ED such that $A\hat{E}D = 90°$. He stops walking when he reaches D, where DAB forms a straight line. He measures and finds that the length of DE is 11.2 m. Find the length of BC.



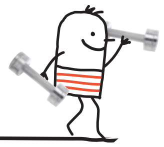
2. To determine the height AB of a tree, Michael places a mirror on the ground at E. From E, he walks backwards to a point D, where he is just able to see the top of the tree in the mirror. CD, FE and AB are perpendicular to the line DEB.



Given that BE = 18 m, ED = 2.1 m and that his eyes at C are 1.4 m above the ground, find the height of the tree.

Problem Solving Tip

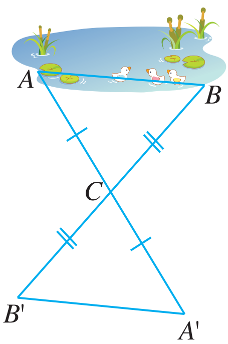
According to the Law of Reflection in Physics, the angle of incidence, $A\hat{E}F$, is equal to the angle of reflection, $C\hat{E}F$.



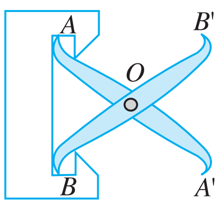
Exercise 9C

BASIC LEVEL

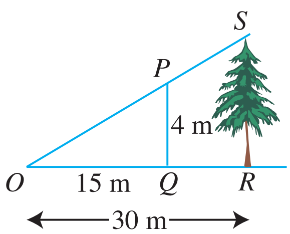
1. The diagram illustrates how the length AB (which cannot be measured directly) of a pond is measured. Choose a point C and measure the length of AC and the length of BC. Produce AC and BC to A' and B' respectively, so that $CA' = AC$ and $CB' = BC$. By measuring the length of $B'A'$, we will be able to find the length of AB. Why is this so?



2. To measure the width of the internal trough, AB, of a machine tool which cannot be measured directly, we make use of a device as shown in the diagram. The device is made up of two parts, AA' and BB', hinged halfway at O. By measuring the distance between A' and B', we will be able to obtain the length of AB. Why is this so?

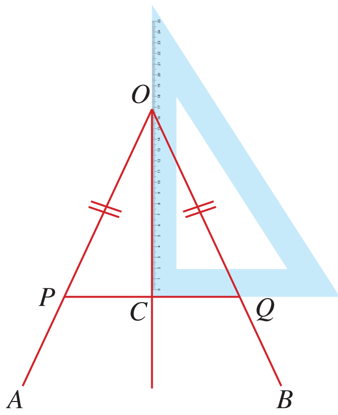


3. The figure shows a tree SR and a pole PQ casting shadows of lengths 30 m and 15 m respectively. If the height of the pole is 4 m, find the height of the tree.

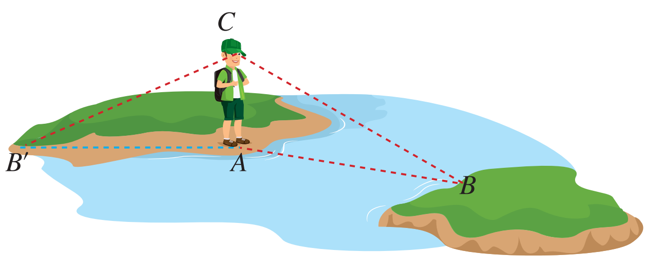


INTERMEDIATE LEVEL

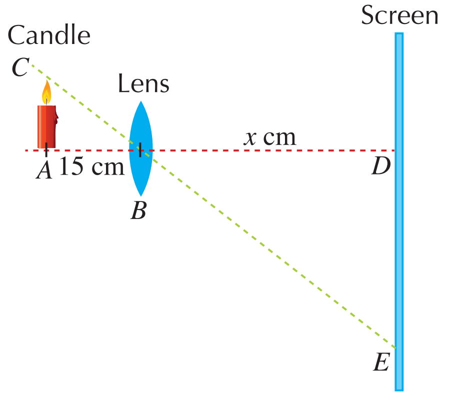
4. In the diagram, P lies on OA and Q lies on OB such that $OP = OQ$. Place a set square with one side along PQ and another side passing through O, as shown in the diagram. Explain why OC is the angle bisector of $A\hat{O}B$.



5. The diagram shows Ethan standing at a point A along a river bank. He looks directly across to the opposite bank, adjusting his cap so that his line of vision CB passes through the lowest point at the rim of his cap and falls on the point B. He then turns around without moving his head. His new line of vision CB' through the lowest point at the rim of his cap now falls on a point B' on the same side of the river. State which measurement he can make in order to find the width AB of the river. Explain your answer.

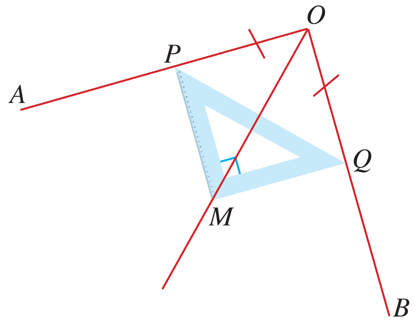


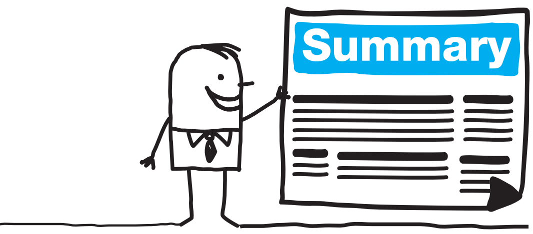
6. A candle is placed 15 cm from a lens and a screen is placed at a distance of x cm from the lens as shown in the diagram. The image of the candle, DE, captured on the screen, is inverted and is 3 times the length of the candle. Find the value of x.



ADVANCED LEVEL

7. Using a set square, we can bisect a given angle. In the diagram, P and Q are marked along the arms, OA and OB of $A\hat{O}B$ respectively, such that $OP = OQ$. Move a 90°-45°-45° set square away from O until the 45° edges coincide with P and Q as shown in the diagram. Explain why OM is the angle bisector of $A\hat{O}B$.





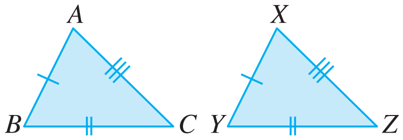
1. The following shows 4 congruence tests and 3 similarity tests.

Congruence Tests		**Similarity Tests**
SSS Congruence Test	↔	SSS Similarity Test
SAS Congruence Test	↔	SAS Similarity Test
AAS Congruence Test	↔	AA Similarity Test (Since AA is enough, there is no need to use the AAS Similarity Test.)
RHS Congruence Test		

2. In general, SSA is not a congruence test.
An exception is the RHS congruence test.

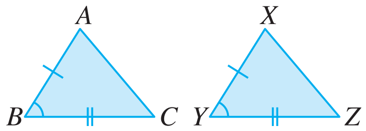
3. Congruence Tests

SSS Congruence Test



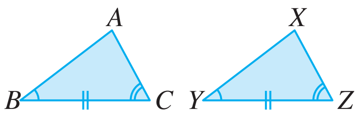
3 corresponding sides are equal,
i.e. $AB = XY$, $BC = YZ$, $AC = XZ$

SAS Congruence Test



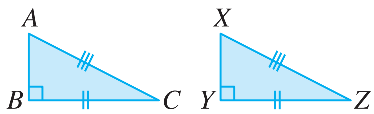
2 corresponding sides and the included angle are equal,
i.e. $AB = XY$, $BC = YZ$, $A\hat{B}C = X\hat{Y}Z$

AAS Congruence Test



1 corresponding side and 2 corresponding angles are equal,
i.e. $BC = YZ$, $A\hat{B}C = X\hat{Y}Z$, $A\hat{C}B = X\hat{Z}Y$

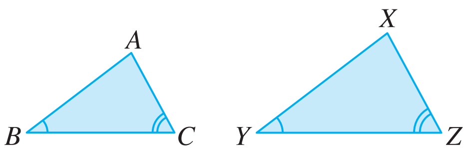
RHS Congruence Test



1 side and the hypotenuse of the right-angled triangle are equal,
i.e. $BC = YZ$, $AC = XZ$, $A\hat{B}C = X\hat{Y}Z = 90°$

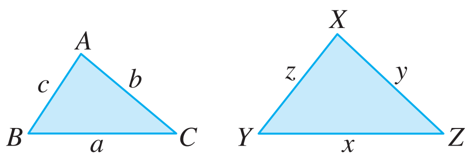
4. Similarity Tests

AA Similarity Test



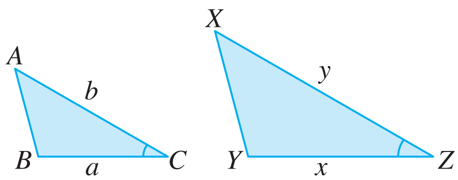
2 corresponding angles are equal,
i.e. $A\hat{B}C = X\hat{Y}Z$, $A\hat{C}B = X\hat{Z}Y$

SSS Similarity Test

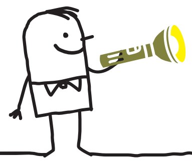


The 3 ratios of the corresponding sides are equal,
i.e. $\frac{a}{x} = \frac{b}{y} = \frac{c}{z}$ or $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$

SAS Similarity Test



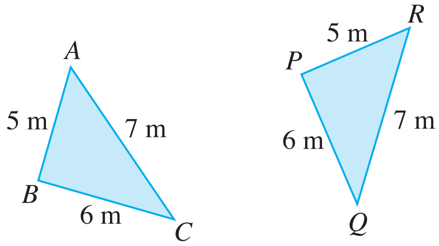
2 ratios of the corresponding sides and the included angle are equal,
i.e. $\frac{a}{x} = \frac{b}{y}$ or $\frac{x}{a} = \frac{y}{b}$, $A\hat{C}B = X\hat{Z}Y$



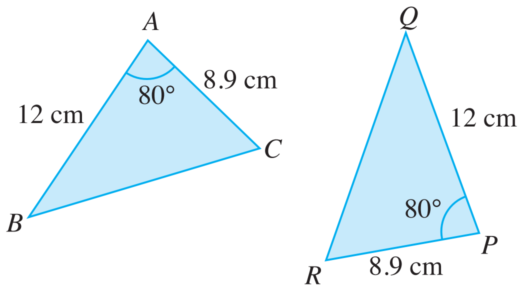
Review Exercise 9

1. Determine whether each of the following pairs of triangles are congruent. If they are congruent, state the congruence test.

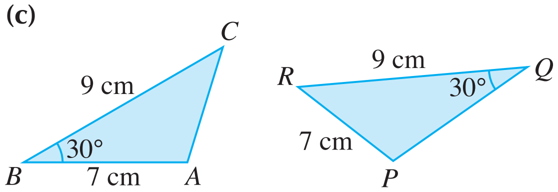
(a)



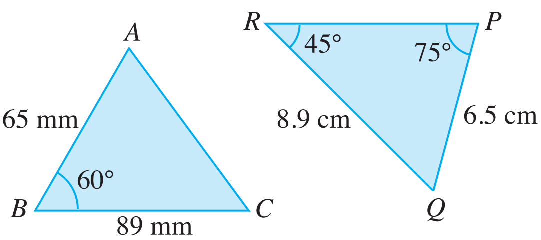
(b)



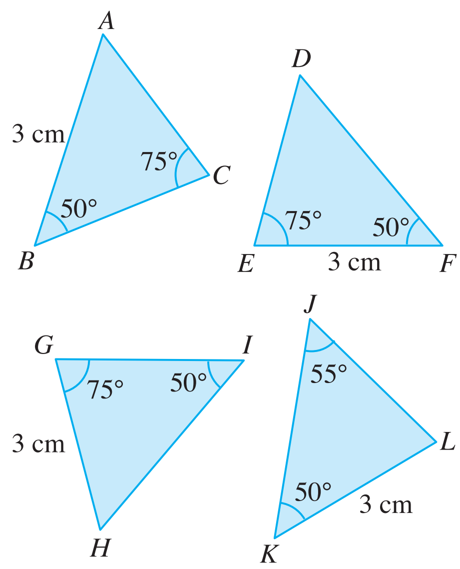
(c)



(d)

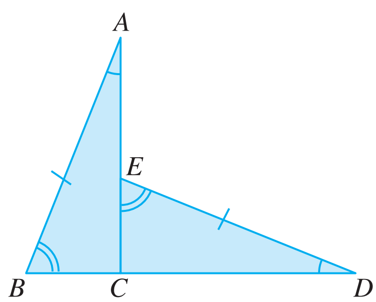


2. Identify a pair of congruent triangles and state the congruence test.

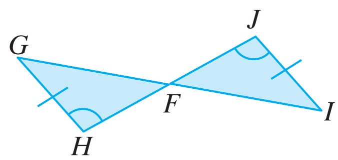


3. Which of the following pairs of triangles are congruent? If they are congruent, state the congruence test and name the other three pairs of equal measurements.

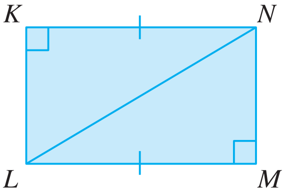
(a)



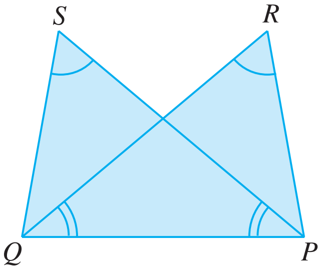
(b)



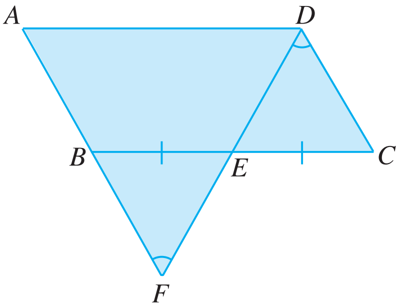
(c)



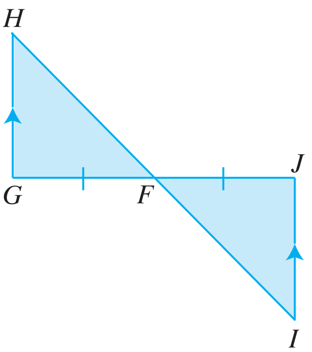
(d)



(e)

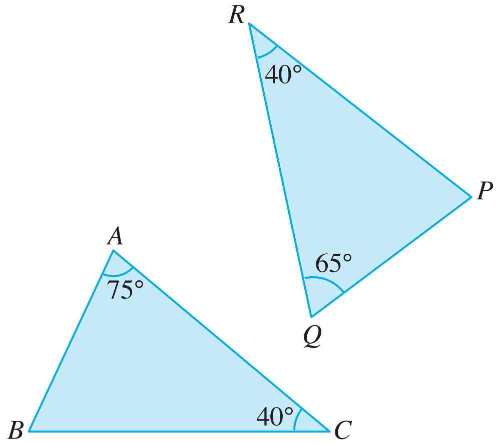


(f)

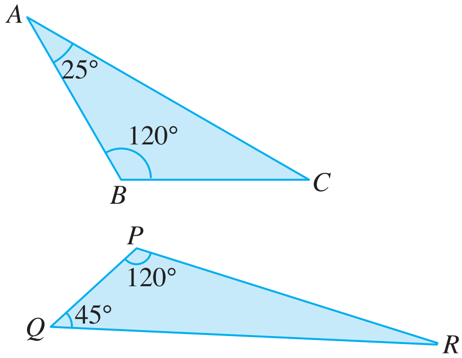


4. Which of the following pairs of triangles are similar? If they are similar, state the reason for similarity. All lengths are given in cm.

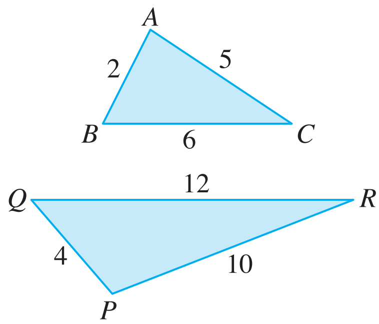
(a)



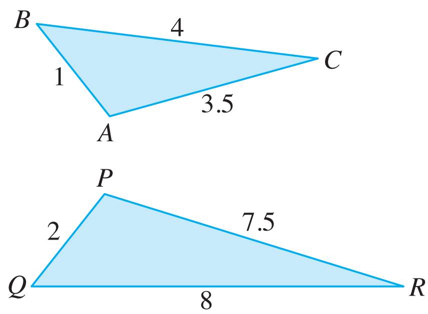
(b)



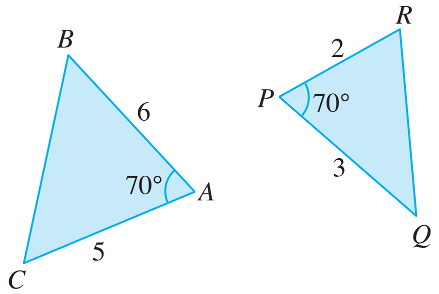
(c)



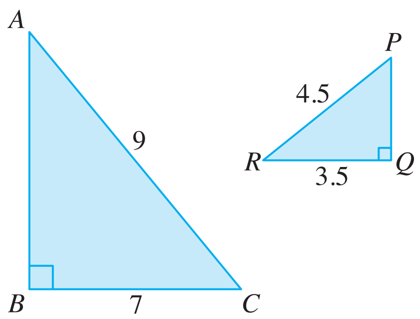
(d)



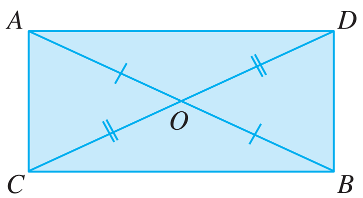
(e)



(f)



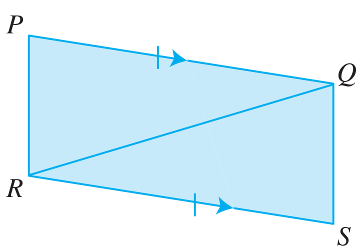
5. In the diagram, AOB and COD are straight lines, $AO = BO$ and $CO = DO$.



(i) Identify a pair of congruent triangles and state the congruence test.

(ii) Write down two pairs of equal angles.

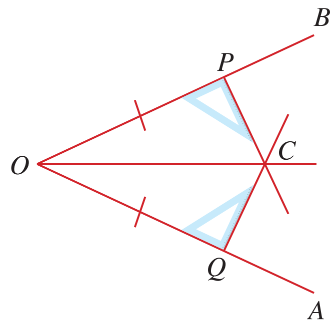
6. In the diagram, PQ is equal and parallel to RS, $PR = 5$ cm and $Q\hat{S}R = 50°$.



(i) Identify a pair of congruent triangles and state the congruence test.

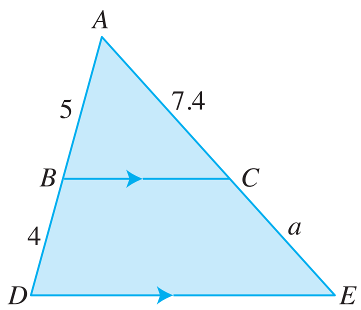
(ii) Find the length of QS and $Q\hat{P}R$.

7. In the diagram, P and Q are points along the arms OA and OB of $A\hat{O}B$ respectively such that $OP = OQ$. A set square is used to construct perpendiculars to OA and OB at P and Q respectively. The perpendiculars meet at C. Explain why OC is the angle bisector of $A\hat{O}B$.

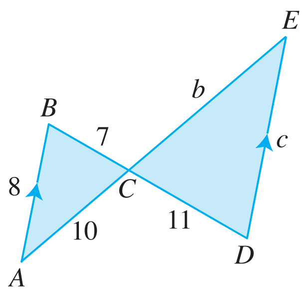


8. In each of the following figures, find the value of each of the unknowns. All lengths given are in cm.

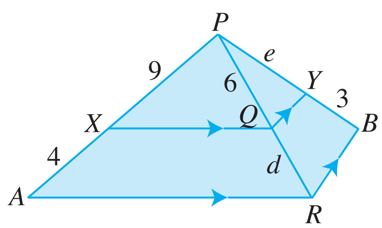
(a)



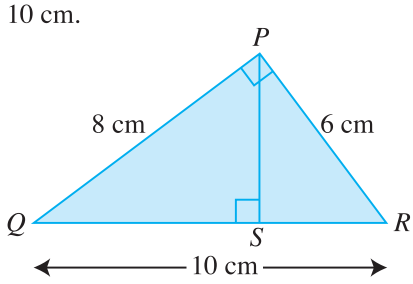
(b)



(c)

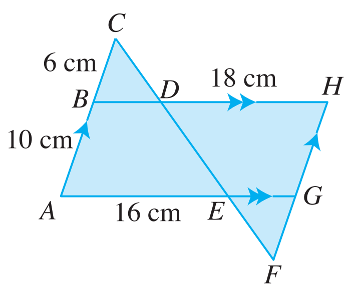


9. In the figure, the angle QPR is a right angle, PS is perpendicular to QR, PQ = 8 cm, PR = 6 cm and QR = 10 cm.



(i) Name a triangle similar to ΔPQS.
(ii) Calculate the length of QS.

10. In the diagram, ABC, $CDEF$, FGH, BDH and AEG are straight lines. BH is parallel to AG, AC is parallel to FH, AB = 10 cm, BC = 6 cm, AE = 16 cm and DH = 18 cm.

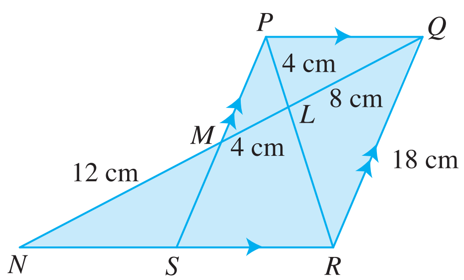


(a) **(i)** Identify two triangles similar to ΔBCD.
(ii) Calculate the length of BD.

(b) Find the length of EG and of FH.

(c) Prove that ΔACE and ΔHFD are similar.

11. In the diagram, PLR and $QLMN$ are straight lines, PQ is parallel to NR, SP is parallel to RQ, QL = 8 cm, LM = PL = 4 cm, MN = 12 cm and QR = 18 cm.

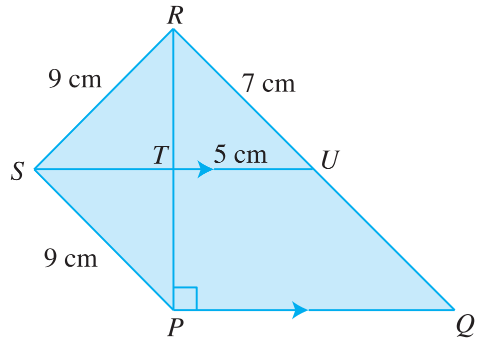


(a) **(i)** Name a triangle similar to ΔPLQ.
(ii) Calculate the length of LR.

(b) **(i)** Name the triangle similar to ΔNQR.
(ii) Calculate the length of MS.

(c) Name three other pairs of similar triangles.

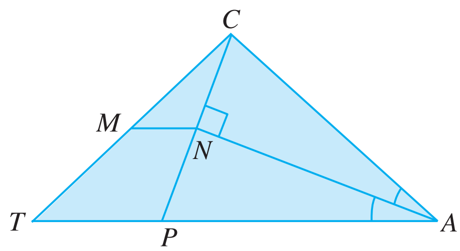
12. In the diagram, STU, RTP and RUQ are straight lines, SU is parallel to PQ, $R\hat{P}Q = 90°$, $SR = SP$ = 9 cm, TU = 5 cm and RU = 7 cm.



(i) Identify two triangles which are congruent.

(ii) Find the length of UQ and of PQ.

13. In ΔCAT, M is the midpoint of CT, $C\hat{A}N = P\hat{A}N$ and CP is a straight line that is perpendicular to NA.

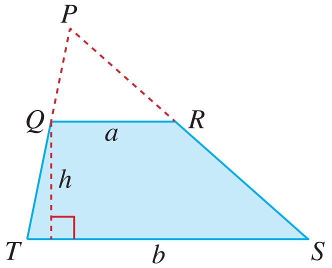


(i) Explain why ΔCAN is congruent to ΔPAN.

(ii) Hence, or otherwise, explain why $MTAN$ is a trapezium.

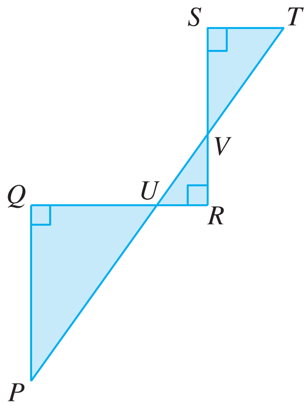
Challenge Yourself

1. The figure shows a trapezium $QRST$, where $QR = a$ units and $ST = b$ units. The height of the trapezium is h units.



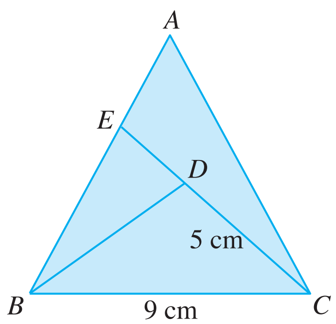
By using similar triangles, show that the area of the trapezium is given by $\frac{1}{2}(a+b)h$.

2. In the diagram, $P\hat{Q}R = Q\hat{R}S = R\hat{S}T = 90°$, $PQ = QR = RS = 5$ cm and $ST = 1$ cm.



Given that PUT and RS intersect at V, find the length of QU.

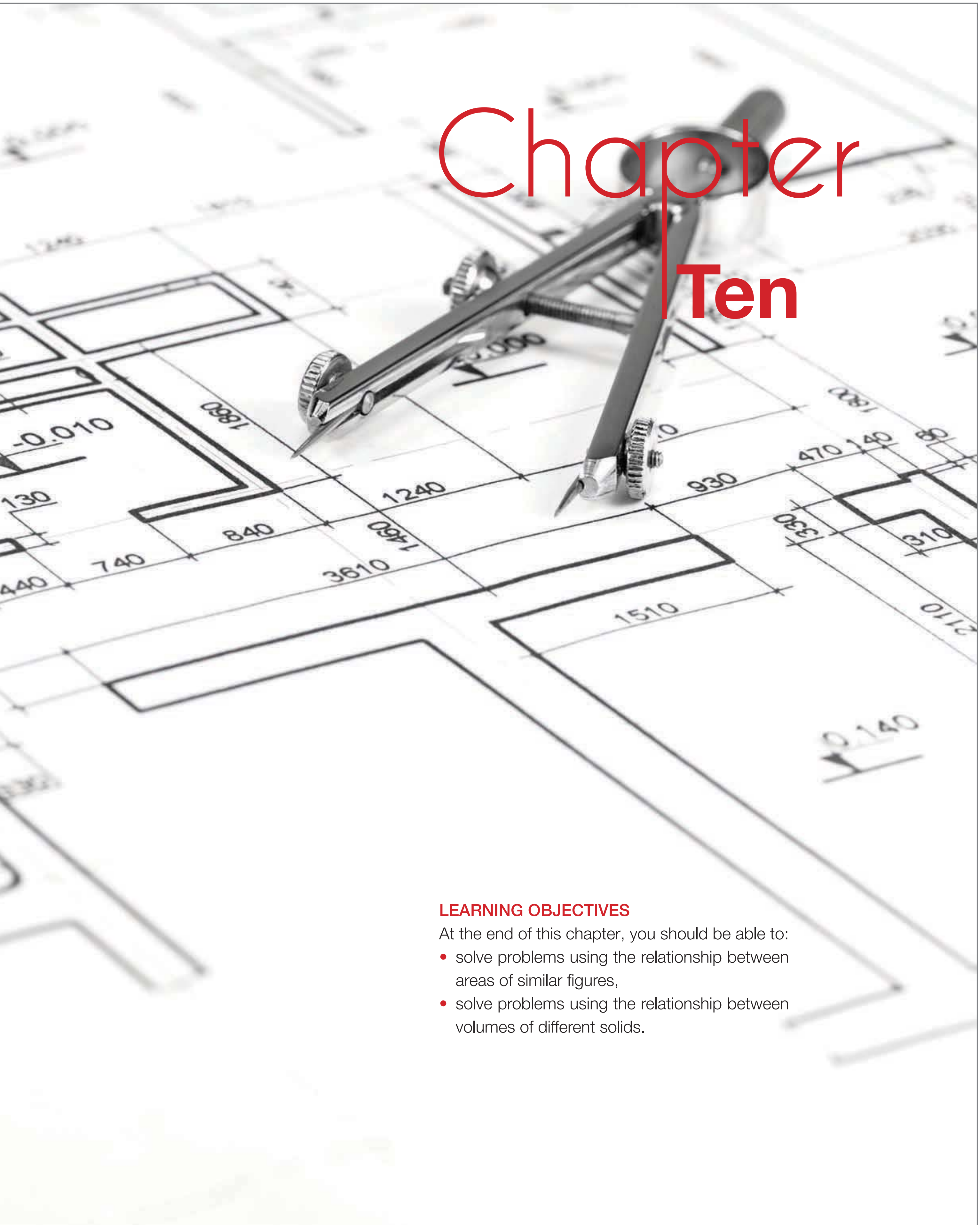
3. In the diagram, $AB = AC$, $CB = CE$, $BD = BE$, $BC = 9$ cm and $CD = 5$ cm.



Find the length of AC.

Area and Volume of Similar Figures and Solids

The scale drawing of a structure is a representation of the actual size of the structure. This drawing shows all the dimensions which are necessary to build the structure. From the scale drawing, we are able to make use of the relationship between the lengths, the areas and the volumes of similar objects to calculate the actual surface area and volume of the structure.



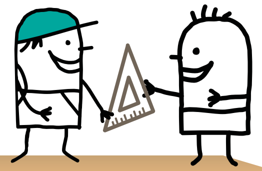
Chapter Ten

LEARNING OBJECTIVES

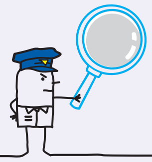
At the end of this chapter, you should be able to:

- solve problems using the relationship between areas of similar figures,
- solve problems using the relationship between volumes of different solids.

10.1 Area of Similar Figures



In Book 2, we have learnt about similar triangles. In this chapter, we will learn how to find the area of similar figures and the volume of similar solids.



Investigation

Areas of Similar Figures

1. Table 10.1 shows three squares. Are they similar? Explain your answer.

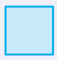
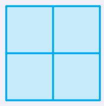
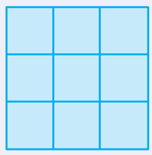
Square			
Length of Square	1 cm	2 cm	3 cm
Area of Square			

Table 10.1

2. Complete Table 10.1 to find the area of each square.

3. **(a)** The length of the second square is double that of the first square. What is the relationship between their areas?

(b) The length of the third square is three times that of the first square. What is the relationship between their areas?

4. Let the length and the area of a square be l_1 and A_1 respectively.
Let the length and the area of a second square be l_2 and A_2 respectively.
Note that the two squares are **similar**.
Express the following ratio of areas in terms of l_1 and l_2.

$$\boxed{\frac{A_2}{A_1} = \quad}$$

5. Is the formula in Question 4 always true?
Let us investigate what happens if we have similar triangles instead.

Table 10.2 shows three **similar** triangles.

Triangle			
Length of Corresponding Side of Triangle	1 unit	2 units	3 units
Area of Triangle	1 square unit		

Table 10.2

6. **(a)** The length of a side of the second triangle is double that of the corresponding side of the first triangle.
What is the relationship between their areas?

(b) The length of a side of the third triangle is three times that of the corresponding side of the first triangle.
What is the relationship between their areas?

7. Let the length and the area of a triangle be l_1 and A_1 respectively.
Let the length and the area of a second *similar* triangle be l_2 and A_2 respectively.
Express the following ratio of areas in terms of l_1 and l_2.

$$\boxed{\frac{A_2}{A_1} = \qquad}$$

In general, the ratio of the areas of two **similar** figures is the square of the ratio of their corresponding lengths, i.e.

$$\boxed{\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2}$$

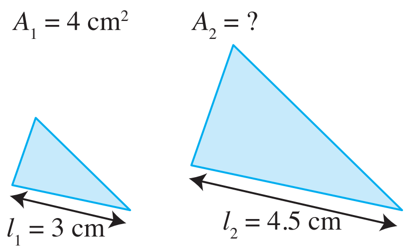
where A_1 and l_1 are the area and the length of the first figure respectively,
and A_2 and l_2 are the area and the length of the second similar figure respectively.

Worked Example 1

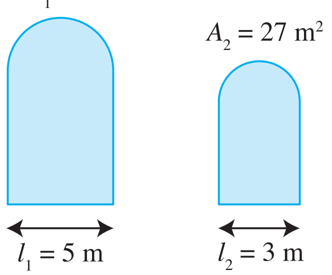
(Finding the Area of Similar Figures)

Find the unknown area of each of the following pairs of similar figures.

(a)



(b)



Solution:

(a)

$$\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2$$

$$\frac{A_2}{4} = \left(\frac{4.5}{3}\right)^2$$

$$= \frac{9}{4}$$

$$\therefore A_2 = \frac{9}{4} \times 4$$

$$= 9 \text{ cm}^2$$

(b)

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

$$\frac{A_1}{27} = \left(\frac{5}{3}\right)^2$$

$$= \frac{25}{9}$$

$$\therefore A_1 = \frac{25}{9} \times 27$$

$$= 75 \text{ m}^2$$

INFORMATION

To simplify $\left(\frac{4.5}{3}\right)^2$ without using a calculator, we have

$$\left(\frac{4.5}{3}\right)^2 = \left(\frac{4.5 \times 2}{3 \times 2}\right)^2 = \left(\frac{9}{6}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

Problem Solving Tip

For **(b)**, write the unknown A_1 first. It will help in subsequent algebraic manipulations.

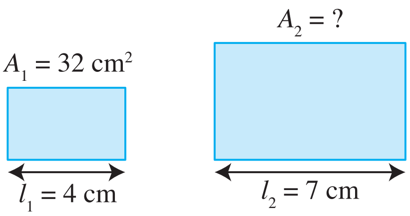
PRACTISE NOW 1

SIMILAR QUESTIONS

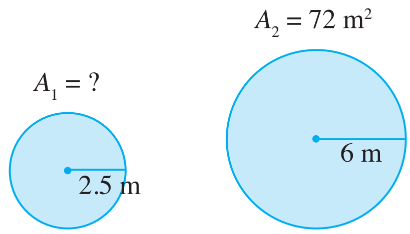
Exercise 10A Questions 1(a)-(f), 2, 3, 5-9

Find the unknown area of each of the following pairs of similar figures.

(a)



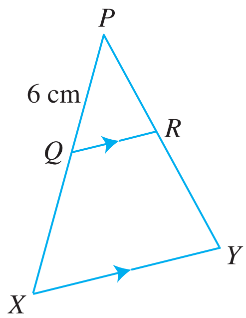
(b)



Worked Example 2

(Finding the Length of Similar Figures)

In the figure, QR is parallel to XY, PQ = 6 cm and the areas of ΔPQR and ΔPXY are 9 cm^2 and 64 cm^2 respectively. Find the length of QX.



Solution:

Since QR is parallel to XY, ΔPQR and ΔPXY are similar.

$$\left(\frac{PX}{PQ}\right)^2 = \frac{\text{Area of } \Delta PXY}{\text{Area of } \Delta PQR}$$

$$\left(\frac{PX}{6}\right)^2 = \frac{64}{9}$$

$$\frac{PX^2}{36} = \frac{64}{9}$$

$$PX^2 = \frac{64}{9} \times 36$$

$$= 256$$

$$PX = \sqrt{256}$$

$$= 16 \text{ cm} \quad (-16 \text{ is rejected because } PX > 0)$$

$$QX = PX - PQ$$

$$= 16 - 6$$

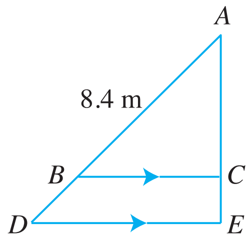
$$= 10 \text{ cm}$$

ATTENTION

In ΔPQR and ΔPXY,
$\angle P$ is a common angle,
$\angle PQR = \angle PXY$ (corr. $\angle$s) and
$\angle PRQ = \angle PYX$ (corr. $\angle$s).

PRACTISE NOW 2

In the figure, BC is parallel to DE, AB = 8.4 m and the areas of ΔABC and ΔADE are 49 m^2 and 100 m^2 respectively. Find the length of BD.



SIMILAR QUESTIONS

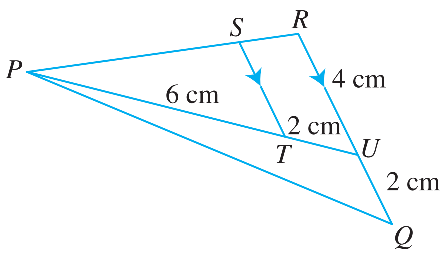
Exercise 10A Questions 4(a)-(d), 10-12

Worked Example 3

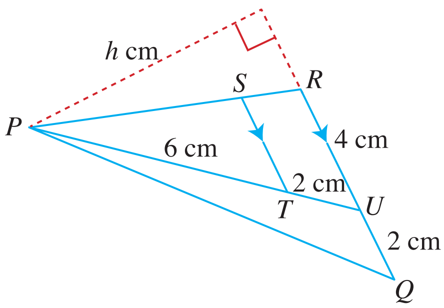
(Problem involving Similar Figures)

In the figure, ST is parallel to RU, RU = 4 cm, UQ = 2 cm, PT = 6 cm and TU = 2 cm. Given that the area of ΔPUR is 12 cm^2, find the area of

(i) ΔPQU, **(ii)** ΔPTS.



Solution:



(i) Notice that ΔPUR and ΔPQU have a common height corresponding to the bases RU and UQ respectively.

Let the common height be h cm.

$$\frac{\text{Area of } \Delta PQU}{\text{Area of } \Delta PUR} = \frac{\frac{1}{2} \times UQ \times h}{\frac{1}{2} \times RU \times h}$$

$$= \frac{UQ}{RU}$$

$$\frac{\text{Area of } \Delta PQU}{12} = \frac{2}{4}$$

$$\text{Area of } \Delta PQU = \frac{2}{4} \times 12$$

$$= 6 \text{ cm}^2$$

(ii) Since ST is parallel to RU, ΔPST and ΔPRU are similar.

$$\frac{\text{Area of } \Delta PTS}{\text{Area of } \Delta PUR} = \left(\frac{PT}{PU}\right)^2$$

$$\frac{\text{Area of } \Delta PTS}{12} = \left(\frac{6}{8}\right)^2$$

$$= \frac{9}{16}$$

$$\text{Area of } \Delta PTS = \frac{9}{16} \times 12$$

$$= 6.75 \text{ cm}^2$$

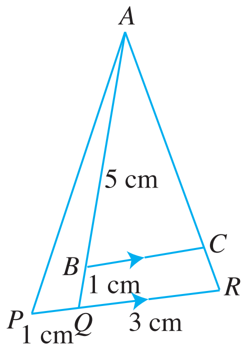
ATTENTION

In ΔPST and ΔPRU,
$\angle P$ is a common angle,
$\angle PST = \angle PRU$ (corr. $\angle$s) and
$\angle PTS = \angle PUR$ (corr. $\angle$s).

PRACTISE NOW 3

In the figure, BC is parallel to QR, $AB = 5$ cm, $BQ = 1$ cm, $PQ = 1$ cm and $QR = 3$ cm. Given that the area of ΔAQR is 21 cm², find the area of

(i) ΔAPQ, **(ii)** ΔABC.

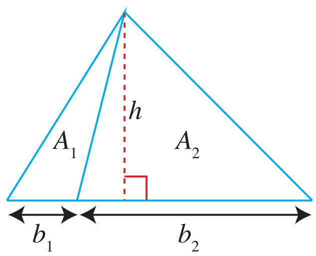


SIMILAR QUESTIONS

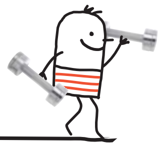
Exercise 10A Questions 13-15

From Worked Example 3, we can conclude that the ratio of the areas of two triangles having a *common height* h is equal to the ratio of the lengths of the bases of the two triangles, i.e.

$$\frac{A_1}{A_2} = \frac{b_1}{b_2},$$



where A_1 and b_1 are the area and the length of the base of the first triangle respectively, and A_2 and b_2 are the area and the length of the base of the second triangle respectively.

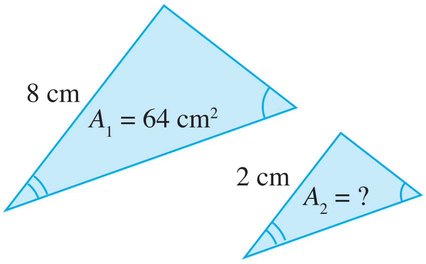


Exercise 10A

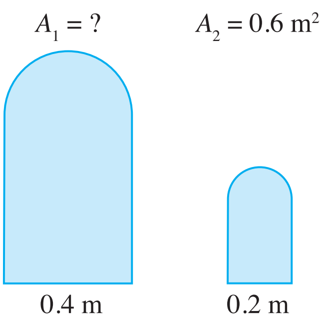
BASIC LEVEL

1. Find the unknown area of each of the following pairs of similar figures.

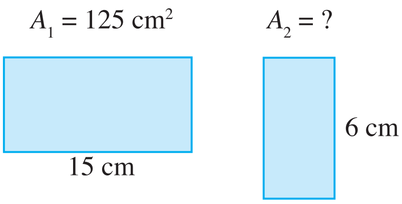
(a)



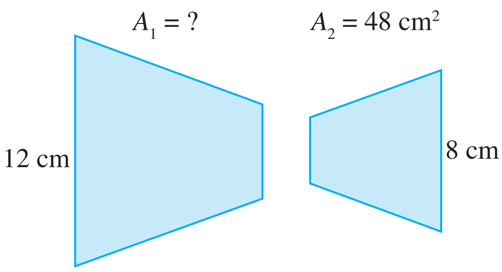
(b)



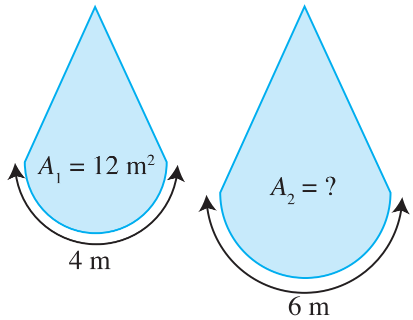
(c)



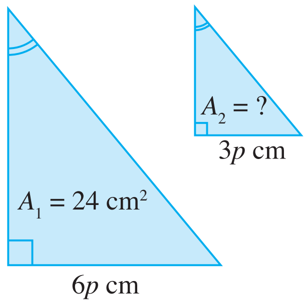
(d)



(e)

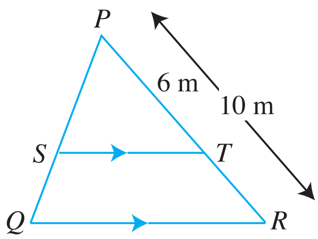


(f)



2. Find the ratio of the areas of two circles whose radii are 4 cm and 7 cm.

3. A triangular plot of land *PQR* is such that *PT* = 6 m and *PR* = 10 m. *ST* and *QR* are two water pipes that are parallel to each other. The area of ΔPST is 24 m².

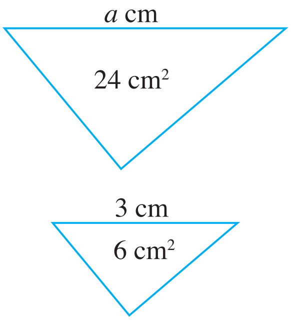


Find the area of the land occupied by

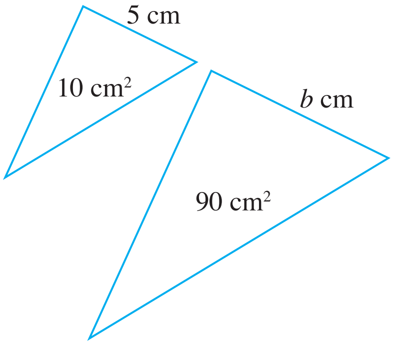
(i) ΔPQR, **(ii)** *SQRT*.

4. Find the unknown value in each of the following pairs of similar figures.

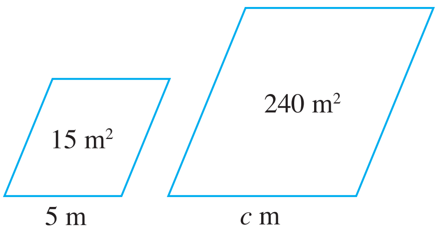
(a)



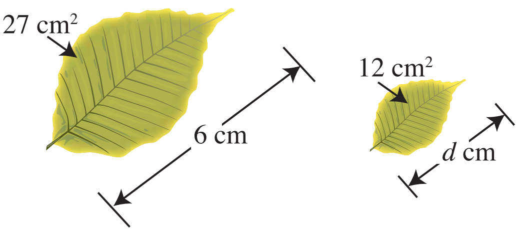
(b)



(c)



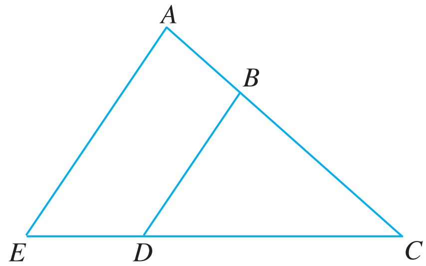
(d)



INTERMEDIATE LEVEL

5. The perimeters of two similar regular hexagons are 10 m and 8 m. Given that the area of the larger hexagon is 200 m², find the area of the smaller hexagon.

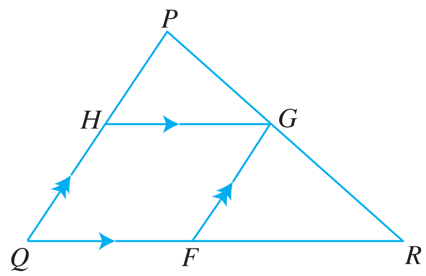
6. In the figure, ΔCAE is an enlargement of ΔCBD with a scale factor of $\frac{4}{3}$.



Given that the area of ΔCBD is 9 cm², find the area of $ABDE$.

7. In a scale drawing of a house, the width, 150 cm, of a door is represented by a line 30 mm long. Find the actual land area, in square metres, occupied by the house if the corresponding area on the plan is 3250 cm².

8. In the figure, HG is parallel to QR, GF is parallel to PQ and $QF : FR = p : q$.



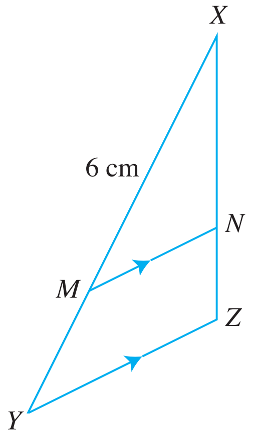
Find the ratio of the area of ΔPHG to that of ΔPQR in terms of p and q.

9. Two solid cones are geometrically similar and the height of one cone is 1.5 times that of the other. Given that the height of the smaller cone is 12 cm and its surface area is 124 cm², find

(i) the height, **(ii)** the surface area,

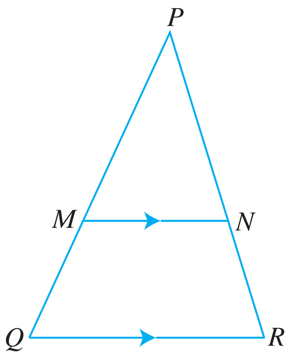
of the larger cone.

10. In the figure, ΔXYZ is an enlargement of ΔXMN.



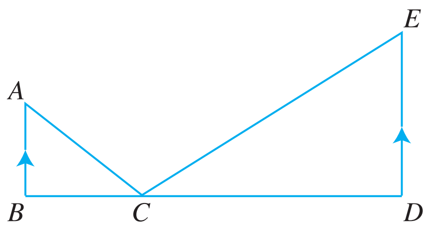
Given that $XM = 6$ cm and that the areas of ΔXMN and $MYZN$ are 14 cm^2 and 22 cm^2 respectively, find the length of MY.

11. In the figure, MN is parallel to QR.



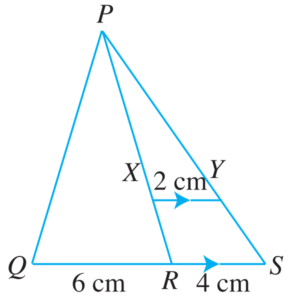
If the areas of ΔPMN and trapezium $MQRN$ are in the ratio 9 : 16, find the ratio $MN : QR$.

12. In the figure, BCD is a straight line and BA is parallel to DE. The areas of ΔABC and ΔCDE are 25 cm^2 and 64 cm^2 respectively.



Given further that CD is 4.5 cm longer than BC, find the length of BC.

13. In the figure, XY is parallel to RS, $XY = 2$ cm, $QR = 6$ cm and $RS = 4$ cm.

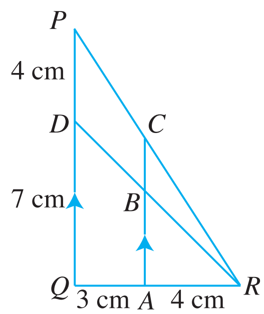


Given that the area of ΔPXY is 10 cm^2, find the area of

(i) ΔPRS, **(ii)** ΔPQR.

ADVANCED LEVEL

14. In the figure, AC is parallel to QP, $AR = 4$ cm, $QA = 3$ cm, $DQ = 7$ cm and $PD = 4$ cm.



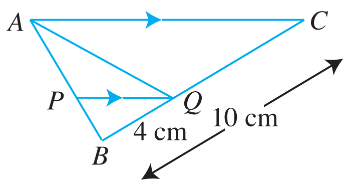
Find

(i) the length of BC,

(ii) the ratio of the area of ΔARB to that of ΔBRC,

(iii) the ratio of the area of ΔBRC to $ABDQ$.

15. In the figure, PQ is parallel to AC.

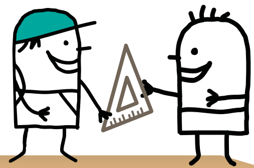


Given that $BQ = 4$ cm, $BC = 10$ cm and the area of ΔBPQ is 8 cm^2, find the area of

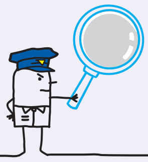
(i) ΔABC, **(ii)** ΔPQC,

(iii) ΔAQC.

10.2 Volume of Similar Solids



In Section 10.1, we have learnt how to find the area of similar figures. In this section, we will learn how to find the volume of similar solids.



Investigation

Volume and Mass of Similar Solids

1. Table 10.3 shows three cubes. Are they similar? Explain your answer.

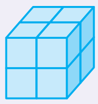
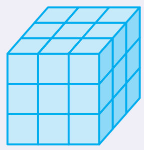
Cube			
Length of Cube	1 cm	2 cm	3 cm
Volume of Cube			

Table 10.3

2. Complete Table 10.3 to find the volume of each cube.

3. **(a)** The length of the second cube is double that of the first cube. What is the relationship between their volumes?

(b) The length of the third cube is three times that of the first cube. What is the relationship between their volumes?

4. Let the length and the volume of a cube be l_1 and V_1 respectively.
Let the length and the volume of a second cube be l_2 and V_2 respectively.
Note that the two cubes are **similar**.
Express the following ratio of volumes in terms of l_1 and l_2.

$$\boxed{\frac{V_2}{V_1} = \quad}$$

5. Is the formula in Question 4 always true?

Let us investigate what happens if we have similar cylinders instead.

Fig. 10.1 shows two **similar** cylinders: they have exactly the same shape, i.e. the ratio of the corresponding lengths is a constant k, called the **scale factor** of the solids.

For example,

$$\boxed{\frac{h_2}{h_1} = k}$$

where h_1 and h_2 are the heights of the first and second cylinders respectively.

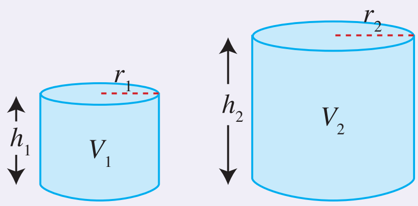


Fig. 10.1

The radii of the circular cross sections of the two cylinders are r_1 and r_2 respectively.

What is the value of $\frac{r_2}{r_1}$?

6. The volume of the first cylinder is $V_1 = \pi r_1^2 h_1$.

(a) Find the volume of the second cylinder, V_2, in terms of r_1 and h_1.

(b) Hence, find the volume of the second cylinder, V_2, in terms of V_1.

7. Express the following ratio of volumes in terms of k, then in terms of h_1 and h_2, and then in terms of r_1 and r_2.

$$\boxed{\frac{V_2}{V_1} = \qquad\qquad}$$

8. If two similar solids are made of the same material, what is the relationship between their masses m_1 and m_2?

Let the density of the material be d. Then

$$\boxed{d = \frac{m_1}{V_1} = \frac{m_2}{V_2},}$$

where V_1 and V_2 are the volumes of the solids respectively.

The density of a material is defined as the mass per unit volume of the material.

Express the following in terms of V_1 and V_2.

$$\boxed{\frac{m_2}{m_1} = \qquad}$$

In general, the ratio of the volumes of two similar solids is the cube of the ratio of their corresponding lengths, and the ratio of their masses is equal to the ratio of their volumes, i.e.

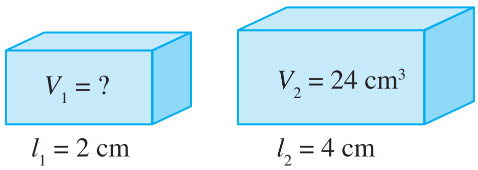
$$\boxed{\frac{V_2}{V_1} = \left(\frac{l_2}{l_1}\right)^3 \text{ and } \frac{m_2}{m_1} = \frac{V_2}{V_1},}$$

where V_1, l_1 and m_1 are the volume, length and mass of the first solid respectively, and V_2, l_2 and m_2 are the volume, length and mass of the second similar solid respectively.

Worked Example 4

(Finding the Volume of Similar Solids)

The figure shows two toy blocks which take the shape of a pair of similar cuboids.



Find the volume, V_1, of the smaller block.

Solution:

$$\frac{V_1}{V_2} = \left(\frac{l_1}{l_2}\right)^3$$

$$\frac{V_1}{24} = \left(\frac{2}{4}\right)^3$$

$$= \left(\frac{1}{2}\right)^3$$

$$= \frac{1}{8}$$

$$\therefore V_1 = \frac{1}{8} \times 24$$

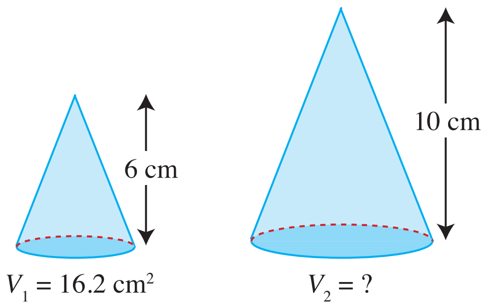
$$= 3 \text{ cm}^3$$

PRACTISE NOW 4

SIMILAR QUESTIONS

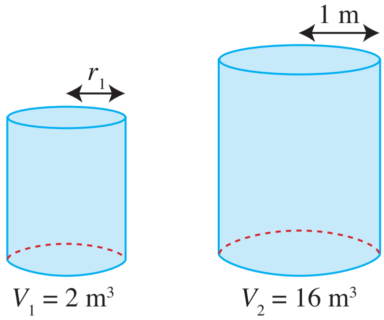
Exercise 10B Questions 1(a)-(e), 2, 3, 4(a)-(d), 5, 9

1. The figure shows two chocolate hats which take the shape of a pair of similar cones.



Find the volume, V_2, of the larger chocolate hat.

2. Find the unknown radius, r_1, for the following pair of similar cylinders.



Worked Example 5

(Finding the Mass of Similar Solids)

Two solid spheres of diameters 4 m and 5 m are made of the same material. Given that the smaller sphere has a mass of 120 kg, find the mass of the larger sphere.

Solution:

Let m_1, V_1 and l_1 be the mass, volume and diameter of the smaller sphere respectively, and m_2, V_2 and l_2 be the mass, volume and diameter of the larger sphere respectively.

$$\frac{m_2}{m_1} = \frac{V_2}{V_1}$$
$$= \left(\frac{l_2}{l_1}\right)^3$$
$$\frac{m_2}{120} = \left(\frac{5}{4}\right)^3$$
$$m_2 = \left(\frac{5}{4}\right)^3 \times 120$$
$$= 234\frac{3}{8}$$

∴ The mass of the larger sphere is $234\frac{3}{8}$ kg.

PRACTISE NOW 5

SIMILAR QUESTIONS

Exercise 10B Questions 6-8, 10-12, 15

1. Two similar solid triangular prisms have heights 5 cm and 8 cm. Given that the smaller prism has a mass of 80 g, find the mass of the larger prism, giving your answer correct to the nearest integer.

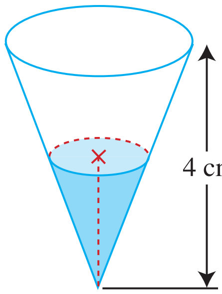
2. The figure shows a statue with a height of 20 cm and a mass of 3 kg. Michael wishes to make a similar statue with a height of 2 m using the same material.

Find the mass of the statue made by Michael.

Worked Example 6

(Problem involving Similar Solids)

The figure shows an inverted conical container of height 4 cm. It contains a volume of water which is equal to one-eighth of its full capacity.



Find

(i) the depth of the water,

(ii) the ratio of the area of the top surface of the water to the area of the top surface of the container.

Solution:

(i) Let V_1 and h_1 be the volume and height of the smaller cone respectively, and V_2 and h_2 be the volume and height of the larger cone respectively.

$$\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3$$

$$\frac{1}{8} = \left(\frac{h_1}{4}\right)^3 \quad \text{(Since } V_1 = \frac{1}{8}V_2\text{, then } \frac{V_1}{V_2} = \frac{1}{8}\text{.)}$$

$$\left(\frac{h_1}{4}\right)^3 = \frac{1}{8}$$

$$\frac{h_1}{4} = \sqrt[3]{\frac{1}{8}}$$

$$= \frac{1}{2}$$

$$h_1 = \frac{1}{2} \times 4$$

$$= 2$$

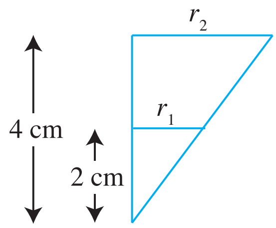
$\therefore$ The depth of the water is 2 cm.

(ii) The top surface of the water and that of the container are circles.

Let r_1 and r_2 be the radii of the smaller circle and the larger circle respectively.

Using similar triangles,

$$\frac{r_1}{r_2} = \frac{h_1}{h_2}$$
$$= \frac{2}{4}$$
$$= \frac{1}{2}$$



Let A_1 and A_2 be the areas of the smaller circle and larger circle respectively.

$$\frac{A_1}{A_2} = \left(\frac{r_1}{r_2}\right)^2$$
$$= \left(\frac{1}{2}\right)^2$$
$$= \frac{1}{4}$$

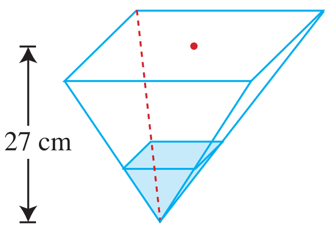
∴ The ratio of the area of the top surface of the water to the area of the top surface of the container is 1 : 4.

PRACTISE NOW 6

SIMILAR QUESTIONS

Exercise 10B Questions 13, 14

The figure shows a container in the shape of an inverted right pyramid of height 27 cm. It contains a volume of vegetable oil which is equal to one-sixth of its full capacity.



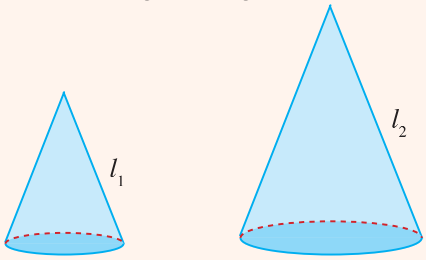
Find

(i) the depth of the vegetable oil,

(ii) the ratio of the area of the top surface of the vegetable oil to the area of the top surface of the container, giving your answer in the form $1 : n$.

Thinking Time

1. Two similar cones with slant heights are given as shown.



Express the ratio of the total surface area of the smaller cone to that of the larger cone in terms of l_1 and l_2. Explain your answer.

2. The similarity ratio formulae, $\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2$ and $\frac{m_2}{m_1} = \frac{V_2}{V_1} = \left(\frac{l_2}{l_1}\right)^3$, have some real-life implications.

For example, why is it not possible for a human to be a giant with a height of about 20 m?

Journal Writing

Search on the Internet to find out the locations of 5 different Merlions in Singapore which are recognised by the Singapore Tourism Board.

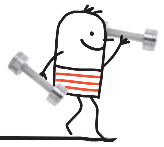
(a) Find the ratio of the heights of these 5 Merlions.

(b) Hence, find the ratio of

  (i) the total surface area of these Merlions,

  (ii) the volume of material used to construct these Merlions.

State any assumptions that you have made.

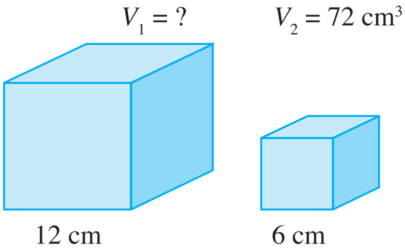


Exercise 10B

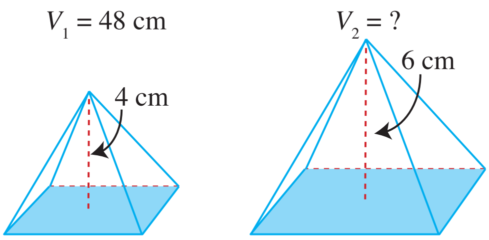
BASIC LEVEL

1. Find the unknown volume of each of the following pairs of similar solids.

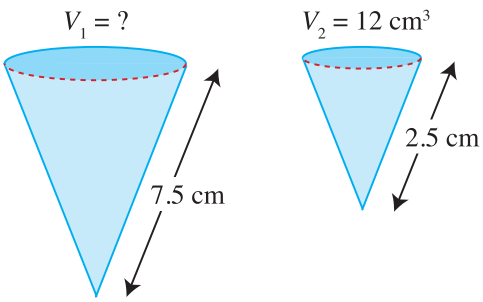
(a)



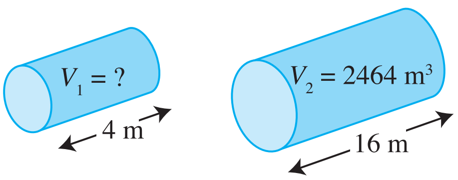
(b)



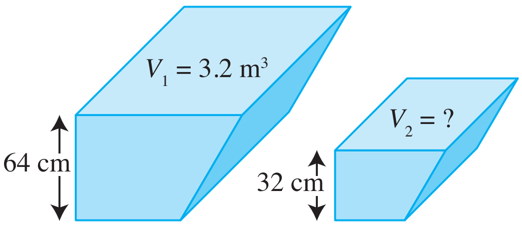
(c)



(d)



(e)



2. Find the ratio of the volumes of

(a) two similar solid cylinders of circumferences 10 cm and 8 cm,

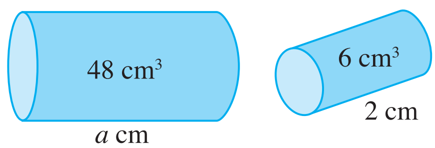
(b) two similar solid cones of heights 9 cm and 12 cm,

(c) two solid spheres of radii 4 cm and 6 cm.

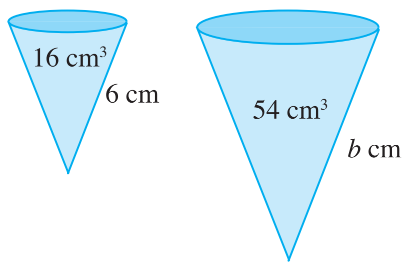
3. In a restaurant, a Junior glass has a height of 6 cm and a Senior glass has a height of 9 cm. Given that the capacity of a Senior glass is 540 cm^3, find the capacity of the Junior glass.

4. Find the unknown value in each of the following pairs of similar solids.

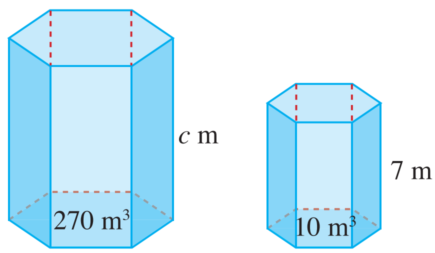
(a)



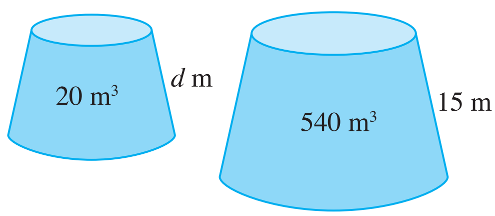
(b)



(c)



(d)



5. The areas of the bases of two similar cones are in the ratio 9 : 16.
   (i) Find the ratio of the heights of the cones.
   (ii) Given that the volume of the larger cone is 448 cm^3, find the volume of the smaller cone.

6. The masses of two spheres of the same material are 640 kg and 270 kg. Find the ratio of their diameters.

7. A certain brand of chilli flakes comes in similar bottles of two sizes – 'mini' and 'ordinary'. The 'mini' bottle has a mass of 280 g and a height of 15 cm. Given that the 'ordinary' bottle has a mass of 750 g, find its height.

8. Two similar solid candy canes have heights 4 cm and 7 cm.
   (i) Find the ratio of the total surface areas of the candy canes.
   (ii) Given that the smaller candy cane has a mass of 10 g, find the mass of the larger candy cane.

INTERMEDIATE LEVEL

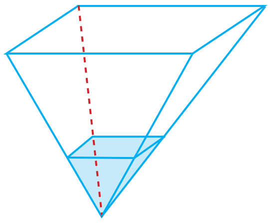
9. The volume of one sphere is 4 times that of another sphere. Given that the radius of the smaller sphere is 3 cm, find the radius of the larger sphere.

10. The mass of a glass figurine of height 6 cm is 500 g. Find the mass of a similar glass figurine if it has a height of 4 cm.

11. A train is 10 m long and has a mass of 72 tonnes. A similar model, made of the same material, is 40 cm long. (1 tonne = 1000 kg)
    (i) Find the mass of the model.
    (ii) Given that the tank of the model train contains 0.85 litres of water when it is full, find the capacity of the tank of the train, giving your answer correct to the nearest integer.

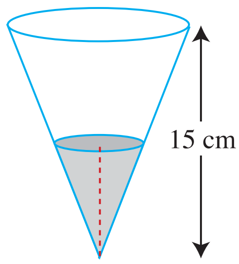
12. The masses of two similar plastic boxes are 8.58 kg and 4.29 kg. Given that the first box has a base area of 12.94 m^2, find the base area of the second box.

13. The figure shows a container in the shape of an inverted right pyramid which contains some water. The area of the top surface of the container is 63 cm^2 and the area of the top surface of the water is 28 cm^2.



Find
(i) the depth of the water if its volume is 336 cm^3,
(ii) the ratio of the depth of the water to the height of the container,
(iii) the capacity of the container.

14. The figure shows an inverted conical container of height 15 cm found in a laboratory. It contains a volume of mercury which is equal to $\frac{8}{27}$ of its full capacity.

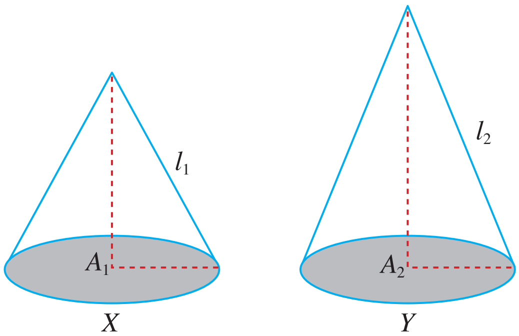


Find
(i) the depth of the mercury,
(ii) the area of the mercury that is exposed to the air if the area of the top surface of the container is 45 cm^2,
(iii) the capacity of the container.

ADVANCED LEVEL

15. A clay model has a mass of x^2 kg and a height of 30 cm. A similar clay model has a mass of $(x + 0.3)$ kg and a height of 20 cm. Find the value of x.

Summary



Volume of X is V_1 Volume of Y is V_2

If X and Y are two similar solids, then

- Ratio of their corresponding lengths is $\frac{l_2}{l_1}$
- Ratio of their areas, $\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2$
- Ratio of their volumes, $\frac{V_2}{V_1} = \left(\frac{l_2}{l_1}\right)^3$

Review Exercise 10

1. Find the ratio of the areas of two similar triangles if the lengths of their corresponding sides are

(a) 3 cm and 5 cm, **(b)** 4.5 m and 9 m,

(c) 2 mm and 3 mm.

2. Find the ratio of the areas of two similar triangles if their perimeters are 294 cm and 336 cm.

3. **(i)** If the length of each side of a regular pentagon is doubled, what will happen to its area?

(ii) Given that the length of each side of a pentagon, whose area is 25 cm^2, is doubled, find the area of the enlarged pentagon.

(iii) If the length of each side of a regular n-sided polygon is tripled, what will happen to its area?

4. The volumes of two similar cylinders are in the ratio 8 : 27. Find the ratio of the base radii of the cylinders.

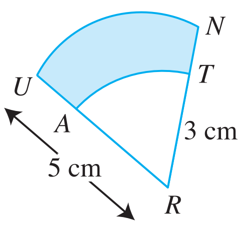
5. The volumes of two similar water jugs are in the ratio 27 : 64. Find

(i) the ratio of the heights of the jugs,

(ii) the ratio of the total surface areas of the jugs.

6. A model of a marble statue of height 3.2 m is made of the same material as the statue. Given that the height of the model is 40 cm and its mass is 12 kg, find the mass of the statue in tonnes.
(1 tonne = 1000 kg)

7. In the figure, the radii of the sectors *RAT* and *RUN* are 3 cm and 5 cm respectively.



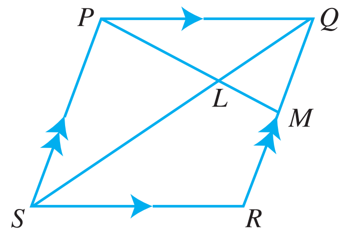
Find the ratio of the area of the shaded region to that of the area of sector *RUN*.

8. Two artificial ponds are similar in every aspect. The perimeter of the surface of the larger pond is three times that of the smaller pond.

(i) Write down the ratio of their surface areas.

(ii) Given that the larger pond contains 10 800 litres of water, find the amount of water contained in the smaller pond.

9. In the figure, *PQRS* is a parallelogram. *PLM* and *SLQ* are straight lines and *M* is the midpoint of *QR*.



(a) Identify a triangle similar to ΔQLM.

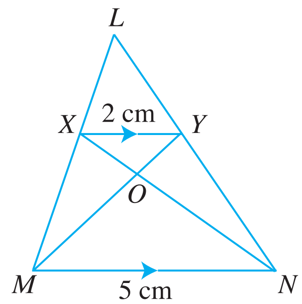
(b) Write down the numerical value of

(i) $\dfrac{\text{area of } \Delta QLM}{\text{area of } \Delta SLP}$,

(ii) $\dfrac{LS}{QS}$,

(iii) $\dfrac{\text{area of } \Delta PLS}{\text{area of } \Delta PQS}$.

10. In the figure, *XY* is parallel to *MN* where $XY = 2$ cm and $MN = 5$ cm. *XN* and *YM* meet at *O*.



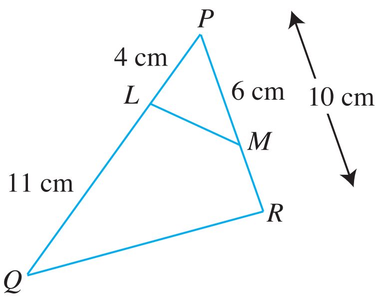
(a) Find the value of $\dfrac{\text{area of quadrilateral } XMNY}{\text{area of } \Delta LMN}$.

(b) **(i)** Identify a triangle similar to ΔXOY.

(ii) Write down the value of $\dfrac{YO}{OM}$.

(iii) Show that the area of ΔXOY : area of ΔXOM : area of $\Delta MON = 4 : 10 : 25$.

11. In the figure, PLQ and PMR are straight lines, $PL = 4$ cm, $LQ = 11$ cm, $PM = 6$ cm and $PR = 10$ cm.

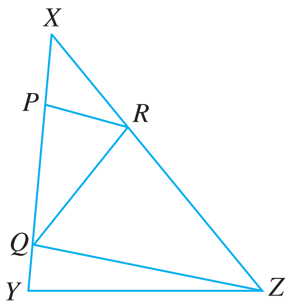


(i) Show that ΔPQR and ΔPML are similar.

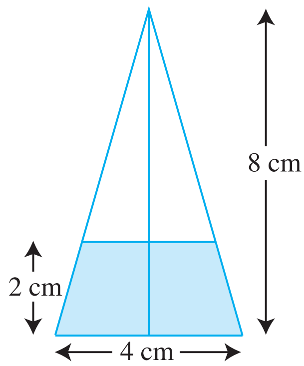
(ii) Write down the value of $\dfrac{\text{area of } \Delta PQR}{\text{area of } \Delta PML}$.

(iii) Given that the area of ΔPML is 6 cm^2, find the area of the quadrilateral $LMRQ$.

12. In the figure, $XP = \frac{1}{2}PQ$, $QY = \frac{1}{3}PQ$ and $XR = \frac{2}{5}XZ$. Find the ratio of the area of ΔQRZ to that of ΔQYZ.



13. The figure shows a vertical section through the axis of a solid paper weight made in the shape of a right circular cone. Its height is 8 cm and the diameter of its base is 4 cm. The shaded portion is made of lead 2 cm thick while the unshaded portion is made of wood. The density of lead and wood are 11.3 g/cm^3 and 0.9 g/cm^3 respectively.



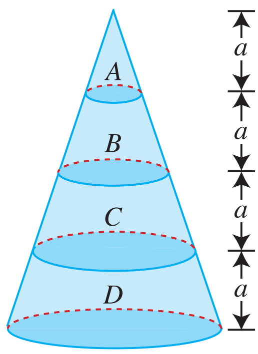
Find

(i) the total volume of the paper weight,

(ii) the volume of the wooden portion,

(iii) the total mass of the paper weight.

14. A right circular cone is divided into 4 portions, A, B, C and D, by planes parallel to the base. The height of each portion is a units.

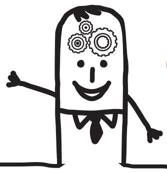


Find

(i) the ratio of the volume of A to that of B,

(ii) the ratio of the volume of B to that of C,

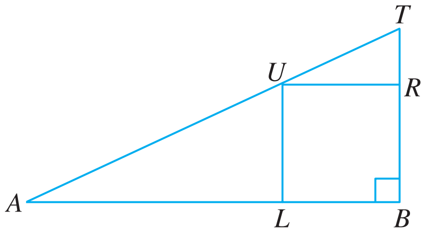
(iii) the ratio of the sum of the volumes of A, B and C to that of D.



Challenge Yourself

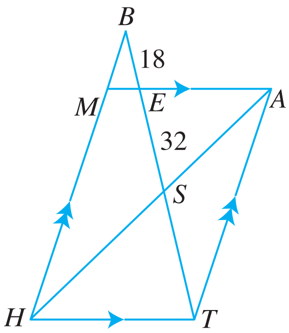
1. In the figure, BAT is a right-angled triangle with $AB = 2BT$. A square $BLUR$ is inscribed in ΔBAT such that L, U and R lie on the sides of ΔBAT.

Hint: Let $BT = x$ and $BR = y$.



Find the ratio of the area of the square $BLUR$ to that of ΔBAT.

2. In the figure, $MATH$ is a parallelogram. BMH and $BEST$ are straight lines, $BE = 18$ cm and $ES = 32$ cm.



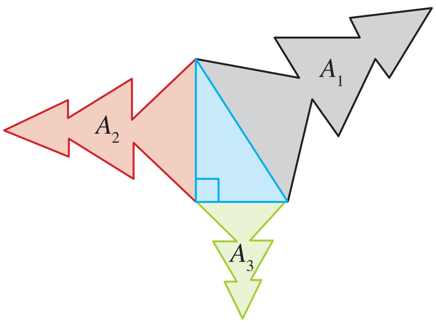
(i) Explain why ΔBME and ΔBHT are similar.

(ii) Name 2 other pairs of similar triangles.

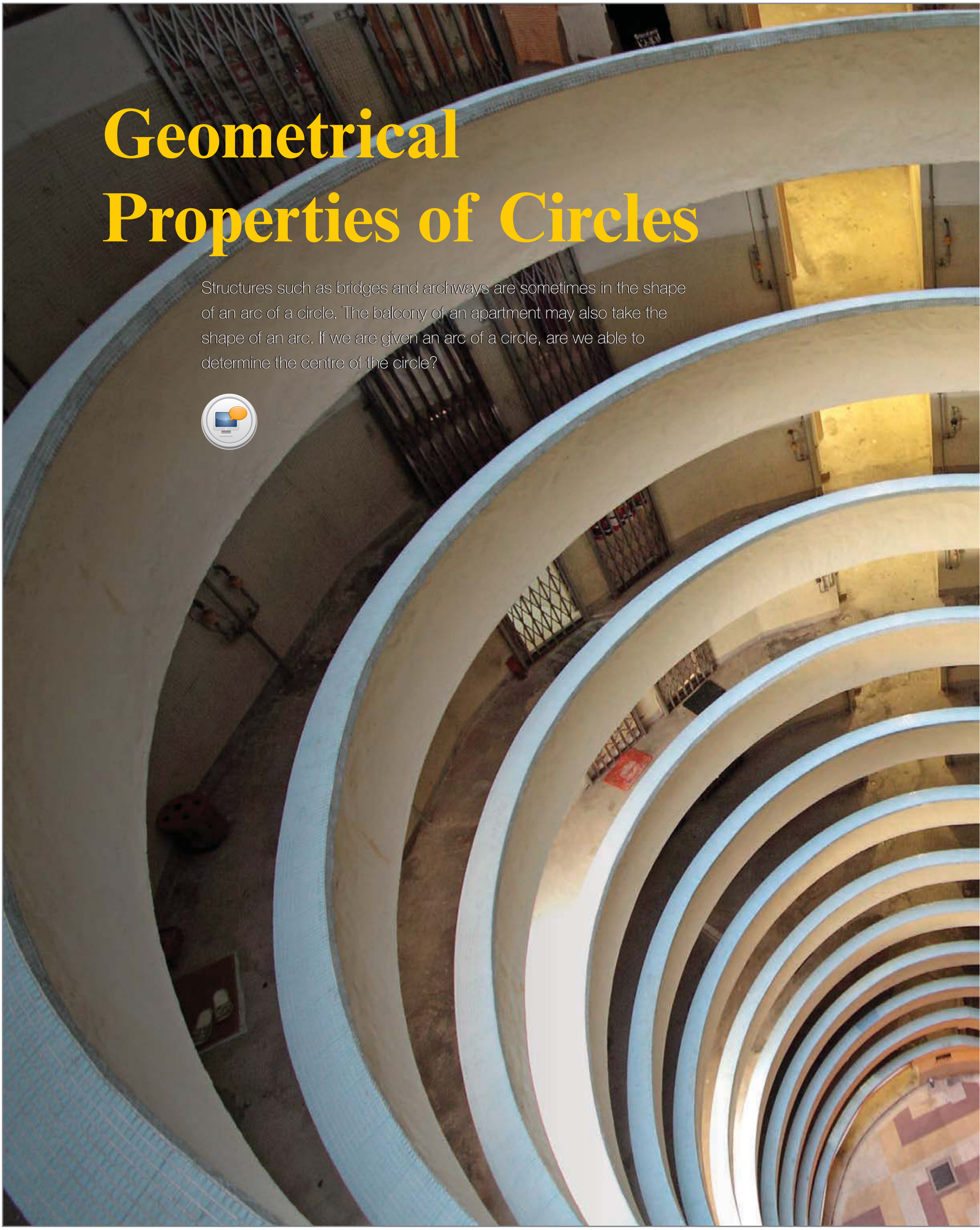
(iii) Find the length of ST.

(iv) Find the ratio of the areas of ΔBME and ΔBHT.

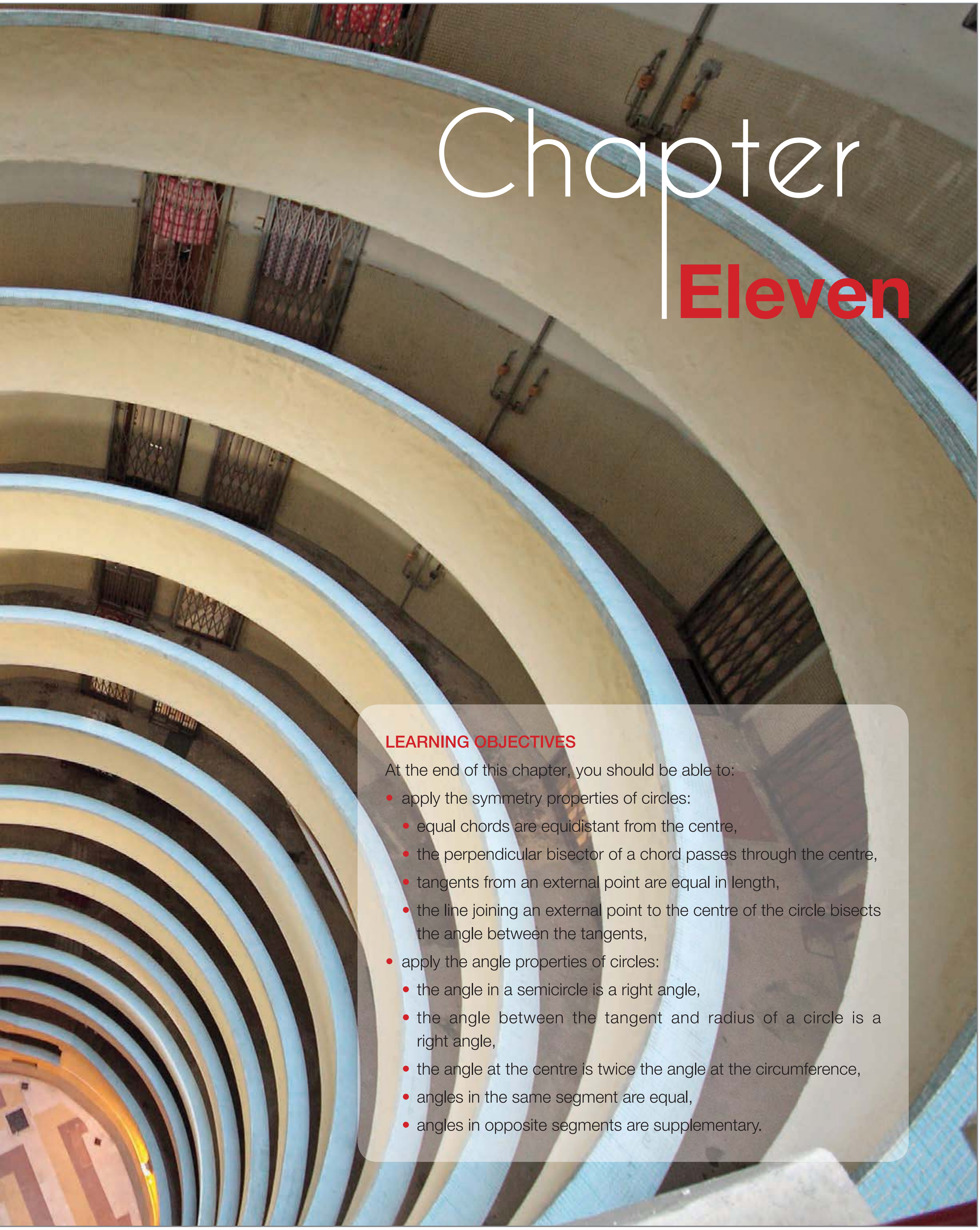
3. The diagram shows a right-angled triangle. The three shaded figures are similar. Their areas are A_1, A_2 and A_3 as indicated. Prove that $A_1 = A_2 + A_3$. This is called the Generalised Pythagoras' Theorem.



Geometrical Properties of Circles



Structures such as bridges and archways are sometimes in the shape of an arc of a circle. The balcony of an apartment may also take the shape of an arc. If we are given an arc of a circle, are we able to determine the centre of the circle?



Chapter Eleven

LEARNING OBJECTIVES

At the end of this chapter, you should be able to:

- apply the symmetry properties of circles:
  - equal chords are equidistant from the centre,
  - the perpendicular bisector of a chord passes through the centre,
  - tangents from an external point are equal in length,
  - the line joining an external point to the centre of the circle bisects the angle between the tangents,
- apply the angle properties of circles:
  - the angle in a semicircle is a right angle,
  - the angle between the tangent and radius of a circle is a right angle,
  - the angle at the centre is twice the angle at the circumference,
  - angles in the same segment are equal,
  - angles in opposite segments are supplementary.

11.1 Symmetric Properties of Circles

In this section, we will learn four symmetric properties of circles – two of them on chords and the other two on tangents.

Perpendicular Bisector of a Chord

Investigation

Circle Symmetric Property 1

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Symmetric Property 1 as shown below.

There are three conditions:

Condition A: The line l (or OM) passes through the centre O of the circle.

Condition B: The line l (or OM) is perpendicular to the chord AB.

Condition C: The line l (or OM) **bisects** the chord AB.

Note that the chord AB must *not* be the diameter of the circle.

In this investigation, you will learn that **any two** of the above three conditions will imply the third one.

INFORMATION

'Bisect' means 'cut into two equal parts.'

Part 1

1. The template shows a circle with centre O and the line OM perpendicular to the chord AB. Which two of the above three conditions are given?

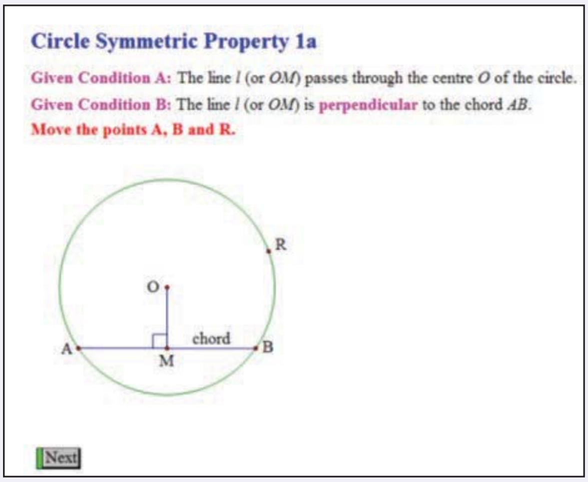


Fig. 11.1

2. Click and drag point A or B to change the chord.

   Click and drag point R to change the size of the circle.

   (a) What do you notice about the length of AM and of MB?

   (b) What do you call the point M?

Part 2

3. Click on the 'Next' button. The next page of the template shows a circle with centre O, and the line OM bisecting the chord AB.

   Which two of the three conditions on the previous page are given?

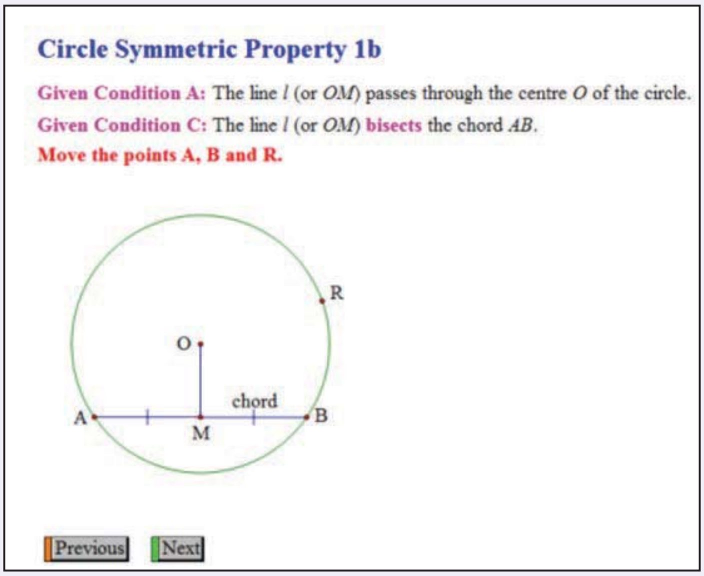


Fig. 11.2

4. Click and drag point A or B to change the chord.

   Click and drag point R to change the size of the circle.

   What do you notice about the size of $\angle AMO$ and $\angle BMO$?

Part 3

5. Use a sheet of paper to draw and cut out a circle. To find the centre of a circle, fold the circle into two equal halves, and then fold again into two equal halves as shown in Fig. 11.3**(a)**.

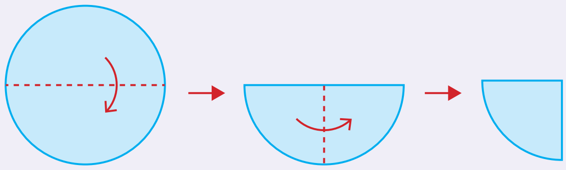


Fig. 11.3(a)

Open up the paper as shown in Fig. 11.3**(b)**, where the dotted lines indicate the lines obtained from the above paper folding.

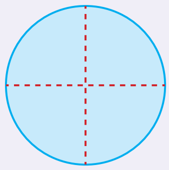


Fig. 11.3(b)

Mark the centre of the circle as O in Fig. 11.3**(b)**. Why is this the centre of the circle?

6. Using the same circle as in Question 5, fold along a chord AB that is not a diameter of the circle and then fold it into two equal halves as shown in Fig. 11.3**(c)**.

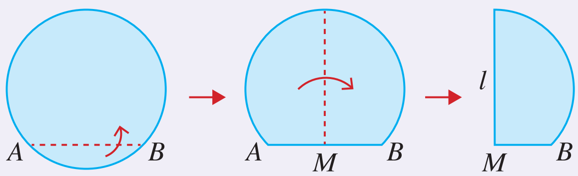


Fig. 11.3(c)

Open up the paper as shown in Fig. 11.3**(d)**, where the dotted lines indicate the lines obtained from the above paper folding.

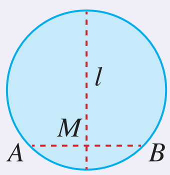


Fig. 11.3(d)

As the paper is folded into two equal halves, the line l bisects the chord AB and $\angle AMB$. Since $\angle AMB = 180°$, l is perpendicular to the chord AB.

(a) Which two of the three conditions on page 361 are satisfied?

(b) Does the line l pass through the centre O of the circle that you have marked in Question 5?

INFORMATION

Can you think of other ways of folding to obtain the centre of the circle?

From the investigation, there are three parts to **Circle Symmetric Property 1** (any two of the three conditions will imply the third one):

(i) If a line l **passes through the centre** of a circle and is **perpendicular to a chord** AB (which is not the diameter) of the circle, then the line l **bisects the chord** AB.

(ii) If a line l **passes through the centre** of a circle and **bisects a chord** AB (which is not the diameter) of the circle, then the line l is **perpendicular to the chord** AB.

(iii) If a line l **bisects the chord** AB of a circle and is **perpendicular to the chord** AB (i.e. l is the perpendicular bisector of the chord AB), then the line l **passes through the centre** of the circle. (This is also true if the chord is a diameter of the circle.)

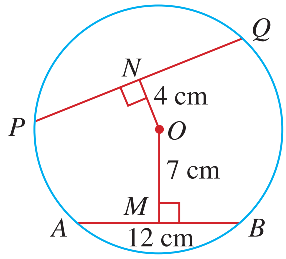
Thinking Time

1. If a line l passes through the centre of a circle and is perpendicular to a chord AB (which is not the diameter) of the circle, by using congruent triangles, prove that the line l bisects the chord AB.
2. If a line l passes through the centre of a circle and bisects a chord AB (which is not the diameter) of the circle, by using congruent triangles, prove that the line l is perpendicular to the chord AB.

Worked Example 1

(Application of Circle Symmetric Property 1)

In the figure, AB and PQ are chords of the circle with centre O. The point M lies on AB such that OM is perpendicular to AB and the point N lies on PQ such that ON is perpendicular to PQ.



Given that $AB = 12$ cm, $OM = 7$ cm and $ON = 4$ cm, find the length of the chord PQ, giving your answer correct to 2 decimal places.

Solution:

OM bisects AB (perpendicular bisector of chord).

$\therefore AM = MB$

$= \dfrac{12}{2}$

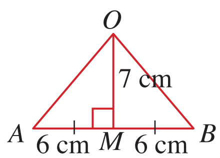
$= 6$ cm

Consider ΔOMA.

$OA^2 = AM^2 + OM^2$ (Pythagoras' Theorem)

$= 6^2 + 7^2$

$= 85$



Since $OP = OA$ (radii of circle),
then $OP^2 = OA^2 = 85$.

Consider ΔONP.

$OP^2 = ON^2 + PN^2$ (Pythagoras' Theorem)

$OA^2 = 4^2 + PN^2$ (since $OA = OP$, radii of circle)

$85 = 16 + PN^2$

$\therefore PN^2 = 85 - 16$

$= 69$

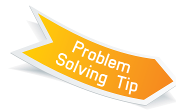
i.e. $PN = \sqrt{69}$ (since length $PN > 0$)

ON bisects PQ. (perpendicular bisector of chord)

$\therefore PQ = 2 \times PN$

$= 2 \times \sqrt{69}$

$= 16.61$ cm (to 2 d.p.)



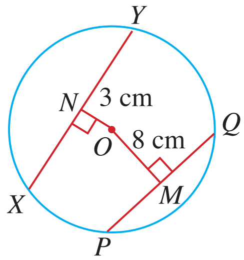
To find PQ, we first try to relate PQ to the information given in the question, i.e. OM, ON, the perpendicular distance of chords from the centre and the length of chord AB.
Since $OP = OA$ (radii of circle) and we have two right-angled triangles OMA and ONP, can we make use of Pythagoras' Theorem to find PN?

PRACTISE NOW 1

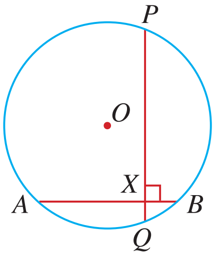
SIMILAR QUESTIONS

Exercise 11A Questions 1(a)-(c), 2-4, 6-8, 11

1. In the figure, PQ and XY are chords of the circle with centre O. The point M lies on PQ such that OM is perpendicular to PQ and the point N lies on XY such that ON is perpendicular to PQ. Given that $XY = 26$ cm, $OM = 8$ cm and $ON = 3$ cm, find the length of the chord PQ, giving your answer correct to 2 decimal places.



2. The figure shows a circle with centre O and radius 7 cm. The chords AB and PQ have lengths 11 cm and 13 cm respectively, and intersect at right angles at X. Find the length of OX.

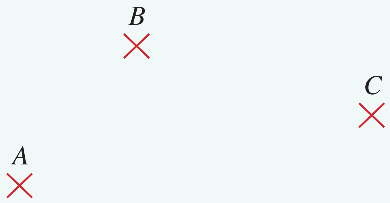


Class Discussion

Application of Circle Symmetric Property 1

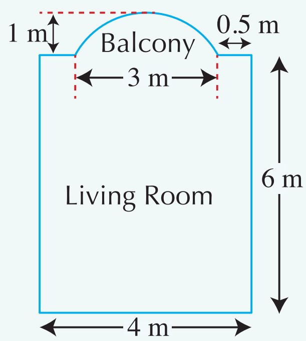
Work in pairs.

1. Construct a circle that passes through the three given points A, B and C.



2. The diagram shows the plan of a living room with a balcony (not drawn to scale). The living room is rectangular (6 m by 4 m) and the balcony is an arc of a circle (see dimensions in diagram). Using a scale of 2 cm to represent 1 m, draw an accurate scale drawing of the living room with the balcony.

Hint: Use the method in Question 1 to draw the arc of the balcony.



Equal Chords

Investigation

Circle Symmetric Property 2

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Symmetric Property 2 as shown below.

Part 1

1. The template shows a circle with centre O and two equal chords.

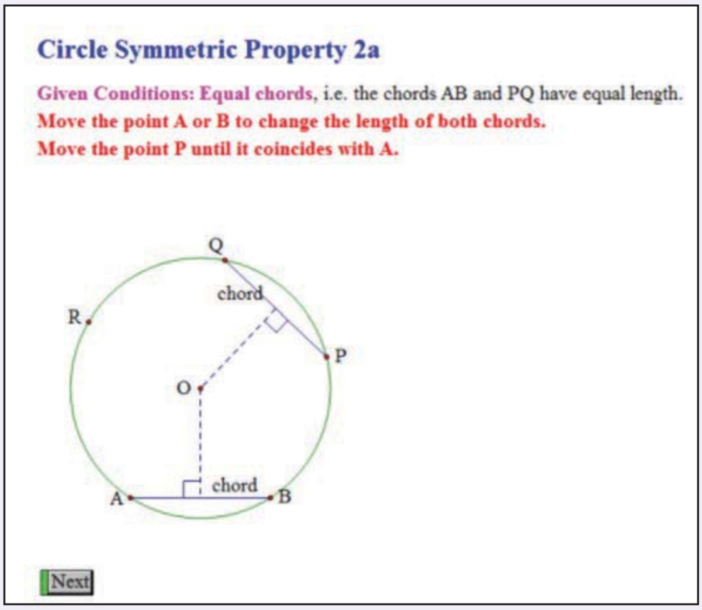


Fig. 11.3

2. Click and drag point A or B to change the lengths of both chords.
Click and drag point R to change the size of the circle.
Click and drag point P until it coincides with the point A.
What do you notice about the distance of both chords from the centre O?

3. Copy and complete the following sentence.
In general, equal chords of a circle are ______________ from the centre of the circle.

ATTENTION

The distance of a point from a line is the perpendicular distance of the point from the line. This distance is also the shortest distance from the point to the line.

Part 2

4. Click on the 'Next' button. The next page of the template shows two chords of a circle that are equidistant from its centre O.

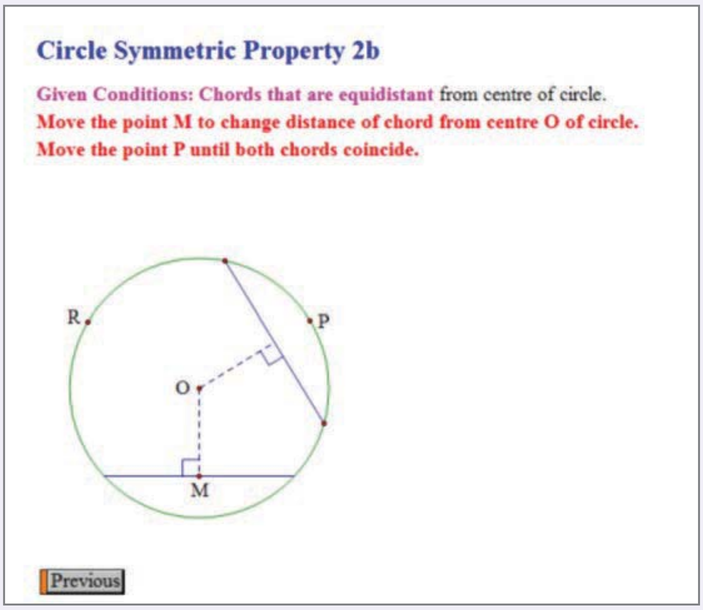


Fig. 11.4

5. Click and drag point M to change the distance of both chords from the centre O.
Click and drag point R to change the size of the circle.
Click and drag point P until both chords coincide.
What do you notice about the lengths of both chords?

6. Copy and complete the following sentence.
In general, chords that are equidistant from the centre of a circle are ______________ in length.

From the investigation, there are two parts to **Circle Symmetric Property 2**:

(i) Equal chords of a circle are equidistant from the centre of the circle.
(ii) Chords that are equidistant from the centre of a circle are equal (in length).

Worked Example 2

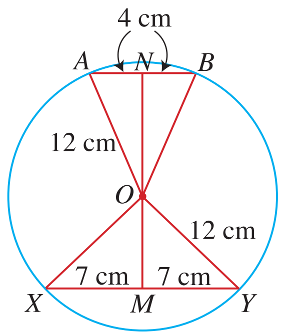
(Application of Circle Symmetric Property 2)

The lengths of two parallel chords of a circle of radius 12 cm are 8 cm and 14 cm respectively. Find the distance between the chords.

Solution:

There are two possible cases about the positions of the two chords AB and XY (equal chords). Let $AB = 8$ cm and $XY = 14$ cm.

Case 1: The chords are on opposite sides of the centre O.



In ΔAON,

$ON^2 = 12^2 - 4^2$ (Pythagoras' Theorem)

$= 128$

$ON = \sqrt{128}$

$= 11.31$ cm (to 4 s.f.)

In ΔYOM,

$OM^2 = 12^2 - 7^2$ (Pythagoras' Theorem)

$= 95$

$OM = \sqrt{95}$

$= 9.747$ cm (to 4 s.f.)

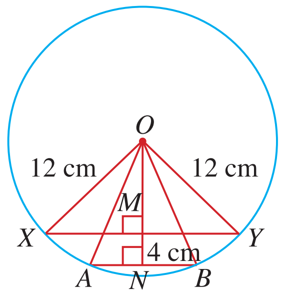
Distance between the chords $= MN$

$= NO + OM$

$= 11.31 + 9.747$

$= 21.1$ cm (to 3 s.f.)

Case 2: The chords are on the same side of the centre O.



Distance between the chords $= MN$

$= ON - OM$

$= 11.31 - 9.747$

$= 1.56$ cm (to 3 s.f.)

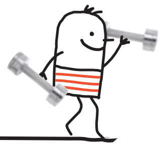
$\therefore$ The distance between the chords can either be 21.1 cm or 1.56 cm.

PRACTISE NOW 2

The lengths of two parallel chords of a circle of radius 20 cm are 10 cm and 30 cm respectively. Find the distance between the chords.

SIMILAR QUESTIONS

Exercise 11A Questions 5(a),(b), 9, 10

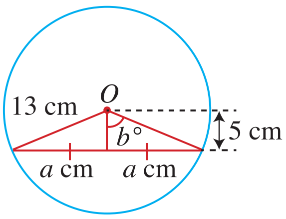


Exercise 11A

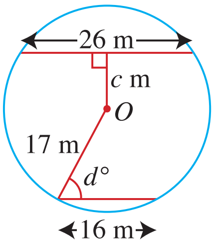
BASIC LEVEL

1. Given that O is the centre of each of the following circles, find the values of the unknowns.

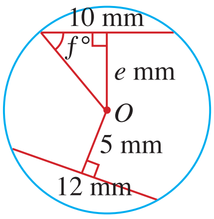
(a)



(b)



(c)



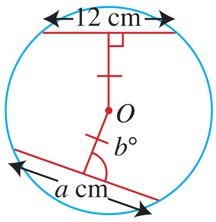
2. AB is a chord of a circle, centre O and with radius 17 cm. Given that AB = 16 cm, find the perpendicular distance from O to AB.

3. A chord of length 24 m is at a distance of 5 m from the centre of a circle. Find the radius of the circle.

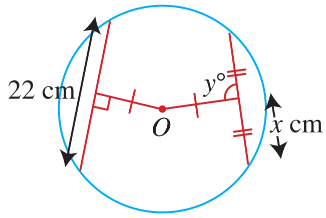
4. A chord of a circle of radius 8.5 cm is 5 cm from the centre. Find the length of the chord.

5. Given that O is the centre of each of the following circles, find the values of the unknowns.

(a)

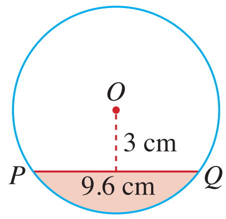


(b)



INTERMEDIATE LEVEL

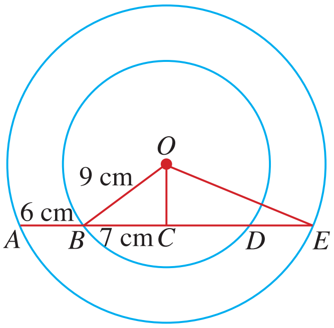
6. The figure shows the cross section of a circular water pipe. The shaded region shows the water flowing through the pipe.



Given that PQ = 9.6 cm and that the surface of the water is 3 cm below the centre O of the circle, find the cross-sectional area of the water pipe.

7. The perpendicular bisector of a chord XY cuts XY at N and the circle at P. Given that XY = 16 cm and NP = 2 cm, calculate the radius of the circle.

8. The figure shows two concentric circles with centre O. The points A and E lie on the circumference of the larger circle while the points B and D lie on the circumference of the smaller circle.



Given that $ABCDE$ is a straight line, OB = 9 cm, AB = 6 cm, BC = 7 cm and AC = CE, find

(i) the length of OC,

(ii) the length of OE.

9. The lengths of two parallel chords of a circle of radius 5 cm are 6 cm and 8 cm respectively. Find the distance between the chords.

10. Two parallel chords PQ and MN are 3 cm apart on the same side of a circle where PQ = 7 cm and MN = 14 cm. Calculate the radius of the circle.

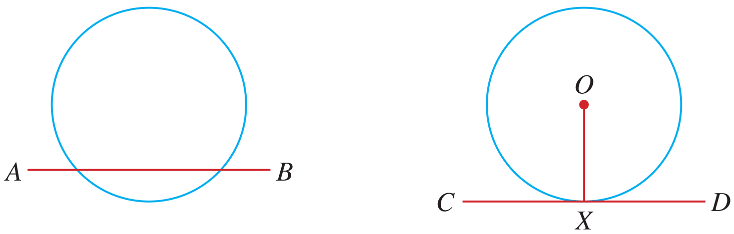
ADVANCED LEVEL

11. The radius of a circle is 17 cm. A chord XY lies 9 cm from the centre and divides the circle into two segments. Find the perimeter of the minor segment.

Radius of a Circle and Tangent to a Circle

A straight line cutting a circle at two distinct points is called a **secant**.

In Fig. 11.5**(a)**, AB is a secant.



(a) **(b)**

Fig. 11.5

If a straight line and a circle have only one point of contact, then that line is called a **tangent**.

In Fig. 11.5**(b)**, CD is a tangent and X is the point of contact.

Investigation

Circle Symmetric Property 3

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Symmetric Property 3 as shown below.

1. The template shows a circle with centre O and radius OP, which is perpendicular to the chord at A.

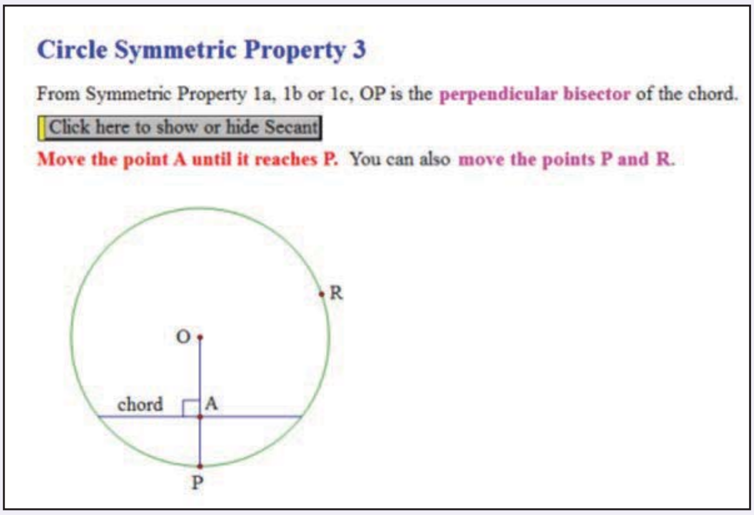


Fig. 11.6

2. Click on the button 'Click here to show or hide Secant'. It will reveal a secant that coincides with the chord, i.e. the secant is also perpendicular to the radius OP. Unlike a chord which is a *line segment* with two end points, a secant is a *line* that cuts the circle at two different points.

3. Click and drag point P to move the radius OP and the secant around the circle.

Click and drag point R to change the size of the circle.

Click and drag point A until it coincides with with the point P.

(a) What do you notice about the secant? What has it become?

(b) What is the angle between the tangent at the point of contact P and the radius of the circle?

4. Copy and complete the following sentence.

In general, the tangent at the point of contact is ______________ to the radius of the circle.

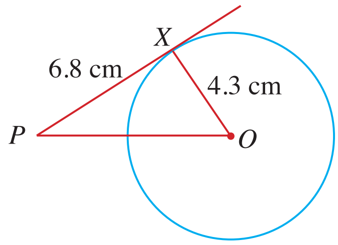
From the investigation, **Circle Symmetric Property 3** states that:

The tangent at the point of contact is perpendicular to the radius of the circle.

Worked Example 3

(Application of Circle Symmetric Property 3)

In the figure, PX is a tangent to the circle, centre O.



Given that $PX = 6.8$ cm and $OX = 4.3$ cm, find

(i) $\angle OPX$,

(ii) the length of OP,

(iii) the area of ΔOPX.

Solution:

(i) $\angle OXP = 90°$ (tangent $\perp$ radius)

In ΔOPX,

$$\tan \angle OPX = \frac{OX}{PX} = \frac{4.3}{6.8}$$

$$\angle OPX = \tan^{-1}\frac{4.3}{6.8} = 32.3° \text{ (to 1 d.p.)}$$

(ii) In ΔOPX,

$$OP^2 = 6.8^2 + 4.3^2 \text{ (Pythagoras' Theorem)}$$
$$= 64.73$$
$$OP = \sqrt{64.73}$$
$$= 8.05 \text{ cm (to 3 s.f.)}$$

ATTENTION

For **(ii)**, trigonometric ratios may be used to find the length of OP, i.e. $\sin 32.31° = \frac{4.3}{OP}$ before solving for OP.

(iii) Area of $\Delta OPX = \frac{1}{2} \times PX \times OX$ (use $\frac{1}{2}$ × base × height)

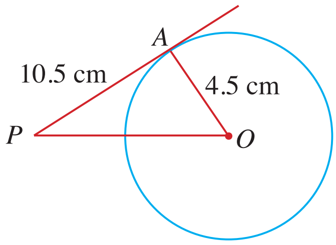
$$= \frac{1}{2} \times 6.8 \times 4.3$$
$$= 14.62 \text{ cm}^2$$

PRACTISE NOW 3

SIMILAR QUESTIONS

Exercise 11B Questions 1-3, 5-9, 12

1. In the figure, PA is a tangent to the circle, centre O.



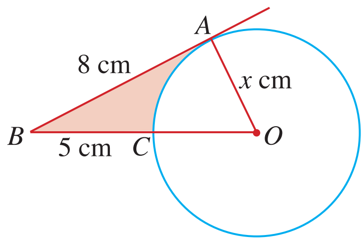
Given that $PA = 10.5$ cm and $OA = 4.5$ cm, find

(i) $\angle OPA$,

(ii) the length of OP,

(iii) the area of ΔOPA.

2. In the figure, AB is a tangent to the circle, centre O.

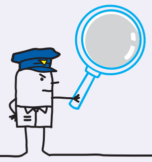


Given that $AB = 8$ cm, $BC = 5$ cm and $OA = x$ cm, find

(i) the value of x,

(ii) $\angle AOB$,

(iii) the area bounded by AB, BC and the minor arc AC.



Investigation

Circle Symmetric Property 4

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Symmetric Property 4 as shown below.

1. The template shows a circle with centre O and two tangents from an external point P touching the circle at A and B respectively.

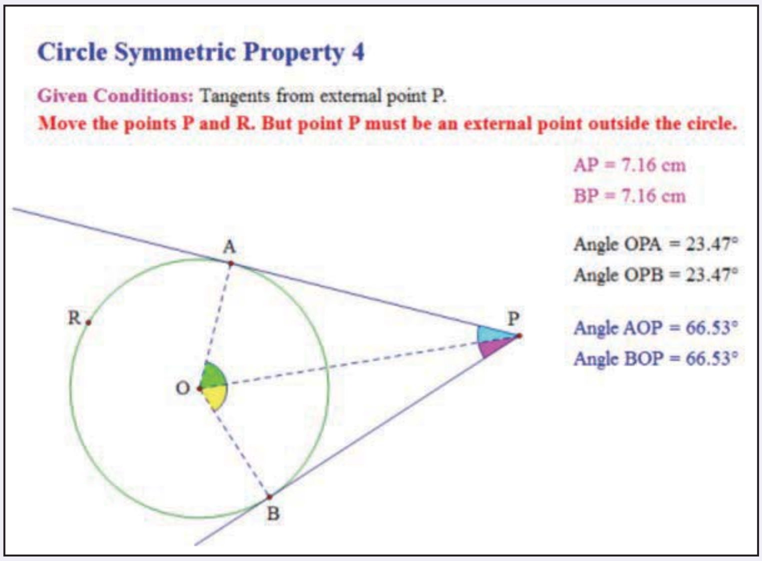


Fig. 11.7

2. Click and drag point P to change the position of the external point, but P must remain outside the circle.
Click and drag point R to change the size of the circle.

(a) What do you notice about the length of AP and of BP?

(b) What do you notice about $\angle OPA$ and $\angle OPB$?

3. Copy and complete the following sentences.
In general,

(a) tangents from an external point are __________ (in length);

(b) the line from the centre of a circle to an external point __________ the angle between the two tangents.

4. Prove the two results in Question 3.
Hint: For Question 3, how are ΔOAP and ΔOBP related?

ATTENTION

Although a tangent is a line (i.e. without any endpoints) and so its length is infinite, the length of a tangent from an external point is the distance between the external point and the point of contact with the circle. In this case, the lengths of the two tangents are AP and BP.

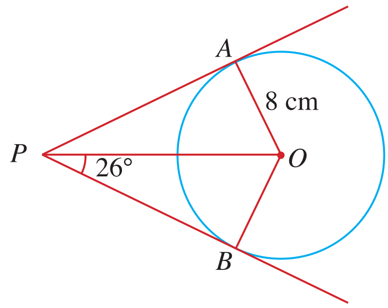
From the investigation, there are two parts to **Circle Symmetric Property 4**:

(i) Tangents from an external point are equal (in length).
(ii) The line from the centre of a circle to an external point bisects the angle between the two tangents from the external point.

Worked Example 4

(Application of Circle Symmetric Property 4)

In the figure, PA and PB are tangents to the circle with centre O.



Given that $OA = 8$ cm and $\angle OPB = 26°$, find

(i) $\angle AOB$,

(ii) the length of AP,

(iii) the area of the quadrilateral $APBO$.

Solution:

(i) $\angle OBP = \angle OAP = 90°$ (tangent ⊥ radius)

$\angle OPA = \angle OPB = 26°$ (symmetric properties of tangents to circle)

$\angle AOB = 360° - \angle OAP - \angle OBP - \angle APB$ (∠sum of a quadrilateral)

$= 360° - 90° - 90° - (26° + 26°)$

$= 128°$

(ii) In ΔOAP,

$$\tan \angle APO = \frac{OA}{AP}$$

$$\tan 26° = \frac{8}{AP}$$

$$AP = \frac{8}{\tan 26°}$$

$= 16.4$ cm (to 3 s.f.)

In Worked Example 4, $PA = PB$ (equal tangents).

(iii) Area of $\Delta OAP = \frac{1}{2} \times AP \times OA$ (use $\frac{1}{2}$ × base × height)

$= \frac{1}{2} \times 16.40 \times 8$

$= 65.61 \text{ cm}^2$

Area of quadrilateral $APBO = 2 \times$ area of ΔOAP

$= 2 \times 65.61$

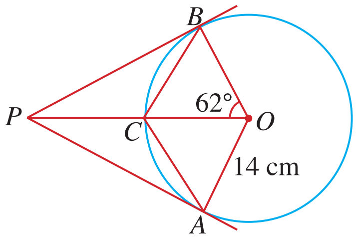
$= 131 \text{ cm}^2$ (to 3 s.f.)

PRACTISE NOW 4

SIMILAR QUESTIONS

Exercise 11B Questions 4(a)-(f), 10, 11

1. In the figure, PA and PB are tangents to the circle with centre O.



Given that $OA = 14$ cm and $\angle BOP = 62°$, find

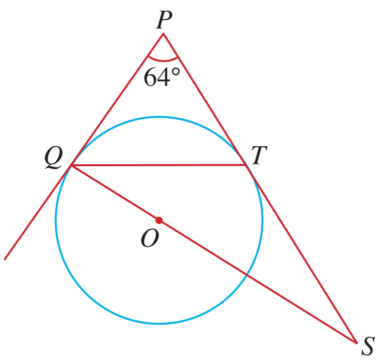
(i) $\angle OPB$,

(ii) $\angle OAC$,

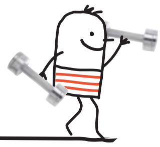
(iii) the length of BP,

(iv) the area of the quadrilateral $APBO$.

2. In the figure, PQ and PT are tangents to the circle, centre O, at the points Q and T respectively. PT produced meets QO produced at S.



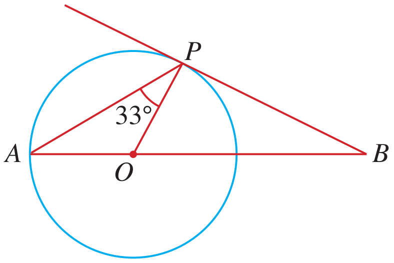
Given that $\angle QPT = 64°$, find $\angle SQT$.



Exercise 11B

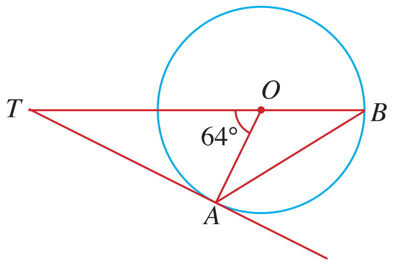
BASIC LEVEL

1. In the figure, BP is a tangent to the circle with centre O.



Given that $\angle APO = 33°$, find $\angle PBA$.

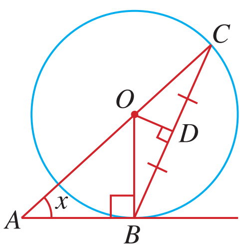
2. In the figure, O is the centre of the circle passing through the points A and B. TA is a tangent to the circle at A and TOB is a straight line.



Given that $\angle AOT = 64°$, find

(i) $\angle ATB$, **(ii)** $\angle TAB$.

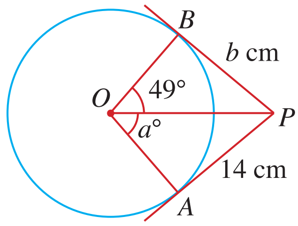
3. In the figure, AB is a tangent to the circle with centre O. D is the midpoint of the chord BC.



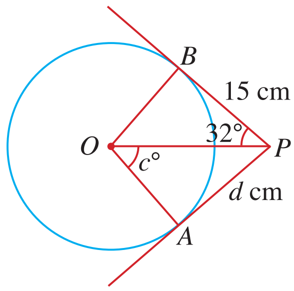
Given that $\angle BAC = x$, find $\angle COD$ in terms of x.

4. Given that PA and PB are tangents to each of the following circles with centre O, find the values of the unknowns.

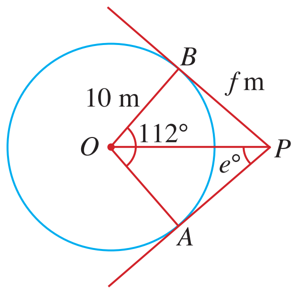
(a)



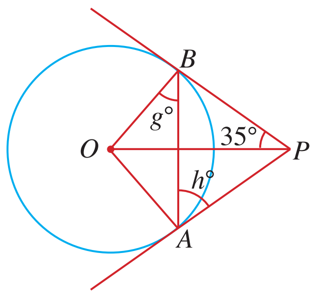
(b)



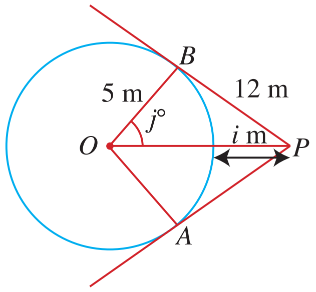
(c)



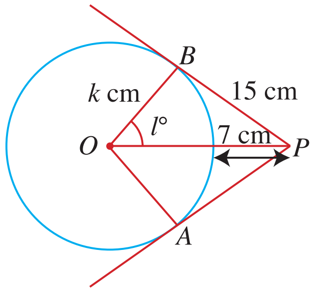
(d)



(e)

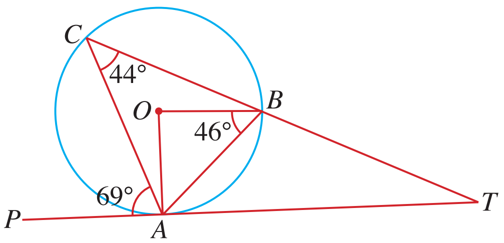


(f)



INTERMEDIATE LEVEL

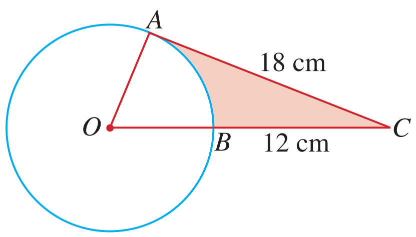
5. In the figure, PAT is a tangent to the circle, centre O, at A. C is a point on the circle such that TBC is a straight line and $\angle ACB = 44°$.



Given that $\angle OBA = 46°$ and $\angle PAC = 69°$, find

(i) $\angle BAT$, **(ii)** $\angle PTC$.

6. The figure shows a circle, centre O. AC is a tangent to the circle at A and OBC is a straight line.



Given that $AC = 18$ cm and $BC = 12$ cm, find

(i) the radius of the circle,

(ii) $\angle AOB$,

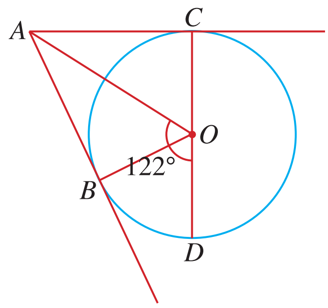
(iii) the area of the shaded region.

7. PQ is a chord of a circle with centre O. Given that $\angle POQ = 84°$, find the obtuse angle between PQ and the tangent at P.

8. The tangent from a point P touches a circle at N. Given that the radius of the circle is 5.6 cm and that P is 10.6 cm away from the centre, find the length of the tangent PN.

9. A point T is 9.1 m away from the centre of a circle. The tangent from T to the point of tangency is 8.4 m. Find the diameter of the circle.

10. In the figure, AB and AC are tangents to the circle at B and C respectively. O is the centre of the circle and $\angle AOD = 122°$.



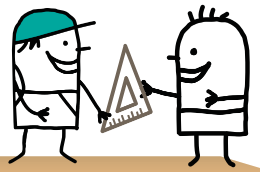
Find $\angle BAC$.

11. The tangents from a point T touch a circle, centre O, at the points A and B. Given that $\angle AOT = 51°$, find $\angle BAT$.

ADVANCED LEVEL

12. Two concentric circles have radii 12 cm and 25.5 cm respectively. A tangent to the inner circle cuts the outer circle at the points H and K. Find the length of HK.

11.2 Angle Properties of Circles



In this section, we will learn the angle properties of circles.

Angles at Centre and Angles at Circumference

Fig. 11.8**(a)** shows a circle with centre O.

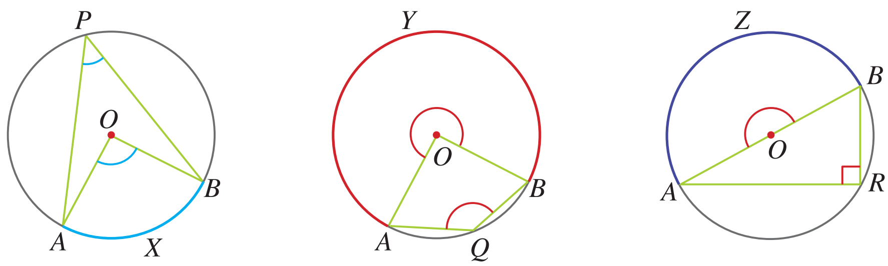
$\angle AOB$ is an angle **subtended at the centre** of the circle by the (blue) *minor* arc AXB.

$\angle APB$ is an angle **subtended at the circumference** of the circle by the same *minor* arc AXB.

Fig. 11.8**(b)** shows another circle with centre O.

$\angle AOB$ is an angle subtended at the centre of the circle by the (red) *major* arc AYB.

$\angle AQB$ is an angle subtended at the circumference of the circle by the same *major* arc AYB.



(a) **(b)** **(c)**

Fig. 11.8

One way to recognise which angle is subtended by which arc is to look at the shape of the arc. For example, the shape of the blue arc indicating $\angle APB$ in Fig. 11.8**(a)** is the same shape as that of the blue minor arc AXB which subtends the angle; and the shape of the red arc indicating $\angle AOB$ in Fig. 11.8**(b)** is the same shape as that of the red major arc AYB which subtends the angle.

Consider Fig. 11.8**(c)**. Can you identify the angle subtended at the centre of the circle and the angle subtended at the circumference by the semicircle AZB?

Class Discussion

Identifying Angles at the Centre and at the Circumference

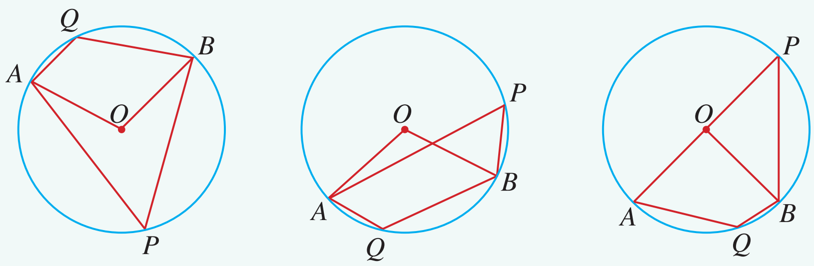
Work in pairs to identify each of the following by using a different coloured pencil or pen to draw the angle in each circle.

(a) Angle at centre subtended by the minor arc AQB

(b) Angle at circumference subtended by the minor arc AQB

(c) Angle at centre subtended by the major arc APB

(d) Angle at circumference subtended by the major arc APB



Investigation

Circle Angle Property 1

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Angle Property 1 as shown below.

1. The template shows a circle with centre O. $\angle AOB$ is an angle at the centre while $\angle APB$ is an angle at the circumference subtended by the same (minor or major) arc AB.

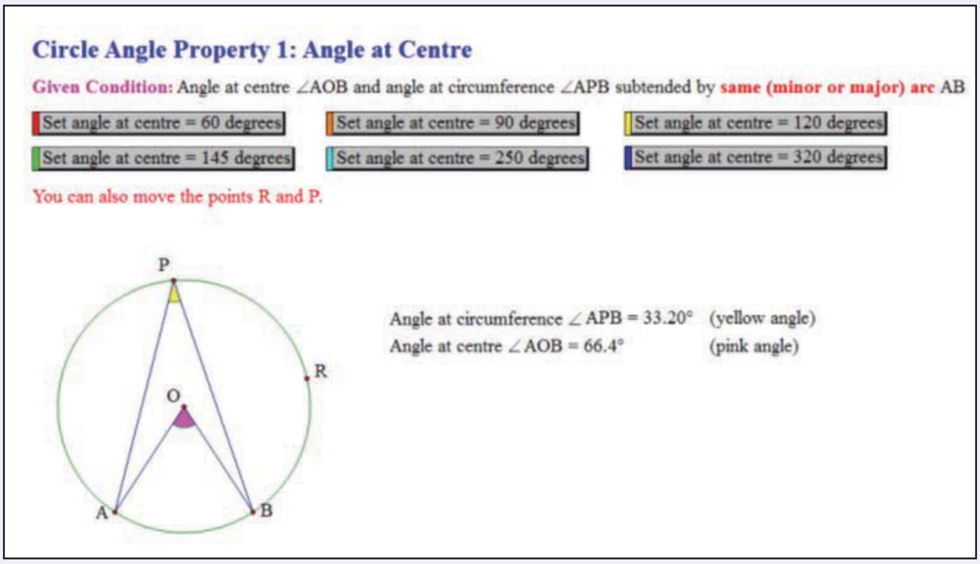


Fig. 11.9

2. Click on the action buttons in the template to set $\angle AOB$ to the values below. You can also move the point R to change the size of the circle, and the point P to change $\angle APB$. Copy and complete Table 11.1 below.

$\angle AOB$	60°	90°	120°	145°	250°	320°
$\angle APB$						
$\dfrac{\angle AOB}{\angle APB}$						

Table 11.1

3. What is the relationship between $\angle AOB$ and $\angle APB$?

4. Copy and complete the following sentence.

In general, an angle at the centre of a circle is ______________ that of any angle at the circumference subtended by the same arc.

From the investigation, **Circle Angle Property 1** states that:

> An angle at the centre of a circle is **twice** that of any angle at the circumference subtended by the same arc.

To prove Circle Angle Property 1, we have to consider 4 cases.

The following proof applies to Case 1 and Case 2.

Case 1 and Case 2:

Fig. 11.10 shows Case 1 (angles subtended by minor arc AB) and Case 2 (angles subtended by major arc AB). The proofs for both cases are actually the same.

Join P to O and produce PO to cut the circle at X.

Let $\angle APX = a$ and $\angle BPX = b$.

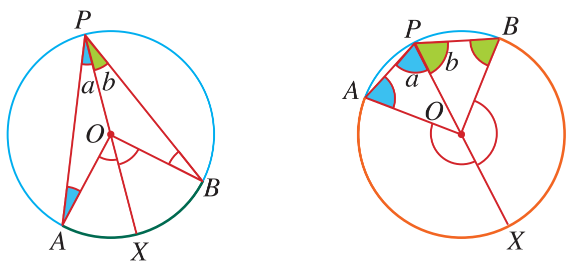


Fig. 11.10

Since $OA = OP$ (radii),

ΔAOP is an isosceles triangle.

Then $\angle OAP = \angle OPA$ (base $\angle$s of isos. Δ)

$= a$

$\therefore$ $\angle AOX = \angle OAP + \angle OPA$ (ext. $\angle$ of Δ)

$= 2a$

Similarly, ΔBOP is an isosceles triangle,

$\angle OBP = \angle OPB = b$ and $\angle BOX = 2b$.

$\therefore$ $\angle AOB = \angle AOX + \angle BOX$

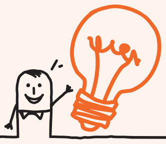
$= 2a + 2b$

$= 2(a + b)$

$= 2 \times (\angle APX + \angle BPX)$

$= 2 \times \angle APB$ (proven)

The Thinking Time on the next page considers the next 2 cases.



Thinking Time

Fig. 11.11 shows Case 3 and Case 4 (two special cases of angles subtended by minor arc *AB*). In each case, prove Circle Angle Property 1.

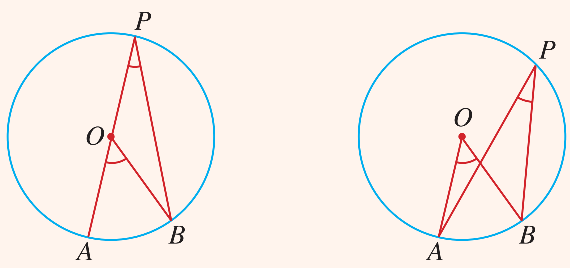


Fig. 11.11

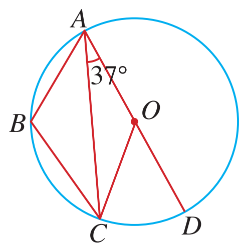
Worked Example 5

(Application of Circle Angle Property 1)

A, *B*, *C* and *D* are four points on a circle with centre *O*. Given that *AOD* is a diameter of the circle and $\angle CAD = 37°$, find

(i) $\angle COD$,

(ii) $\angle ABC$.



Solution:

(i) $\angle COD = 2 \times \angle CAD$ ($\angle$ at centre $= 2\ \angle$ at $\odot^{ce}$)

$= 2 \times 37°$

$= 74°$

(ii) Reflex $\angle AOC = 180° + 74°$

$= 254°$

$\therefore \angle ABC = \dfrac{254°}{2}$ ($\angle$ at centre $= 2\ \angle$ at $\odot^{ce}$)

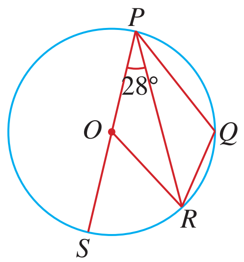
$= 127°$

PRACTISE NOW 5

SIMILAR QUESTIONS

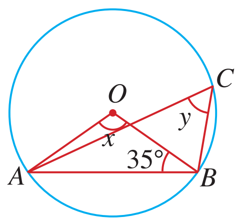
Exercise 11C Questions 1(a)-(h), 9, 10

1. P, Q, R and S are four points on a circle with centre O.
Given that POS is a diameter of the circle and $\angle OPR = 28°$, find

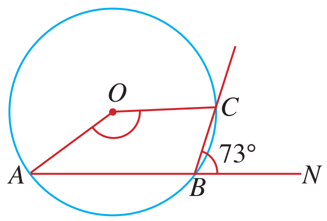


(i) $\angle SOR$, **(ii)** $\angle PQR$.

2. Given that O is the centre of the circle and $\angle ABO = 35°$, find the angles marked x and y.



3. In the figure, O is the centre of the circle and A and B lie on the circumference such that ABN is a straight line.



Given that C lies on the circumference such that $\angle NBC = 73°$, find the obtuse angle AOC.

Angle in a Semicircle

Investigation

Circle Angle Property 2

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Angle Property 2 as shown below.

1. The template shows a circle with centre O. $\angle AOB$ is an angle at the centre while $\angle APB$ is an angle at the circumference.

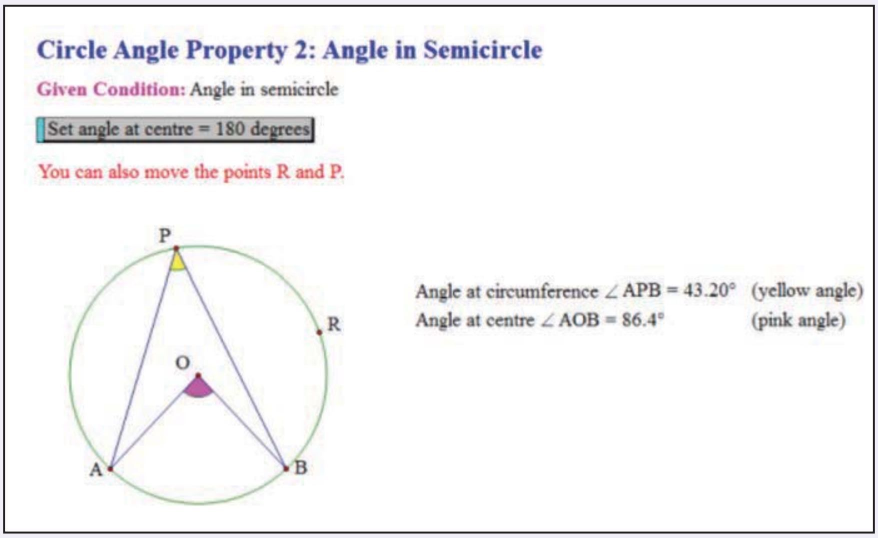


Fig. 11.12

2. Click on the action button in the template to set $\angle AOB = 180°$. You can also move the point R to change the size of the circle, and the point P to change $\angle APB$.
   (a) What is $\angle APB$ equal to?
   (b) What is the special name given to the sector APB when $\angle AOB = 180°$?
3. Copy and complete the following sentence.
   In general, an angle in a semicircle is always equal to ______________.
4. Prove the angle property in Question 3.

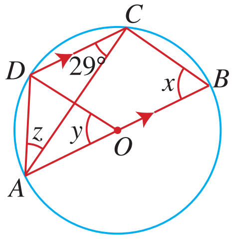
From the investigation, **Circle Angle Property 2** states that:

An angle in a semicircle is always equal to 90°.

Worked Example 6

(Application of Circle Angle Property 2)

A, B, C and D are four points on a circle with centre O. Given that AOB is a diameter of the circle, DC is parallel to AB and $\angle DCA = 29°$, find the angles marked x, y and z.



Solution:

$\angle ACB = 90°$ (rt. ∠ in a semicircle)

$\angle CAB = 29°$ (alt. ∠s, AB // DC)

$\therefore x = 180° - 90° - 29°$ (∠ sum of a Δ)

$= 61°$

$y = 2 \times \angle ACD$ (∠ at centre = 2 ∠ at ⊙ce)

$= 2 \times 29°$

$= 58°$

$\angle ADO = \angle DAO$ (base ∠ of isos. Δ)

$= \frac{1}{2}(180° - 58°)$

$= 61°$

$\therefore z = \angle ADO - \angle CAB$

$= 61° - 29°$

$= 32°$

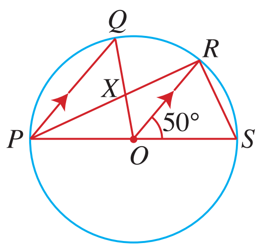
PRACTISE NOW 6

P, Q, R and S are four points on a circle with centre O. Given that POS is a diameter of the circle, PQ is parallel to OR and $\angle ROS = 50°$, find

(i) $\angle OPR$,

(ii) $\angle QOR$,

(iii) $\angle PXQ$.

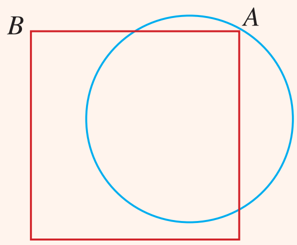


SIMILAR QUESTIONS

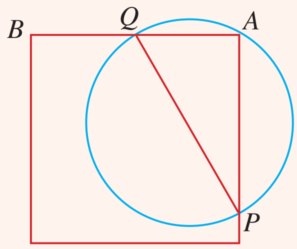
Exercise 11C Questions 2(a)-(d), 11-13

Do you know how the centre of a circle can be determined? Follow the instructions given and discover the answer yourself.

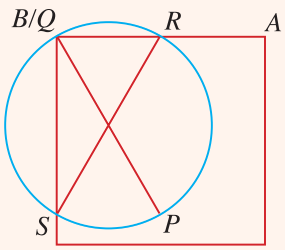
1. Place a rectangular sheet of paper under a circle such that one of its corners touches the circle, say at the point A.



2. Join the two points, P and Q, as shown.



3. Move the same sheet of paper such that another of its corners touches the circle, say at the point B. Join the two points R and S as shown.

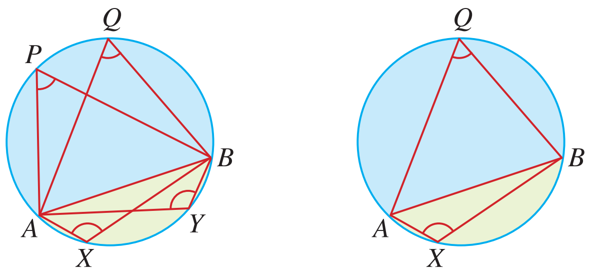


The result would show that the point of intersection of PQ and RS gives the centre of the circle. Explain why this is true.

Angles in Same or Opposite Segments

Fig. 11.13**(a)** shows a circle with a chord AB that divides the circle into two segments.

The larger segment $APQB$ is called the major segment (shaded blue) while the smaller segment $AXYB$ is called the minor segment (shaded green). $\angle APB$ and $\angle AQB$ are angles subtended at the circumference of the circle by the same minor arc AB. Since $\angle APB$ and $\angle AQB$ lie in the same (major) segment, they are called **angles in the same segment**. $\angle AXB$ and $\angle AYB$ are angles subtended at the circumference of the circle by the same major arc AB. Since $\angle AXB$ and $\angle AYB$ lie in the same (minor) segment, they are called angles in the same segment.



(a) **(b)**

Fig. 11.13

Fig. 11.13**(b)** shows a circle with a chord AB that divides the circle into two segments. The segment AQB and the segment AXB are called **opposite segments** (*not* different segments). $\angle AQB$ and $\angle AXB$ are angles subtended at the circumference of the circle by the minor arc AB and by the major arc AB respectively. Since $\angle AQB$ and $\angle AXB$ lie in opposite segments, they are called **angles in opposite segments**.

ATTENTION

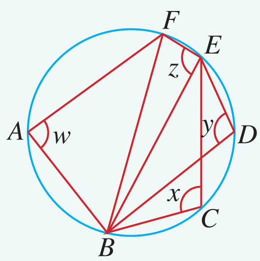
Opposite segments must be formed by the *same chord*.

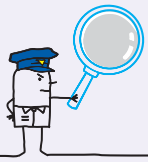
Class Discussion

Angles in Same or Opposite Segments

Work in pairs.

The figure on the right shows a circle with four angles labelled w, x, y and z. Work in pairs to identify which pairs of the four angles are in the same segment and which pairs of the four angles are in opposite segments. For each case, specify the chord that forms the segment(s). In particular, are $\angle w$ and $\angle y$ angles in opposite segments? Explain your answer.





Investigation

Circle Angle Property 3

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Angle Property 3 as shown below.

1. The template shows a circle with centre O. $\angle APB$ and $\angle AQB$ are angles in the same (minor or major) segment.

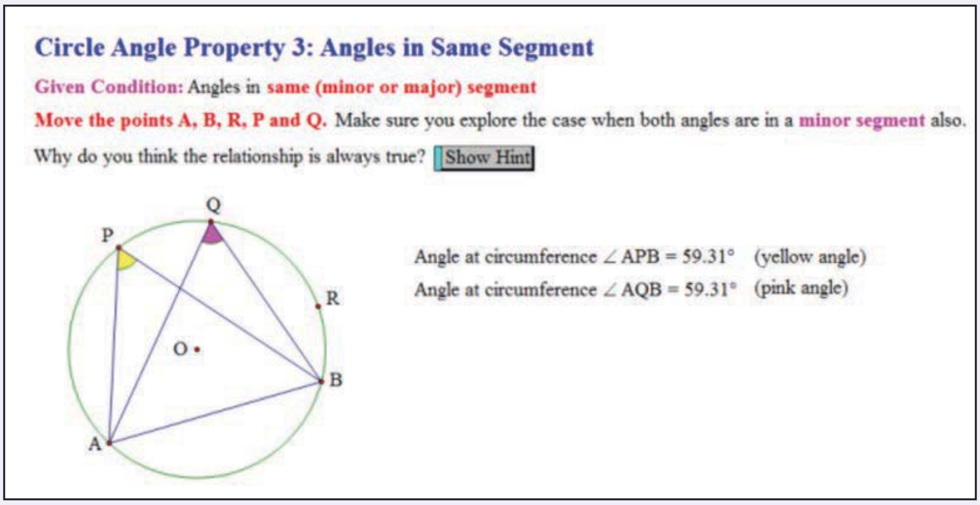


Fig. 11.14

2. Click and drag point A or B to change the size of $\angle APB$ and of $\angle AQB$.
Click and drag point R to change the size of the circle.
Click and drag point P or Q to change the position of $\angle APB$ and of $\angle AQB$.
What do you notice about $\angle APB$ and $\angle AQB$?

3. Copy and complete the following sentence.
In general, angles in the same segment are ______________.

4. Prove the angle property in Question 3. You can also click on the button 'Show Hint' in the template.

ATTENTION

To adjust $\angle APB$ and $\angle AQB$ until they are in the same minor segment, click and drag point A or B until arc $APQB$ is a minor arc.

From the investigation, Circle Angle Property 3 states that:

> Angles in the same segment are equal.

Worked Example 7

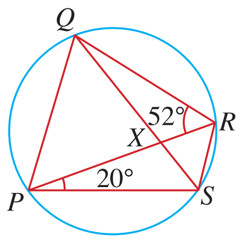
(Application of Circle Angle Property 3)

In the figure, P, Q, R and S are points on the circumference of a circle. Given that PR and QS intersect at the point X, $\angle RPS = 20°$ and $\angle PRQ = 52°$, find

(i) $\angle SQR$,

(ii) $\angle QSP$,

(iii) $\angle PXQ$.



Solution:

(i) $\angle SQR = \angle RPS$ ($\angle$s in same segment)

$= 20°$

(ii) $\angle QSP = \angle PRQ$ ($\angle$s in same segment)

$= 52°$

(iii) $\angle PXQ = \angle SQR + \angle PRQ$ (ext. $\angle$ = sum of int. opp. $\angle$s)

$= 20° + 52°$

$= 72°$

PRACTISE NOW 7

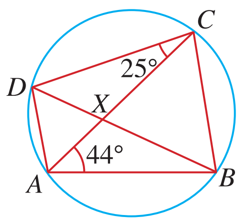
SIMILAR QUESTIONS

Exercise 11C Questions 3(a), (b), 4, 5, 14, 15, 23

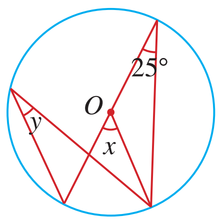
1. In the figure, A, B, C and D are points on the circumference of a circle. Given that AC and BD intersect at the point X, $\angle BAC = 44°$ and $\angle ACD = 25°$, find

(i) $\angle CDX$, **(ii)** $\angle ABX$,

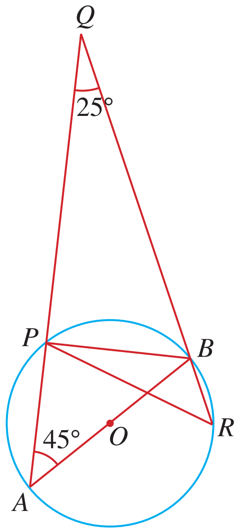
(iii) $\angle CXB$.



2. Given that O is the centre of the circle, find the angles marked x and y.



3. In the figure, AB is a diameter of the circle with centre O. APQ and RBQ are straight lines. Find $\angle BPR$.



Investigation

Circle Angle Property 4

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Circle Angle Property 4 as shown below.

1. The template shows a circle with centre O. $\angle APB$ and $\angle AQB$ are angles in opposite (minor or major) segments.

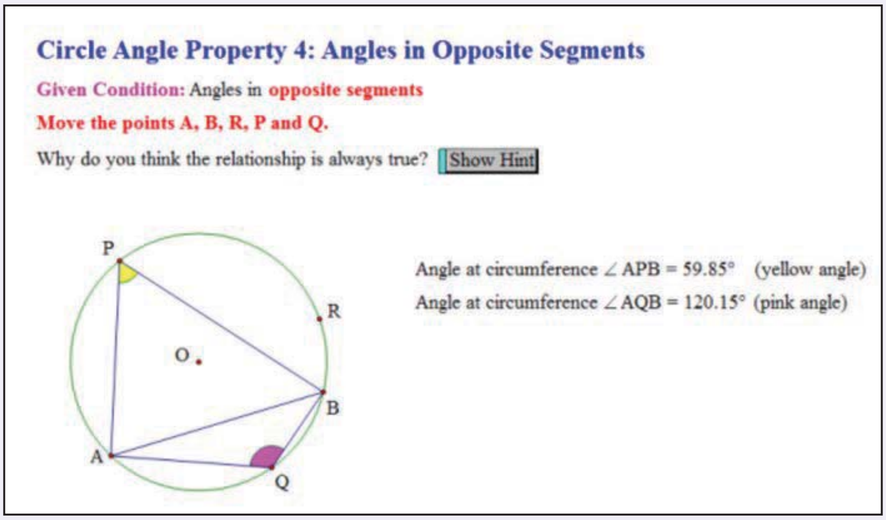


Fig. 11.15

2. Click and drag the point A or B to change the size of $\angle APB$ and of $\angle AQB$.
   Click and drag the point R to change the size of the circle.
   Click and drag the point P or Q to change the position of $\angle APB$ and of $\angle AQB$.
   What do you notice about $\angle APB$ and $\angle AQB$?
3. Copy and complete the following sentence.
   In general, angles in opposite segments are **supplementary**, i.e. they add up to _________.
4. Prove the angle property in Question 3. You can also click on the button 'Show Hint' in the template.

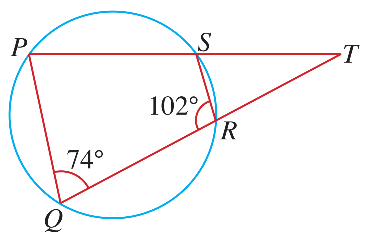
From the investigation, **Circle Angle Property 4** states that:

> Angles in opposite segments are *supplementary*, i.e. they add up to 180°.

Worked Example 8

(Application of Circle Angle Property 4)

In the figure, P, Q, R and S are points on the circumference of the circle. PST and QRT are straight lines, $\angle PQR = 74°$ and $\angle QRS = 102°$.



Find

(i) $\angle QPS$,

(ii) $\angle RTS$,

(iii) $\angle RST$.

Solution:

(i) $\angle QPS = 180° - 102°$ ($\angle$s in opp. segments)
$= 78°$

(ii) $\angle RTS = 180° - 74° - 78°$ ($\angle$ sum of a Δ)
$= 28°$

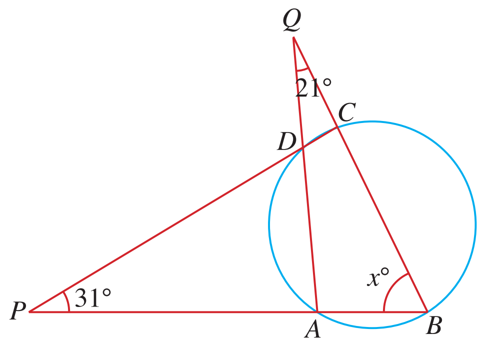
(iii) $\angle RST = 102° - 28°$ (ext. $\angle$ = sum of int. opp. $\angle$s)
$= 74°$

PRACTISE NOW 8

SIMILAR QUESTIONS

Exercise 11C Questions 6(a)-(d), 7, 8, 16, 17, 24

1. In the figure, A, B, C and D are points on the circumference of a circle. PAB, QCB, PDC and QDA are straight lines.



Given that $\angle BPC = 31°$, $\angle AQB = 21°$ and $\angle PBQ = x°$, find

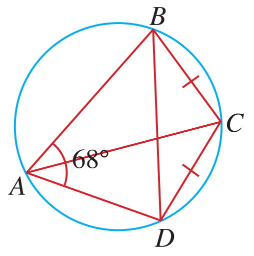
(i) $\angle BAD$ in terms of x,

(ii) $\angle BCD$ in terms of x,

(iii) the value of x,

(iv) $\angle PAD$.

2. In the figure, A, B, C and D are points on the circumference of the circle and $BC = CD$.

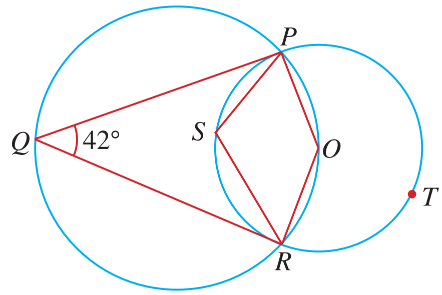


Given that $\angle BAD = 68°$, find $\angle BAC$.

Worked Example 9

(Application of Circle Properties)

In the figure, O is the centre of the smaller circle passing through the points P, S, R and T. The points P, Q, R and O lie on the larger circle.



Given that $\angle PQR = 42°$, find $\angle PSR$.

Solution:

$\angle POR = 180° - 42°$ ($\angle$s in opp. segments)
$= 138°$

Reflex $\angle POR = 360° - 138°$ ($\angle$s at a point)
$= 222°$

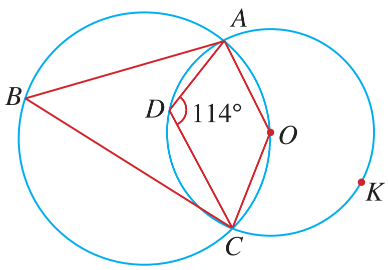
$\angle PSR = \frac{222°}{2}$ ($\angle$ at centre $= 2\ \angle$ at $\odot^{ce}$)
$= 111°$

PRACTISE NOW 9

SIMILAR QUESTIONS

Exercise 11C Questions 18-20

In the figure, O is the centre of the smaller circle passing through the points A, D, C and K. The points A, B, C and O lie on the larger circle.

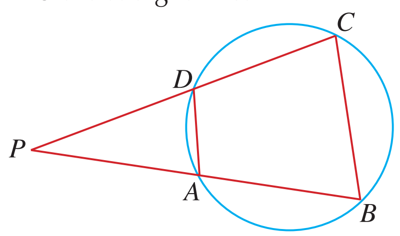


Given that $\angle ADC = 114°$, find $\angle ABC$.

Worked Example 10

(Application of Circle Properties)

In the figure, A, B, C and D are points on the circle. PAB and PDC are straight lines.



(i) Show that ΔPAD is similar to ΔPCB.

(ii) Given also that PA = 12 cm, AD = 7 cm and PC = 28 cm, find the length of BC.

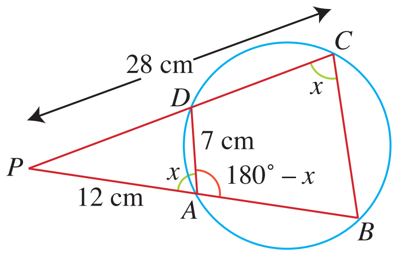
Solution:

(i) Let $\angle BCD = x$.

Then $\angle BAD = 180° - x$ ($\angle$s in opp. segments)

i.e. $\angle PAD = 180° - (180° - x)$

$= x$



In ΔPAD and ΔPCB,

$\angle P$ is a common angle.

$\angle PAD = \angle PCB$

$\therefore$ ΔPAD is similar to ΔPCB. (2 pairs of corr. $\angle$s equal)

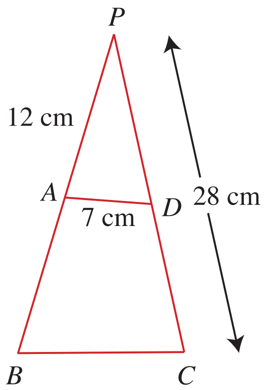
(ii) Using similar triangles,

$$\frac{BC}{DA} = \frac{PC}{PA}$$

$$\frac{BC}{7} = \frac{28}{12}$$

$$BC = \frac{28}{12} \times 7$$

$$= 16\frac{1}{3} \text{ cm}$$



PRACTISE NOW 10

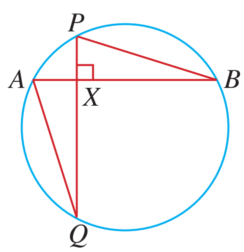
SIMILAR QUESTIONS

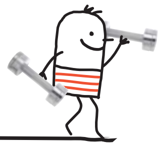
Exercise 11C Questions 21, 22, 25

In the figure, A, P, B and Q are points on the circle. The chords AB and PQ intersect at right angles at X.

(i) Show that ΔAXQ is similar to ΔPXB.

(ii) Given also that AX = 5 cm, QX = 10.5 cm and PX = 3.4 cm, find the length of BX.



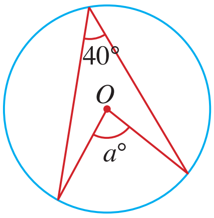


Exercise 11C

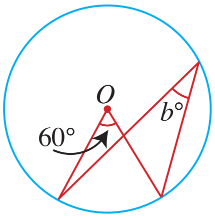
BASIC LEVEL

1. Given that O is the centre of each of the following circles, find the value of each of the unknowns.

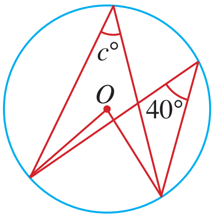
(a)



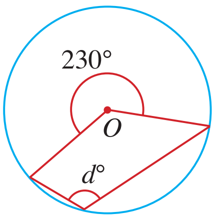
(b)



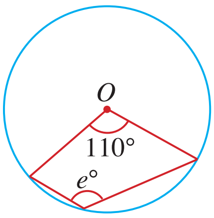
(c)



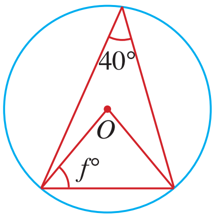
(d)



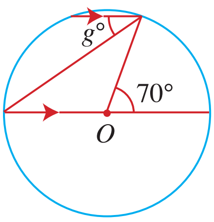
(e)



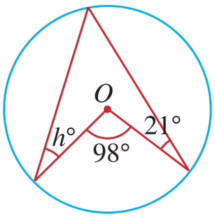
(f)



(g)

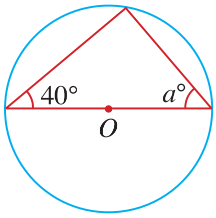


(h)

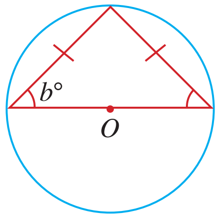


2. Given that O is the centre of each of the following circles, find the value of each of the unknowns.

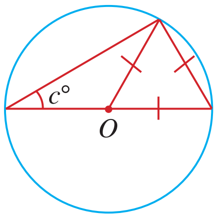
(a)



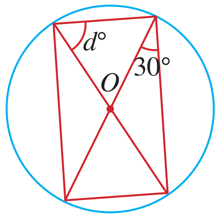
(b)



(c)

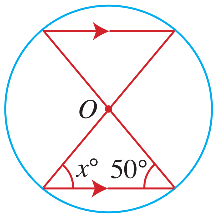


(d)

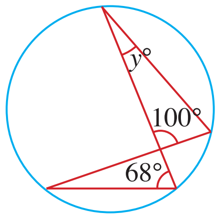


3. Find the value of each of the unknowns.

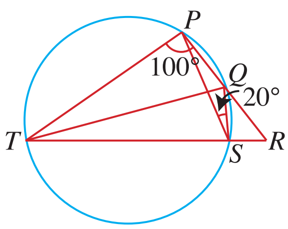
(a)



(b)

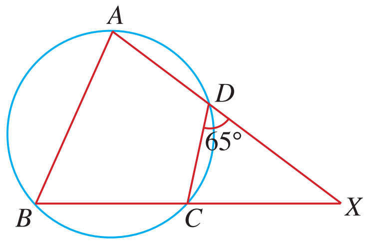


4. In the figure, $\angle TPQ = 100°$ and $\angle PSQ = 20°$.



Find $\angle PQT$.

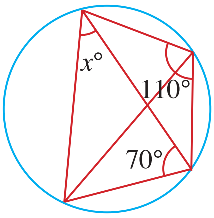
5. In the figure, A, B, C and D are points on the circle such that AD produced meets BC produced at X.



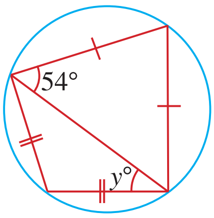
Given that $\angle CDX = 65°$, find $\angle ABC$.

6. Find the values of the unknowns.

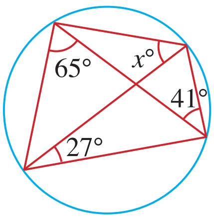
(a)



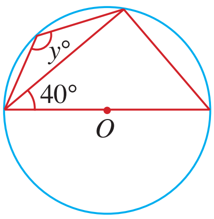
(b)



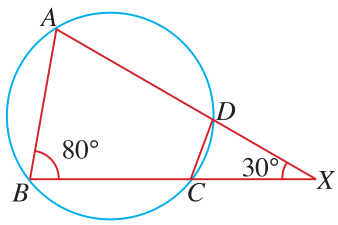
(c)



(d)



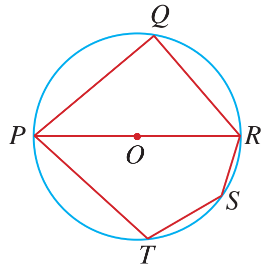
7. In the figure, A, B, C and D are points on the circle such that AD produced meets BC produced at X.



Given that $\angle ABC = 80°$ and $\angle AXB = 30°$, find

(i) $\angle BAD$, **(ii)** $\angle XCD$.

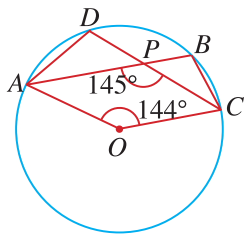
8. In the figure, O is the centre of the circle.



Find the sum of $\angle PQR$, $\angle PRS$ and $\angle PTS$.

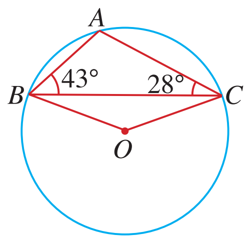
INTERMEDIATE LEVEL

9. In the figure, O is the centre of the circle, $\angle AOC = 144°$ and $\angle APC = 145°$.



Find $\angle BAD$.

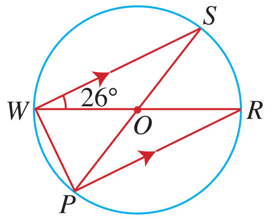
10. In the figure, O is the centre of the circle, $\angle ABC = 43°$ and $\angle ACB = 28°$.



Find

(i) $\angle OBA$, **(ii)** $\angle OCA$.

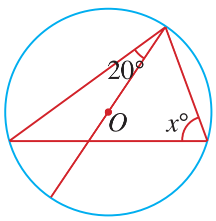
11. In the figure, O is the centre of the circle, $\angle SWR = 26°$. WS is parallel to PR.



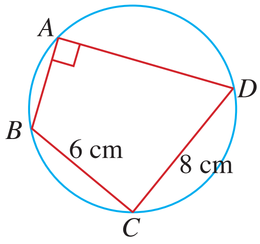
Find

(i) $\angle PWR$, **(ii)** $\angle SPW$.

12. Given that O is the centre of the circle, find the value of x.

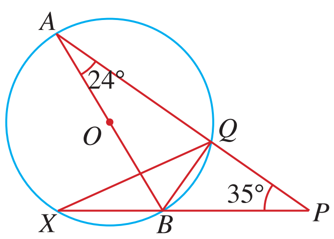


13. In the figure, $\angle BAD = 90°$, $BC = 6$ cm and $CD = 8$ cm.



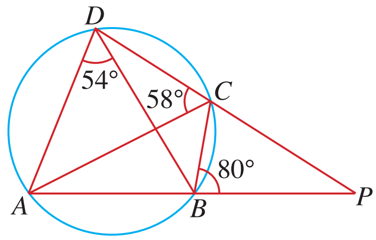
Find the area of the circle.

14. In the figure, A, Q, B and X are points on the circle. AB is a diameter of the circle.



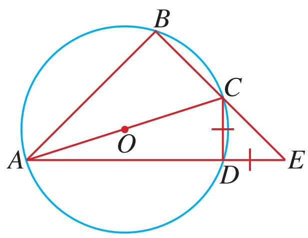
Given that $\angle BAP = 24°$ and $\angle BPA = 35°$, find $\angle BQX$.

15. In the figure, $\angle ADB = 54°$, $\angle ACD = 58°$ and $\angle CBP = 80°$.



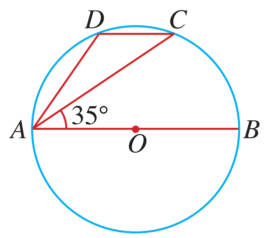
Find $\angle APD$.

16. In the figure, O is the centre of the circle.



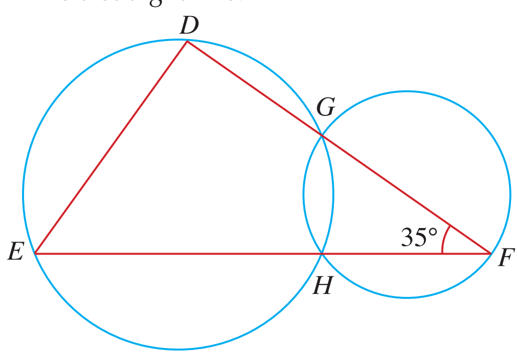
Given that $CD = DE$, find $\angle BAD$.

17. In the figure, AB is a diameter of the circle.



Given that $\angle CAB = 35°$, find $\angle ADC$.

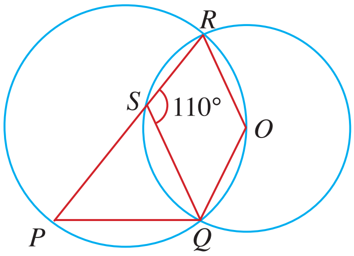
18. In the figure, two circles intersect at the points G and H. GF is a diameter of the circle GHF and $\angle GFH = 35°$. ED is a chord in the larger circle and EHF is a straight line.



Find

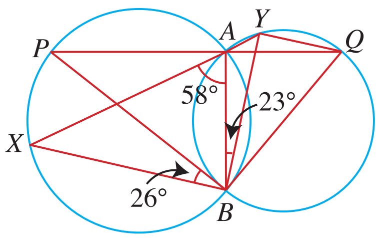
(i) $\angle EDG$, **(ii)** $\angle DEF$.

19. In the figure, two circles intersect at the points Q and R. O is the centre of the circle SQR, $\angle RSQ = 110°$ and PSR is a chord in the larger circle.



Find $\angle QPS$.

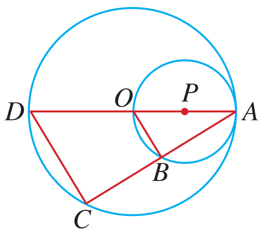
20. In the figure, points P, A, B and X lie on the larger circle and Q, B, A and Y lie on the smaller circle. PAQ and XAY are straight lines, $\angle BAX = 58°$, $\angle PBX = 26°$ and $\angle ABY = 23°$.



Find

(i) $\angle AQB$, **(ii)** $\angle AYQ$.

21. In the figure, O is the centre of the larger circle passing through the points A, C and D with DOA as a diameter. P is the centre of the smaller circle through points O, B and A, with OPA as a diameter.

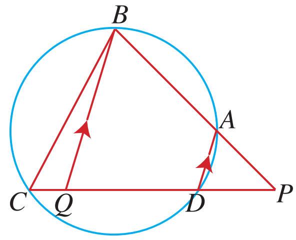


(a) Show that ΔABO is similar to ΔACD.

(b) Given also that $AP = 4$ cm and $OB = 4.5$ cm, find the length of

(i) OC, **(ii)** CD.

22. In the figure, A, B, C and D are points on the circle. PAB and $PDQC$ are straight lines. QB is parallel to DA.

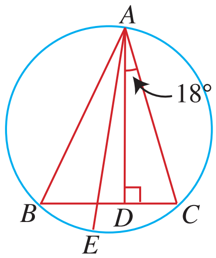


(i) Show that ΔPAD is similar to ΔPBQ.

(ii) Name another triangle that is similar to ΔPAD. Explain your answer.

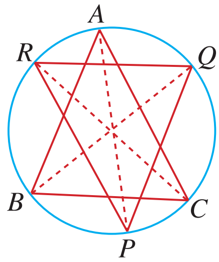
ADVANCED LEVEL

23. In the figure, A, B, E and C are points on the circle. AE is the diameter of the circle and AD is the height of ΔABC.



Given that $\angle CAD = 18°$, find $\angle BAE$.

24. In the figure, A, Q, C, P, B and R are points on the circle. AP, BQ and CR are the angle bisectors of $\angle A$, $\angle B$ and $\angle C$ respectively.

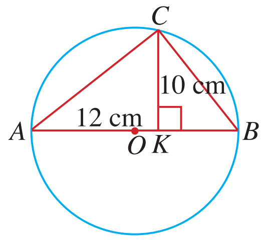


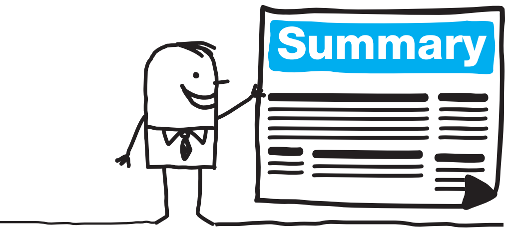
Given that $\angle A = 50°$, $\angle B = 70°$ and $\angle C = 60°$, find $\angle P$, $\angle Q$ and $\angle R$.

25. In the figure, AOB is a diameter of the circle, centre O. C is a point on the circumference such that CK is perpendicular to AB.

(i) Show that ΔACK is similar to ΔCBK.

(ii) Given also that $AK = 12$ cm and $CK = 10$ cm, find the radius of the circle.

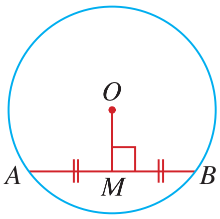




Symmetric Properties of Circle

Property 1:

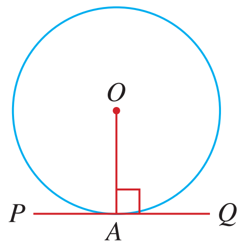
Perpendicular bisector of chord



The perpendicular bisector of a chord of a circle passes through the centre of the circle, i.e. $AM = MB \Leftrightarrow OM \perp AB$

Property 3:

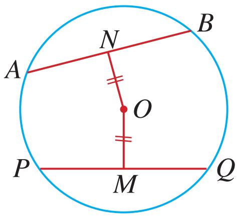
Tangent perpendicular to radius



The tangent at the point of contact is perpendicular to the radius of a circle, i.e. $PQ \perp OA$

Property 2:

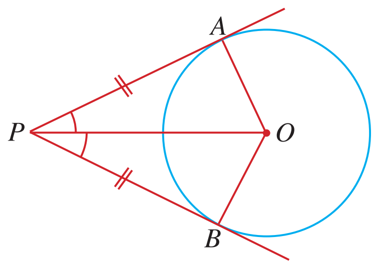
Equal chords



Chords that are equidistant from the centre of a circle are equal in length, i.e. $PQ = AB \Leftrightarrow OM = ON$

Property 4:

Equal tangents

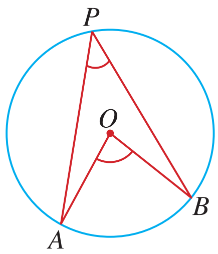


Tangents from an external point are equal in length. The line from the centre of a circle to an external point bisects the angle between the two tangents from the external point, i.e. $PA = PB$.

Angle Properties of Circle

Property 1:

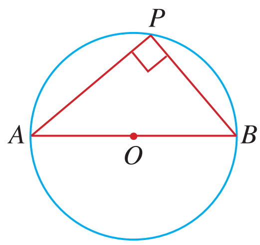
Angle at centre = 2 × Angle at circumference



An angle at the centre of a circle is twice that of any angle at the circumference subtended by the same arc, i.e. $\angle AOB = 2 \times \angle APB$

Property 2:

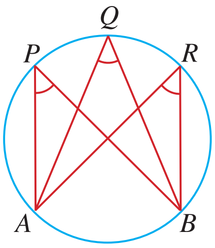
Right angle in semicircle



An angle is a semicircle is always equal to 90°, i.e. AOB is a diameter $\Leftrightarrow \angle APB = 90°$

Property 3:

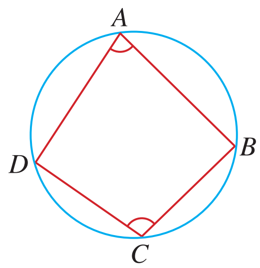
Angles in same segment are equal



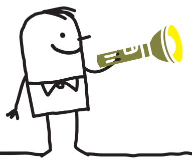
Angles in the same segment are equal, i.e. $\angle APB = \angle AQB = \angle ARB$

Property 4:

Angles in opposite segments are supplementary



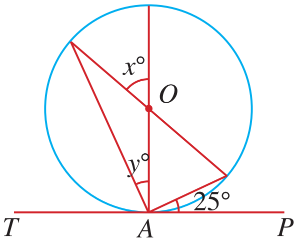
Angles in opposite segments are supplementary, i.e. $\angle DAB + \angle DCB = 180°$



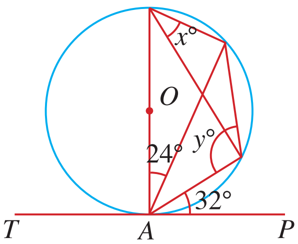
Review Exercise 11

1. Given that PAT is a tangent to each of the following circles with centre O, find the values of the unknowns.

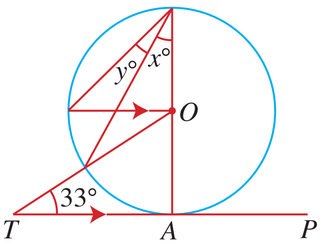
(a)



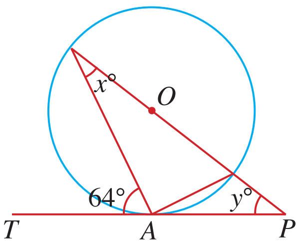
(b)



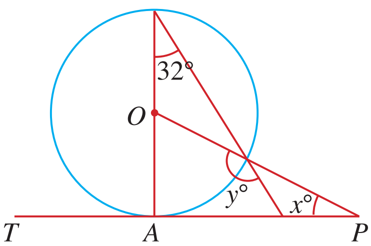
(c)



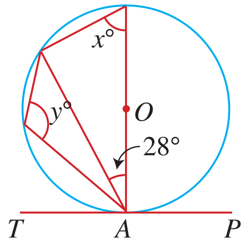
(d)



(e)

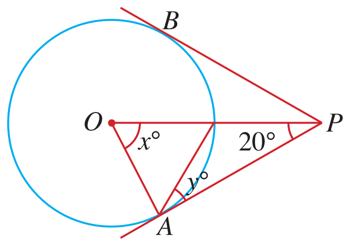


(f)

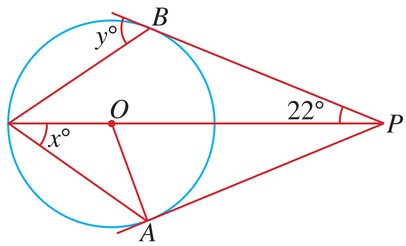


2. Given that PA and PB are tangents to each of the following circles with centre O, find the values of the unknowns.

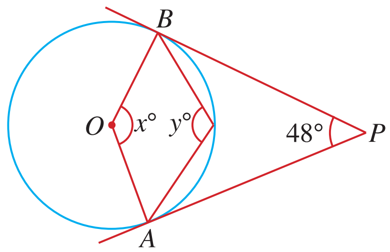
(a)



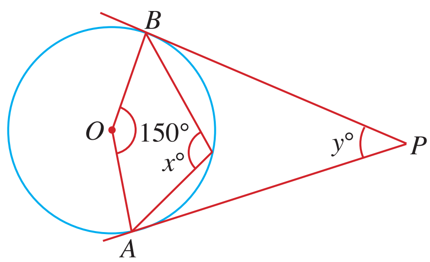
(b)



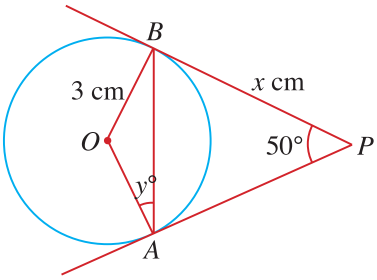
(c)



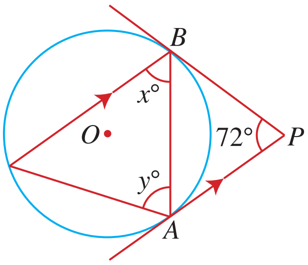
(d)



(e)

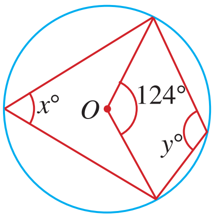


(f)

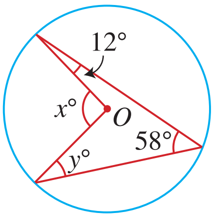


3. Given that O is the centre of each of the following circles, find the value of each of the unknowns.

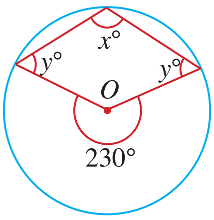
(a)



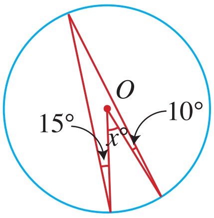
(b)



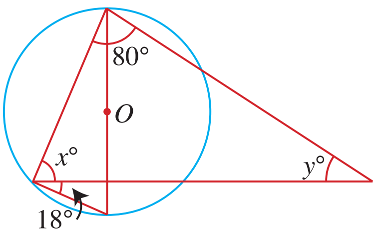
(c)



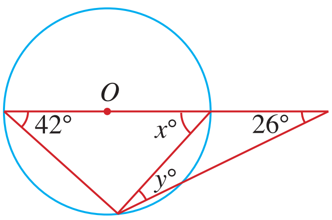
(d)



(e)

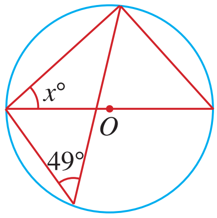


(f)

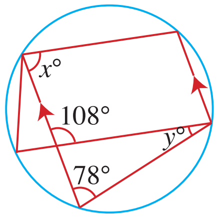


4. Given that O is the centre of each of the following circles, find the values of the unknowns.

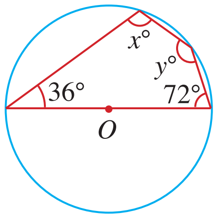
(a)



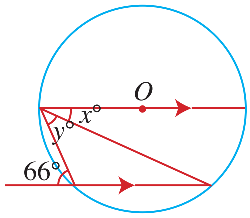
(b)



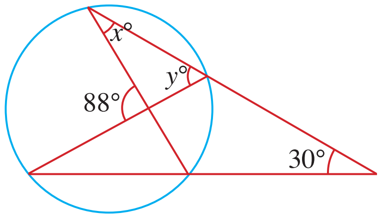
(c)



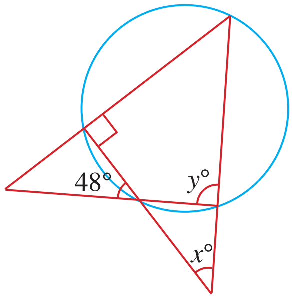
(d)



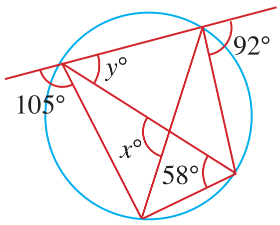
(e)



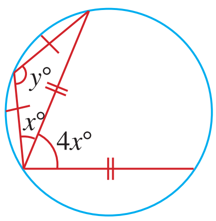
(f)



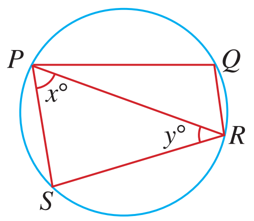
(g)



(h)

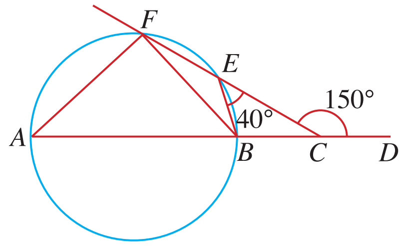


5. In the figure, P, Q, R and S are points on the circle.



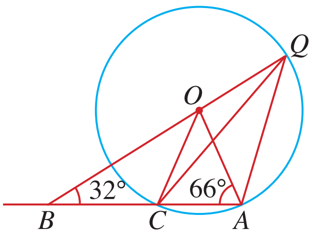
Express $\angle PQR$ in terms of x and y.

6. In the figure, A, B, E and F are points on the circle. AB is the diameter of the circle. $ABCD$ and CEF are straight lines, $\angle BEC = 40°$ and $\angle FCD = 150°$.



Find $\angle EBF$.

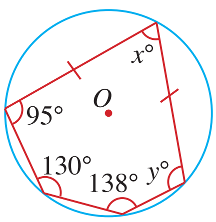
7. In the figure, Q, A and C are points on the circle, centre O. BOQ and BCA are straight lines, $\angle OAC = 66°$ and $\angle OBC = 32°$.



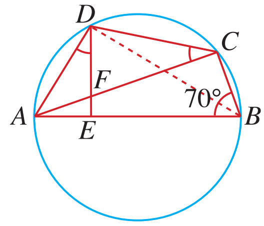
Find

(i) $\angle CQA$, **(ii)** $\angle QCA$.

8. Find the value of x and of y in the following figure.



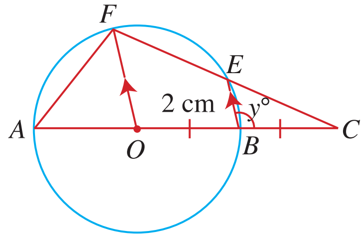
9. The figure shows a circle with AB as a diameter.



Given that $\angle ADE = \angle DCA$ and $\angle CBA = 70°$, find

(i) $\angle FEB$, **(ii)** $\angle EFC$.

10. In the figure, O is the centre of the circle. FEC and $AOBC$ are two straight lines. BE is parallel to OF, $OB = BC = 2$ cm and $\angle CBE = y°$.

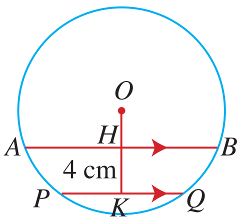


Find

(i) the length of BE,

(ii) $\angle FAO$ in terms of y.

11. In the figure, AB and PQ are parallel chords in a circle, centre O. H and K are the midpoints of AB and PQ respectively.



Given that $AB = 26$ cm, $PQ = 22$ cm and $HK = 4$ cm, find

(i) the length of OH,

(ii) the radius of the circle.

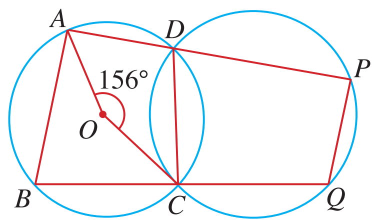
12. PA is a diameter of a circle and PT is a tangent. S is a point on the circle such that $\angle SPT = 46°$.

    (i) Find $\angle PAS$.

    (ii) Hence, find $\angle PRS$, where R is any other point on the minor arc PS of the circle.

13. L, M and N are three points on a circle. The tangents at L and M intersect at P. Given that $\angle LPM = 58°$, find $\angle LNM$.

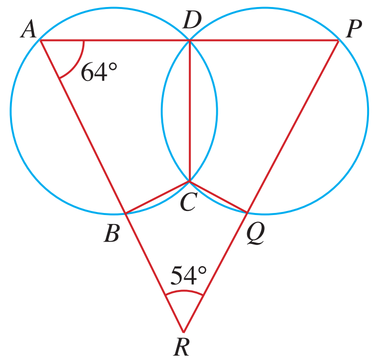
14. In the figure, O is the centre of the circle passing through the points A, B, C and D. The points C, D, P and Q lie on the circumference of another circle. ADP and BCQ are straight lines and $\angle AOC = 156°$.



    Find

    (i) $\angle PDC$, **(ii)** $\angle PQC$.

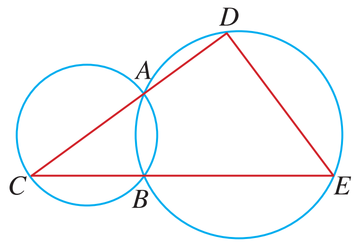
15. The figure shows two circles $ABCD$ and $CDPQ$ intersecting at C and D.



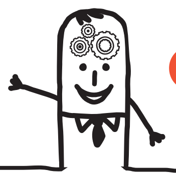
    Given that ADP, ABR and PQR are straight lines, $\angle PAR = 64°$ and $\angle ARP = 54°$, find

    (i) $\angle APR$, **(ii)** $\angle BCQ$.

16. In the figure, CAD and CBE are straight lines.

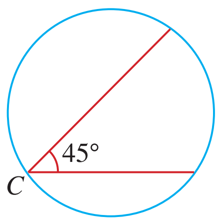


    Given that CA is a diameter of the circle ABC, determine if $\angle ADE$ is a right angle.



Challenge Yourself

The figure shows the plan of a circular hall of a jewellery exhibition. C is a hidden video camera which scans an angle of 45°. How many more such video cameras must be installed on the walls of the hall so that they will cover the entire hall? Indicate the position where each video camera must be mounted.

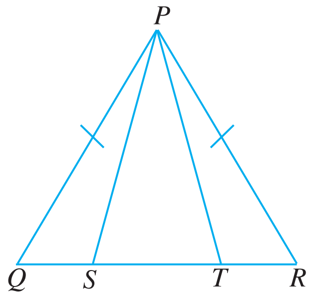


How many video cameras are required if each one can scan an angle of

(a) 35°? **(b)** 60°? **(c)** 90°? **(d)** 100°?

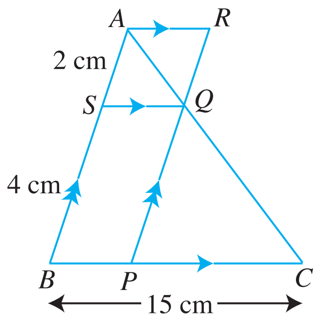
D1 Revision Exercise

1. In the figure, $PQ = PR$ and $\angle PST = \angle PTS$.



Determine if ΔPQS is congruent to ΔPRT.

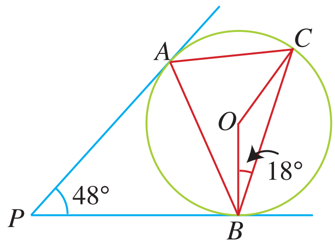
2. In the figure, AR, SQ and the straight line BPC are parallel. BSA is a straight line and is parallel to PQR.



(a) Show that ΔASQ is congruent to ΔQRA.
(b) Name a triangle similar to ΔABC.
(c) Given that $AS = 2$ cm, $SB = 4$ cm and $BC = 15$ cm, find the length of SQ.
(d) Name two triangles similar to ΔPCQ.
(e) Given that the area of ΔABC is 36 cm^2, find the area of
  (i) ΔPCQ,
  (ii) ΔBPQ,
  (iii) quadrilateral $ASQR$.

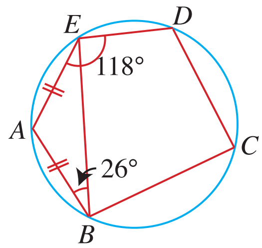
3. A scale model of a warehouse is 45 cm high whereas the actual warehouse has a height of 30 m.
(i) Find the scale of the model.
(ii) Given that the floor area of the model is 810 cm^2, find the actual floor area of the house in m^2.
(iii) If the volume of one of the rooms in the model is 162 cm^3, find the volume of the corresponding room in the actual house in m^3.

4. PA and PB are tangents to the circle with centre O.



Given that $\angle APB = 48°$ and $\angle OBC = 18°$, find
(i) $\angle BAC$, **(ii)** $\angle ABC$.

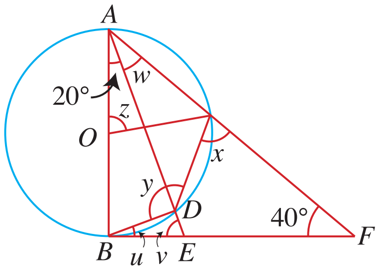
5. A, B, C, D and E are points on a circle.



Given that $AB = AE$, $\angle ABE = 26°$ and $\angle AED = 118°$, find
(i) $\angle BAE$, **(ii)** $\angle BCD$.

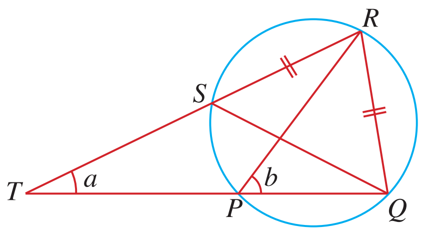
D1 Revision Exercise

6. In the figure, AB is a diameter of the circle with centre O. BEF is a tangent to the circle at B, $\angle BAE = 20°$ and $\angle AFB = 40°$.



Find the angles u, v, w, x, y and z.

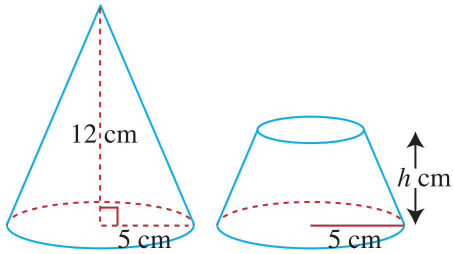
7. In the figure, TPQ and TSR are straight lines and $SR = QR$.



Given that $\angle QTR = a$ and $\angle QPR = b$, express each of the following in terms of a and b.

(i) $\angle RQS$ **(ii)** $\angle PRS$
(iii) $\angle PST$ **(iv)** $\angle QRS$
(v) $\angle PSQ$

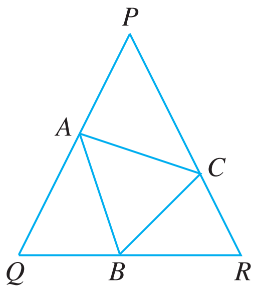
8. The figure shows a wax candle in the shape of a right circular cone with base radius 5 cm and height 12 cm. It takes 1 hour 40 minutes to burn completely. After $12\frac{1}{2}$ minutes of burning, the candle is reduced to a frustrum with a height of h cm.



Find
(i) the total surface area of the cone before burning starts,
(ii) the value of h,
(iii) the total surface area of the frustrum.

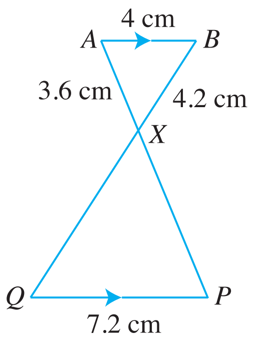
D2 Revision Exercise

1. In the figure, ΔPQR is an equilateral triangle with sides of length 16 cm. A, B and C are points on PQ, QR and PR respectively such that $PA = QB = RC = 4$ cm.



(i) Show that ΔAPC is congruent to ΔBQA.
(ii) Name the third triangle which is congruent to ΔAPC and ΔBQA and show that ΔABC is an equilateral triangle.

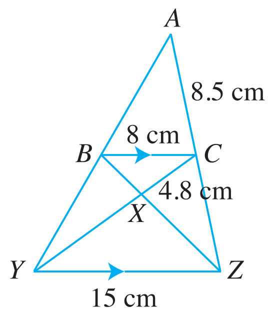
2. The figure shows two triangles ABX and PQX. AB is parallel to QP, $AB = 4$ cm, $AX = 3.6$ cm, $BX = 4.2$ cm and $QP = 7.2$ cm.



Find
(i) the length of PX and of QX,
(ii) the ratio of the area of ΔABX to that of ΔPQX.

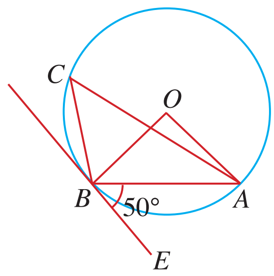
3. The surface area of two cups are in the ratio 9 : 64. If the smaller cup has a height of 25 cm and a volume of 2400 cm^3, find
(i) the height of the larger cup,
(ii) the exact volume of the larger cup.

4. In the figure, $BC = 8$ cm, $CX = 4.8$ cm, $AC = 8.5$ cm and $YZ = 15$ cm. BC is parallel to YZ.



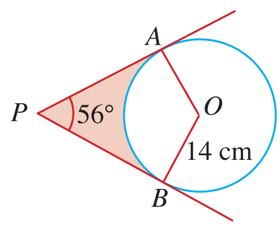
Find
(i) the length of XY and of CZ,
(ii) the ratio of the area of ΔAYZ to the area of trapezium $BCZY$.

5. In the figure, EB is the tangent to the circle with centre O at B.



Given that $\angle ABE = 50°$, find
(i) $\angle AOB$,
(ii) $\angle ACB$.

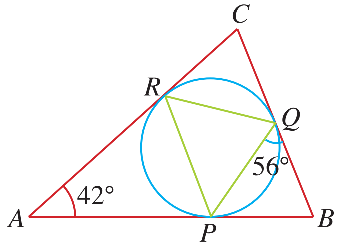
6. In the figure, PA and PB are tangents to the circle, centre O.



Given that the radius of the circle is 14 cm and $\angle APB = 56°$, find the area of the shaded region.

D2 Revision Exercise

7. The tangents to a circle at P, Q and R intersect at A, B and C as shown.



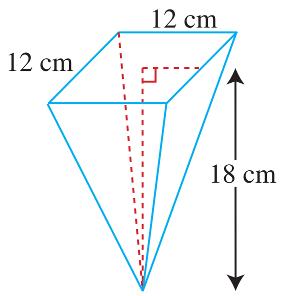
Given that $\angle BAC = 42°$ and $\angle PQB = 56°$, find

(i) $\angle ACB$,

(ii) $\angle PQR$,

(iii) $\angle RPQ$.

8. The figure shows an empty inverted pyramid with a square base of length 12 cm and height 18 cm. A pipe can fill the pyramid with water in 4 minutes.



(i) Find the height of water in the pyramid after 30 seconds.

(ii) Calculate the ratio of the surface area in contact with the water to that of the surface area which is not in contact with the water after 30 seconds.

Problems in Real-World Contexts

PROBLEM 1: The Singapore Sports Hub

The Singapore Sports Hub is a newly-built sports complex in Singapore. Standing at 82.5 m tall, the hub is built in the shape of a dome with a diameter of 310 m.

(a)

(b)

Consider the 2-dimensional shape of the dome. It is similar to that of a parabola, which can be modelled by a quadratic equation $y = ax^2 + bx + c$, where x m and y m represent the horizontal and vertical distances respectively away from the point O (see Fig. **(b)**) and a, b and c are real numbers.

(a) Based on the information given, state the coordinates of two points (other than the point O) on the dome.

(b) Sketch the graph representing the dome and label the coordinates from your answer in **(a)**.

(c) Using the general form of a quadratic function, $y = ax^2 + bx + c$ and the coordinates stated in **(a)**,

(i) write a system of three equations to solve for the three unknowns a, b and c,

(ii) find the values of a, b and c,

(iii) state the equation of the graph representing the dome.

(d) Suggest another way to orientate the coordinate axes to find the quadratic equation modelling the dome.

PROBLEM 2: Road Tax for Singapore-registered Vehicles

The road tax of a Singapore-registered car is calculated based on the car's engine capacity. The following table shows the formula of calculating the road tax per annum for petrol cars.

Engine Capacity (EC) in cc	6-Monthly Road Tax Formula (From 1 July 2008) in $
$EC \leqslant 600$	200×0.782
$600 < EC \leqslant 1000$	$[200 + 0.125 \times (EC - 600)] \times 0.782$
$1000 < EC \leqslant 1600$	$[250 + 0.375 \times (EC - 1000)] \times 0.782$
$1600 < EC \leqslant 3000$	$[475 + 0.75 \times (EC - 1600)] \times 0.782$
$EC > 3000$	$[1525 + 1 \times (EC - 3000)] \times 0.782$

(a) Calculate the 6-monthly road tax of a petrol car with an engine capacity of 1400 cc.

(b) How much road tax is payable per annum for a car with an engine capacity of 3000 cc?

(c) Using a suitable scale, draw the graph of road tax against engine capacity.

PROBLEM 3: High-Speed Chase

During a routine operation along an expressway one night, a car drove through a police road block without stopping. The police signalled for the car to stop but it accelerated and the police gave chase.

The speed and the time of the speeding car and the police car during the 3-minute high-speed chase along the expressway are recorded in the table.

Time	Speed of Speeding Car (km/h)	Speed of Police Car (km/h)
1st minute	110	95
2nd minute	145	140
3rd minute	160	185

(i) Based on the information given, using a distance-time graph, determine whether the police car will be able to overtake the speeding car and arrest the driver during the high-speed chase. Show how you arrive at your conclusion.

(ii) Are there any assumptions that you may have to make?

PROBLEM 4: Credit Card Debts

Many people have a credit card. Having a credit card allows people to track their expenses, enjoy promotions jointly offered by the merchant and the credit card companies and save them the hassle of having to carry a large amount of cash around. However, if the credit card is not being used wisely, an unpaid credit card bill may chalk up a debt that could snowball and cripple their finances.

Credit card companies typically impose a credit charge of 2% per month or 24% per annum for any unpaid bill plus a penalty of $50 per month if the bill is not paid or not paid in full.

Suppose a man owed a credit card company $100 at the beginning of January 2014 and did not pay a single cent to clear his debt.

The following table illustrates the working to obtain the amount of debt he owes at the end of 3 months. Complete the table using the formula given for each column below.

Month	Amount owed at the beginning of the month, A	Additional fee, $(A + 50)$	Interest, $I = \frac{2}{100}(A + 50)$	Amount owed at the end of the month, $D = A + 50 + I$
January	$100.00	$150.00	$3.00	$153.00
February	$153.00	$203.00	$4.06	$207.06
March	$207.06	$257.06	$5.14	$262.20
April				
May				
June				

(a) Using the above set of data and a spreadsheet (as described below), obtain different functions for estimating the amount of debt that the man will owe at the end of

(i) 1 year, **(ii)** 3 years.

Open a spreadsheet and type in 2 columns of data. For the first column, use 1, 2, 3, 4, 5 and 6 to replace the months of January to June respectively. For the second column, enter the amount, $$D$, owed at the end of each of the months. Select the entire table and insert a scatterplot with only the markers (or points). Right-click on the points and add a trendline to model or best fit the data: choose one of the functions given, i.e. exponential, linear, etc. For the forecast, select forward 36 periods, and choose to display the equation of the trendline on the scatterplot.

Use the equation of the trendline to obtain the estimated values of D at the end of 1 year and 3 years.

(b) The formula for D is given by

$$\boxed{D = A(1.02)^n + 2550(1.02)^n - 2550,}$$

where A is the initial amount that the man owes the credit card company and n is the number of months that he did not pay a single cent to the company.

Use the formula to obtain the value of D at the end of
(i) 1 year, **(ii)** 3 years.
Give your answers correct to the nearest cent.

(c) Compare the estimated values of D obtained using the different models in the spreadsheet with the actual values obtained using the formula. Which is a better model for the estimation of values for the different periods of time?

PROBLEM 5: Prices of Watermelons

A fresh fruit stall holder in a school canteen sells slices of watermelon or juice. The following table shows the sizes and prices of the watermelons that are on sale at a wholesale market.

Size	 small	 medium	 large
Diameter (cm)	24	28	32
Price ($)	4.10	5.80	7.60

Which size of watermelon should the stall holder buy in order to maximise his profit? Show your working to support your answer and state any assumptions made.

PROBLEM 6: The Broken Plate

An archaeologist found a part of a broken circular plate, drawn to scale of 1 cm to represent 4 cm, as shown.

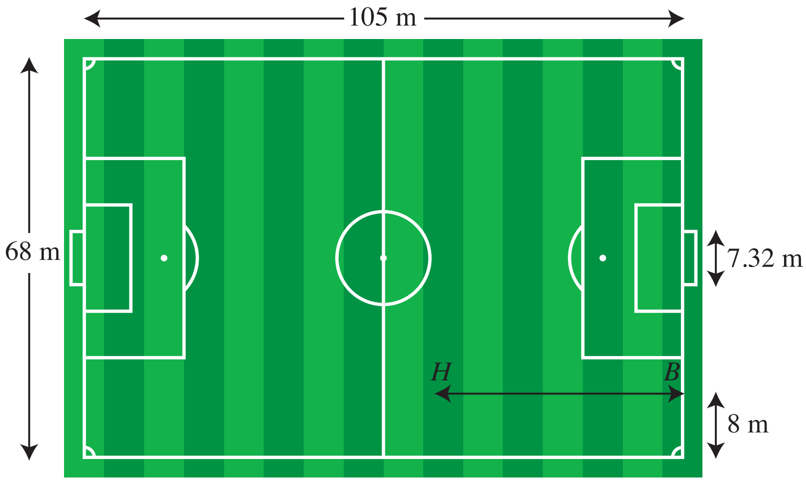
(i) Explain how you can help to find the diameter of the plate using geometrical properties of circles.

(ii) Using the method you have explained in **(i)**, find the diameter of the original circular plate.

(iii) Hence, find the area of the original circular plate.

PROBLEM 7: Penalty Shootout

The figure shows a soccer field measuring 105 m by 68 m and a goal post on each side with a width of 7.32 m.



Problem Solving Tip

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry template Soccer.

(i) For a penalty shootout, the ball is placed at a distance of 11 m from the centre of the goal post. Find the widest angle for shooting that the player can make before the ball misses the goal post.
Hint: Let O be the point at which the ball is placed, and P and Q be the two ends of the goal post. Find angle POQ.

(ii) A winger dribbles a ball 8 m from the side of the field along the line HB. At which point along HB should he strike the ball so that he will get the widest angle for shooting?

(iii) Explain how the angle for shooting varies as the winger strikes the ball at different points along HB.

Problem 8: Constructing A 3-Dimensional Star

To raise funds for charity, your class has been tasked to make and sell star-shaped paper lanterns. You will have to design 3-dimensional stars by creating a 2-dimensional template of each star and determining the amount of materials that are required.

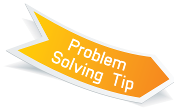

You may have to consider the following:

- Creation of a 2-dimensional template
- Materials for lantern, e.g. coloured paper, glue, string, lightbulbs
- Size and colour of paper to be used and amount of paper needed
- Amount of money to be raised

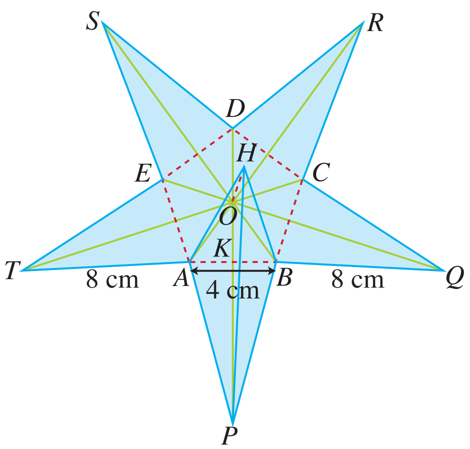
Guiding Questions:

1. How can we obtain the dimensions of the 2-dimensional template? Consider a star which is about 20 cm wide and 6 cm deep, with cross section and a 3-dimensional section *APBH* as shown in Fig. **(a)**. $TA = BQ = 8$ cm, $AB = 4$ cm and height of star above centre O, $OH = 3$ cm. Note that *TABQ* is not a straight line.

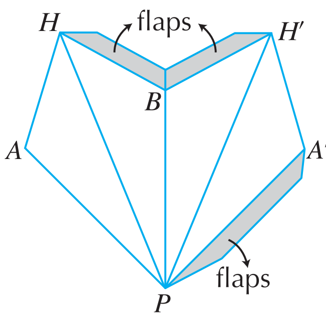
Fig. **(b)** shows the template used to make one part of the 3-dimensional star with cross section *OAPB*. You can fold the template along the lines and glue the flap on side *PA′* to *PA* to make this part of the 3-dimensional star. You need to make 5 such parts and glue each to the other along the remaining 2 flaps to make the 3-dimensional star.



The depth of the star is 6 cm because $OH = 3$ cm and $OH' = 3$ cm, where H' is the point directly below centre O on the other surface of the 3-dimensional star.



(a)



(b)

2. How do you calculate the amount of paper needed for each star? What size of paper is appropriate and available? How many sheets of paper would be needed?

3. What is the total cost of materials?

4. Propose a price for a lantern. Is the price proposed reasonable? How many lanterns have to be sold to raise the amount of money?

Practise Now Answers

CHAPTER 1

Practise Now 1

(a) -8 or 1 **(b)** $1\frac{1}{3}$ or $-2\frac{1}{2}$

Practise Now 2

(a) 3 or -17 **(b)** 8.32 or 1.68

Practise Now 3

(a) $(x + 10)^2 - 100$

(b) $\left(x - \frac{7}{2}\right)^2 - \frac{49}{4}$

(c) $\left(x + \frac{1}{10}\right)^2 - \frac{1}{100}$

(d) $(x + 3)^2 - 18$

Practise Now 4

1. **(a)** 0.61 or -6.61 **(b)** -0.81 or -6.19
(c) 1.62 or -0.62

2. 4.72 or -5.72

Practise Now 5

(a) 1.27 or -2.77 **(b)** 1.72 or -0.117

(c) 4.73 or 1.27 **(d)** 5.24 or 0.764

Practise Now 6

1. **(i)** $15, 5, -1, -3, -1, 5, 15$
(iii) 2.2 or -0.2

2. -2.3 or 1

Practise Now 7

1. **(i)** $16, 9, 4, 1, 0, 1, 4, 9$
(iii) 3

2. 4

Practise Now 8

1. **(a)** -1 or -6 **(b)** 0.703 or -2.37

2. 2.35 or -0.851

Practise Now 9

1. **(a)** 3.46 or 2.14 **(b)** 3.58 or 1.92

2. 3 or 2.5

Practise Now 10

(i) $(9 - x)$ cm **(iii)** 7.928 or 1.072

(iv) 4.25 cm^2

Practise Now 11

(ii) 126.16 or -133.16 **(iii)** 4.51 h

Practise Now 13

1. **(i)** $(-1, 0), (5, 0); (0, 5)$
(ii) $(2, 9)$
(iv) $x = 2$

2. **(i)** $(-2, 0), (0, 0); (0, 0)$
(ii) $(-1, -1)$
(iv) $x = -1$

Practise Now 14

1. **(i)** $(x - 3)^2 - 3$
(ii) $(3, -3)$
(iv) $x = 3$

2. **(i)** $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$
(ii) $\left(-\frac{1}{2}, \frac{3}{4}\right)$
(iv) $x = -\frac{1}{2}$

CHAPTER 2

Practise Now 1

(a) $x \geqslant 10$ **(b)** $y < \frac{1}{2}$

Practise Now 2

$x > 14$

(i) 17 **(ii)** 27

Practise Now 3

1. **(a)** $x < 1\frac{2}{5}$ **(b)** $y \geqslant \frac{3}{29}$
(c) $z \leqslant 26$

2. 4

Practise Now 4

60

Practise Now 5

6

Practise Now 6

$-1 < x \leqslant 5$

Practise Now 7

1. no solution

2. $-13 < y \leqslant 7$

CHAPTER 3

Practise Now 1

(a) 4^{12} **(b)** $(-3)^7$

(c) a^{20} **(d)** $6x^6y^7$

Practise Now 2

(a) 9^4 **(b)** $(-4)^7$

(c) a^4 **(d)** $3x^3y$

Practise Now 3

1. **(a)** 6^{12} **(b)** k^{45}
(c) 3^{10q}

2. 2

Practise Now 4

(a) 24^7 **(b)** $125b^{12}$

(c) $-32c^{10}d^{25}$ **(d)** $m^{28}n^{19}$

(e) $\frac{p^5q^4}{27}$

Practise Now 5

(a) 3^3 **(b)** 2^{15}

(c) $\frac{p^{11}}{q^{10}}$ **(d)** x^{11}

Practise Now 6

1. **(a)** 1 **(b)** 1
(c) 3 **(d)** 1

2. **(a)** 3 **(b)** 10

Practise Now 7

(a) $\frac{1}{36}$ **(b)** $-\frac{1}{8}$

(c) $1\frac{61}{64}$ **(d)** 9

Practise Now 8

(a) a^4 **(b)** $\frac{b^{15}}{c^6}$

(c) $\frac{2d}{e^2}$ **(d)** $\frac{5f^4}{3}$

(e) $\frac{6}{g^2}$ **(f)** $-2h^4$

Practise Now 9

(a) 4 **(b)** 4

(c) $\frac{2}{3}$

Practise Now 10

(a) 6 **(b)** $\frac{1}{2}$

(c) $-\frac{1}{5}$

Practise Now 11

1. (a) 16 **(b)** $\frac{1}{8}$

(c) 1000

2. (a) $a^{\frac{n}{3}}$ **(b)** $x^{-\frac{2}{5}}$

Practise Now 12

(a) m^2 **(b)** $\frac{1}{m^{\frac{7}{15}}}$

(c) $\frac{m}{n^{\frac{5}{3}}}$ **(d)** $\frac{m^{\frac{11}{3}}}{n^{\frac{11}{12}}}$

(e) $\frac{5m^7}{n^{\frac{14}{5}}}$ **(f)** $\frac{m^3}{n^{\frac{8}{7}}}$

Practise Now 13

(a) 3 **(b)** –2

(c) $1\frac{1}{3}$

Practise Now 14

1. \$646.52

2. (a) \$60.60 **(b)** \$61.16

3. 3%

Practise Now 15

1. (a) 5.3×10^6 **(b)** 6×10^8

(c) 4.8×10^{-5} **(d)** 2.1×10^{-10}

2. (a) 1 325 000 **(b)** 0.0044

Practise Now 16

(a) 2.54×10^{-5} m

(b) 2.34×10^2 cm

(c) 4.0×10^{12} bytes

Practise Now 17

(a) 5.20×10^9 **(b)** 1.09×10^{-1}

(c) 4×10^3 **(d)** 2.5×10^{-13}

(e) 1.60×10^5 **(f)** -2.56×10^6

(g) 6.57×10^{-9} **(h)** 4.21×10^4

Practise Now 18

800

CHAPTER 4

Practise Now 1

(a) $\frac{2}{3}$ **(b)** –1

(c) 0

Practise Now 2

12

Practise Now 3

(a) 5 units **(b)** 11.4 units

(c) 6 units

Practise Now 4

(a) $\left(0, 2\frac{1}{4}\right)$ **(b)** (–3, 0)

$3\frac{3}{8}$ units^2

Practise Now 5

1. $\angle DEF$ **2.** No

Practise Now 6

1. 7 **2.** 32

Practise Now 7

(a) $y = \frac{2}{7}x + \frac{11}{7}$ **(b)** $y = 4$

(c) $x = -3$

CHAPTER 5

Practise Now 1

(i) 17.5, –17.5 **(ii)** 2.35, –2.55

Practise Now 2

(i) 1.20 **(ii)** –2.5

Practise Now 3

(i) –0.9 **(ii)** 0.8 or –0.8

Practise Now 4

(i) 0.3 **(ii)** –0.3

Practise Now 5

(a) $a = 5, b = -3$ **(c)** 1.6

(d) (ii) $h = 2, k = -4$

Practise Now 6

(b) 3.5 minutes **(c)** 27 km/h

Practise Now 7

(i) $2\frac{2}{3}$ m/s^2 **(ii)** 4.89 m/s

(iii) 2 m/s^2

Practise Now 8

(a) $a = 2, b = 17$

(c) (i) 0.7 or 3.8 **(ii)** $t = 2.25$

(iii) 9 m/s^2 **(iv)** $0.25 < t < 4.2$

Practise Now 9

Carpark X

Practise Now 10

(a) 60 beats/minute

(b) 6 beats/minute^2

(c) 1 beat/minute^2

CHAPTER 6

Practise Now 1

1. **(a)** 0.995 **(b)** 0.629

2. 0.905

Practise Now 2

1. **(a)** $\frac{3}{5}$ **(b)** $-\frac{4}{5}$

(c) $1\frac{1}{3}$

2. **(a)** 13 units

(b) **(i)** $\frac{12}{13}$ **(ii)** $-\frac{12}{13}$

(iii) $2\frac{2}{5}$

Practise Now 3

(a) 24.5° or 155.5°

(b) 103.5°

(c) 84.0°

Practise Now 4

298 m^2

Practise Now 5

1. 3.59 **2.** 53.1°

Practise Now 6

(i) 82.3° **(ii)** 8.01 cm

(iii) 10.7 cm

Practise Now 7

1. **(i)** 34.7° **(ii)** 103.3°

(iii) 17.5 cm

2. **(i)** 52.1° **(ii)** 31.1°

(iii) 8.11 cm

Practise Now 8

83.2°, 50.8°, 7.65 cm or
96.8°, 37.2°, 5.96 cm

Practise Now 9

(i) 16.1 cm **(ii)** 69.5°

(iii) 39.5°

Practise Now 10

96.8°

CHAPTER 7

Practise Now 1

1. 63.8 m **2.** 19.0 m

Practise Now 2

1. **(i)** 354 m **(ii)** 14.3°

2. 45.2 m

Practise Now 3

1. **(a)** 050° **(b)** 330°

(c) 230° **(d)** 150°

2. **(a)** 123° **(b)** 231°

(c) 303° **(d)** 051°

Practise Now 4

1. 208 m

2. **(i)** 245° **(ii)** 310 m, 317 m

Practise Now 5

(i) 49.4 km **(ii)** 199.3°

Practise Now 6

(i) 200° **(ii)** 2.92 km

(iii) 4.92 km **(iv)** 1.25 km

Practise Now 7

1. **(i)** 58.0° **(ii)** 74.2°

(iii) 28.1°

2. **(i)** 14.6 cm **(ii)** 28.3°

Practise Now 8

1. **(i)** 32.0° **(ii)** 35.3°

(iii) 23.8°

2. **(i)** 33.7° **(ii)** 53.1°

(iii) 30.8°

Practise Now 9

(i) 170 m **(ii)** 8.4°

Practise Now 10

68.8 m

CHAPTER 8

Practise Now 1

1. **(i)** 99.5 cm **(ii)** 108 cm

2. $\left(\frac{21}{2}\pi + 18\right)$ cm **3.** 6.65

Practise Now 2

1. 13.2 cm **2.** 34.0 cm

Practise Now 3

(i) 56.6 cm **(ii)** 155 cm

Practise Now 4

(ii) 459 cm^2

Practise Now 5

(ii) 122 m^2

Practise Now 6

(a) 12° **(b)** 270°

(c) 174.2° **(d)** 458.4°

Practise Now 7

(a) 0.628 rad **(b)** 5.03 rad

(c) 3.45 rad **(d)** 6.98 rad

Practise Now 8

(a) 0.605 **(b)** 0.973

(c) 2.82

Practise Now 9

(a) 1.06 rad **(b)** 1.22 rad

(c) 0.722 rad

Practise Now 10

(i) 7.00 cm **(ii)** 11.9 cm

Practise Now 11

(i) 0.772 rad **(ii)** 3.23 cm

(iii) 5.30 cm

Practise Now 12

4.71 cm

Practise Now 13

1. **(i)** 0.75 rad **(ii)** 54 m^2

2. 22 cm

Practise Now 14

(ii) 15.8 m **(iii)** 93.4 m^2

Practise Now 15

(i) 12.96 cm **(ii)** 43.4 cm^2

Practise Now 16

(ii) $(9.6\pi - 11.5)$ cm **(iii)** 52.5 cm^2

CHAPTER 9

Practise Now 1

1. E, F, D; EF, FD, 11, ED; EFD, SSS

2. $\Delta WXY \equiv \Delta WZY$

Practise Now 2

2. No

Practise Now 3

1. **(ii)** 25°

2. **(i)** $\Delta PQS \equiv \Delta RSQ$

(ii) 7 cm, 140°

Practise Now 6

1. **(a)** Yes **(b)** No

(c) Yes **(d)** Yes

2. **(ii)** $DE = 10.5$ cm, $BD = 3$ cm

(iii) $\frac{AB}{BD} = \frac{AC}{CE}$

Practise Now 7

(a) Yes **(b)** No

Practise Now 8

(a) Yes **(b)** No

Practise Now 9

1. **(iii)** 2 cm

2. $BQ = 20$ cm, $AC = 28$ cm

Practise Now 11

1. 56 m **2.** 12 m

CHAPTER 10

Practise Now 1

(a) 98 cm^2 **(b)** 12.5 m^2

Practise Now 2

3.6 m

Practise Now 3

(i) 7 cm^2 **(ii)** 14.6 cm^2

Practise Now 4

1. 75 cm^3 **2.** 0.5 m

Practise Now 5

1. 328 g **2.** 3000 kg

Practise Now 6

(i) 14.9 cm **(ii)** 1 : 3.30

CHAPTER 11

Practise Now 1

1. 21.35 cm **2.** 5.05 cm

Practise Now 2

32.6 cm or 6.14 cm

Practise Now 3

1. **(i)** 23.2° **(ii)** 11.4 cm

(iii) 23.625 cm^2

2. **(i)** 3.9 **(ii)** 64.0°

(iii) 7.10 cm^2

Practise Now 4

1. **(i)** 28° **(ii)** 59°

(iii) 26.3 cm **(iv)** 369 cm^2

2. 32°

Practise Now 5

1. **(i)** 56° **(ii)** 118°

2. $x = 110°, y = 55°$

3. 146°

Practise Now 6

(i) 25° **(ii)** 50°

(iii) 105°

Practise Now 7

1. **(i)** 44° **(ii)** 25°

(iii) 69°

2. $x = 50°, y = 25°$

3. 20°

Practise Now 8

1. **(i)** $(159 - x)°$ **(ii)** $(149 - x)°$

(iii) 64 **(iv)** 85°

2. 34°

Practise Now 9

48°

Practise Now 10

(ii) 7.14 cm

Answers

CHAPTER 1

Exercise 1A

1. **(a)** 1 or $-3\frac{1}{2}$ **(b)** 2 or $-\frac{3}{4}$
(c) 2 or -9 **(d)** 1 or -4
(e) 1 or $-\frac{2}{3}$ **(f)** 2 or $-1\frac{2}{3}$

2. **(a)** 2 or -4 **(b)** $1\frac{1}{2}$ or $-2\frac{1}{2}$
(c) $2\frac{3}{5}$ or -1 **(d)** $2\frac{7}{12}$ or $2\frac{1}{12}$
(e) 0.32 or -6.32 **(f)** 3.90 or -0.90
(g) 7.65 or 2.35 **(h)** 3.66 or -2.66

3. **(a)** $(x+6)^2-36$
(b) $(x-3)^2-8$
(c) $\left(x+\frac{3}{2}\right)^2-\frac{17}{4}$
(d) $\left(x+\frac{9}{2}\right)^2-\frac{85}{4}$
(e) $\left(x+\frac{1}{4}\right)^2-\frac{1}{16}$
(f) $\left(x-\frac{1}{9}\right)^2-\frac{1}{81}$
(g) $(x+0.1)^2-0.01$
(h) $(x-0.7)^2-0.49$

4. **(a)** 1.45 or -3.45 **(b)** 1.61 or -18.61
(c) 11.20 or 0.80 **(d)** 5.85 or -0.85
(e) 1.61 or -1.86 **(f)** 0.81 or 0.05
(g) 0.74 or -1.34 **(h)** 4.34 or 0.46

5. **(a)** 8.12 or -0.12 **(b)** 4.79 or 0.21
(c) 8.22 or -1.22 **(d)** 7.80 or -1.80

6. $y=\frac{a\pm\sqrt{a^2+24}}{2}$

Exercise 1B

1. **(a)** -0.268 or -3.73 **(b)** 0.155 or -2.15
(c) 3.19 or 0.314 **(d)** 3.36 or -1.69
(e) 0.922 or -3.25 **(f)** 1.77 or 0.225

2. **(a)** 2.72 or -7.72 **(b)** 1.96 or -0.764
(c) 1.07 or -0.699 **(d)** 1.60 or -1.10
(e) 1.67 or -1.07 **(f)** 9.73 or 6.27

3. **(a)** 0.618 or -1.62 **(b)** 2.70 or -0.370
(c) 3.73 or 0.268 **(d)** 4.54 or -1.54
(e) 2.30 or -1.30 **(f)** 0.468 or -0.468

4. **(a)** 1 **(b)** -2.90 or 0.230
(c) no real solution **(d)** no real solution

Exercise 1C

1. **(i)** $8, 1, -2, -1, 4, 13$
(iii) 2.3 or 0.20

2. **(i)** $-5, 5, 9, 7, -1, -15$
(iii) 0.90 or -2.55

3. **(i)** $4, 1, 0, 1, 4, 9$ **(iii)** -3

4. **(ii)** 0.80 or -2.10

5. 1.45 or -3.45

6. **(ii)** -1.5

7. 5

8. **(b)** 7.5

9. **(b)** **(i)** 6.4 **(ii)** 5.6 m
(c) 6.4

Exercise 1D

1. **(a)** 1.77 or -2.27 **(b)** 0.818 or -4.48
(c) 3.73 or 0.268 **(d)** 8.14 or 0.860
(e) 5.54 or -0.541 **(f)** -2 or $-\frac{2}{3}$

2. **(ii)** -4 or 3 **(iii)** 3 and 4

3. **(a)** -2 or $\frac{3}{5}$ **(b)** 15.6 or 1.41

4. 10.6 or -0.141

5. **(a)** 2 or -4 **(b)** 2.70 or -3.70
(c) 6.5 or 2 **(d)** 0.775 or -0.775
(e) 6 or $-\frac{2}{3}$ **(f)** 6.43 or -2.43
(g) 5.14 or 1.36 **(h)** 7.16 or 0.838

6. **(i)** $(56-x)$ cm **(iii)** 41.67 or 14.33
(iv) 44.1 cm

7. **(ii)** 0 or $\frac{1}{3}$ **(iii)** 4 cm

8. **(i)** $\frac{60}{x}$ **(ii)** $\frac{60}{x+2}$
(iv) $\frac{1}{2}$ or $-1\frac{2}{3}$ **(v)** 6 minutes

9. **(i)** $\frac{350}{x}$ **(ii)** $\frac{350}{x+0.15}$
(iv) \$1.40

10. **(i)** $\frac{2}{x}+\frac{8}{x+1}$ **(iii)** -4 or 3
(iv) 2 hours 40 minutes

11. **(ii)** 88.08 or 11.92 **(iii)** 7.95 hours

12. **(i)** $\frac{1500}{x}$ **(ii)** $\frac{1500}{x+50}$
(iv) 363.10 or -413.10
(v) 3 minutes 38 seconds

13. **(i)** $\frac{2000}{x}$ **(ii)** $\frac{1000}{x+0.05}$
(iv) 1.25 or -0.03 **(v)** US\$1 = S\$1.30

14. **(a)** $-2\frac{6}{7}$ **(b)** $\frac{1}{2}$ or $-\frac{1}{3}$
(c) 5.12 or -3.12 **(d)** 7.54 or 1.46

15. 15 km/h

Exercise 1E

2. **(a)** $(0, 2), x=0$ **(b)** $(0, -6), x=0$
(c) $(3, 1), x=3$ **(d)** $(-1, -3), x=-1$
(e) $(-2, 3), x=-2$ **(f)** $(4, -1), x=4$

3. **(i)** $x\left(x+\frac{3}{4}\right)$

5. **(i)** $(x+3)(x-2)$

7. **(i)** $(x-4)^2-11$
(iii) $(4, -11)$
(iv) $x=4$

8. $\left(x+\frac{3}{2}\right)^2-\frac{5}{4}, \left(-1\frac{1}{2}, -1\frac{1}{4}\right)$

9. **(i)** $h=-\frac{1}{2}, k=\frac{3}{4}$

10. $p=5, q=21$; $(5, 21)$

Review Exercise 1

1. **(a)** -0.683 or -7.32 **(b)** 0.405 or -7.41
(c) 11.6 or -0.603 **(d)** 0.566 or -1.77

2. **(a)** -0.177 or -2.82 **(b)** 2.59 or -0.257
(c) -1 or $1\frac{1}{4}$ **(d)** 1.85 or -0.180

3. **(a)** $3\frac{2}{5}$ or $2\frac{3}{5}$ **(b)** 7.46 or 0.536
(c) 2.61 or -4.61 **(d)** 2.58 or -6.58

4. **(i)** 2.78 or 0.72 **(ii)** 3.78 or 1.72

5. **(a)** $7x^2-20x+12=0$
(b) $6x^2+7x+2=0$

6. **(a)** 1.58 or -7.58 **(b)** 0.245 or -12.2
(c) 0.171 or -1.17 **(d)** 5.68 or 1.32
(e) 3.22 or -0.621 **(f)** 4.91 or -0.577
(g) 19 or $-1\frac{1}{2}$ **(h)** 3.44 or -0.436

7. (i) $(x-3)(x-4)$

8. (i) $-\left(x-\frac{5}{2}\right)^2+\frac{9}{4}$

9. 3 and 4

10. (i) $\frac{65000}{x-5}-\frac{65000}{x}=20$

(ii) \$1625

11. (i) $\frac{40}{x}$ h (ii) $\frac{40}{x-30}$ h

(iv) 101.17 or –71.17

(v) 33.7 minutes

12. (i) $\frac{6000}{x}$ (ii) $\frac{6000}{x+10}$

(iv) 200 or –210 (v) 16.2

13. (i) $(35-2x)$ m, $(22-2x)$ m

(iii) 24.76 or 3.74

(iv) 3.74 m

14. (a) (i) –2.2 or 27.2

(c) (i) 216 m

(ii) 6.5 m or 18.5 m

15. 20 minutes, 25 minutes

16. 17.4 km/h

Challenge Yourself

1. 15 or 24 2. $-\frac{b}{a}, \frac{c}{a}$

CHAPTER 2

Exercise 2A

1. (a) $<$ (b) $<$

2. (a) $a<1$ (b) $b\geqslant 7$

(c) $c<-2$ (d) $d\geqslant 0$

(e) $e\geqslant -1\frac{1}{2}$ (f) $f<-\frac{2}{5}$

(g) $g\geqslant 4$ (h) $h>4$

(i) $j<1\frac{1}{10}$ (j) $k\geqslant -\frac{5}{8}$

(k) $m\leqslant 4$ (l) $n<-1$

3. $x\leqslant 4\frac{1}{2}$

(i) 4 (ii) 4

4. $x<3$

(i) 2 (ii) Yes

5. (a) $p<1\frac{1}{7}$ (b) $q\leqslant \frac{11}{13}$

6. (a) $a\geqslant 1\frac{1}{2}$ (b) $b>5$

(c) $c<8$ (d) $d>-1$

(e) $e<-10$ (f) $f\leqslant 2$

(g) $g\geqslant -1\frac{8}{9}$ (h) $h>108$

7. –10

8. (i) $x\geqslant -4$ (ii) 0

Exercise 2B

1. 19

2. $x=24, y=1; x=12, y=2; x=8, y=3; x=6, y=4$

3. (a) $-2\leqslant x\leqslant 7$

(b) $-1\frac{1}{3}<x<5$

4. (a) $-2, -1, 0$

(b) $3, 4, 5, 6, 7, 8$

5. (a) $x\geqslant 2$ (b) $3<x\leqslant 5$

(c) $-9<x<-3$ (d) $\frac{1}{2}<x\leqslant 6$

6. 10

7. 15 625

8. 18

9. 11

10. 29, 31, 37, 41, 43, 47

11. 18

12. 5

13. (a) $3\frac{1}{2}\leqslant a\leqslant 4\frac{1}{3}$

(b) $1<b<4$

(c) $1\frac{1}{3}<c<6$

(d) $0\leqslant d<3$

14. (a) $-4\leqslant a\leqslant 4$

(b) $b\leqslant -6$

(c) $\frac{5}{6}\leqslant c<1$

(d) $-2\leqslant d<8\frac{1}{3}$

15. (a) 8, 9, 10

(b) 5

(c) 2, 3

(d) 4, 5, 6, 7, 8

16. (a) 12 (b) –5

(c) 35 (d) 0

(e) 0, 49

17. (a) –10 (b) 5

(c) 2 (d) 2

(e) 1, 16 (f) 40

18. (a) False (b) True

(c) True

Review Exercise 2

1. (a) $a\leqslant 5$ (b) $b<\frac{2}{3}$

(c) $c\geqslant -2$ (d) $d>6$

(e) $e\geqslant 3\frac{1}{2}$ (f) $f\leqslant 6\frac{2}{3}$

(g) $g<1\frac{1}{3}$ (h) $h>2\frac{3}{4}$

2. (a) $a<-24$ (b) $b<7\frac{1}{5}$

(c) $c\leqslant -\frac{9}{20}$ (d) $d>-28\frac{3}{8}$

(e) $e\geqslant -3$ (f) $f\geqslant 2\frac{4}{9}$

3. (a) no solution

(b) $1\frac{2}{3}<b<8$

(c) $-\frac{1}{2}\leqslant c<\frac{3}{8}$

(d) $-1\leqslant d<10$

4. (a) 14 (b) 13

(c) $14\frac{1}{2}$

5. (a) $9\frac{1}{2}$ (b) 10

6. (a) 10, 11 (b) –6, –5

7. (a) 7, –3 (b) $2\frac{1}{2}, -\frac{1}{2}$

(c) $12\frac{1}{2}$, 0

8. (a) –13 (b) $1\frac{3}{4}$

(c) 33 (d) 37

9. 410.1 cm^2

10. 5

11. 18 years

12. 30

13. 1

Challenge Yourself

1. $12\leqslant z\leqslant 40$

2. $x>1\frac{2}{3}$

CHAPTER 3

Exercise 3A

1. (a) 2^{10} (b) $(-4)^{11}$

(c) x^{11} (d) $24y^9$

2. (a) 5^3 (b) $(-7)^7$

(c) $6x^4$ (d) $-3y^5$

3. (a) 9^8 (b) h^{10}

(c) 15^{14} (d) 18^3

(e) $8k^{18}$ (f) $81x^{24}y^8$

4. **(a)** 2^{13} **(b)** 3^{20}
(c) $\frac{m^5}{32}$ **(d)** $\frac{27}{n^6}$
(e) $\frac{p^{24}}{q^6}$ **(f)** $\frac{x^4}{y^8}$

5. **(a)** $h^{13}k^{10}$ **(b)** $-4m^{18}n^{12}$
(c) $22p^9q^{17}$ **(d)** h^4k^2
(e) $5m^6n^6$ **(f)** $5x^4y^3$

6. **(a)** a^{11} **(b)** b^{41}
(c) $-c^{28}$ **(d)** $\frac{9d^3}{8}$
(e) e^7 **(f)** $-8f^9$

7. **(a)** $8a^9b^9$ **(b)** $25c^8d^8$
(c) $64ef^3$ **(d)** $-\frac{2h}{g}$

8. **(a)** $\frac{2a^4}{b^5}$ **(b)** $\frac{c^9}{4d^8}$
(c) $3e^3f^3$ **(d)** $-\frac{8}{27g^3h^{12}}$

9. **(a)** $\frac{5x^6y^8}{2}$ **(b)** $\frac{32x^8y^2}{9}$
(c) $2y^5$ **(d)** $2x^2y^2$

10. $a = 2, b = 6$

Exercise 3B

1. **(a)** 1 **(b)** 1
(c) 4 **(d)** –8
(e) 1 **(f)** 7

2. **(a)** 16 **(b)** 7
(c) –63 **(d)** 211

3. **(a)** $\frac{1}{343}$ **(b)** $-\frac{1}{5}$
(c) $1\frac{7}{9}$ **(d)** $\frac{3}{5}$

4. **(a)** 1 **(b)** $\frac{24}{25}$
(c) $2\frac{2}{3}$ **(d)** 16

5. **(a)** 14 **(b)** 5
(c) $\frac{1}{2}$ **(d)** $\frac{2}{3}$

6. **(a)** $\sqrt{81}, 9$ **(b)** $\sqrt[3]{-27}, -3$
(c) $\frac{1}{\sqrt[4]{16}}, \frac{1}{2}$ **(d)** $(\sqrt{4})^3, 8$
(e) $\frac{1}{(\sqrt[3]{8})^5}, \frac{1}{32}$ **(f)** $(\sqrt[3]{-1000})^2, 100$

7. **(a)** $a^{\frac{1}{4}}$ **(b)** $b^{\frac{2}{3}}$
(c) $c^{\frac{4}{5}}$ **(d)** $d^{-\frac{1}{6}}$
(e) $e^{-\frac{1}{2}}$ **(f)** $f^{-\frac{5}{3}}$

8. **(a)** 3 **(b)** –7
(c) $2\frac{1}{2}$ **(d)** –2

9. \$1298.56

10. **(a)** $15a^9$ **(b)** $-2\frac{2}{3}$
(c) $\frac{d^{10}}{c^6}$ **(d)** $\frac{2}{e^{24}}$
(e) $\frac{144}{h^6}$ **(f)** $\frac{k^{12}}{j^{12}}$
(g) m^3n **(h)** $61p^3$

11. **(a)** $a^{\frac{5}{6}}$ **(b)** $b^{\frac{1}{2}}$
(c) $c^{\frac{17}{10}}$ **(d)** $\frac{1}{d^{\frac{6}{5}}}$
(e) $\frac{e^{\frac{3}{2}}}{f^2}$ **(f)** $\frac{g}{h^{\frac{6}{5}}}$

12. **(a)** $\frac{a^{\frac{4}{3}}}{b^{\frac{3}{2}}}$ **(b)** $\frac{c^{10}}{d^{\frac{16}{5}}}$
(c) $\frac{e^{\frac{11}{3}}}{f^{\frac{11}{12}}}$ **(d)** $\frac{5g}{h}$
(e) $\frac{j^2k}{h^3}$ **(f)** $\frac{m^{\frac{56}{5}}n^{\frac{3}{5}}}{2}$

13. **(a)** \$21 074.13 **(b)** \$20 991.14

14. 3.01%

15. \$2264.09

16. **(a)** $\frac{z^{13}}{x^{53}y^4}$ **(b)** $\frac{x^2z^{11}}{y^{10}}$
(c) $\frac{ac^{2n+1}}{b^3}$ **(d)** $\frac{a}{c(a+b)^3}$

17. **(i)** Company *B* **(ii)** \$103.41

Exercise 3C

1. **(a)** 8.53×10^4 **(b)** 5.27×10^7
(c) 2.3×10^{-4} **(d)** 9.4×10^{-8}

2. **(a)** 9600 **(b)** 400 000
(c) 0.000 28 **(d)** 0.000 001

3. **(i)** 3×10^2 MHz **(ii)** 3×10^5 MHz

4. **(i)** 7×10^{-11} m **(ii)** 7.4×10^{-11} m
(iii) 35 : 37

5. 3.94×10^5%

6. **(a)** 1.67×10^2 **(b)** 1.41×10^{-9}
(c) 3.35×10^{-1} **(d)** 3.33×10^3
(e) 3.36×10^4 **(f)** -3.04×10^7
(g) 1.53×10^{-1} **(h)** 3.35×10^{-5}

7. **(a)** 2.46×10^{-12} **(b)** 6.94×10^7
(c) 1.1×10^4 **(d)** 2.1×10^2
(e) 5.41×10^{-2} **(f)** 1.99×10^5

8. **(a)** 3.15×10^9 **(b)** 4.5×10^4

9. 7.6×10^{-3}

10. **(a)** 1.6×10^{14} **(b)** 6.4×10^{-2}

11. 3.33×10^{-7}

12. **(i)** 3×10^8 m/s
(ii) 43 minutes 15 seconds

13. **(i)** 1.44×10^6 km **(ii)** 400 days

14. **(i)** 1.25×10^8 **(ii)** 2.15
(iii) 1.69

Review Exercise 3

1. **(a)** a^7b^4 **(b)** $3a^2b^2$
(c) $-27a^9b^{15}$ **(d)** $\frac{a^2b}{2}$

2. **(a)** 5^{16} **(b)** 5^{-3}
(c) $5^{\frac{1}{5}}$

3. **(a)** 125 **(b)** $\frac{5}{36}$
(c) $2\frac{1}{9}$ **(d)** $1\frac{37}{125}$

4. **(a)** 3 **(b)** $\frac{3}{5}$
(c) 64 **(d)** $\frac{1}{64}$

5. **(a)** $\frac{x^4}{81}$ **(b)** $3x^3$

6. **(a)** y^3 **(b)** $\frac{4y^{10}}{x^6}$
(c) $\frac{y^{33}}{x^7}$ **(d)** $\frac{4}{x^2}$

7. **(a)** $2p^{\frac{14}{15}}$ **(b)** $\frac{p^{\frac{22}{5}}}{q^{\frac{12}{5}}}$
(c) $\frac{p^{\frac{20}{3}}}{q}$ **(d)** $\frac{3q^{\frac{32}{3}}}{p^{\frac{8}{3}}}$

8. **(a)** 6 **(b)** $\frac{1}{7}$
(c) 8

9. **(a)** $\frac{3}{4}$ **(b)** 0
(c) –1 **(d)** 16

10. **(a)** \$16 969.85 **(b)** \$16 952.14

11. \$45 972

12. **(a)** 3.26×10^4 **(b)** 3.1×10^{-10}
(c) 2.58×10^5 **(d)** 3.64×10^{-3}
(e) -2.47×10^6 **(f)** 7.42×10^{-4}

13. **(a)** 9.79×10^7 **(b)** 1.1×10^5
(c) 2.94×10^{-10} **(d)** 6.36×10^{-5}

14. **(i)** 2.20×10^{-8} m **(ii)** 3.85×10^{-17} m^2

15. **(i)** 1.496×10^{11} m **(ii)** 499 s

16. **(i)** 2.4×10^8 m **(ii)** 1920 s

17. **(i)** 2.99×10^{-23} **(ii)** 9.36×10^{24}

Challenge Yourself

1. 2^{3^4}
2. 7
3. 2

REVISION EXERCISE A1

1. (a) 8 (b) 125 (c) $\frac{4}{5}$ (d) 3 (e) $1\frac{1}{3}$
2. (a) a^7 (b) $\frac{1}{b^{3.5}}$ (c) $\frac{c^{25}}{d^{15}}$
3. (a) 0 (b) 11
4. (i) $p < -2.8$ (ii) –3
5. 5, 6, 7, 8
6. (i) 2.76×10^{-8} m (ii) 6.08×10^{-17} m^2
7. 55 cm, 30 cm
8. 5.5 h, 6 h
9. (i) $y^2 + 16, y^2 - 16y + 73$ (iii) 2 or 6 (iv) 15 cm^2 or 13 cm^2
10. (ii) 15 or 2.6 (iii) 2.6 cm

REVISION EXERCISE A2

1. (a) $8a^3b^6$ (b) c (c) $\frac{d^2}{4}$ (d) $\frac{1}{2}e^3$
2. 64
3. (a) –3 (b) ±9 (c) 4.5
4. (a) 6 (b) $\frac{1}{6}$
5. $-5 \leqslant q < 8$
6. (a) 11 (b) 10 (c) $9\frac{3}{4}$
7. (i) $\frac{1}{2}(a+1)(5a+3)$ (iii) 1.11 or –2.71 (iv) 5.32
8. 2 h, 4 h
9. (ii) 6.84 cm, 5.84 cm
10. 20 km/h

CHAPTER 4

Exercise 4A

1. (a) $-\frac{1}{2}$ (b) –10 (c) $-\frac{4}{3}$ (d) –3 (e) $\frac{11}{4}$ (f) 0
2. $0, \frac{1}{2}, -\frac{5}{6}, -\frac{1}{6}$
3. $-2\frac{4}{5}$
4. $1\frac{5}{6}$
5. –6 or 3
6. 9
7. 3
8. $1\frac{1}{2}$ or -1
9. (i) $-\frac{5}{2}, \frac{2}{5}, -\frac{5}{2}, \frac{2}{5}$ (ii) They are equal.

Exercise 4B

1. (a) 8.06 units (b) 8.54 units (c) 11.4 units (d) 10.8 units
2. ±7.07
3. (a) $\left(0, -8\frac{1}{3}\right)$ (b) $\left(2\frac{3}{11}, 0\right)$
4. $\left(0, -3\frac{21}{26}\right)$
5. (i) 32 units, 48 units^2 (ii) 9.6 units
6. (i) 4.5 units^2 (ii) 5.83 units (iii) (0, 4) (iv) 11 or –5
7. (i) 3 units, 4.12 units, 4.47 units (ii) 6 units^2 (iii) 6 or –8
8. –2 or 1
9. (ii) 9 units^2
10. 2.57 units
11. 3.48 units

Exercise 4C

1. 3
2. 15
3. (a) $y = -x$ (b) $y = 2x + 1$ (c) $y = \frac{1}{4}x + \frac{7}{2}$ (d) $y = \frac{9}{10}x + \frac{2}{5}$ (e) $y = -x - 6$ (f) $y = \frac{2}{3}x - \frac{1}{3}$ (g) $y = 0$ (h) $x = 0$
4. (a) $y = \frac{1}{3}x$ (b) $y = 3x - 2$ (c) $y = -3x + 1$ (d) $y = -\frac{1}{2}x + \frac{19}{2}$ (e) $y = 4$ (f) $y = ax + a$
5. $y = 2x$
6. (a) 0, 1; $y = 1$ (b) undefined; no y-intercept, $x = 1.5$ (c) 1, –1; $y = x - 1$ (d) $-\frac{1}{2}$, 1; $y = -\frac{1}{2}x + 1$
7. (i) 4 units^2 (ii) $\frac{1}{2}$ (iii) $y = x$
8. –4 or 3
9. (i) $y = -\frac{2}{3}x + 2$ (ii) $-\frac{2}{3}$ (iii) (3, 0)
10. (i) $y = -\frac{2}{3}x + 3$ (ii) 0
11. $y = \frac{5}{2}x - \frac{19}{2}$
12. (i) $y = 3x - 8$ (ii) (4, 4)
13. (i) (–6, 0) (ii) $\left(2, -6\frac{2}{3}\right)$ (iii) $y = -\frac{5}{6}x + \frac{3}{2}$ (iv) $y = -1$
14. (a) (i) 3 (ii) $y = 3x + 3$ (b) (6, 3)
15. $m = 0$; $n = 0$
16. (i) $A(8, 0)$, $B(0, 6)$ (ii) 10 units (iii) $\left(3\frac{3}{7}, 3\frac{3}{7}\right)$ (iv) $y = x$
17. (i) $\left(-8\frac{1}{2}, 0\right)$ (ii) (2, 7) (iii) 7.28 units (iv) 3.85 units

Review Exercise 4

1. (i) $y = 2x - 3$ (ii) 5
2. (i) –3 (ii) 14
3. $y = -8x$
4. (i) $\frac{3}{4}$ (ii) 10 units (iii) $y = -\frac{3}{4}x - 6$
5. (i) $y = -\frac{3}{4}x + 6$ (ii) $\left(2\frac{6}{7}, 3\frac{6}{7}\right)$ (iii) $y = 3\frac{6}{7}$ (iv) $x = 2\frac{6}{7}$
6. (i) $\frac{2}{3}$ (ii) $y = \frac{2}{3}x + \frac{5}{3}$ (iii) 10 units^2 (iv) 4.12 units
7. (i) (–4, 0) (ii) (2, 4) (iii) 16 units^2
8. (i) $y = 3x + 3$ (ii) 10 (iii) –12 (iv) (2, 1)
9. (i) $P(5, 0)$, $Q(0, 12)$ (ii) 13 units (iii) $y = -\frac{12}{5}x - 2$ (iv) (–5, 0)
10. $t = 15 - h$
11. (i) $y = -4$ (ii) 7.2 units^2 (iii) 41.76 (iv) 2.23 units
12. (i) $y = -0.1x + 10$

CHAPTER 5

Exercise 5A

1. **(a)** –27, –1, 0, 8
(c) **(i)** 3.5 **(ii)** 2.3

2. **(a)** 2.75
(c) **(i)** –0.5 **(ii)** 1.75

3. **(a)** 8, 4, 1.3, 0.8
(c) **(i)** 1.1 **(ii)** 2.65

4. **(a)** $a = 1.1, b = 0.4$
(c) **(i)** 1.3 **(ii)** 1.5

5. **(i)** 1.5 **(ii)** 2.4

6. **(a)** **(i)** 9.5 **(ii)** 15
(iii) 27
(b) **(i)** 0.8 **(ii)** 3.5
(iii) 4.15

7. **(i)** –1.8 **(ii)** 3.35

8. **(a)** $h = 2, k = 5.5$
(c) **(i)** –0.3 **(ii)** 0.9

9. **(i)** 0.65 **(ii)** 2.45

10. **(i)** 5.5 **(ii)** 1, 2.75

11. **(a)** –1, –0.6, 1.6
(b) **(i)** –1.5, –0.35, 1.9
(ii) –1.5, –0.35, 1.9

12. **(a)** 1.7 **(c)** no solution

13. **(a)** –4.2
(b) **(i)** 1.5 or 4
(ii) 1.6 or 5.8
(iii) 0.6 or 4.3

Exercise 5B

1. **(a)** 0.5, 8, 16, 32
(c) **(i)** 12 **(ii)** –0.65

2. **(a)** 3, 4.2, 8.5
(c) **(i)** 4.9, 14.8 **(ii)** –0.25, 1.30

3. **(i)** –7.4 **(ii)** 1.1

4. **(a)** $a = 2.5, b = 7.7$
(c) **(i)** 2.6, 8.5 **(ii)** 1.725, 2.45

5. **(a)** 1.6
(b) **(i)** (–1.15, 0.3)
(ii) –1.15

6. 3

7. **(a)** 5, 8, 5 **(c)** 4
(d) **(ii)** $h = 1, k = 9$

8. **(b)** –14 **(c)** 10

9. **(a)** **(ii)** –0.85

10. **(c)** (1, 2)

Exercise 5C

1. **(ii)** 1153

2. **(i)** 15 km **(iii)** 65 minutes

3. **(i)** 1 h **(ii)** 30 km/h
(iii) 34.3 km/h

4. **(i)** 5 m/s^2 **(ii)** 7.5 m/s

5. **(ii)** 7 m/s

6. **(ii)** 3.54 m/s **(iii)** $t = 25.5$

7. **(b)** **(i)** 2.3 minutes
(ii) 0.44 km/minute
(iii) 2.75 minutes

8. **(b)** **(i)** 1 h 11 minutes
(ii) 1 h and 1 h 22 minutes

9. **(b)** **(i)** 1011 **(ii)** 8.3 km

11. **(i)** 9 **(ii)** 30 m/s

12. **(i)** 1.5 m/s^2 **(ii)** 100 s

13. **(a)** $a = 4, b = 10$
(c) **(i)** 1.7, 5.3 **(ii)** 3.5 s
(iii) –3 **(iv)** $2.4 < t < 4.6$

14. **(ii)** 9.5 m/s, 34 m/s
(iii) 2.5 m/s^2, 5 m/s^2

15. **(a)** $h = 8, k = 10$
(c) **(i)** 2.8 s **(ii)** 10 km/h^2
(iii) $1.65 < t < 4$
(d) 0.4

16. $6\frac{2}{3}$ m/s

17. **(a)** 55 cents **(b)** Company *B*

18. **(a)** 30 minutes **(b)** 20 km/h^2
(d) 1648

20. **(b)** **(i)** 3 cm/s^2, –1 cm/s^2
(ii) $3.3 < t < 6.7$
(c) $a = -0.5, b = 5, c = 0$

22. **(b)** 12 m/s

Review Exercise 5

1. **(a)** –12, –8, –12, –8
(c) **(i)** –9.5 **(ii)** 3.1

2. **(a)** 3, 0, 15
(c) **(i)** –2.75 **(ii)** 2.4
(iii) –2, 0, 2

3. **(a)** **(i)** 3.85 **(ii)** –0.375, –1.375
(b) (–0.7, 3.8)

4. **(i)** 0.625 **(ii)** (0.55, –0.2)

5. **(i)** $y = 1.45, x = 1.75$
(ii) $0.9 < x < 3.275$
(iii) 0.67 **(iv)** 3.575

6. **(ii)** 27.7 km/h

7. **(a)** **(i)** $1 \leqslant t \leqslant 2$ **(ii)** $13\frac{1}{3}$ km/h
(iii) 40 km/h **(iv)** 17.1 km/h

8. **(i)** 320 000
(ii) 16, 33, 40; 3200, 512
(iii) 350 units

9. **(a)** 5°C/minute **(c)** 2°C/minute

REVISION EXERCISE B1

1. $3y = 5x - 4$

2. **(i)** 10 units^2 **(ii)** (0, 6)
(iii) (8, 6)

3. **(i)** $A(8, 0), B(0, 6)$ **(ii)** 39 units^2
(iii) 1.2 **(iv)** 7.8 units

4. **(i)** $13\frac{1}{3}$ km/h **(ii)** 1030, 20 km
(iii) 1 h **(iv)** 10 km

5. **(i)** 16 **(ii)** 24

6. **(a)** $h = 3.5, k = 2.7$
(c) **(i)** 1.5 **(ii)** 0.9
(d) 0.5 **(e)** 1.4

REVISION EXERCISE B2

1. $5x + 7y + 25 = 0$

2. **(i)** $P(-4, 0), Q(0, 2)$
(ii) $\left(-1\frac{1}{3}, 1\frac{1}{3}\right)$
(iii) $2\frac{2}{3}$ units^2

3. **(i)** (3, 4) **(ii)** (0, –3)
(iii) 21 units^2 **(iv)** 3.61 units
(v) $3y = 2x - 9$

4. **(b)** **(i)** 1233, 77 km

(ii) 60 km

5. **(i)** 1 minute 35 seconds

6. **(i)** 6.5 **(iii)** 2.15 or 5.35

(iv) –2.5

(v) $y = 14.1, x = 3.75$

(vi) 3.75 cm by 3.75 cm, square

CHAPTER 6

Exercise 6A

1. **(a)** sin 70° **(b)** sin 4°

(c) sin 82° **(d)** –cos 81°

(e) –cos 73° **(f)** –cos 5°

2. **(a)** 0.530 **(b)** 0.819

3. **(a)** 3.535 **(b)** 0.707

(c) –2.121

4. **(a)** $\frac{4}{5}$ **(b)** $-\frac{3}{5}$

(c) $1\frac{1}{3}$

5. **(a)** 9

(b) **(i)** $\frac{9}{41}$ **(ii)** $-\frac{40}{41}$

(iii) $\frac{9}{40}$

6. **(a)** 31.3° **(b)** 48.6°

(c) 61.0° **(d)** 20.2°

7. **(a)** 148.7° **(b)** 131.4°

(c) 119.0° **(d)** 159.8°

8. **(a)** 47.9° **(b)** 40.9°

(c) 60° **(d)** 9.9°

9. **(a)** 48.9° or 131.1° **(b)** 72.2° or 107.8°

(c) 28.1° or 151.9° **(d)** 103.8°

(e) 141.5° **(f)** 58.4°

10. **(a)** $\frac{8}{17}$ **(b)** $-\frac{15}{17}$

(c) $\frac{8}{15}$

11. **(a)** $\frac{3}{5}$ **(b)** $-\frac{4}{5}$

(c) $\frac{3}{8}$

12. **(a)** $\frac{5}{13}$ **(b)** $-\frac{12}{13}$

(c) $\frac{5}{27}$

13. 27 or 153

14. **(a)** 18.0° or 142.0° **(b)** 134.1°

Exercise 6B

1. **(a)** 34.2 cm^2 **(b)** 29.4 cm^2

(c) 41.5 m^2 **(d)** 31.7 m^2

(e) 27.4 cm^2 **(f)** 70.7 m^2

2. 117 cm^2 **3.** 9040 cm^2

4. **(i)** 633 cm^2 **(ii)** 29.5 cm

5. 23 000 m^2

6. **(i)** 27.5° **(ii)** 10.5 cm

(iii) 6.22 cm^2

7. 3.23

8. **(i)** 30° **(ii)** 10 cm

(iii) 346 cm^2

9. 116 cm^2 **10.** 22.7 cm

11. 10.2°, 169.8°

12. **(i)** 10.8 cm^2 **(ii)** 104.5°

Exercise 6C

1. **(a)** $\angle C = 62°, b = 10.7$ cm, $c = 9.76$ cm

(b) $\angle F = 79.3°, d = 4.43$ m, $f = 6.96$ m

(c) $\angle H = 38°, g = 11.5$ mm, $i = 5.29$ mm

2. 11.8 cm **3.** 15.6 cm

4. **(a)** $\angle B = 26.9°, \angle C = 61.1°$, $c = 13.4$ cm

(b) $\angle A = 55.6°, \angle C = 26.4°, c = 7.81$ m

(c) $\angle B = 31.7°, \angle A = 113.3°$, $a = 15.2$ cm

5. **(i)** 39.2° **(ii)** 39.8°

(iii) 13.6 cm

6. **(i)** 39.6° **(ii)** 49.4°

(iii) 8.80 cm

7. **(i)** 12.5 m **(ii)** 41.7°

(iii) 13.4 m

8. **(i)** 6.92 m **(ii)** 40.1 m^2

9. **(i)** 9.40 cm **(ii)** 5 cm

(iii) 4.92 cm

10. **(i)** No **(ii)** 9.47 cm

(iii) 5.62 cm

11. 14.7 km^2

12. **(i)** 2.64 cm **(ii)** 55.8°

(iii) 49.4°

13. **(a)** No **(b)** Yes

(c) No **(d)** Yes

(e) Yes **(f)** No

14. **(a)** Not possible **(b)** Possible

(c) Not possible **(d)** Possible

15. $\angle ABC = 68.9°, \angle ACB = 53.1°$, $AB = 13.2$ cm or $\angle ABC = 111.1°$, $\angle ACB = 10.9°, AB = 3.12$ cm

16. **(i)** 9.18 cm **(ii)** 0.734 km

(iii) 0.321 km^2

17. **(i)** 49.9° or 130.1° **(ii)** $6\frac{2}{3}$ cm

Exercise 6D

1. 6.24 cm

2. 4.57 cm

3. 9.45 cm

4. $\angle X = 48.2°, \angle Y = 58.4°, \angle Z = 73.4°$

5. 34.5°

6. 88.5°

7. **(i)** 9 m **(ii)** 15.1 m

8. **(i)** 6.12 m **(ii)** 7 m

9. **(i)** 3.46 cm **(ii)** 5.29 cm

(iii) 90°

10. **(i)** 22.6° **(ii)** 4.84 m

(iii) 6.86 m

11. **(i)** 7.94 **(ii)** 81.0°

12. **(i)** 73.4° **(ii)** 1.92 cm

(iii) 2.18 cm

13. **(i)** 20 km **(ii)** 89.6°

(iii) 225 km^2

14. 93.8°, 9.29 cm

15. **(i)** No **(ii)** $\frac{131}{144}$

(iii) 6.78

16. **(i)** $-\frac{1}{20}$ **(ii)** 6.57 cm

17. 7.09 cm

Review Exercise 6

1. **(a)** 25 cm

(b) **(i)** $\frac{7}{24}$ **(ii)** $-\frac{24}{25}$

(iii) $\frac{24}{25}$

2. **(a)** 37 cm

(b) **(i)** $\frac{12}{37}$ **(ii)** $-\frac{35}{37}$

(iii) $3\frac{109}{420}$

3. (i) $-\frac{1}{\sqrt{5}}$ or -0.4472
(ii) $\frac{2}{\sqrt{5}}$ or 0.8944
(iii) $\frac{3}{4}$

4. (i) 13 units, 14.4 units
(ii) $\frac{12}{13}, -\frac{5}{13}$
(iii) 18 units^2
(iv) 2.50 units

5. (a) 24.8° or 155.2° (b) 21.3°
(c) 26.7° (d) 108.5°

6. (b) (i) $-\frac{35}{37}, \frac{12}{37}$ (ii) 5.7°

7. (i) $\frac{21}{44}$ (ii) 20.1°
(iii) 13.2 cm^2 (iv) 5.04 cm

8. (i) 2.96 m (ii) 1.19 m
(iii) 6.00 m^2 (iv) 2.52 m
(v) 56.6°

9. (i) 377 m (ii) 200 m
(iii) 232 m (iv) 85 900 m^2

10. (i) 157 cm (ii) 45.7°
(iii) 2.33 m^2

11. (i) 1810 m^2 (ii) 66.8 m

12. (i) 43 100 m^2 (ii) 277 m
(iii) 193 m

Challenge Yourself

(a) 12.9 cm, 20.9 cm (b) 64.5 cm^2

CHAPTER 7

Exercise 7A

1. 119 m 2. 52.1 m
3. 27.6° 4. 63.1 m
5. 36.3 m 6. 68.7°
7. 9.74 m 8. 40.3 m
9. 35.0° 10. 210
11. 63.5 m 12. 10.3 m
13. 28.0
14. (i) 81.3 m (ii) 93.5 m

Exercise 7B

1. (a) 033° (b) 118°
(c) 226° (d) 321°

2. (a) 055° (b) 165°
(c) 317° (d) 235°
(e) 345° (f) 137°

3. (a) 036° (b) 216°
(c) 073° (d) 253°
(e) 296° (f) 116°

4. (a) 34.6 km (b) 35.5 km

5. 40.2 km

6. (a) 310° (b) 270°
(c) 220°

7. (a) 315° (b) 003° or 267°
(c) 238° or 032°

8. 028° or 216°

9. (a) 218 m (b) 180 m
(c) 436 m

10. 7.97 km

11. (i) 696 m (ii) 038.9°

12. (i) 126.6° (ii) 16.9 m
(iii) 305.8°

13. (i) 553 m (ii) 40.5°
(iii) 184.5° (iv) 331 m

14. 31.2 km, 080.2°

15. (a) (i) 65.9 km (ii) 139.0°
(b) 1009

Exercise 7C

1. (i) 9.22 cm (ii) 6 cm
(iii) 33.1° (iv) 45°

2. (i) 13 cm (ii) 53.1°
(iii) 67.4°

3. (i) 33.7° (ii) 10 cm
(iii) 21.8°

4. (i) 56.3° (ii) 5.55 m
(iii) 10.6 m (iv) 52.0°

5. 38.6 m

6. (i) 14.1 cm (ii) 28.7 cm
(iii) 63.8°

7. (i) 113 cm (ii) 15.7°
(iii) 37.1°

8. (i) 22.4° (ii) 44.8°

9. (i) 26.6° (ii) 57.5°

10. (i) 053.1° (ii) 61.2 m
(iii) 4.7°

11. (a) (i) 227° (ii) 120°
(iii) 330°
(b) 16.6 m

12. (a) (i) 053.1° (ii) 326.3°
(b) 12.1 m, 7.7°

13. (a) (i) 380 m (ii) 33 400 m^2
(iii) 176 m
(b) 7.8°

14. (i) 53.1° (ii) 57.0°

15. (i) 72.0° (ii) 55.5 cm^2

16. 39.3 m, 46.8 m

17. (i) 023.5° (ii) 38.6°

Review Exercise 7

1. 40.2 m 2. 124 m

3. 207 m

4. (i) 14.8 cm (ii) 46.6°
(iii) 22.5 cm

5. (i) 17.1 m (ii) 16.8 m
(iii) 35.5°

6. (i) 24 m (ii) 45.2°
(iii) 24.7 m

7. 14.8 m, 18.1°

8. (a) (i) 269 m (ii) 66.5°
(iii) 011.2° (iv) 56 700 m^2
(b) 8.8°

9. (a) (i) 100° (ii) 126°
(iii) 378 m (iv) 370 m
(b) 12.2°

10. (i) 9.02 km (ii) 098°
(iii) 5.76 km (iv) 2.3°
(v) 4.22 km

Challenge Yourself

1. $\sin\alpha\cos\beta$

2. $\frac{h\tan\alpha}{\cos\beta}$

CHAPTER 8

Exercise 8A

1. (a) 11.4 cm (b) 32.7 cm
(c) 63.5 cm (d) 53.7 cm

2. (a) (i) 11.9 cm (ii) 62.6 cm
(b) (i) 31.3 cm (ii) 101 cm
(c) (i) 44.5 cm (ii) 101 cm
3. (a) 16.0 cm (b) 28.0 cm
4. (a) 49° (b) 80°
(c) 263° (d) 346°
5. 1.18 m
6. 191.0°
7. (a) 17.0 cm (b) 37.0 cm
8. $\left(18 + \frac{25}{3}\pi\right)$ cm
9. (i) 105° (ii) 12.8 cm
10. (i) 61.8° (ii) 30.5 cm
11. (i) 67.7 cm (ii) 198 cm
12. (ii) 48.2 cm
13. (i) 27.3° (ii) 54.5°
(iii) 32.4 cm
14. 44.4 cm
15. 43.0 cm

Exercise 8B

1. (a) 8.80 cm, 30.8 cm^2, 22.8 cm
(b) 108.0°, 66 mm, 1150 mm^2
(c) 28.0 mm, 132 mm, 188 mm
(d) 84.0 cm, 9240 cm^2, 388 cm
(e) 225.1°, 385 m^2, 83 m
(f) 15.3 cm, 20.1 cm, 50.8 cm
2. (a) (i) 17.7 cm (ii) 12.8 cm^2
(b) (i) 8.22 cm (ii) 2.14 cm^2
(c) (i) 26.7 cm (ii) 44.0 cm^2
3. (a) 14.7 cm, 103 cm^2
(b) 24.2 cm, 169 cm^2
(c) 30.8 cm, 216 cm^2
(d) 52.8 cm, 370 cm^2
4. (a) 385 cm^2, 22.0 cm
(b) 898 cm^2, 51.3 cm
(c) 1150 cm^2, 66.0 cm
(d) 2120 cm^2, 121 cm
5. (a) 9.33 cm (b) 12.0 cm
6. (a) 60.3° (b) 165.8°
(c) 303.5° (d) 26.7°
7. (i) 43.6 cm, 118 cm^2
(ii) 33.2 cm, 40.8 cm^2
(iii) 263 cm, 1640 cm^2
8. (i) 100° (ii) 42.0 cm
9. 84 cm^2
10. (ii) 32.5 cm^2 11. $1.47p^2$
12. (i) 132° (ii) 13.0 cm
(iii) 92.7 cm (iv) 200 cm^2
13. (i) 8.49 cm (ii) 21.4 cm
(iii) 20.5 cm^2

Exercise 8C

1. (a) 150° (b) 25.7°
(c) 183.3° (d) 146.7°
2. (a) 0.653 rad (b) 1.38 rad
(c) 2.48 rad (d) 5.38 rad
3. (a) $\frac{\pi}{12}$ rad (b) $\frac{\pi}{10}$ rad
(c) $\frac{5\pi}{12}$ rad (d) $\frac{5\pi}{4}$ rad
4. (a) 0.717 (b) 0.856
(c) 14.1 (d) 0.383
(e) 0.156 (f) 3.08
5. (a) 0.833 (b) 1.40
(c) 0.448 (d) 0.148
(e) 0.694 (f) 0.898
6. (i) 11.7 cm (ii) 14.6 cm
7. (i) 6.79 m (ii) 14.4 m
8. (i) 9.36 cm (ii) 14.0 cm
9. (i) 0.159 rad (ii) 11.2 m
(iii) 3.34 m
10. 13.8 m

Exercise 8D

1. (a) 9.6 cm (b) 3.5 cm
(c) 43.75 m (d) 9 mm
2. (a) 70.4 cm^2 (b) 66.47 cm^2
(c) 108.9 m^2 (d) 2650.8 mm^2
3. (i) $1\frac{1}{8}$ rad (ii) 144 cm^2
4. (a) 5 cm, 10 cm^2 (b) 1.5 rad, 27 cm^2
(c) 12 m, 57.6 m^2 (d) 10 m, 12 m
(e) 2 rad, 16 mm (f) 9 mm, $\frac{2}{3}$ rad
5. (i) 81.2 cm (ii) 243 cm^2
6. 32 cm 7. 117 m
8. (i) 22.5 cm^2 (ii) $(2\pi - 1.8)$ rad
(iii) $(10\pi - 9)$ cm
9. (i) $\frac{1}{2}r^2\theta = 8, 2r + r\theta = 18$
(ii) $r = 8, \theta = \frac{1}{4}$
10. (i) 15.6 cm (ii) 100 cm^2
11. 292 cm^2
12. (i) $\frac{\pi}{3}$ (ii) 12.6 cm
(iii) 43.5 cm^2
13. (i) 18 cm (ii) 72.1 cm^2
14. (i) 6.4 cm (ii) 19.7 cm^2
15. (i) 6 (ii) 3.80 cm^2
16. (i) 800 cm^2 (ii) 774 cm^2
17. (i) 4 rad (ii) 10.9 m
(iii) 85.6 m^2
18. (i) 4.80 cm (ii) 73.0 cm^2
19. (i) 2.09 rad (ii) 22.1 cm^2
20. 76.8 cm
21. (i) 1.32 (ii) 6.30 cm^2
22. (i) 1.04 (ii) 44.7 cm^2
23. (i) 0.284 rad (ii) 2.83 cm^2
24. (i) 14.4 cm (ii) 19.3 cm^2
(iii) 13.6 cm (iv) 35.8 cm^2
25. (i) $\frac{3}{14}\pi$ (ii) 164 cm^2

Review Exercise 8

1. (i) 50.3 cm (ii) 302 cm^2
2. (i) 60.8 cm^2 (ii) $(2\pi - 1.9)$ rad
(iii) $(16\pi - 15.2)$ cm
3. (i) 0.453 (ii) 3.80 cm^2
4. (i) 1.55 rad (ii) 27.6 cm^2
5. 22%
6. (ii) 2.91×10^{-3}
7. (i) 23.9 cm (ii) 77.7 cm^2
(iii) 98.6 cm^2 (iv) 17.1 cm
8. (i) 4.73 cm, 12.5 cm
(ii) 54.1 cm^2
9. $0.433r^2$
10. $50(\pi - 1)$ cm^2
11. (a) (i) 17.6 cm (ii) 141 cm^2
(iii) 107 cm^2
(b) 80.0 cm^2

Challenge Yourself

1. (a) $(7\pi + 14)$ cm (b) $(7\pi + 14)$ cm
   (c) $(7\pi + 14)$ cm (d) $(7\pi + 14)$ cm
2. (i) 3 (ii) 0.644 rad
   (iii) 6.99 cm^2
3. (ii) 22.1 cm^2
4. (i) 16 (ii) 175 cm^2

REVISION EXERCISE C1

1. 0.352, 2.79
2. (i) 1.9 rad (ii) 60.8 cm^2
3. (i) 13.2 cm (ii) 59.7 cm^2
   (iii) 8.57 cm^2
4. 35.3
5. (i) 5.14 cm (ii) 69.1°
   (iii) 9.89 cm
6. (i) 97.2° (ii) 17.9 cm^2
   (iii) 64.6 cm^2 (iv) 101.4°
   (v) 10.8 cm
7. (i) 005.4° (ii) 078.6°
   (iii) 11.9 km
8. (i) 60° (ii) 10.4 cm
   (iii) 46.1°
9. 56.3 m

REVISION EXERCISE C2

1. 2.09
2. (a) 0.643 (b) –0.966
3. 17.4 cm^2
4. $x = 7.37, y = 6.52$
5. (i) 9.77 cm (ii) 13.1 cm
   (iii) 59.2° (iv) 19.2°
6. (i) 7.81 cm (ii) 33.7°
   (iii) 42.6° (iv) 12.0 cm
   (v) 46.8 cm^2
7. (a) (i) 072° (ii) 108°
   (iii) 036°
   (b) 1.101×10^6 m^2
8. (i) 7.21 cm (ii) 64.3°
   (iii) 76.5°
9. (i) 3.56 m (ii) 10.6 m
   (iii) 70.4° (iv) 30.7°

CHAPTER 9

Exercise 9A

1. (a) (ii) and (vii) (b) (iii) and (v)
   (c) (i) and (ix) (d) (vi) and (viii)
2. (a) P, Q, R; PQ, QR, PR, 6; PQR, SSS
   (b) Z, Y, X; $ZY, Z\hat{Y}X, YX$, 5; ZYX, SAS
   (c) W, V, U; $W\hat{V}U, W\hat{U}V$, 70, VU, 7; NML, AAS
   (d) U, T, S; $U\hat{T}S$, 90, US, TS, 5; ΔIHG, RHS
3. (a) No (b) No
   (c) No (d) No
4. (a) $\Delta ABD \equiv \Delta CBD$
   (b) $\Delta ABD \equiv \Delta CDB$
   (c) $\Delta ABC \equiv \Delta EDC$
   (d) $\Delta ABC \equiv \Delta CDA$
   (e) $\Delta ADE \equiv \Delta CDB$
   (f) $\Delta BCD \equiv \Delta EFD$
   (g) $\Delta ABD \equiv \Delta CBD$
   (h) $\Delta ABC \equiv \Delta CDA$
5. (ii) 4 cm
   (iii) 80°
   (iv) RS is parallel to UV.
6. (i) ΔJIH (ii) 80°
7. (a) $\Delta ABC \equiv \Delta CDA$
   (b) $\Delta EFG \equiv \Delta GHE$
   (c) $\Delta IJK \equiv \Delta KLI$
   (d) $\Delta MNO \equiv \Delta OPM$
   (e) $\Delta QRS \equiv \Delta STQ$
   (f) $\Delta UVW \equiv \Delta WXU$

Exercise 9B

1. (a) (i) and (iii), (v) and (vii)
   (b) (ii) and (vi)
   (c) (iv) and (viii)
2. (a) 60; S, T, U; $T\hat{S}U, S\hat{T}U$, 60; STU, 3, ∠s
   (b) N, M, L; 8, 3; NL, $\frac{21}{7}$, 3; ML, $\frac{15}{5}$, 3; NML, 3, sides
   (c) G, I, H; $G\hat{I}H$, 90; $\frac{6}{9}$, $\frac{2}{3}$, IH, $\frac{4}{6}$, $\frac{2}{3}$; IH; GIH, 2, sides, included
3. (a) No (b) No
   (c) No

4. (a) ΔABC and ΔEDC
   (b) ΔIJH and ΔIFG
   (c) ΔPQR and ΔTSR
   (d) ΔUVW and ΔUXY
5. (a) $x = 9, y = 18$
   (b) $x = 4.8, y = 7.5$
   (c) $x = 12, y = 15$
   (d) $x = 3.2, y = 7.5$
6. (ii) 110.5°
7. $XY = 30$ cm, $WZ = 27$ cm
8. 16.2 cm
9. (i) ΔBAC, ΔDBC and ΔDAB
   (ii) $BC = 6\frac{2}{3}$ m, $CD = 5\frac{1}{3}$ m
10. (i) $RT = 6$ cm, $PR = 12$ cm
    (ii) 2 : 1
11. (ii) ΔART and ΔMNT
    (iii) 3.75 cm

Exercise 9C

3. 8 m
5. Length of AB'
6. 45

Review Exercise 9

1. (a) Yes (b) Yes
   (c) No (d) Yes
2. $\Delta DEF \equiv \Delta JLK$, AAS
3. (a) $\Delta ABC \equiv \Delta DEC$
   $A\hat{C}B = D\hat{C}E, BC = EC, AC = DC$
   (b) $\Delta FGH \equiv \Delta FIJ$
   $F\hat{G}H = F\hat{I}J, FG = FI, FH = FJ$
   (c) $\Delta KLN \equiv \Delta MNL$
   $K\hat{L}N = M\hat{N}L, K\hat{N}L = M\hat{L}N, KL = MN$
   (d) $\Delta SQP \equiv \Delta RPQ$
   $S\hat{Q}P = R\hat{P}Q, SQ = RP, SP = RQ$
   (e) $\Delta EBF \equiv \Delta ECD$
   $E\hat{B}F = E\hat{C}D, BF = CD, EF = ED$
   (f) $\Delta FHG \equiv \Delta FIJ$
   $F\hat{H}G = F\hat{I}J, FH = FI, GH = JI$
4. (a) Yes (b) No
   (c) Yes (d) No
   (e) No (f) Yes
5. (i) $\Delta OAD \equiv \Delta OBC$
   (ii) $A\hat{O}D = B\hat{O}C, O\hat{A}D = O\hat{B}C$

6. (i) $\Delta PQR \equiv \Delta SRQ$, SAS
(ii) 5 cm, 50°

8. (a) $a = 5.92$
(b) $b = 15\frac{5}{7}$, $c = 12\frac{4}{7}$
(c) $d = 2\frac{2}{3}$, $e = 6\frac{3}{4}$

9. (i) ΔRQP (ii) 6.4 cm

10. (a) (i) ΔACE and ΔGFE
(ii) 6 cm
(b) $EG = 8$ cm, $FH = 18$ cm

11. (a) (i) ΔRLN (ii) 8 cm
(b) (i) ΔNMS (ii) 9 cm
(c) ΔPLM and ΔRLQ, ΔPQM and ΔSNM, ΔPQM and ΔRNQ

12. (i) $\Delta STR \equiv \Delta STP$
(ii) $UQ = 7$ cm, $PQ = 10$ cm

Challenge Yourself

2. 3 cm 3. 13.5 cm

CHAPTER 10

Exercise 10A

1. (a) 4 cm^2 (b) 2.4 m^2
(c) 20 cm^2 (d) 108 cm^2
(e) 27 m^2 (f) 6 cm^2

2. 16 : 49

3. (i) $66\frac{2}{3}$ m^2 (ii) $42\frac{2}{3}$ m^2

4. (a) 6 (b) 15
(c) 20 (d) 4

5. 128 m^2 6. 7 cm^2

7. 812.5 m^2 8. $\frac{p^2}{(p+q)^2}$

9. (i) 18 cm (ii) 279 cm^2

10. 3.62 cm 11. 3 : 5

12. 7.5 cm

13. (i) 40 cm^2 (ii) 60 cm^2

14. (i) $2\frac{2}{7}$ cm (ii) 7 : 4
(iii) 64 : 231

15. (i) 50 cm^2 (ii) 12 cm^2
(iii) 30 cm^2

Exercise 10B

1. (a) 576 cm^3 (b) 162 cm^3
(c) 324 cm^3 (d) 38.5 m^3
(e) 0.4 m^3

2. (a) 125 : 64 (b) 27 : 64
(c) 8 : 27

3. 160 cm^3

4. (a) 4 (b) 9
(c) 21 (d) 5

5. (i) 3 : 4 (ii) 189 cm^3

6. 4 : 3 7. 20.8 cm

8. (i) 16 : 49 (ii) 53.6 g

9. 4.76 cm 10. 148 g

11. (i) 4.61 kg (ii) 13 281 l

12. 8.15 m^2

13. (i) 36 cm (ii) 2 : 3
(iii) 1134 cm^2

14. (i) 10 cm (ii) 20 cm^2
(iii) 225 cm^3

15. 3.65

Review Exercise 10

1. (a) 9 : 25 (b) 1 : 4
(c) 4 : 9

2. 49 : 64

3. (ii) 100 cm^2 4. 2 : 3

5. (i) 3 : 4 (ii) 9 : 16

6. 6.144 tonnes 7. 16 : 25

8. (i) 9 : 1 (ii) 400 l

9. (a) ΔSLP
(b) (i) $\frac{1}{4}$ (ii) $\frac{2}{3}$
(iii) $\frac{2}{3}$

10. (a) $\frac{21}{25}$
(b) (i) ΔNOM (ii) $\frac{2}{5}$

11. (ii) $\frac{25}{4}$ (iii) 31.5 cm^2

12. 27 : 10

13. (i) 33.5 cm^3 (ii) 14.1 cm^3
(iii) 232 g

14. (i) 1 : 7 (ii) 7 : 19
(iii) 27 : 37

Challenge Yourself

1. 4 : 9

2. (ii) ΔEAS and ΔTHS, ΔTSA and ΔBSH
(iii) 40 cm
(iv) 1 : 25

CHAPTER 11

Exercise 11A

1. (a) $a = 12, b = 67.4$
(b) $c = 11.0, d = 61.9$
(c) $e = 6, f = 50.2$

2. 15 cm 3. 13 m

4. 13.7 cm

5. (a) $a = 12, b = 90$ (b) $x = 11, y = 90$

6. 18.0 cm^2 7. 17 cm

8. (i) 5.66 cm (ii) 14.2 cm

9. 1 cm or 7 cm 10. 8.39 cm

11. 63.3 cm

Exercise 11B

1. 24°

2. (i) 26° (ii) 122°

3. $45° + \frac{x}{2}$

4. (a) $a = 49, b = 14$
(b) $c = 58, d = 15$
(c) $e = 34, f = 14.8$
(d) $g = 35, h = 55$
(e) $i = 8, j = 67.4$
(f) $k = 12.6, l = 50.0$

5. (i) 44° (ii) 25°

6. (i) 7.5 cm (ii) 67.4°
(iii) 34.4 cm^2

7. 138° 8. 9 cm

9. 7 m 10. 64°

11. 51° 12. 45 cm

Exercise 11C

1. (a) 80 (b) 30
(c) 40 (d) 115
(e) 125 (f) 50
(g) 35 (h) 28

2. **(a)** 50 **(b)** 45
(c) 30 **(d)** 60

3. **(a)** 50 **(b)** 12

4. 60 **5.** 65

6. **(a)** 40 **(b)** 36
(c) 47 **(d)** 130

7. **(i)** 70° **(ii)** 70°

8. 270° **9.** 37°

10. **(i)** 62° **(ii)** 47°

11. **(i)** 64° **(ii)** 64°

12. 70 **13.** 78.5 cm^2

14. 31° **15.** 32°

16. 45° **17.** 125°

18. **(i)** 90° **(ii)** 55°

19. 40°

20. **(i)** 35° **(ii)** 131°

21. **(b)** **(i)** 8 cm **(ii)** 9 cm

22. **(ii)** ΔPCB **23.** 18°

24. $\angle P = 65°$, $\angle Q = 55°$, $\angle R = 60°$

25. **(ii)** $10\frac{1}{6}$ cm

Review Exercise 11

1. **(a)** $x = 50, y = 25$
(b) $x = 34, y = 114$
(c) $x = 28.5, y = 16.5$
(d) $x = 26, y = 38$
(e) $x = 26, y = 148$
(f) $x = 62, y = 118$

2. **(a)** $x = 70, y = 35$
(b) $x = 34, y = 56$
(c) $x = 132, y = 114$
(d) $x = 105, y = 30$
(e) $x = 6.43, y = 25$
(f) $x = 54, y = 72$

3. **(a)** $x = 62, y = 118$
(b) $x = 116, y = 46$
(c) $x = 115, y = 57.5$
(d) $x = 50$
(e) $x = 72, y = 28$
(f) $x = 48, y = 22$

4. **(a)** $x = 41$
(b) $x = 78, y = 30$
(c) $x = 108, y = 144$
(d) $x = 24, y = 42$
(e) $x = 29, y = 59$
(f) $x = 42, y = 90$
(g) $x = 103, y = 45$
(h) $x = 22.5, y = 135$

5. $x + y$ **6.** 20°

7. **(i)** 24° **(ii)** 49°

8. $x = 74, y = 103$

9. **(i)** 90° **(ii)** 110°

10. **(i)** 1 cm **(ii)** $\frac{1}{2}y$

11. **(i)** 4 cm **(ii)** 13.6 cm

12. **(i)** 46° **(ii)** 134°

13. 61° or 119°

14. **(i)** 78° **(ii)** 102°

15. **(i)** 62° **(ii)** 126°

16. Yes

Challenge Yourself

(a) 6 **(b)** 3
(c) 2 **(d)** 2

REVISION EXERCISE D1

1. Yes

2. **(b)** ΔASQ **(c)** 5 cm
(d) ΔBCA, ΔRAQ
(e) **(i)** 16 cm^2 **(ii)** 8 cm^2
(iii) 8 cm^2

3. **(i)** 3 : 200 **(ii)** 360 m^2
(iii) 48 m^3

4. **(i)** 72° **(ii)** 42°

5. **(i)** 128° **(ii)** 88°

6. $u = 20°, v = 70°, w = 30°, x = 70°,$
$y = 130°, z = 80°$

7. **(i)** b **(ii)** $b - a$
(iii) $2b - a$ **(iv)** $180° - 2b$
(v) $180° + a - 3b$

8. **(i)** 283 cm^2 **(ii)** 6
(iii) 251 cm^2

REVISION EXERCISE D2

1. **(ii)** ΔCRB

2. **(i)** $PX = 6.48$ cm, $QX = 7.56$ cm
(ii) 25 : 81

3. **(i)** $66\frac{2}{3}$ cm **(ii)** $45\,511\frac{1}{9}$ cm^3

4. **(i)** 9 cm, $7\frac{7}{16}$ cm **(ii)** 225 : 161

5. **(i)** 50° **(ii)** 100°

6. 157 cm^2

7. **(i)** 70° **(ii)** 69°
(iii) 55°

8. **(i)** 9 cm **(ii)** 1 : 3

PROBLEMS IN REAL-WORLD CONTEXTS

1. **(a)** (310, 0), (155, 82.5)
(c) **(i)** $c = 0,$
$155^2a + 155b = 82.5,$
$310^2a + 310b = 0$
(ii) $a = -\frac{33}{9610}, b = 1\frac{2}{31}, c = 0$
(iii) $y = -\frac{33}{9610}x^2 + 1\frac{2}{31}x$

2. **(a)** S\$312.80
(b) [S\$475 + 0.75 × (EC − 1600)] × 0.782;
S\$2385.10

3. **(i)** Yes

4. **(b)** **(i)** \$810.84 **(ii)** \$2855.70

5. The large watermelon

6. **(ii)** 20 cm **(iii)** 314 cm^2

7. **(i)** 36.8°

New Syllabus Mathematics (NSM) is a series of textbooks where the inclusion of valuable learning experiences, as well as the integration of real-life applications of learnt concepts serve to engage the hearts and minds of students sitting for the GCE O-level examination in Mathematics. The series covers the MOE Syllabus for Mathematics implemented from 2013.

Special Features

- Chapter Opener to arouse students' interest and curiosity
- Learning Objectives for students to monitor their own progress
- Investigation, Class Discussion, Thinking Time, Journal Writing and Performance Task for students to develop requisite skills, knowledge and attitudes
- Worked Examples to show students the application of concepts
- Practise Now for immediate practice
- Similar Questions for teachers to choose questions that require similar application of concepts
- Exercise classified into Basic, Intermediate and Advanced to cater to students with different learning abilities
- Summary to help students consolidate concepts learnt
- Review Exercise to consolidate the learning of concepts
- Challenge Yourself to challenge high-ability students
- Revision Exercises to help students assess their learning after every few chapters

ISBN 978 981 237 932 0

9 789812 379320

shinglee publishers pte ltd