CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Stabilization Of Underactuated Mechanical System Using Adaptive Sliding Mode Control

by

Qaiser Khan

A thesis submitted in partial fulfillment for the degree of Master of Science

in the

Faculty of Engineering
Department of Electrical Engineering

2018

I am dedicating this thesis to my Parents, Family, Friends and Teachers for their endless love, care and inspiration.

CAPITAL UNIVERSITY OF SCIENCE & TECHNOLOGY
ISLAMABAD

CERTIFICATE OF APPROVAL

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MEE-151006

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Acknowledgements

First and foremost I would like to thank Allah almighty, who gave me the courage and support to continue my graduate studies. The determination granted by Allah helped me to tolerate the hard time to produce this thesis. I cannot forget the ideal man of the world and most respectable personality for whom Allah created the whole universe, Prophet Mohammed (Peace Be Upon Him).

I would like to thank to my supervisor, Dr. Fazal-ur-Rehman, with whom it has been an honor to work. He allowed me tremendous freedom in choosing my area of research and gave me outstanding guidance all along the way. His ”outside of the box” thinking has taught me to do the same and for that I am grateful to him.

I am highly grateful to my loving parents and family who kept me motivated, guided and focused throughout my Master. Their help in various regards contributed in keeping my moral high. I extend my heartiest thanks to my colleagues in control system research group, especially Mr. Ibrahim Shah, Mr. Sami ud Din ,Mr. Sarfaraz Ahmad , Mr. Waseem Abbasi and Mr. Nazim Siddique. They always guided me whenever I was in need of that.I also thankful to my hostel roomate Mr. Habibullah Somroo, Mr. Yasir Ahmed and Mr.Haris.

Abstract

In the past decade, there has been increasing interest in underactuated mechanical systems. These systems have many practical and diverse applications in modern science and engineering. The broad application areas of underactuated systems include robotics, industry, and aerospace systems. This thesis presents a simple stabilizing control algorithm for a class of underactuated mechanical systems with n degrees of freedom (n DOF). The methodology is based on adaptive sliding mode control. Firstly, the system is transformed into a special structure through input transformation, containing nominal part plus some unknown term. The unknown term is computed adaptively. Later the transformed system is stabilized using adaptive sliding mode control. The adapted laws are derived in such a way that the time derivative of a Lyapunov function becomes strictly negative. The effectiveness of the proposed method is applied to different underactuated systems with 2 DOF and 3DOF: Inverted pendulum, TORA system, The Pendubot and Acrobot, Ball and Beam system and double inverted pendulum. Computer simulation results show the effectiveness of the proposed control algorithm on these systems.

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Abbreviations

DOF	Degree of Freedom
RP	Reaching Phase
SMC	Sliding Mode Control
TORA	Translational Oscillator with Rotational Actuator
UMS	Underactuated Mechanical System
UMSs	Underactuated Mechanical Systems

Chapter 1

Introduction

1.1 Introduction

In the past decade, there has been increasing interest in underactuated systems. Underactuated mechanical systems are having lesser number of independent control actuators than the degrees of freedom. These systems have diverse applications in modern science and engineering. The broad areas of applications regarding underactuated mechanical systems include robotics and aerospace systems. Apart from practical applications, these systems have been of great importance and interest in research as a prototype framework for complex nonlinear systems. The reasons for the underactuation include dynamics, of the system by nature like aircraft, spacecraft, helicopters, and underwater vehicles, underactuation can be introduced by design for the reduction of weight, this phenomenon can be imposed artificially to create complex low order nonlinear systems for gaining insight in control of high order underactuated systems, such as inverted pendulum system, ball on a beam system, and other systems of such kind.

Considering the applications of the class mentioned above, from the clan of robotics UMSs, includes flexible-link joints, mobile robots, and many other kinds of manipulators. The aforesaid class also incorporate aerial and underwater vehicles such as surface vessels [1] , twin rotor system [2], etc.

Control of UMS is the challenging and active field of research due to the benefits of the underactuation property. There are several control methods available for fully actuated mechanical systems (where a number of control inputs are equal to degree of freedom) which includes, partial feedback linearization collocated [3] and non-collocated [4], passivity [5], robust and adaptive control [6], sliding mode control [7] [8] fuzzy logic [9], and backstepping [10] . Aforesaid strategies are unable to imply on the underactuated mechanical systems, due to non-holonomic nature of such kind of systems (also, they are not fully linearizable) [11].

In this research work stabilization of underactuated mechanical systems is being focused. In this regard, the technique of the adaptive sliding mode control is considered. In this work our emphasis is to bring the system toward equilibrium (considered at the origin) from an initial condition, to perform this task, an appropriate sliding surface is being considered followed by a Lyapunov function. Besides, the presentation of stability analysis is very appealing. Five systems are considered to prove the effectiveness of the proposed technique, which includes TORA system [12] ball and beam balancer [13] , inverted pendulum [14], acrobot [3], pendubot [15] and double inverted pendulum [16].The efficiency of the suggested algorithm is verified through simulation studies using MATLAB.

1.2 Motivation

Importance of underactuated mechanical systems is already established in engineering and military applications due to its applicability toward aerial and underwater vehicles in the sense of weight reduction. Second and principal reason for the flourishing of research in this domain is the provisional of backup control capability in the case of failure/damage of fully actuated system. The aspect of stabilization of underactuated system will always appear likely or unlikely in the scenarios mentioned above, which is discussed in this thesis and becomes the notion for this work.

1.3 Application of Research

If ones look around within a small circle, we live in a world of machines/equipment. These equipment/ machines can be underactuated or fully actuated (considering with the perspective of proposed work). If fully actuated system left working by any means, we have no choice to deal it as underactuated systems, in this regard world is quite rich with the application perspective of the underactuated system starting from our home. If we talk on a larger scale, every industry is equipped with the machinery based on the underactuation phenomenon in some capacity. Moreover, its application also includes spacecraft and marine vehicles.

1.4 Thesis Organization

The outline of this thesis is as follows.

Chapter 2 -Literature Review:

This chapter will give us a review of the literature published about the underactuated mechanical systems. This literature establishes proposed work along with selected examples.

Chapter 3 -Proposed Control Algorithm :

This chapter describes the problem and controllability issue regarding underactuated mechanical systems. It contains the proposed algorithm for the underactuated mechanical system

Chapter 4 -Application To Proposed Control Algorithm:

The efficiency of the suggested control strategy is applied to different underactuated mechanical systems such as TORA system, ball and beam balancer, Pendubot, and Acrobot.

Chapter 5 -Conclusion and Future Work:

This chapter covers the judgment and upcoming work.

Chapter 2

Literature Review

This chapter presents the literature review of underactuated mechanical systems (UMS) controlled via sliding mode. This study will help to decide a control strategy for UMS. The proposed control strategy will ensure that the UMS will remain stable within a specific bound of disturbances.

2.1 Underactuated Mechanical Systems

A system having few independent control actuators than its degree of freedom is said to be UMS. It is an active research field due to broad applications in different disciplines.
Examples of such systems include a mobile robot, helicopter, underactuated manipulator, space robot, spacecraft, surface vessel and an underwater vehicle. Fully actuated systems don't have such challenges as in UMS systems. In the literature, diverse control techniques have been presented, including backstepping as investigated in [17],energy and passivity-built regulator as in [18], hybrid and switched control in [19], and intelligent and fuzzy control as in [20]. It is hard to pinpoint the general concept that permits to conduct a routine investigation for all UMSs because of the variety and extensive research on this topic. Therefore, these systems are dealt on a case to case basis most of the time.

Spong made the first generalization for underactuated systems in [21], where it was proved that UMSs could be partly linearized by feedback (locally). According to actuation variable, he proposed changes in the input that convert nonlinear models into partly linear ones. Nevertheless, the new control comes in both converted subsystems. Seto gave the first classification for UMSs in [22]. He showed the way for generalized forces to be transmitted through degrees of freedom.
Later on, Olfati-Saber [4] gave a second classification for UMS based on some system structural properties like integrable normalized generalized momentums, kinetic symmetry, actuation mode, and interacting inputs. Some of the most recognized works with respect to energy point of view include Astrom [23], Bloch [24], and Spong [25]. In the same way, passivity-built procedures consist in swinging or routing the previous systems, but to bring them to the homoclinic path. Jankovic [26] and Sepulchre [27] also proposed passivity-based control and introduced the systems transformation in a cascaded form. Kolesnichenko [28] also posed such work for Translational Oscillator with Rotational Actuator (TORA) and pendubot. Hauser [13] posed approximate linearization methodology for the ball and beam balancer. Other samples that are largely studied in the name of UMS are Inertia wheel pendulums and cranes. It has been investigated widely because of their extensive use in industry, in [29] and in [30] respectively.

2.2 Examples Underactuated Mechanical Systems

The UMS include Translational Oscillator with Rotational Actuator (TORA) system [31], the beam and ball system [13], the Acrobot [3], the Pendubot [15], the cart-pole system [32], the crane system [33], and the double inverted pendulum [34]. This system will be introduced briefly in touch coming study.

2.2.1 Acrobot and Pendubot

Acrobot [3] and Pendubot [15] are two-link manipulators with a single actuator at elbow and shoulder, respectively. Both manipulators have same equations of motion and are graphically alike. The stabilization of the two-link manipulator to its upright equilibrium point ($q_1 = \frac{\pi}{2}$ *and* $q_2 = 0$) from any initial condition is their control task.

Energy-based control is one of the famous control approach used for stabilization [15]. Lai in [35] gave a complete unified control technique for its stabilization. An impulse momentum approach provided a new idea to swing up control by Albahkali in [36] and Jafari in [37].

2.2.2 Cart-Pole System

This system is a benchmark underactuated system. It is used as a testbed for nonlinear control study. The control objective is to swing the pendulum from its steady downward equilibrium state. ($q_1 = 0$ *and* $q_2 = \pi$) to vertical unbalanced equilibrium point ($q_2 = 0$), while retaining the cart at its original point ($q_1 = 0$). Considerable work has been done in the past via fuzzy control (FC) and energy based strategies for the under consideration cart-pole system in [38].

2.2.3 Ball and Beam System

The ball and beam system [13] consists of a beam able to move up and downward via a motor connected at one end (whereas the other end of the beam is kept fixed). As this beam is made of metal and iron ball is allowed to move freely on it. The control task is to stabilize the ball on the desired position on the beam, starting from an initial condition on the beam. The Lyapunov-based method [39] control works on this system.

2.2.4 Translational Oscillator with Rotational Actuator System (TORA)

The TORA system is a non-linear benchmark example for different control techniques. The system contains an oscillating translational stage and an eccentric

revolving pendulum. To make sure the horizontal displacement $q_1 \to 0$ in the presence of any exterior disturbance is the control task of TORA. [40].
Since our main focus will be on the SMC, therefore, in the next section SMC is introduced briefly.

2.3 Sliding Mode Control

It is a robust non-linear control design methodology with inherent robustness properties against disturbances, perturbations, and parametric variations. Sliding mode control (SMC) is recognized as a discontinuous type of control procedure because of its controller's nature. SMC structure is comparatively easier to design and use. Apart from being a control systems method, it is also employed for the disturbance estimation and rejection.

SMC [41] is a variable structure control systems design procedure. The very fundamental notion of sliding mode control is described in [42]. SMC occurs in two phases namely, sliding phase and reaching phase briefly discussed in [43]. Upcoming subsection explains about the sliding surface design for SMC.

2.3.1 Sliding Surface

For the employment of SMC, at first step, switching surface design is required. The switching surface can also be named as a sliding surface. When the sliding surface is defined, then the above mentioned two phases come in to place in specific order. Reaching phase is attained first, and it is responsible for the attraction of system states from an initial condition on the switching surface. When reaching phased is achieved, and the system lies on the sliding surface, then sliding phase into place, and the system's stats slide towards the origin (equilibrium point) using a discontinuous control action (which also ensures robustness). Fig. 2.1 shows the reaching phase (RP), sliding mode (SM) and sliding surface (SS) in a pictorial way.

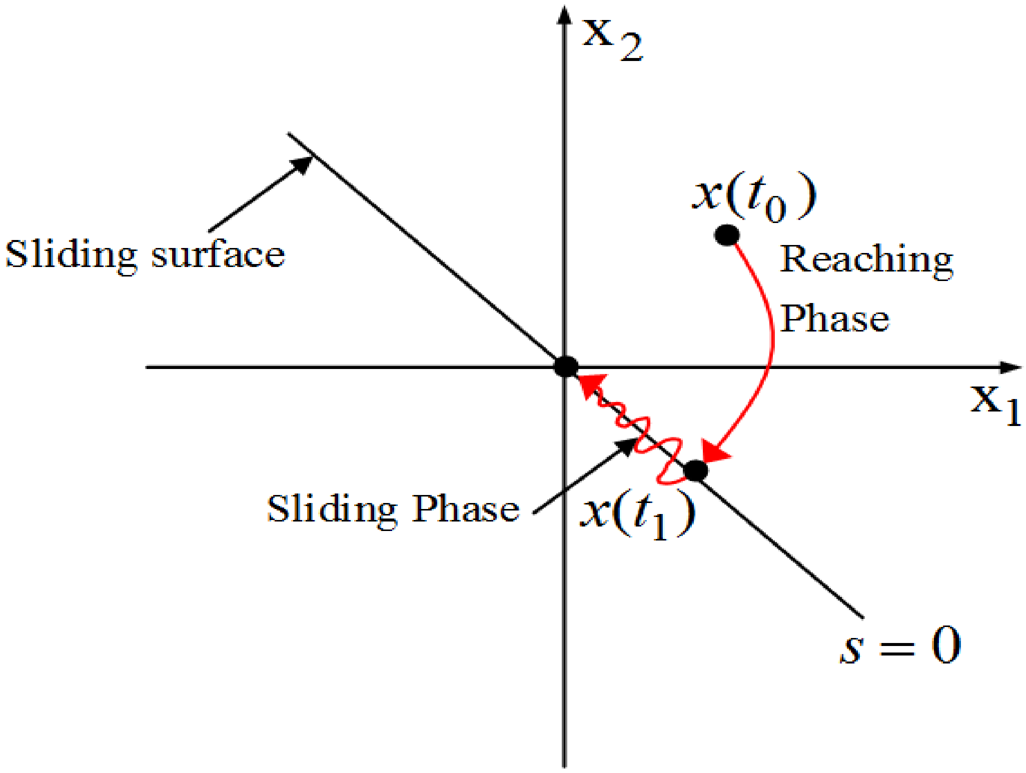


FIGURE 2.1: The Sliding Phase, Reaching Phase and Sliding Surface

2.3.2 Chattering Phenomenon

Due to the discontinuous control, chattering will generate in the system, which is considered to be dangerous for the system's mechanical and electromechanical parts. Such chattering has much adverse effect in real-world applications. This phenomenon may lead to considerable unwanted oscillations that degrade the performance of the system. To avoid chattering effect, various solutions to this problem have been proposed. A new design scheme based on the estimation of sliding variable was presented [44]. The method based on the describing function approach was developed for a chattering analysis of the structure in the existence of the un-modeled dynamics. Another way to reduce chattering effect is using High Order Sliding Mode (HOSM) control techniques.

2.4 Summary

In this chapter literature review of underactuated mechanical systems (UMS) is presented. In a literature review, different techniques were seen that was previously used to stabilize UMSs. Different UMSs were discussed with the control task. The sliding mode control is also discussed.

Chapter 3

Proposed Control Algorithm

This chapter presents a simple stabilizing control algorithm for a class of UMSs with n degrees of freedom (n DOF). The control algorithm is based on adaptive sliding mode control technique. The general approach to the control problem makes it simple and easy to apply to the whole class of systems. Once it is known that a particular system belongs to the considered class and can be written in the general form, the designed control algorithm can be applied to it.

3.1 Dynamical model of under-actuated mechanical systems

(A) - General n degrees of freedom(n DOF)under-actuated mechanical systems: The general dynamical equations of motion for a simple mechanical control system with n degrees of freedom (n DOF) are given by Euler-Lagrange equation as follows [4]:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F(q)u \tag{3.1}$$

In Eq.(3.1), $q \in R^n$ is the generalized configuration vector, $u \in R^m$ is the control input vector and $F(q) = [f_1(q), \dots f_m(q)]^T$ is the matrix of external forces.

where L is the Lagrangian function which is defined by, the difference of kinetic energy "K" and potential energy "V". Its mathematical expression appears as follows.

$$L(q,\dot{q}) = K - V = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q) \tag{3.2}$$

In vector form, the dynamics (3.1) can be written as:

$$M(q)\ddot{q} + C(q,\dot{q})q + G(q) = F(q)u \tag{3.3}$$

For the general case $F(q) = [F_1(q), F_2(q)]^T$ and partitioning $q = [q_1, q_2]^T$ according to $F(q)$, where $q_1 \in R^{n-m}$ and $q_2 \in R^m$, dynamics (3.3) can be written as [4]:

$$\begin{aligned} m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + c_1 + g_1 &= F_1(q)u \\ m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + c_2 + g_2 &= F_2(q)u \end{aligned} \tag{3.4}$$

M(q)= $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ is the symmetrical positive definite inertia matrix, $c_1(q,\dot{q}) \in R^{(n-m)}$ and $c_2(q,\dot{q}) \in R^m$ are the centrifugal and coriolis terms, $g_1(q) \in R^{n-m}$ and $g_2(q) \in R^m$ are the gravitational terms, and $u \in R^m$ is the vector of control inputs produced by m actuators. For $F_1(q) = 0$ and $F_2(q) = 1$, one may have

$$\begin{aligned} m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + c_1 + g_1 &= 0 \\ m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + c_2 + g_2 &= u \end{aligned} \tag{3.5}$$

Where as for $F_1(q) = 1$ and $F_2(q) = 0$, one may have

$$\begin{aligned} m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + c_1 + g_1 &= u \\ m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + c_2 + g_2 &= 0 \end{aligned} \tag{3.6}$$

In this thesis we are considering both the cases when $F_1(q) = 0$ and $F_2(q) = 1$, $F_1(q) = 1$ and $F_2(q) = 0$. Solving the first system in (3.5) for $\ddot{q_1}$ and $\ddot{q_2}$, and then substituting the result in the second equation, (3.5) can be written as:

$$\bar{m_{11}}\ddot{q}_1 + \bar{c_1} + \bar{g_1} = u \tag{3.7a}$$

$$\bar{m_{22}}\ddot{q}_2 + \bar{c_2} + \bar{g_2} = u \tag{3.7b}$$

where

$$\begin{aligned}
\bar{m}_{11}(q) &= m_{21} - m_{22}m_{12}^{-1}m_{11} \\
\bar{c}_1(q,\dot{q}) &= c_2 - m_{22}m_{12}^{-1}c_1 \\
\bar{g}_1(q) &= g_2 - m_{22}m_{12}^{-1}g_1 \\
\bar{m}_{22}(q) &= m_{22} - m_{21}m_{11}^{-1}m_{12} \\
\bar{c}_2(q,\dot{q}) &= c_2 - m_{21}m_{11}^{-1}c_1 \\
\bar{g}_2(q) &= g_2 - m_{21}m_{11}^{-1}g_1
\end{aligned} \tag{3.8}$$

Since $q_1 \in R^{n-m}$ and $q_2 \in R^m$, dynamics (3.7) is a set of two second order systems in state variables. The state space representation of (3.7) can be written as [28]

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_1 + b_2(x)u \\
&\vdots \\
\dot{x}_{2n-1} &= x_2 \\
\dot{x}_{2n} &= f_n + b_n(x)u
\end{aligned} \tag{3.9}$$

Here $x = [x_1, x_2 \ldots, x_{2n-1}, x_{2n}]^T$ is the state vector $f_i(x)$ and $b_i(x)$, $i = 1, 2, ..., n$ are the nonlinear functions of the states and u is the single control input. For $n = 2$, the equation (3.9) can be written as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_1 + b_2(x)u
\end{aligned} \tag{3.10}$$

The state space models of the Pendubot and single inverted pendulum systems are represented by (3.10). For $n = 3$ the equation (3.9) gives:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_1 + b_1(x)u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= f_2 + b_2(x)u \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= f_3 + b_3(x)u\end{aligned} \tag{3.11}$$

The state space model of the parallel double inverted pendulum system is represented by (3.11). The problem formulation is almost complete. In the next section the control design will be presented.

3.2 The control problem

The control problem is to find U such that all the states of the system converge to zero. For set point regulation of 2DOF underactuated mechanical systems we take $x_{des} = [0, 0, 0, 0]^T$ which can be achieved by an appropriate translation of the frame of reference.

3.3 The proposed algorithm for general case

In this section we aim to present the control strategy for a general case. This wil be done in a series of steps.

Step 1:

The system (3.9) written into the following form.

$$\begin{aligned} \ddot{x}_1 &= f_1 + b_1(x)u \\ \ddot{x}_3 &= f_2 + b_2(x)u \\ &\vdots \\ \ddot{x}_{2n-1} &= f_n + b_n(x)u \end{aligned} \tag{3.12}$$

By defying $Y = [x_1\ x_3 \ldots x_{2n-1}]^T,\ \dot{Y} = [\dot{x}_1\ \dot{x}_3\ \ldots \dot{x}_{2n-1}]^T,\ \ddot{Y} = [\ddot{x}_1\ \ddot{x}_3\ \ldots \ddot{x}_{2n-1}]^T$ $F(x) = [f_1(x)\ f_2(x)\ \ldots f_n(x)]^T, B(x) = [b_1(x)\ b_2(x)\ \ldots b_n(x)]^T$, the system (3.12) may be written as:

$$\ddot{Y} = F(x) + B(x)u \tag{3.13}$$

Step 2:

By adding and subtracting $v = [0\ u_2\ \ldots u_n]^T$ in (3.13) on may get

$$\ddot{Y} = F(x) + G(x)w - v \tag{3.14}$$

where, G(x)= $\begin{bmatrix} b_1 & 0 & 0 & \ldots & 0 \\ b_2 & 1 & 0 & \ldots & 0 \\ b_3 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \\ b_n & 0 & 0 & 0 & 1 \end{bmatrix}$ such that $G^{-1}(x)$ exist and , w = $\begin{bmatrix} u \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$.

Notethat $v = [0\ u_2\ \ u_3\ \ \ldots u_n]^T$ is unknown inputs vector and can be computed adaptively. Let $\hat{v}$ be the estimate of v and let $\tilde{v} = v - \hat{v}$ be the error of estimation. Then system (3.14) becomes

$$\ddot{Y} = F(x) + G(x)w - \tilde{v} - \hat{v} \tag{3.15}$$

Step 3:

Choose a sliding surface: S = $\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \\ \vdots \\ x_{n-1} + x_n \end{bmatrix} = Y + \dot{Y}$

Then $\dot{S} = \dot{Y} + \ddot{Y} = \dot{Y} + F(x) + G(x)w - \tilde{v} - \hat{v}$

By choosing $w = -\{G^{-1}(x)\dot{Y} + F(x) - \hat{v} - KS - Ksign(S)\}$

$K = diag\{k_1, k_2, ..., k_n\}, k_i, > 0, i = 1, 2, ..., n$, one has

$$\dot{S} = -KS - Ksign(S) - \tilde{v} \tag{3.16}$$

Step 4:

Choose a Lyapunov function $V = \frac{1}{2}S^T S + \frac{1}{2}\tilde{v}^T \Gamma^{-1} \tilde{v}$, where Γ is $n \times n$ diagonal positive definite matrix. Then

$\dot{V} = S^T \dot{S} + \tilde{v}^T \Gamma^{-1} \dot{\tilde{v}} = S^T\{-KS - Ksign(S) - \tilde{v}\} + \tilde{v}^T \Gamma^{-1} \dot{\tilde{v}}$

$\dot{V} = -KS^T S - K|S| + \tilde{v}^T\{\Gamma^{-1}\dot{\tilde{v}} - S\}$

By using

$\dot{\tilde{v}} = \Gamma S$

$\dot{\hat{v}} = -\Gamma S$

We have

$\dot{V} = S^T \dot{S} + \tilde{v}^T \dot{\tilde{v}} = S^T\{-KS - Ksign(S) - \tilde{v}\} + \tilde{v}^T \Gamma^{-1} \dot{\tilde{v}}$

$\dot{V} = S^T\{-KS - Ksign(S) - \tilde{v}\} + \tilde{v}^T\{\Gamma^{-1}\dot{\tilde{v}} - S\}$

$$\dot{V} = -KS^T S - K|S| \leq 0 \tag{3.17}$$

Eq.(3.17) confirms that $S \to 0$ in finite time. Since S = $\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \\ \vdots \\ x_{n-1} + x_n \end{bmatrix} \to 0$

and S is Hurwitz

therefore $x_i \to 0$, for $i = 1, ...n$

3.3.1 Proposed algorithm for the case when $n = 2$;

For $n = 2$ the system (3.9) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_1 + b_1(x)u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= f_1 + b_2(x)u \end{aligned} \tag{3.18}$$

Step 1:

Write the system (3.17) as:

$$\begin{aligned} \ddot{x}_1 &= f_1 + b_1(x)u \\ \ddot{x}_3 &= f_2 + b_2(x)u \end{aligned} \tag{3.19}$$

By defying $Y = [x_1 \ x_3]^T, \ \dot{Y} = [\dot{x}_1 \ \dot{x}_3\,]^T, \ \ddot{Y} = [\ddot{x}_1 \ \ddot{x}_3\,]^T$
$F(x) = [f_1(x) \ f_2(x)\,]^T, B(x) = [b_1(x) \ b_2(x)\,]^T$, the system (3.18) can be written as:

$$\ddot{Y} = F(x) + B(x)u \tag{3.20}$$

Step 2:

By adding and subtracting $v = [0 \ u_2]^T$ in (3.19) we have

$$\ddot{Y} = F(x) + G(x)w - v \tag{3.21}$$

Where, G(x)= $\begin{bmatrix} b_1 & 0 \\ b_2 & 1 \end{bmatrix}$, w = $\begin{bmatrix} u \\ u_2 \end{bmatrix}$ and $G^{-1}(x)$ exist.

Assume $v = [0 \ u_2]^T$ is unknown input vector.Then system (3.19) becomes:

$$\ddot{Y} = F(x) + G(x)w - \tilde{v} - \hat{v} \tag{3.22}$$

Choose a sliding surface: S = $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \end{bmatrix}$=$Y + \dot{Y}$

Then $\dot{S} = \dot{Y} + \ddot{Y} = \dot{Y} + F(x) + G(x)w - \tilde{v} - \hat{v}$

By choosing $w = -\{G^{-1}(x)\dot{Y} + F(x) - \hat{v} - KS - Ksign(S)\}$ where,

$K = diag\{k_1, k_2\}, k_1k_2, > 0$, one has

$$\dot{S} = -KS - Ksign(S) - \tilde{v} \tag{3.23}$$

Step 4:

Choose a Lyapunov function $V = \frac{1}{2}S^TS + \frac{1}{2}\tilde{v}^T\Gamma^{-1}\tilde{v}$, where Γ is 2×2 diagonal positive matrix. Then

$\dot{V} = S^T\dot{S} + \tilde{v}^T\dot{\tilde{v}} = S^T\{-KS - Ksign(S) - \tilde{v}\} + \tilde{v}^T\Gamma^{-1}\dot{\tilde{v}}$

$\dot{V} = -KS^TS - K|S| + \tilde{v}^T\{\Gamma^{-1}\dot{\tilde{v}} - S\}$

By using

$\dot{\tilde{v}} = \Gamma S$

$\dot{\hat{v}} = -\Gamma S$

We have

$\dot{V} = S^T\dot{S} + \tilde{v}^T\dot{\tilde{v}} = S^T\{-KS - Ksign(S) - \tilde{v}\} + \tilde{v}^T\Gamma^{-1}\dot{\tilde{v}}$ $\dot{V} = S^T\{-KS - Ksign(S) - \tilde{v}\} + \tilde{v}^T\{\Gamma^{-1}\dot{\tilde{v}} - S\}$

$\dot{V} = -KS^TS - K|S| \leq 0$

From this we conclude that $S \rightarrow 0$. Since S = $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \end{bmatrix} \rightarrow 0$ and S is Hurwitz therefore $x_i \rightarrow 0$, for $i = 1, ...4$

3.3.2 Proposed algorithm for the case when $n = 3$

For $n = 3$ the system (3.9) becomes as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2(x)u \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= f_3 + b_3(x)u
\end{aligned} \tag{3.24}$$

Write the system (3.23) as:

$$\begin{aligned}
\ddot{x}_1 &= f_1 + b_1(x)u \\
\ddot{x}_3 &= f_2 + b_2(x)u \\
\ddot{x}_5 &= f_3 + b_3(x)u
\end{aligned} \tag{3.25}$$

By defying $Y = [x_1\ x_3 \ldots x_5]^T,\ \dot{Y} = [\dot{x}_1\ \dot{x}_3\ \ldots \dot{x}_5]^T,\ \ddot{Y} = [\ddot{x}_1\ \ddot{x}_3\ \ldots \ddot{x}_5]^T$ $F(x) = [f_1(x)\ f_2(x)\ \ldots f_3(x)]^T, B(x) = [b_1(x)\ b_2(x)\ \ldots b_3(x)]^T$, the system (3.24) can be written as:

$$\ddot{Y} = F(x) + B(x)u \tag{3.26}$$

Step 2:

By adding and subtracting $v = [0\ u_2\ \ldots u_3]^T$ in (3.24) we have

$$\ddot{Y} = F(x) + G(x)w - v \tag{3.27}$$

Where, G(x)= $\begin{bmatrix} b_1 & 0 & 0 \\ b_2 & 1 & 0 \\ b_3 & 0 & 1 \end{bmatrix}$, w = $\begin{bmatrix} u \\ u_2 \\ u_3 \end{bmatrix}$ and $G^{-1}(x)$ exist.

$v = [0\ u_2\ \ u_3\,]^T$ is unknown input vector. Then system (3.26) becomes:

$$\ddot{Y} = F(x) + G(x)w - \tilde{v} - \hat{v} \tag{3.28}$$

Step 3:

Choose a Sliding surface: S = $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \\ x_5 + x_6 \end{bmatrix}$=$Y + \dot{Y}$

Then $\dot{S} = \dot{Y} + \ddot{Y} = \dot{Y} + F(x) + G(x)w - \tilde{v} - \hat{v}$

By choosing $w = -\{G^{-1}(x)\dot{Y} + F(x) - \hat{v} - KS - Ksign(S)\}$ where,

$K = diag\{k_1, k_2, k_3\}, k_1k_2k_3, > 0$, we have

$$\dot{S} = -KS - Ksign(S) - \tilde{v} \tag{3.29}$$

Step 4:

Choose a Lyapunov function $V = \frac{1}{2}S^TS + \frac{1}{2}\tilde{v}^T\Gamma^{-1}\tilde{v}$, where Γ is 3×3 diagonal positive matrix. Then

$\dot{V} = S^T\dot{S} + \tilde{v}^T\dot{\tilde{v}} = S^T\{-KS - Ksign(S) - \tilde{v}\}$+$\tilde{v}^T\Gamma^{-1}\dot{\tilde{v}}$

$\dot{V} = -KS^TS - K|S| + \tilde{v}^T\{\Gamma^{-1}\dot{\tilde{v}} - S\}$

By using

$\dot{\tilde{v}} = \Gamma S$

$\dot{\hat{v}} = -\Gamma S$

We have

$\dot{V} = S^T\dot{S} + \tilde{v}^T\dot{\tilde{v}} = S^T\{-KS - Ksign(S) - \tilde{v}\}$+$\tilde{v}^T\Gamma^{-1}\dot{\tilde{v}}$

$\dot{V} = S^T\{-KS - Ksign(S) - \tilde{v}\} + \tilde{v}^T\{\Gamma^{-1}\dot{\tilde{v}} - S\}$

$\dot{V} = -KS^TS - K|S| \leq 0$

From this we conclude that $S \to 0$. Since S = $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \\ x_5 + x_6 \end{bmatrix} \to 0$ and S is Hurwitz

therefore $x_i \to 0$, for $i = 1, ...6$

3.4 Summary

In this chapter dynamical model of an underactuated mechanical system is presented. Different mathematical terms were discussed related to UMSs. The proposed algorithm is explained for a general case. Also, the proposed method is explained for the 2DOF and 3DOF UMSs. In the next chapter, the proposed algorithm will be applied on different UMSs.

Chapter 4

Applications Of Proposed Algorithm

In this chapter, different underactuated mechanical systems are considered to verify the proposed control strategy. These systems are supposed to have 2 DOF and 3 DOF, i.e., Inverted pendulum, TORA system, Ball, and Beam system, The Pendubot, Acrobat, and Double inverted pendulum. Computer simulation is carried out to show the validity of the proposed control algorithm.

4.1 The Inverted Pendulum System with a nonlinear spring

The proposed control scheme is now used to stabilize an inverted pendulum with a nonlinear spring as considered in [14]. The system consists of two masses m_1 and m_2. The mass m_1 on a horizontal surface and an inverted pendulum m_2 supported by a massless rod shown in fig. 4.1. At one end the mass is connected to the wall, and at the other end, it is connected to the inverted pendulum. The displacement of mass m_1 is x whereas from the vertical the angle of the pendulum is θ. Both

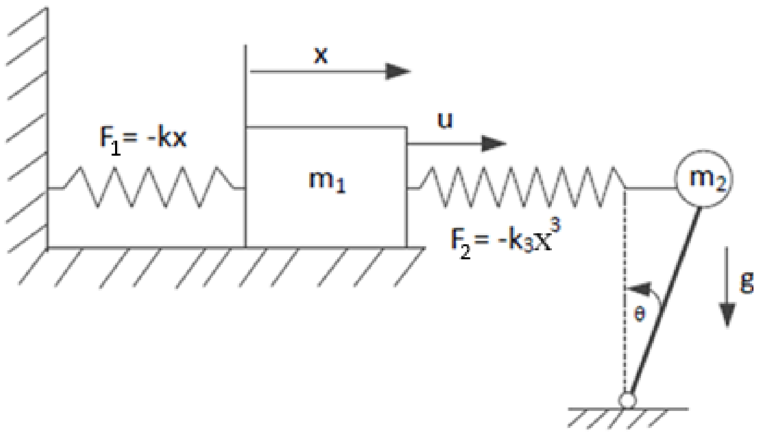


FIGURE 4.1: The inverted pendulum with a nonlinear spring

linear and nonlinear springs are unstretched at $x = 0$ and $\theta = 0$. The control force acts on mass m_1 whereas m_2 is passively controlled.

The dynamic of the system given in [14] appears as follows:
$\ddot{\theta} = \frac{g}{l}\sin\theta + \frac{k}{s}(x - \sin\theta)^3\cos\theta$
$\ddot{x} = \frac{k}{m_1} - \frac{k_s}{m_1}(x - l\sin\theta)^3\frac{u}{m_1}$
which can be transformed into the following system [14]

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \frac{g}{l}\sin x_1 + x_3^3\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= u\end{aligned} \tag{4.1}$$

where x_1, x_2, x_3 and x_4 are the state variables defined as; $x_1 = x, x_2 = \dot{x}, x_3 = \theta$, and $x_4 = \dot{\theta}$, u is the control input, g is the acceleration due gravity, and l is the pendulum's length.By choosing $w = -\{G^{-1}(x)\dot{Y} + F(x) - \hat{v} - KS - Ksign(S)\}$ System (4.1) can also be written as:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= f_1(x)\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= u\end{aligned} \tag{4.2}$$

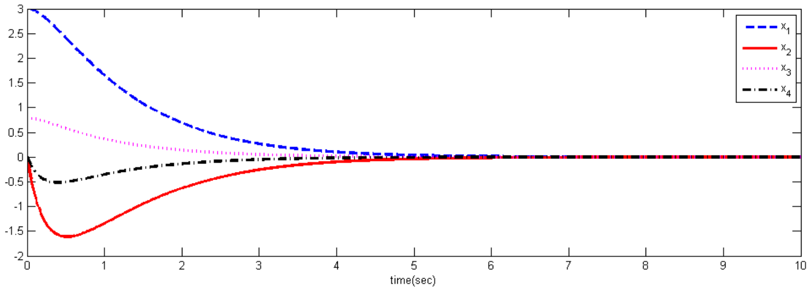
where $f_1(x) = x_3^3 + \frac{g}{l}\sin x_1$ Apply the above algorithm on this system. In simulations we used $g = 9.8$ and $l = 1$

For two different initial conditions fig. 4.2 and fig. 4.3 show the simulation results.

4.1.1 Results and Discussion

The simulation of the inverted pendulum with a nonlinear spring is done for the different initial condition. The time history of angle $\theta(t)$, angular velocity $\dot{\theta}$, the displacement $x(t)$ and velocity $\dot{x}(t)$ of the controlled system are shown in fig 4.2(a) and fig 4.3(a). For different initial condition both the figure show that the system asymptotically approach to zero. From fig 4.2(b) and fig 4.3(b) it can be seen that the control force exponentially decay as time increases. So the closed loop system is exponentially stable.

shows the simulation result When the disturbance is added to the equation (4.2), it is seen clearly that all the states of system convergence to zero as shown in fig. 4.4.



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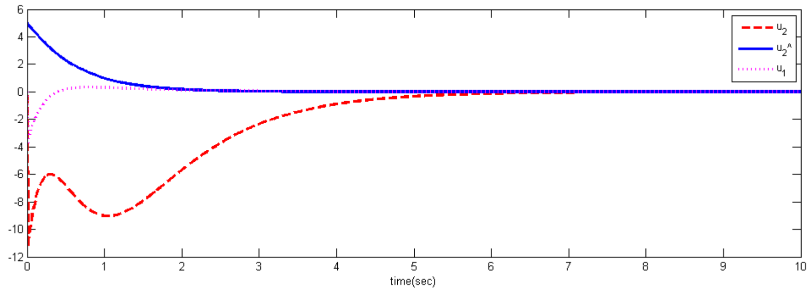

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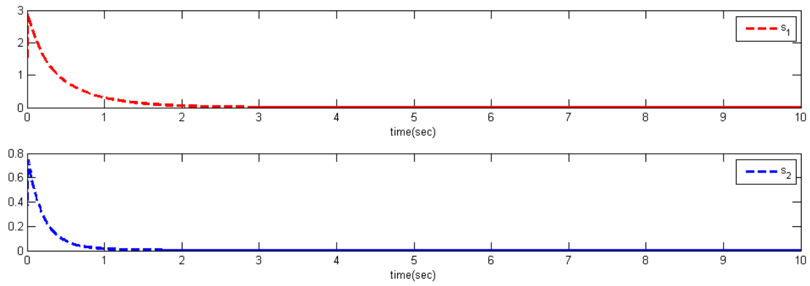
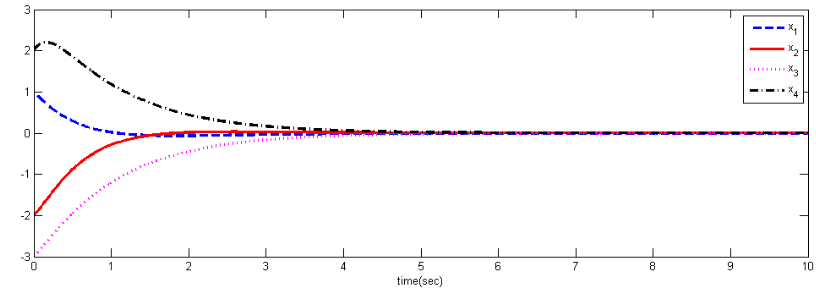
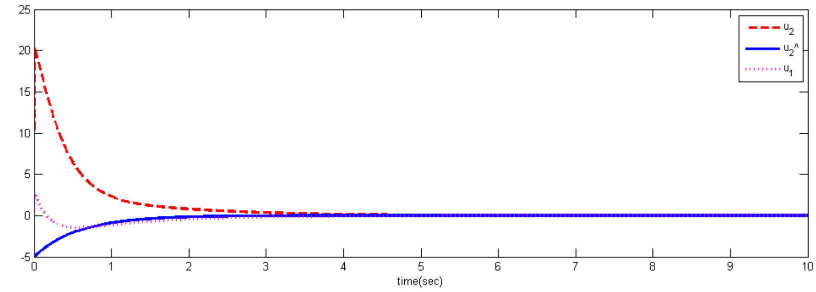


(c)

FIGURE 4.2: *Closed loop response of the Inverted Pendulum system with a nonlinear spring ,(a) Time response of system states corresponding to initial condition* $(x_1(0), ..., x_4(0)) = (3, 0, \frac{\pi}{4}, 0)$ *(b) Time response of* u_2, $\hat{u}_2$ *and control effort* $u = u_1$ *(c)Time history of the sliding surfaces* s_1 *and* s_2



(a)



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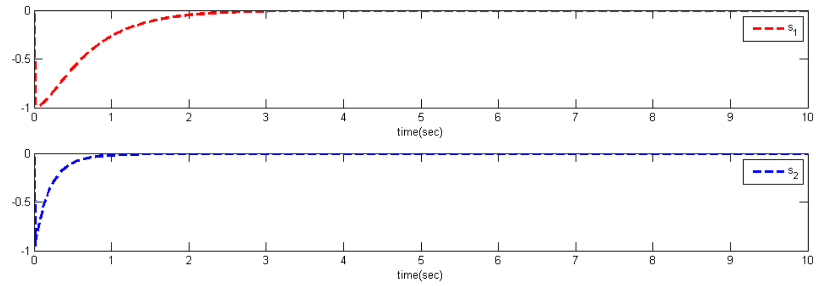
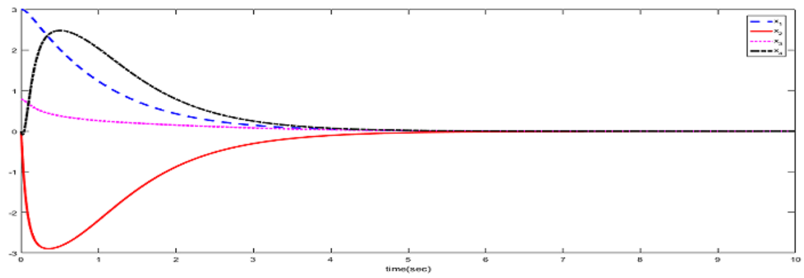
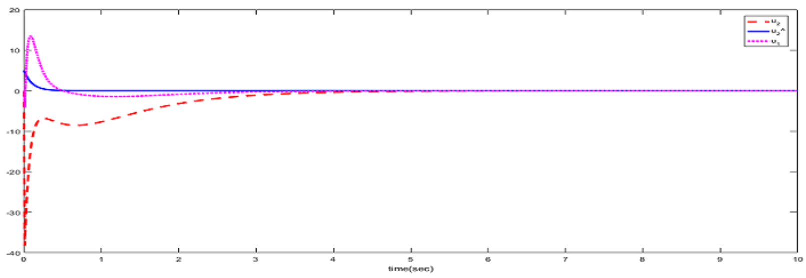


(c)

FIGURE 4.3: *Closed loop response of the Inverted Pendulum system with a nonlinear spring ,(a) Time response of system states corresponding to initial condition* $(x_1(0), ..., x_4(0)) = (1, -2, -3, 2)$*, (b) Time response of* u_2*,* $\hat{u}_2$ *and control effort* $u = u_1$*, (c)Time history of the sliding surfaces* s_1 *and* s_2



(a)



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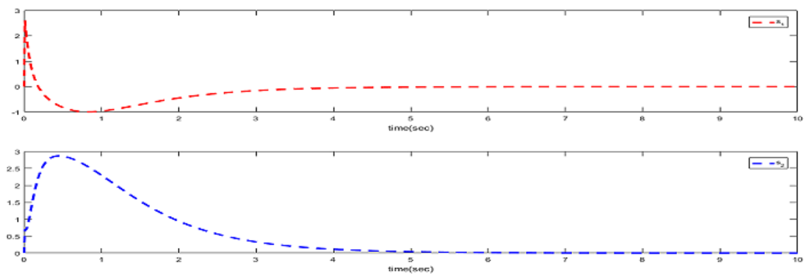


(c)

FIGURE 4.4: *Closed loop response of the Inverted Pendulum system with a nonlinear spring ,(a) Time response of system states corresponding to initial condition* $(x_1(0), ..., x_4(0)) = (3, 0, \frac{\pi}{4}, 0)$ *(b) Time response of* u_2, $\hat{u}_2$ *and control effort* $u = u_1$ *(c)Time history of the sliding surfaces* s_1 *and* s_2

4.2 Translational Oscillator with Rotational Actuator (TORA) System

The system as shown in the following figure 4.5 has been considered as a benchmark system to estimate the performance of numerous nonlinear systems [26]. To make sure the horizontal displacement $q_1 \to 0$ in the occurrence of any exterior disturbance is the control task of TORA. The dynamic of the TORA system as given in [45] is:

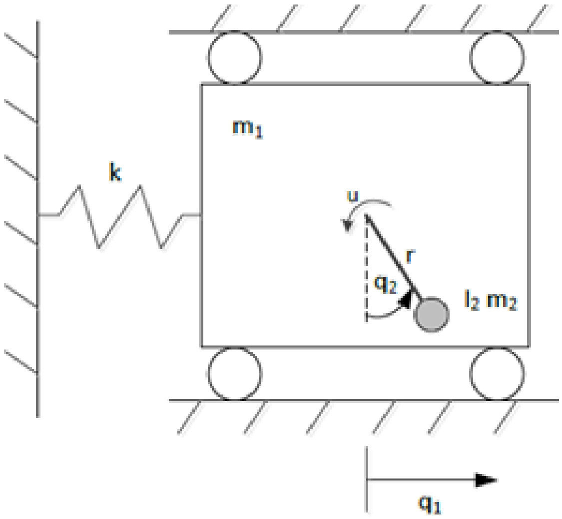


FIGURE 4.5: TORA System

$$\begin{aligned}
\dot{q}_1 &= p_1 \\
\dot{p}_1 &= \frac{-q_1 + \epsilon p_2^2 \sin q_2}{1 - \epsilon^2 \cos^2 q_2} - \frac{\epsilon \cos q_2}{1 - \epsilon^2 \cos^2 q_2} + v \\
\dot{q}_2 &= p_2 \\
\dot{p}_2 &= \frac{\epsilon \cos q_2 (q_1 - \epsilon p_2^2 sin q_2)}{1 - \epsilon^2 \cos^2 q_2} - \frac{1}{1 - \epsilon^2 \cos^2 q_2} + v
\end{aligned} \tag{4.3}$$

where q_1 be the normalized displacement of the platform from the equilibrium position, $\dot{q}_1 = p_1$, q_2 be the angle of the rotor and $\dot{q}_2 = p_2$, v is the control input and ϵ a constant parameter. Using the following coordinate transformation as given in [45]:

$$
\begin{aligned}
x_1 &= q_1 + \epsilon \sin q_2 \\
x_2 &= p_1 + \epsilon p_2 \cos q_2 \\
x_3 &= q_2 \\
x_4 &= p_2
\end{aligned}
\tag{4.4}
$$

The TORA

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \epsilon \sin x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u
\end{aligned}
\tag{4.5}
$$

The System (4.5) can be further modified as:

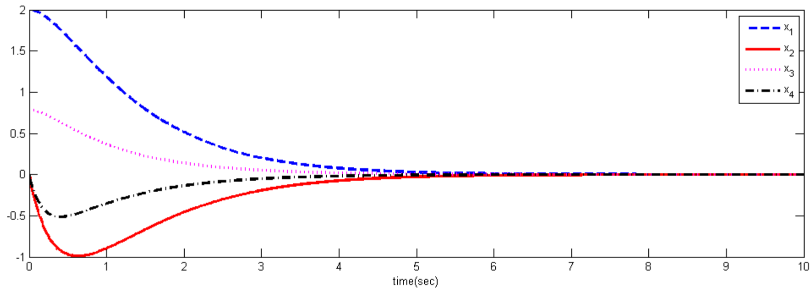
$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1(x) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u
\end{aligned}
\tag{4.6}
$$

where $f_1(x) = -x_1 + \epsilon \sin x_3$. By choosing $w = -\{G^{-1}(x)\dot{Y} + F(x) - \hat{v} - KS - Ksign(S)\}$ Now apply the proposed algorithm on this system. In simulation we used $\epsilon = 2$ and $f_1(x) = -x_1 + \epsilon \sin x_3$. Simulation results are shown in fig. 4.6 and 4.7 for two different initial conditions.

4.2.1 Results and Discussion

The simulation of the TORA system is done for the different initial condition. The time history of angle $q_2(t)$, angular velocity $\dot{q}_2(t)$, the displacement $q_1(t)$ and velocity $\dot{q}_1(t)$ of the controlled system are shown in fig 4.5(a) and fig 4.6(a). For different initial condition both the figure show that the system asymptotically approach to zero. From fig 4.6(b) and fig 4.7(b) it can be seen that the control force exponentially decay as time increases. So the closed loop system is exponentially stable. Also fig 4.6(c) and fig 4.7(c) shows that the sliding surfaces s_1 and s_2 are asymptotically stable.

Fig. 4.8 shows the simulation result when the disturbance is added to the equation (4.6). It is seen clearly that all the states of system convergence to zero.



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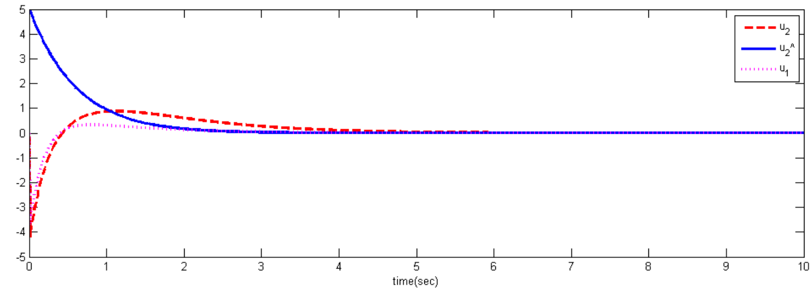

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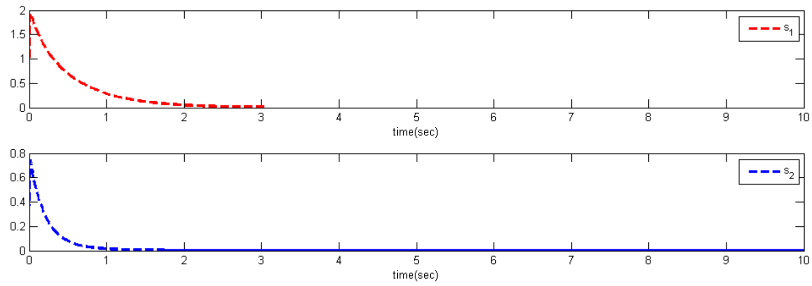
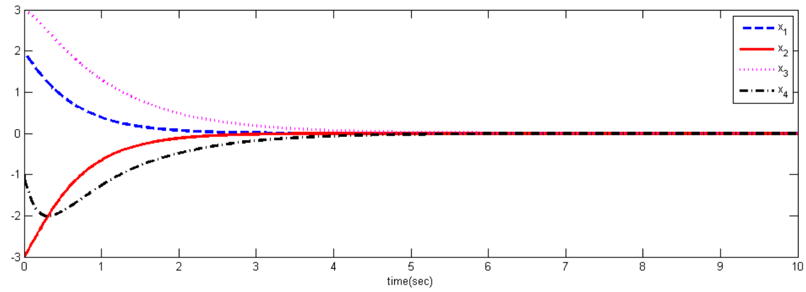


(c)

FIGURE 4.6: *Closed loop response of the TORA system ,(a) Time response of system's states corresponding to initial condition* $(x_1(0), ..., x_4(0)) = (2, 0, \frac{\pi}{4}, 0)$, *(b) Time response of* u_2, $\hat{u}_2$ *and control effort* $u = u_1$, *(c) Time history of the sliding surfaces* s_1 *and* s_2



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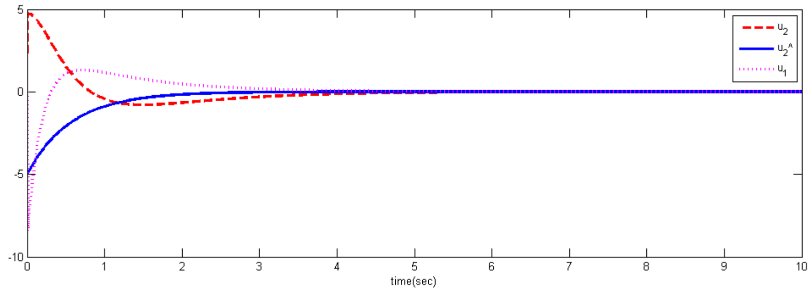

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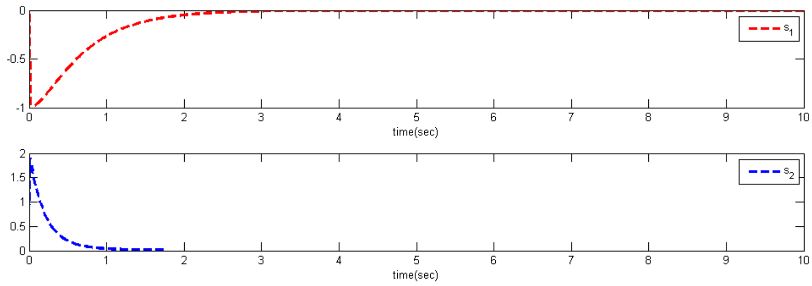
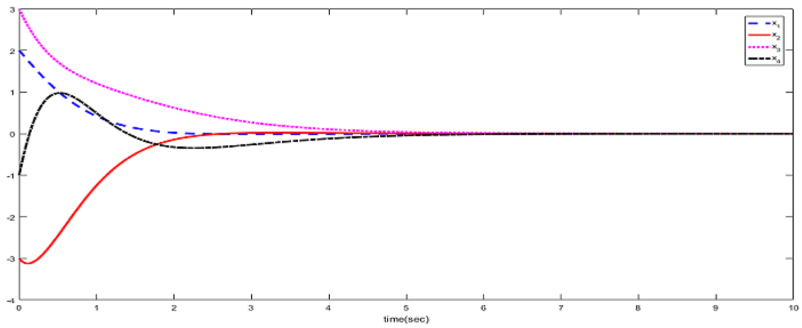
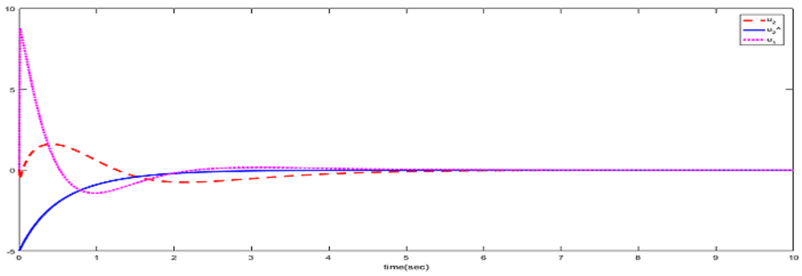


(c)

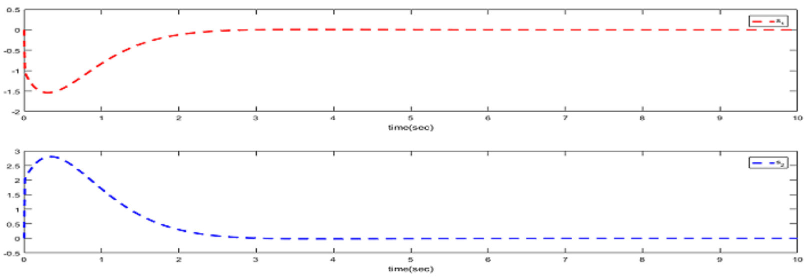
FIGURE 4.7: *Closed loop response of the TORA System , (a) Time response of system's states corresponding to initial condition* $(x_1(0), ..., x_4(0)) = (2, -3, 3, -1)$*, (b) Time response of* u_2*,* $\hat{u}_2$ *and control effort* $u = u_1$*, (c) Time history of the sliding surfaces* s_1 *and* s_2



(a)



(b)



(c)

FIGURE 4.8: *Closed loop response of the Inverted Pendulum system with a nonlinear spring ,(a) Time response of system states corresponding to initial condition* $(x_1(0), ..., x_4(0)) = (3, 0, \frac{\pi}{4}, 0)$ *(b) Time response of* u_2, $\hat{u}_2$ *and control effort* $u = u_1$ *(c)Time history of the sliding surfaces* s_1 *and* s_2

4.3 The Ball and Beam System

The system is shown in the following fig 4.9. The mathematical model describing the ball and beam system as given in [8]. The control task is to stabalize the ball on the desired position on the beam, starting from any initial condition on the beam.

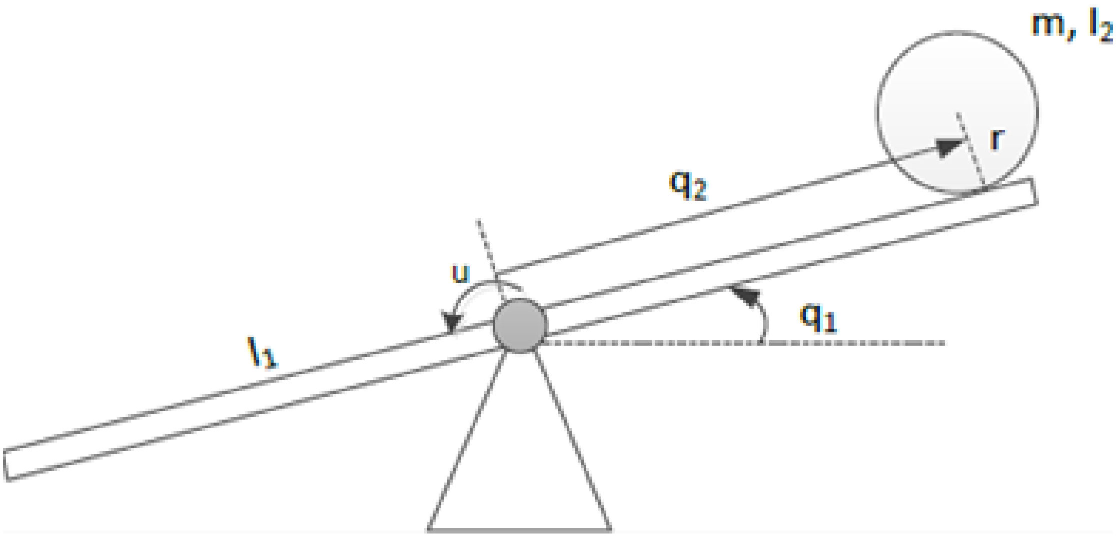


FIGURE 4.9: Ball and Beam System

$$
\begin{aligned}
&(mq_2^2 + k_1)\ddot{q}_1 + (2mq_2\dot{q}_2) + (mgq_2 + \frac{1}{2}LMg)\cos q_1 = x_2 \\
&k_4\ddot{q}_2 - q_2\dot{q}_1 + g\sin q_1 = 0
\end{aligned} \tag{4.7}
$$

By defining the state variables as: $x_1 = q_2, x_2 = \dot{q}_2, x_3 = q_1, x_4 = \dot{q}_1$, the system as given in (4.7) can be written as:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{k_4}(x_1x_4^2 - g\sin x_3) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{1}{mx_1^2 + k_1}[(2mx_1x_2 + k_2)x_4 - \frac{1}{k_4}(mgx_1 + \frac{1}{2}Mg)\cos x_3]
\end{aligned} \tag{4.8}
$$

By choosing $u = \bar{u}(mx_1^2 + k_1)[(2mx_1x_2 + k_2)x_4 + \frac{1}{k4}(mgx_1 + \frac{1}{2}Mg)\cos x_3]$, where $\bar{u}$ is the new input, then system (4.8) can be written as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{k4}(x_1x_4^2 - g\sin x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \bar{u} \end{aligned} \tag{4.9}$$

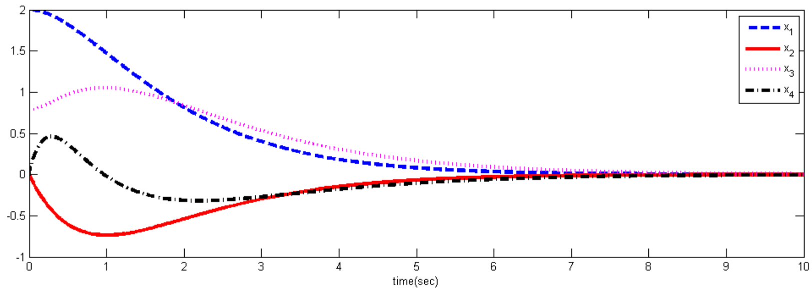
The system (4.9) can be written as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_1(x) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= v \end{aligned} \tag{4.10}$$

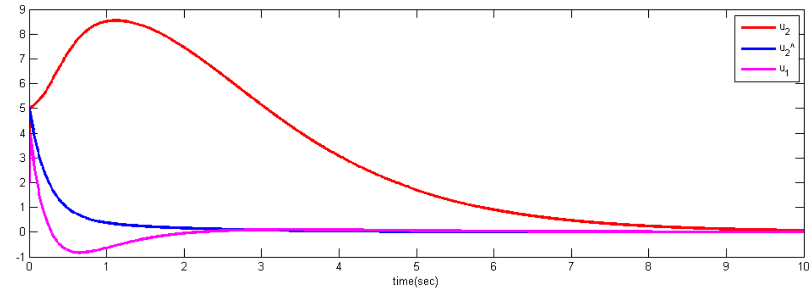
The proposed algorithm is applied on this system. In simulation it is used $k_4 = 1$ and $g = 9.8$, $f_1(x) = \frac{1}{k4}(x_1x_4^2 - g\sin x_3)$ Simulation results are shown in fig 4.10 and 4.11 for two different initial conditions.

4.3.1 Results and Discussion

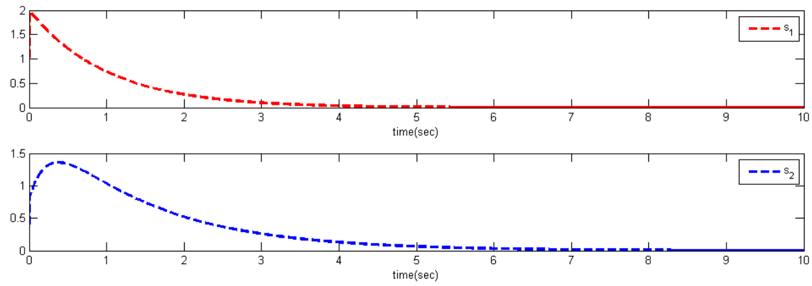
he primary objective was to steer the system to the desired state starting from an initial state. The proposed control law is tested on the Ball and Beam System. The objective has been achieved for two different initial conditions as shown by the simulation results.



(a)

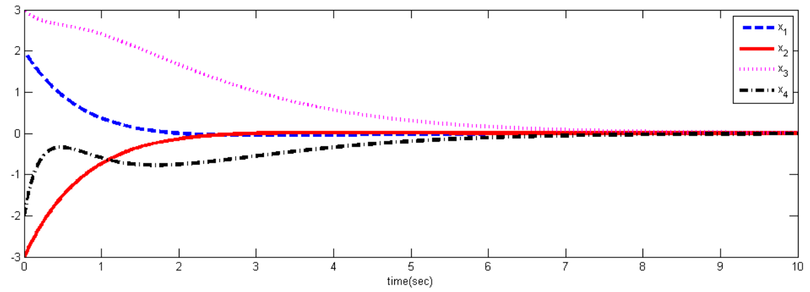


(b)

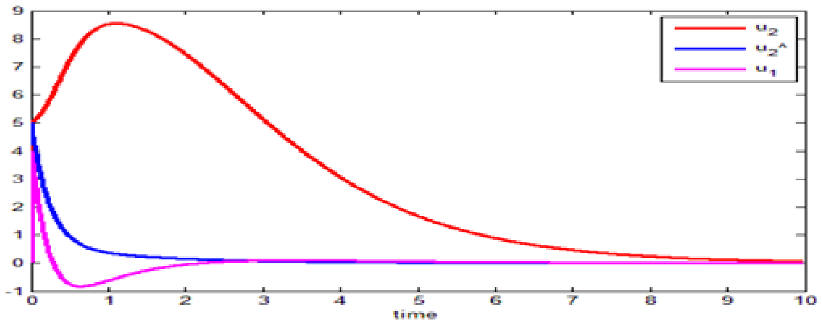


(c)

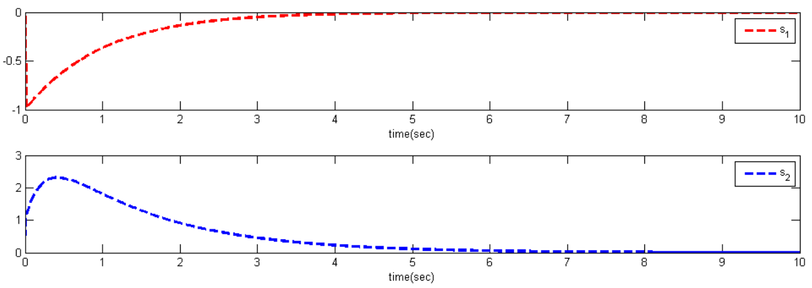
FIGURE 4.10: *Closed loop response of the Ball and Beam System ,(a) Time response of system's states corresponding to initial condition* $(x_1(0), ..., x_4(0)) = (2, -2, \frac{\pi}{5}, -\frac{\pi}{6})$, *(b) Time response of* u_2, $\hat{u}_2$ *and control effort* $u = u_1$, *(c)Time history of the sliding surfaces* s_1 *and* s_2



(a)



(b)



(c)

FIGURE 4.11: *Closed loop response of the Ball and Beam System , (a) Time response of system's states corresponding to initial condition* $(x_1(0), ..., x_4(0)) = (2, 0, \frac{\pi}{4}, 0)$, *(b) Time response of* u_2, $\hat{u}_2$ *and control effort* $u = u_1$, *(c) Time history of the sliding surfaces* s_1 *and* s_2

4.4 The Acrobot and The Pendubot System

The Acrobot :

The Acrobot is a two link manipulator with a single actuator at the elbow as shown in fig. 4.12(a).

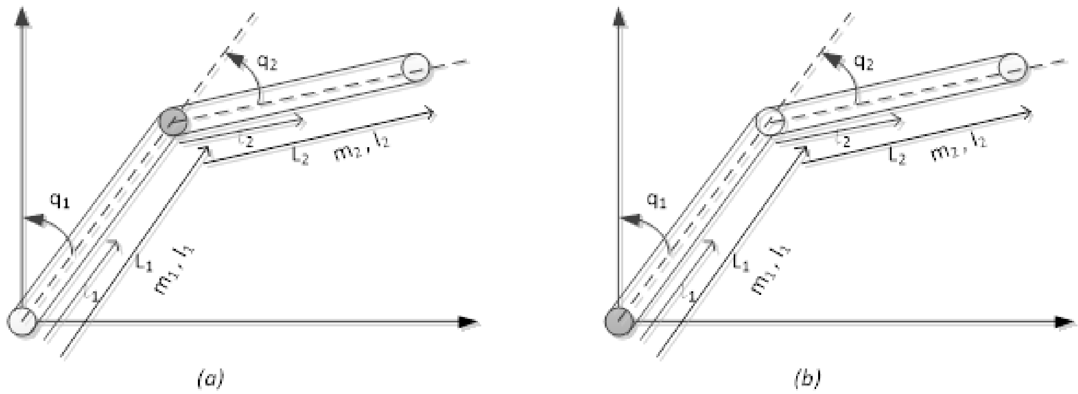


FIGURE 4.12: The Acrobot and the Pendubot

For the dynamical model of the Acrobot, $F(q) = [F_1(q), F_2(q)]^T = [0, 1]^T$ in (3.4) and the equations of motion are given by (3.5) as:

$$\begin{aligned} m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + c_1 + g_1 &= 0 \\ m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + c_2 + g_2 &= u \end{aligned} \tag{4.11}$$

with $m_{11}, m_{12}, m_{21}, m_{22}, c_1, g_1, c_2$,and g_2 as follows:

$m_{11}(q_2) = I_1 + I_2 + m_1 l_1^2 + m_2(L_1^{2+} l_2^2) + 2m_2 L_1 l_2 \cos(q_2)$

$m_{12}(q_2) = I_2 + m_2 l_2^2 + m_2 L_1 l_2 \cos(q_2)$ $m_{21}(q_2) = m_{12}(q_2)$

$m_{22}(q_2) = I_2 + m_2 l_2^2$

$c_1(q, \dot{q}) = -m_2 L_1 l_2 \sin(q_2)(2\dot{q}_1\dot{q}_2 + q_2^2)$

$c_2(q, \dot{q}) = m_2 L_1 l_2 \sin(q_2)(2\dot{q}_1\dot{q}_2 + \dot{q}_2^2)$

$g_1(q_1, q_2) = -(m_1 l_1 + m_2 L_1) g \sin(q_1) - m_2 l_2 g \sin(q_1 + q_2)$

$g_2(q_1, q_2) = -m_2 l_2 g \sin(q_1 + q_2)$

The system can be written as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_1 + b_2(x)u
\end{aligned} \tag{4.12}$$

where

$f_1(x) = -\bar{m}_{11}^{-1}(\bar{c}_1 + g_1)$

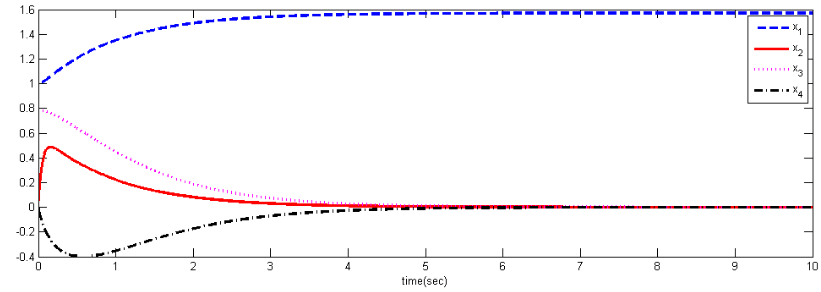
$b_1(x) = \bar{m}_{11}^{-1}$

$f_2(x) = -\bar{m}_{22}^{-})(\bar{c}_2 + \bar{g}_2)$

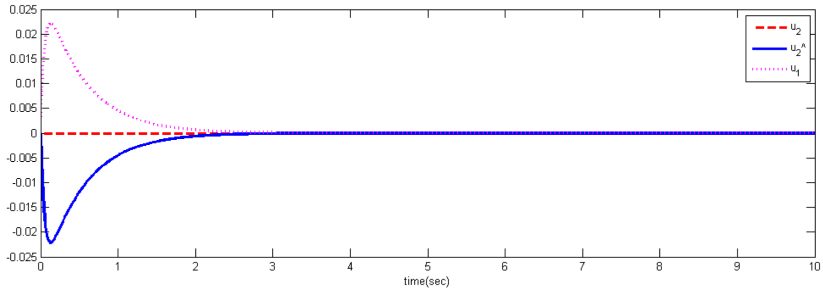
$b_2(x) = \bar{m}_{22}^{-1}$

We chose the physical parameters of the Acrobot, adopted from [35], as:
$m_1 = 1(kg)$, $m_2 = 1(kg)$, $L_1 = 1(m)$, $L_2 = 2(m)$, $l_1 = 0.5(m)$, $l_2 = 1(m)$, $I_1 = 0.0833(kg.m^2)$, $I_2 = 0.33(kg.m^2)$, and $g = 9.8(m.sec^{-2})$).

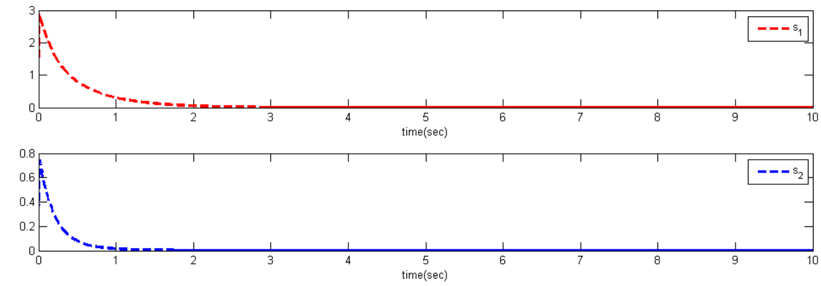
The control task, is to swing the Acrobot from the downward stable equilibrium state $(q_1 = \frac{\pi}{2}, q_2 = 0)$ to the upright unstable equilibrium state $(q_1 = 0, q_2 = 0)$. fig. 4.13 shows the simulation results for the Acrobot with the proposed control algorithm.



(a)



(b)



(c)

FIGURE 4.13: *Closed loop response of the Acrobot ,(a) Time response of system's states corresponding to initial condition* $(x_1(0), ..., x_4(0)) = (1, 0, \frac{\pi}{4}, 0)$, *(b) Time response of* u_2, $\hat{u}_2$ *and control effort* $u = u_1$, *(c)Time history of the sliding surfaces* s_1 *and* s_2

The Pendubot :

The Pendubot is also a two link manipulator with a single actuator at the base as shown in fig 4.12(b) . For the dynamical model of the Pendubot, $F(q) = [F_1(q), F_2(q)]^T = [1, 0]^T$ in (3.4) and the equations of motion are given by (3.6) as:

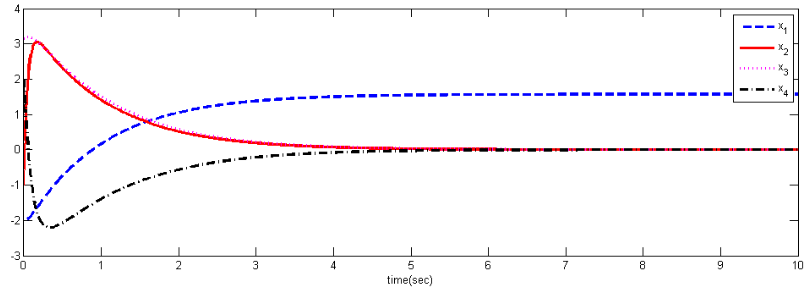
$$\begin{aligned} m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + c_1 + g_1 = u \\ m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + c_2 + g_2 = 0 \end{aligned} \tag{4.13}$$

with m_{11}, m_{12}, m_21, m_{22}, c_1, g_1, c_2, g_2, f_1, f_2, b_1 and b_2 the same as for the Acrobot, given by (4.11). We chose the physical parameters of the Pendubot, adopted from [35], as:

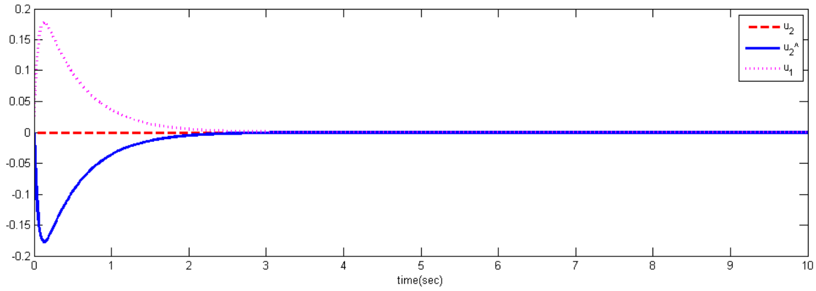
The control task, is to swing the Acrobot from the downward stable equilibrium state $(q_1 = \frac{\pi}{2}, q_2 = 0)$ to the upright unstable equilibrium state $(q_1 = 0, q_2 = 0)$. fig. 4.14 shows the simulation results for the pendubot with the proposed control algorithm.

4.4.1 Results and Discussion

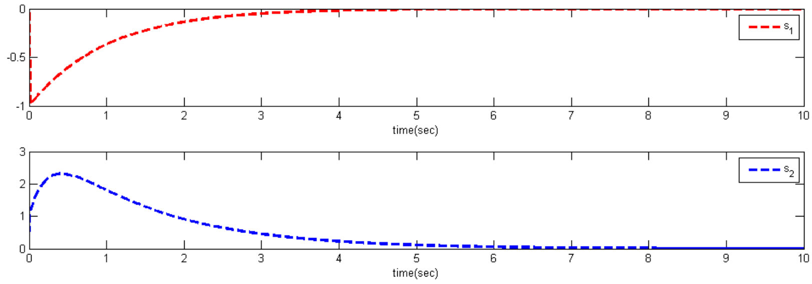
Simulation results in fig 4.12 and fig 4.13 show the performance of the proposed control algorithm. The settling time is less than 10 seconds in both the cases. The response of the state trajectories is smooth and less oscillatory as compared to other methods, for example in [11]. The control effort is also smooth and within reasonable limits. All these achievements show the effectiveness of the proposed control algorithm.



(a)



(b)



(c)

FIGURE 4.14: *Closed loop response of the Pendubot ,(a) Time response of system's states corresponding to initial condition* $(x_1(0), ..., x_4(0)) = (-2, -1, \pi, 2)$*, (b) Time response of* u_2*,* $\hat{u}_2$ *and control effort* $u = u_1$*, (c)Time history of the sliding surfaces* s_1 *and* s_2

4.5 The Double Inverted Pendulum System

The double inverted pendulum system is made up of two links pendulum on a moving cart as is shown in fig 4.15. In this system there are three subsystems. First subsystem is pendulum 1 , second subsytem is pendulum 2, and the third subsystem is cart. The control task is to move the cart on a rail origin by balancing both of the pendulum upright. [16].

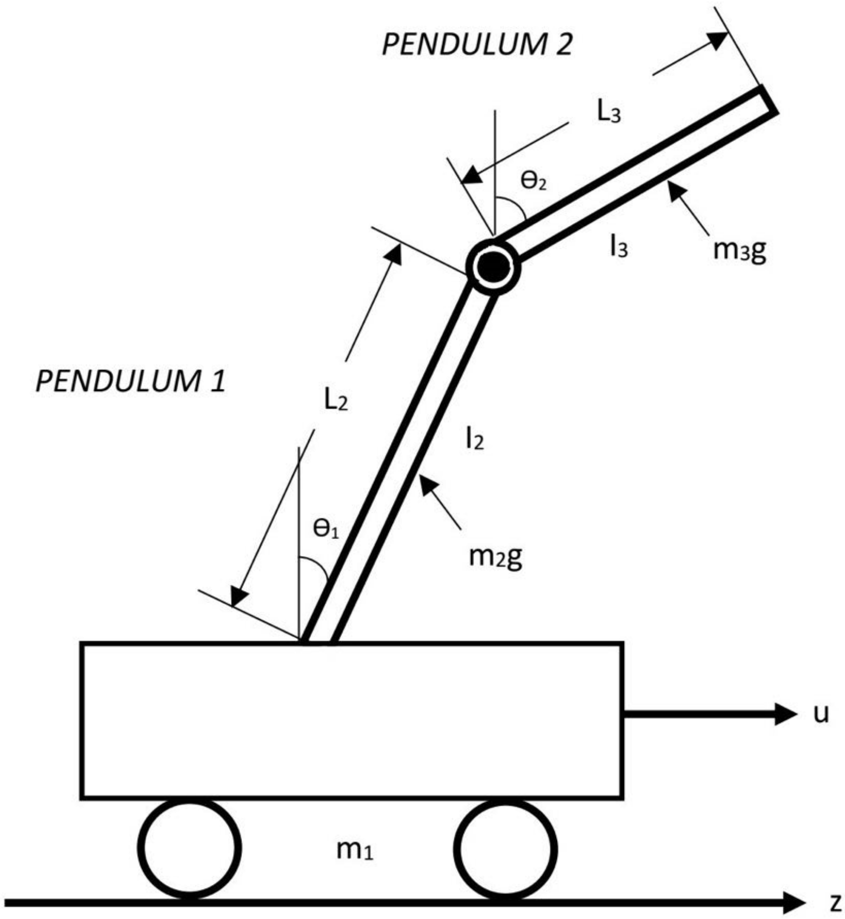


Figure 4.15: Double Inverted Pendulum System

With respect to the vertical line, θ_1 and θ_2 be the angle of pendulum 1 and pendulum 2. Whereas, for origin z be the cart position. u be the control force. Let $m_i, i = 1, 2, 3$ be the masses of the cart, the pendulum 1 and the pendulum 2. Let L_i $i = 2, 3$ be the respective length of the lower and upper pendulums and $l_i, i = 2, 3$ be the lengths from their center of masses respectively. Let $I_i, i = 2, 3$

be the respective inertia of the pendulums.

By defining the coordinates of the center of masses as: $z_1 = \begin{bmatrix} z \\ 0 \end{bmatrix}$, $z_2 = \begin{bmatrix} z + l_2 \sin\theta_1 \\ l_2 \cos\theta_1 \end{bmatrix}$ $z_3 = \begin{bmatrix} z + L_2 \sin\theta_1 + l_3 \sin(\theta_1 + \theta_2) \\ L_2 \cos\theta_1 l_3 \cos(\theta_1 + \theta_2) \end{bmatrix}$ and $\theta = [z \;\; \theta_1 \;\; \theta_2]^T$, then the total kinetic energy is:

$K = \frac{1}{2}\dot{\theta}^T\dot{\theta}$, where ,$M$ is the 3×3 symmetric matrix and its elements are given as:

$m_{11} = m_1 + m_2 + m_3,$

$m_{22}(\theta_2) = I_2 + I_3 + m_2 l_2^2 + m_3 L_2^2 + m_3 l_3^2 + 2m_3 L_2 l_3 \cos\theta_2$

$m_{33} = I_3 + m_3 l_3^2 // \; m_{12}(\theta_1, \theta_2) = m_{21} = (m_2 l_2 + m_3 L_2)\cos\theta_1 + m_3 l_3 \cos(\theta_1 + \theta_2)$

$m_{13}(\theta_1, \theta_2) = m_{31} = m_3 l_3 \cos(\theta_1 + \theta_2)$

$m_{23}(\theta_2) = m_{32} = (m_3 l_3^2 + m_3 l_3 L_2)\cos\theta_2 + I_3$

Therefore

$K = \frac{1}{2}m_{11}\dot{z}^2 + \frac{1}{2}m_{22}(\theta_2)\dot{\theta_1}^2 + \frac{1}{2}m_{33}\dot{\theta_2}^2 + m_{12}(\theta_1, \theta_2)\dot{x}\dot{\theta_1} + m_{13}(\theta_1, \theta_2)\dot{x}\dot{\theta_2} + m_{23}(\theta_2)\dot{\theta_1}\dot{\theta_2}$

The total potential energy is:

$V = (m_2 l_2 + m_3 L_2) g \cos\theta_1 + m_3 l_3 g \cos(\theta_1 + \theta_2)$

Then the Lagrangian is:

$L = K - V = K = \frac{1}{2}m_{11}\dot{z}^2 + \frac{1}{2}m_{22}(\theta_2)\dot{\theta_1}^2 + \frac{1}{2}m_{33}\dot{\theta_2}^2 + m_{12}(\theta_1, \theta_2)\dot{x}\dot{\theta_1} + m_{13}(\theta_1, \theta_2)\dot{x}\dot{\theta_2} + m_{23}(\theta_2)\dot{\theta_1}\dot{\theta_2} - (m_2 l_2 + m_3 L_2) g \cos\theta_1 + m_3 l_3 g \cos(\theta_1 + \theta_2)$

Then Euler-Lagrange equation of motion : $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = F$ gives that:

$$m_{11}\ddot{z} + m_{12}\ddot{\theta_1} + m_{13}\ddot{\theta_2} = u \tag{4.14}$$

$$\begin{aligned} &m_{12}\ddot{z} + m_{22}\ddot{\theta_1} + m_{23}\ddot{\theta_2} + \{k \sin\theta_1 + m_3 l_3 \sin(\theta_1 + \theta_2)\}\dot{z}\dot{\theta}1 \\ &+ \{m_3 l_3 \sin(\theta_1 + \theta_2)\}\dot{z}\dot{\theta_2} + kg \sin\theta_1 + m_3 l_3 g \sin(\theta_1 + \theta_2) = 0 \\ &where \quad k = (m_2 l_2 + m_3 L_2) \end{aligned} \tag{4.15}$$

$$m_{13}\ddot{z} + m_{23}\ddot{\theta}_1 + m_{33}\ddot{\theta}_2 + \{2m_3l_3L_2\sin\theta_2\}\dot{\theta}_1^2 + \{m_3l_3\sin(\theta_1+\theta_2)\}\dot{z}\dot{\theta}_1$$
$$+ \{m_3l_3\sin(\theta_1+\theta_2)\}\dot{z}\dot{\theta}_2 + \{m_3l_3\sin(\theta_1+\theta_2)\}\dot{\theta}_1\dot{\theta}_2 + m_3l_3g\sin(\theta 1+\theta 2) = 0 \tag{4.16}$$

Equation (4.14)-(4.16) can be expressed as:

$$\begin{aligned} m_{11}\ddot{z} + m_{12}\ddot{\theta}_1 + m_{13}\ddot{\theta}_2 &= u \\ m_{12}\ddot{z} + m_{22}\ddot{\theta}_1 + m_{23}\ddot{\theta}_2 + c_2 + g2 &= 0 \\ m_{13}\ddot{z} + m_{23}\ddot{\theta}_1 + m_{33}\ddot{\theta}_2 + c_2 + g2 &= 0 \end{aligned} \tag{4.17}$$

Where

$c_2 = \{k\sin\theta_1 + m_3l_3\sin(\theta_1+\theta_2)\}\dot{z}\dot{\theta}1 + \{m_3l_3\sin(\theta_1+\theta_2)\}\dot{z}\dot{\theta}_2$

$g_2 = kg\sin\theta_1 + m_3l_3g\sin(\theta_1+\theta_2$, where, $k = (m_2l_2 + m_3L_2)$

$c_3 = \{2m_3l_3L_2\sin\theta_2\}\dot{\theta}_1^2 + \{m_3l_3\sin(\theta_1+\theta_2)\}\dot{z}\dot{\theta}_1 + \{m_3l_3\sin(\theta_1+\theta_2)\}\dot{z}\dot{\theta}_2$

$+ \{m_3l_3\sin(\theta_1+\theta_2)\}\dot{\theta}_1\dot{\theta}_2$

$g_3 = m_3l_3g\sin(\theta_1+\theta_2)$

Solving the equation (4.17) we have

$$\begin{aligned} \ddot{z} &= -\frac{m_{12}}{m_{11}}\ddot{\theta}_1 - \frac{m_{13}}{m_{11}}\ddot{\theta}_2 + \frac{1}{m_{11}}u \\ \bar{m}_{11}\ddot{\theta}_1 + \bar{m}_{12}\ddot{\theta}_2 &= -\frac{m_{12}}{m_{11}}u - c_2 - g_2 \\ \bar{m}_{12}\ddot{\theta}_1 + \bar{m}_{22}\ddot{\theta}_2 &= -\frac{m_{13}}{m_{11}}u - c_3 - g_3 \end{aligned} \tag{4.18}$$

where $\bar{m}_{11} = (m_{22} - \frac{{m_{12}}^2}{m_{11}})$, $\bar{m}_{12} = (m_{23} - \frac{m_{12}m_{13}}{m_{11}})$ and $\bar{m}_{22} = (m_{33} - \frac{{m_{13}}^2}{m_{11}})$

Equation (4.18) can be written further as:

$$\begin{aligned} \ddot{z} &= -\frac{m_{12}}{m_{11}}\ddot{\theta}_1 - \frac{m_{13}}{m_{11}}\ddot{\theta}_2 + \frac{1}{m_{11}}u \\ \bar{\bar{m}}_{11}\ddot{\theta}_1 &= u(\frac{m_{12}\bar{m}_{22} - \bar{m}_{12}m_{13}}{m_{11}}) + \bar{c}_2 + \bar{g}_2 - \bar{c}_3 - \bar{g}_3 \\ \bar{\bar{m}}_{11}\ddot{\theta}_2 &= u(\frac{\bar{m}_{11}m_{13} - \bar{m}_{12}m_{12}}{m_{11}}) - \bar{\bar{c}}_2 - \bar{\bar{g}}_2 + \bar{\bar{c}}_3 + \bar{\bar{g}}_3 \end{aligned} \tag{4.19}$$

where

$\bar{c}_2 = \bar{m}_{22}c_2$, $\bar{g}_2 = \bar{m}_{22}g_2$, $\bar{c}_3 = \bar{m}_{12}c_3$ and $\bar{g}_3 = \bar{m}_{12}g_3$

$\bar{\bar{c}}_2 = \bar{m}_{11}c_2$, $\bar{\bar{g}}_2 = \bar{m}_{12}g_2$, $\bar{\bar{c}}_3 = \bar{m}_{11}c_3$, $\bar{\bar{c}}_3 = \bar{m}_{11}c_3$, $\bar{\bar{g}}_3 = \bar{m}_{11}g_3$ and $\bar{\bar{m}}_{11} = \bar{m}_{12}^2 -$

$\bar{m}_{22}\bar{m}_{11}$

From (4.19) we have the following expressions:

$$\begin{aligned}
\ddot{z} &= \frac{m_{12}}{m_{11}\bar{\bar{m}}_{11}}\{-\bar{c}_2 - \bar{g}_2 + \bar{c}_3 + \bar{g}_3\} + \frac{m_{13}}{m_{11}\bar{\bar{m}}_{11}}\{\bar{\bar{c}}_2 + \bar{\bar{g}}_2 - \bar{\bar{c}}_3 - \bar{\bar{g}}_3\} \\
&+ \frac{1}{m_{11}}\{1 + m_{12}\bar{m}_{22} - \bar{m}_{12}m_{13} + \bar{m}_{11}m_{13} - \bar{m}_{12}m_{12}\}u \\
\ddot{\theta}_1 &= \frac{1}{\bar{\bar{m}}_{11}}\{\bar{c}_2 + \bar{g}_2 - \bar{c}_3 - \bar{g}_3\} + u(\frac{m_{12}\bar{m}_{22} - \bar{m}_{12}m_{13}}{m_{11}\bar{\bar{m}}_{11}}) \\
\ddot{\theta}_2 &= \frac{1}{\bar{\bar{m}}_{11}}\{-\bar{\bar{c}}_2 - \bar{\bar{g}}_2 + \bar{\bar{c}}_3 + \bar{\bar{g}}_3\} + u(\frac{\bar{m}_{11}m_{13} - \bar{m}_{12}m_{12}}{m_{11}\bar{\bar{m}}_{11}})
\end{aligned} \tag{4.20}$$

Define the state variables as: $x_1 = z,\ x_2 = \dot{z},\ x_3 = \theta,\ x_4 = \dot{\theta}_1,\ x_5 = \theta_2$ *and* $x_6 = \dot{\theta}_2\ x = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6]^T$. By using the following notations:

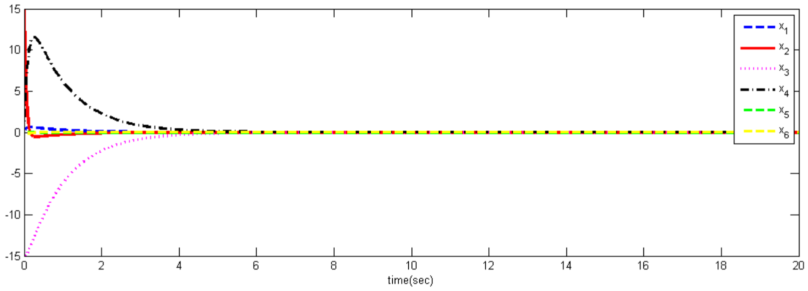
$$\begin{aligned}
f_1(x) &= \frac{m_{12}}{m_{11}\bar{\bar{m}}_{11}}\{-\bar{c}_2 - \bar{g}_2 + \bar{c}_3 + \bar{g}_3\} + \frac{m_{13}}{m_{11}\bar{\bar{m}}_{11}}\{\bar{\bar{c}}_2 + \bar{\bar{g}}_2 - \bar{\bar{c}}_3 - \bar{\bar{g}}_3\} \\
b_1(x) &= \frac{1}{m_{11}}\{1 + m_{12}\bar{m}_{22} - \bar{m}_{12}m_{13} + \bar{m}_{11}m_{13} - \bar{m}_{12}m_{12}\} \\
f_2(x) &= \frac{1}{\bar{\bar{m}}_{11}}\{\bar{c}_2 + \bar{g}_2 - \bar{c}_3 - \bar{g}_3\} \\
b_2(x) &= (\frac{m_{12}\bar{m}_{22} - \bar{m}_{12}m_{13}}{m_{11}\bar{\bar{m}}_{11}}) \\
f_3(x) &= \frac{1}{\bar{\bar{m}}_{11}}\{-\bar{\bar{c}}_2 - \bar{\bar{g}}_2 + \bar{\bar{c}}_3 + \bar{\bar{g}}_3\} \\
b_3(x) &= (\frac{\bar{m}_{11}m_{13} - \bar{m}_{12}m_{12}}{m_{11}\bar{\bar{m}}_{11}})
\end{aligned} \tag{4.21}$$

The state space representation of the equation is:

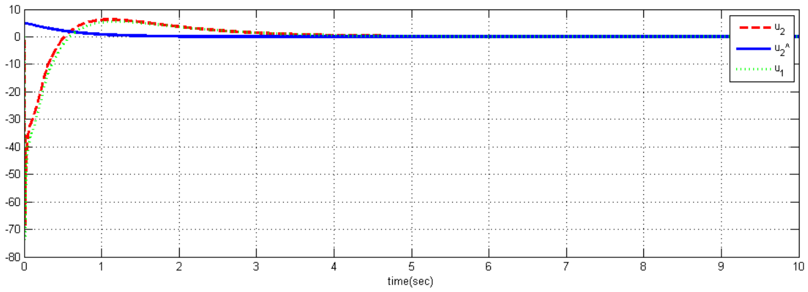
$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2(x)u \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= f_3 + b_3(x)u
\end{aligned} \tag{4.22}$$

4.5.1 Results and Discussion

The physical parameters of the Double Inverted Pendulum as:
$m_1 = m_2 = m_3 = 1$, $l_2 = l_3 = 0.75$, $I_2 = \frac{4}{3}m_2 l_2^2$, $I_3 = \frac{4}{3}m_3 l_3^2$, $g = 9.8$
The initial and the desired is chosen as in [16]. fig 4.16 and 4.17 show the simulation result of double inverted pendulum system. Both the pendulum and cart move to its equilibrium point when the force acts only on the cart. The simulation is done for different initial condition.

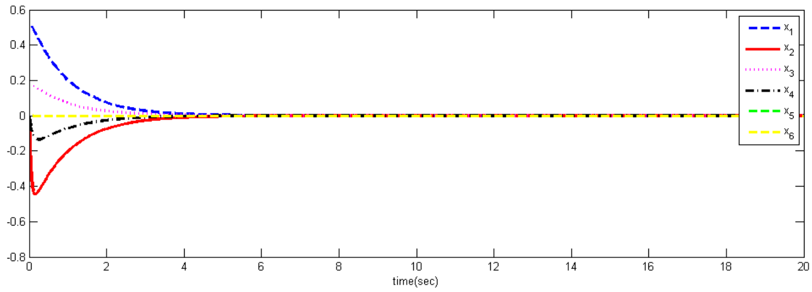


(a)

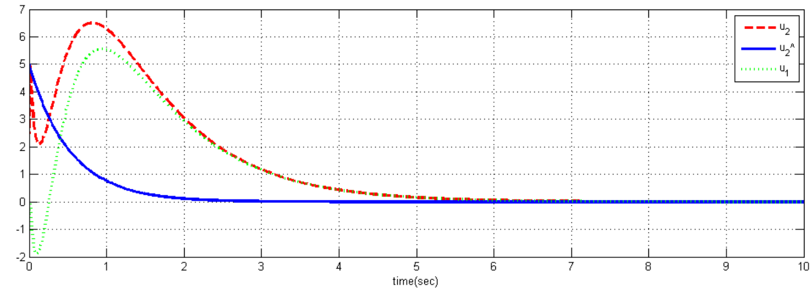


(b)

FIGURE 4.16: *Closed loop response of the Double Inverted Pendulum System ,(a) Time response of system's states corresponding to initial condition* $(x_1(0), ..., x_6(0)) = (0, 15, -15, 0, 0, 0)$, *(b) Time response of* u_2, $\hat{u}_2$ *and control effort* $u = u_1$,



(a)



(b)

FIGURE 4.17: *Closed loop response of the Double Inverted Pendulum System ,(a) Time response of system's states corresponding to initial condition* $(x_1(0), ..., x_6(0)) = (\frac{\pi}{6}, 0, \frac{\pi}{18}, 0, 0, 0)$, *(b) Time response of* u_2, $\hat{u}_2$ *and control effort* $u = u_1$,

4.6 Summary

In this chapter, the proposed algorithm is applied on different UMSs. The algorithm is applied on five UMSs having 2DOF and one UMS having 3DOF. Every system has different control task. The control task is achieved which show the effectiveness of the proposed algorithm.

Chapter 5

Conclusion and Future work

5.1 Conclusion

In the past few decades there has been an increasing interest in the underactuated mechanical systems. These systems have many practical and diverse applications in modern science and engineering. The broad application areas of underactuated systems include robotics, industry, and aerospace systems. This thesis presented a simple stabilizing control algorithm for a class of UMS with n degrees of freedom (n DOF). The methodology is based on an adaptive sliding mode control. A system is transformed to a special structure through input transformation, containing nominal part plus some unknown term. The unknown term is computed adaptively. Later the transformed system is stabilized using the designed adaptive sliding mode control. The adapted laws were derived in such a way that the time derivative of a Lyapunov function becomes strictly negative. The effectiveness of the proposed method is applied to different underactuated systems having 2 DOF and 3DOF: Inverted pendulum, TORA system, The Pendubot and Acrobot, Ball and Beam system and double inverted pendulum. Computer simulation results show the validity of the proposed control scheme on these systems which confirmed the sliding mode enforcement in finite time along with asymptotic convergence of system's states.

5.2 Future Work

The productive completion of this thesis, some suggestion is created that should be examine in the future. The work carried out in this thesis can be extended in future. To other UMS the proposed technique is applicable. There are some areas which need future attention. The future work is to implement on the practical UMS. After implementing compare the result with the the proposed technique.

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