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Planar Symmetric Concave Central Configuration in Newtonian Four Body Problem

by

Qurat-ul-Ain

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First of all, I dedicate this research project to Allah Almighty, The most merciful and beneficent, creator and Sustainer of the earth

And

Dedicated to Prophet Muhammad (peace be upon him) whom, the world where we live and breathe owes its existence to his blessings

And

Dedicated to my parents and brother, who pray for me and always pave the way to success for me

And

Dedicated to my teachers, who are a persistent source of inspiration and encouragement for me

CERTIFICATE OF APPROVAL

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(Qurat-ul-Ain)

Abstract

In this thesis we discuss non-collinear central configurations for four masses which are symmetric about the center of mass. Basically we consider a problem in which we are given a symmetric concave configuration of four bodies under those conditions in which it is permissible to select positive masses in order to make it central. We get four equations of motions for four massive bodies. The arrangement of masses are such that m_1, m_2 and m_3 lies on the vertices of a tetrahedron such that all distances between the masses are equal and m_4 be any positive real number greater then 0 lie either inside the triangle or on the symmetrical axis. We also configure that no central configuration exist for positive masses in some localities. Contrarily, if we consider any configuration in the complement of union of these region, we may choose any possible positive masses in order to make these configuration central.

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Abbreviations

2BP	Two-Body Problem
3BP	Three-Body Problem
4BP	Four-Body Problem
CC	Central Configuration
CFBP	Collinear four-body problem
CM	Celestial Mechanics
2-D	Two dimentinal
3-D	Three dimentinal
$\mathbf{M}_s$	Mass of the Sun
NBP	N-body problem
NFBP	Newtonian four-body problem
R	Region
SI	System International

Symbols

G	Universal gravitational constant	$m^3 kg^{-1} s^{-2}$
$\mathbf{F}$	Gravitational force	Newton
r	Distance	Meter
$\mathbf{P}$	Linear momentum	$kg\ m\ s^{-1}$
$\mathbf{L}$	Angular momentum	$kg\ m^2\ s^{-1}$
m_i	Point masses	kg
$\mathbb{R}$	Real number	
$\ni$	Such that	
$\forall$	For all	
$\in$	Belongs to	

Chapter 1

Introduction

Celestial Mechanics is defined as the science of studying the motion of celestial bodies. Basically it is that branch of astronomy which deals with the motion of heavenly bodies in space. Historically, celestial mechanics applies standards of physics (classical mechanics) to astronomical bodies, which includes stars and planets. It is the science devoted to the study of the motion of the celestial bodies on the basis of the laws of gravitation [1].

1.1 Historical background

Celestial Mechanics (CM) has its beginnings in early cosmology where the movements of the Sun, the Moon, and the five planets perceptible to the unassisted eye-Mercury, Venus, Mars, Jupiter, and Saturn were determined and investigated. The word planet is plagiarized from the Greek word "wanderer". In the $18th$-$19th$ centuries, celestial mechanics was progressing with perpetual achievement in developing highly precise theories of the movements of the planets and the Moon.
In the end of 19th century, Poincare (who contributed so much to the improvement of celestial mechanics) developed the aim of celestial mechanics to be the solution of the question that whether Newtons law of gravitation alone is adequate to clarify the entirety of the observed movements of celestial bodies.

Poincare has indeed received general acknowledgment in pure mathematics and theoretical physics. However, this formulation of aim of celestial mechanics exhibits that Poincare has contributed an essential part to the conspiracy of astronomical observations with the consequences of mathematical and physical theories. Modern celestial mechanics initiated with Isaac Newton's Principia of Mathematica in 1687 [2].

Johannes Kepler (1571 – 1630), the first astronomer to closely accommodate the predictive geometrical astronomy, that have been prevalent from "Ptolemy" in the 2nd century to "Copernicus", with physical conception to yield a new astronomy, based upon Causes or Celestial Physics in 1609. His work depend upon the modern laws of planetary orbits, which he refined using his physical assumptions and the planetary conclusions made by Tycho Brahe. Kepler's model significantly enhanced the accuracy of predictions of planetary motion, years earlier than Isaac Newton evolved his law of gravitation in 1686.

Isaac Newton (1642 – 1727) is one of the most famous mathematician who introduces the idea that the movements of bodies in the heavens, such as the Sun, the Moon and planets, and the movement of bodies on the ground, similar to cannon balls and falling apples, could be characterized by the similar arrangement of physical laws. In this way he combined celestial and terrestrial dynamics. Kepler's Laws for the case of a circular orbit is easy to prove by simple using "Newton's law of universal gravitation". Elliptical orbits contain more complicated calculations, that Newton defined in his "Principia of Mathematica" [3].

After Newton, **Lagrange** (1736 – 1813) attempted to solve the 3BP, analyzed the stability of planetary orbits, and found the presence of the Lagrangian points. Lagrange additionally reformulated the standards of classical mechanics, emphasizing energy more than force and building up a strategy to utilize a single polar coordinate equation to describe any orbit, even those that are parabolic and hyperbolic. It is helpful for computing the conduct of planets and comets. More recently, it has also become valuable to ascertain spacecraft trajectories.

Simon Newcomb (1835 – 1909) was a Canadian-born mathematician who also work on mathematical astronomy. He revised Peter Andreas Hansen's table of lunar positions. In 1877, under "George William Hill" (American astronomer) counseling, he recalculated all the major astronomical constants.
Newcombs most important work appeared in the astronomical papers prepared for the use of the American Ephemeris and Nautical Almanac, a series of memoirs that he founded in 1879 with the object of giving a systematic determination of the constants of astronomy from the best existing data, a re-investigation of the theories of the celestial motions, and the preparation of tables, formulae, and perceptions for the construction of ephemerides, and for other applications of the same results. Among them were his tables of the Sun, Mercury, Venus, Mars, Uranus, and Neptune, along with tables of Jupiter and Saturn that were devised by G. W. Hill. These tables were used throughout most of the world for calculating daily positions of the objects from 1901 to 1959.

The 2-body problem (2BP) is to predict the motion of two massive objects which are abstractly viewed as point particles. The 2BP [4] is most common in the case of a gravity that occurs in astronomy to determine orbits of objects such as satellites, planets and stars. Newton solved 2BP by using his fundamental law of gravity. Newtonian mechanics is a mathematical model whose purpose is to explore the motions of the various objects in the universe.
The basic concept of this model were first enunciated by sir Isaac Newton in a work entitled "***Philosophiae Naturalis Principia Mathematica***". This work was published in 1687. The problem has no significant solution for $N \geq 3$. Although we have a R3BP that gives a particular solution. The 3BP is the problem of taking the initial positions and velocities of three point masses and solving for their subsequent motion according to Newton's laws of motion and Newton's law of universal gravitation. The 3BP is a special case of the N-body problem. The 3BP deals with gravitationally interacting astronomical bodies and intends to predict their motions. The 3BP have been studied over three hundred years. It arose in an attempt to understand the Sun's effect on the motion of the Moon around the

Earth. NBP also known as many body problem [5].

The many body problem was first formulated precisely by Newton. In its form where the object involve point masses: "it may be stated as given at any time the position and velocities of three or more massive particles moving under their mutual gravitational forces, the mass also being known, calculated their positions and velocities at any other time".

The NBP [6] which predicts the individual motion of a system of celestial bodies that gravitationally attract with each other. Mathematicians and astronomers continued to work on the NBP over the last four centuries. First of all, Kepler in his planetary motion laws [7] defining the elliptical trajectories of planets around the Sun. Most important works in science history in which Newton derived and formulated Kepler's law. Newton turned his attention to comparatively more difficult systems, after the justification of Kepler's laws. Although, he was unable to achieve any breakthrough in (3BP) throughout his life after much struggle.

1.2 Newtonian Four Body Problem (NFBP)

The NFBP is to determine the motion of four point masses that interacts each other only by Newtons law of gravitation. In other words, they behave gravitationally like four particles at a distance apart equal to the distance between their centers. These four masses will be assumed to be sufficiently isolated from each other in the universe so that only force acting is the inverse square force of their mutual attraction along the line joining their centers.

Dziobek for the first time, discussed central configuration with relative distances as coordinates. **Moulton** focused on the straight line solutions in the problem of N bodies.

MacMillan and **Bartky** studied permanent configurations in the problem of four bodies. **Albouy**, **Fu** and **Sun** have focused on the symmetry of planar four-body convex central configurations. **Bernat**, **Llibre** and **Perez** have discussed the planar central configurations of the 4BP with three equal masses. **Pina** and **Lonngi** have studied central configurations for the planar Newtonian four-body problem. [8]

1.2.1 Equation of Motion for NFBP

Let the system consists of four particles of masses m_1, m_2, m_3 and m_4 situated at P_1, P_2, P_3 and P_4 respectively. Let P_i $(i = 1, 2, 3, 4)$ be the position vectors of the four particles respectively with respect to an inertial frame with O as the origin. Considering the masses spherically symmetrical with homogeneous layers so that they attract one another like point masses. The only forces acting are the mutual Newtonian gravitational attractions between the bodies [8].
The equations of motion of the four particles given by Pina and Lonngi [9] can be written as

$$\begin{aligned}
m_1\ddot{\rho}_1 &= +\frac{Gm_1m_2(\rho_2-\rho_1)}{\rho_{12}^3} + \frac{Gm_1m_3(\rho_3-\rho_1)}{\rho_{13}^3} + \frac{Gm_1m_4(\rho_4-\rho_1)}{\rho_{14}^3} \\
m_2\ddot{\rho}_2 &= -\frac{Gm_2m_1(\rho_1-\rho_2)}{\rho_{21}^3} + \frac{Gm_2m_3(\rho_3-\rho_2)}{\rho_{23}^3} + \frac{Gm_2m_4(\rho_4-\rho_2)}{\rho_{24}^3} \\
m_3\ddot{\rho}_3 &= -\frac{Gm_3m_1(\rho_1-\rho_3)}{\rho_{31}^3} + \frac{Gm_3m_2(\rho_2-\rho_3)}{\rho_{32}^3} + \frac{Gm_3m_4(\rho_4-\rho_3)}{\rho_{34}^3} \\
m_4\ddot{\rho}_4 &= -\frac{Gm_4m_1(\rho_1-\rho_4)}{\rho_{41}^3} + \frac{Gm_4m_2(\rho_2-\rho_4)}{\rho_{42}^3} + \frac{Gm_4m_3(\rho_3-\rho_4)}{\rho_{43}^3}
\end{aligned}$$

where $\rho_{ij} = \rho_{ji}$ $(i \neq j = 1, 2, 3, 4)$ are the distances between the i^{th} and j^{th} particles, $m_j (j = 1, 2, 3, 4)$ denotes the masses of four particles and G is the gravitational constant.
The right hand sides of the above equations are the gravitational forces which are derived from the potential energy given by

$$U = -\frac{Gm_1m_2}{\rho_{12}} - \frac{Gm_1m_3}{\rho_{13}} - \frac{Gm_1m_4}{\rho_{14}} - \frac{Gm_2m_3}{\rho_{23}} - \frac{Gm_2m_4}{\rho_{24}} - \frac{Gm_3m_4}{\rho_{34}}$$

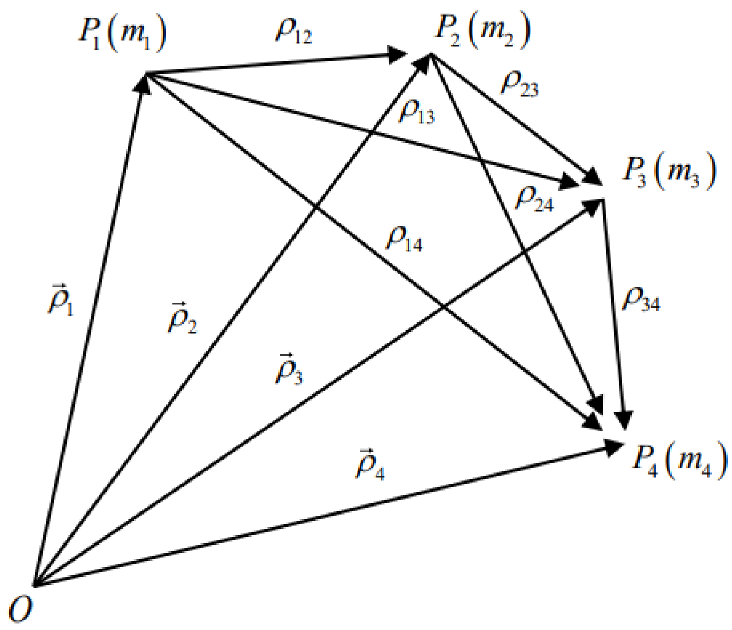

FIGURE 1.1: Newtonian Four Body Problem

1.3 Central Configuration (CC)

In CM and the mathematics of the NBP, a **central configuration** (CC) is an arrangement of point masses with the property that each mass is pulled by the combined gravitational force of the system directly towards the center of mass, with acceleration proportional to its distance from the center. CC's play an important role in the study of the **Newtonian N-body problem**. For example, they lead to the only explicit solutions of the equations of motion, they govern the behavior of solutions near collisions, and they influence the topology of the integral manifolds.[10]

1.3.1 Importance of CC's

CC of the NBP are important because:

- They allow to compute all the homographic solutions [11].

- If the N bodies are going to a simultaneous collision, then the particles tend to a central configuration [12].
- If the N bodies are going simultaneously at infinity in parabolic motion (i.e. the radial velocity of each particle tends to zero as the particle tends to infinity), then the particles tend to a CC [13].
- There is a relation between central configurations and the bifurcations of the hyper-surfaces of constant energy and angular momentum [14].

1.4 N-Body Problem (NBP)

The NBP consists in studying the motion of N point-like masses, interacting among themselves through no other forces than their mutual gravitational attraction according to Newtons gravitational law.

The 2BP deals with much of the essential work of astrodynamics, but by using other bodies, we sometimes need to model the universe. The next logical move, then, is to build 3BP formulas. NBP is another generalization of the 3BP. Generally, it takes a fixed number of integration constants to solve general differential equations of motions in NBP.

Pizzetti [15] in 1904 proved that the configuration of the N bodies in a homographic solution is central at any instant of time. It is important to note that homographic solutions with rotation and eventually with a dilatation only exist for planar central configurations. For spatial central configurations all the homographic solutions only have a dilation [11].

Consider a simple gravity problem in which we have constant acceleration over time, $x(t) = x_0$. After integration, we get the velocity, $\dot{x}(t) = x_0 t + x_1$. Integrating again gives, $\ddot{x}(t) = x_2 + x_1 t + \frac{1}{2} x_0 t^2$. We need to know the initial conditions to complete the solution. This example is a straight forward analytical approach that uses initial values and constants of integration, called motion integrals. Unfortunately, this is not always a straightforward scenario. When initial conditions itself do not have a solution, motion integrals will minimize the order of nonlinear equations, also known as the dimensions of the dynamic system.

Preferably, if the number of integrals is equal to the order of nonlinear equations, it should be reduced to zero. These integrals are the fixed functions of the initial conditions, as well as the direction and velocity of the object at any moment, so they are named as constants of the motion.

1.4.1 Equation of Motion in the NBP

The equations of motion of a system of N particles are described by a set of N ordinary differential equations, each of which is simply Newton's second law of motion applied to an individual particle.

First, here we set up the equations of motions of N large particles of masses $m_k(k = 1, 2, 3, 4...N)$ whose radius vectors are $\mathbf{r}_k$ from a non-accelerated point O while their mutual radius vectors are $\mathbf{r}_{kj}$ and

$$\mathbf{r}_{kj} = \mathbf{r}_j - \mathbf{r}_k \tag{1.1}$$

The equations of motion of the NBP are

$$m_k\ddot{\mathbf{r}}_k = G\sum_{j=1 j\neq k}^{N} \frac{m_k m_j}{\mathbf{r}_{kj}^3}\mathbf{r}_{kj}, \tag{1.2}$$

for $k = 1, ..., N$ where G is the gravitational constant, $\mathbf{r}_k \in \mathbb{R}^3$ is the position vector of the punctual mass m_k in an inertial system, and $\mathbf{r}_{jk}$ is the Euclidean distance between the masses m_j and m_k, thus

$$\mathbf{r}_{kj} = -\mathbf{r}_{jk} \tag{1.3}$$

1.5 Four Body Central Configuration

The four-body classification of CC's is much more difficult and generally unsolved than the three-body case. The equilateral triangle in the three-body case, is a relative equilibrium for any mass choice. No configurations are central configurations for each mass vector in the $N = 4$ planar problem, although it is true that any N masses arranged in a normal $N - 1$ simplex in $\mathbb{R}^{N-1}$ form a central configuration [16]. Note that the Lagrange's configuration is a specific example of Saari's $N = 3$; the $N = 4$ case was actually first discovered in 1891 by Lehmann-Filhes.
There are two special cases in which the central four-body configurations are understood. Gannaway [17] and Arensdorf [18] have studied the limiting case in which one of the four masses goes to zero, often referred to as the $3 + 1$ case, and the case in which all four masses are equal has recently been classified by Albouy [19].
There are only four classes of equivalence of planar central configurations for the equal mass case: an equilateral triangle with a mass at its center, the square, a collinear configuration, and a unique isosceles triangle with another mass on its symmetry axis. The last type of central configuration will be referred to as an isosceles configuration. Notice that all the CC's have at least one axis of symmetry under reflection for the equal mass case [20].

1.6 Thesis Contribution

In this thesis [21] we review and investigate a concave non-collinear NFBP which involves symmetrical arrangements for four masses such that either one is present inside the triangle formed by other three to form an equilateral triangle or it is located at the center of the triangle to form an isosceles triangle. The masses are m_1, m_2, m_3 and m_4. These four masses are arranged in a way that $m_1 = m_2 = m_3 > 0$ and m_4 be any positive real number that is greater then 0.
In the first part we study the CC of all the four given masses, i.e m_1 to m_4. Secondly we proved some theorems related to our problem. In the last we draw

possible regions in st-plane so that the mass functions are positive and investigate all the four point masses graphically .

1.7 Dissertation Outlines

We split this dissertation into five chapters.

In **Chapter 1** we introduce the given problem and briefly discuss the purpose of this research.

Chapter 2 includes some basic definitions related to CM like momentum, symmetry, central configuration etc and important terms related to research are also defined..

In **Chapter 3** the paper [21] is comprehensively reviewed.

Chapter 4 includes the graphical analysis of all the four point masses m_1, m_2, m_3 and m_4.

Chapter 5 summarizes the entire research with concluding remarks.

References used in the thesis are mentioned in **Bibliography**.

Chapter 2

Preliminaries

This chapter includes some basic definitions and basic concepts that will help us in better understanding of our objective research.

2.1 Basic definitions

2.1.1 Mechanics

"The area of physics concerned with the study of motion is known as mechanics." [22]

2.1.2 Dynamics

"The branch of mechanics that deals with the motion of bodies under some action of forces is known as Dynamics." [22]

2.1.3 Kinematics

"The mechanical branch concerned with the object movement without reference to the focus which induces motion is known as Kinematics." [22]

2.1.4 Statics

"The mechanical branch concerned with those conditions under which no motion is evident." [22]

2.1.5 Motion

"The phenomenon in which an object changes its location over time is motion. Motion is mathematically defined in terms of displacement, distance, velocity, acceleration, speed, and time."[22]

2.1.6 Velocity

"In relation to a frame of reference, the velocity of an object is the rate of change of its location and is a function of time." [22]

2.1.7 Acceleration

"Acceleration is a vector quantity defined as the rate at which the object changes its velocity." [22]

2.1.8 Momentum

"Momentum can be defined in short as "mass in motion." As all objects contains mass so if an object is in motion, then it has momentum. Momentum depends upon the two variables mass and velocity. Mathematically, momentum can be defined as

$$P = mv$$

where m is the mass and v is the velocity. Above expression illustrates that momentum is directly proportional to an object's mass and velocity." [22]

2.1.9 Angular Momentum

"Angular momentum for a point-like particle of mass m with linear momentum $\mathbf{D}$ about a point O, defined by the equation

$$\mathbf{B} = \mathbf{r} \times \mathbf{D},$$

where $\mathbf{r}$ is the vector from the point O to the particle. The torque about the point O acting on the particle is equal to the rate of change of the angular momentum about the point O of the particle i.e.,

$$\boldsymbol{\tau} = \frac{d\mathbf{B}}{dt}.\text{''}[22]$$

2.1.10 Conservation of Angular Momentum

" It states that if the net external torque acting on a particle is zero, the angular momentum will remain unchange. Mathematically

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{D}) = 0 \qquad or \qquad \mathbf{r} \times \mathbf{D} = \mathbf{constant}.\text{''}[22]$$

2.1.11 Conservation of Energy

"In a conservative force field the total energy (i.e the sum of kinetic energy and potential energy) remains constant. This is known as the principle of conservation of energy. Mathematically

$$U = W + Q,$$

where

- U is the total energy of a system.
- Q is the kinetic energy of a system.
- W is the potential energy of a system."[22]

2.1.12 Conservation of Momentum

"If we put $\mathbf{F} = 0$ in Newton's second law, we find

$$\frac{d}{dt}(m\mathbf{v}) = 0 \qquad or \qquad m\mathbf{v} = \mathbf{constant}$$

This leads to the following statement that if the net external force acting on a particle is zero, its momentum will remain unchange. It is known as *Principle of Conservation of Momentum*." [22]

2.1.13 Conservative Force Field

" A force field $\mathbf{F}$ is conservative if and only if there exists a continuously differentiable scalar field V such that $\mathbf{F} = -\nabla V$. If there is no scalar function V such that $\mathbf{F} = \nabla V$, then $\mathbf{F}$ is called Non-conservative force field." [22]

2.1.14 Center of mass

"The **center of mass** is a position defined relative to an object or system of objects. It is the average position of all the parts of the system, weighted according to their masses. In general the center of mass can be found by vector addition of the weighted position vectors which point to the center of mass of each object in a system. Mathematically, it is defined as

$$\mathbf{c} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + ... + m_n\mathbf{x}_n}{m_1 + m_2 + ... + m_n}."$$ [22]

2.1.15 Moment of Inertia

" Moment of inertia is defined as the product of mass of section and the square of the distance between the reference axis and the centroid of the section . Moment of inertia I is defined as the ratio of the net angular momentum L of a system to its angular velocity ω around a principal axis that is

$$I = \frac{L}{\omega}.\text{''}[23]$$

2.1.16 Orthogonal transformation

"Basically an **orthogonal transformation** is a linear transformation $T : V \to V$ on a real inner product space V, that preserves the inner product. That is, for each pair u, v of elements of V, we have

$$\langle u, v\rangle = \langle Tu, Tv\rangle.\text{''}[24]$$

2.1.17 Open and Closed Region in $\mathbb{R}^2$

"A region is open if it consists entirely of interior points whereas a region is closed if it also contains all of its boundary points."

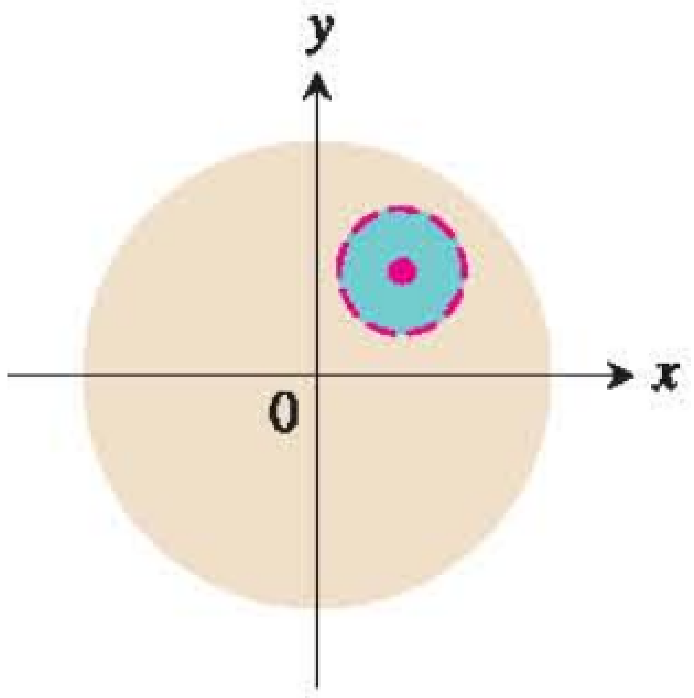

FIGURE 2.1: $x^2 + y^2 < 1$(Open unit disk)

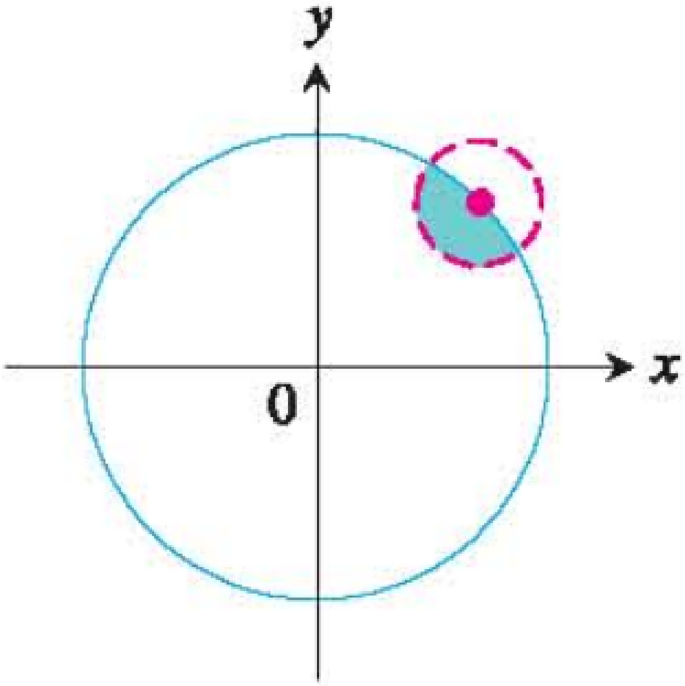

FIGURE 2.2: $x^2 + y^2 = 1$(Boundary of unit disk)

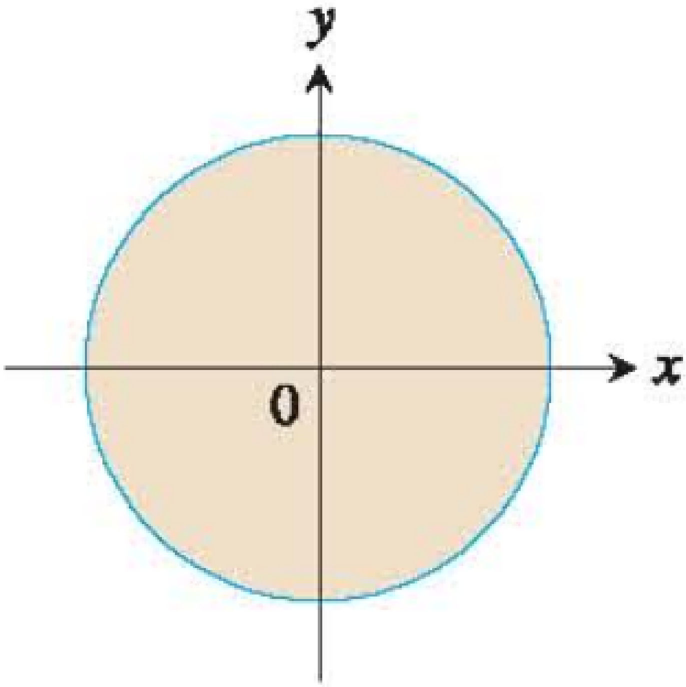

FIGURE 2.3: $x^2 + y^2 \leq 1$(Closed unit disk)

"A region in the plane is **bounded** if it lies inside a disk of fixed radius and a region is **unbounded** if it is not bounded."

2.1.18 Symmetry

"In general, two points P_1 and P_2 are said to be symmetric to a line if the line is the perpendicular bisector of the line segment joining the two points. In a similar fashion a graph is said to symmetric to a line if all points of the graph can be grouped into pairs which are symmetric to the line and then the line is called the axis of symmetry of the graph. A point of symmetry occurs if all points on the graph can be grouped into pairs so that all the line segments joining the pairs are then bisected by the same point."[25]

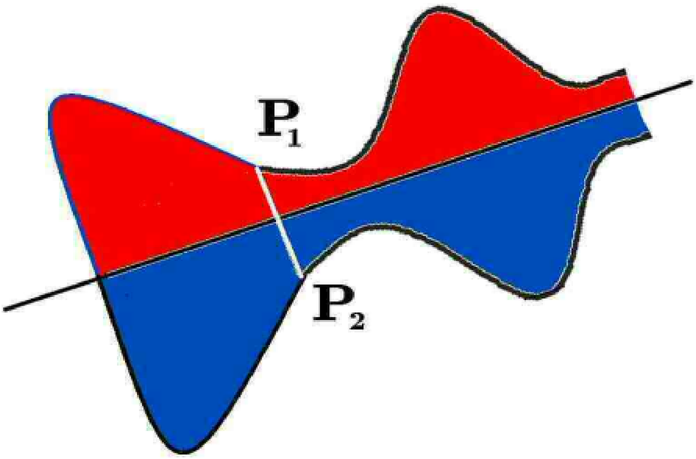

FIGURE 2.4: Symmetry about a line

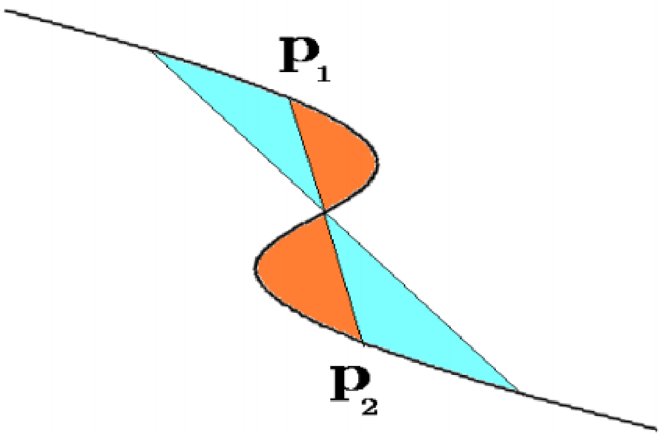

FIGURE 2.5: Symmetry about a point

2.1.19 Planes and Axis of Symmetry

"If any homogeneous body is symmetrical with respect to any plane, the center of mass will be in that plane, because each element of mass on one side of the plane can be paired with the corresponding element of mass on the other side and the whole body can be divided into such paired elements. This plane is called a **plane of symmetry**. Whereas if a homogeneous body is symmetrical with respect to two planes, then the center of mass is in the line of their intersection. Such a line is called **axis of symmetry**."[25]

2.1.20 Newtonian Potential

"In mathematics, the **Newtonian potential or Newton potential** is an operator in vector calculus that acts as the inverse to the negative Laplacian, on functions that are smooth and decay rapidly enough at infinity. In its general nature, it is a singular integral operator, defined by convolution with a function having a mathematical singularity at the origin, the Newtonian kernel which is the fundamental solution of the Laplace equation. It is named for Isaac Newton, who first discovered it and proved that it was a harmonic function in the special case of three variables, where it served as the fundamental gravitational potential in Newton's law of universal gravitation. In modern potential theory, the Newtonian potential is instead thought of as an electrostatic potential."

2.1.21 Planar Central Configuration

"The planar central configurations of the 4-BP are classified as convex or concave. Thus a central configuration is **convex** if none of the bodies is located in the interior of the triangle formed by the other three. A central configuration is **concave** if one of the bodies is in the interior of the triangle formed by the other three." [26]

2.2 Kepler's Laws

Johannes Kepler, an astronomer and mathematician, discovered three laws concerning the motion of the planets. He discovered these laws from experimental data without the aid of calculus or vector analysis. Newton, using calculus, verified these laws with the model for the inverse square law of attraction. Kepler's three laws can be defined as follows:

1. "Keplers first law states that all the planets of the solar system describe elliptical paths with the sun at one focus.

$$r = \frac{p}{1 + k\cos\theta},$$

where $p = \frac{h^2}{GM}$ and $k = \frac{C}{GM}$."

2. "Kepler's second law states that the position vector **r** sweeps out equal areas in equal time intervals. Consider the area swept out by the position vector of a planet during a time interval $\triangle t$. This element of area, in polar coordinates, is written as

$$dW = \frac{1}{2}\mathbf{r}^2 d\theta."$$

3. "Keplers third law depicts the fact that the square of the period of one revolution is proportional to the cube of the semi-major axis of the elliptical orbit.

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) a^3,$$

where T is the time period, a is the semi major axis, M_s is the mass of Sun and G is the universal gravitational constant."[25]

2.3 Newton's Laws of Motion

The following three laws of motion given by Newton are considered the axioms of mechanics:

1. **First law of motion**: "Every particle persists in a state of rest or of uniform motion in a straight line unless acted upon by a force."

2. **Second law of motion**: " If $\mathbf{F}$ is the external force acting on a particle of mass m which as a reaction is moving with velocity $\mathbf{v}$, then

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}.$$

If m is independent of time this becomes

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a},$$

where $\mathbf{a}$ is the acceleration of the particle."

3. **Third law of motion**: "For every action, there is an equal and opposite reaction."

2.4 Newton's Universal Law of Gravitation

"Every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. In equation form, this is

$$F = G\frac{mM}{r^2}$$

where F is the magnitude of the gravitational force. G is the gravitational constant, given by $G = 6.674 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$ [27]."

2.5 Two Body Problem(2BP)

The 2BP, first explored and resolved by Newton, states: Assume that at any time t, given the positions and velocities of two massive bodies moving under their mutual gravitational force, then what will be their position and velocities for any other time t, if the masses are given? For-example a planet orbiting around a star and two stars orbiting around each other (Earth-Sun, Moon-Earth).

2.5.1 The Solution to the 2BP

"Between any two bodies of masses m_1 and m_2 at a distance s from each other there exist attractive forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ directed from one body to the other. The two forces have equal magnitude which is directly proportional to the product of the masses and inversely proportional to the square of the distance". Now Newton's universal gravitational law can be rewritten as:

$$\mathbf{F}_{12} = \mathbf{F}_{21} = G\frac{m_1 m_2}{\mathbf{s}^3}\mathbf{s}, \tag{2.1}$$

and

$$\mathbf{F}_{12} = -\mathbf{F}_{21}, \tag{2.2}$$

where the G is the universal gravitational constant that depends only on the chosen system of units.

Let us now consider, in a given inertial reference system, two bodies of mass m_1 and m_2 respectively, $\mathbf{s}_1$ and $\mathbf{s}_2$ are their distance vectors from the origin of the system. The equations of motion for 2BP are given by

$$m_1\ddot{\mathbf{s}}_1 = m_1\frac{d^2\mathbf{s}_1}{dt^2} = G\frac{m_1 m_2}{s^3}\mathbf{s}, \tag{2.3}$$

and

$$m_2\ddot{\mathbf{s}}_2 = m_2\frac{d^2\mathbf{s}_2}{dt^2} = -G\frac{m_1 m_2}{s^3}\mathbf{s}, \tag{2.4}$$

where $\mathbf{s} = |\mathbf{s}_2 - \mathbf{s}_1|$. Now if we multiply the equation (2.1) by m_1 and m_2 and then after some simplification we obtain,

$$m_1\ddot{\mathbf{s}}_1 + m_2\ddot{\mathbf{s}}_2 = 0 \tag{2.5}$$

and, by defining

$$\mathbf{c} = \frac{m_1\mathbf{s}_1 + m_2\mathbf{s}_2}{m_1 + m_2} \tag{2.6}$$

By integrating, we immediately get

$$m_1\dot{\mathbf{s}}_1 + m_2\dot{\mathbf{s}}_2 = \mathbf{k}_1, \tag{2.7}$$

that means the total linear momentum of the system remains constant. Integrating again implies that

$$m_1\dot{\mathbf{s}}_1 + m_2\dot{\mathbf{s}}_2 = \mathbf{k}_1 t + \mathbf{k}_2, \tag{2.8}$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ are constant vectors.

Now by using the equation (2.6), $\mathbf{S}$ is defined as:

$$\begin{aligned}(m_1 + m_2)\mathbf{S} &= m_1\mathbf{s}_1 + m_2\mathbf{s}_2, \\ m_t\mathbf{S} &= m_1\mathbf{s}_1 + m_2\mathbf{s}_2,\end{aligned} \tag{2.9}$$

where $m_t = m_1 + m_2$.

After taking derivative of equation (2.9) and comparing with equation (2.7), we get

$$m_t\dot{\mathbf{S}} = \mathbf{k}_1 \quad \Rightarrow \quad \dot{\mathbf{S}} = \frac{\mathbf{k}_1}{m_t} = constant$$

. Solving the equations (2.3) and (2.4), we get:

$$\ddot{\mathbf{s}}_1 - \ddot{\mathbf{s}}_2 = \frac{Gm_2}{s^3}\mathbf{s} + \frac{Gm_1}{s^3}\mathbf{s}, \tag{2.10}$$

$$\Rightarrow \ddot{\mathbf{s}} = \alpha\frac{\mathbf{s}}{s^3}$$

$$\Rightarrow \ddot{\mathbf{s}} + \alpha\frac{\mathbf{s}}{s^3} = \mathbf{0}, \tag{2.11}$$

where $\alpha = G(m_1 + m_2)$.

Now taking the cross product of $\mathbf{s}$ with equation (2.11), we get:

$$\mathbf{s} \times \alpha\ddot{\mathbf{s}} + \frac{\alpha^2}{s^3}\mathbf{s} \times \mathbf{s} = \mathbf{0}$$

$$\Rightarrow \mathbf{s} \times \ddot{\mathbf{s}} = \mathbf{0}, \tag{2.12}$$

$$\Rightarrow \mathbf{s} \times \alpha\ddot{\mathbf{s}} = \mathbf{0},$$

$$\Rightarrow \mathbf{s} \times \mathbf{F} = \mathbf{0}, \tag{2.13}$$

where $\mathbf{F} = \alpha\ddot{\mathbf{s}} = \alpha\mathbf{a}$ (α is reduced mass i.e. constant). After integrating above equation (2.12), we get:

$$\mathbf{s} \times \dot{\mathbf{s}} = \mathbf{T}, \tag{2.14}$$

where $\mathbf{T}$ is a constant vector.

2.6 Three-Body Problem (3BP)

"The 3BP is the problem of taking the initial positions and velocities (or momenta) of three point masses and solving for their subsequent motion according to Newton's laws of motion and Newton's law of universal gravitation." It originated as an effort to understand the influence of the Sun on the movement of the Moon around the Earth.

The 3BP led to the discovery of the planet Neptune, describing the position and stability of the planet and has strengthened our knowledge of the solar system's stability. In the past, many physicists, astronomers and mathematicians attempted unsuccessfully to find closed form solutions to the 3BP.

Such solutions do not exist because motions of the three bodies are in general unpredictable, which makes the 3BP one of the most challenging problems in the history of science.

2.6.1 The Solution to the 3BP

We assume a problem having 3-mass points moving under their mutual gravitational attraction.

So the equation of motion for the 3BP will be

$$m_a \frac{d^2\mathbf{r}_a}{dt^2} = \sum_{b \neq a} G m_a m_b \frac{\mathbf{r}_b - \mathbf{r}_a}{|\mathbf{r}_b - \mathbf{r}_a|^3}, \qquad a = 1, 2, 3. \tag{2.15}$$

where m_a are the point masses with position vectors $\mathbf{r}_a$ for $a = 1, 2, 3$.

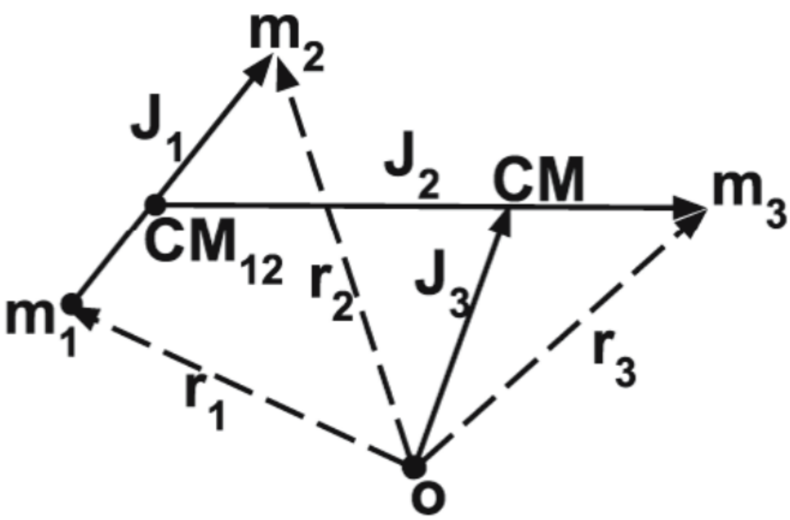

Figure 2.6: Three Body problem

where O is the origin of the coordinate system and $\mathbf{J}_1, \mathbf{J}_2$ and $\mathbf{J}_3$ are Jacobi vectors defined as

$$\mathbf{J}_1 = \mathbf{r}_2 - \mathbf{r}_1, \tag{2.16}$$

$$\mathbf{J}_2 = \mathbf{r}_3 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \tag{2.17}$$

$$\mathbf{J}_3 = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3}. \tag{2.18}$$

$\mathbf{J}_3$ is the coordinate of the CM (center of mass), $\mathbf{J}_1$ the position vector of m_2 relative to m_1 and $\mathbf{J}_2$ that of m_3 relative to the CM of m_1 and m_2 (see figure (2.6)). The important thing related to Jacobi vectors is that $T = \frac{1}{2}\sum_{a=1,2,3} m_a\dot{\mathbf{r}}_a^2$ (kinetic energy) and $I = \sum_{a=1,2,3} m_a\mathbf{r}_a^2$ (moment of inertia) remains diagonal:

$$T = \frac{1}{2}\sum_{a=1,2,3} m_a\dot{\mathbf{J}}_a^2,$$

$$I = \sum_{a=1,2,3} m_a\mathbf{J}_a^2.$$

Here the potential energy V (in terms of $\mathbf{J}_1$ and $\mathbf{J}_2$) can be expressed as

$$V = -\frac{Gm_1m_2}{|\mathbf{J}_1|} - \frac{Gm_2m_3}{|\mathbf{J}_2 - \mu\mathbf{J}_1|} - \frac{Gm_3m_1}{|\mathbf{J}_2 + \mu\mathbf{J}_1|}, \tag{2.19}$$

where $\mu_1 = \frac{m_1}{m_1+m_2}$ and $\mu_2 = \frac{m_2}{m_1+m_2}$. An instantaneous configuration of the three bodies defines a triangle with masses at its vertices. The moment of inertia about the center of mass $I_{CM} = M_1\mathbf{J}_1^2 + M_2\mathbf{J}_2^2$ defines the size of the triangle.[28]

2.6.2 Restricted Three-Body Problem(R3BP)

"The R3BP is a simplified version of the 3BP where one of the mass m_3 is assumed much smaller than the other two m_1 and m_2. Thus, m_1 and m_2 move in Keplerian orbits which are not affected by m_3. In the planar circular R3BP, m_1 and m_2 move in fixed circular orbits around their common CM with angular speed $\wedge = (G(m_1+m_2)/d^3)^{1/2}$ defined by Kepler's third law and m_3 moves in the same plane as m_1 and m_2. The two masses m_1 and m_2 are separated by d."

This system has 2 degrees of freedom associated with the planar motion of m_3 and therefore, a 4-D phase space just like the planar Kepler problem for the reduced mass. However, unlike the latter which has three conserved quantities and is exactly solvable, the planar R-3BP has only one known conserved quantity "Jacobi integral", which is the energy of m_3 in the non-inertial frame of the primaries (i.e m_1 and m_2):

$$E = \left(\frac{1}{2}M_3\dot{r}^2 + \frac{1}{2}m_3 r^2\dot{\phi}^2\right) - \frac{1}{2}m_3 \wedge^2 r^2 - Gm_3\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) \equiv T + V_{eff}. \quad (2.20)$$

Here (r, ϕ) are the plane polar coordinates of m_3 in the co-rotating frame of the primaries (m_1 and m_2) with origin located at their CM while r_1 and r_2 are the distances of m_3 from m_1 and m_2. [28]

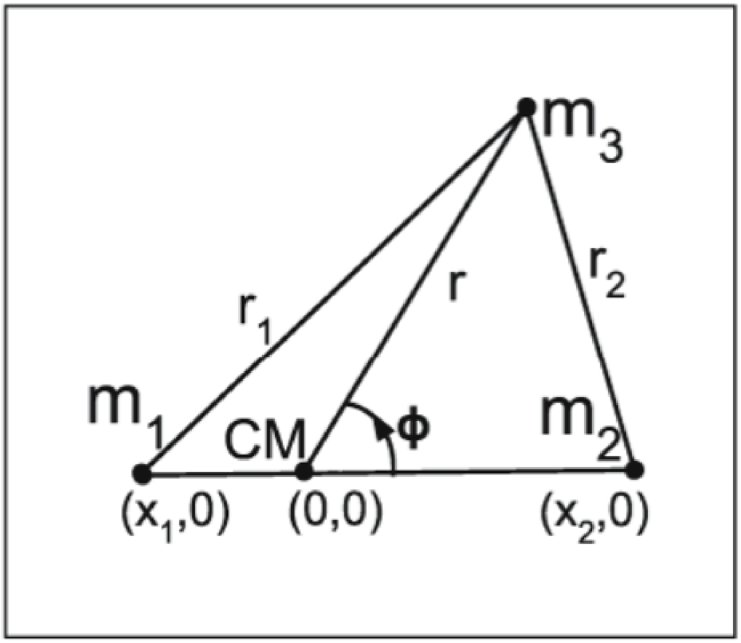

FIGURE 2.7: The secondary m_3 in the co-rotating frame of primaries m_1 and m_2 in the R3BP. The origin is located at the CM of m_1 and m_2 which coincides with the CM of the system since $m_3 << m_1$ and m_2.

Chapter 3

Planar Symmetric Concave Central Configuration in Newtonian Four Body Problem

3.1 Introduction and Important Results

The Newtonian N-Body problem [29–37] involves the motion of N points with masses $m_i \in \mathbb{R}^+, i = 1, ...N$.
The equation of motion for N-positive masses is directed by Gravitational law and Newton's second law:

$$m_i \ddot{\mathbf{q}}_i = \sum_{k \neq i}^{N} m_k m_i \frac{\mathbf{q}_k - \mathbf{q}_i}{|\mathbf{q}_k - \mathbf{q}_i|^3}, \qquad i = 1, 2, ...N. \tag{3.1}$$

Here $\mathbf{q}_i \in \mathbb{R}^d, (d = 1, 2, 3)$ is the position vector of the ith body, m_i is the mass of the ith body.

The above equation (3.1) can be written as

$$m_i \ddot{\mathbf{q}}_i = \frac{\partial U(\mathbf{q})}{\partial \mathbf{q}_i}, \qquad i = 1, 2, ...N \tag{3.2}$$

and

$$\begin{aligned} U(q) &= U(q_1, q_2, q_3, ..., q_N) \\ &= \sum_{1 \le k < j \le i} \frac{m_k m_i}{|q_k - q_j|}, \end{aligned} \tag{3.3}$$

where $U(q)$ is the Newtonian potential of the above system)3.1). The first momentum, total mass and center of mass is defined by

$$\begin{aligned} C &= m_1 \mathbf{q}_1 + m_2 \mathbf{q}_2 + ... + m_N \mathbf{q}_N, \\ M &= m_1 + m_2 + m_3 + ... + m_N, \\ c &= \frac{C}{M} \end{aligned}$$

respectively.

The set defined by $\Delta = \{\mathbf{q} \in (\mathbb{R}^d)^N : \mathbf{q}_i = \mathbf{q}_j, i \neq j\}$ is the set of collision configurations, where a configuration $q = (\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N) \in (\mathbb{R}^d)^N \setminus \Delta$ is called a central configuration [11].

"A CC is basically a particular configuration of N-bodies where the acceleration vector of each body is proportional to its position vector, and the constant of proportionality is the same for N bodies." Now if there exit some positive constant λ then equation (3.1) can be written as

$$\sum_{j=1, j\neq i}^{N} \frac{m_j(\mathbf{q}_j - \mathbf{q}_i)}{|\mathbf{q}_j - \mathbf{q}_i|^3} = -\lambda(\mathbf{q}_i - c), \qquad i = 1, 2, 3, ..., N. \tag{3.4}$$

Moreover, it can easily be classified [38] that $\lambda = \frac{U}{I}$ where $I = \sum_{i=1}^{N} m_i |\mathbf{q}_i - c|^2$ denotes the inertial of the system (3.1). The set Δ of central configuration remains unchange under some transformations on $(\mathbb{R}^d)^N$. These transformations are named as translation, scaling and orthogonal transformation.

Different forms of NBP persuade the analysis of CC. We can have a look in these papers [16, 39–42], that allow to calculate the homographic solutions, and if the

N-bodies are describing for a simultaneous collision, then the bodies turn to CC. These configurations also appear as a key point when stating the topological reforms of the integral augment.

Few important results related to CC's for NBP's along with their masses are greatly explained by some authors. Specially, Hampton-Moeckal [43] proved that the number of relative equilibrium of the NFBP is finite with any given positive masses, up to symmetry. They also show that this number always lie between 32 and 8472. But, except for the collinear one, the shapes of CC's are very difficult to research and it is still an important development about Smale-Wintner's finiteness conjecture of CC for NBP with given positive masses up to symmetry. Smale's problems are a list of 18 unsolved math problems suggested by Steve Smale in 1998. [11, 42]

Another scientist Cors-Roberts [44] also worked on four-body problem. They basically investigated four-body co-circular central configurations, where the bodies are assumed to lie on a common circle with positive masses. They also proved that it is a set of 2-D surface and the configuration has a line of symmetry if any two of the masses of four-body co-circular configurations are equal. They also studied that there are four-body central configurations with positive masses and no symmetries.

Let m_1, m_2, m_3, m_4 be the four mass points. If one of these point mass lies inside the triangle formed by the remaining three point mass then a configuration $q = q_1, q_2, q_3, q_4$ is said to be concave. Long and Sun [45] proved that:

3.1.1 Lemma

"*Let $\alpha, \beta > 0$ be any two given real numbers. Let $q = (q_1, q_2, q_3, q_4) \in (\mathbb{R}^2)^4$ be a concave non-collinear central configuration with masses $(\beta, \alpha, \beta, \beta)$ respectively, and with q_2 located inside the triangle formed by q_1, q_3 and q_4. Then the configuration q must posses a symmetry, so either q_1, q_2 and q_4 form an equilateral triangle and q_2 is located at the center of the triangle, or q_1, q_3 and q_4 form an* *isosceles triangle, and q_2 is on the symmetrical axis of the triangle.*"

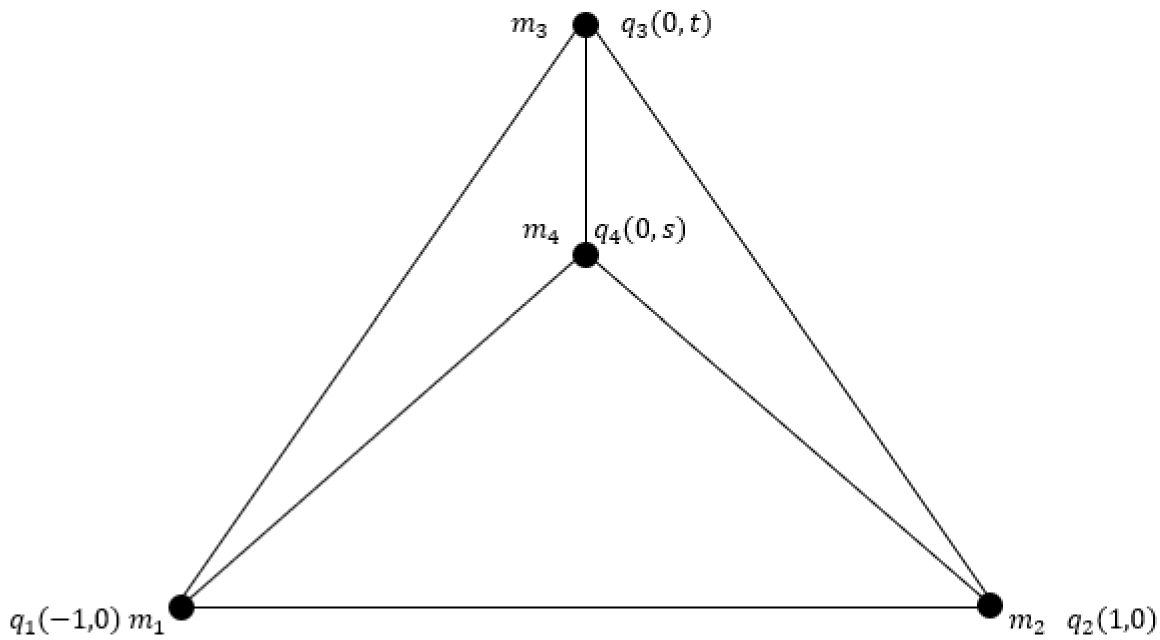

FIGURE 3.1: The Symmetric Concave Configuration

In order to solve any problem or any theorem, first we consider an inverse of that problem. Let a configuration is given, we have to calculate mass vectors to prove that the given configuration is CC. In order to understand this problem we can see that Moultan [39] and Wintner [11] take the inverse problem to solve for collinear NBP. Moultan proved the two closely related problem for collinear NBP. One is that, for N arbitrary positive masses we have to determine the ratios of the distances in order to show that the bodies will always remain collinear under proper initial projection. Second problem is that of determining, if they are placed at N arbitrary collinear points for N masses, they will always remain in a straight line under proper initial projection.

Moeckel and Albouy [29] proved that for a given central configuration, each of them determines a two-parameter family of masses to make it central where negative masses are also permitted. Similarly, Xie and Ouyang prove that it is often possible to pick positive masses for any configuration in the complement of the

compact region to make the configuration central. They also show that for any given configuration, it can be a CC for infinitely many masses. A configuration is said to be a super central configuration if it is a CC for m mass vector and another mass vector m', where m' is the permutation of m and $m' \neq m$.
Xie [46] also proved the super central configuration also exist in the collinear 4BP by considering the inverse problem of CC of NBP. Their results were also used to examine the number of central configurations under geometric equivalence in [47]. Marshall Hampton study few results of concave CC's for 4BP in theorem 6, 7 and 8 in paper [48]. They proved that for a strictly admissible triangle, there is a special CC, for which the triangle is the outer triangle with suitable parameter and the forth point traces out a simple curve from the circumcenter to the equilateral point.

In the current study, we take an inverse problem that is for a given planar symmetric concave configuration (see figure 3.1), we have to find a positive mass vectors to make the configuration a central configuration.
Now, for 4 bodies, substituting $N = 4$ in equation (3.4), we get equations for CC for general 4BP that are given below:

$$\left.\begin{aligned}
m_2\frac{\mathbf{q}_2-\mathbf{q}_1}{|\mathbf{q}_2-\mathbf{q}_1|^3}+m_3\frac{\mathbf{q}_3-\mathbf{q}_1}{|\mathbf{q}_3-\mathbf{q}_1|^3}+m_4\frac{\mathbf{q}_4-\mathbf{q}_1}{|\mathbf{q}_4-\mathbf{q}_1|^3} &= -\lambda(\mathbf{q}_1-\mathbf{c}),\\
m_1\frac{\mathbf{q}_1-\mathbf{q}_2}{|\mathbf{q}_1-\mathbf{q}_2|^3}+m_3\frac{\mathbf{q}_3-\mathbf{q}_2}{|\mathbf{q}_3-\mathbf{q}_2|^3}+m_4\frac{\mathbf{q}_4-\mathbf{q}_2}{|\mathbf{q}_4-\mathbf{q}_2|^3} &= -\lambda(\mathbf{q}_2-\mathbf{c}),\\
m_1\frac{\mathbf{q}_1-\mathbf{q}_3}{|\mathbf{q}_1-\mathbf{q}_3|^3}+m_2\frac{\mathbf{q}_2-\mathbf{q}_3}{|\mathbf{q}_2-\mathbf{q}_3|^3}+m_4\frac{\mathbf{q}_4-\mathbf{q}_3}{|\mathbf{q}_4-\mathbf{q}_3|^3} &= -\lambda(\mathbf{q}_3-\mathbf{c}),\\
m_1\frac{\mathbf{q}_1-\mathbf{q}_4}{|\mathbf{q}_1-\mathbf{q}_4|^3}+m_2\frac{\mathbf{q}_2-\mathbf{q}_4}{|\mathbf{q}_2-\mathbf{q}_4|^3}+m_3\frac{\mathbf{q}_3-\mathbf{q}_4}{|\mathbf{q}_3-\mathbf{q}_4|^3} &= -\lambda(\mathbf{q}_4-\mathbf{c}).
\end{aligned}\right\} \tag{3.5}$$

Chunhua Deng and Shiqing Zhang proved the following theorems:

3.1.2 Theorem 1

Let $q_1 = (-1, 0), q_2 = (1, 0), q_3 = (0, t), q_4 = (0, s)$ *where* $0 < s < t$ *and consider that* $\mathbf{c} = \frac{C}{M} = (c_x, c_y) = q_4$ *is the center of mass. The symmetric concave configuration* $q = (q_1, q_2, q_3, q_4)$ *can be a central configuration iff* $t = \sqrt{3}, s = \frac{\sqrt{3}}{3}$*. The* *masses of q_1, q_2 and q_4 are all equal, i.e $m_1 = m_2 = m_3 > 0$, where the mass of q_4 can be any positive number $m_4 > 0$.*

3.1.3 Theorem 2

Let us consider $q_1 = (-1,0), q_2 = (1,0), q_3 = (0,t), q_4 = (0,s)$ where $0 < s < t$ and consider that $\mathbf{c} = \frac{C}{M} = (c_x, c_y) \neq q_4$ is the center of mass. Then there exist two open bounded regions E and F which can be seen in the figure (3.8), the configuration $q = (q_1, q_2, q_3, q_4)$ can be a central configuration with positive masses:

$$m_1 = m_2 = \lambda \frac{4\sqrt{1+t^2}^3 (t-c_y)}{t\sqrt{1+s^2}^3 (t-s)^3} \frac{(t-s)^3 - \sqrt{1+s^2}^3}{\left(\frac{2}{\sqrt{1+s^2}}\right)^3 - \left(\frac{\sqrt{1+t^2}}{t-s}\right)^3}, \tag{3.6}$$

$$m_3 = \frac{\lambda s\sqrt{1+t^2}^3}{\sqrt{1+s^2}^6 (t-s)^3} \frac{\beta}{\gamma}, \tag{3.7}$$

$$m_4 = \frac{\lambda(t-c_y)}{(t-s)} \frac{(8 - \sqrt{1+t^2}^3)}{\left(\left(\frac{2}{\sqrt{1+s^2}}\right)^3 - \left(\frac{\sqrt{1+t^2}}{t-s}\right)^3\right)}, \tag{3.8}$$

where $\beta = (\sqrt{1+s^2}^3 - 8)(\sqrt{1+s^2}^3 - (t-s)^3)$ and
$\gamma = \left(\frac{t-s}{(t-s)^3} + \frac{s}{\sqrt{1+s^2}^3} - \frac{t}{\sqrt{1+t^2}^3}\right)$
$\left(\left(\frac{2}{\sqrt{1+s^2}}\right)^3 - \left(\frac{\sqrt{1+t^2}}{t-s}\right)^3\right).$

3.2 Isosceles Concave Central Configuration with Four Bodies

As we are given the values of $q_1 = (-1,0), q_2 = (1,0), q_3 = (0,t), q_4 = (0,s)$ where $0 < s < t$, then by using all these values in the the equation (3.5), it can be divided into two parts:

For x-axis:

$$\left.\begin{aligned} \frac{m_2}{4}+\frac{1}{\sqrt{1+t^2}^3}m_3+\frac{1}{\sqrt{1+s^2}^3}m_4=\lambda(1+c_x)\\ \frac{m_2}{4}+\frac{-1}{\sqrt{1+t^2}^3}m_3+\frac{-1}{\sqrt{1+s^2}^3}m_4=-\lambda(1-c_x)\\ \frac{-1}{\sqrt{1+t^2}^3}m_1+\frac{1}{\sqrt{1+t^2}^3}m_2=\lambda c_x\\ \frac{-1}{\sqrt{1+s^2}^3}m_1+\frac{1}{\sqrt{1+s^2}^3}m_2=\lambda c_x. \end{aligned}\right\} \tag{3.9}$$

For y-axis:

$$\left.\begin{aligned} \frac{t}{\sqrt{1+t^2}^3}m_3+\frac{s}{\sqrt{1+s^2}^3}m_4=\lambda c_y\\ \frac{t}{\sqrt{1+s^2}^3}m_3+\frac{s}{\sqrt{1+s^2}^3}m_4=\lambda c_y\\ \frac{-t}{\sqrt{1+t^2}^3}m_1+\frac{-t}{\sqrt{1+t^2}^3}m_2+\frac{s-t}{(t-s)^3}m_3=-\lambda(t-c_y)\\ \frac{-s}{\sqrt{1+s^2}^3}m_1+\frac{-s}{\sqrt{1+s^2}^3}m_2+\frac{t-s}{(t-s)^3}m_3=-\lambda(s-c_y). \end{aligned}\right\} \tag{3.10}$$

As we can see that the first two equations equation (3.10) are similar.

Now, solving the third and forth equation in equation (3.9), we get the following results

$$\left(\frac{1}{\sqrt{1+s^2}^3}-\frac{1}{\sqrt{1+t^2}^3}\right)m_1+\left(\frac{1}{\sqrt{1+t^2}^3}-\frac{1}{\sqrt{1+s^2}^3}\right)m_2=0,$$

$$\left(\frac{1}{\sqrt{1+s^2}^3}-\frac{1}{\sqrt{1+t^2}^3}\right)(m_2-m_1)=0, \tag{3.11}$$

for $0<s<t$, we have $m_1=m_2$.

Adding the first two equations in equation (3.9) with $m_1=m_2$. We get the following result:

$$c_x=0$$

Now, using all the above values and combining equation (3.9) and (3.10), we get a system of equations for central configuration.

$$\left.\begin{aligned} \frac{m_2}{4} + \frac{1}{\sqrt{1+t^2}^3}m_3 + \frac{1}{\sqrt{1+s^2}^3}m_4 = \lambda \\ \frac{t}{\sqrt{1+t^2}^3}m_3 + \frac{s}{\sqrt{1+s^2}^3}m_4 = \lambda c_y \\ \frac{-2t}{\sqrt{1+t^2}^3}m_2 + \frac{s-t}{(t-s)^3}m_4 = -\lambda(t-c_y) \\ \frac{-2s}{\sqrt{1+s^2}^3}m_2 + \frac{t-s}{(t-s)^3}m_3 = -\lambda(s-c_y). \end{aligned}\right\} \tag{3.12}$$

3.3 Proof of Theorem 1

In this part, we will find the solutions of all the four masses m_1, m_2, m_3, m_4 with two given parameters s and t for the given 4 body CC.

We consider that the center of masses is denoted by c and its value can be determined by $c = \frac{C}{M} = q_4$ i.e $c_y = s$. Now using all these assumptions, the system of equation (3.12) for central configuration becomes

$$\left.\begin{aligned} \frac{m_2}{4} + \frac{1}{\sqrt{1+t^2}^3}m_3 + \frac{1}{\sqrt{1+s^2}^3}m_4 = \lambda \\ \frac{t}{\sqrt{1+t^2}^3}m_3 + \frac{s}{\sqrt{1+s^2}^3}m_4 = \lambda s \\ \frac{-2t}{\sqrt{1+t^2}^3}m_2 + \frac{s-t}{(t-s)^3}m_4 = -\lambda(t-s) \\ \frac{-2s}{\sqrt{1+s^2}^3}m_2 + \frac{t-s}{(t-s)^3}m_3 = 0. \end{aligned}\right\} \tag{3.13}$$

From the above system of equation, we can rewrite the forth equation as

$$\frac{2s}{\sqrt{1+s^2}^3}m_2 = \frac{t-s}{(t-s)^3}m_3,$$

which implies,

$$m_2 = \frac{\sqrt{1+s^2}^3}{2s} \frac{t-s}{(t-s)^3} m_3. \tag{3.14}$$

Now using equation (3.14) in the third equation of system (3.13), we get

$$\frac{-2t}{\sqrt{1+t^2}^3} \frac{\sqrt{1+s^2}^3}{2s} \frac{t-s}{(t-s)^3} m_3 + \frac{s-t}{(t-s)^3} m_4 = -\lambda(t-s), \tag{3.15}$$

for $0 < s < t$, we have

$$\frac{-t}{\sqrt{1+t^2}^3} \frac{\sqrt{1+s^2}^3}{s} \frac{1}{(t-s)^2} m_3 - \frac{1}{(t-s)^2} m_4 = -\lambda(t-s),$$

which implies

$$\frac{t}{\sqrt{1+t^2}^3} \frac{\sqrt{1+s^2}^3}{s} \frac{1}{(t-s)^3} m_3 + \frac{1}{(t-s)^3} m_4 = \lambda. \tag{3.16}$$

From the second equation of system (3.13) we get

$$m_3 = \left(\lambda s - \frac{s}{\sqrt{1+s^2}^3} m_4 \right) \frac{\sqrt{1+t^2}^3}{t}$$

Now using the above value of m_3 in the equation (3.16), we get

$$\frac{t}{\sqrt{1+t^2}^3} \frac{\sqrt{1+s^2}^3}{s(t-s)^3} \left(\lambda s - \frac{s}{\sqrt{1+s^2}^3} m_4 \right) \frac{\sqrt{1+t^2}^3}{t} + \frac{1}{(t-s)^3} m_4 = \lambda,$$

which implies

$$t - s = \sqrt{1+s^2}. \tag{3.17}$$

Using equation (3.17) in equation (3.14) and third equation of system (3.13), we get

$$m_2 = \frac{t-s}{2s} m_3$$

and

$$m_4 = \lambda \sqrt{1+s^2}^3 - \frac{2t}{\sqrt{1+t^2}^3} \frac{\sqrt{1+s^2}^3}{t-s} m_2.$$

Therefore, the last three equations in system (3.13) will be

$$\left.\begin{aligned} t - s &= \sqrt{1+s^2} \\ m_4 = \lambda\sqrt{1+s^2}^3 - \frac{2t}{\sqrt{1+t^2}^3} &\frac{\sqrt{1+s^2}^3}{t-s} m_2 \\ m_3 &= \frac{2s}{t-s} m_2 \end{aligned}\right\} \tag{3.18}$$

Now, using all the above values of equation (3.18) in the first equation of system (3.13), we get

$$\frac{m_2}{4} + \frac{1}{\sqrt{1+t^2}^3} \frac{2s}{t-s} m_2 + \lambda - \frac{2t}{\sqrt{1+t^2}^3} \frac{\sqrt{1+s^2}^3}{t-s} m_2 = \lambda,$$

$$m_2 \left(\frac{1}{4} + \frac{1}{\sqrt{1+t^2}^3} \frac{2s}{t-s} - \frac{2t}{\sqrt{1+t^2}^3} \frac{\sqrt{1+s^2}^3}{t-s} \right) = 0.$$

After simplifying the above equation we get the following values of parameters s and t.

$$t = \sqrt{3}, \qquad s = \frac{\sqrt{3}}{3}. \tag{3.19}$$

As we knew the values of s and t, so we have $m_1 = m_2 = m_3$ and $m_4 = \frac{8}{9}\sqrt{3}\lambda - \frac{\sqrt{3}}{3} m_2$.

Moreover, for any positive mass $m_4 > 0$, we can take any value of $\lambda > 0$ such that $m_4 = \frac{8}{9}\sqrt{3}\lambda - \frac{\sqrt{3}}{3}m_2$. Hence we have proved theorem 1.

3.4 The Proof of Theorem 2

Now, in this part, we consider that the center of mass is given by $c = \frac{C}{M} \neq q_4$ i.e $c_y \neq s$.

Now combining the second and third equation in system (3.12) and then eliminating m_4, we get the following results:

$$\frac{s-t}{(t-s)^3}\frac{t}{\sqrt{1+t^2}^3}m_3 + \frac{s}{\sqrt{1+s^2}^3}\frac{2t}{\sqrt{1+t^2}^3}m_2 = \lambda(ac_y + b), \tag{3.20}$$

where $a = \frac{s-t}{(t-s)^3} - \frac{s}{\sqrt{1+s^2}^3}$ and $b = \frac{st}{\sqrt{1+s^2}^3}$.

Now taking the forth equation in (3.12) and multiply both sides by $\frac{t}{\sqrt{1+t^2}^3}$, we get

$$\frac{t}{\sqrt{1+t^2}^3}\frac{2s}{\sqrt{1+s^2}^3}m_2 + \frac{s-t}{(t-s)^3}\frac{t}{\sqrt{1+t^2}^3}m_3 = \lambda(c_y - s)\frac{t}{\sqrt{1+t^2}^3}. \tag{3.21}$$

Now by solving equation (3.20) and (3.21), we get the following suitable conditions for solving the system of equation (3.12).

$$\lambda\left(\left(\frac{s-t}{(t-s)^3} - \frac{s}{\sqrt{1+s^2}^3}\right)c_y + \frac{ts}{\sqrt{1+s^2}^3}\right) - \lambda(c_y - s)\frac{t}{\sqrt{1+t^2}^3} = 0,$$
$$\left(\frac{s-t}{(t-s)^3} - \frac{s}{\sqrt{1+s^2}^3}\right)c_y + \frac{ts}{\sqrt{1+s^2}^3} = (c_y - s)\frac{t}{\sqrt{1+t^2}^3}. \tag{3.22}$$

By solving the above equation (3.22), we get the value of c_y, i.e

$$\left(\frac{s-t}{(t-s)^3} - \frac{s}{\sqrt{1+s^2}^3}\right)c_y - \frac{t}{\sqrt{1+t^2}^3}c_y = \frac{st}{\sqrt{1+s^2}^3} - \frac{st}{\sqrt{1+t^2}^3},$$

then

$$c_y = \left(\frac{st}{\sqrt{1+s^2}^3} - \frac{st}{\sqrt{1+t^2}^3} \right) / \left(\frac{t-s}{(t-s)^3} + \frac{s}{\sqrt{1+s^2}^3} - \frac{t}{\sqrt{1+t^2}^3} \right). \tag{3.23}$$

Now using the above, the system (3.12) becomes

$$\left. \begin{aligned} c_y = \left(\frac{st}{\sqrt{1+s^2}^3} - \frac{st}{\sqrt{1+t^2}^3} \right) / \left(\frac{t-s}{(t-s)^3} + \frac{s}{\sqrt{1+s^2}^3} - \frac{t}{\sqrt{1+t^2}^3} \right) \\ \frac{m_2}{4} + \frac{1}{\sqrt{1+t^2}^3} m_3 + \frac{1}{\sqrt{1+s^2}^3} m_4 = \lambda \\ \frac{t}{\sqrt{1+t^2}^3} m_3 + \frac{s}{\sqrt{1+s^2}^3} m_4 = \lambda c_y \\ \frac{-2t}{\sqrt{1+t^2}^3} m_2 + \frac{s-t}{(t-s)^3} m_4 = -\lambda(t-c_y) \end{aligned} \right\} \tag{3.24}$$

Hence, third and forth equation in system (3.24) can be re-written as

$$\frac{t}{\sqrt{1+t^2}^3} m_3 = \lambda c_y - \frac{s}{\sqrt{1+s^2}^3} m_4,$$

$$\frac{1}{\sqrt{1+t^2}^3} m_3 = \frac{1}{t} \left(\lambda c_y - \frac{s}{\sqrt{1+s^2}^3} m_4 \right) \tag{3.25}$$

and

$$\frac{2t}{\sqrt{1+t^2}^3} m_2 + \frac{t-s}{(t-s)^3} m_4 = \lambda(t-c_y),$$

$$\frac{m_2}{4} = \left(\lambda(t-c_y) - \frac{t-s}{(t-s)^3} m_4 \right) \frac{\sqrt{1+t^2}^3}{8t}. \tag{3.26}$$

Substituting the above equation (3.25) and (3.26) into the second equation of system (3.24), we get

$$\frac{\sqrt{1+t^2}^3}{8t}X + \frac{1}{t}Y + \frac{1}{\sqrt{1+S^2}^3}m_4 = \lambda, \tag{3.27}$$

where $X = \lambda(t - c_y) - \dfrac{t-s}{(t-s)^3}m_4$ and $Y = \lambda c_y - \dfrac{s}{\sqrt{1+s^2}^3}m_4$. So

$$m_4 = \frac{\lambda(t-c_y)}{(t-s)} \frac{(8-\sqrt{1+t^2}^3)}{\left(\left(\frac{2}{\sqrt{1+s^2}}\right)^3 - \left(\frac{\sqrt{1+t^2}}{t-s}\right)^3\right)} \tag{3.28}$$

Now, using the value of m_4 in equation (3.25) and equation (3.26) we get the following results.

$$m_3 = \frac{\sqrt{1+t^2}^3}{t}\left(\lambda c_y - \frac{s}{\sqrt{1+s^2}^3}\left(\frac{\lambda(t-c_y)}{(t-s)} \frac{(2^3-\sqrt{1+t^2}^3)}{\left(\left(\frac{2}{\sqrt{1+s^2}}\right)^3 - \left(\frac{\sqrt{1+t^2}}{t-s}\right)^3\right)}\right)\right). \tag{3.29}$$

$$m_2 = \left(\lambda(t-c_y) - \frac{t-s}{(t-s)^3}m_4\right)\frac{\sqrt{1+t^2}^3}{2t}. \tag{3.30}$$

After some manipulation, we get the final results.

$$m_3 = \frac{\lambda s\sqrt{1+t^2}^3}{\sqrt{1+s^2}^6(t-s)^3}\frac{\beta}{\gamma}, \tag{3.31}$$

where $\beta = (\sqrt{1+s^2}^3 - 8)(\sqrt{1+s^2}^3 - (t-s)^3)$ and $\gamma = \left(\frac{t-s}{(t-s)^3} + \frac{s}{\sqrt{1+s^2}^3} - \frac{t}{\sqrt{1+t^2}^3}\right)$ $\left(\left(\frac{2}{\sqrt{1+s^2}}\right)^3 - \left(\frac{\sqrt{1+t^2}}{t-s}\right)^3\right)$.

Similarly

$$m_2 = \lambda\frac{4\sqrt{1+t^2}^3(t-c_y)}{t\sqrt{1+s^2}^3(t-s)^3}\frac{((t-s)^3 - \sqrt{1+s^2}^3)}{((\frac{2}{\sqrt{1+s^2}})^3 - (\frac{\sqrt{1+t^2}}{t-s})^3)}. \tag{3.32}$$

As it is given that $m_1 = m_2$. So,

$$m_1 = m_2 = \lambda \frac{4\sqrt{1+t^2}^3 (t - c_y)}{t\sqrt{1+s^2}^3 (t-s)^3} \frac{((t-s)^3 - \sqrt{1+s^2}^3)}{\left(\left(\frac{2}{\sqrt{1+s^2}}\right)^3 - \left(\frac{\sqrt{1+t^2}}{t-s}\right)^3\right)}. \tag{3.33}$$

Hence we apply the condition (3.23) for the solution of masses and find the solution of all the four masses accuracy in the series of equation (3.28-3.33).

In the following equation we will compute the mass functions and compute the most possible region in st-plane where we get the positive mass functions.

3.5 Lemma 1

The region in which $m_4 > 0$ for $0 < s < t$ is the union of A and B in figure (3.11), surrounded by curves $t = \sqrt{3}, 2(t-s) - \sqrt{1+t^2}\sqrt{1+s^2} = 0$ and $t - s = 0$, i.e. $A = \{(s,t)|0 < s < t < \sqrt{3}, p_2 > 0\}$ and $B = \{(s,t)|0 < s < t, t > \sqrt{3}, p_2 < 0\}$ (shown in figure 3.11).

Proof

By simple calculations, one can calculate the center of masses i.e.

$$c = (c_x, c_y) = (0, \frac{sm_4 + tm_3}{m_1 + m_2 + m_3 + m_4}). \tag{3.34}$$

then from equation (3.28) we have $t - c_y > 0$ for $0 < s < t$. For ease, let us denote $p_1 = 8 - \sqrt{1+t^2}$ and $p_2 = \left(\frac{2}{\sqrt{1+s^2}}\right)^3 - \left(\frac{\sqrt{1+t^2}}{t-s}\right)^3$. So it implies that, if $m_4 > 0$

then $\frac{p_1}{p_2} > 0$.

In the following graphs it is shown that by taking $p_2 = 0$ (figure (3.2)), we get a smooth monotone increasing curve above the curve $t = s$ and bounded by $s = \sqrt{3}$ from right.

The graphs of $p_2 > 0$ and $p_2 < 0$ are separately shown in figure (3.3 and 3.4) respectively. In figure (3.3 and 3.7) the shaded region shows that $t > s$ and $t < s$ respectively.

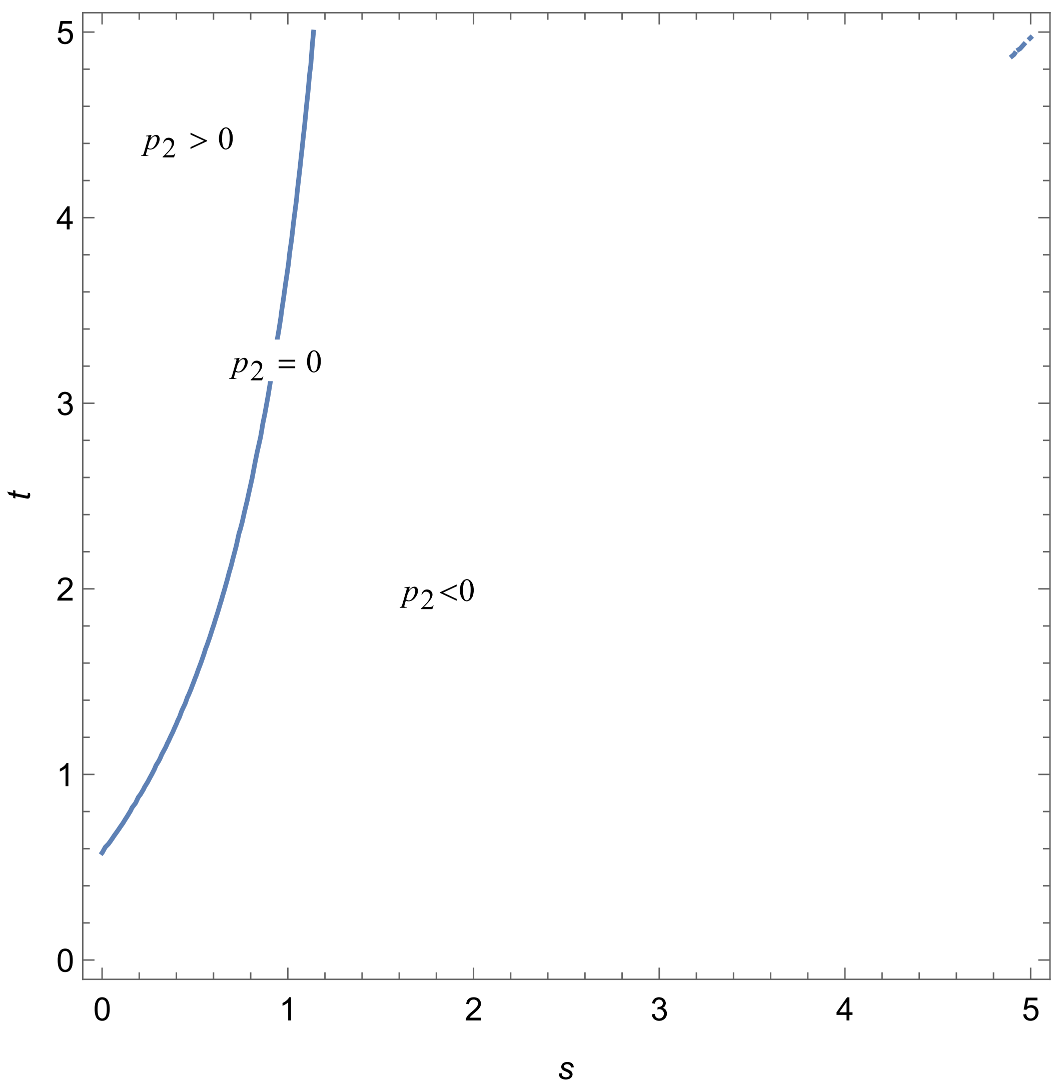

FIGURE 3.2: The sign of p_2

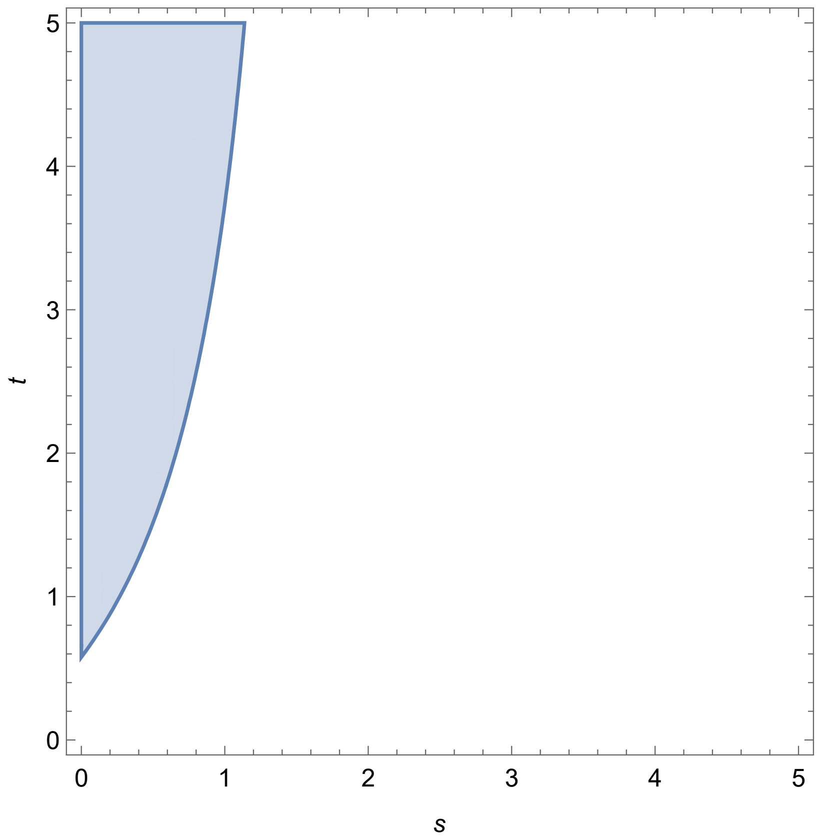

FIGURE 3.3: $p_2 > 0$ (shaded part)

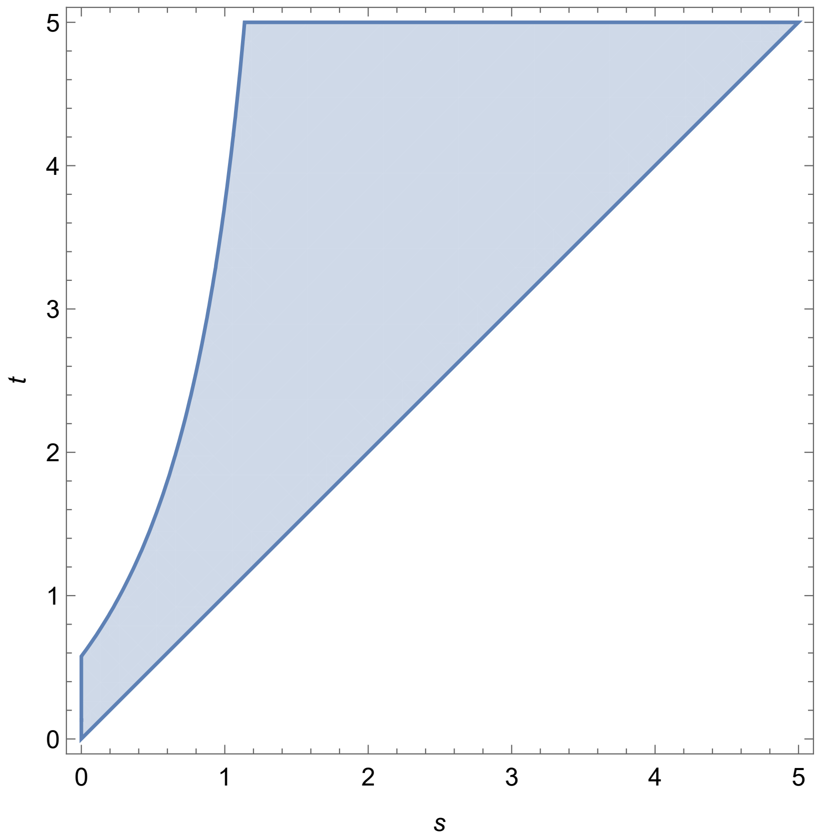

FIGURE 3.4: $p_2 < 0$ (shaded part)

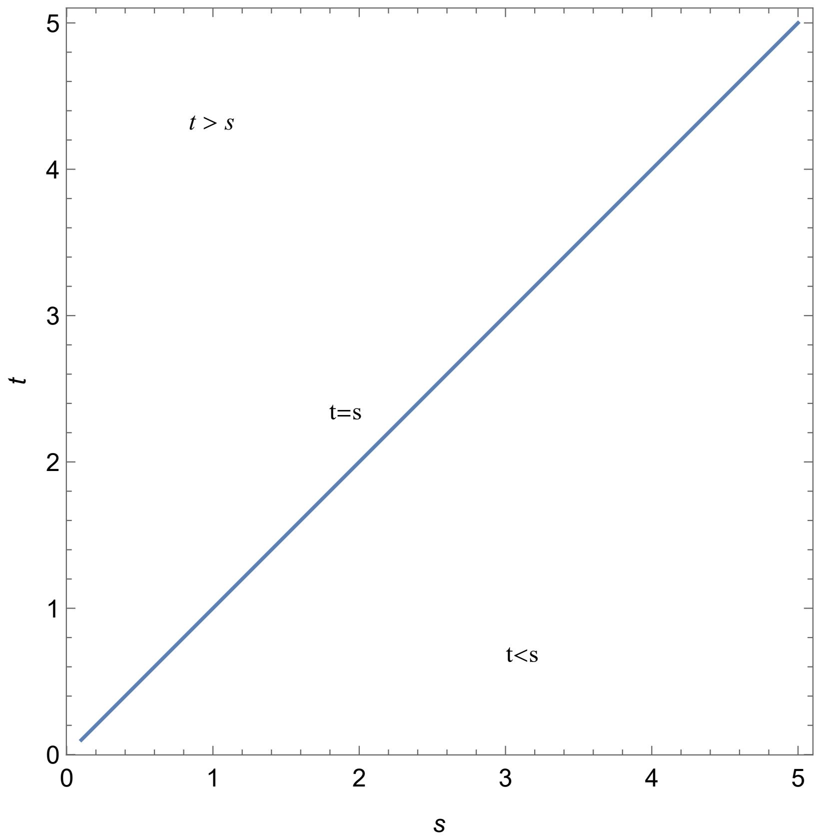

FIGURE 3.5: The sign of $t = s$

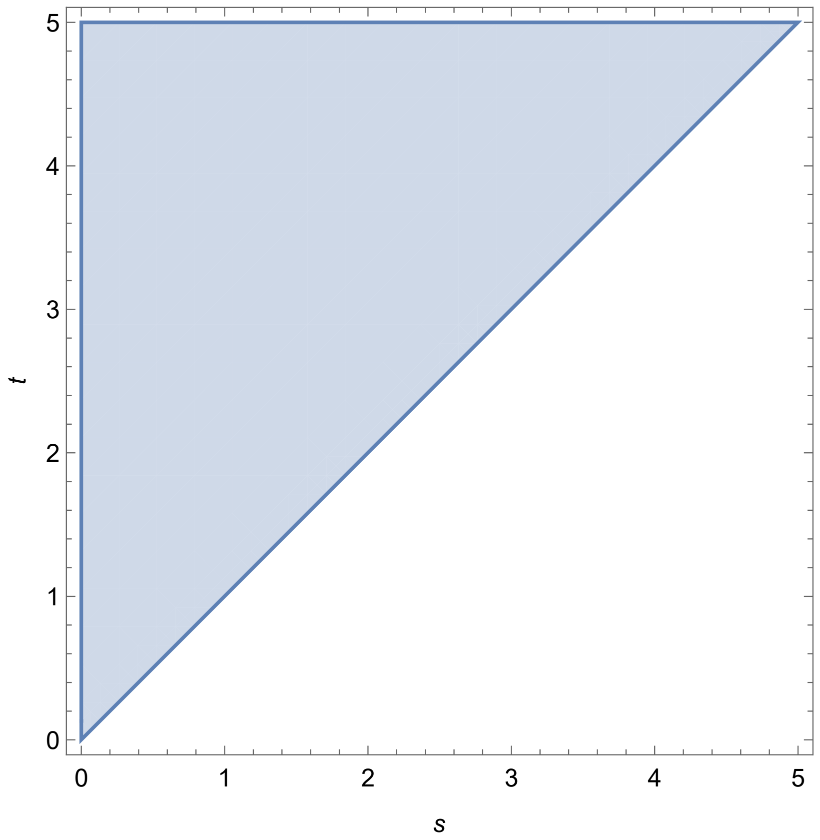

FIGURE 3.6: $t > s$ (shaded part)

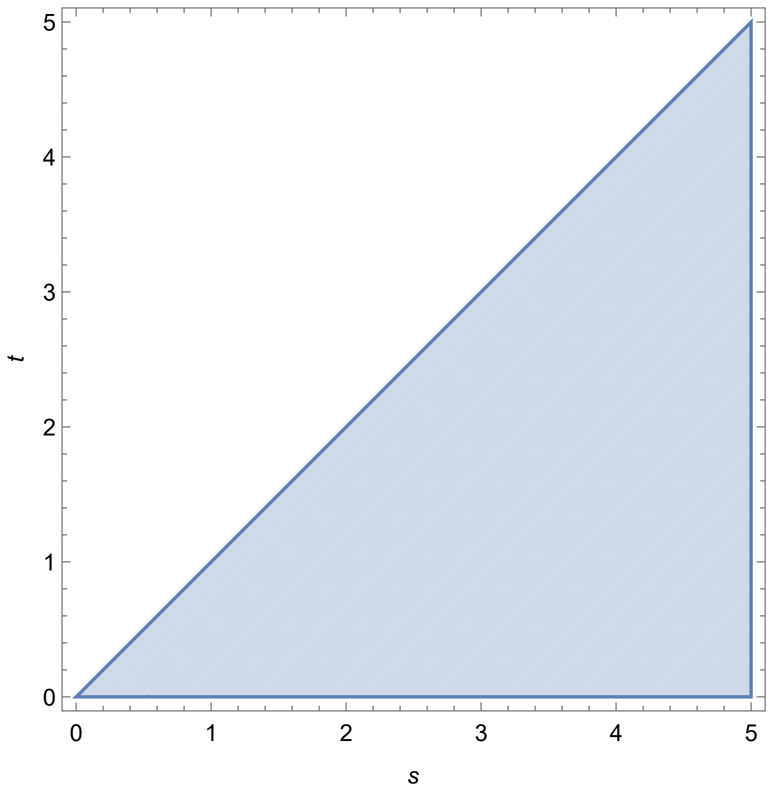

FIGURE 3.7: $t < s$ (shaded part)

Now we can see that if $p_2 = 0$ then $\left(\frac{2}{\sqrt{1+s^2}}\right)^3 - \left(\frac{\sqrt{1+t^2}}{t-s}\right)^3 = 0$, implies that $\sqrt{1+s^2}\sqrt{1+t^2} = 2(t-s)$.

We can clearly examine that

$$\sqrt{1+s^2}\sqrt{1+t^2} = 2(t-s) < 2t \tag{3.35}$$

then

$$\sqrt{1+s^2} < \frac{2t}{\sqrt{1+t^2}} < 2.$$

Hence

$$s < \sqrt{3} \tag{3.36}$$

Moreover, by applying limit on $\sqrt{1+s^2}\sqrt{1+t^2} = 2(t-s)$, we get

$$\lim_{t\to+\infty} 2\left(1 - \frac{s}{t}\right) = \lim_{t\to+\infty} \sqrt{1+s^2}\sqrt{1+\frac{1}{t^2}},$$

then

$$\lim_{t \to +\infty} s = \sqrt{3}. \tag{3.37}$$

Now, let us take the derivative of equation $\sqrt{1+s^2}\sqrt{1+t^2} = 2(t-s)$ with respect to s. We have

$$\left(2 - \frac{t\sqrt{1+s^2}}{\sqrt{1+t^2}}\right)\frac{dt}{ds} = 2 + \frac{s\sqrt{1+t^2}}{\sqrt{1+s^2}}. \tag{3.38}$$

As we know that

$$2 - \frac{t\sqrt{1+s^2}}{\sqrt{1+t^2}} > 2 - \frac{2t}{\sqrt{1+t^2}} > 0.$$

Therefore

$$\frac{dt}{ds} > 0.$$

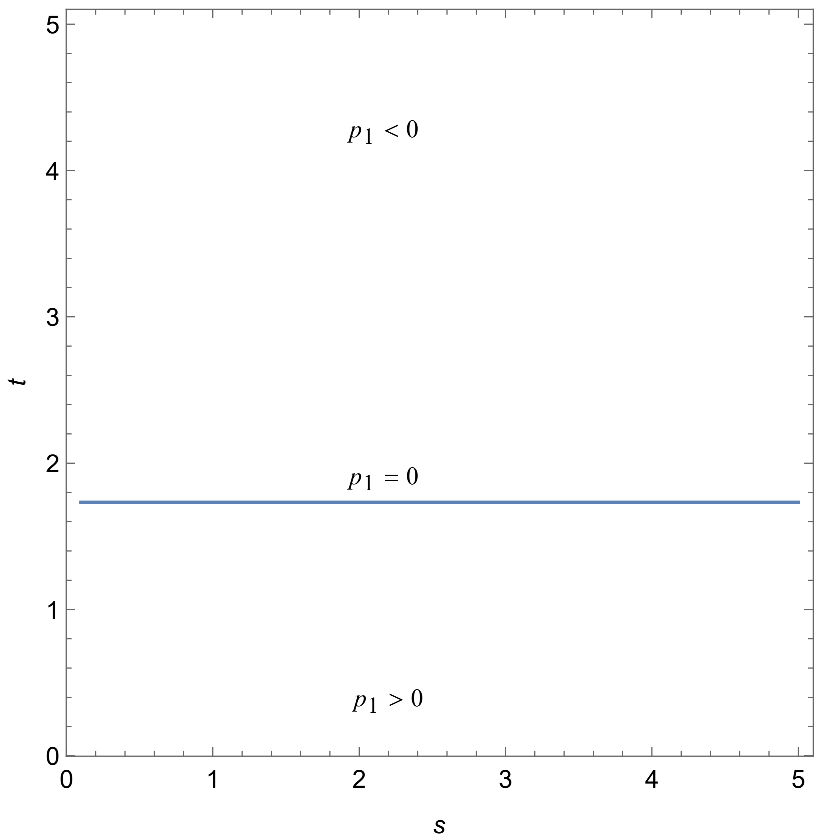

FIGURE 3.8: The sign of p_1

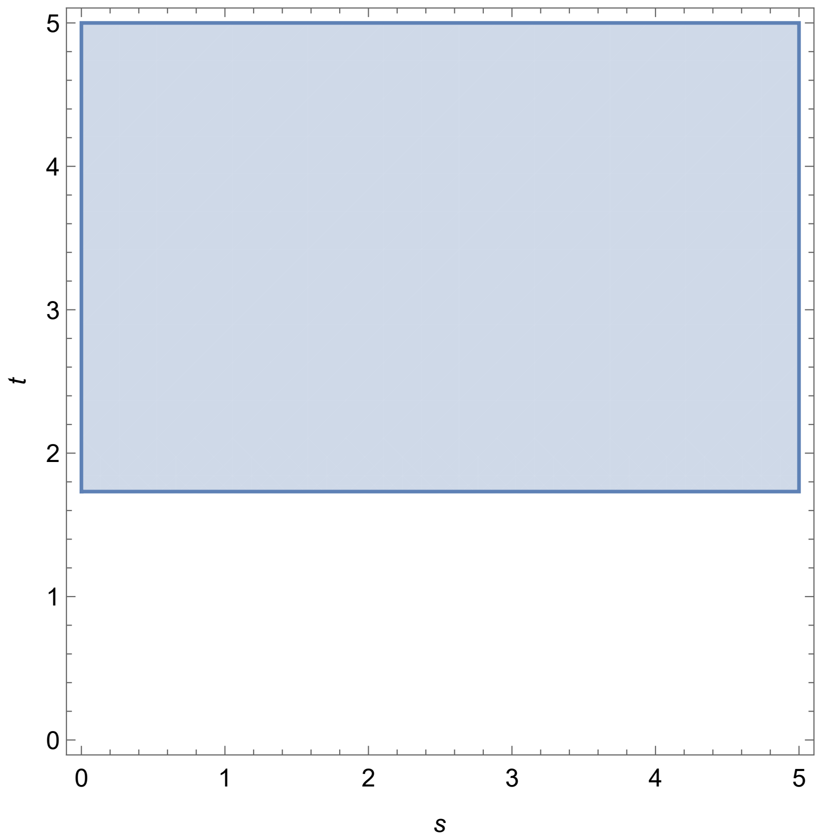

FIGURE 3.9: $p_1 < 0$ (shaded part)

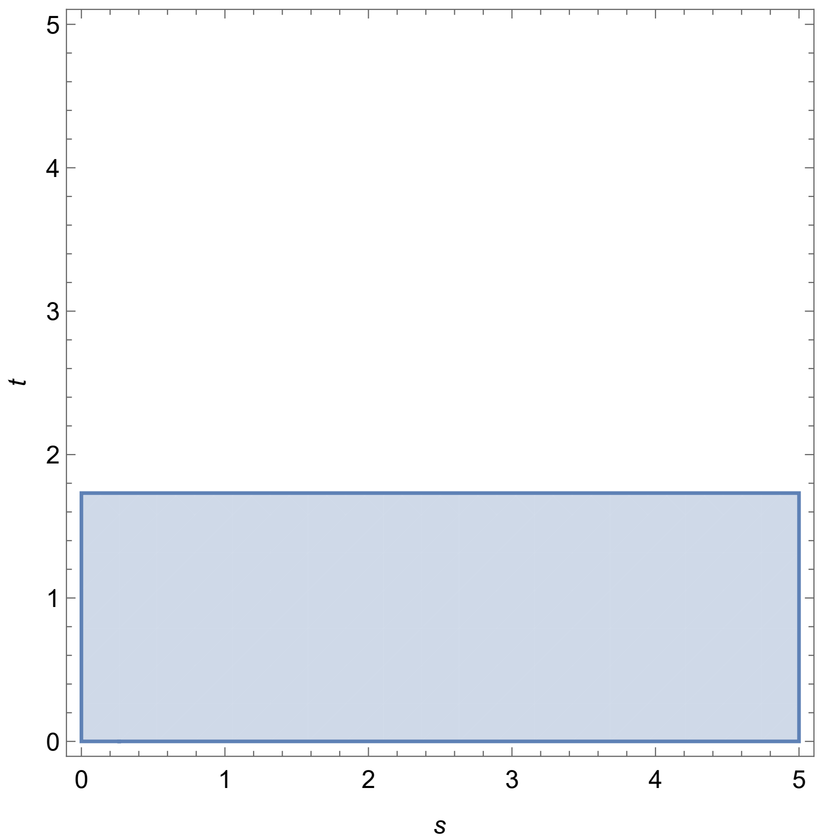

FIGURE 3.10: $p_1 > 0$ (shaded part)

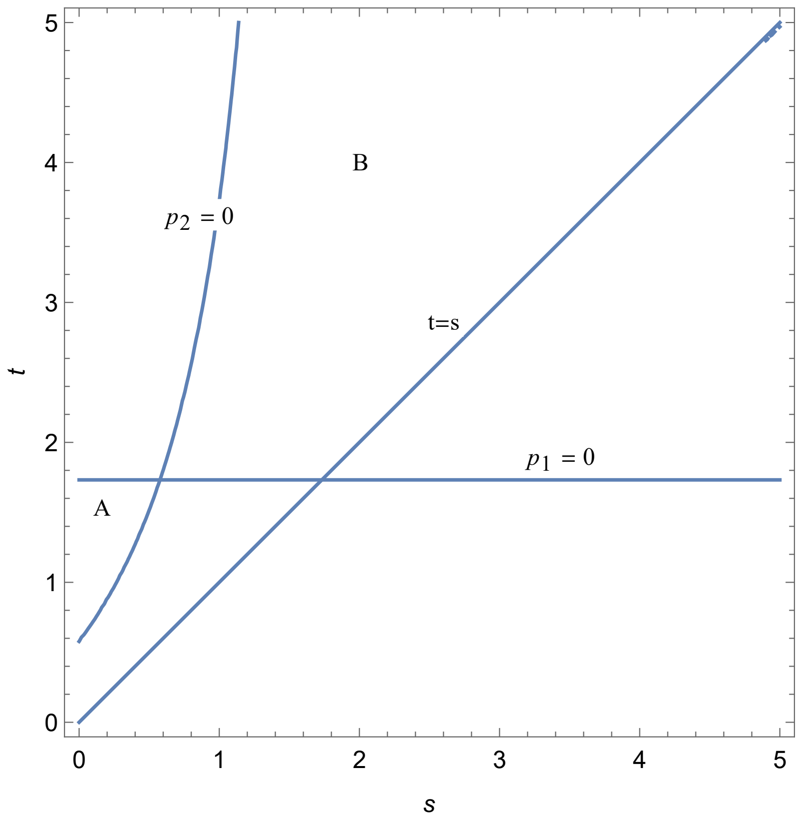

FIGURE 3.11: The region of A and B for $m_4 > 0$

The signs of p_1, p_2 are also shown in the mentioned figure (3.2) and (3.8).

It also configures that the region of $m_4 > 0$ is the union of two non-empty open sets A, B shown in figure (3.11).

The shaded part in figure (3.9, 3.10) shows that $p_1 < 0$ and $p_1 > 0$ respectively.

3.6 Lemma 2

The region in which $m_3, m_4 > 0$ for $0 < s < t$ is the union of E, F and G in figure (3.21), where $E = \{(s,t)|0 < s < \frac{\sqrt{3}}{3}, \sqrt{1+t^2} + s < t < \sqrt{3}\}$, $F = \{(s,t)|\frac{\sqrt{3}}{3} < s < \sqrt{3}, \sqrt{3} < t < \sqrt{1+s^2} + s\}$ and $G = \{(s,t)|\sqrt{3} < s < +\infty, \sqrt{1+s^2} + s < t < +\infty\}$.

Proof

For our easiness, let us denote

$$p_3 = \sqrt{1+s^2}^3 - 8,$$
$$p_4 = \sqrt{1+s^2}^3 - (t-s)^3,$$
$$p_5 = \frac{t-s}{(t-s)^3} + \frac{s}{\sqrt{1+s^2}^3} - \frac{t}{\sqrt{1+t^2}^3}.$$

By using these substitution in equation 3.31, we have $m_3 > 0$ which is equal to $\frac{p_3 p_4}{p_5 p_2} > 0$.

When $t > s$ and $t - s < \sqrt{1+t^2}$, we get

$$\begin{aligned} p_5 &= \frac{t-s}{(t-s)^3} + \frac{s}{\sqrt{1+s^2}^3} - \frac{t}{\sqrt{1+t^2}^3} \\ &= t\left(\frac{1}{(t-s)^3} - \frac{1}{\sqrt{1+t^2}^3}\right) + s\left(\frac{1}{\sqrt{1+s^2}^3} - \frac{1}{(t-s)^2}\right) \\ &> s\left(\frac{1}{(t-s)^3} - \frac{1}{\sqrt{1+t^2}^3}\right) + s\left(\frac{1}{\sqrt{1+s^2}^3} - \frac{1}{(t-s)^2}\right) \\ &= s\left(\frac{1}{\sqrt{1+s^2}^3} - \frac{1}{\sqrt{1+t^2}^3}\right) \end{aligned}$$

Therefore

$$p_5 > 0. \tag{3.39}$$

At $p_5 = 0$, we get a curve line (see figure (3.18)). In figure (3.19 and 3.20) the shaded region shows that $p_5 > 0$ and $p_5 < 0$ respectively. When $p_3 = 0$, we get a straight line i.e $s = \sqrt{3}$ in the st-plane (see figure (3.12))and $p_3 > 0$ in the first quadrant (see shaded part in figure (3.13)) and for $p_3 < 0$ see shaded region in figure (3.14). When $p_4 = 0$, we get a smooth monotone increasing curve at $t = s + \sqrt{1+s^2}$ (see figure (3.15)). Also $p_4 < 0$ above this curve (shown in figure (3.15)).

The shaded part in figure (3.16 and 3.17) shows that $p_4 > 0$ and $p_4 < 0$ respectively. After some calculations, we can get the implicit curves $p_1 = 0, p_2 = 0$ and $p_4 = 0$. They have only one common point i.e $(\frac{\sqrt{3}}{3}, \sqrt{3})$ with given domain $0 < s < t$.

It can be shown in figure (3.21). Hence, the locality of $m_3, m_4 > 0$ is the union of the three given non-empty open sets E, F and G as shown in the figure (3.21).

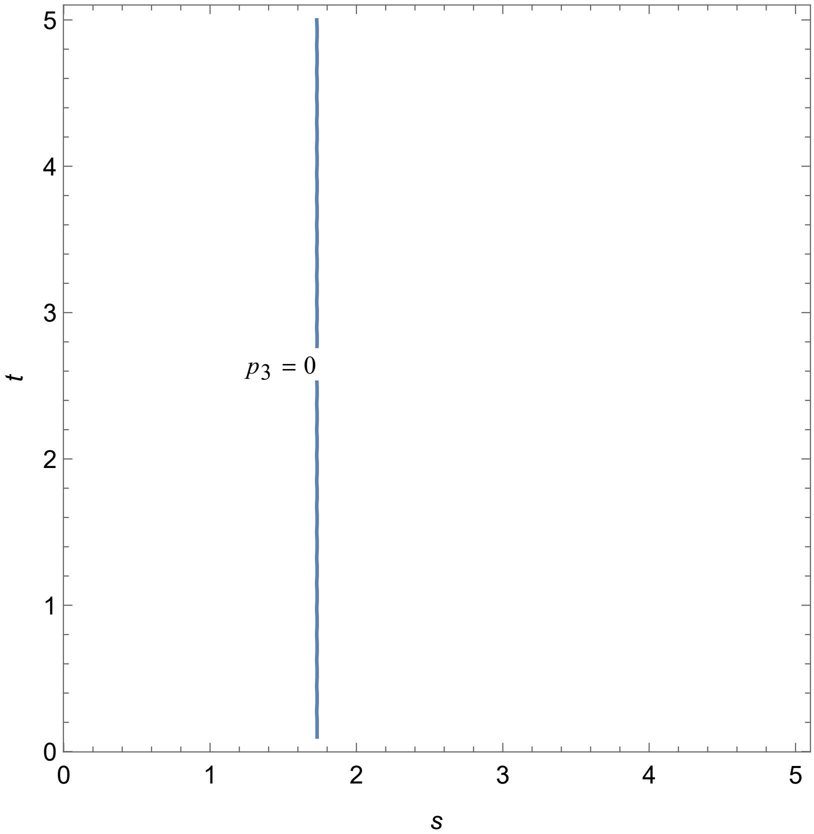

FIGURE 3.12: The sign of p_3

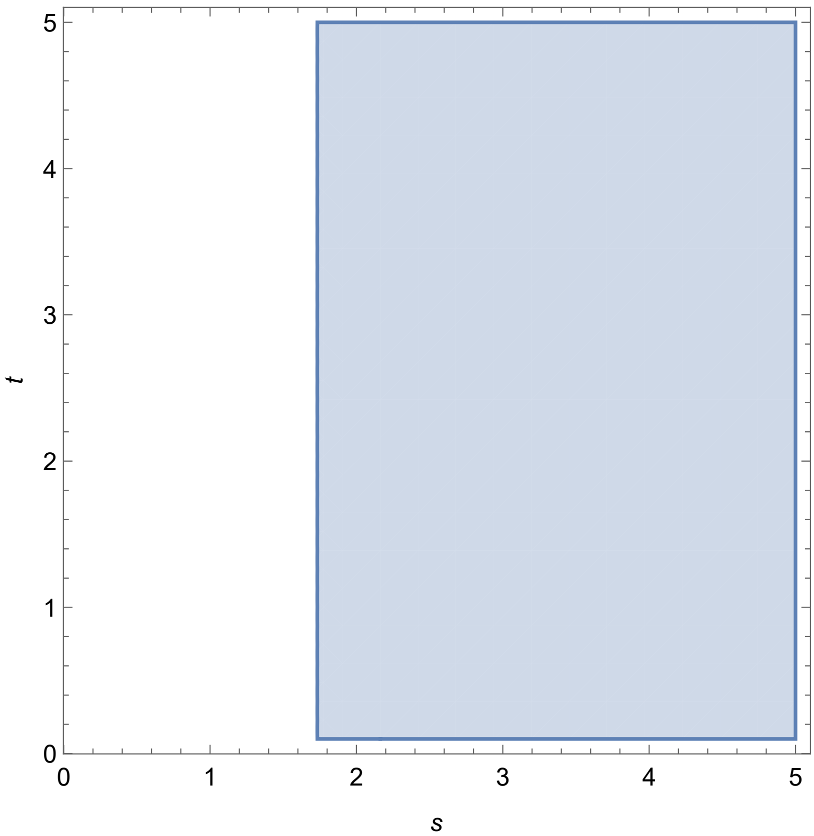

FIGURE 3.13: $p_3 > 0$ (shaded part)

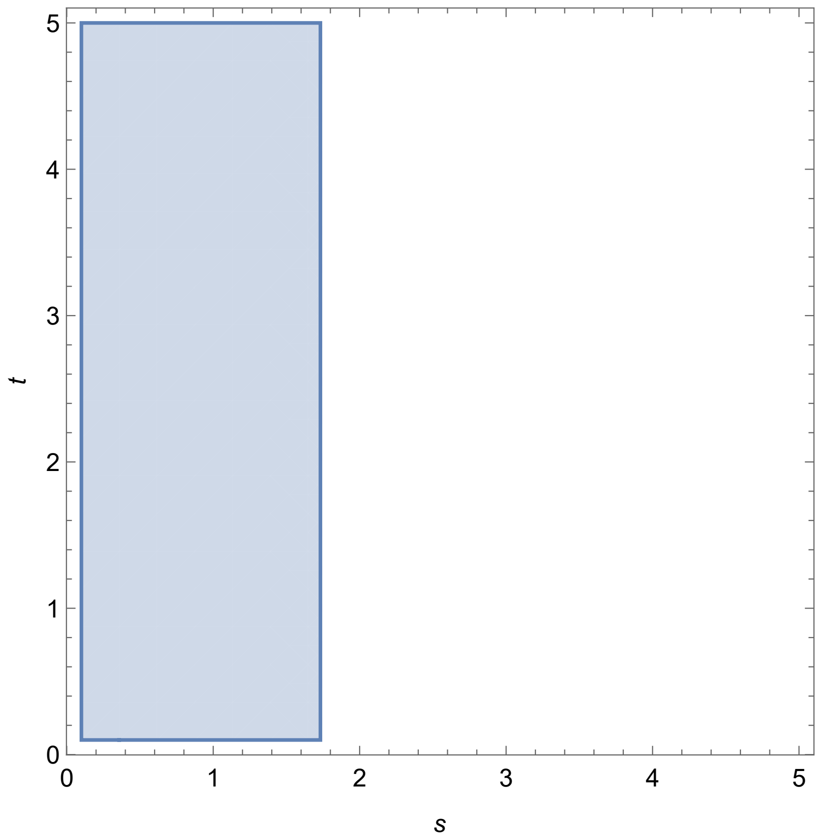

FIGURE 3.14: $p_3 < 0$ (shaded part)

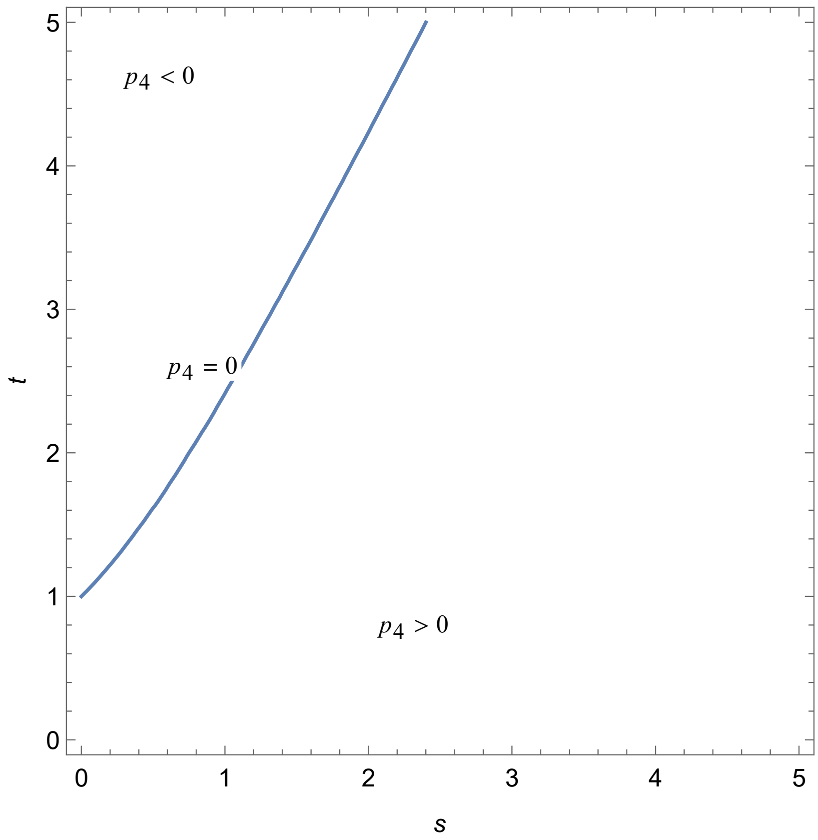

FIGURE 3.15: The sign of p_4

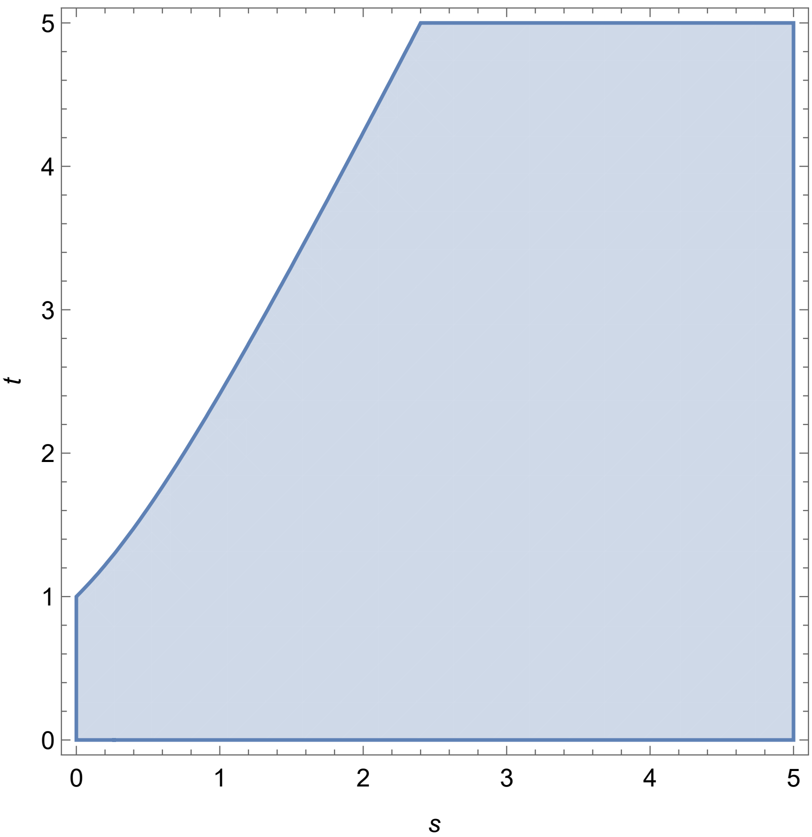

FIGURE 3.16: $p_4 > 0$ (shaded part)

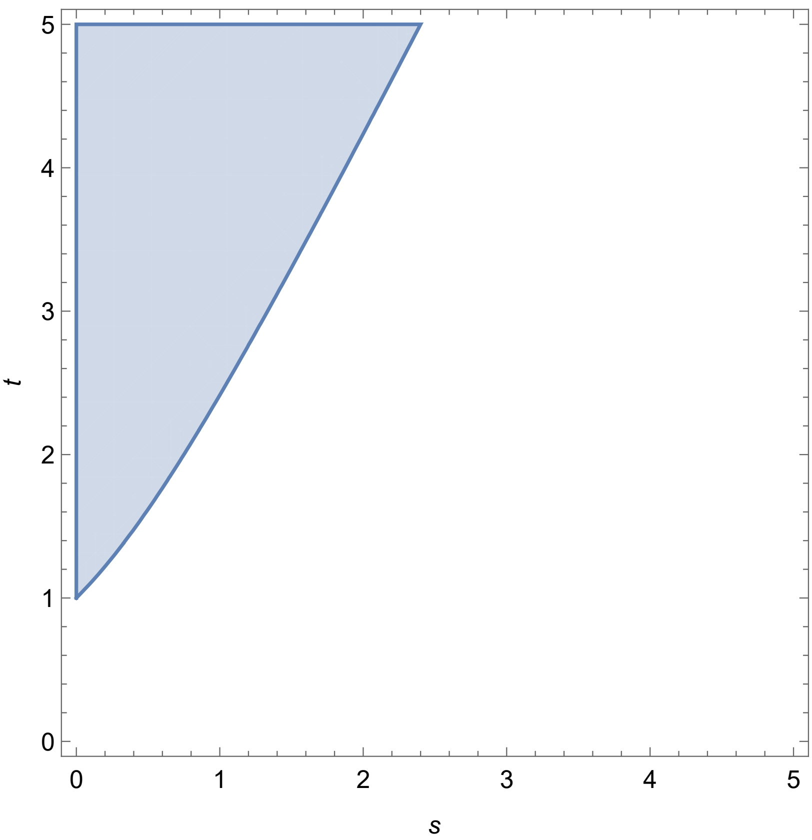

FIGURE 3.17: $p_4 < 0$ (shaded part)

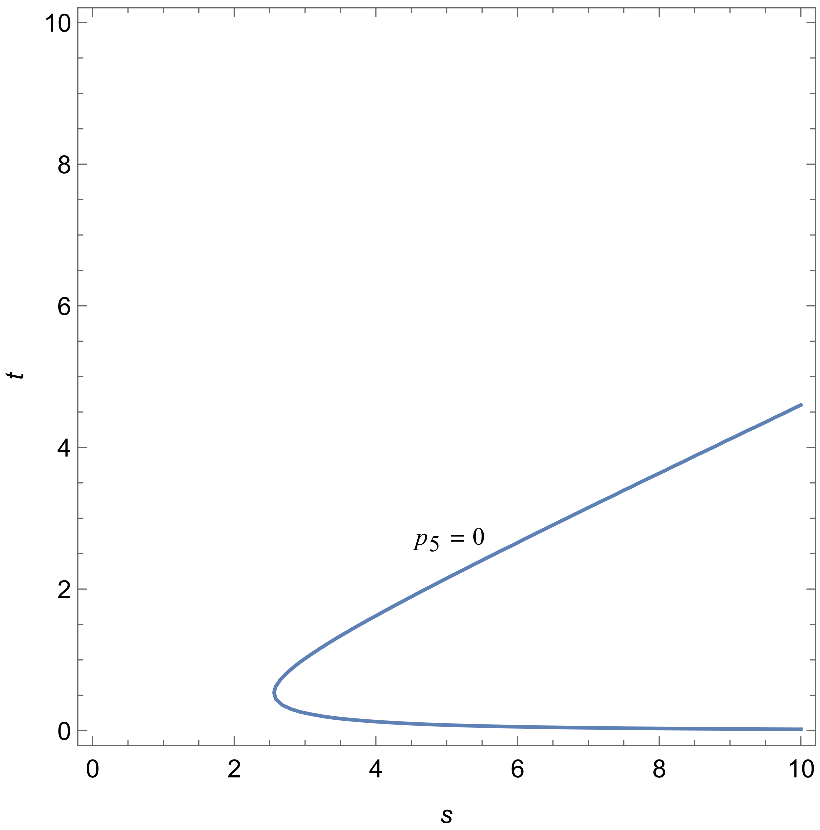

FIGURE 3.18: The sign of p_5

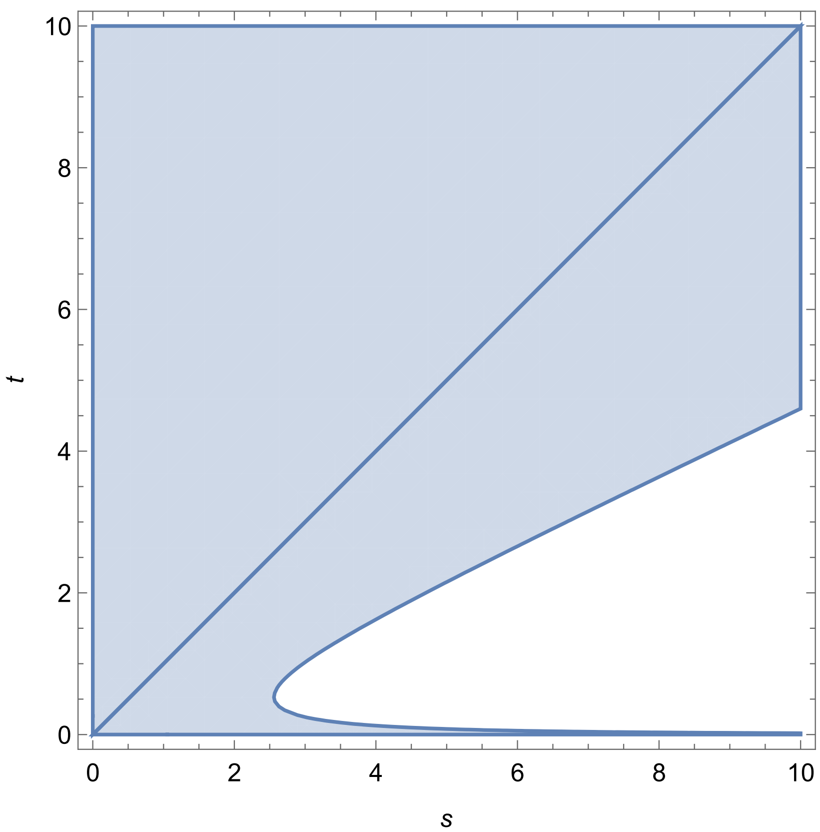

FIGURE 3.19: $p_5 > 0$ (shaded part)

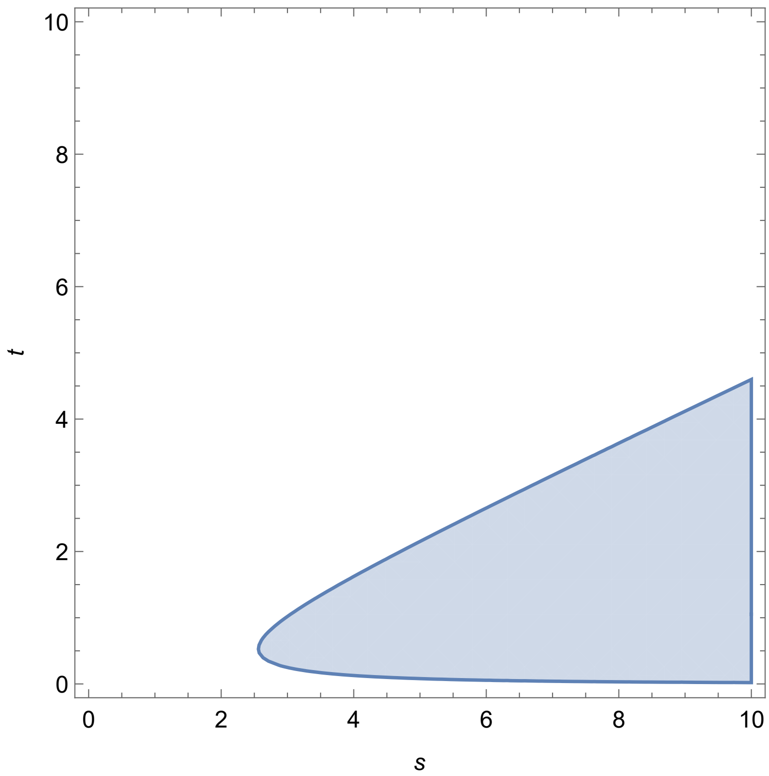

FIGURE 3.20: $p_2 < 0$ (shaded part)

3.7 Lemma 3

The region in which $m_i > 0$, where $i = 1, 2, 3, 4$ for $0 < s < t$ is the union of non-empty open sets E and G shown in figure (3.21).

Proof

From theorem 2, we know that

$$m_1 = m_2 = \lambda \frac{4\sqrt{1+t^2}^3 (t - c_y)}{t\sqrt{1+s^2}^3 (t-s)^3} \frac{\left((t-s)^3 - \sqrt{1+s^2}^3\right)}{\left(\left(\frac{2}{\sqrt{1+s^2}}\right)^3 - \left(\frac{\sqrt{1+t^2}}{t-s}\right)^3\right)}.$$

By using the substitutions in the above equation from Lemma 1 and Lemma 2, we get the following result.

$$m_1 = m_2 = -\lambda \frac{4\sqrt{1+t^2}^3 (t - c_y)}{t\sqrt{1+s^2}^3 (t-s)^3} \frac{p_4}{p_2}. \tag{3.40}$$

The signs of p_2 and p_4 indicate the sign of m_i for $i = 1, 2$. We obtained $p_2, p_4 > 0$ for $(s, t) \in G$ and therefore $m_1 = m_4 < 0$. If $(s, t) \in E \cup F$, we can easily find that $m_1 = m_2 > 0$. By this we get a complete proof of Lemma 3.

We get a complete proof of Theorem 2 by using all the above three Lemmas (1-3).

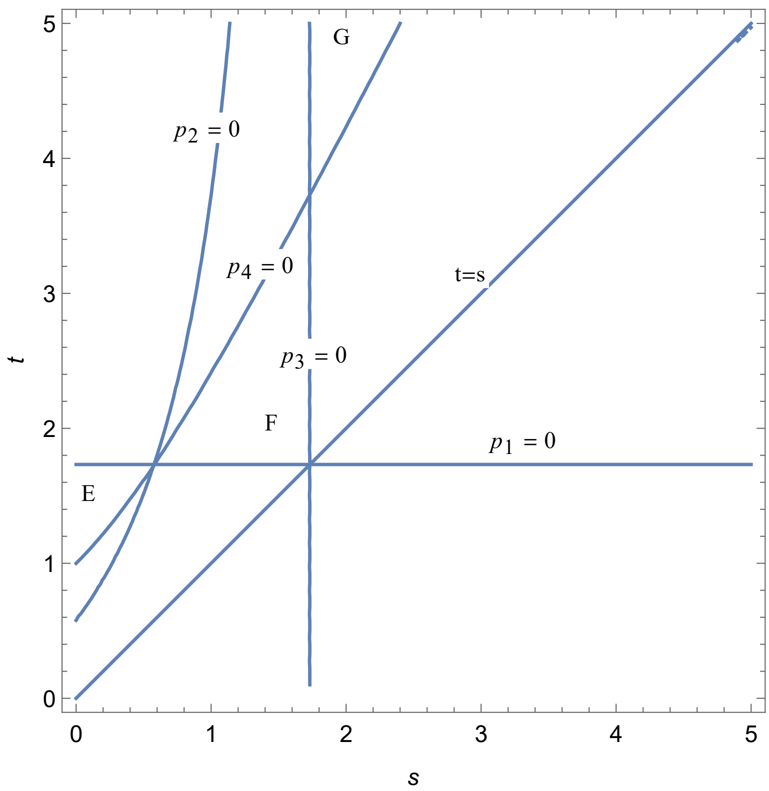

FIGURE 3.21: The E, F and G for $m_3, m_4 > 0$

Chapter 4

Graphical Analysis of Newtonian Four-Body Problem

4.1 Introduction

In this section, the graphical structure of all the solutions of Newtonian four-body problem i.e m_1, m_2, m_3 and m_4 will be discussed.
By taking the solutions of all these four masses mentioned in the previous chapter 3 named (3.28, 3.31, 3.33), we will discuss them graphically in detail.

Positivity analysis of all the mass functions m_1, m_2, m_3 and m_4 is also discussed in this section. There are different cases of theorem 2 (mentioned in chapter 3) which will be discussed in this section.

1. When $m_1 > 0$ for $0 < s < t$ and $\lambda > 0$.
2. When $m_2 > 0$ for $0 < s < t$ and $\lambda > 0$.
3. When $m_3 > 0$ for $0 < s < t$ and $\lambda > 0$.
4. When $m_4 > 0$ for $0 < s < t$ and $\lambda > 0$.
5. When $m_k > 0$ $(k = 1, 2, 3, 4)$ for $0 < s < t$ and $\lambda > 0$.

4.2 Case-1

In this case, the graphical representation of m_1 is discussed. Here m_1 is the function of s and t.

So by using the values of s and t, we will show that $m_1 > 0$.

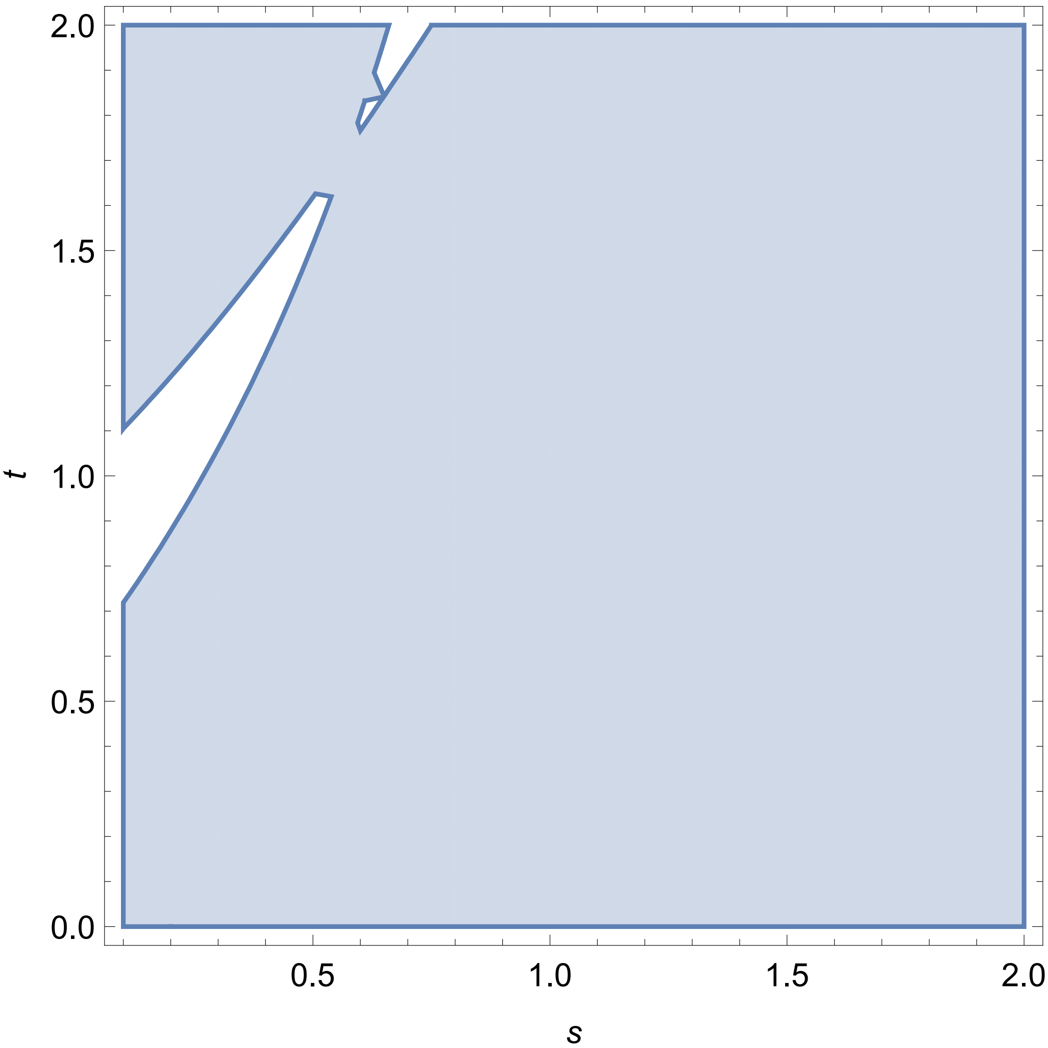

FIGURE 4.1: $m_1 > 0$ (Shaded part)

In the graph above, the shaded region shows that all the values of m_1 are greater then zero for $\lambda = 1$. The intervals of s and t are shown in the above graph.

4.2.1 Positivity of m_1

In the following table, we basically choose some selective points (from shaded region) and check the positivity of m_1 as

Coordinate points (s, t)	**Values of** $m_1 > 0$
(0.1745, 1.804)	2.5504
(0.4425, 1.843)	2.95371
(0.9733, 1.876)	3.07018
(1.105, 1.34)	3.98971
(1.462, 1.03)	4.07596
(1.541, 0.5486)	4.71417

TABLE 4.1: Positivity for m_1

4.2.2 3-D Graphical Representation of m_1

The following graph shows the graphical structure of m_1 in 3-D i.e s, t and λ. In this graph the shaded region defines that all the values of m_1 are positive for $t > s > 0$ and $\lambda > 0$. By taking the intervals of $s \in [0.1, 2], t \in [1, 2]$ and $\lambda \in [0, 2]$, one can get a 3-D graph of $m_1 > 0$.

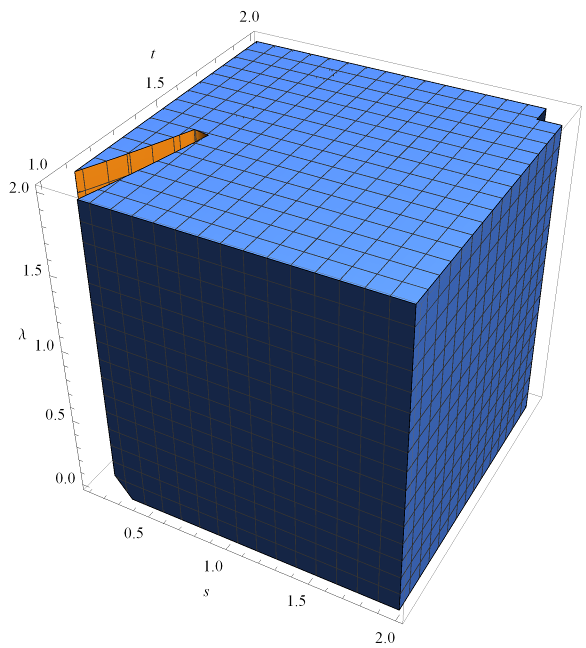

FIGURE 4.2: $m_1 > 0$ (shaded part)

4.3 Case-2

In the second case, the graphical representation of m_2 is discussed. Here m_2 is the function of s and t.

So by using the values of s and t, we will show that that region in which $m_2 > 0$.

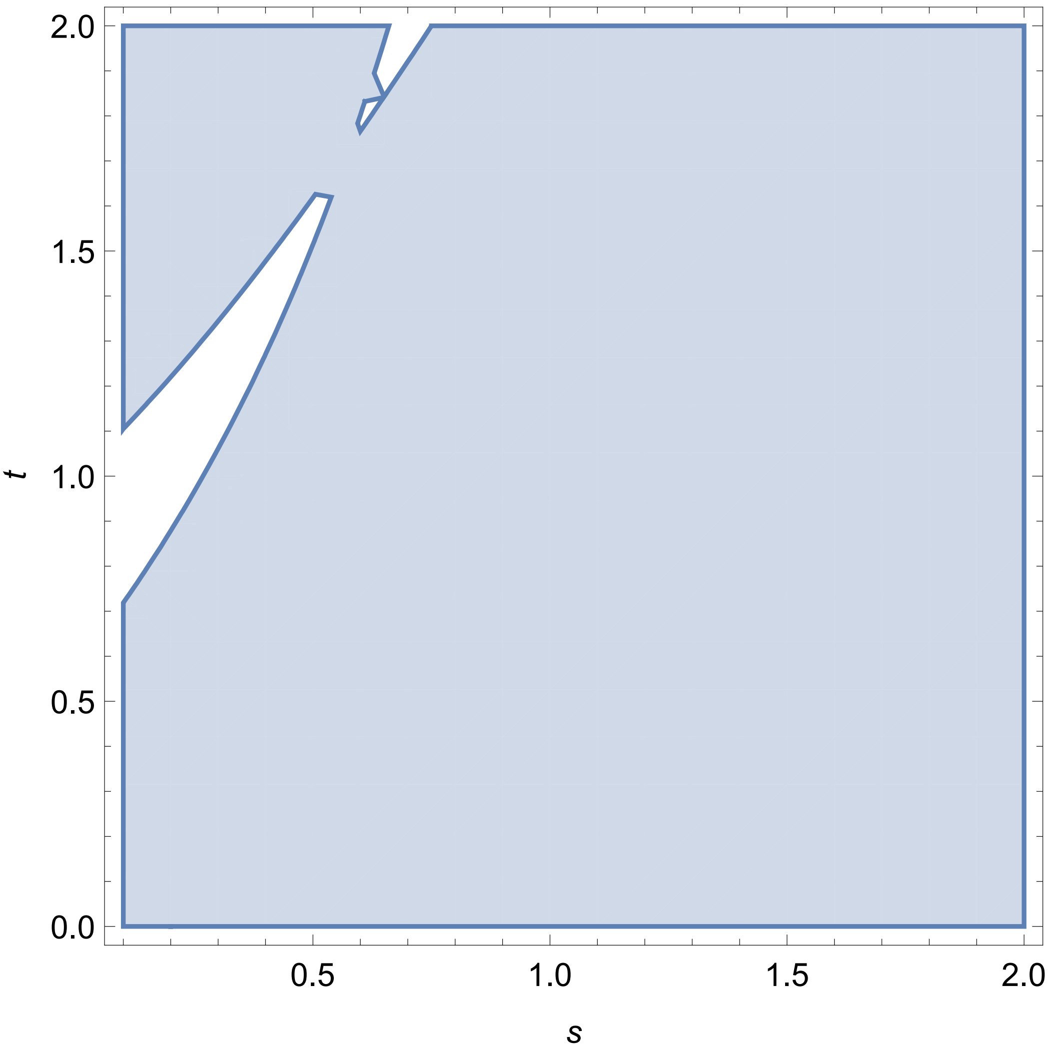

FIGURE 4.3: $m_2 > 0$ (Shaded part)

In the graph above, the shaded region shows that all the values of m_2 are greater then zero for $\lambda = 1$.

The intervals of s and t can be seen in the above graph.

4.3.1 Positivity of m_2

In the following table, we basically choose some selective points (from shaded part) and check the positivity of m_2 on those points.

Coordinate points (s, t)	**Values of** $m_1 > 0$
(0.1745, 1.804)	3.09751
(0.4425, 1.843)	3.96809
(0.9733, 1.876)	3.95661
(1.105, 1.34)	4.0094
(1.462, 1.03)	4.14323
(1.541, 0.5486)	4.71417

TABLE 4.2: Positivity for m_2

4.3.2 3-D Graphical Representation of m_2

The following graph shows the graphical structure of m_2 in 3-D i.e s, t and λ. In this graph the shaded region defines that all the values of m_2 are positive for $t > s > 0$ and $\lambda > 0$. By taking the intervals of $s \in [0.1, 2], t \in [1, 2]$ and $\lambda \in [0, 2]$, one can get a 3-D graph of $m_2 > 0$.

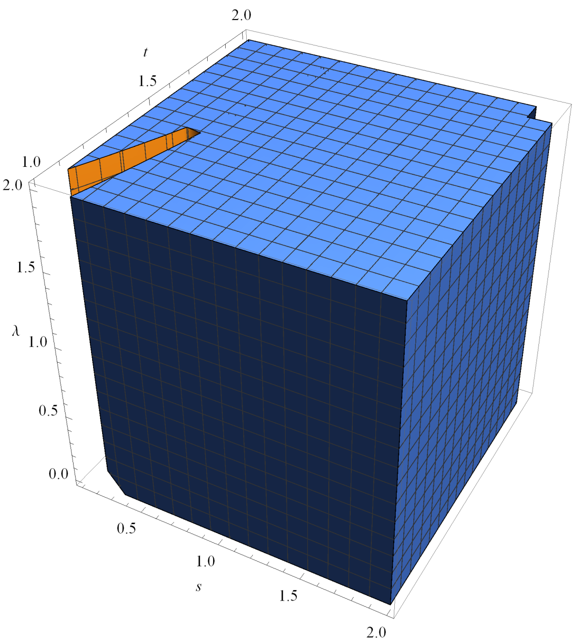

FIGURE 4.4: $m_2 > 0$ (shaded part)

4.4 Case-3

In the third case, m_3 is graphically analyze. It can be clearly seen from the solutions mentioned in chapter 3 that m_3 is the function of s and t. So by putting the values of s and t, one can show that $m_3 > 0$.

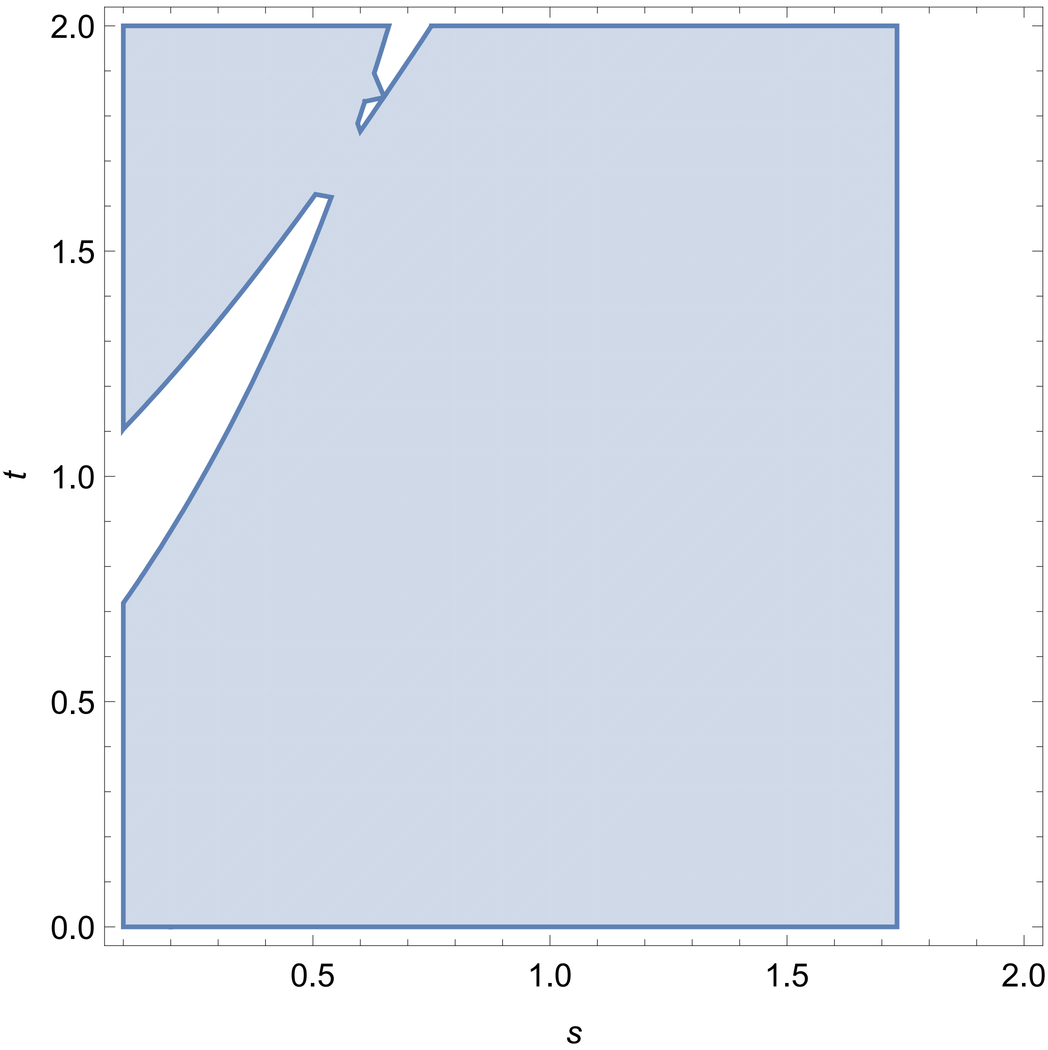

FIGURE 4.5: $m_3 > 0$ (Shaded part)

In the above graph, the shaded region shows that all the values of m_3 are positive for $\lambda = 1$.

The intervals of s and t are clearly shown in the above graph.

4.4.1 Positivity of m_3

The following table defines the positivity of the mass function m_3 by taking some selective points of the coordinates s and t.

Coordinate points (s, t)	**Value of** $m_3 > 0$
(0.1745, 1.804)	0.030859
(0.4425, 1.843)	0.040617
(0.9733, 1.876)	0.0498802
(1.105, 1.34)	0.0180938
(1.462, 1.03)	0.1158434
(1.541, 0.5486)	0.313208

TABLE 4.3: Positivity for m_3

4.4.2 3-D Graphical Representation of m_3

The following graph shows the graphical structure of m_3 in 3-D i.e s, t and λ. In this graph the shaded region defines that all the values of m_3 are positive for $t > s > 0$ and $\lambda > 0$. By taking the intervals of $s \in [0.1, 2], t \in [1, 2]$ and $\lambda \in [0, 2]$, one can get a 3-D graph of $m_3 > 0$.

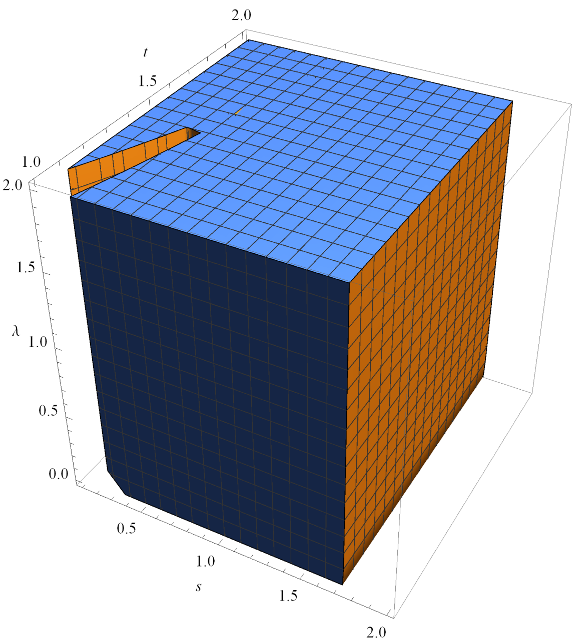

FIGURE 4.6: $m_3 > 0$ (shaded part)

4.5 Case-4

In the fourth case, the graphical representation of m_4 is discussed briefly. Here m_4 is the function of s and t. So by using the values of s and t, one can get a result that $m_4 > 0$.

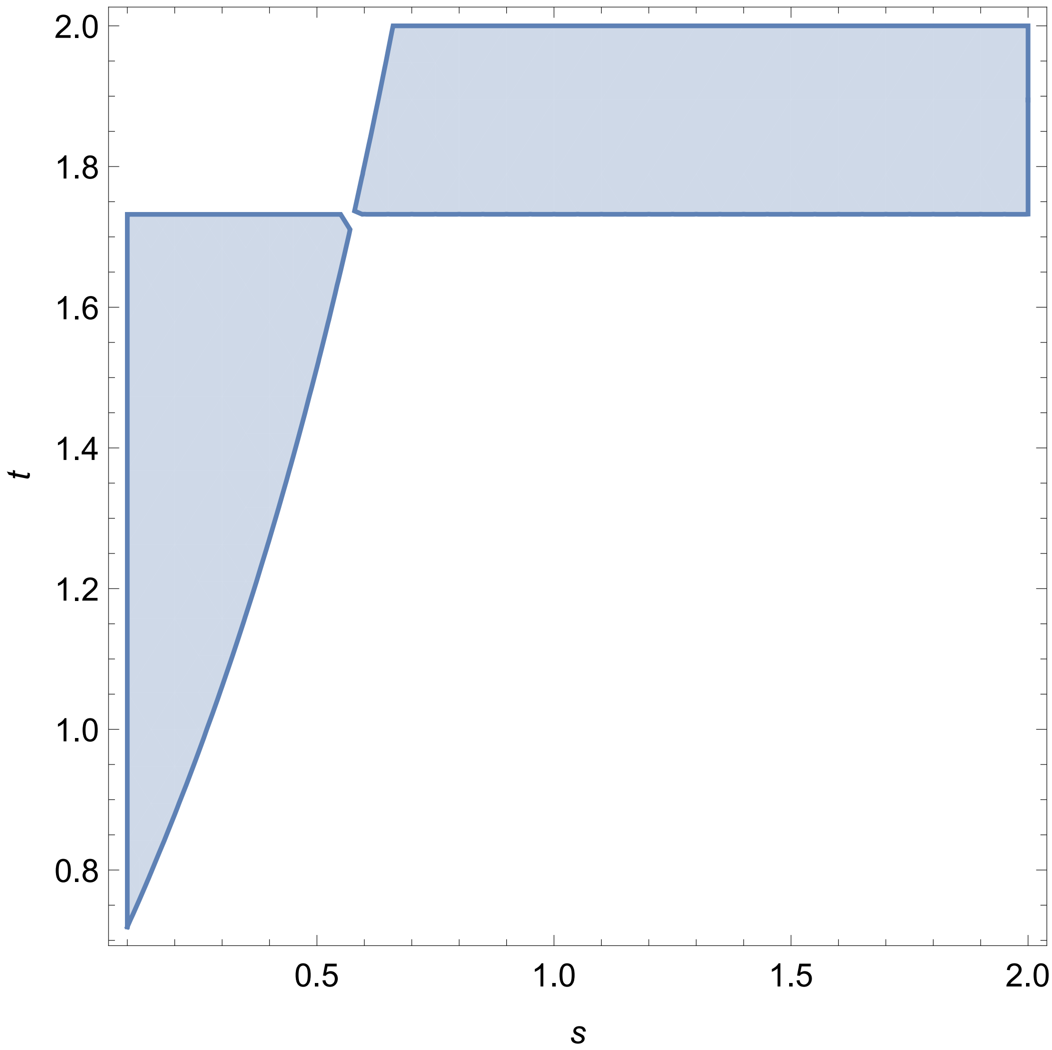

FIGURE 4.7: $m_4 > 0$ (Shaded part)

In the above graph, the shaded region shows that all the values of m_4 are greater then zero for $\lambda = 1$.

The values of s and t are taken from the above mentioned intervals of s and t.

4.5.1 Positivity of m_4

In the following table, we basically choose some points of coordinates s and t (from shaded part) and put in the given mass function m_4 to check its positivity.

Coordinate points (s, t)	**Values of** m_4
(0.1745, 1.804)	0.0109978
(0.4425, 1.843)	0.0221466
(0.9733, 1.876)	0.035059
(1.935, 1.975)	0.0966863
(0.9062, 1.911)	0.526793
(o.5238, 1.702)	0.00198729

TABLE 4.4: Positivity for m_4

4.5.2 3-D Graphical Representation of m_4

The following graph shows the graphical structure of m_4 in 3-D i.e s, t and λ. In this graph the shaded region defines that all the values of m_4 are positive for $t > s > 0$ and $\lambda > 0$. By taking the intervals of $s \in [0.1, 2], t \in [1, 2]$ and $\lambda \in [0, 2]$, one can get a 3-D graph of $m_4 > 0$.

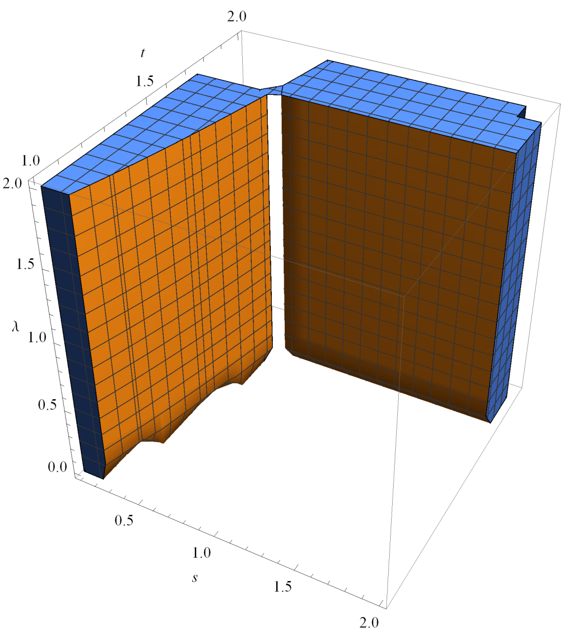

FIGURE 4.8: $m_4 > 0$ (shaded part)

4.6 Case-5

Here m_1, m_2, m_3 and m_4 are the functions of s and t. So by using the values of s and t, one can show that the values of all the masses are positive at some locality.

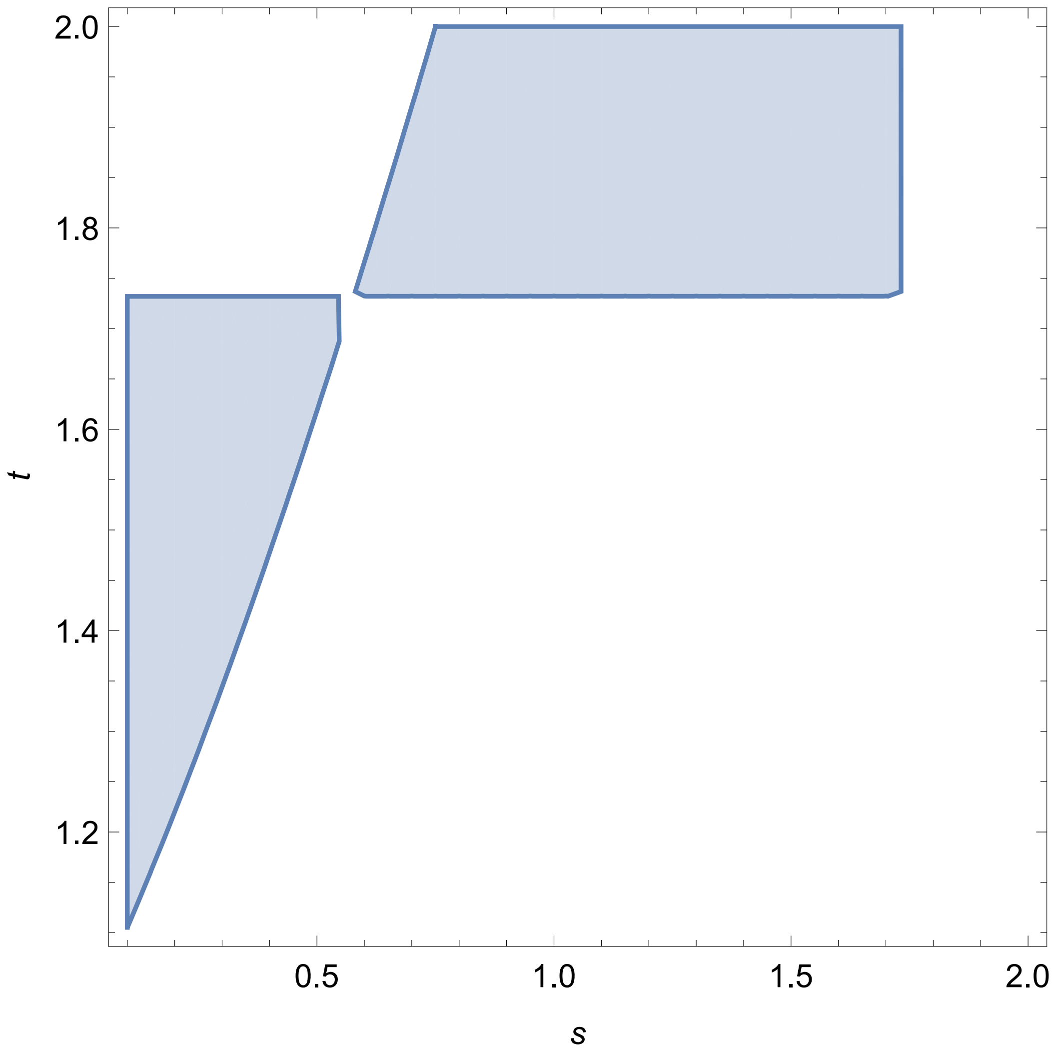

FIGURE 4.9: $m_k > 0$ (Shaded part for $k = 1, 2, 3, 4$)

In the graph above, the shaded region shows that all the values of m_1, m_2, m_3 and m_4 are greater then zero for $\lambda = 1$.

The intervals of s and t are shown in the above graph. In the graph above, the shaded region shows that all the values of m_1 are greater then zero for $\lambda = 1$. The intervals of s and t are shown in the above graph.

4.6.1 Positivity of $m_k > 0$ $(k = 1, 2, 3, 4)$

In the following table, we basically choose some values of coordinates s and t (from shaded part)in the given mass functions $(m_1, m_2, m_3$ and $m_4)$ to check either they satisfy the given condition (i.e $m_1, m_2, m_3, m_4 > 0$) or not.

Coordinate points (s, t)	**Positivity condition for** $m_k > 0$ $(k = 1, 2, 3, 4)$
(1.016, 1.893)	True
(0.7972, 1.79)	True
(0.2589, 1.639)	True
(1.052, 1.784)	True
(1.047, 1.948)	True
(1.448, 1.806)	True

TABLE 4.5: Positivity for $m_k > 0$ $(k = 1, 2, 3, 4)$

4.6.2 3-D Graphical Representation of $m_k > 0$ $(k = 1, 2, 3, 4)$

The following graph shows the graphical structure of all the four masses in 3-D i.e s, t and λ.

In this graph the shaded region defines that all the values of the four masses m_1, m_2, m_3 and m_4 are positive for $t > s > 0$ and $\lambda > 0$.

By taking the intervals of $s \in [0.1, 2], t \in [1, 2]$ and $\lambda \in [0, 2]$, one can get a 3-D graph of $m_k > 0$ (i.e $m_1 > 0, m_2 > 0, m_3 > 0$ and $m_4 > 0$).

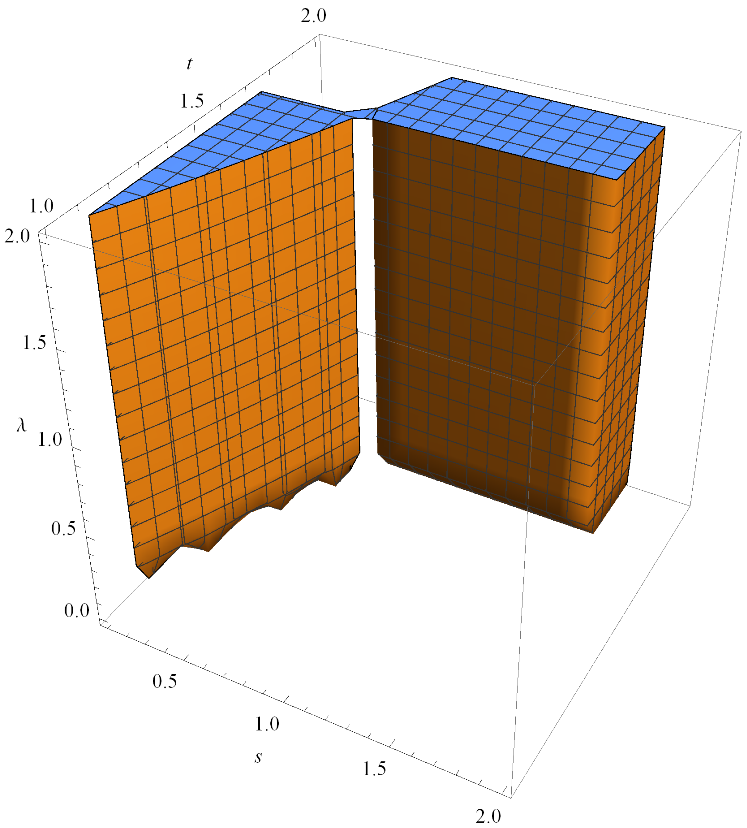

FIGURE 4.10: $m_k > 0$ $(k = 1, 2, 3, 4)$ (shaded part)

Chapter 5

Conclusion

The work of Chunhua Deng and Shiqing Zhang [21] on "Planar symmetric concave central configuration in Newtonian four body problem" is discussed and elaborated to represent the complete analysis of the article [21] in this thesis.
We basically used the CC condition with this arrangement of masses (see equation (3.4)) to find the equations of motion for four massive bodies (m_1, m_2, m_3 and m_4). We then solved these equations for the two cases mentioned in theorem 1 and 2. After finding the solutions of all the masses, we analyzed the mass functions and found all the possible regions in st plane such that the mass functions are positive by using three Lemmas $(3.5, 3.6$ and $3.7)$. We have also drawn the graphs of these regions where their value is zero. Lastly, we analyzed the graphical structure of all the four point masses m_1, m_2, m_3 and m_4 (see chapter 4) and investigated that there are some possible regions where the given masses m_1, m_2, m_3 and m_4 are positive. We have also shown the graphical representation of all the four given masses in 3-D.

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