CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Best Proximity Point for Multi-Valued $(\alpha\mathcal{F}, \mathtt{b}, \breve{\phi})$-Contraction on Partially Ordered $\mathtt{b}$-Metric Spaces

by

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A thesis submitted in partial fulfillment for the degree of Master of Philosophy

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Dedicated to my Parents

CERTIFICATE OF APPROVAL

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Abstract

Recently, Jain et al. established certain best proximity point results for multi-valued generalized contraction on partially ordered complete metric spaces accompanying the notion of altering distance function. In this thesis, the idea of generalized $(\alpha\mathcal{F}, \mathtt{b}, \breve{\phi})$-contraction in the setting of $\mathtt{b}$-metric is introduced. The main results of the research are about the existence of fixed points for multi-valued $(\alpha\mathcal{F}, \mathtt{b}, \breve{\phi})$-contractions on partially ordered $\mathtt{b}$-metric space. Furthermore, examples are provided for the verification of the main result. Eventually, the existence of the solution to a second-order differential equation and a fractional differential equation is analyzed using the proven results' axioms. It is worth mentioning that the results of Jain et al. are the special cases of the theorems proved in the present research. Several Corollaries are elaborated to show that our results generalize many existing fixed-point results.

Contents

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Abbreviations

ADF	Altering Distance Function
BCP	Banach Contraction Principle
bMS	b-Metric Space
BPP	Best Proximity Point
FP	Fixed Point
MS	Metric space
POCMS	Partially Ordered Complete Metric Space
POMS	Partially Ordered Metric Space

Symbols

$\wp$	Metric space
$\wp_{\mathsf{b}}$	b-metric space
$\mathbb{J}$	A non-empty set
$\mathcal{S}$	The mapping
$\breve{\phi}$	Altering distance function
$\mathcal{C}[a,b]$	Collection of continuous functions
$\mathbb{N}$	The set of natural number
$\mathbb{R}$	The set of real number
$\in$	Belongs to
$\exists$	There exist
$\times$	Cartesian product

Chapter 1

Introduction

One of the significant mathematical achievements of the first part of the twentieth century was the introduction of functional analysis, which has a wide range of applications. Functional analysis is a field of mathematics that evolved from classical analysis. Now a days, the functional analytic approach and its results have value in many fields of mathematics. The functional analysis concerns functionals, functions, and functions in infinite dimensional spaces. The rapid development of functional analysis techniques began about a century ago. The outstanding result of that development is a fixed point(FP)theory. In multiple disciplines of applied or pure mathematics, as well as other quantitative sciences, in particular, economics, engineering, and so on, fixed point results have proved extremely useful in determining and establishing the existence of solutions to various issues. Fixed point theory ensures a solution to non-linear problems by demonstrating the presence of fixed points. First of all, Poincare [1] started some preliminary work on fixed point theory in 1866. He presented his primary fixed point theorem without providing any proof.

Brouwer [2] in 1912, was the first to prove the fixed point theorem on the unit sphere, and it is regarded as one of the early approaches that Kakutani [3] further pursued. Stefan Banach [4] presented the Banach contraction principle(BCP) in 1922, a fundamental theorem in fixed point theory in the context of metric space and the most influential mathematical concepts. The BCP provided not only the

requirements for the existence of a unique fixed point of a contraction defined on a complete metric space but also the procedure for finding the desired fixed point, which Brouwers fixed point theorem lacked. Many similar theorems were given over the next few decades, depending on the sort of mapping and space. Later, BCP was generalized under other mapping flavors; Edelstein [5] provided the first generalization in 1962 by altering the contraction requirement. Edelstein used continuous mapping on a compact space for the existence of a fixed point.
Kasahara [6] conducted additional research on BCP in premetric spaces in 1968, and Kannan [7] highlighted certain advances in the continuity of contraction condition of BCP in the same year. Nadlers [8]] generalized BCP for multi-valued functions in 1969 by using Hausdroff metric over the family of nonempty closed bounded subsets of a complete metric space. In 1972, Chatterjea [9] generalized the BCP as, every Chatterjea type contration on a complete metric space has a unique fixed point.

In 1975, Dass and Gupta [10] gave the fixed point theorem of new rational contraction to generalized the BCP.

The notion of metric space was introduced by Frechet [11] in 1906. Later, the concept of the metric was developed and generalized in many different directions in mathematics and fundamental sciences; such generalizations were created by altering, modifying, adding, and eliminating metric space features and conditions. In this prospect, Bakhtin [12] introduced a new notion, namely b-metric. He accomplished this goal by altering the metric space triangle inequality. This new idea is a fascinating generalization of the metric space and an intriguing direction for researchers. We can observe that in the previous several years, many new structures in b-metric spaces have been constructed by mathematicians.
Czerwik [13] generalized a fixed point result employing the weaker triangular state. Dikranjan [14] and Heinonen [15] also established new results by using complete bMS for single-valued mappings and then for set-valued mappings. In this context, several new findings are demonstrated by mathematicians employing the complete b-metric space layout for self mappings and, eventually, multi-valued mappings.
Among the critical challenges within metric fixed point theory is estimating the

solution of fixed point problem. It compels scholars to use contractive conditions over cardinal functions to ensure a fixed points existence. When non-self mappings are involved, the issue becomes more exciting and complicated. The idea of non-self maps explores the idea of the best proximity point(BPP) along with associated theorems. Basha [16] discoverd the best proximity point using the Banach contraction principle in 2010. Basha et al. [17] considered non-self mappings on metric spaces and analyzed the existence of best proximity point. Karapinar and Erhan [18] investigated the ideal proximity for various contractions. The notion of fixed points for multi-valued mappings is vital in confirming the presence of solutions according to the theory concerning integral inclusions. Nadler pioneered the investigation within fixed point theory for multi-valued mappings. Strict contractive criteria, either for self-mappings or multi-valued non-self mappings, do not assure the presence of fixed points in the setting of metric spaces, as shown in [19]. Recently, Wardwoski [20] suggested the concept of $\mathcal{F}$-contraction as a generalized contraction. Klim et al. [21] studied and demonstrated fixed point theorems involving $\mathcal{F}$-contractions for dynamic processes.
Sagheer et al. [22] developed the concept of $(\alpha, \mathcal{F})$-contractive multi-valued mappings on uniform spaces in 2022. Recently, jain et al. [23] gave a novel idea for multi-valued $\mathcal{F}$- contraction on partially ordered metric space (POMS) using an approach concerning altering distance function to guarantee the occurrence of best proximity point via best proximity theorem.

The format of the thesis is slightly presented here.

Chapter 2 This chapter offers precise definitions and examples for illuminating the essential notions of metric spaces. We also go through several forms of mappings, fixed points, and the fixed point theorems.

Chapter 3 provides a detailed review of the article by Jain et al. [23]. The authors explained the contraction, which is formed by combining the notion of the $\mathcal{F}$- function and altering distance function. This idea is called multi-valued $\mathcal{F}$-contraction based on altering distance function. $\mathcal{F}$-contraction is used to generalize multiple fixed-point results. Examples and applications are provided.

Chapter 4 provides an extension of the results given in [23]. The b-metric space platform is used for this purpose. The condition is further generalized by involving an α function. Examples are also provided for the better understanding of the proved theorem. Applications are constructed for the authentication purpose of our main result.

Chapter 5 include our research analysis in well defined manner in this chapter.

Chapter 2

Basic Material

This chapter includes some basic definitions, examples, and results that are significant enough to be used in subsequent chapters. The first section of this chapters presents introduction with some crucial definitions from the metric space. The subject of the following section is b-metric space, and the final section provides a historical overview of fixed point theorems.

2.1 Metric Spaces

In 1906, M. Frechet presented the idea of metric space, which is the generalization of natural distance. Later, these spaces served as a platform between topological spaces and real analysis for the foundation of metric fixed point theory.

Definition 2.1.1. Metric Space

"A metric space is a pair $(\mathbb{J}, \wp)$, where $\mathbb{J}$ is a set and $\wp$ is a metric on $\mathbb{J}$ (or distance function on $\mathbb{J}$), that is, a function define on $\mathbb{J} \times \mathbb{J}$ such that $\forall\ \mathrm{a}, \mathrm{b}, \mathrm{m} \in \mathbb{J}$ we have

(M1): $\wp$ is real-valued, finite and non-negative,

(M2): $\wp(\mathrm{a}, \mathrm{b}) = 0$ if and only if $\mathrm{a} = \mathrm{b}$,

(M3): $\wp(\mathrm{a}, \mathrm{b}) = \wp(\mathrm{b}, \mathrm{a})$, (Symmetry)

(M4): $\wp(\mathrm{a}, \mathrm{m}) \leq \wp(\mathrm{a}, \mathrm{b}) + \wp(\mathrm{b}, \mathrm{m})$. (Triangular inequality)"[24]

Example 2.1.2.

Let $\mathbb{J} = \ell^{\infty}$, be the set of all bounded real or complex sequences. Define a metric function;

$$\wp(\mathrm{a}, \mathrm{b}) = \max_{i \in \mathbb{N}} \{|\mathrm{a}_i - \mathrm{b}_i|\}; \quad \forall\ \mathrm{a}, \mathrm{b} \in \ell^{\infty} \quad \text{where;} \quad \mathrm{a} = \{\mathrm{a}_i\}, \mathrm{b} = \{\mathrm{b}_i\}.$$

The first three axioms are straightforward. To prove the triangular inequality, we continue as follows:

$$\begin{aligned} \wp(\mathrm{a}, \mathrm{b}) &= \max\{|\mathrm{a}_i - \mathrm{b}_i|\} \\ &= \max\{|\mathrm{a}_i - \mathrm{c}_i + \mathrm{c}_i - \mathrm{b}_i|\} \\ &\leq \max\{|\mathrm{a}_i - \mathrm{c}_i|\} + \max\{|\mathrm{c}_i - \mathrm{b}_i|\} \\ &\leq \wp(\mathrm{a}, \mathrm{c}) + \wp(\mathrm{c}, \mathrm{b}). \end{aligned}$$

Hence $(\ell^{\infty}, \wp)$ is metric space.

Example 2.1.3.

Consider a real number $p \geq 1$ and define a set of real sequences as

$$\ell^p = \{\{\mathrm{a}_n\} : |\mathrm{a}_1|^p + |\mathrm{a}_2|^p + ... < \infty\}.$$

Define $\wp : \ell^p \times \ell^p \to \mathbb{R}$ as

$$\wp(\mathrm{b}, \mathrm{c}) = \left(\sum_{i=1}^{\infty} |\mathrm{b}_i - \mathrm{c}_i|^p \right)^{\frac{1}{p}},$$

here first three properties are trivially satisfied. One can easily prove the triangular inequality by using Minkowski inequality.

Definition 2.1.4. Continuous Mapping

"Let $(\mathbb{J}_1, \wp_1)$ and $(\mathbb{J}_2, \wp_2)$ be metric spaces. A mapping

$$\mathcal{S} : \mathbb{J}_1 \to \mathbb{J}_2$$

is said to be continuous at a point $\mathrm{b} \in \mathbb{J}_1$, if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\wp_2(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) < \epsilon \quad \forall\ \mathrm{a} \quad \text{whenever} \quad \wp_1(\mathrm{a}, \mathrm{b}) < \delta.$$

$\mathcal{S}$ is said to be continuous if it is continuous at every point of $\mathbb{J}_1$." [24]

Example 2.1.5.

Assume $\mathbb{J} = \mathbb{R}$ and $\wp$ is a usual metric. The widely known quadratic function is continuous. We consider one such mapping, $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ defined by

$$\mathcal{S}\mathrm{a} = 4\mathrm{a}^2$$

For any $\delta > \wp(\mathrm{a}, \mathrm{b}) = |\mathrm{a} - \mathrm{b}|$, consider

$$\begin{aligned} \wp(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) &= |4\mathrm{a}^2 - 4\mathrm{b}^2| \\ &= 4|\mathrm{a} - \mathrm{b}||\mathrm{a} + \mathrm{b}| \\ &= 4\wp(\mathrm{a}, \mathrm{b})|\mathrm{a} + \mathrm{b}| \\ &< 4\delta|\mathrm{a} + \mathrm{b}| \end{aligned}$$

so if we choose $\epsilon = 4\delta|\mathrm{a} + \mathrm{b}|$, then we have

$$\wp(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) < \epsilon \quad \text{whenever} \quad \wp(\mathrm{a}, \mathrm{b}) < \delta.$$

Definition 2.1.6. Convergence

"A sequence $\{\mathrm{a}_n\}$ in a metric space $\mathbb{J} = (\mathbb{J}, \wp)$ is said to converge or to be convergent if there is an $\mathrm{a} \in \mathbb{J}$ such that

$$\lim_{n\to\infty} \wp(\mathrm{a}_n, \mathrm{a}) = 0.$$

a is called limit of $\{\mathrm{a}_n\}$ and we write

$$\lim_{n\to\infty} \mathrm{a}_n = \mathrm{a} \quad \text{or} \quad \mathrm{a}_n \to \mathrm{a}.$$

We say that $\{\mathrm{a}_n\}$ converges to a or has the limit a. If $\{\mathrm{a}_n\}$ does not converges, then it is said to be divergent."[24]

Example 2.1.7.

Consider the set of real numbers $\mathbb{R}$ with usual metric which is defined as;

$$\wp(\mathrm{a}, \mathrm{b}) = |\mathrm{a} - \mathrm{b}|$$

then, the sequence $\{a_n\} = \{\frac{1}{n}\}$ in $\mathbb{J}$ is a convergent sequence.

Definition 2.1.8. Cauchy Sequence

"A sequence $\{\mathrm{a}_n\}$ in a metric space $\mathbb{J} = (\mathbb{J}, \wp)$ is said to be Cauchy (or fundamental) if for every $\epsilon > 0$ there is an $\mathbb{N} = \mathbb{N}(\epsilon)$ such that

$$\wp(\mathrm{a}_m, \mathrm{b}_n) < \epsilon \qquad \forall\, m, n > \mathbb{N}.\text{" [24]}$$

Definition 2.1.9. Complete Space

"A space $\mathbb{J}$ is said to be complete if every Cauchy sequence in $\mathbb{J}$ converges (that is, has a limit which is an element of $\mathbb{J}$)."[24]

Example 2.1.10.

With usual metric on $\mathbb{R}$ the closed interval $[0, 1]$ is complete.

For our main result it is necessary to define the distance between two sets. For this purpose we define the following concept.

Definition 2.1.11. Distance of a Point and a Set.

"The distance $\wp(\mathrm{a}, A)$ from a point a to a non-empty subset A of $(\mathbb{J}, \wp)$ is defined to be

$$\wp(\mathrm{a}, A) = \inf_{a \in A} \wp(\mathrm{a}, a).\text{"[24]}$$

Definition 2.1.12. Distance between two Sets.

"The distance $D(\mathbb{K}, \mathbb{L})$ between two non-empty subsets $\mathbb{K}$ and $\mathbb{L}$ of a metric space $(\mathbb{J}, \wp)$ is defined to be

$$D(\mathbb{K}, \mathbb{L}) = \inf \{\wp(\mathrm{a}, \mathrm{b}) : \mathrm{a} \in \mathbb{K}, \mathrm{b} \in \mathbb{L}\}.\text{"[24]}$$

Definition 2.1.13. Hausdorff Distance

"Let $(\mathbb{J}, \wp)$ be a metric space and $\mathcal{CB}(\mathbb{J})$ denotes the collection of all non-empty closed and bounded subsets of $\mathbb{J}$. For $\mathbb{K}, \mathbb{L} \in \mathcal{CB}(\mathbb{J})$ define

$$H(\mathbb{K}, \mathbb{L}) = \max\left\{\sup_{\mathrm{a}\in\mathbb{K}} \wp(\mathrm{a}, \mathbb{L}), \sup_{\mathrm{b}\in\mathbb{L}} \wp(\mathrm{b}, \mathbb{K})\right\},$$

where $\wp(\mathrm{a}, \mathbb{L})$ is distance of a to the set $\mathbb{L}$. It is known that H is a metric on $\mathcal{CB}(\mathbb{J})$, called the Hausdorff metric induced by the metric $\wp$." [25]

2.2 b-Metric Space

In 1989, Bakhtin [12] proposed the idea of b-metric space (bMS). It is accurately described as an initial extension of a MS. In current section, some definitions, examples and various facts pertaining to b-metric spaces are presented.

Definition 2.2.1. b-metric Space

"Let $\mathbb{J}$ be a non-empty set and let $\mathtt{b} \geq 1$ be a given real number. A function $\wp_b \colon \mathbb{J} \times \mathbb{J} \to [0, \infty)$ is called a b-metric if for all $\mathrm{a}, \mathrm{b}, \mathrm{m} \in \mathbb{J}$ the following conditions are satisfied,

(b_1) : $\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}) = 0 \iff \mathrm{a} = \mathrm{b}$,

(b_2) : $\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}) = \wp_{\mathtt{b}}(\mathrm{b}, \mathrm{a})$,

(b_3) : $\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{m}) \leq \mathtt{b}[\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}) + \wp_{\mathtt{b}}(\mathrm{b}, \mathrm{m})]$.

The pair $(\mathbb{J}, \wp_{\mathtt{b}})$ is called a b-metric space." [26]

Remark

1. Every bMS is a metric with $\mathtt{b} = 1$.
2. Class of bMS is larger than the class of MS.

Example 2.2.2.

Assume $(\mathbb{J}, \wp)$ is a metric space. Then for a real number $s > 1$, we define a function

$$\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}) = (\wp(\mathrm{a}, \mathrm{b}))^s$$

then, $\wp_{\mathtt{b}}$ is a $\mathtt{bMS}$ along $\mathtt{b} = 2^{s-1}$.

To prove this following following inequality is required:

$$\left(\frac{\mathrm{a}+\mathrm{b}}{2}\right)^s \leq \frac{\mathrm{a}^s+\mathrm{b}^s}{2} \quad \text{for a, b} > 0.$$

Let us check the third axiom

$$\wp_{\mathtt{b}}(\mathrm{a},\mathrm{c}) \leq 2^{s-1}\left(\wp_{\mathtt{b}}(\mathrm{a},\mathrm{b}) + \wp_{\mathtt{b}}(\mathrm{b},\mathrm{c})\right)$$

$$\begin{aligned}
\frac{(\mathrm{a}+\mathrm{b})^s}{2^s} &\leq \frac{(\mathrm{a}^s+\mathrm{b}^s)}{2} \\
\Rightarrow (\mathrm{a}+\mathrm{b})^s &\leq 2^s\frac{(\mathrm{a}^s+\mathrm{b}^s)}{2} \\
\Rightarrow (\mathrm{a}+\mathrm{b})^s &\leq 2^{s-1}\left(\mathrm{a}^s+\mathrm{b}^s\right) \\
&\leq 2^{s-1}\left[(\wp(\mathrm{a},\mathrm{b}))^s + (\wp(\mathrm{b},\mathrm{c}))^s\right] \\
\Longrightarrow \wp(\mathrm{a},\mathrm{c}) &\leq 2^{\mathrm{s}-1}\left(\wp(\mathrm{a},\mathrm{b}) + \wp(\mathrm{b},\mathrm{c})\right)
\end{aligned}$$

hence, $\wp$ is a $\mathtt{b}$-metric with $\mathtt{b} = 2^{s-1}$

Definition 2.2.3. Cauchy Sequence in $\mathtt{b}$-metric space

"Let $(\mathbb{J}, \wp)$ be a $\mathtt{b}$-metric space. Then a sequence $\{\mathrm{a}_n\}$ in $\mathbb{J}$ is called Cauchy sequence if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for each $n, m \geq n(\epsilon)$ we have

$$\wp(\mathrm{a}_n, \mathrm{a}_m) < \epsilon.\text{"}[27]$$

Definition 2.2.4. Convergence in $\mathtt{b}$-metric space

"Let $(\mathbb{J}, \wp)$ be a $\mathtt{b}$-metric space. Then a sequence $\{\mathrm{a}_n\}$ in $\mathbb{J}$ is called convergent sequence if and only there exists $\mathrm{a} \in \mathbb{J}$ such that there exists $n(\epsilon) \in \mathbb{N}$ such that for all $n \geq n(\epsilon)$ we have

$$\wp(\mathrm{a}_n, \mathrm{a}) < \epsilon.$$

In this case we write $\lim\limits_{n\to\infty} \mathrm{a}_n = \mathrm{a}$." [27]

Definition 2.2.5. Completeness in $\mathtt{b}$-metric space

"The $\mathtt{b}$-metric space is complete if every Cauchy sequence convergent." [27]

2.3 Fixed Point and Contractions

The burgeoning area of fixed point theory began, with the crucial work of poincare at the end of the nineteenth and early twentieth centuries.
In multiple discipline of applied and pure mathematics as well as other quantitative sciences fixed point results have proved extremely useful in determining the existence of solutions. In this section, definition of fixed point and various types of contractions with examples are presented in well defined manner.

Definition 2.3.1. Fixed Point
"A fixed point of a mapping $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ of a set $\mathbb{J}$ into itself is an $\mathrm{a} \in \mathbb{J}$ which is mapped onto itself (is "kept fixed" by $\mathcal{S}$), that is,

$$\mathcal{S}\mathrm{a} = \mathrm{a},$$

the image $\mathcal{S}\mathrm{a}$ coincides with a."[24]

Geometrically, The presence of a fixed point for a real-valued function, expressed as $\mathrm{b} = \mathcal{S}(\mathrm{a})$, is determined by the intersection of the function's graph and the real line $\mathrm{b} = \mathrm{a}$.

Example 2.3.2.
Suppose $\mathbb{J} = \mathbb{R}$. A self mapping $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ such that

$$\mathcal{S}(\mathrm{a}) = \mathrm{a}^2,$$

has two fixed points that are $= 0, 1$ (Fig 2.1).

Example 2.3.3. If $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ defined as $\mathcal{S}(\mathrm{a}) = \mathrm{a} - \ln(1 + e^{\mathrm{a}})$, then there is no fixed point of $\mathcal{S}$.

Example 2.3.4.
Consider the following trigonometric function.

$$\mathcal{S}(\mathrm{a}) = \tan(\mathrm{a}),$$

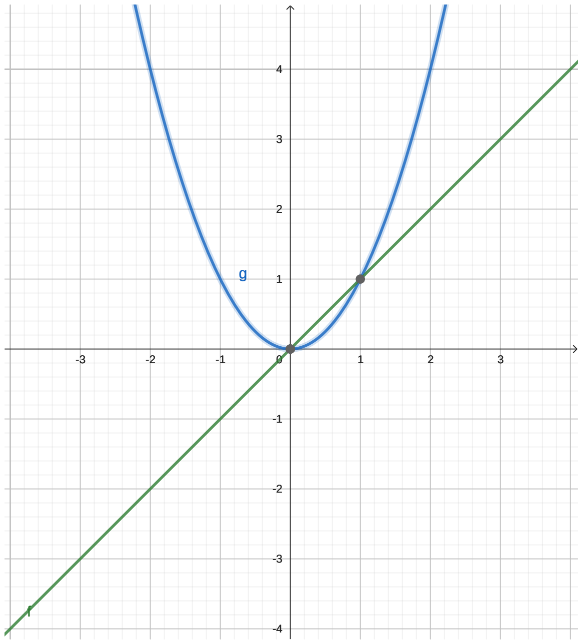

FIGURE 2.1: Two Fixed Points

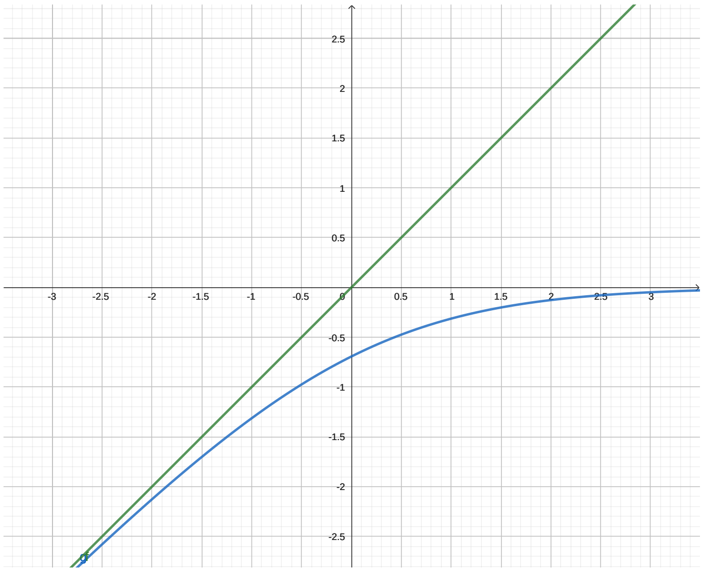

FIGURE 2.2: No Fixed Point

This function has infinite number of fixed points.

Definition 2.3.5. Lipschitzian Mapping

"Let $(\mathbb{J}, \wp)$ be a metric space. A mapping $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ is said to be Lipschitzain if there exists a constant $\nu \geq 0$ with,

$$\wp(\mathcal{S}(\mathrm{a}), \mathcal{S}(\mathrm{b})) \leq \nu \wp(\mathrm{a}, \mathrm{b}) \quad \forall\, \mathrm{a}, \mathrm{b} \in \mathbb{J}.$$

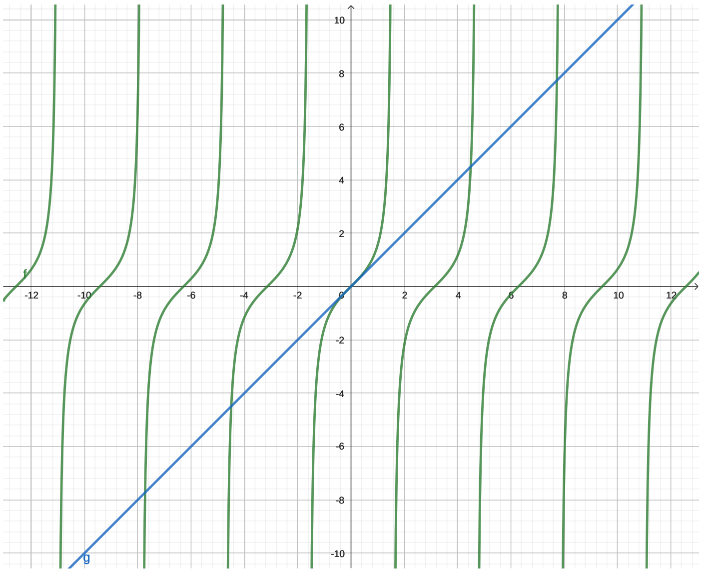

FIGURE 2.3: Infinitely Many Fixed Points

The smallest ν for which this condition holds is said to be the Lipschitzian constant for $\mathcal{S}$." [28]

Example 2.3.6.

Consider $(\mathbb{R}, \wp)$ with usual metric. Define a self map in $\mathcal{S} : \mathbb{R} \to \mathbb{R}$ as

$$\begin{aligned} \mathcal{S}(\mathrm{a}) &= 2\mathrm{a} + 7, \\ \implies \wp(\mathcal{S}(\mathrm{a}), \mathcal{S}(\mathrm{b})) &= |2\mathrm{a} + 7 - 2\mathrm{b} - 7| \\ &= |2\mathrm{a} - 2\mathrm{b}| \\ &= |2||\mathrm{a} - \mathrm{b}| \\ &= 2\wp(\mathrm{a}, \mathrm{b}), \end{aligned}$$

showing that $\mathcal{S}$ is Lipschitzian map with Lipchitz constant 2.

Definition 2.3.7. Contraction Mapping

"Let $(\mathbb{J}, \wp)$ be a complete metric space and mapping $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ is called contraction mapping on $\mathbb{J}$ if, $\exists\, \nu \in [0, 1)$ such that

$$\wp(\mathcal{S}_{\mathrm{a}}, \mathcal{S}_{\mathrm{b}}) \leq \nu\, \wp(\mathrm{a}, \mathrm{b}) \qquad \forall\, \mathrm{a}, \mathrm{b} \in \mathbb{J}.\text{"}[29]$$

Example 2.3.8. Let $\mathbb{J} = [0, 1]$ with usual metric.

Define $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ by

$$\begin{aligned}
\mathcal{S}(\mathrm{a}) &= \frac{1}{2+\mathrm{a}} \\
\wp\left(\mathcal{S}(\mathrm{a}), \mathcal{S}(\mathrm{b})\right) &= |\frac{1}{2+\mathrm{a}} - \frac{1}{2+\mathrm{b}}| \\
&= \left|\frac{\mathrm{b}-\mathrm{a}}{(2+\mathrm{a})(2+\mathrm{b})}\right| \\
&\leq \frac{1}{4}\wp(\mathrm{a}, \mathrm{b})
\end{aligned}$$

is contraction mapping with contraction constant $\frac{1}{4}$.

Definition 2.3.9. Contractive Mapping

"A mapping $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ is said to be contractive if for $\mathrm{a} \neq \mathrm{b}$, we have,

$$\wp(\mathcal{S}(\mathrm{a}), \mathcal{S}(\mathrm{b})) < \wp(\mathrm{a}, \mathrm{b}),$$

for all $\mathrm{a}, \mathrm{b} \in \mathbb{J}$."[30]

Example 2.3.10.

Consider $\mathbb{J} = [1, \infty)$ with usual metric, define $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ by

$$\mathcal{S}(\mathrm{a}) = \mathrm{a} + \frac{1}{\mathrm{a}}$$

$$\begin{aligned}
\wp\left(\mathcal{S}(\mathrm{a}), \mathcal{S}(\mathrm{b})\right) &= \left|\left(\mathrm{a} + \frac{1}{\mathrm{a}}\right) - \left(\mathrm{b} + \frac{1}{\mathrm{b}}\right)\right| \\
&= \left|(\mathrm{a}-\mathrm{b}) - \left(-\frac{1}{\mathrm{a}} + \frac{1}{\mathrm{b}}\right)\right|
\end{aligned}$$

$$\begin{aligned}
&= \left|(\mathrm{a}-\mathrm{b}) - \left(\frac{-\mathrm{b}+\mathrm{a}}{\mathrm{ab}}\right)\right| \\
&= \left|(\mathrm{a}-\mathrm{b}) - \left(\frac{\mathrm{a}-\mathrm{b}}{\mathrm{ab}}\right)\right| \\
&= |\ \mathrm{a}-\mathrm{b}\ ||\ 1 - \frac{1}{\mathrm{ab}}\ | \\
&< |\ \mathrm{a}-\mathrm{b}\ | \qquad \text{since} \lim_{\mathrm{a}\to\infty} |\ 1 - \frac{1}{\mathrm{ab}}\ | = 1
\end{aligned}$$

Therefore $\mathcal{S}$ is contractive.

One of the most fundamental fixed point theorem, called the Banach Contraction Principle BCP, was given by the stefan Banach in 1922. As obvious from name, the BCP is applied on contraction mappings defined on complete metric spaces. Many extension of the crucial BCP are constructed for other types of mappings, some milestone on fixed point theory are discussed below.

Theorem 2.3.11. Banach Contraction Principle

"Let $(\mathbb{J}, \wp)$ be a complete metric space and $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ be a contraction mapping, then $\mathcal{S}$ admits a unique fixed point in $\mathbb{J}$." [4]

Theorem 2.3.12. Edelstein Theorem

"Let $\mathbb{J}$ be a metric space and $\mathcal{S}$ a mapping of $\mathbb{J}$ into it self; $\mathcal{S}$ will be said to be a globally contractive mapping if the condition

$$\wp(\mathcal{S}(\mathrm{a}), \mathcal{S}(\mathrm{b})) < \nu\wp(\mathrm{a}, \mathrm{b})$$

with constant ν, $0 \leq \nu < 1$ holds for every $\mathrm{a}, \mathrm{b} \in \mathbb{J}$. [31]

Definition 2.3.13. Multi-valued Mapping

"Suppose $(\mathbb{J}, \wp)$ be a CMS with contraction mapping $\mathcal{S} : \mathbb{J} \to \mathbb{J}$. Let $\mathbb{J}$ and $\mathbb{K}$ be non-empty sets. $\mathcal{S}$ is said to be multi-valued mapping from $\mathbb{K}$ to $\mathbb{J}$ if $\mathcal{S}$ is function for $\mathbb{K}$ to the power set of $\mathbb{J}$. We denote a multi-valued mapping by

$$\mathcal{S} : \mathbb{K} \to 2^{\mathbb{J}}.\text{"}[32]$$

Definition 2.3.14. Fixed Point of Multi-valued

"Let $\mathbb{J}$ be any non-empty set. An element $\mathrm{a} \in \mathbb{J}$ is said to be a fixed point of a multi-valued mappings $\mathcal{S} : \mathbb{J} \to 2^{\mathbb{J}}$ if

$$\mathrm{a} \in \mathcal{S}\mathrm{a}.\text{"}[25]$$

Definition 2.3.15. Multi-valued Contraction

"A multi-valued mapping $\mathcal{S} : \mathbb{J} \to \mathcal{CB}(\mathbb{K})$ is said to be contraction if

$$H(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) \leq \nu\wp(\mathrm{a}, \mathrm{b})$$

for all $\mathrm{a}, \mathrm{b} \in \mathbb{J}$ and for some $\nu \in [0,1)$." [25]

Remark: Multi-valued Contraction Mappings are Continuous.

The following is Nadler's well known FP theorem for multi-valued mappings

Theorem 2.3.16. Nadler's Fixed Point Theorem

"Let $(\mathbb{J}, \wp)$ be a complete metric space. If $\mathcal{S} : \mathbb{J} \to \mathcal{CB}(\mathbb{J})$ is a multi-valued contraction mapping, then $\mathcal{S}$ has a fixed point." [8]

Chapter 3

Existence of Best Proximity Point Results, for Multi-Valued $\mathcal{F}$-Contraction with Applications

This chapter includes the detailed review of Jain et al. [23], who establish BPP theorems by defining a novel concept of multivalued $\mathcal{F}$-contraction over partially ordered complete metric space (POCMS) with the assumption of altering distance function (ADF).

3.1 Preliminaries

Firstly, some symbols are introduced that are used in main result. Assume $\mathbb{J}$ is a non-empty set and $(\mathbb{J}, \wp, \preceq)$ is a POMS. Suppose $\mathbb{K}$ and $\mathbb{L}$ are non-empty subsets of the metric space $(\mathbb{J}, \wp)$ and $\mathcal{CB}(\mathbb{J})$ represents the family of closed and bounded non-empty subsets of $\mathbb{J}$. Now, define the following;

$$\mathcal{D}(\mathrm{a}, \mathbb{L}) = \inf\{\wp(\mathrm{a}, \mathrm{b}) : \mathrm{b} \in \mathbb{L}, \quad \forall \quad \mathrm{a} \in \mathbb{J}\}$$

$$\delta(\mathbb{K}, \mathbb{L}) = \sup\{\wp(\mathrm{a}, \mathrm{b}) : \mathrm{a} \in \mathbb{K}, \quad \text{and} \quad \mathrm{b} \in \mathbb{L}\}$$

$$\wp(\mathbb{K}, \mathbb{L}) = \inf\{\wp(\mathrm{a}, \mathrm{b}) : \mathrm{a} \in \mathbb{K}, \quad \text{and} \quad \mathrm{b} \in \mathbb{L}\}$$

$$\mathbb{K}_0 = \{\mathrm{a} \in \mathbb{K} : \wp(\mathrm{a}, \mathrm{b}) = \wp(\mathbb{K}, \mathbb{L}), \quad \text{for some} \quad \mathrm{b} \in \mathbb{L}\}$$
$$\mathbb{L}_0 = \{\mathrm{b} \in \mathbb{L} : \wp(\mathrm{a}, \mathrm{b}) = \wp(\mathbb{K}, \mathbb{L}), \quad \text{for some} \quad \mathrm{a} \in \mathbb{K}\}.$$

Definition 3.1.1. Best Proximity Point

Suppose $\mathbb{K}$ and $\mathbb{L}$ are non-empty subsets of a metric space $(\mathbb{J}, \wp)$ and $\mathcal{S} : \mathbb{K} \to 2^{\mathbb{L}}$ is a multivalued mapping. Then, $\exists$ a point $\mathrm{a} \in \mathbb{J}$ is the `BPP` for $\mathcal{S}$ if

$$\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}) = \wp(\mathbb{K}, \mathbb{L}). \text{ [19]}$$

Remark 3.1.2.

If we take self-mapping, then `BBP` turn into a fixed point.

Khan et al. [33] presented the idea of altering distance function (`ADF`) as:

Definition 3.1.3. Altering Distance Function (ADF)

A function $\breve{\phi} : \mathbb{R}^+ \to \mathbb{R}^+$ is known as `ADF` if it fulfil the conditions given below:

(D1). $\breve{\phi}$ is continuous,

(D2). $\breve{\phi}$ is monotonically increasing,

(D3). $\breve{\phi}(b) > 0 \ \ \forall \ \mathrm{b} > 0$.

In 2012, Wardowski [20] introduced the notion of $\mathcal{F}$-contraction.

Definition 3.1.4. $\mathcal{F}$-Mapping

A mapping $\mathcal{F} : (0, +\infty) \to \mathbb{R}$ is known as $\mathcal{F}$-contraction if:

$(\mathbb{F}1)$: $\mathcal{F}$ is increasing, which implies $\forall \ a_1, b_1 \in (0, \infty)$, so that

$$a_1 < b_1 \implies \mathcal{F}(a_1) \leq \mathcal{F}(b_1);$$

$(\mathbb{F}2)$: Every sequence $\{\Im_n\}$ of positive numbers,

$$\lim_{n\to\infty} \Im_n = 0 \text{ iff } \lim_{n\to\infty} \mathcal{F}(\Im_n) = -\infty;$$

$(\mathbb{F}3)$: $\exists \ \nu \in (0, 1)$ such that

$$\lim_{\Im \to 0} \Im^{\nu} \mathcal{F}(\Im) = 0.$$

Family of all such $\mathcal{F}$-function is denoted by $\bar{F}$.

Example 3.1.5.

Define $\mathcal{F}$: $(0, \infty) \to \mathbb{R}$ with $\nu \in (0,1)$ and $\mathrm{a} \in \mathbb{R}^{+}$ as

$$\mathcal{F}(\mathrm{a}) = \frac{-1}{\sqrt{\mathrm{a}}} \quad \text{for} \quad \mathrm{a} > 0,$$

one can easily verify that all the conditions of $\mathcal{F}$-mapping for any constant $\nu \in (0,1)$ are satisfied.

Definition 3.1.6. $\mathcal{F}$-contraction

A mapping $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ is called $\mathcal{F}$-contraction if for $\check{\tau} > 0$

$$\wp(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) > 0 \implies \check{\tau} + \mathcal{F}\left(\wp\left(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}\right)\right) \leq \mathcal{F}\left(\wp\left(\mathrm{a}, \mathrm{b}\right)\right) \quad \forall\ \mathrm{a}, \mathrm{b} \in \mathbb{J}, \quad [20] \tag{3.1}$$

for some $\mathcal{F} \in \bar{F}$.

Remark 3.1.7.

Every $\mathcal{F}$-contraction is necessarily continuous.

Example 3.1.8.

For an $\mathcal{F}$ mapping $\mathcal{F} : (0, +\infty) \to \mathbb{R}$ defined as:

$$\mathcal{F}(\mathrm{a}) = \ln \mathrm{a} + \mathrm{a},$$

with $\mathrm{a} > 0$, and constant $\nu \in \left(\frac{1}{2}, 1\right)$ the contraction condition (3.1) takes the following form:

$$\frac{\wp\left(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}\right)}{\wp\left(\mathrm{a}, \mathrm{b}\right)} e^{\wp(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) - \wp(\mathrm{a}, \mathrm{b})} \leq e^{-\check{\tau}},$$

for all $\mathrm{a}, \mathrm{b} \in \mathbb{R}^{+}$, $\mathcal{S}\mathrm{a} \neq \mathcal{S}\mathrm{b}$.

Raj [34] initially presented $\mathcal{P}$-property as;

Definition 3.1.9. **$\mathcal{P}$-Property**

Suppose $(\mathbb{K}, \mathbb{L})$ be the pair of non-empty subsets of metric space $\mathbb{J}$ such that $\mathbb{K}_0$ is non-empty. Then the pair $(\mathbb{K}, \mathbb{L})$ have $\mathcal{P}$-property iff,

$$\left.\begin{aligned} \wp(\mathrm{a}_1, \mathrm{b}_1) = \wp(\mathbb{K}, \mathbb{L}) \\ \wp(\mathrm{a}_2, \mathrm{b}_2) = \wp(\mathbb{K}, \mathbb{L}) \end{aligned}\right\} \implies \wp(\mathrm{a}_1, \mathrm{a}_2) = \wp(\mathrm{b}_1, \mathrm{b}_2),$$

where $\mathrm{a}_1, \mathrm{a}_2 \in \mathbb{K}_0$ and $\mathrm{b}_1, \mathrm{b}_2 \in \mathbb{L}_0$.

3.2 Multivalued $\mathcal{F}$-Contraction

Pragadeeswarar et al. [35] established some BPP results regarding multivalued mappings in the setting of POMS. These results are further explained by Jain et al. [23] on partially ordered complete metric spaces. For better understanding of those results following definitions are necessary:

Definition 3.2.1. **Multivalued $\mathcal{F}$-Contraction with Altering Distance Function**

Suppose $\mathbb{K}$ and $\mathbb{L}$ are two non-empty closed subsets of $(\mathbb{J}, \wp)$. A multivalued mapping $\mathcal{S} : \mathbb{K} \to \mathcal{CB}(\mathbb{L})$ is called $\mathcal{F}$-contraction with ADF $\breve{\phi}$ in order that $\mathcal{S}\mathrm{a}_0 \subseteq \mathbb{L}_0$, $\forall$ $\mathrm{a}_0 \in \mathbb{K}_0$ it satisfying:

$$\breve{\tau} + \mathcal{F}\left(\breve{\phi}\left(\delta\left(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}\right)\right)\right) \leq \mathcal{F}\left(\breve{\phi}\left(\mathcal{N}\left(\mathrm{a}, \mathrm{b}\right)\right) - \breve{\phi}\left(\wp\left(\mathbb{L}, \mathbb{M}\right)\right)\right) \quad \forall \ \mathrm{a} \leq \mathrm{b} \in \mathbb{L}, \tag{3.2}$$

Where,

$$\mathcal{N}(\mathrm{a}, \mathrm{b}) = \max\left\{\wp(\mathrm{a}, \mathrm{b}), \mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}), \mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{b}), \frac{\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{b}) + \mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a})}{2}\right\}$$

the function $\breve{\phi}$ with $\breve{\phi}(\mathrm{a}, \mathrm{b}) \leq \breve{\phi}(\mathrm{a}) + \breve{\phi}(\mathrm{b})$ $\forall$ $\mathrm{a}, \mathrm{b} \in [0, +\infty)$.

By Choosing $\mathcal{F}(\mathrm{a}) = \ln \mathrm{a}$ in (3.2) is knwon as contraction by Pragadeeswarar et al. [35]:

$$\begin{aligned}
&\check{\tau} + \ln\left(\breve{\phi}\left(\delta\left(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}\right)\right)\right) \leq \ln\left(\breve{\phi}\left(\mathcal{N}\left(\mathrm{a}, \mathrm{b}\right)\right) - \breve{\phi}\left(\wp\left(\mathbb{L}, \mathbb{M}\right)\right)\right)\\
&\Leftrightarrow \ln e^{\check{\tau}} + \ln\left(\breve{\phi}\left(\delta\left(\mathcal{S}\mathrm{a}.\mathcal{S}\mathrm{b}\right)\right)\right) \leq \ln\left(\breve{\phi}\left(\mathcal{N}\left(\mathrm{a}, \mathrm{b}\right)\right) - \breve{\phi}\left(\wp\left(\mathbb{K}, \mathbb{L}\right)\right)\right)\\
&\Leftrightarrow \ln \breve{\phi}\left(\delta\left(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}\right)\right) \leq \ln\left\{\frac{\breve{\phi}\left(\mathcal{N}\left(\mathrm{a}, \mathrm{b}\right)\right) - \breve{\phi}\left(\wp(\mathrm{a}, \mathrm{b})\right)}{e^{\check{\tau}}}\right\}\\
&\Leftrightarrow \breve{\phi}\left(\delta\left(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}\right)\right) \leq \frac{1}{e^{\check{\tau}}}\left(\breve{\phi}\left(\mathcal{N}\left(\mathrm{a}, \mathrm{b}\right)\right)\right) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L}))) \qquad \because \frac{1}{e^{\check{\tau}}} = \nu\\
&\Leftrightarrow \breve{\phi}\left(\delta\left(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}\right)\right) \leq \nu\left(\breve{\phi}\left(\mathcal{N}\left(\mathrm{a}, \mathrm{b}\right)\right)\right) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L}))).
\end{aligned}$$

Theorem 3.2.2.

Consider a `POCMS` $(\mathbb{J}, \preceq, \wp)$. Suppose $\mathbb{K}$ and $\mathbb{L}$ be non-empty closed subset of the `MS` $(\mathbb{J}, \wp)$ in such a way that $\mathbb{K}_0$ is non empty and the pair $(\mathbb{K}, \mathbb{L})$ has $\mathcal{P}$-property. Suppose $\mathcal{S} : \mathbb{K} \to \mathcal{CB}(\mathbb{L})$ be a multivalued $\mathcal{F}$-contraction with `ADF` $\breve{\phi}$, satisfies :

(Q1) : $\exists$ two elements $\mathrm{a}_0, \mathrm{a}_1 \in \mathbb{K}_0$ and $\mathrm{b}_0 \in \mathcal{S}\mathrm{a}_0$ such that

$\wp(\mathrm{a}_1, \mathrm{b}_0) = \wp(\mathbb{K}, \mathbb{L})$ and $\mathrm{a}_0 \preceq \mathrm{a}_1$.

(Q2) : $\forall$ $\mathrm{a}, \mathrm{b} \in \mathbb{K}_0, \mathrm{a} \preceq \mathrm{b} \implies \mathcal{S}\mathrm{a} \subset \mathcal{S}\mathrm{b}$.

(Q3) : If $\{\mathrm{a}_n\}$ is a non decreasing sequence in $\mathbb{K}$ such that $\mathrm{a}_n \to \mathrm{a}$, then $\mathrm{a}_n \preceq \mathrm{a}, \ \forall\ n$.

Then, $\exists$ $\mathrm{a} \in \mathbb{K}$ such that $\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}) = \wp(\mathbb{K}, \mathbb{L})$.

Proof. Using $(Q1)$ $\exists$ $\mathrm{a}_0, \mathrm{a}_1$ in $\mathbb{K}_0$ and $\mathrm{b}_0 \in \mathcal{S}\mathrm{a}_0$ such that $\wp(\mathrm{a}_1, \mathrm{b}_0) = \wp(\mathbb{K}, \mathbb{L})$ and $\mathrm{a}_0 \preceq \mathrm{a}_1$.

By $(Q2) \implies \mathcal{S}\mathrm{a}_0 \subset \mathcal{S}\mathrm{a}_1$, so $\exists$ $\mathrm{b}_1 \in \mathcal{S}\mathrm{a}_1$ with $\wp(\mathrm{a}_2, \mathrm{b}_1) = \wp(\mathbb{K}, \mathbb{L})$ such that $\mathrm{a}_1 \preceq \mathrm{a}_2$. Generally, in each case $n \in \mathbb{N}$, $\exists$ $\mathrm{a}_{n+1} \in \mathbb{K}_0$ and $\mathrm{b}_n \in \mathcal{S}\mathrm{a}_n$ such that $\wp(\mathrm{a}_{n+1}, \mathrm{b}_n) = \wp(\mathbb{K}, \mathbb{L})$. Thus,

$$\wp(\mathrm{a}_{n+1}, \mathrm{b}_n) = \mathcal{D}(\mathrm{a}_{n+1}, \mathcal{S}\mathrm{a}_n) = \wp(\mathbb{K}, \mathbb{L}) \quad \forall\ n \in \mathbb{N}. \tag{3.3}$$

Now,

$$\mathrm{a}_0 \preceq \mathrm{a}_1 \preceq \mathrm{a}_3 \preceq ... \preceq \mathrm{a}_n \preceq \mathrm{a}_{n+1}...$$

If $\exists\, n_0$ such that $\mathrm{a}_{n_0} = \mathrm{a}_{n_0+1}$ then $\wp(\mathrm{a}_{n_0+1}, \mathrm{b}_{n_0}) = \mathcal{D}(\mathrm{a}_{n_0}, \mathcal{S}\mathrm{a}_{n_0}) = \wp(\mathbb{K}, \mathbb{L})$. It follows that a_{n_0} is the best proximity point of $\mathcal{S}$ and we are done.

Now, assume $\mathrm{a}_n \neq \mathrm{a}_{n+1}\ \forall\ n$. Since $\wp(\mathrm{a}_{n+1}, \mathrm{b}_n) = \wp(\mathbb{K}, \mathbb{L})$ and $\wp(\mathrm{a}_n, \mathrm{b}_{n-1}) = \wp(\mathbb{K}, \mathbb{L})$ and $(\mathbb{K}, \mathbb{L})$ has the $\mathcal{P}$-property

$$\wp(\mathrm{a}_n, \mathrm{a}_{n+1}) = \wp(\mathrm{b}_{n-1}, \mathrm{b}_n) \quad \forall\ n \in \mathbb{N}. \tag{3.4}$$

Given $\mathrm{a}_{n-1} \prec \mathrm{a}_n$, so

$$\begin{aligned}
\mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))) &= \mathcal{F}(\breve{\phi}(\wp(\mathrm{b}_{n-1}, \mathrm{b}_n))) \\
\mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))) &\leq \mathcal{F}(\breve{\phi}(\delta(\mathcal{S}\mathrm{a}_{n-1}, \mathcal{S}\mathrm{a}_n))) \\
&\leq \mathcal{F}(\breve{\phi}(\mathcal{N}(\mathrm{a}_{n-1}, \mathrm{a}_n)) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L}))) - \breve{\tau}.
\end{aligned} \tag{3.5}$$

Now

$$\begin{aligned}
\mathcal{N}(\mathrm{a}_{n-1}, \mathrm{a}_n) &= \max\Big\{\wp(\mathrm{a}_{n-1}, \mathrm{a}_n), \mathcal{D}(\mathrm{a}_{n-1}, \mathcal{S}\mathrm{a}_{n-1}), \mathcal{D}(\mathrm{a}_n, \mathcal{S}\mathrm{a}_n), \frac{\mathcal{D}(\mathrm{a}_{n-1}, \mathcal{S}\mathrm{a}_n) + \mathcal{D}(\mathrm{a}_n, \mathcal{S}\mathrm{a}_{n-1})}{2}\Big\} \\
&\leq \max\Big\{\wp(\mathrm{a}_{n-1}, \mathrm{a}_n), \wp(\mathrm{a}_{n-1}, l_{n-1}), \wp(\mathrm{a}_n, l_n), \frac{\wp(\mathrm{a}_{n-1}, l_n) + \wp(\mathrm{a}_n, l_{n-1})}{2}\Big\} \\
&\leq \max\Big\{\wp(\mathrm{a}_{n-1}, \mathrm{a}_n), \wp(\mathrm{a}_{n-1}, \mathrm{b}_{n-2}) + \wp(\mathrm{b}_{n-2}, \mathrm{b}_{n-1})\wp(\mathrm{a}_n, \mathrm{b}_{n-1}) + \wp(\mathrm{b}_{n-1}, \mathrm{b}_n), \\
&\quad \frac{\wp(\mathrm{a}_{n-1}, \mathrm{b}_{n-2}) + \wp(\mathrm{b}_{n-2}, \mathrm{b}_{n-1}) + \wp(\mathrm{b}_{n-1}, \mathrm{b}_n) + \wp(\mathrm{a}_n, l_{n-1})}{2}\Big\} \\
&\leq \max\Big\{\wp(\mathrm{a}_{n-1}, \mathrm{a}_n), \wp(\mathbb{K}, \mathbb{L}) + \wp(\mathrm{a}_{n-1}, \mathrm{a}_n), \wp(\mathbb{K}, \mathbb{L}) + \wp(\mathrm{a}_n, \mathrm{a}_{n-1}), \\
&\quad \frac{\wp(\mathbb{K}, \mathbb{L}) + \wp(\mathrm{a}_{n-1}, \mathrm{a}_n) + \wp(\mathrm{a}_n, \mathrm{a}_{n+1}) + \wp(\mathbb{K}, \mathbb{L})}{2}\Big\} \\
&\leq \max\Big\{\wp(\mathbb{K}, \mathbb{L}) + \wp(\mathrm{a}_{n-1}, \mathrm{a}_n), \wp(\mathbb{K}, \mathbb{L}) + \wp(\mathrm{a}_n, \mathrm{a}_{n+1})\Big\}.
\end{aligned}$$

Using Equation (3.4)

$$\begin{aligned}
\mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}) &\leq \mathcal{F}(\breve{\phi} \max\Big\{\wp(\mathbb{K}, \mathbb{L}) + \wp(\mathrm{a}_{n-1}, \mathrm{a}_n), \wp(\mathbb{K}, \mathbb{L}) + \wp(\mathrm{a}_n, \mathrm{a}_{n+1})\Big\} \\
&\quad - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L}))) - \breve{\tau}.
\end{aligned} \tag{3.6}$$

If $\wp(\mathrm{a}_n, \mathrm{a}_{n+1}) > \wp(\mathrm{a}_{n-1}, \mathrm{a}_n)$ from (3.5)

$$\begin{aligned}
\mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}) &\leq \mathcal{F}(\breve{\phi}(\wp(\mathbb{K}, \mathbb{L}) + \wp(\mathrm{a}_n, \mathrm{a}_{n+1})), -\breve{\phi}(\wp(\mathbb{K}, \mathbb{L}))) - \breve{\tau} \\
\mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}) &\leq \mathcal{F}(\breve{\phi}(\wp(\mathbb{K}, \mathbb{L})) + \breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))), -\breve{\phi}(\wp(\mathbb{K}, \mathbb{L}))) - \breve{\tau}
\end{aligned}$$

$$\mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1})) \leq \mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))) - \check{\tau},$$

which leads to contradiction. So,

$$\wp(\mathrm{a}_n, \mathrm{a}_{n+1}) \leq (\mathrm{a}_{n-1}, \mathrm{a}_n). \tag{3.7}$$

Since, the sequence $\{\wp(\mathrm{a}_n, \mathrm{k_{n+1}})\}$ is monotonically, non-increasing and bounded below, so, $\exists$ s ≥ 0 such that,

$$\lim_{n\to\infty} \wp(\mathrm{a}_n, \mathrm{a}_{n+1}) = \mathrm{s} \geq 0. \tag{3.8}$$

Let $\lim_{n\to\infty} \wp(\mathrm{a}_n, \mathrm{a}_{n+1}) = \mathrm{s} \geq 0$ using (3.7), equation (3.8) becomes

$$\begin{aligned}\mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))) &\leq \mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_{n-1}, \mathrm{a}_n))) - \check{\tau} \\ &\Rightarrow \mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))) \leq \mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_{n-2}, \mathrm{a}_{n-1}))) - 2\check{\tau}.\end{aligned}$$

Continuing in the same manner, following is obtained

$$\mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))) \leq \mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_0, \mathrm{a}_1))) - n\check{\tau}. \tag{3.9}$$

$$\Longrightarrow \lim_{n\to\infty} \mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))) = -\infty$$

$$\Longrightarrow \lim_{n\to\infty} \breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1})) = 0 \tag{3.10}$$

using $(F3)$ $\exists\, \gamma \in (0,1)$ such that,

$$\lim_{\wp(\mathrm{a}_n,\mathrm{a}_{n+1})\to 0} (\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1})))^{\gamma} \mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))) = 0$$

$$\Longrightarrow \lim_{n\to\infty} (\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1})))^{\gamma} \mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))) = 0 \tag{3.11}$$

Now, by (3.9),

$$\mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))) \leq \mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_0, \mathrm{a}_1))) - n\check{\tau}$$

$$\mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))) - \mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_0, \mathrm{a}_1)) \leq -n\check{\tau}$$
$$(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1})))^{\gamma}\mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))) - \mathcal{F}(\breve{\phi}(\wp(\mathrm{a}_0, \mathrm{a}_1))) \leq -(\breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1})))^{\gamma}n\check{\tau} \leq 0.$$

Denoting $\psi_n = \breve{\phi}(\wp(\mathrm{a}_n, \mathrm{a}_{n+1}))$, we have

$$(\psi_n)^{\gamma}(\mathcal{F}(\psi_n) - \mathcal{F}(\psi_0)) \leq -(\psi_n)^{\gamma}n\check{\tau},$$

applying $n \to \infty$, the above equation, (3.10) and (3.11) gives.

$$\lim_{n\to\infty}(\psi_n)^{\gamma}(\mathcal{F}(\psi_n) - \mathcal{F}(\psi_0)) \leq \lim_{n\to\infty} -(\psi_n)^{\gamma}n\check{\tau} \leq 0$$
$$\lim_{n\to\infty} n(\psi_n)^{\gamma} = 0. \tag{3.12}$$

Now, note that from (3.12) for any value of $\epsilon > 0$ $\exists$ $n_1 \in \mathbb{N}$ such that

$$|n(\psi_n)^{\gamma} - 0| < \epsilon \qquad \forall n \geq n_1,$$
$$\implies |n(\psi_n)^{\gamma}| < \epsilon,$$
$$\implies (\psi_n) < \frac{\epsilon}{n^{\frac{1}{\gamma}}} \qquad \forall n \geq n_1.$$

consider $\{a_n\}$ is Cauchy,

so assume $\mathrm{m}, \mathrm{n} \in \mathbb{N} \ni \mathrm{m} > \mathrm{n} > \mathrm{n}_1$. Hence,

$$\breve{\phi}(\wp(\mathrm{a}_m, \mathrm{a}_n)) \leq \breve{\phi}(\wp(\mathrm{a}_m, \mathrm{a}_{m-1})) + \breve{\phi}(\wp(\mathrm{a}_{m-1}, \mathrm{a}_{m-2})) + \ldots + \breve{\phi}(\wp(\mathrm{a}_{n+1}, \mathrm{a}_n))$$
$$\leq \psi_{m-1} + \psi_{m-2} + \ldots + \psi_n$$
$$< \sum_{i=n}^{\infty}\psi_i \leq \sum_{i=n}^{\infty}\frac{\epsilon}{n^{\frac{1}{\gamma}}}.$$

Given $\gamma \in (0, 1)$ so, $\frac{1}{\gamma} > 1$. Consequentially, by using the P-series test, $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{\gamma}}}$ is convergent for $\frac{1}{\gamma} > 1$. Therefore, $\{\mathrm{a}_n\}$ is a Cauchy sequence in $\mathbb{K}$. Given that $\mathbb{K}$ is complete, so $\exists$ $\mathrm{a} \in \mathbb{K}$ such that,

$$\lim_{n\to\infty} \mathrm{a}_n = \mathrm{a} \quad \text{or} \quad \mathrm{a}_n \to \mathrm{a}.$$

Since $\wp(\mathrm{a}_n, \mathrm{a}_{n+1}) = \wp(\mathrm{b}_{n-1}, \mathrm{b}_n)$. Hence $\{\mathrm{b}_n\}$ is Cauchy sequence in $\mathbb{K}$ and convergent.
Assume that $\mathrm{b}_n \to \mathrm{b}$. By the relation $\wp(\mathrm{a}_{n+1}, \mathrm{b}_n) = \wp(\mathbb{K}, \mathbb{L}) \quad \forall\ n$.
We conclude that $\wp(\mathrm{a}, \mathrm{b}) = \wp(\mathbb{K}, \mathbb{L})$. Now, suppose that $\mathrm{b} \in \mathcal{S}\mathrm{a}$. Given a_n is an increasing sequence in $\mathbb{K}$ and $\mathrm{a}_n \to \mathrm{a}$ according to the axiom $(Q3)$, $\mathrm{a}_n \preceq \mathrm{a}$ for all n.
Suppose that $\mathrm{b} \notin \mathcal{S}\mathrm{a}$. Consider the contraction condition (3.2),

$$\begin{aligned}
&\mathcal{F}(\breve{\phi}(\mathcal{D}(\mathrm{b}_n, \mathcal{S}\mathrm{a})))\\
\leq &\mathcal{F}(\breve{\phi}(\delta(\mathcal{S}\mathrm{a}_n, \mathcal{S}\mathrm{a})))\\
\leq &\mathcal{F}\left(\breve{\phi}\left(\max\left\{\wp(\mathrm{a}_n, \mathrm{a}), \mathcal{D}(\mathrm{a}_n, \mathcal{S}\mathrm{a}_n), \mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}), \frac{\mathcal{D}(\mathrm{a}_n), \mathcal{S}\mathrm{a}) + \mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}_n}{2}\right\}\right) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L})\right) - \check{\tau}\\
\leq &\mathcal{F}\left(\breve{\phi}\left(\max\left\{\wp(\mathrm{a}_n, \mathrm{a}), \wp(\mathrm{a}_n, \mathcal{S}\mathrm{a}_n), \mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}), \frac{\mathcal{D}(\mathrm{a}_n), \mathcal{S}k) + \wp(\mathrm{a}, \mathcal{S}\mathrm{a}_n}{2}\right\}\right) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L})\right) - \check{\tau},
\end{aligned}$$

applying $n \to \infty$ on the above inequality by using $\mathrm{a}_n \to \mathrm{a}, \mathrm{b}_n \to \mathrm{b}$ and $\wp(\mathrm{a}, \mathrm{b}) = \wp(\mathbb{K}, \mathbb{L})$, we have

$$\begin{aligned}
&\mathcal{F}(\breve{\phi}(\mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a})))\\
&\leq \mathcal{F}\left(\breve{\phi}\left(\max\left\{0, \wp(\mathrm{a}_n, \mathrm{a}), \mathcal{D}(\mathrm{a}_n, \mathcal{S}\mathrm{a}_n), \mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}), \frac{\mathcal{D}(\mathrm{a}_n), \mathcal{S}\mathrm{a}) + \mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}_n}{2}\right\}\right) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L})\right) - \check{\tau}\\
&\leq \mathcal{F}(\breve{\phi}(\wp(\mathbb{K}, \mathbb{L}) + \mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a})) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L}))) - \check{\tau}\\
&\leq \mathcal{F}(\breve{\phi}(\wp(\mathbb{K}, \mathbb{L}) + \wp(\mathrm{b}, \mathcal{S}\mathrm{a})) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L}))) - \check{\tau}\\
&\mathcal{F}(\breve{\phi}(\mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a}))) \leq \mathcal{F}(\breve{\phi}(\mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a}))) - \check{\tau},
\end{aligned}$$

which is contradiction.

This means that $\mathrm{b} \in \mathcal{S}\mathrm{a}$, and hence, $\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}) = \wp(\mathbb{K}, \mathbb{L})$.

Hence, a is `BBP` of $\mathcal{S}$. □

3.3 Consequences

This section includes some consequences of Theorem 3.2.2. Following theorem follows analogously by Theorem 3.2.2. Here, some results of `BPP` and `FP` for multi-valued and self mappings in `POCMS` are given.

Theorem 3.3.1.

Consider $(\mathbb{J}, \preceq, \wp)$ be a POCMS. Suppose $\mathbb{K}$ and $\mathbb{L}$ are non-empty closed subset of the MS $(\mathbb{J}, \wp)$ in such a way that $\mathbb{K}_0$ is non-empty and $\mathbb{K}, \mathbb{L}$ satisfies $\mathcal{P}$-property. Assume $\mathcal{S} : \mathbb{K} \to \mathcal{CB}(\mathbb{L})$ be a multi-valued mapping along ADF $\breve{\phi}$, satisfies the following axioms,

(A1): $\exists$ two elements $\mathrm{a}_0, \mathrm{a}_1 \in \mathbb{K}_0$ and $\mathrm{b}_0 \in \mathcal{S}\mathrm{a}_0$ such that
$\wp(\mathrm{a}_1, \mathrm{b}_0) = \wp(\mathbb{K}, \mathbb{L})$ and $\mathrm{a}_0 \preceq \mathrm{a}_1$;

(A2): $\mathcal{S}(\mathbb{K}_0) \subseteq \mathbb{L}_0$ and $\mathcal{F}(\breve{\phi}(\delta(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}))) \leq \mathcal{F}(\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b}))) - \check{\tau} \quad \forall\, \mathrm{a} \preceq \mathrm{b} \in \mathbb{L}$, where

$$\mathcal{N}(\mathrm{a}, \mathrm{b}) = \max\left\{\wp(\mathrm{a}, \mathrm{b}), \mathcal{D}(\mathrm{a}, \mathrm{b}) - \wp(\mathbb{K}, \mathbb{L}), \mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{b}) - \wp(\mathbb{K}, \mathbb{L}), \frac{\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{b}) + \mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a})}{2} - \wp(\mathbb{K}, \mathbb{L})\right\}$$

$\forall\ (\mathrm{a}, \mathrm{b}) \in [0, +\infty)$;

(A3): For all $(\mathrm{a}, \mathrm{b}) \in \mathbb{L}_0, \mathrm{a} \preceq \mathrm{b} \implies \mathcal{S}\mathrm{a} \subset \mathcal{S}\mathrm{b}$;

(A4): If $\{\mathrm{a}_n\}$ is a non-decreasing sequence in $\mathbb{K} \ni \mathrm{a}_n \to \mathrm{a}$, then $\mathrm{a}_n \preceq \mathrm{a} \quad \forall\, n$.

Then, $\exists$ an element a in $\mathbb{K}$ such that $\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}) = \wp(\mathbb{K}, \mathbb{L})$.

Proof. It follows from Theorem 3.2.2 . □

Here are some corollaries, which are deduced from Theorem (3.2.2) by incorporating the self mapping.

Corollary 3.3.2.

Consider a POCMS $(\mathbb{J}, \preceq, \wp)$. Assume $\mathbb{K}$ and $\mathbb{L}$ be a non-empty closed subsets of MS $(\mathbb{J}, \wp) \ni \mathbb{K}_0$ is non-empty and $(\mathbb{K}, \mathbb{L})$ satisfies the $\mathcal{P}$-property. Suppose $\mathcal{S} : \mathbb{K} \to \mathbb{L}$ is a self mapping satisfying:

(A1): $\exists$ two elements $(\mathrm{a}_0, \mathrm{a}_1)$ in $\mathbb{K}_0$ and $\mathrm{b}_0 \in \mathcal{S}\mathrm{a}_0$ such that
$\wp(\mathrm{a}_1, \mathrm{b}_0) = \wp(\mathbb{K}, \mathbb{L})$ and $\mathrm{a}_0 \preceq \mathrm{a}_1$;

(A2): $\mathcal{S}(\mathbb{K}_0) \subset \mathbb{L}_0$ and $\mathcal{F}(\breve{\phi}(\wp(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}))) \leq \mathcal{F}(\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b}))) - \check{\tau} \quad \forall\ \mathrm{a} \preceq \mathrm{b} \in \mathbb{K}$, where

$$\mathcal{N}(\mathrm{a}, \mathrm{b}) = \max\Big\{\wp(\mathrm{a}, \mathrm{b}), \wp(\mathrm{a}, \mathcal{S}\mathrm{a}) - \wp(\mathbb{K}, \mathbb{L}) - \wp(\mathrm{b}, \mathcal{S}\mathrm{b}) - \wp(\mathbb{K}, \mathbb{L}), \frac{\wp(\mathrm{a}, \mathcal{S}\mathrm{b}) + \wp(\mathrm{b}, \mathcal{S}\mathrm{a})}{2} - \wp(\mathbb{K}, \mathbb{L})\Big\}$$

and $\breve{\phi}$ is an ADF such that $\breve{\phi}(\mathrm{a} + \mathrm{b}) \leq \breve{\phi}(\mathrm{a}) + \breve{\phi}(\mathrm{b})$,
$\forall\ (\mathrm{a}, \mathrm{b}) \in [0, \infty)$;

(A3): $\forall\ \mathrm{a}, \mathrm{b} \in \mathbb{K}_0, \mathrm{a} \preceq \mathrm{b} \implies \mathcal{S}\mathrm{a} \preceq \mathcal{S}\mathrm{b}$;

(A4): If $\{\mathrm{a}_n\}$ is a non-decreasing sequence in $\mathbb{K}$, $\ni \mathrm{a}_n \to \mathrm{a}$, then $\mathrm{a}_n \preceq \mathrm{a} \quad \forall\ n$.
so, $\exists\ \mathrm{a} \in \mathbb{K}$ such that $\wp(\mathrm{a}, \mathcal{S}\mathrm{a}) = \wp(\mathbb{K}, \mathbb{L})$.

Proof. Follows from Theorem (3.2.2) □

If we consider $\mathbb{K} = \mathbb{L}$ in Theorem 3.2.2 and Theorem 3.3.1 following results are obtained.

Corollary 3.3.3.

Consider a POCMS $(\mathbb{J}, \preceq, \wp)$. Suppose $\mathbb{K}$ is non-empty closed subset of MS $\mathbb{J}$. Consider a multi-valued $\mathcal{F}$-contraction
$\mathcal{S} : \mathbb{K} \to \mathcal{CB}(\mathbb{K})$ along ADF $\breve{\phi}$ satisfies the following axioms:

(A1): $\exists$ two elements $\mathrm{a}_0, \mathrm{a}_1$ in $\mathbb{K}$ and $\mathrm{b}_o \in \mathcal{S}\mathrm{a}_0$ such that $\wp(\mathrm{a}_1, \mathrm{b}_0) = 0$ and $\mathrm{a}_0 \preceq \mathrm{a}_1 = \mathrm{b}_0$;

(A2): $\forall\ \mathrm{a}, \mathrm{b} \in \mathbb{K}, \mathrm{a} \preceq \mathrm{b} \implies \mathcal{S}\mathrm{a} \preceq \mathcal{S}\mathrm{b}$;

(A3): If $\{\mathrm{a}_n\}$ is a non- decreasing sequence in $\mathbb{K} \ni \mathrm{a}_n \to \mathrm{a}$, then $\mathrm{a}_n \preceq \mathrm{a}\ \forall$ n.
Then, $\exists$ a in $\mathbb{K}$ such that $\wp(\mathrm{a}, \mathcal{S}\mathrm{a}) = \mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}) = 0$. a is a fixed point in $\mathcal{S}$.

Corollary 3.3.4.

Consider a POCMS, $(\mathbb{J}, \preceq, \wp)$. Let $\mathbb{K}$ is a non-empty closed subset of a metric space $\mathbb{J}$ and $\mathcal{S} : \mathbb{K} \to \mathbb{K}$ is a self mapping and $\breve{\phi}$ is an ADF satisfying:

(A1): $\exists$ two elements $\mathrm{a}_0, \mathrm{a}_1$ in $\mathbb{K}$ and $\mathrm{b}_o \in \mathcal{S}\mathrm{a}_0$ such that $\wp(\mathrm{a}_1, \mathcal{S}\mathrm{a}_0) = 0$ and $\mathrm{a}_0 \preceq \mathrm{a}_1$;

(A2): $\mathcal{S}$ satisfies $\mathcal{F}(\breve{\phi}(\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b}))) \leq \mathcal{F}(\breve{\phi}(\mathcal{N}(\mathrm{a},\mathrm{b}))) - \check{\tau}$ for all $\mathrm{a} \preceq \mathrm{b}$ in $\mathbb{K}$, where

$$\mathcal{N}(\mathrm{a},\mathrm{b}) = \max\left\{\wp(\mathrm{a},\mathrm{b}), \wp(\mathrm{a},\mathcal{S}\mathrm{a}), \wp(\mathrm{b},\mathcal{S}\mathrm{b}), \frac{\wp(\mathrm{a},\mathcal{S}\mathrm{b}) + \wp(\mathrm{b},\mathcal{S}\mathrm{a})}{2}\right\}$$

and $\breve{\phi}$ is an `ADF` such that $\breve{\phi}(\mathrm{a}+\mathrm{b}) \leq \breve{\phi}(\mathrm{a}) + \breve{\phi}(l) \ \ \forall \ \ \mathrm{a},\mathrm{b} \in [0,\infty)$;

(A3): $\forall \ \mathrm{a},\mathrm{b} \in \mathbb{K}, \mathrm{a} \preceq \mathrm{b} \implies \mathcal{S}\mathrm{a} \preceq \mathcal{S}\mathrm{b}$;

(A4): If $\{\mathrm{a}_n\}$ is a non-decreasing sequence in $\mathbb{K}$, $\ni \mathrm{a}_n \to \mathrm{a}$, then $\mathrm{a}_n \preceq \mathrm{a} \ \forall$ n. Then, $\exists$ a in $\mathbb{K}$ such that $\wp(\mathrm{a},\mathcal{S}\mathrm{a}) = 0$ i.e. a is a `FP` of mapping $\mathcal{S}$.

Following corollaries are obtained by further choosing $\breve{\phi}$, an identity function $\breve{\phi}(\mathtt{r}) = \mathtt{r}$ for all $\mathtt{r} \in (0,\infty)$.

Corollary 3.3.5.

Consider a partially ordered metric space $(\mathbb{J},\preceq,\wp)$. Suppose $\mathbb{K}$ is non-empty closed subsets of $\mathbb{J}$, and $\mathcal{S} : \mathbb{K} \to \mathbb{L}$ be a self mapping satisfying:

(A1): $\exists$ two elements $\mathrm{a}_0, \mathrm{a}_1$ in $\mathbb{K}$ and $\mathrm{b}_0 \in \mathcal{S}\mathrm{a}_0$ such that $\wp(\mathrm{a}_1,\mathcal{S}\mathrm{a}_0) = 0$ and $\mathrm{a}_0 \preceq \mathrm{a}_1$;

(A2): $\mathcal{S}$ satisfies $\mathcal{F}(\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b}))) \leq \mathcal{F}(\mathcal{N}(\mathrm{a},\mathrm{b}))) - \check{\tau} \quad \forall \ \mathrm{a} \preceq \mathrm{b} \in \mathbb{K}$, where

$$\mathcal{N}(\mathrm{a},\mathrm{b}) = \max\left\{\wp(\mathrm{a},\mathrm{b}), \wp(\mathrm{a},\mathcal{S}\mathrm{a}), \wp(\mathrm{b},\mathcal{S}\mathrm{b}), \frac{\wp(\mathrm{a},\mathcal{S}\mathrm{b}) + \wp(\mathrm{b},\mathcal{S}\mathrm{a})}{2}\right\}$$

(A3): $\forall \ \mathrm{a},\mathrm{b} \in \mathbb{K}, \mathrm{a} \preceq \mathrm{b} \implies \mathcal{S}\mathrm{a} \preceq \mathcal{S}\mathrm{b}$;

(A4): If $\{\mathrm{a}_n\}$ is a non-decreasing sequence in $\mathbb{K}$ such that $\mathrm{a}_n \to \mathrm{a}$, then $\mathrm{a}_n \preceq \mathrm{a} \ \ \forall \ n \in \mathbb{N}$.
Then, $\exists$ an element a in $\mathbb{K}$ such that $\wp(\mathrm{a},\mathcal{S}\mathrm{a}) = 0$ i.e. a is a `FP` of the mapping $\mathcal{S}$.

Example 3.3.6.

Consider $\mathbb{J} = \mathbb{R}^2$ and the order $(\mathrm{a},\mathrm{b}) \preceq (\mathrm{m},\mathrm{q}) \iff \mathrm{a} \leq \mathrm{m}$ and $\mathrm{b} \leq \mathrm{q}$, here $\leq$ is usual order within $\mathbb{R}$. As a result, $(\mathbb{J},\preceq)$ is a partially ordered set. Furthermore,

$(\mathbb{J}, \wp)$ `MS` with the metric specified as:

$$\wp\left(\left(\mathrm{a}_1, \mathrm{b}_1\right), \left(\mathrm{a}_2, \mathrm{b}_2\right)\right) = \mid \mathrm{a}_1 - \mathrm{a}_2 \mid + \mid \mathrm{b}_1 - \mathrm{b}_2 \mid .$$

Suppose $\mathbb{K} = \{(-7,0), (0,-7), (0,5)\}$ and $\mathbb{L} = \{(-2,0), (0,-2), (0,0), (-2,2), (2,2)\}$ is a closed subset of $\mathbb{J}$.

Consider the following calculations:

$$\begin{aligned}
\wp(\mathbb{K}, \mathbb{L}) = &\inf\{\wp(\mathrm{a}, \mathrm{b}) : \mathrm{a} \in \mathbb{K} \quad \text{and} \quad \mathrm{b} \in \mathbb{L}\} \\
= &\inf\{\wp((-7,0), (-2,0)), \wp((-7,0), (0,-2)), \\
&\wp((-7,0), (0,0)), \wp((-7,0), (-2,2)), \wp((-7,0), (2,2)), \\
&\wp((0,-7), (-2,0)), \wp((0,-7), (0,-2)), \wp((0,-7), (0,0)), \\
&\wp((0,-7), (0,0)), \wp((0,-7), (-2,2)), \wp((0,-7), (2,2)) \\
&, \wp((0,5), (-2,0)), \wp((0,5), (0,-2)), \wp((0,5), (0,0)), \\
&\wp((0,5), (-2,2)), \wp((0,5), (2,2))\}
\end{aligned}$$

$$\begin{aligned}
\wp(\mathbb{K}, \mathbb{L}) = &\inf\{|5| + |0|, |7| + |2|, |7| + |0|, |5| + |2|, |9| + |2|, \\
&|2| + |7|, |0| + |5|, |0| + |7|, |2| + |9|, |2| + |9|, \\
&|2| + |5|, |0| + |7|, |0| + |5|, |2| + |3|, |2| + |3|\} \\
= &\inf\{5, 9, 7, 7, 11, 9, 5, 7, 11, 11, 7, 7, 5, 5, 5\} \\
= &5,
\end{aligned}$$

and $\mathbb{K} = \mathbb{K}_0$ and $\mathbb{L} = \mathbb{L}_0$. Let $\mathcal{S} : \mathbb{K} \to \mathcal{CB}(\mathbb{L})$ is defined as,

$$\mathcal{S}(\mathrm{a}, \mathrm{b}) = \begin{cases} \{(0,-2), (0,0)\}, & \text{if } (\mathrm{a}, \mathrm{b}) = (-7,0) \\ \{(2,2), (-2,2)\}, & \text{if } (\mathrm{a}, \mathrm{b}) = (0,-7) \\ \{(-2,2), (0,0), (0,-2), (2,2)\}, & \text{if } (\mathrm{a}, \mathrm{b}) = (0,5). \end{cases}$$

As, there are two components $(-7,0), (0,5) \in \mathbb{K}_0$ and $(0,0) \in \mathcal{S}(-7,0)$ such that,

$$\wp((0,5), (0,0)) = \wp(\mathbb{K}, \mathbb{L}) = 5 \quad \text{and} \quad (-7,0) \preceq (0,5).$$

As a result, the first condition is satisfied. Now, condition (3.2) of Theorem 3.2.2 be proven. It is straightforward to demonstrate that $\mathcal{S}a_0$ is a component of $\mathbb{L}_0\ \ \forall\ a_0 \in \mathbb{K}$.

Since " $\preceq$ " is defined in $\mathbb{K}$, there are two cases, and each of them give the following $\delta(\mathbb{K}, \mathbb{L}) = \sup\{\wp(a, b) : a \in \mathbb{K}, b \in \mathbb{L}\}$.

$\mathcal{S}a_1 = \{(0, -2), (0, 0)\}, \mathcal{S}a_2 = \{(2, 2), (-2, 2)\}, \mathcal{S}a_3 = \{(-2, 2), (0, 0), (0, -2), (2, 2)\}$.

Now, we calculate

$$\begin{aligned}
\delta(\mathcal{S}a_1, \mathcal{S}a_2) &= \sup\{\wp(a, b) : a \in \mathcal{S}a_1, b \in \mathcal{S}a_2\} \\
&= \sup\{6, 6, 4, 4\} \\
&= 6 \\
\delta(\mathcal{S}a_1, \mathcal{S}a_3) &= \sup\{\wp(a, b) : a \in \mathcal{S}a_1, b \in \mathcal{S}a_3\} \\
&= \sup\{6, 2, 0, 6, 4, 0, 2, 4\} \\
&= 6 \\
\delta(\mathcal{S}a_2, \mathcal{S}a_3) &= \sup\{\wp(a, b) : a \in \mathcal{S}a_2, b \in \mathcal{S}a_3\} \\
&= \sup\{4, 4, 6, 4, 0, 4, 6, 4\} \\
&= 6 \\
\delta(\mathcal{S}a, \mathcal{S}b) &= 6.
\end{aligned}$$

Now

$$\mathcal{N}(a, b) = \max\left\{\wp(a, b), \mathcal{D}(a, \mathcal{S}a), \mathcal{D}(b, \mathcal{S}b), \frac{\mathcal{D}(a, \mathcal{S}b) + \mathcal{D}(b, \mathcal{S}a}{2})\right\}$$

$$\begin{aligned}
\wp(a, b) &= (-7, 0), (0, 5) \\
&= |-7-0| + |0-5| \\
&= |7| + |5| \\
&= 12 \\
\mathcal{D}(a, \mathcal{S}a) &= (-7, 0), (0, 0) \\
&= |-7-0| + |0-0|
\end{aligned}$$

$$
\begin{aligned}
&= |7| + |0| \\
&= 7 \\
\mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{b}) &= (0,5),(2,2) \\
&= |0-2| + |5-2| \\
&= |2| + |3| \\
&= 5 \\
\frac{1}{2}\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{b}) + \mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a}) &= \frac{1}{2}\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{b}) + \mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a}) \\
&= \frac{1}{2}\left[(\mathcal{D}(-7,0),(0,-2)) + (\mathcal{D}(0,5),(0,-2))\right] \\
&= \frac{1}{2}\left[|7+2| + |0+7|\right] \\
&= \frac{1}{2}\left[9+2\right] \\
&= 8 \\
\mathcal{N}(\mathrm{a}, \mathrm{b}) &= \max\{12, 7, 5, 8\} \\
\mathcal{N}(\mathrm{a}, \mathrm{b}) &= 12 \\
\text{and} \quad \wp(\mathbb{K}, \mathbb{L}) &= 5.
\end{aligned}
$$

Suppose $\mathcal{F}$ is defined as $\mathcal{F}(\alpha) = \ln\alpha + \alpha$ and $\check{\tau} = 1$.
For $\breve{\phi}(\mathrm{q}) = 2\mathrm{q}$, we get $\breve{\phi}(\delta(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b})) = 2 \times 6 = 12$, also,
$\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b})) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L})) = 2(12) - 2(5) = 24 - 10 = 14$.
Thus,

$$
\begin{aligned}
&\frac{\breve{\phi}(\delta(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}))}{\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b})) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L}))} e^{\breve{\phi}(\delta(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b})) - (\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b})) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L}))}\\
&= \frac{12}{14}\left(e^{12-14}\right) = \frac{12}{14}e^{-2} < e^{-1}.
\end{aligned}
$$

So, $\mathcal{S}$ meets the requirement (3.2). One can easily verify $(Q2)$ and $(Q3)$. Hence, all the hypotheses of the Theorem (3.2.2) are fulfilled. It's also clear that $(0,5)$ is BPP of $\mathcal{S}$, i.e.

$$\mathcal{D}((0,5), \mathcal{S}(0,5)) = \wp(\mathbb{K}, \mathbb{L}) = 5$$

.

Example 3.3.7.

Suppose $\mathbb{J} = \{0, 1, 2, 3...\}$ is a partial order set having usual order $\preceq$ and suppose $\wp : \mathbb{J} \times \mathbb{J} \to \mathbb{R}$ be given as

$$\wp(\mathrm{a}, \mathrm{b}) = \begin{cases} 0; & \mathrm{a} = \mathrm{b}, \\ \mathrm{a} + \mathrm{b}; & \mathrm{a} \neq \mathrm{b}. \end{cases}$$

Then $(\mathbb{J}, \wp)$ is a complete metric space. Suppose $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ be defined as,

$$\mathcal{S}(\mathrm{a}) = \begin{cases} 0 & \text{if } \mathrm{a} = 0, \\ \mathrm{a} - 1 & \text{if } \mathrm{a} \neq 0. \end{cases}$$

So, we show how $\mathcal{S}$ is $\mathcal{F}$-contraction, with $\mathcal{F}(\alpha) = \ln \alpha + \alpha$ and $\check{\tau} = 1$.
Let us consider the following five cases:
Case 1: Assume $\mathrm{a} > \mathrm{b}$ along $\mathrm{b} \neq 0$, then

$$\begin{aligned} \wp(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) &= \wp(\mathrm{a}-1, \mathrm{b}-1) = \mathrm{a}+\mathrm{b}-2 \\ \mathcal{N}(\mathrm{a}, \mathrm{b}) &= \max\left\{\wp(\mathrm{a}, \mathrm{b}), \wp(\mathrm{a}, \mathcal{S}\mathrm{a}), \wp(\mathrm{b}, \mathcal{S}\mathrm{b}), \frac{\wp(\mathrm{a}, \mathcal{S}\mathrm{a}) + \wp(\mathrm{b}, \mathcal{S}\mathrm{b})}{2}\right\} \\ &= \max\left\{\wp(\mathrm{a}, \mathrm{b}), \wp(\mathrm{a}, \mathrm{a}-1), \wp(\mathrm{b}, \mathrm{b}-1), \frac{\wp(\mathrm{a}, \mathrm{b}-1) + \wp(\mathrm{b}, \mathrm{a}-1)}{2}\right\} \\ &= \max\{\mathrm{a}+\mathrm{b}, 2\mathrm{a}-1, 2\mathrm{b}-1, \mathrm{a}+\mathrm{b}, \mathrm{a}+\mathrm{b}-1\} \\ &= 2\mathrm{a}-1 \\ \frac{\wp(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b})}{\mathcal{N}(\mathrm{a}, \mathrm{b})} e^{\wp(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) - \mathcal{N}(\mathrm{a}, \mathrm{b})} &= \frac{\mathrm{a}+\mathrm{b}-2}{2\mathrm{a}-1} e^{\mathrm{a}+\mathrm{b}-2-(2\mathrm{a}-1)} \\ &= \frac{\mathrm{a}+\mathrm{b}-2}{2\mathrm{a}-1} e^{-\mathrm{a}+\mathrm{b}-1} < e^{-1}. \end{aligned}$$

Case 2: If $\mathrm{b} > \mathrm{a}$ and $\mathrm{a} \neq 0$, then

$$\begin{aligned} \wp(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) &= \wp(\mathrm{a}-1, \mathrm{b}-1) = \mathrm{a}+\mathrm{b}-2 \\ \mathcal{N}(\mathrm{a}, \mathrm{b}) &= \max\left\{\wp(\mathrm{a}, \mathrm{b}), \wp(\mathrm{a}, \mathcal{S}\mathrm{a}), \wp(\mathrm{b}, \mathcal{S}\mathrm{b}), \frac{\wp(\mathrm{a}, \mathcal{S}\mathrm{a}) + \wp(\mathrm{b}, \mathcal{S}\mathrm{b})}{2}\right\} \\ &= \max\left\{\wp(\mathrm{a}, \mathrm{b}), \wp(\mathrm{a}, \mathrm{a}-1), \wp(\mathrm{b}, \mathrm{b}-1), \frac{\wp(\mathrm{a}, \mathrm{b}-1) + \wp(\mathrm{b}, \mathrm{a}-1)}{2}\right\} \\ &= \max\{\mathrm{a}+\mathrm{b}, 2\mathrm{a}-1, 2\mathrm{b}-1, \mathrm{a}+\mathrm{b}, \mathrm{a}+\mathrm{b}-1\} \\ &= 2\mathrm{a}-1 \end{aligned}$$

$$\frac{\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})}{\mathcal{N}(\mathrm{a},\mathrm{b})}e^{\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})-\mathcal{N}(\mathrm{a},\mathrm{b})} = \frac{\mathrm{a}+\mathrm{b}-2}{2\mathrm{a}-1}e^{\mathrm{a}+\mathrm{b}-2-(2\mathrm{b}-1)}$$
$$= \frac{\mathrm{a}+\mathrm{b}-2}{2\mathrm{b}-1}e^{\mathrm{a}-\mathrm{b}-1} < e^{-1}.$$

Case 3: If a > b and b = 0, then

$$\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b}) = \wp(\mathrm{a}-1,0) = \mathrm{a}-1$$
$$\mathcal{N}(\mathrm{a},\mathrm{b}) = \max\left\{\wp(\mathrm{a},\mathrm{b}),\wp(\mathrm{a},\mathcal{S}\mathrm{a}),\wp(\mathrm{b},\mathcal{S}\mathrm{b}),\frac{\wp(\mathrm{a},\mathcal{S}\mathrm{a})+\wp(\mathrm{b},\mathcal{S}\mathrm{b})}{2}\right\}$$
$$= \max\left\{\wp(\mathrm{a},0),\wp(\mathrm{a},\mathrm{a}-1),\wp(0,0),\frac{\wp(\mathrm{a},0)+\wp(0,\mathrm{a}-1)}{2}\right\}$$
$$= \max\left\{\mathrm{a},2\mathrm{a}-1,0,\mathrm{a}-\frac{1}{2}\right\}$$
$$= 2\mathrm{a}-1$$
$$\frac{\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})}{\mathcal{N}(\mathrm{a},\mathrm{b})}e^{\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})-\mathcal{N}(\mathrm{a},\mathrm{b})} = \frac{\mathrm{a}-1}{2\mathrm{a}-1}e^{\mathrm{a}-1-(2\mathrm{a}-1)}$$
$$= \frac{\mathrm{a}-1}{2\mathrm{a}-1}e^{\mathrm{a}} < e^{-1}.$$

Case 4: If b > a and a = 0, then

$$\wp(0,\mathcal{S}\mathrm{b}) = \wp(0,\mathrm{b}-1) = \mathrm{b}-1$$
$$\mathcal{N}(\mathrm{a},\mathrm{b}) = \max\left\{\wp(\mathrm{a},\mathrm{b}),\wp(\mathrm{a},\mathcal{S}\mathrm{a}),\wp(\mathrm{b},\mathcal{S}\mathrm{b}),\frac{\wp(\mathrm{a},\mathcal{S}\mathrm{a})+\wp(\mathrm{b},\mathcal{S}\mathrm{b})}{2}\right\}$$
$$= \max\left\{\wp(0,\mathrm{b}),\wp(0,\mathrm{a}-0),\wp(\mathrm{b},\mathrm{b}-1),\frac{\wp(0,\mathrm{b}-1)+\wp(\mathrm{b},0)}{2}\right\}$$
$$= \max\left\{\mathrm{a},2\mathrm{a}-1,0,\mathrm{a}-\frac{1}{2}\right\}$$
$$= 2\mathrm{b}-1$$
$$\frac{\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})}{\mathcal{N}(\mathrm{a},\mathrm{b})}e^{\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})-\mathcal{N}(\mathrm{a},\mathrm{b})} = \frac{\mathrm{b}-1}{2\mathrm{b}-1}e^{\mathrm{b}-1-(2\mathrm{b}-1)}$$
$$= \frac{\mathrm{b}-1}{2\mathrm{b}-1}e^{\mathrm{b}} < e^{-1}.$$

Case 5: If b = a, then

$$\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b}) = \wp(\mathrm{a}-1,\mathrm{a}-1) = 0$$
$$\mathcal{N}(\mathrm{a},\mathrm{b}) = \max\left\{\wp(\mathrm{a},\mathrm{b}),\wp(\mathrm{a},\mathcal{S}\mathrm{a}),\wp(\mathrm{b},\mathcal{S}\mathrm{b}),\frac{\wp(\mathrm{a},\mathcal{S}\mathrm{a})+\wp(\mathrm{b},\mathcal{S}\mathrm{b})}{2}\right\}$$
$$= \max\left\{\wp(0,\wp(\mathrm{a},\mathrm{a}-1),\wp(\mathrm{a},\mathrm{a}-1),\frac{\wp(\mathrm{a},\mathrm{a}-1)+\wp(\mathrm{a},\mathrm{a}-1)}{2}\right\}$$
$$= \max\{0,2\mathrm{a}-1,2\mathrm{a}-1,2\mathrm{a}-1\}$$
$$= 2\mathrm{a}-1$$
$$\frac{\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})}{\mathcal{N}(\mathrm{a},\mathrm{b})}e^{\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})-\mathcal{N}(\mathrm{a},\mathrm{b})} = \frac{0}{2\mathrm{a}-1}e^{0-(2\mathrm{a}-1)} < e^{-1}.$$

Hence, all the conditions of Corollary (3.3.5) are met and 0 is a fixed point of $\mathcal{S}$.

3.4 Applications

In this section, two applications of the main results are provided.

3.4.1 Application Regarding Equation of Motion

Suppose $\wp : \mathcal{C}[0,1] \times \mathcal{C}[0,1] \to \mathbb{R}$ is a MS defined as

$$\wp(\mathrm{a}, \mathrm{b}) = \parallel \mathrm{a} - \mathrm{b} \parallel_\infty = \max_{\grave{t} \in [0,1]} |\mathrm{a}(\grave{t}) - \mathrm{b}(\grave{t})|.$$

and $(\mathcal{C}[0,1], \wp)$ is a CMS.

A body with mass m started its motion at time $\grave{t} = 0$ and distance x. A force f act on it in the direction x and its velocity increases from 0 to 1 instantly after $\grave{t} = 0$. The problem is to explore a function for position in terms of time $\grave{t}$.

The governing equation for this problem is

$$\mathrm{m}\frac{d^2\mathrm{x}}{d\grave{t}^2} = \mathrm{f}(\grave{t}, \mathrm{x}(\grave{t})) \quad \text{together with} \quad \mathrm{x}(0) = 0, \mathrm{x}'(1) = 0, \tag{3.13}$$

here f is a real valued function with domain in $[0,1] \times \mathbb{R}$.

Green′s function of (3.13) is ;

$$\mathcal{G}(\grave{t}, \xi) = \begin{cases} (-1+\xi)\grave{t}, & \grave{t} \leq \xi \\ -\xi(1-\grave{t}), & \grave{t} \geq \xi. \end{cases}$$

Assume $\breve{\phi} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is function along subsequent constraints:

1. $\mid \mathrm{f}(\grave{t}, q) - \mathrm{f}(\grave{t}, r) \mid \leq \max_{q,r \in \mathbb{R}} \mid q - r \mid \quad \forall\, \grave{t} \in [0,1]$ having $q, r \in \mathbb{R}$ with $\breve{\phi}(q, r) \geq 0$;

2. $\exists\ \mathrm{x}_0 \in \mathcal{C}[0,1] \ni \breve{\phi}(\mathrm{x}_0(\mathtt{t}, \mathcal{S}\mathrm{x}_0(\grave{\mathtt{t}})) \geq 0 \quad \forall\ \grave{\mathtt{t}} \in [0,1]$, where $\mathcal{S}$ is self mapping on $[0,1]$.

Theorem 3.4.1. Let $\mathbb{J} = \mathcal{C}[0,1]$. Consider a mapping $\mathcal{S} : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ defined as

$$\mathcal{S}\mathrm{x}(\grave{\mathtt{t}}) = \int_0^1 \mathcal{G}(\grave{\mathtt{t}}, \xi)\mathrm{f}(\xi, \mathrm{x}(\xi))d\xi, \quad \grave{\mathtt{t}} \in [0,1],$$

satisfying the above assumptions (1) and (2). Then the equation (3.13) has a solution.

Proof. Let $\mathrm{x} \in \mathbb{C}([0,1])$. is a solution of integral equation,

$$\mathrm{x}(\grave{\mathtt{t}}) = \int_0^1 \mathcal{G}(\grave{\mathtt{t}}, \xi)\mathrm{f}(\xi, \mathrm{x}(\xi))d\xi, \quad \grave{\mathtt{t}} \in [0,1].$$

Let $\mathrm{x}, \mathrm{y} \in \mathcal{C}[0,1]$ in order that $\breve{\phi}(\mathrm{x}(\grave{\mathtt{t}}), \mathrm{y}(\grave{\mathtt{t}})) \geq 0 \quad \forall\ \grave{\mathtt{t}} \in [0,1]$.
Suppose,

$$|\ \mathcal{S}(\mathrm{x}(\grave{\mathtt{t}})) - \mathcal{S}(\mathrm{y}(\grave{\mathtt{t}}))\ | = \left|\int_0^1 \mathcal{G}(\grave{\mathtt{t}}, \xi)\mathrm{f}(\xi, \mathrm{x}(\xi))d\xi - \int_0^1 \mathcal{G}(\grave{\mathtt{t}}, \xi)\mathrm{f}(\xi, \mathrm{y}(\xi))d\xi\right|$$

$$\begin{aligned}
|\ \mathcal{S}(\mathrm{x}(\grave{\mathtt{t}})) - \mathcal{S}(\mathrm{y}(\grave{\mathtt{t}}))\ | &\leq \int_0^1 \mathcal{G}(\grave{\mathtt{t}}, \xi)\ |\ \mathrm{f}(\xi, \mathrm{x}(\xi)) - \mathrm{f}(\xi, \mathrm{y}(\xi))\ |\ d\xi \\
&\leq \int_0^1 \mathcal{G}(\grave{\mathtt{t}}, \xi) \max |\ \mathrm{x}(\xi) - \mathrm{y}(\xi)\ |\ d\xi \\
&\leq \|\mathrm{x} - \mathrm{y}\|_\infty \sup_{\grave{\mathtt{t}} \in [0,1]} \left\{ \int_0^1 \mathcal{G}(\grave{\mathtt{t}}, \xi) d\xi \right\}.
\end{aligned}$$

As

$$\begin{aligned}
\int_0^1 \mathcal{G}(\grave{\mathtt{t}}, \xi) d\xi &= \int_0^1 (\grave{\mathtt{t}} - 1)\xi d\xi + \int_{\grave{\mathtt{t}}}^1 (\xi - 1)\grave{\mathtt{t}} d\xi \\
&= (\grave{\mathtt{t}} - 1)\xi d\xi|_0^{\grave{\mathtt{t}}} + \grave{\mathtt{t}} \left(\frac{\xi^2}{2} - \xi\right)|_{\grave{\mathtt{t}}^1} \\
&= (\grave{\mathtt{t}} - 1)\frac{\grave{\mathtt{t}}^2}{2} + \grave{\mathtt{t}} \left(\frac{-1}{2}\right) - \frac{\grave{\mathtt{t}}^3}{2} + \grave{\mathtt{t}}^2
\end{aligned}$$

$$= \frac{\mathtt{t}^2}{2} - \frac{\mathtt{t}}{2} \quad \forall \mathtt{t} \in [0,1].$$

So, $\sup_{\mathtt{t}\in[0,1]} \left\{ \int_0^1 \mathcal{G}(\mathtt{t},\xi)d\xi \right\} = \frac{1}{8}$ it follows that $\|\mathcal{S}\mathrm{x} - \mathcal{S}\mathrm{y}\|_\infty \leq \frac{1}{8}\|\mathrm{x}-\mathrm{y}\|_\infty$.

Taking natural log on both sides

$$\ln(\|\mathcal{S}\mathrm{x} - \mathcal{S}\mathrm{y}\|_\infty) \leq \ln(\|\mathrm{x}-\mathrm{y}\|_\infty) - \ln 8,$$
$$\ln 8 + \ln(\|\mathcal{S}\mathrm{x} - \mathcal{S}\mathrm{y}\|_\infty) \leq \ln(\|\mathrm{x}-\mathrm{y}\|_\infty).$$

Suppose that the function $\mathcal{F} : (0,\infty) \to \mathbb{R}$ described as $\mathcal{F}(\mathrm{x}) = \ln \mathrm{x}$.

Since $\ln 8 + \mathcal{F}(\wp(\mathcal{S}\mathrm{x}, \mathcal{S}\mathrm{y})) \leq \mathcal{F}(\wp(\mathrm{x},\mathrm{y}))$.

And $\mathcal{F}(\wp(\mathrm{x},\mathrm{y})) \leq \mathcal{F}(\mathcal{N}(\mathrm{x},\mathrm{y})) \implies \ln 8 + \mathcal{F}(\wp(\mathcal{S}\mathrm{x}, \mathcal{S}\mathrm{y})) \leq \mathcal{F}(\mathcal{N}(\mathrm{x},\mathrm{y}))$.

Mapping $\mathcal{S}$ is $\mathcal{F}$-contraction. According to Corollary (3.3.5) $\mathcal{S}$ possess fixed point x in $\mathcal{C}^2([0,1])$, such that

$$\begin{aligned} \mathcal{S}(\mathrm{x}(\mathtt{t}) &= \mathrm{x}(\mathtt{t}) \\ &= \int_0^1 \mathcal{G}(\mathtt{t},\xi)\mathrm{f}(\xi,\mathrm{x}(\xi))d\xi, , \quad \mathtt{t} \in [0,1]. \end{aligned}$$

which is the solution to (3.14). □

3.4.2 Application Regarding Fractional Calculus

$\wp : \mathcal{C}([0,1]) \times \mathcal{C}([0,1]) \to \mathbb{R}$ is a `MS` define as

$$\wp(\mathrm{a},\mathrm{b}) = \| \mathrm{a} - \mathrm{b} \|_\infty = \max_{\mathtt{t}\in[0,1]} |\mathrm{a}(\mathtt{t}) - \mathrm{b}(\mathtt{t})|.$$

The Caputo fractional derivative of α order of continuous function $\mathrm{q} : [0,+\infty) \to \mathbb{R}$ described as:

$${}^{\nu}\mathcal{D}^{\alpha}(\mathrm{q}(\mathtt{t})) = \frac{1}{\Gamma(\mathrm{b}-\alpha)} \int_0^{\mathtt{t}} (\mathtt{t}-\xi)^{\mathrm{b}-\alpha-1} \mathrm{j}^{(\mathrm{b})}(\xi)d\xi \quad (\mathrm{b}-1 < \alpha < n, \mathrm{b} = [\alpha]+1).$$

Here Γ represents gamma function and α represents the integral component of real integer. We illustrate the existence of a solution to a non-linear fractional differential equation in this section.

$$^{\nu}\mathcal{D}^{\alpha}(\mathrm{a}(\mathtt{\grave{t}}) + \mathrm{f}(\mathtt{\grave{t}}, \mathrm{a}(\mathtt{\grave{t}})) = 0 \quad (0 \leq \mathtt{\grave{t}} \leq 1, 1 < \alpha \leq 2), \tag{3.14}$$

using $\mathrm{a}(0) = \mathrm{a}(1) = 0$ where $\mathtt{f}$ is a real valued function with domain in $[0,1] \times \mathbb{R}$. Green's function [36] of Equation (3.14), is defined as

$$\mathcal{G}(\mathtt{\grave{t}}, \xi) = \begin{cases} \frac{[\mathtt{\grave{t}}(1-\xi)^{\alpha-1} - (\mathtt{\grave{t}}-\xi)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq \xi \leq \mathtt{\grave{t}} \leq 1, \\ \frac{[\mathtt{\grave{t}}(1-\xi)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq \mathrm{t} \leq \xi \leq 1. \end{cases}$$

Assume the following conditions are met:

1. $\mid \mathrm{f}(\mathtt{\grave{t}}, \mathrm{a}) - \mathrm{f}(\mathtt{\grave{t}}, \mathrm{b}) \mid \leq e^{-\check{\tau}} \mathbb{J}(\mathrm{a}, \mathrm{b}) \quad \forall \quad \mathtt{\grave{t}} \in [0,1]$ also $\mathrm{a}, \mathrm{b} \in \mathbb{R}$, such that
$$\mathbb{J}(\mathrm{a}, \mathrm{b}) = \max\left\{\mid \mathrm{a} - \mathrm{b} \mid, \mid \mathrm{a} - \mathcal{S}\mathrm{a} \mid, \mid \mathrm{b} - \mathcal{S}\mathrm{b} \mid, \frac{\mid \mathrm{a} - \mathcal{S}\mathrm{b} \mid + \mid \mathrm{b} - \mathcal{S}\mathrm{a} \mid}{2}\right\};$$
2. $\exists\ \mathrm{a}_0 \in \mathcal{C}[0,1]$ such that $\breve{\phi}(\mathrm{a}_0(\mathtt{\grave{t}}), \mathcal{S}\mathrm{a}_0(\mathtt{\grave{t}}) \geq 0 \quad \forall \ \mathtt{\grave{t}} \in [0,1]$.

Theorem 3.4.2.

Consider a mapping $\mathcal{S} : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ defined as

$$\mathcal{S}(\mathrm{a}(\mathtt{\grave{t}})) = \int_0^1 \mathcal{G}(\mathtt{\grave{t}}, \xi)\mathrm{f}(\xi, \mathrm{a}(\xi))d\xi,$$

satisfying the above assumptions (1) and (2). Then the fractional differential equation (3.14) has a solution.

Proof. It is obvious that the solution of (3.14) is equivalent to,

$$\mathrm{a}(\mathrm{t}) = \int_0^1 \mathcal{G}(\mathtt{\grave{t}}, \xi)\mathrm{f}(\xi, \mathrm{a}(\xi))d\xi \quad \text{for all } \mathtt{\grave{t}} \in [0,1].$$

Consider

$$\mid \mathcal{S}\mathrm{a}(y) - \mathcal{S}\mathrm{b}(y) \mid = \left| \int_0^1 \mathcal{G}(y, \xi)\mathrm{f}\,(\xi, \mathrm{a}(\xi))\, d\xi - \int_0^1 \mathcal{G}(y, \xi)\mathrm{f}\,(\xi, \mathrm{b}(\xi))\, d\xi \right|$$

$$\leq \int_0^1 |\mathcal{G}(y,\xi)\left(\mathrm{f}\left(\xi,\mathrm{a}(\xi)\right)-\mathrm{f}\left(\xi,\mathrm{b}(\xi)\right)\right)d\xi|$$
$$\leq \int_0^1 ||\mathcal{G}(y,\xi)||\left(\mathrm{f}\left(\xi,\mathrm{a}(\xi)\right)-\mathrm{f}\left(\xi,\mathrm{b}(\xi)\right)\right)|d\xi$$
$$\leq \int_0^1 |\mathcal{G}(y,\xi)|e^{-\check{\tau}}\mathbb{J}(\mathrm{a},\mathrm{b})d\xi$$
$$\leq \int_0^1 |\mathcal{G}(y,\xi)|e^{-\check{\tau}}\max\left\{|\mathrm{a}-\mathrm{b}|,|\mathrm{a}-\mathcal{S}\mathrm{a}|,|\mathrm{b}-\mathcal{S}\mathrm{b}|,\frac{|\mathrm{a}-\mathcal{S}\mathrm{b}|+|\mathrm{b}-\mathcal{S}\mathrm{a}|}{2}\right\}d\xi$$

$$\leq e^{-\check{\tau}}\max\left\{\wp(\mathrm{a},\mathrm{b}),\wp(\mathrm{a},\mathcal{S}\mathrm{a}),\wp(\mathrm{b},\mathcal{S}\mathrm{b}),\frac{\wp(\mathrm{a},\mathcal{S}\mathrm{b})+\wp(\mathrm{b},\mathcal{S}\mathrm{a})}{2}\right\}\int_0^1\left(\mathcal{G}(y,\xi)\right)d\xi$$
$$\leq e^{-\check{\tau}}\max\left\{\wp(\mathrm{a},\mathrm{b}),\wp(\mathrm{a},\mathcal{S}\mathrm{a}),\wp(\mathrm{b},\mathcal{S}\mathrm{b}),\frac{\wp(\mathrm{a},\mathcal{S}\mathrm{b})+\wp(\mathrm{b},\mathcal{S}\mathrm{a})}{2}\right\}\times\sup_{y\in[0,1]}\left(\int_0^1\left(\mathcal{G}(y,\xi)\right)d\xi\right).$$

Since

$$\int_0^1 \mathcal{G}(y,\xi)d\xi = \frac{1}{\gamma\Gamma(\gamma)}\left[\mathrm{t}^{\gamma-1}-\mathrm{t}^{\gamma}\right] \quad y\in[0,1].$$

Then

$$\sup_{y\in[0,1]}\left(\int_0^1\left(\mathcal{G}(y,\xi)\right)d\xi\right)\leq 1.$$

As a result of this

$$|\ \mathcal{S}\mathrm{a}(y)-\mathcal{S}\mathrm{b}(y)\ |\leq e^{-\check{\tau}}\mathcal{N}(\mathrm{a},\mathrm{b})$$

where

$$\mathcal{N}(\mathrm{a},\mathrm{b})=\max\left\{\wp(\mathrm{a},\mathrm{b}),\wp(\mathrm{a},\mathcal{S}\mathrm{a}),\wp(\mathrm{b},\mathcal{S}\mathrm{b}),\frac{\wp(\mathrm{a},\mathcal{S}\mathrm{a})+\wp(\mathrm{b},\mathcal{S}\mathrm{a})}{2}\right\}.$$

Hence , $\forall\ \mathrm{a},\mathrm{b}\in\mathbb{J}$ and $\forall\ y\in[0,1]$,

we have

$$\wp(\mathcal{S}\mathrm{a}-\mathcal{S}\mathrm{b})\leq\left(e^{-\check{\tau}}\mathcal{N}(\mathrm{a},\mathrm{b})\right).$$

Using logarithm on both sides, we have

$$\ln\left(\wp(\mathcal{S}\mathrm{a}-\mathcal{S}\mathrm{b})\leq\ln\left(e^{-\check{\tau}}\mathcal{N}(\mathrm{a},\mathrm{b})\right)\right)-\check{\tau}.$$

Let's now suppose that the function $\mathcal{F}:(0,+\infty)\to(0,+\infty)$ identified as $\mathcal{F}(y)=\ln y$.

$$\check{\tau}+\mathcal{F}(\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b}))\leq\mathcal{F}(\wp(\mathcal{N}(\mathrm{a},\mathrm{b}))).$$

Hence, mapping $\mathcal{S}$ known as $\mathcal{F}$-contraction. According to the Corollary (3.3.5) $\mathcal{S}$ having fixed point a in $\mathcal{C}([0,1])$, such that

$$\begin{aligned}\mathcal{S}(\mathrm{a}(\check{\mathtt{t}}) &= \mathrm{a}(\check{\mathtt{t}}) \\ &= \int_0^1 \mathcal{G}(\check{\mathtt{t}}, \xi)\mathrm{f}(\xi, \mathrm{a}(\xi))d\xi. \quad \forall\, \grave{t} \in [0,1]\end{aligned}$$

which is the solution to (3.14) □

Chapter 4

Best Proximity Point for Multi-valued $(\alpha\mathcal{F}, \mathtt{b}, \breve{\phi})$-Contractions on Partially Ordered $\mathtt{b}$-Metric Spaces

This chapter provides certain fixed point results generalizing the result of Jain et al. [23]. This task is achieved by using $\mathtt{b}$-metric space as the base space and incorporating α function in the contraction. Two examples are provided to justify the required axioms of the theorems. For application purpose existence of the solution to a fractional differential equation is established using the proven result.

4.1 Preliminaries

Following notations and assumptions are used throughout this chapter.
Suppose $\mathbb{J}$ is a non-empty set and $(\mathbb{J}, \wp_{\mathtt{b}}, \preceq)$ is a partially ordered $\mathtt{bMS}$. Let $\mathbb{K}$ and $\mathbb{L}$ be non-empty subsets of the $\mathtt{bMS}$ and $\mathcal{CB}(\mathbb{J})$ represents the family of closed and bounded non-empty subsets of $\mathbb{J}$. Then;

$$\begin{aligned}
\mathcal{D}(\mathrm{a}, \mathbb{L}) &= \inf\{\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}) : \mathrm{b} \in \mathbb{L} \quad \forall \quad \mathrm{a} \in \mathbb{J}\} \\
\delta(\mathbb{K}, \mathbb{L}) &= \sup\{\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}) : \mathrm{a} \in \mathbb{K} \quad \text{and} \quad \mathrm{b} \in \mathbb{L}\} \\
\wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}) &= \inf\{\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}) : \mathrm{a} \in \mathbb{K} \quad \text{and} \quad \mathrm{b} \in \mathbb{L}\} \\
\mathbb{K}_0 &= \{\mathrm{a} \in \mathbb{K} : \wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}) \quad \text{for some} \quad \mathrm{b} \in \mathbb{L}\} \\
\mathbb{L}_0 &= \{\mathrm{b} \in \mathbb{L} : \wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}) \quad \text{for some} \quad \mathrm{a} \in \mathbb{K}\}.
\end{aligned}$$

Definition 4.1.1. **α-Admissible Mapping**

A mapping $\mathcal{S} : \mathbb{K} \to \mathcal{CB}(\mathbb{K})$ is said to be α-admissible mapping for $\alpha : \mathbb{K} \times \mathbb{K} \to [0, \infty)$, such that

$$\alpha(\mathrm{a}_0, \mathrm{a}_1) \geq 1 \implies \alpha(\mathrm{v}, \mathrm{w}) \geq 1,$$

for $\mathrm{v} \in \mathcal{S}\mathrm{a}_0$ and $\mathrm{w} \in \mathcal{S}\mathrm{a}_1$. [37]

Definition 4.1.2. **Multivalued $(\alpha\mathcal{F}, \mathtt{b}, \breve{\phi})$-Contraction**

Let $(\mathbb{J}, \wp_{\mathtt{b}})$ be a partially ordered `bMS` and $\mathbb{K}$ and $\mathbb{L}$ are two non-empty closed subsets of $\mathbb{J}$ such that $\mathcal{S}\mathrm{a}_0 \subset \mathbb{L}_0$, $\mathrm{a}_0 \in \mathbb{L}_0$. A mapping $\mathcal{S} : \mathbb{K} \to \mathcal{CB}(\mathbb{L})$ is called $(\alpha\mathcal{F}, \mathtt{b}, \breve{\phi})$-contraction with $\breve{\phi}$ an `ADF` if it satisfies

$$\check{\tau} + \mathcal{F}\left(\alpha(\mathrm{a}_0, \mathrm{a}_1)(\mathtt{b}^2(\breve{\phi}((\delta(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}))\right) \leq \mathcal{F}\left(\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b})) - \breve{\phi}(\wp_{\mathtt{b}}(\mathbb{L}, \mathbb{M}))\right) \quad \forall\ \mathrm{a} \leq \mathrm{b} \quad \in \mathbb{L}, \quad (4.1)$$

where $\alpha : \mathbb{J} \times \mathbb{J} \to [0, \infty)$ and

$$\mathcal{N}(\mathrm{a}, \mathrm{b}) = \max\left\{\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}), \mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}), \mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{b}), \frac{\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{b}) + \mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a})}{2\mathtt{b}}\right\}.$$

Also $\breve{\phi}(\mathrm{a}, \mathrm{b}) \leq \breve{\phi}(\mathrm{a}) + \breve{\phi}(\mathrm{b}) \quad \forall\ \mathrm{a}, \mathrm{b} \in [0, +\infty)$.

Now, by choosing $\mathcal{F}(\mathrm{a}) = \ln \mathrm{a}$, in (4.1), the contraction condition takes the following form

$$\begin{aligned}
&\check{\tau} + \ln\left(\alpha(\mathrm{a}_0, \mathrm{a}_1)(\mathtt{b}^2(\breve{\phi}(\delta(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}))\right) \leq \ln\left(\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b})) - \breve{\phi}(\wp_{\mathtt{b}}((\mathbb{L}, \mathbb{M}))\right) \\
&\Leftrightarrow \ln e^{\check{\tau}} + \ln\left((\alpha(\mathrm{a}_0, \mathrm{a}_1)(\mathtt{b}^2(\breve{\phi}(\delta(\mathcal{S}\mathrm{a}.\mathcal{S}\mathrm{b}))\right) \leq \ln\left(\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b})) - \breve{\phi}(\wp_{\mathtt{b}}((\mathbb{L}, \mathbb{M}))\right)
\end{aligned}$$

$$\Leftrightarrow \ln(\alpha(\mathrm{a}_0, \mathrm{a}_1)(\mathtt{b}^2(\breve{\phi}(\delta(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b})) \leq \ln\left\{\frac{\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b})) - \breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}))}{e^{\check{\tau}}}\right\}$$

$$\Leftrightarrow (\alpha(\mathrm{a}_0, \mathrm{a}_1)(\mathtt{b}^2(\breve{\phi}(\delta(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b})) \leq \frac{1}{e^{\check{\tau}}}\left(\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b}))\right) - \breve{\phi}(\wp_{\mathtt{b}}(\mathbb{L}, \mathbb{M}))) \quad \because \frac{1}{e^{\check{\tau}}} = \nu$$

$$\Leftrightarrow (\alpha(\mathrm{a}_0, \mathrm{a}_1)(\mathtt{b}^2(\breve{\phi}(\delta(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b})) \leq \nu\left(\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b}))\right) - \breve{\phi}(\wp_{\mathtt{b}}(\mathbb{L}, \mathbb{M}))).$$

4.2 Main Theorem

Theorem 4.2.1.

Consider a partially ordered complete metric space $(\mathbb{J}, \preceq, \wp_{\mathtt{b}})$. Suppose $\mathbb{K}$ and $\mathbb{L}$ are non-empty closed subset of the MS $(\mathbb{J}, \wp_{\mathtt{b}})$ in such a way that $\mathbb{K}_0$ is non-empty and $(\mathbb{K}, \mathbb{L})$ possesses $\mathcal{P}$-property. Let $\mathcal{S} : \mathbb{K} \to \mathcal{CB}(\mathbb{L})$ be an α-admissible multivalued $(\alpha\mathcal{F}, \mathtt{b}, \breve{\phi})$-contraction such that the conditions given below are satisfied:

(Q1) . $\exists\ \mathrm{a}_0, \mathrm{a}_1 \in \mathbb{K}$ such that $\alpha(\mathrm{a}_0, \mathrm{a}_1) \geq 1 \implies \alpha(\mathrm{v}, \mathrm{w}) \geq 1$ for $\mathrm{v} \in \mathcal{S}\mathrm{a}_0$, $\mathrm{w} \in \mathcal{S}\mathrm{a}_1$

(Q2) . $\exists$ two elements $\mathrm{a}_0, \mathrm{a}_1 \in \mathbb{K}_0$ and $\mathrm{b}_0 \in \mathcal{S}\mathrm{a}_0$ such that $\wp_{\mathtt{b}}(\mathrm{a}_1, \mathrm{b}_0) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$ with $\mathrm{a}_0 \preceq \mathrm{a}_1$

(Q3) . $\forall\ \mathrm{a}, \mathrm{b} \in \mathbb{K}_0$ with $\mathrm{a} \preceq \mathrm{b} \implies \mathcal{S}\mathrm{a} \subset \mathcal{S}\mathrm{b}$;

(Q4) . If $\{\mathrm{a}_n\}$ is a non decreasing sequence in $\mathbb{K}$ such that $\mathrm{a}_n \to \mathrm{a}$, then $\mathrm{a}_n \preceq \mathrm{a}\ \ \forall\ n$.

Then, $\exists\ \mathrm{a} \in \mathbb{K}$ such that $\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$.

Proof.

By using condition (Q1) and α-admissibility of $\mathcal{S}$, we have

$$\alpha(\mathrm{a}_n, \mathrm{a}_{n+1}) \geq 1 \quad \forall\ n = 0, 1, 2...$$

According to (Q2), $\exists\ \mathrm{a}_0, \mathrm{a}_1$ in $\mathbb{K}_0$ and $\mathrm{b}_0 \in \mathcal{S}\mathrm{a}_0$ such that $\wp_{\mathtt{b}}(\mathrm{a}_1, \mathrm{b}_0) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$ and $\mathrm{a}_0 \leq \mathrm{a}_1$

(Q3) $\implies$ $\mathcal{S}\mathrm{a}_0 \subset \mathcal{S}\mathrm{a}_1$, so $\exists\ \mathrm{b}_1 \in \mathcal{S}\mathrm{a}_1$ with $\wp_{\mathtt{b}}(\mathrm{a}_2, \mathrm{b}_1) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$ such that

$a_1 \preceq a_2$. Generally, for each $n \in \mathbb{N}$, $\exists\ a_{n+1} \in \mathbb{K}_0$ and $b_n \in \mathcal{S}a_n$ such that $\wp_{\mathtt{b}}(a_{n+1}, b_n) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$. Thus,

$$\wp_{\mathtt{b}}(a_{n+1}, b_n) = \mathcal{D}(a_{n+1}, \mathcal{S}a_n) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}), \quad \forall\ n \in \mathbb{N} \tag{4.2}$$

where $a_0 \preceq a_1 \preceq a_3 \preceq ... \preceq a_n \preceq a_{n+1}...$

If there exist n_0 such that $a_{n_0} = a_{n_0+1}$ then $\wp_{\mathtt{b}}(a_{n_0+1}, b_{n_0}) = \mathcal{D}(a_{n_0}, \mathcal{S}a_{n_0}) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$, then a_{n_0} is the best proximity point of $\mathcal{S}$ and hence we are done with proof. Assume that $a_n \neq a_{n+1}\ \forall\, n$. Since $\wp_{\mathtt{b}}(a_{n+1}, b_n) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$ and $\wp_{\mathtt{b}}(a_n, b_{n-1}) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$ and $(\mathbb{K}, \mathbb{L})$ has the $\mathcal{P}$-property

$$\wp_{\mathtt{b}}(a_{n+1}, a_n) = \wp_{\mathtt{b}}(b_n, b_{n-1}). \quad \forall\ n \in \mathbb{N}. \tag{4.3}$$

Given $a_{n-1} \prec a_n$, so

$$\begin{aligned}\mathcal{F}(\mathtt{b}^2(\breve{\phi}(\wp_{\mathtt{b}}(a_n, a_{n+1}))) &\leq \check{\tau} + \mathcal{F}(\alpha(a_n, a_{n+1})(\mathtt{b}^2(\breve{\phi}(\wp_b(a_n, a_{n+1}))) \\ &= \check{\tau} + \mathcal{F}(\alpha(a_n,\ a_{n+1})(\mathtt{b}^2(\breve{\phi}(\wp_{\mathtt{b}}(b_{n-1}, b_n))) \\ &\leq \mathcal{F}(\alpha(a_n,\ a_{n+1})(\mathtt{b}^2(\breve{\phi}(\delta(\mathcal{S}a_{n-1}, \mathcal{S}a_n))) \\ &\leq \mathcal{F}(\breve{\phi}(\mathcal{N}(a_{n-1}, a_n)) - \breve{\phi}(\wp_b(\mathbb{K}, \mathbb{L}))) - \check{\tau}.\end{aligned} \tag{4.4}$$

Now

$$\begin{aligned}\mathcal{N}(a_{n-1}, a_n) &= \max\left\{\wp_b(a_{n-1}, a_n), \mathcal{D}(a_{n-1}, \mathcal{S}a_{n-1}), \mathcal{D}(a_n, \mathcal{S}a_n), \frac{\mathcal{D}(a_{n-1}, \mathcal{S}a_n) + \mathcal{D}(a_n, \mathcal{S}a_{n-1})}{2\mathtt{b}}\right\} \\ &\leq \max\left\{\wp_{\mathtt{b}}(a_{n-1}, a_n), \wp(a_{n-1}, b_{n-1}), \wp(a_n, b_n), \frac{\wp(a_{n-1}, b_n) + \wp(a_n, b_{n-1})}{2\mathtt{b}}\right\} \\ &\leq \max\Big\{\wp_{\mathtt{b}}(a_{n-1}, a_n), (\mathtt{b}\wp_{\mathtt{b}}(a_{n-1}, b_{n-2}) + \wp_{\mathtt{b}}(b_{n-2}, b_{n-1})), (\mathtt{b}\wp_{\mathtt{b}}(a_n, b_{n-1}) + \wp_{\mathtt{b}}(b_{n-1}, b_n)), \\ &\quad \frac{\mathtt{b}\wp_{\mathtt{b}}(a_{n-1}, b_{n-2}) + \mathtt{b}^2\wp_{\mathtt{b}}(b_{n-2}, b_{n-1}) + \mathtt{b}^2\wp_{\mathtt{b}}(b_{n-1}, b_n) + \wp_{\mathtt{b}}(a_n, b_{n-1})}{2\mathtt{b}}\Big\} \\ &\leq \max\Big\{\wp_{\mathtt{b}}(a_{n-1}, a_n), (\mathtt{b}\wp_{\mathtt{b}}(a_{n-1}, b_{n-2}) + \wp_{\mathtt{b}}(b_{n-2}, b_{n-1})), (\mathtt{b}\wp_{\mathtt{b}}(a_n, b_{n-1}) + \wp_{\mathtt{b}}(b_{n-1}, b_n)), \\ &\quad \frac{\mathtt{b}\wp_{\mathtt{b}}(a_{n-1}, b_{n-2}) + \mathtt{b}^2\wp_{\mathtt{b}}(b_{n-2}, b_{n-1}) + \mathtt{b}^2\wp_{\mathtt{b}}(b_{n-1}, b_n) + \mathtt{b}\wp_{\mathtt{b}}(a_n, b_{n-1})}{2\mathtt{b}}\Big\} \\ &\leq \max\Big\{\wp_{\mathtt{b}}(a_{n-1}, a_n), (\mathtt{b}\wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}) + \mathtt{b}\wp_{\mathtt{b}}(a_{n-1}, a_n)\,, (\mathtt{b}\wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}) + \mathtt{b}\wp_{\mathtt{b}}(a_n, a_{n+1})\,, \\ &\quad \frac{\wp_{\mathtt{b}}(a_n, a_{n+1}) + \mathtt{b}\wp_{\mathtt{b}}(b_{n-2}, b_{n-1}) + \mathtt{b}\wp_{\mathtt{b}}(b_{n-1}, b_n) + \wp_{\mathtt{b}}(a_n, b_{n-1})}{2}\Big\} \\ &\leq \max\left\{\wp_{\mathtt{b}}(a_{n-1}, a_n), \mathtt{b}(\wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}) + \wp_{\mathtt{b}}(a_{n-1}, a_n)), \mathtt{b}(\wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}) + \wp_{\mathtt{b}}(a_n, a_{n+1})),\right.\end{aligned}$$

$$\left.\frac{\mathtt{b}(\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L})+\wp_{\mathtt{b}}(\mathrm{a}_{n-1},\mathrm{a}_n))+\mathtt{b}(\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1})+\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L}))}{2}\right\}$$
$$\leq \max\Big\{\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L})+\wp_{\mathtt{b}}(\mathrm{a}_{n-1},\mathrm{a}_n), \wp_{\mathtt{b}}(\mathbb{K},\mathbb{L})+\mathtt{b}(\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1}))\Big\}.$$

Using Equation (4.4) $\implies$

$$\mathcal{F}(\mathtt{b}^2(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1}) \leq \mathcal{F}(\breve{\phi}\max\Big\{\mathtt{b}(\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L})+\wp_{\mathtt{b}}(\mathrm{a}_{n-1},\mathrm{a}_n)), \mathtt{b}(\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L})+\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1}))\Big\}$$
$$-\breve{\phi}(\mathtt{b}(\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L}))) - \check{\tau}. \tag{4.5}$$

If $\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1}) > \wp_{\mathtt{b}}(\mathrm{a}_{n-1},\mathrm{a}_n)$ from (4.5) we have

$$\mathcal{F}(\mathtt{b}^2(\breve{\phi}\,(\wp(\mathrm{a}_n,\mathrm{a}_{n+1}))) \leq \mathcal{F}(\mathtt{b}(\breve{\phi}(\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L}))+(\mathtt{b}\,(\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1}))) - (\mathtt{b}(\breve{\phi}(\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L}))) - \check{\tau}$$
$$\mathcal{F}(\mathtt{b}^2(\breve{\phi}(\wp(\mathrm{a}_n,\mathrm{a}_{n+1}))) \leq \mathcal{F}(\mathtt{b}(\breve{\phi}(\wp(\mathrm{a}_n,\mathrm{a}_{n+1}))) - \check{\tau},$$

which leads to contradiction. So,

$$\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1}) \leq \wp_{\mathtt{b}}(\mathrm{a}_{n-1},\mathrm{a}_n), \tag{4.6}$$

since, the sequence $\{\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1})\}$ is monotonically, non increasing and bounded below, so, $\exists\ \ \mathrm{s} \geq 0$,

$$\lim_{n\to\infty} \wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1}) = \mathrm{s} \geq 0. \tag{4.7}$$

Let $\lim\limits_{n\to\infty} \wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1}) = \mathrm{s} \geq 0$ using Equation (4.6), Equation (4.5) becomes

$$(\mathtt{b}^2(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1})))) \leq (\mathtt{b}(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_{n-1},\mathrm{a}_n)))) - \check{\tau}$$
$$\mathcal{F}(\mathtt{b}(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1})))) \leq \mathcal{F}(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_{n-1},\mathrm{a}_n)))) - \check{\tau}.$$

Take $(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1})) = \psi_n$ and subsituting in above equation.

$$\mathcal{F}(\mathtt{b}(\psi_n) \leq \mathcal{F}(\psi_{n-1})) - \check{\tau}.$$

Iteratively,

$$\Rightarrow \mathcal{F}(\mathtt{b}^n(\psi_n) \leq \mathcal{F}(\mathtt{b}^{n-1}(\psi_{n-1}) - \check{\tau} \leq \mathcal{F}(\mathtt{b}^{n-2}(\psi_{n-2}) - 2\check{\tau}.... \leq \mathcal{F}(\psi_0) - n\check{\tau}. \quad (4.8)$$

$$\implies \lim_{n\to\infty} \mathcal{F}(\mathtt{b}^n(\psi_n)) = -\infty$$

$$\implies \lim_{n\to\infty} \mathtt{b}^n(\psi_n)) = 0. \quad \text{by}(F2) \quad (4.9)$$

Using $(F3)$, $\exists\ \gamma \in (0, 1)$ such that,

$$\implies \lim_{n\to 0}(\mathtt{b}^n\psi_n)^\gamma \mathcal{F}(\mathtt{b}^n\psi_n) = 0 \quad \forall\ n \in \mathbb{N}.$$

$$(4.8) \implies \lim_{n\to\infty}(\mathtt{b}^n\psi_n)^\gamma(\mathcal{F}(\mathtt{b}^n\psi_n) - \mathcal{F}(\psi_0)) \leq \lim_{n\to\infty}(\mathtt{b}^n\psi_n)^\gamma n\check{\tau} \leq 0. \quad (4.10)$$

$$0 \leq \lim_{n\to\infty}(\mathtt{b}\psi_n)^\gamma n\check{\tau} \leq 0.$$

Now, as $\check{\tau} > 0$, we have

$$\lim_{n\to\infty}(\mathtt{b}\psi_n)^\gamma n = 0.$$

So, $\exists\ n_1 \in \mathcal{N}$ such that

$$(\mathtt{b}^n\psi_n)^\gamma\, n \leq 1 \quad \forall\ n \geq n_1$$

$$\implies \mathtt{b}^n\psi_n \leq \frac{1}{n^{\frac{1}{\gamma}}}. \quad (4.11)$$

We have to show that $\{\mathrm{a}_n\}$ is Cauchy.

So, assume $\mathrm{n}, \mathrm{m} \in \mathbb{N}$ such that $\mathrm{n} > \mathrm{m} \geq \mathrm{n}_1$ and $\mathtt{b} \geq 1$. Hence, by triangular inequality,

$$\begin{aligned}\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_n, \mathrm{a}_m)) &\leq \mathtt{b}(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_n, \mathrm{a}_{n+1}))) + \mathtt{b}(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_{n+1}, \mathrm{a}_m))) \\ &\leq \mathtt{b}(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_n, \mathrm{a}_{n+1})) + \mathtt{b}^2(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_{n+1}, \mathrm{a}_{n+2})) + \mathtt{b}^2(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_{n+2}, \mathrm{a}_m)) \\ &\leq \mathtt{b}(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_n, \mathrm{a}_{n+1})) + \mathtt{b}^2(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_{n+1}, \mathrm{a}_{n+2})) + ... + \mathtt{b}^{m-n}(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_{m-1}, \mathrm{a}_m)).\end{aligned} \quad (4.12)$$

Equation (4.12)

$$\begin{aligned}
\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_m)) &\leq \mathtt{b}\psi_n+\mathtt{b}^2\psi_{n+1}+\mathtt{b}^3\psi_{n+2}...+\mathtt{b}^{m-n}\ \psi_m\\
&=\sum_{i=n}^{m-1}\mathtt{b}^{i-n+1}(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_i,\mathrm{a}_{i+1})))\\
&\leq\sum_{i=n}^{\infty}\mathtt{b}^{i}(\breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_i,\mathrm{a}_{i+1})))\\
&\leq\sum_{i=n}^{\infty}\mathtt{b}^{i}(\psi_i)\\
\implies \qquad \breve{\phi}(\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_m)) &\leq\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{\gamma}}}.
\end{aligned}$$

Given $\gamma\in(0,1)$ so $\frac{1}{\gamma}>1$. Consequently, by using the $\mathtt{P}$-series test $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{\gamma}}}$ is convergent for $\frac{1}{\gamma}>1$. Therefore, $\{\mathrm{a}_n\}$ is a Cauchy sequence in $\mathbb{K}$. Given that, $\mathbb{K}$ is complete so $\exists$ $\mathrm{a}\in\mathbb{K}$ such that

$$\lim_{n\to\infty}\mathrm{a}_n=\mathrm{a}\quad\text{or}\quad\mathrm{a}_n\to\mathrm{a}.$$

Since $\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}_{n+1})=\wp_{\mathtt{b}}(\mathrm{b}_{n-1},\mathrm{b}_n)$. The sequence $\{\mathrm{b}_n\}$ in $\mathbb{K}$ is Cauchy and then convergent.
Assume that $\mathrm{b}_n\to\mathrm{b}$. By the relation $\wp_{\mathtt{b}}(\mathrm{a}_{n+1},\mathrm{b}_n)=\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L})\ \ \forall\ n$.
We conclude that $\wp_{\mathtt{b}}(\mathrm{a},\mathrm{b})=\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L})$. Now, suppose that $\mathrm{b}\in\mathcal{S}\mathrm{a}$. Given $\{\mathrm{a}_n\}$ is an increasing sequence in $\mathbb{K}$ and $\mathrm{a}_n\to\mathrm{a}$ according to the axiom $(Q3)$, $\mathrm{a}_n\preceq\mathrm{a}$ for all $n\in\mathbb{N}$.
Suppose $\mathrm{b}\notin\mathcal{S}\mathrm{a}$. Consider the contraction condition (4.1)

$$\begin{aligned}
\mathcal{F}(\mathtt{b}^2(\breve{\phi}(\mathcal{D}(\mathrm{b}_n,\mathcal{S}\mathrm{a}))) &\leq \mathcal{F}(\mathtt{b}^2(\breve{\phi}(\delta(\mathcal{S}\mathrm{a}_n,\mathcal{S}\mathrm{a})))\\
&\leq\mathcal{F}\bigg(\breve{\phi}\bigg(\max\bigg\{\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a}),\mathcal{D}(\mathrm{a}_n,\mathcal{S}\mathrm{a}_n),\mathcal{D}(\mathrm{a},\mathcal{S}\mathrm{a}),\frac{\mathcal{D}(\mathrm{a}_n),\mathcal{S}\mathrm{a})+\mathcal{D}(\mathrm{a},\mathcal{S}\mathrm{a}_n)}{2\mathtt{b}}\bigg\}\bigg)\bigg)-\\
&\qquad\breve{\phi}(\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L})\bigg)-\breve{\tau}\\
&\leq\mathcal{F}\bigg(\breve{\phi}\bigg(\max\bigg\{(\wp_{\mathtt{b}}(\mathrm{a}_n,\mathrm{a})),\mathcal{D}(\mathrm{a}_n,\mathcal{S}\mathrm{a}_n)),\mathcal{D}(\mathrm{a},\mathcal{S}\mathrm{a})),\frac{\mathcal{D}(\mathrm{a}_n,\mathcal{S}k)+\wp(\mathrm{a},\mathcal{S}\mathrm{a}_n)}{2\mathtt{b}}\bigg\}\bigg)-\\
&\qquad\breve{\phi}(\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L})\bigg)-\breve{\tau}
\end{aligned}$$

applying $n\to\infty$ on the above inequality by using $\mathrm{a}_n\to\mathrm{a}$, $\mathrm{b}_n\to\mathrm{b}$ and $\wp_{\mathtt{b}}(\mathrm{a},\mathrm{b})=\wp_{\mathtt{b}}(\mathbb{K},\mathbb{L})$ we have

$$
\begin{aligned}
&\mathcal{F}(\mathtt{b}^2(\breve{\phi}(\mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a})))\\
&\quad \le \mathcal{F}\Bigg(\breve{\phi}\Bigg(\max\Bigg\{0,(\wp(\mathrm{a}_n, \mathrm{a})), (\mathcal{D}(\mathrm{a}_n, \mathcal{S}\mathrm{a}_n)), (\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a})), \frac{\mathcal{D}(\mathrm{a}_n), \mathcal{S}\mathrm{a}) + \mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}_n}{2\mathtt{b}}\Bigg\}\Bigg) -\\
&\qquad\qquad \breve{\phi}(\wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}))\Bigg) - \breve{\tau}\\
&\quad \le \mathcal{F}(\breve{\phi}(\wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})) + (\mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a}))) - \breve{\phi}(\wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}))) - \breve{\tau}\\
&\quad \le \mathcal{F}(\breve{\phi}(\wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})) + \phi(\mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a}))) - \breve{\phi}(\wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}))) - \breve{\tau}\\
\implies &\mathcal{F}(\mathtt{b}^2(\breve{\phi}(\mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a}))) \le \mathcal{F}(\breve{\phi}(\mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a}))) - \breve{\tau},
\end{aligned}
$$

which is a contradiction.

This means that $\mathrm{b} \in \mathcal{S}\mathrm{a}$, and hence, $\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{a}) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$. This implies a is the `BPP` of $\mathcal{S}$. □

Theorem 4.2.2.

Consider a `POCMS` $(\mathbb{J}, \preceq, \wp_{\mathtt{b}})$. Suppose $\mathbb{K}$ and $\mathbb{L}$ are non-empty closed subset of `MS` $(\mathbb{J}, \wp_{\mathtt{b}})$ in such a way that $\mathbb{K}_0$ is non-empty and $(\mathbb{K}, \mathbb{L})$ satisfies $\mathcal{P}$-property. Let $\mathcal{S} : \mathbb{K} \to \mathcal{CB}(\mathbb{L})$ be an α-admissible multivalued $(\alpha\mathcal{F}, \mathtt{b}, \breve{\phi})$-contraction such that the conditions given below are satisfied:

(A1). $\exists\ \mathrm{a}_0, \mathrm{a}_1 \in \mathbb{K}$ such that $\alpha(\mathrm{a}_0, \mathrm{a}_1) \ge 1 \implies \alpha(\mathrm{v}, \mathrm{w}) \ge 1$ for $\mathrm{v} \in \mathcal{S}\mathrm{a}_0$, $\mathrm{w} \in \mathcal{S}\mathrm{a}_1$;

(A2). There are two elements $\mathrm{a}_0, \mathrm{a}_1 \in \mathbb{K}_0$ and $\mathrm{b}_0 \in \mathcal{S}\mathrm{a}_0$ such that
$\wp_{\mathtt{b}}(\mathrm{a}_1, \mathrm{b}_0) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$ and $\mathrm{a}_0 \preceq \mathrm{a}_1$;

(A3). $\mathcal{S}(\mathbb{K})_0 \subseteq \mathbb{L}_0$ and $\mathcal{F}\left(\alpha(\mathrm{a}_0, \mathrm{a}_1)(\mathtt{b}^2(\breve{\phi}((\delta\,(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}))\right) \le \mathcal{F}\left(\breve{\phi}\,(\mathcal{N}\,(\mathrm{a}, \mathrm{b})) - \breve{\phi}\,(\wp_{\mathtt{b}}\,(\mathbb{L}, \mathbb{M}))\right) - \tau\ \ \forall\ \mathrm{a} \le \mathrm{b}\ \in \mathbb{L}$, where $\alpha : \mathbb{J} \times \mathbb{J} \to [0, \infty)$ and

$$\mathcal{N}(\mathrm{a}, \mathrm{b}) = \max\left\{\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}), \mathcal{D}(\mathrm{a}, \mathrm{b}) - \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}), \mathcal{D}_{\mathtt{b}}(\mathrm{b}, \mathcal{S}\mathrm{b}) - \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}), \frac{\mathcal{D}(\mathrm{a}, \mathcal{S}\mathrm{b}) + \mathcal{D}(\mathrm{b}, \mathcal{S}\mathrm{a})}{2\mathtt{b}} - \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})\right\}$$

$\forall\ (\mathrm{a}, \mathrm{b}) \in [0, +\infty)$;

(A4). $\forall\ \mathrm{a}, \mathrm{b} \in \mathbb{L}_0, \mathrm{a} \preceq \mathrm{b} \implies \mathcal{S}\mathrm{a} \subset \mathcal{S}\mathrm{b}$;

(A5). If $\{\mathrm{a}_n\}$ is a non-decreasing sequence in $\mathbb{K}$ such that $\mathrm{a}_n \to \mathrm{a}$, then
$\mathrm{a}_n \preceq \mathrm{a},\ \ \forall\ n$.

Then, $\exists$ an element a in $\mathbb{K}$ such that $\mathcal{D}_{\mathtt{b}}(\mathrm{a}, \mathcal{S}\mathrm{a}) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$.

Proof. It follows from Theorem 4.2.1. □

Here are some corollaries, which are deduced from Theorem 4.2.1 by incorporating the self mapping.

Corollary 4.2.3.

Consider a POCMS $(\mathbb{J}, \preceq, \wp_{\mathtt{b}})$. Suppose $\mathbb{K}$ and $\mathbb{L}$ are non-empty closed subsets of MS $(\mathbb{J}, \wp_{\mathtt{b}})$ such that $\mathbb{K}_0$ is non-empty and $(\mathbb{K}, \mathbb{L})$ satisfies the $\mathcal{P}$-property. Let $\mathcal{S} : \mathbb{K} \to \mathbb{L}$ be α-admissible single-valued mapping $(\alpha\mathcal{F}, \mathtt{b}, \breve{\phi})$-contraction such that the conditions given below are satisfied:

(A1). $\exists\ \mathrm{a}_0, \mathrm{a}_1 \in \mathbb{K}$ such that $\alpha(\mathrm{a}_0, \mathrm{a}_1) \geq 1 \implies \alpha(\mathrm{v}, \mathrm{w}) \geq 1$ for $\mathrm{v} \in \mathcal{S}\mathrm{a}_0$, $\mathrm{w} \in \mathcal{S}\mathrm{a}_1$;

(A2). $\exists$ two elements $\mathrm{a}_0, \mathrm{a}_1$ in $\mathbb{K}_0$ and $\mathrm{b}_o \in \mathcal{S}\mathrm{a}_0$ such that $\wp_{\mathtt{b}}(\mathrm{a}_1, \mathrm{b}_0) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$ and $\mathrm{a}_0 \preceq \mathrm{a}_1$;

(A3). $\forall$, $\mathrm{a}, \mathrm{b} \in \mathbb{K}_0, \mathrm{a} \preceq \mathrm{b} \implies \mathcal{S}\mathrm{a} \preceq \mathcal{S}\mathrm{b}$;

(A4). If $\{\mathrm{a}_n\}$ is a non decreasing sequence in $\mathbb{K}$ such that $\mathrm{a}_n \to \mathrm{a}$, then $\mathrm{a}_n \leq \mathrm{a}\ \forall\ n$. so, $\exists\ \mathrm{a} \in \mathbb{K}$ such that $\wp_{\mathtt{b}}(\mathrm{a}, \mathcal{S}\mathrm{a}) = \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L})$.

Proof. Follows from Theorem 4.2.1 . □

Similarly, if we consider $\mathbb{K} = \mathbb{L}$ in Theorems 4.2.1 and Theorem 4.2.2, following results are obtained.

Corollary 4.2.4.

Consider a POCMS $(\mathbb{J}, \preceq, \wp_{\mathtt{b}})$. Suppose $\mathbb{K}$ is non-empty closed subset of MS $\mathbb{J}$. Let $\mathcal{S} : \mathbb{K} \to \mathcal{C}B(\mathbb{K})$ be an α-admissible multi-valued $(\alpha\mathcal{F}, \mathtt{b}, \breve{\phi})$-contraction such that the conditions given below are satisfied:

(A1). $\exists\ \mathrm{a}_0, \mathrm{a}_1 \in \mathbb{K}$ such that $\alpha(\mathrm{a}_0, \mathrm{a}_1) \geq 1 \implies \alpha(\mathrm{v}, \mathrm{w}) \geq 1$ for $\mathrm{v} \in \mathcal{S}\mathrm{a}_0$, $\mathrm{w} \in \mathcal{S}\mathrm{a}_1$;

(A2). $\exists$ two elements $\mathrm{a}_0, \mathrm{a}_1$ in $\mathbb{K}$ and $\mathrm{b}_o \in \mathcal{S}\mathrm{a}_0$ such that $\wp_{\mathtt{b}}(\mathrm{a}_1, \mathrm{b}_0) = 0$ and $\mathrm{a}_0 \preceq \mathrm{a}_1 = \mathrm{b}_0$;

(A3). $\forall$ a, b $\in \mathbb{K}$, a $\preceq$ b $\implies \mathcal{S}$a $\preceq \mathcal{S}$b;

(A4). If $\{a_n\}$ is a non decreasing sequence in $\mathbb{K}$ such that $a_n \to a$, then $a_n \preceq a\ \forall$ $n \in \mathbb{N}$

Then, $\exists$ a in $\mathbb{K}$ such that $\wp_{\mathtt{b}}(a, \mathcal{S}a) = \mathcal{D}(a, \mathcal{S}a) = 0$. a is FP of mapping $\mathcal{S}$.

Corollary 4.2.5.

Consider a POCMS $(\mathbb{J}, \preceq, \wp_{\mathtt{b}})$. Let $\mathbb{K}$ be a non-empty closed subset of a MS $\mathbb{J}$. Let $\mathcal{S} : \mathbb{K} \to \mathbb{K}$ be an α admissible single-valued $(\alpha\mathcal{F}, \mathtt{b}, \breve{\phi})$-contraction in such a way that the following axioms are satisfies:

(A1). $\exists$ $a_0, a_1 \in \mathbb{K}$ such that $\alpha(a_0, a_1) \geq 1 \implies \alpha(v, w) \geq 1$ for $v \in \mathcal{S}a_0$, $w \in \mathcal{S}a_1$;

(A2). $\exists$ two elements a_0, a_1 in $\mathbb{K}$ and $b_0 \in \mathcal{S}a_0$ such that $\wp_{\mathtt{b}}(a_1, \mathcal{S}a_0) = 0$ and $a_0 \leq a_1$;

(A3). $\forall$ a, b $\in \mathbb{K}$, a $\preceq$ b $\implies \mathcal{S}$a $\leq \mathcal{S}$b;

(A4). If $\{a_n\}$ is a non decreasing sequence in $\mathbb{K}$ such that $a_n \to a$, then $a_n \preceq a$ for all $n \in \mathbb{N}$

Then, $\exists$ a in $\mathbb{K}$ such that $\wp_{\mathtt{b}}(a, \mathcal{S}a) = 0$ i.e. a is a FP of the mapping $\mathcal{S}$.

Following corollary is obtained by further choosing $\breve{\phi}$, an identity function

$$\breve{\phi}(\mathtt{r}) = \mathtt{r}, \quad \forall\ \mathtt{r} \in (0, \infty).$$

Corollary 4.2.6.

Consider a partially ordered $(\mathbb{J}, \preceq, \wp_{\mathtt{b}})$. Suppose $\mathbb{K}$ is non-empty closed subsets of $\mathbb{J}$. Let $\mathcal{S} : \mathbb{K} \to \mathbb{K}$ be an α-admissible single-valued $(\alpha\mathcal{F}, \mathtt{b})$-contraction satisfying the following axioms:

(A1). $\exists$ $a_0, a_1 \in \mathbb{K}$ such that $\alpha(a_0, a_1) \geq 1 \implies \alpha(v, w) \geq 1$ for $v \in \mathcal{S}a_0$, $w \in \mathcal{S}a_1$

(A2). $\exists$ two elements a_0, a_1 in $\mathbb{K}$ and $b_0 \in \mathcal{S}a_0$ such that $\wp_{\mathtt{b}}(a_1, \mathcal{S}a_0) = 0$ and $a_0 \preceq a_1$;

(A3). $\forall$, a, b $\in \mathbb{K}$, a $\preceq$ b $\implies \mathcal{S}$a $\preceq \mathcal{S}$b;

(A4). If $\{a_n\}$ is a non decreasing sequence in $\mathbb{K}$ such that $a_n \to a$, then $a_n \preceq k$ $\forall\ n \in \mathbb{N}$.

Then, $\exists$ a in $\mathbb{K}$ such that $\wp_{\mathtt{b}}(a, \mathcal{S}a) = 0$ i.e. a is a FP of the mapping $\mathcal{S}$

Example 4.2.7.

Consider $\mathbb{J} = \mathbb{R}^2$ and assume the order $(a, b) \preceq (m, q) \iff a \leq m$ and $b \leq q$, here $\leq$ is the usual order within $\mathbb{R}$. As a result, $(\mathbb{J}, \preceq)$ partially ordered set. Furthermore, $(\mathbb{J}, \wp_{\mathtt{b}})$ is a complete bMS, $\mathtt{b} \geq 1$ with the metric specified as

$$\wp_{\mathtt{b}}\left((a_1, b_1), (a_2, b_2)\right) = \left(\mid a_1 - a_2 \mid\right)^2 + \left(\mid b_1 - b_2 \mid\right)^2.$$

Next, define α-admissible $\alpha : \mathbb{K} \times \mathbb{K} \to [0, \infty)$ as

$$\alpha(a, b) = \{(a_1 + b_1) + (a_2 + b_2) + 3\}$$

Suppose

$$\mathbb{K} = \{(-7, 0), (0, -7), (0, 5)\}$$

and

$$\mathbb{L} = \{(-2, 0), (0, -2), (0, 0), (-2, 2), (2, 2)\}$$

is a closed subset of $\mathbb{J}$. consider the following calculations

$$\begin{aligned}
\wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}) &= \inf\{\wp_{\mathtt{b}}(a, b) : a \in \mathbb{K} \text{ and } b \in \mathbb{L}\} \\
&= \inf\{\wp_{\mathtt{b}}((-7, 0), (-2, 0)), \wp_{\mathtt{b}}((-7, 0), (0, -2)), \\
&\quad \wp_{\mathtt{b}}((-7, 0), (0, 0)), \wp_{\mathtt{b}}((-7, 0), (-2, 2)), \wp_{\mathtt{b}}((-7, 0), (2, 2)), \\
&\quad \wp_{\mathtt{b}}((0, -7), (-2, 0)), \wp_{\mathtt{b}}((0, -7), (0, -2)), \wp_{\mathtt{b}}((0, -7), (0, 0)), \\
&\quad \wp_{\mathtt{b}}((0, -7), (0, 0)), \wp_{\mathtt{b}}((0, -7), (-2, 2)), \wp_{\mathtt{b}}((0, -7), (2, 2)) \\
&\quad , \wp_{\mathtt{b}}((0, 5), (-2, 0)), \wp_{\mathtt{b}}((0, 5), (0, -2)), \wp_{\mathtt{b}}((0, 5), (0, 0)), \\
&\quad \wp_{\mathtt{b}}((0, 5), (-2, 2)), \wp_{\mathtt{b}}((0, 5), (2, 2))\} \\
\wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}) &= \inf\{|5|^2 + |0|^2, |7|^2 + |2|^2, |7|^2 + |0|^2, |5|^2 + |2|^2, |9|^2 + |2|^2,
\end{aligned}$$

$$|2|^2+|7|^2, |0|^2+|5|^2, |0|^2+|7|^2, |2|^2+|9|^2, |2|^2+|9|^2,$$
$$|2|^2+|5|^2, |0|^2+|7|^2, |0|^2+|5|^2, |2|^2+|3|^2, |2|^2+|3|^2\}$$
$$= \inf\{25, 53, 49, 29, 85, 53, 25, 49, 85, 85, 29, 49, 25, 13, 13\}$$
$$= 13$$

and $\mathbb{K} = \mathbb{K}_0$ and $\mathbb{K} = \mathbb{K}_0$. Let $\mathcal{S} : \mathbb{K} \to \mathcal{CB}(\mathbb{L})$ is defined as

$$\mathcal{S}(\mathrm{a}, \mathrm{b}) = \begin{cases} \{(0,-2),(0,0)\} & \text{if}(\mathrm{a},\mathrm{b}) = (-7,0) \\ \{(2,2),(-2,2)\} & \text{if}(\mathrm{a},\mathrm{b}) = (0,-7) \\ \{(-2,2),(0,0),(0,-2),(2,2)\} & \text{if}(\mathrm{a},\mathrm{b}) = (0,5). \end{cases}$$

As, there are two elements that are $(-7,0),(0,5) \in \mathbb{K}_0$ and $(0,0) \in \mathcal{S}(-7,0)$ such that

$$\wp_{\mathtt{b}}((0,5),(0,0)) = \wp_{\mathtt{b}}(\mathbb{K},\mathbb{L}) = 25 \quad \text{and} \quad (-7,0) \preceq (0,5)$$

As a result, the first condition is satisfied.
Now, condition (4.1) of Theorem 4.2.1 must be satisfied. It is straightforward to demonstrate that $\mathcal{S}\mathrm{a}_0$ is element of $\mathbb{L}_0$ $\forall$ $\mathrm{a}_0 \in \mathbb{K}$.
Since " $\preceq$ " is defined in $\mathbb{K}$, there are two cases, and each of those given below
$\delta(\mathbb{K},\mathbb{L}) = \sup\{\wp_{\mathtt{b}}(\mathrm{a},\mathrm{b}) : \mathrm{a} \in \mathbb{K}, \mathrm{b} \in \mathbb{L}\}$
$\mathcal{S}\mathrm{a}_1 = \{(0,-2),(0,0)\}, \mathcal{S}\mathrm{a}_2 = \{(2,2),(-2,2)\}, \mathcal{S}\mathrm{a}_3 = \{(-2,2),(0,0),(0,-2),(2,2)\}$
Now we calculate

$$\delta(\mathcal{S}\mathrm{a}_1, \mathcal{S}\mathrm{a}_2) = \sup\{\wp_{\mathtt{b}}(\mathrm{a},\mathrm{b}) : \mathrm{a} \in \mathcal{S}\mathrm{a}_1, \mathrm{b} \in \mathcal{S}\mathrm{a}_2\}$$
$$= \sup\{20, 20, 8, 8\}$$
$$= 20$$
$$\delta(\mathcal{S}\mathrm{a}_1, \mathcal{S}\mathrm{a}_3) = \sup\{\wp_{\mathtt{b}}(\mathrm{a},\mathrm{b}) : \mathrm{a} \in \mathcal{S}\mathrm{a}_1, \mathrm{b} \in \mathcal{S}\mathrm{a}_3\}$$
$$= \sup\{20, 4, 0, 20, 8, 0, 4, 8\}$$
$$= 20$$
$$\delta(\mathcal{S}\mathrm{a}_1, \mathcal{S}\mathrm{a}_3) = \sup\{\wp_{\mathtt{b}}(\mathrm{a},\mathrm{b}) : \mathrm{a} \in \mathcal{S}\mathrm{a}_1, \mathrm{b} \in \mathcal{S}\mathrm{a}_3\}$$
$$= \sup\{20, 4, 0, 20, 8, 0, 4, 8\}$$

$$
\begin{aligned}
&= 20 \\
\delta(\mathcal{S}a_2, \mathcal{S}a_3) &= \sup\{\wp_{\mathtt{b}}(a, b) : a \in \mathcal{S}a_2, b \in \mathcal{S}a_3\} \\
&= \sup\{8, 8, 20, 8, 0, 8, 20, 8\} \\
&= 20 \\
\delta(\mathcal{S}a, \mathcal{S}b) &= 20.
\end{aligned}
$$

Now, $\mathcal{N}(a, b) = \max\left\{\wp(a, b), \mathcal{D}(a, \mathcal{S}a), \mathcal{D}(b, \mathcal{S}b), \frac{\mathcal{D}(a,\mathcal{S}b)+\mathcal{D}(b,\mathcal{S}a}{2})\right\}$

$$
\begin{aligned}
\wp_{\mathtt{b}}(a, b) &= (-7, 0), (0, 5) \\
&= |-7-0|^2 + |0-5|^2 \\
&= |7|^2 + |5|^2 \\
&= 74 \\
\mathcal{D}(a, \mathcal{S}a) &= (-7, 0), (0, 0) \\
&= |-7-0|^2 + |0-0|^2 \\
&= |7|^2 + |0|^2 \\
&= 49 \\
\mathcal{D}(b, \mathcal{S}b) &= (0, 5), (2, 2) \\
&= |0-2|^2 + |5-2|^2 \\
&= |2|^2 + |3|^2 \\
&= 13 \\
\frac{1}{2}\mathcal{D}(a, \mathcal{S}b) + \mathcal{D}(b, \mathcal{S}a) &= \frac{1}{2}\mathcal{D}(a, \mathcal{S}b) + \mathcal{D}(b, \mathcal{S}a) \\
&= \frac{1}{2}\left[(\mathcal{D}(-7, 0), (0, -2)) + (\mathcal{D}(0, 5), (0, -2))\right] \\
&= \frac{1}{2}\left[|7+2|^2 + |0+7|^2\right] \\
&= \frac{1}{2}\left[9^2 + 7^2\right] \\
&= 65 \\
\mathcal{N}(a, b) &= \max\{74, 49, 13, 65\} \\
\mathcal{N}(a, b) &= 74
\end{aligned}
$$

$$\text{and} \quad \wp_{\mathtt{b}}(\mathbb{K}, \mathbb{L}) = 13.$$

Suppose $\mathcal{F}$ is defined as $\mathcal{F}(\alpha) = \ln \alpha + \alpha$, $\check{\tau} = 1$ $\mathtt{b} = 2$

For $\breve{\phi}(\mathrm{t}) = 2\mathrm{t}$, we get $\breve{\phi}(\delta(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b})) = 2 \times 20 = 40$ also

$\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b})) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L})) = 2(74) - 2(13) = 148 - 26 = 122$.

Thus,

$$\frac{\breve{\phi}(\mathtt{b}(\alpha(\mathrm{a}, \mathrm{b})(\delta(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}))}{\breve{\phi}(\mathcal{N}(\mathrm{a}, \mathrm{b})) - \breve{\phi}(\wp(\mathbb{K}, \mathbb{L}))} e^{\breve{\phi}(\mathtt{b}(\alpha(\mathrm{a},\mathrm{b})(\delta(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b}))-(\breve{\phi}(\mathcal{N}(\mathrm{a},\mathrm{b}))-\breve{\phi}(\wp(\mathbb{K},\mathbb{L})))}$$
$$= \frac{80}{122}\left(e^{80-122}\right) = \frac{40}{61}e^{-42} < e^{-1}.$$

So, $\mathcal{S}$ meet requirement (4.1). One can easily verify $(Q2)$ and $(Q3)$. Hence, all the hypotheses of the Theorem 4.2.1 are fulfilled. This also clear that $(0, 5)$ is $\mathtt{BPP}$ of $\mathcal{S}$, i.e.

$$\mathcal{D}((0, 5), \mathcal{S}(0, 5)) = \wp(\mathbb{K}, \mathbb{L}) = 5$$

.

Example 4.2.8.

suppose $\mathbb{J} = \{0, 1, 2, 3...\}$ is a partial order set, having usual order $\preceq$, and suppose $\wp : \mathbb{J} \times \mathbb{J} \to \mathbb{R}$ be given as

$$\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}) = \begin{cases} 0; & \mathrm{a} = \mathrm{b}, \\ (\mathrm{a} + \mathrm{b})^2; & \mathrm{a} \neq \mathrm{b}. \end{cases}$$

Then $(\mathbb{J}, \wp_{\mathtt{b}})$ is a complete $\mathtt{b}$ metric space, $\mathtt{b} > 1$. Suppose $\mathcal{S} : \mathbb{J} \to \mathbb{J}$ be defined as

$$\mathcal{S}(\mathrm{a}) = \begin{cases} 0 & \text{if } \mathrm{a} = 0, \\ \mathrm{a} - 1 & \text{if } \mathrm{a} \neq 0 \end{cases}$$

Consider, $\alpha : \mathbb{J} \times \mathbb{J} \to [0, \infty)$, working as

$$\alpha(\mathrm{a}_0, \mathrm{a}_1) = \begin{cases} 2 & \text{if} \quad \mathrm{a}_0, \mathrm{a}_1 \in \{0, 1\} \\ \frac{1}{2} & \quad \text{otherwise} \end{cases}$$

$\mathrm{a}_0 = 0, \mathrm{a}_1 = 2 \implies \mathcal{S}\mathrm{a}_0 = 0, \mathcal{S}\mathrm{a}_1 = \mathrm{a}_1 - 1 = 1$
$\alpha(\mathcal{S}\mathrm{a}_0, \mathcal{S}\mathrm{a}_1) = \alpha(0,1) = 2 > 1$ $\alpha(\mathcal{S}\mathrm{a}_1, \mathcal{S}\mathrm{a}_0) = \alpha(0,1) = 2 > 1$ showing that $\mathcal{S}$ is an α- admissible map.
Now, we show how $\mathcal{S}$ is $\mathcal{F}$-contraction with $\mathcal{F}(\alpha) = \ln \alpha + \alpha$, $\check{\tau} = 1$.
There are five cases:
Case 1: Let a > b and b $\neq$ 0,then

$$\begin{aligned}
\wp(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) =& \wp(\mathrm{a}-1, \mathrm{b}-1) = (\mathrm{a}+\mathrm{b}-2)^2 \\
\mathcal{N}(\mathrm{a}, \mathrm{b}) =& \max\Big\{ \wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}), \wp_{\mathtt{b}}(\mathrm{a}, \mathcal{S}k), \wp_{\mathtt{b}}(\mathrm{b}, \mathcal{S}\mathrm{b}), \\
& \frac{\wp_{\mathtt{b}}(\mathrm{a}, \mathcal{S}\mathrm{a}) + \wp_{\mathtt{b}}(\mathrm{b}, \mathcal{S}\mathrm{b})}{2\mathtt{b}} \Big\} \\
=& \max\Big\{ \wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}), \wp_{\mathtt{b}}(\mathrm{a}, \mathrm{a}-1), \wp_{\mathtt{b}}(\mathrm{b}, \mathrm{b}-1), \\
& \frac{\wp(\mathrm{a}, \mathrm{b}-1) + \wp_{\mathtt{b}}(\mathrm{b}, \mathrm{a}-1)}{2\mathtt{b}} \Big\} \\
=& \max\left\{ (\mathrm{a}+\mathrm{b})^2, (2\mathrm{a}-1)^2, (2\mathrm{b}-1)^2, (\mathrm{a}+\mathrm{b}-1)^2 \right\} \\
=& (2\mathrm{a}-1)^2 \\
\frac{\mathtt{b}(\alpha(\mathrm{a}_0, \mathrm{a}_1)\wp(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b})}{\mathcal{N}(\mathrm{a}, \mathrm{b})} e^{\mathtt{b}(\alpha(\mathrm{a}_0,\mathrm{a}_1)\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b}) - \mathcal{N}(\mathrm{a},\mathrm{b})} =& \frac{(\mathrm{a}+\mathrm{b}-2)^2}{(2\mathrm{a}-1)^2} e^{(\mathrm{a}+\mathrm{b}-2)^2 - (2\mathrm{a}-1)^2} \\
=& \frac{(\mathrm{a}+\mathrm{b}-2)^2}{(2\mathrm{a}-1)^2} e^{-3\mathrm{a}^2 + \mathrm{b}^2 + 2\mathrm{ab} - 3 + 4\mathrm{a}} < e^{-1}
\end{aligned}$$

Case 2: If b > a and a $\neq$ 0, then

$$\begin{aligned}
\wp_{\mathtt{b}}(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) =& \wp_{\mathtt{b}}(\mathrm{a}-1, \mathrm{b}-1) = (\mathrm{a}+\mathrm{b}-2)^2 \\
\mathcal{N}(\mathrm{a}, \mathrm{b}) =& \max\Big\{ \wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}), \wp_{\mathtt{b}}(\mathrm{a}, \mathcal{S}\mathrm{a}), \wp_{\mathtt{b}}(\mathrm{b}, \mathcal{S}\mathrm{b}), \\
& \frac{\wp_{\mathtt{b}}(\mathrm{a}, \mathcal{S}\mathrm{a}) + \wp_{\mathtt{b}}(\mathrm{b}, \mathcal{S}\mathrm{b})}{2\mathtt{b}} \Big\} \\
=& \max\Big\{ \wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}), \wp_{\mathtt{b}}(\mathrm{a}, \mathrm{a}-1), \wp_{\mathtt{b}}(\mathrm{b}, \mathrm{b}-1), \\
& \frac{\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}-1) + \wp_{\mathtt{b}}(\mathrm{b}, \mathrm{a}-1)}{2\mathtt{b}} \Big\} \\
=& \max\left\{ (\mathrm{a}+\mathrm{b})^2, (2\mathrm{a}-1)^2, (2\mathrm{b}-1)^2, (\mathrm{a}+\mathrm{b}-1)^2 \right\}
\end{aligned}$$

$$
\begin{aligned}
&= (2\mathrm{a}-1)^2\\
\frac{\mathtt{b}(\alpha(\mathrm{a}_0,\mathrm{a}_1)\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})}{\mathcal{N}(\mathrm{a},\mathrm{b})}e^{\mathtt{b}(\alpha(\mathrm{a}_0,\mathrm{a}_1)\wp(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})-\mathcal{N}(\mathrm{a},\mathrm{b})} &= \frac{(\mathrm{a}+\mathrm{b}-2)^2}{(2\mathrm{a}-1)^2}e^{(\mathrm{a}+\mathrm{b}-2)^2-(2\mathrm{b}-1)^2}\\
&= \frac{\mathrm{a}+\mathrm{b}-2}{2\mathrm{b}-1}e^{-3\mathrm{b}^2+\mathrm{a}^2+2\mathrm{ab}+4\mathrm{b}-3} < e^{-1}
\end{aligned}
$$

Case 3: If a > b and b = 0, then

$$
\begin{aligned}
\wp_{\mathtt{b}}(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b}) &= \wp_{\mathtt{b}}(\mathrm{a}-1,0) = (\mathrm{a}-1)^2\\
\mathcal{N}(\mathrm{a},\mathrm{b}) &= \max\Big\{\wp_{\mathtt{b}}(\mathrm{a},\mathrm{b}), \wp_{\mathtt{b}}(\mathrm{a},\mathcal{S}\mathrm{a}), \wp_{\mathtt{b}}(\mathrm{b},\mathcal{S}\mathrm{b}),\\
&\qquad \frac{\wp_{\mathtt{b}}(\mathrm{a},\mathcal{S}\mathrm{a})+\wp_{\mathtt{b}}(\mathrm{b},\mathcal{S}\mathrm{b})}{2\mathtt{b}}\Big\}\\
&= \max\Big\{\wp(\mathrm{a},0), \wp_{\mathtt{b}}(\mathrm{a},\mathrm{a}-1), \wp_{\mathtt{b}}(0,0),\\
&\qquad \frac{\wp_{\mathtt{b}}(\mathrm{a},0)+\wp_{\mathtt{b}}(0,\mathrm{a}-1)}{2\mathtt{b}}\Big\}\\
&= \max\left\{\mathrm{a}^2, (2\mathrm{a}-1)^2, 0, \mathrm{a}^2-\frac{1}{4}\right\}\\
&= (2\mathrm{a}-1)^2\\
\frac{\mathtt{b}(\alpha(\mathrm{a}_0,\mathrm{a}_1)\wp_{\mathtt{b}}(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})}{\mathcal{N}(\mathrm{a},\mathrm{b})}e^{\mathtt{b}(\alpha(\mathrm{a}_0,\mathrm{a}_1)\wp_{\mathtt{b}}(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})-\mathcal{N}(\mathrm{a},\mathrm{b})} &= \frac{(\mathrm{a}-1)^2}{(2\mathrm{a}-1)^2}e^{(\mathrm{a}-1)^2-(2\mathrm{b}-1)^2}\\
&= \frac{(\mathrm{a}-1)^2}{(2\mathrm{a}-1)^2}e^{-3\mathrm{a}^2+2\mathrm{a}} < e^{-1}
\end{aligned}
$$

Case 4: If b > a and a = 0, then

$$
\begin{aligned}
\wp_{\mathtt{b}}(0,\mathcal{S}\mathrm{b}) &= \wp_{\mathtt{b}}(0,\mathrm{b}-1) = (\mathrm{b}-1)^2\\
\mathcal{N}(\mathrm{a},\mathrm{b}) &= \max\Big\{\wp_{\mathtt{b}}(\mathrm{a},\mathrm{b}), \wp_{\mathtt{b}}(\mathrm{a},\mathcal{S}\mathrm{a}), \wp_{\mathtt{b}}(\mathrm{b},\mathcal{S}\mathrm{b}),\\
&\qquad \frac{\wp_{\mathtt{b}}(\mathrm{a},\mathcal{S}\mathrm{a})+\wp_{\mathtt{b}}(\mathrm{b},\mathcal{S}\mathrm{b})}{2\mathtt{b}}\Big\}\\
&= \max\Big\{\wp_{\mathtt{b}}(0,\mathrm{b}), \wp_{\mathtt{b}}(0,\mathrm{a}-0), \wp_{\mathtt{b}}(\mathrm{b},\mathrm{b}-1),\\
&\qquad \frac{\wp_{\mathtt{b}}(0,\mathrm{b}-1)+\wp_{\mathtt{b}}(\mathrm{b},0)}{2\mathtt{b}}\Big\}\\
&= \max\left\{\mathrm{b}^2, (2\mathrm{b}-1)^2, 0, \mathrm{b}^2-\frac{1}{4}\right\}\\
&= (2\mathrm{b}-1)^2\\
\frac{\mathtt{b}(\alpha(\mathrm{a}_0,\mathrm{a}_1)\wp_{\mathtt{b}}(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})}{\mathcal{N}(\mathrm{a},\mathrm{b})}e^{\mathtt{b}(\alpha(\mathrm{a}_0,\mathrm{a}_1)\wp_{\mathtt{b}}(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})-\mathcal{N}(\mathrm{a},\mathrm{b})} &= \frac{(\mathrm{b}-1)^2}{(2\mathrm{b}-1)^2}e^{(\mathrm{a}-1)^2-(2\mathrm{b}-1)^2}\\
&= \frac{(\mathrm{b}-1)^2}{(2\mathrm{b}-1)^2}e^{-3\mathrm{b}^2+2\mathrm{b}} < e^{-1}
\end{aligned}
$$

Case 5: If b = a, then

$$\begin{aligned}
\wp_{\mathtt{b}}(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) =& \wp_{\mathtt{b}}(\mathrm{a}-1, \mathrm{a}-1) = 0 \\
\mathcal{N}(\mathrm{a}, \mathrm{b}) =& \max\left\{\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}), \wp_{\mathtt{b}}(\mathrm{a}, \mathcal{S}\mathrm{a}), \wp_{\mathtt{b}}(\mathrm{b}, \mathcal{S}\mathrm{b}), \frac{\wp_{\mathtt{b}}(\mathrm{a}, \mathcal{S}\mathrm{a}) + \wp_{\mathtt{b}}(\mathrm{b}, \mathcal{S}\mathrm{b})}{2\mathtt{b}}\right\} \\
=& \max\left\{\wp_{\mathtt{b}}(0, \wp_{\mathtt{b}}(\mathrm{a}, \mathrm{a}-1), \wp_{\mathtt{b}}(\mathrm{a}, \mathrm{a}-1),\right. \\
& \left.\frac{\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{a}-1) + \wp_{\mathtt{b}}(\mathrm{a}, \mathrm{a}-1)}{2\mathtt{b}}\right\} \\
=& \max\left\{0, (2\mathrm{a}-1)^2, (2\mathrm{a}-1)^2, (2\mathrm{a}-1)^2\right\} \\
=& (2\mathrm{a}-1)^2
\end{aligned}$$

$$\frac{\mathtt{b}(\alpha(\mathrm{a}_0, \mathrm{a}_1)\wp_{\mathtt{b}}(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b})}{\mathcal{N}(\mathrm{a}, \mathrm{b})} e^{\mathtt{b}(\alpha(\mathrm{a}_0, \mathrm{a}_1)\wp_{\mathtt{b}}(\mathcal{S}\mathrm{a}, \mathcal{S}\mathrm{b}) - \mathcal{N}(\mathrm{a}, \mathrm{b})} = \frac{0}{(2\mathrm{a}-1)^2} e^{0-(2\mathrm{a}-1)^2} < e^{-1}$$

Hence, all the axioms of Corollary 4.2.6 are met and 0 is a `FP` of $\mathcal{S}$.

4.3 Application

In this section, two applications of the main result are provided.

4.3.1 Application Regarding Equation of Motion

Consider $\wp_{\mathtt{b}} : \mathcal{C}[0, 1] \times \mathcal{C}[0, 1] \to \mathbb{R}$ is a `bMS` defined as

$$\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}) = \| \mathrm{a} - \mathrm{b} \|_\infty = \sup_{\grave{\mathtt{t}} \in [0,1]} |\mathrm{a}(\grave{\mathtt{t}}) - \mathrm{b}(\grave{\mathtt{t}})|^2,$$

and $(\mathcal{C}[0, 1], \wp_{\mathtt{b}})$ is a complete `bMS`.

A body with mass m started its motion at time $\grave{\mathrm{t}} = 0$ and $\mathrm{x} = 0$. A force `f` act on it in the direction of x-axis and its velocity increases from 0 to 1 instantly after $\grave{\mathtt{t}} = 0$. The problem is to explore a function for position in terms of time $\grave{\mathtt{t}}$

The governing equation for this problem is

$$\mathrm{m}\frac{d^2\mathrm{x}}{d\grave{\mathtt{t}}^2} = \mathtt{f}(\grave{\mathtt{t}}, \mathrm{x}(\grave{\mathtt{t}})) \quad \text{together with} \quad \mathrm{x}(0) = 0, \mathrm{x}'(1) = 0, \tag{4.13}$$

where `f` is a real valued function with domain $[0, 1] \times \mathbb{R}$.

$\mathcal{G}$reen's function for (4.13), is defined as

$$\mathcal{G}(\check{\mathtt{t}}, \xi) = \begin{cases} (-1+\xi)\check{\mathtt{t}}, & \grave{\mathrm{t}} \leq \xi \\ -\xi(1-\check{\mathtt{t}}), & \check{\mathtt{t}} \geq \xi. \end{cases}$$

Assume $\breve{\phi} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function along subsequent constraints:

1. $\mid \mathtt{f}(\check{\mathtt{t}}, q) - \mathtt{f}(\check{\mathtt{t}}, r) \mid^2 \leq \frac{|q-r|}{\alpha(q,r)^{\frac{1}{2}}} \quad \forall \quad \check{\mathtt{t}} \in [\mathtt{0}, \mathtt{1}]$ having $q, r \in \mathbb{R}$ with $\breve{\phi}(q, r) \geq 0$;
2. $\exists\ \mathrm{x}_0 \in \mathcal{C}[0,1] \ni \breve{\phi}(\mathrm{x}_0(\mathrm{t}), \mathcal{S}\mathrm{x}_0(\check{\mathtt{t}})) \geq 0 \quad \forall \quad \check{\mathtt{t}} \in [\mathtt{0}, \mathtt{1}]$, where $\mathcal{S}$ is self-map on $\mathcal{C}[\mathtt{0}, \mathtt{1}]$

Theorem 4.3.1. Let $\mathbb{J} = \mathcal{C}[0,1]$. Consider $\mathcal{S}$ is self-map on $\mathcal{C}[\mathtt{0}, \mathtt{1}]$ defined as

$$\mathcal{S}\mathrm{x}(\check{\mathtt{t}}) = \int_0^1 \mathcal{G}(\check{\mathtt{t}}, \xi)\mathtt{f}(\xi, \mathrm{x}(\xi))d\xi, \quad \check{\mathtt{t}} \in [0,1],$$

satisfying the above assumptions 1 and 2. Then the Equation (4.13) has a solution.

Proof. The solution of Equation (4.13) is,

$$\mathrm{x}(\check{\mathtt{t}}) = \int_0^1 \mathcal{G}(\check{\mathtt{t}}, \xi)\mathtt{f}(\xi, \mathrm{x}(\xi))d\xi, \quad \check{\mathtt{t}} \in [0,1].$$

Assume $\mathrm{x}, \mathrm{y} \in \mathcal{C}[0,1] \ni \breve{\phi}(\mathrm{x}(\check{\mathtt{t}}), \mathrm{y}(\check{\mathtt{t}})) \geq 0 \quad \forall\, \grave{\mathrm{t}} \in [0,1]$.

Now

$$\mid \mathcal{S}(\mathrm{x}(\check{\mathtt{t}})) - \mathcal{S}(\mathrm{y}(\check{\mathtt{t}})) \mid^2 = \left| \int_0^1 \mathcal{G}(\grave{\mathrm{t}}, \xi)\mathtt{f}(\xi, \mathrm{x}(\xi))d\xi - \int_0^1 \mathcal{G}(\check{\mathtt{t}}, \xi)\mathtt{f}(\xi, \mathrm{y}(\xi))d\xi \right|^2$$

$$\mid \mathcal{S}(\mathrm{x}(\check{\mathtt{t}})) - \mathcal{S}(\mathrm{y}(\check{\mathtt{t}})) \mid^2 \leq \left(\int_0^1 \mathcal{G}(\check{\mathtt{t}}, \xi) \mid \mathtt{f}(\xi, \mathrm{x}(\xi)) - \mathtt{f}(\xi, \mathrm{y}(\xi)) \mid d\xi \right)^2$$

$$\leq \int_0^1 \mathcal{G}(\check{\mathtt{t}}, \xi)^2 \left(\frac{\mid \mathrm{x}(\xi) - \mathrm{y}(\xi) \mid}{\alpha(\mathrm{x}, \mathrm{y})^{\frac{1}{2}}} \right)^2 d\xi.$$

Taking $\sup\limits_{\grave{\mathrm{t}} \in [0,1]} \mid \mathcal{S}(\mathrm{x}(\check{\mathtt{t}})) - \mathcal{S}(\mathrm{y}(\check{\mathtt{t}})) \mid^2 \leq \sup\limits_{\check{\mathtt{t}} \in [0,1]} \frac{\mid \mathrm{x}(\xi) - \mathrm{y}(\xi) \mid^2}{\alpha(\mathrm{x}, \mathrm{y})} \sup\limits_{\check{\mathtt{t}} \in [0,1]} \left\{ \int_0^1 \mathcal{G}(\check{\mathtt{t}}, \xi) d\xi \right\}^2.$

$$\text{As} \quad \int_0^1 (\mathcal{G}(\grave{\mathtt{t}}, \xi)d\xi)^2 = \int_0^1 ((\grave{\mathtt{t}} - 1)\xi)^2 d\xi + \int_{\grave{\mathtt{t}}}^1 ((\xi - 1)\grave{\mathtt{t}})^2 d\xi$$
$$= (\grave{\mathtt{t}} - 1)^2 \xi^2 d\xi|_0^{\grave{\mathtt{t}}} + \grave{\mathtt{t}}\, (\xi - 1)^2|_{\grave{\mathtt{t}}}^1$$
$$= (\grave{\mathtt{t}} - 1)\frac{\grave{\mathtt{t}}^3}{3} + \left(\grave{\mathtt{t}}^2 \frac{(\grave{\mathtt{t}} - 1)^3}{3}\right)$$
$$= \frac{\grave{\mathtt{t}}^4}{3} - \frac{2}{3}\grave{\mathtt{t}}^3 + \frac{\grave{\mathtt{t}}^2}{3} \quad \forall\ \grave{\mathtt{t}} \in [0, 1].$$

So,

$$\sup_{\grave{\mathtt{t}} \in [0,1]} \left\{ \int_0^1 \mathcal{G}(\grave{\mathtt{t}}, \xi)d\xi \right\}^2 = \frac{1}{4}$$

implies

$$\|\mathcal{S}\mathrm{x} - \mathcal{S}\mathrm{y}\|_\infty \leq \frac{1}{4}\frac{\|\mathrm{x} - \mathrm{y}\|_\infty}{\alpha(\mathrm{x}, \mathrm{y})}.$$
$$\Longrightarrow\ \alpha(\mathrm{x}, \mathrm{y})\|\mathcal{S}\mathrm{x} - \mathcal{S}\mathrm{y}\|_\infty \leq \frac{1}{4}\|\mathrm{x} - \mathrm{y}\|_\infty$$

Taking natural log on both sides

$$\ln(\alpha(\mathrm{x}, \mathrm{y})\|\mathcal{S}\mathrm{x} - \mathcal{S}\mathrm{y}\|_\infty) \leq \ln(\|\mathrm{x} - \mathrm{y}\|_\infty) - \ln 4,$$
$$\ln 4 + \ln(\alpha(\mathrm{x}, \mathrm{y})\|\mathcal{S}\mathrm{x} - \mathcal{S}\mathrm{y}\|_\infty) \leq \ln(\|\mathrm{x} - \mathrm{y}\|_\infty).$$
$$\ln 4 + \mathcal{F}(\alpha(\mathrm{x}, \mathrm{y})\wp_{\mathtt{b}}(\mathcal{S}\mathrm{x}, \mathcal{S}\mathrm{y})) \leq \mathcal{F}(\wp_{\mathtt{b}}(\mathrm{x}, \mathrm{y})).$$
$$\mathcal{F}(\wp_{\mathtt{b}}(\mathrm{x}, \mathrm{y})) \leq \mathcal{F}(\mathcal{N}(\mathrm{x}, \mathrm{y}))$$
$$\Longrightarrow\ \ln 4 + \mathcal{F}(\alpha(\mathrm{x}, \mathrm{y})\wp_{\mathtt{b}}(\mathcal{S}\mathrm{x}, \mathcal{S}\mathrm{y})) \leq \mathcal{F}(\mathcal{N}(\mathrm{x}, \mathrm{y})).$$

Mapping $\mathcal{S}$ is with $\mathcal{F}(\mathrm{x}) = \ln \mathrm{x}$. According to Corollary 4.2.6 $\mathcal{S}$ possess fixed point x in $\mathcal{C}^2([0, 1])$, such that

$$\mathcal{S}(\mathrm{x}(\grave{\mathtt{t}}) = \mathrm{x}(\grave{t})$$
$$= \int_0^1 \mathcal{G}(\grave{\mathtt{t}}, \xi)\mathtt{f}(\xi, \mathrm{x}(\xi))d\xi, \quad \grave{\mathtt{t}} \in [0, 1],$$

which is the solution to (4.13). □

4.3.2 Application Regarding Fractional Calculus

Suppose $\wp_{\mathtt{b}} : \mathcal{C}([0,1]) \times \mathcal{C}([0,1]) \to \mathbb{R}$ is a $\mathtt{bMS}$, $\mathtt{b} \geq 1$ define as

$$\wp_{\mathtt{b}}(\mathrm{a}, \mathrm{b}) = \parallel \mathrm{a} - \mathrm{b} \parallel_{\infty} = \max_{\dot{\mathtt{t}} \in [0,1]} |\mathrm{a}(\dot{\mathtt{t}}) - \mathrm{b}(\dot{\mathtt{t}})|^2.$$

The Caputo fractional derivative of α order continuous function $\mathrm{q} : [0, +\infty) \to \mathbb{R}$ described in such a way

$$^{\nu}\mathcal{D}^{\alpha}(\mathrm{q}(\dot{\mathtt{t}})) = \frac{1}{\Gamma(\mathrm{b} - \alpha)} \int_0^{\dot{\mathtt{t}}} (\dot{\mathtt{t}} - \xi)^{\mathrm{b}-\alpha-1} \mathrm{j}^{(\mathrm{b})}(\xi) d\xi \quad (\mathrm{b} - 1 < \alpha < n, \mathrm{b} = [\alpha] + 1).$$

Here Γ represents Gamma function and $[\alpha]$ represents the integral component of real number. We illustrate the existence of a solution to a non-linear fractional differential equation in this section.

$$^{\nu}\mathcal{D}^{\alpha}(\mathrm{a}(\dot{\mathtt{t}}) + \mathtt{f}(\dot{\mathtt{t}}, \mathrm{a}(\dot{\mathtt{t}})) = 0 \quad (0 \leq \dot{\mathtt{t}} \leq 1, 1 < \alpha \leq 2). \tag{4.14}$$

Using $\mathrm{a}(0) = \mathrm{a}(1) = 0$ where $\mathtt{f}$ is real valued function with domain $[\mathtt{0}, \mathtt{1}] \times \mathbb{R}$. Green's function for (4.14) defined as, [36]

$$\mathcal{G}(\dot{\mathtt{t}}, \xi) = \begin{cases} \frac{[\dot{\mathtt{t}}(1-\xi)^{\alpha-1} - (\dot{\mathtt{t}}-\xi)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq \xi \leq \dot{\mathtt{t}} \leq 1, \\ \frac{[\dot{\mathtt{t}}(1-\xi)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq \mathrm{t} \leq \xi \leq 1. \end{cases}$$

Assume the following conditions are met:

1. $| \mathtt{f}(\dot{\mathtt{t}}, \mathrm{a}) - \mathtt{f}(\dot{\mathtt{t}}, \mathrm{b}) |^2 \leq \frac{e^{\frac{-\tilde{\tau}}{2}}}{\alpha(\mathrm{a},\mathrm{b})^{\frac{1}{2}}} \mathbb{J}(\mathrm{a}, \mathrm{b}) \quad \forall \ \dot{\mathtt{t}} \in [0,1]$ also $\mathrm{a}, \mathrm{b} \in \mathbb{R}$, such that

$$\mathbb{J}(\mathrm{a}, \mathrm{b}) = \max\left\{ | \mathrm{a} - \mathrm{b} |, | \mathrm{a} - \mathcal{S}\mathrm{a} |, | \mathrm{b} - \mathcal{S}\mathrm{b} |, \frac{| \mathrm{a} - \mathcal{S}\mathrm{b} | + | \mathrm{b} - \mathcal{S}\mathrm{a} |}{2\mathtt{b}} \right\};$$

2. $\exists\ \mathrm{a}_0 \in \mathcal{C}[0,1] \ni \breve{\phi}(\mathrm{a}_0(\check{\mathtt{t}}), \mathcal{S}\mathrm{a}_0(\check{\mathtt{t}})) \geq 0 \quad \forall\ \check{\mathtt{t}} \in [\mathtt{0}, \mathtt{1}]$.

Theorem 4.3.2.

Consider $\mathcal{S}$ is self-mapping on $\mathcal{C}[0,1]$ defined as

$$\mathcal{S}(\mathrm{a}(\check{\mathtt{t}})) = \int_0^1 \mathcal{G}(\check{\mathtt{t}}, \xi)\mathtt{f}(\xi, \mathrm{a}(\xi))d\xi,$$

satisfying the above assumptions 1 and 2. Then the fractional differential equation (4.14) has a solution.

Proof. It is obvious that solution of (4.14) is ,

$$\mathrm{a}(\mathrm{t}) = \int_0^1 \mathcal{G}(\check{\mathtt{t}}, \xi)\mathtt{f}(\xi, \mathrm{a}(\xi))d\xi \quad \text{for all } \check{\mathtt{t}} \in [0,1].$$

Consider

$$\begin{aligned} \mid \mathcal{S}\mathrm{a}(y) - \mathcal{S}\mathrm{b}(y) \mid^2 &= \left| \int_0^1 \mathcal{G}(y,\xi)\mathtt{f}\,(\xi, \mathrm{a}(\xi))\, d\xi - \int_0^1 \mathcal{G}(y,\xi)\mathtt{f}\,(\xi, \mathrm{b}(\xi))\, d\xi \right|^2 \\ &\leq \int_0^1 |\mathcal{G}(y,\xi)\,(\mathtt{f}\,(\xi, \mathrm{a}(\xi)) - \mathtt{f}\,(\xi, \mathrm{b}(\xi)))\, d\xi|^2 \end{aligned}$$

$$\begin{aligned} &\leq \int_0^1 |\mathcal{G}(y,\xi)|^2 \left(\frac{\mid (\mathtt{f}\,(\xi, \mathrm{a}(\xi)) - \mathtt{f}\,(\xi, \mathrm{b}(\xi))) \mid}{\alpha(\mathrm{a},\mathrm{b})^{\frac{1}{2}}} \right)^2 d\xi \\ &\leq \int_0^1 |\mathcal{G}(y,\xi)|^2 \frac{e^{-\check{\tau}}}{\alpha(\mathrm{a},\mathrm{b})} \mathbb{J}(\mathrm{a},\mathrm{b}) d\xi \\ &\leq \int_0^1 |\mathcal{G}(y,\xi)|^2 \frac{e^{-\check{\tau}}}{\alpha(\mathrm{a},\mathrm{b})} \max\left\{ \mid \mathrm{a}-\mathrm{b} \mid, \mid \mathrm{a} - \mathcal{S}\mathrm{a} \mid, \mid \mathrm{b} - \mathcal{S}\mathrm{b} \mid, \frac{\mid \mathrm{a} - \mathcal{S}\mathrm{b} \mid + \mid \mathrm{b} - \mathcal{S}\mathrm{a} \mid}{2\mathtt{b}} \right\} d\xi \\ &\leq \frac{e^{-\check{\tau}}}{\alpha(\mathrm{a},\mathrm{b})} \max\left\{ \wp(\mathrm{a},\mathrm{b}), \wp(\mathrm{a},\mathcal{S}\mathrm{a}), \wp(\mathrm{b},\mathcal{S}\mathrm{b}), \frac{\wp(\mathrm{a},\mathcal{S}\mathrm{b}) + \wp(\mathrm{b},\mathcal{S}\mathrm{a})}{2\mathtt{b}} \right\} \left(\int_0^1 (\mathcal{G}(y,\xi))\, d\xi \right)^2 \\ &\leq \frac{e^{-\check{\tau}}}{\alpha(\mathrm{a},\mathrm{b})} \max\left\{ \wp(\mathrm{a},\mathrm{b}), \wp(\mathrm{a},\mathcal{S}\mathrm{a}), \wp(\mathrm{b},\mathcal{S}\mathrm{b}), \frac{\wp(\mathrm{a},\mathcal{S}\mathrm{b}) + \wp(\mathrm{b},\mathcal{S}\mathrm{a})}{2\mathtt{b}} \right\} \times \sup_{y\in[0,1]} \left(\int_0^1 (\mathcal{G}(y,\xi))\, d\xi \right)^2. \end{aligned}$$

As

$$\sup_{y\in[0,1]} \left(\int_0^1 (\mathcal{G}(y,\xi))\, d\xi \right) \leq 1.$$

It follows that

$$\mid \mathcal{S}\mathrm{a}(y) - \mathcal{S}\mathrm{b}(y) \mid^2 \leq \frac{e^{-\check{\tau}}}{\alpha(\mathrm{a},\mathrm{b})} \mathcal{N}(\mathrm{a},\mathrm{b})$$

$$\implies \alpha(\mathrm{a},\mathrm{b}) \mid \mathcal{S}\mathrm{a}(y) - \mathcal{S}\mathrm{b}(y) \mid^2 \leq e^{-\check{\tau}},$$

where

$$\mathcal{N}(\mathrm{a},\mathrm{b}) = \max\left\{\wp_{\mathtt{b}}(\mathrm{a},\mathrm{b}), \wp_{\mathtt{b}}(\mathrm{a},\mathcal{S}\mathrm{a}), \wp_{\mathtt{b}}(\mathrm{b},\mathcal{S}\mathrm{b}), \frac{\wp_{\mathtt{b}}(\mathrm{a},\mathcal{S}\mathrm{b}) + \wp_{\mathtt{b}}(\mathrm{b},\mathcal{S}\mathrm{a})}{2\mathtt{b}}\right\}.$$

Hence , $\forall$ $\mathrm{a},\mathrm{b} \in \mathbb{J}$ and $\forall$ $y \in [0,1]$,

we have

$$\alpha(\mathrm{a},\mathrm{b})\wp_{\mathtt{b}}(\mathcal{S}\mathrm{a} - \mathcal{S}\mathrm{b}) \leq \left(e^{-\check{\tau}}\mathcal{N}(\mathrm{a},\mathrm{b})\right).$$

Using natural log on both sides, we have

$$\ln\left(\alpha(\mathrm{a},\mathrm{b})\wp_{\mathtt{b}}(\mathcal{S}\mathrm{a} - \mathcal{S}\mathrm{b}) \leq \ln\left(\mathcal{N}(\mathrm{a},\mathrm{b})\right)\right) - \check{\tau}.$$

Hence

$$\check{\tau} + \mathcal{F}(\alpha(\mathrm{a},\mathrm{b})\wp_{\mathtt{b}}(\mathcal{S}\mathrm{a},\mathcal{S}\mathrm{b})) \leq \mathcal{F}(\wp_{\mathtt{b}}(\mathcal{N}(\mathrm{a},\mathrm{b}))).$$

Mapping $\mathcal{S}$ is with $\mathcal{F}(\mathrm{y}) = \ln(\mathrm{y})$. According to the Corollary 4.2.6 $\mathcal{S}$ having fixed point a in $\mathcal{C}[0,1]$, $\ni$

$$\begin{aligned} \mathcal{S}(\mathrm{a}(\grave{t})) &= \mathrm{a}(\grave{t}) \\ &= \int_0^1 \mathcal{G}(\grave{\mathtt{t}},\xi)\mathtt{f}(\xi,\mathrm{a}(\xi))d\xi. \quad \forall\ \grave{\mathtt{t}} \in [0,1] \end{aligned}$$

which is the solution to (4.14) □

Chapter 5

Conclusions

The primary source of inspiration in the current dissertation is Wardowski's work, which grants the proposal of $\mathcal{F}$-contraction. The abridgement of the thesis dissertation is stated below:

• We give precise definitions and examples to enlighten the essential notions of metric spaces. We also go through several forms of mappings, fixed points, and fixed-point theorems. A brief background is provided for an explanation of fixed-point theory.

• A comprehensive review of the article by Jain et al. [23] is provided. This review explained the idea of Multivalued $\mathcal{F}$-contraction based on altering distance function. Some non-trivial examples are provided for the authentication of the main theorem. The existence of a solution to the ordinary differential equation and Caputo-type fractional differential equation is established by using the fixed-point technique.

• The work of Jain et al. [23] is further extended by using the platform of $\mathtt{b}$-metric space. Furthermore, the multivalued $\mathcal{F}$-contraction is generalized to multivalued $(\alpha\mathcal{F}, \mathtt{b}, \tilde{\phi})$-contraction.

The following strategy is adopted for this purpose:

• Construct an iterative sequence;

• Proof of the fact that this sequence is Cauchy is provided;

• Existence of the BBP is established.

• The established results generalize many existing results in the literature. This fact is assured by providing several corollaries. Several examples are given to validate the assumptions of the theorems.

• The existence of the solution to the ordinary differential equation and Caputo-type fractional differential equation is established using the proven results' axioms.

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