CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

$(\alpha - F - \mathbb{P}_{\triangleleft})$ Suzuki Contraction Mappings on Orthogonal $\mathfrak{b}$-metric Spaces

by

Shabana Kanwal

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

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Dedicated to my family

CERTIFICATE OF APPROVAL

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Abstract

Beg et al. introduced the concept of generalized orthogonal F- Suzuki contraction mapping and proved some fixed point theorems on orthogonal $\mathfrak{b}$-metric spaces. The results given in this article extend some of the well-known results in existing literature. The authors applied the proven result and showed the existence of a unique solution to the first-order ordinary differential equation. In this thesis, we further generalized the contraction condition of Beg et al. by incorporating the α function. Two theorems are established in this research. The first theorem establishes a unique fixed point for $(\alpha - F - \mathbb{P}_{\triangleleft})$ Suzuki contraction. In the second theorem, the existence of a unique fixed point is proved using the orthogonal continuity. One non-trivial example is provided for the validation of proven theorems. Several corollaries are elaborated to exhibit the fact that many existing fixed point results are the special case of those proved in the present research. Furthermore, our results generalize the results proved by Beg et al.

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Abbreviations

ms	Metric Space
$\mathcal{C}ms$	Complete Metric Space
$\mathfrak{b}ms$	$\mathfrak{b}$-Metric Space
$\mathcal{C}\mathfrak{b}ms$	Complete $\mathfrak{b}$-Metric Space
$\mathcal{OC}ms$	Orthogonal Complete Metric Space
$\mathcal{OC}\mathfrak{b}ms$	Orthogonal Complete $\mathfrak{b}$-Metric Space
$\mathcal{F}$-$\mathcal{S}cm$	F- Suzuki Contraction Mapping
α-$\mathcal{F}$-$\mathcal{S}cm$	α- F- Suzuki Contraction Mapping
BCP	Banach Contraction Principle

Symbols

$\mathbb{N}$	Natural Number
$\mathbb{R}$	Real Number
$\Re$	Metric Function
$\mathcal{H}$	Non-empty space
$\mathbb{P}$	Mapping
$\lhd$	Orthogonal Relation
$\mathfrak{S}$	Family of F-Mappings
$\exists$	There exist
$\forall$	For all

Chapter 1

Introduction

Mathematics is the most powerful tool in the history of science and technology. As the mother of all sciences, it has played a vital role in solving daily life problems. In the complexities of this world, mathematics provides the vital tools to find the solutions to problems.Functional analysis is an essential branch of mathematics that arose in the early 19th century. This branch is the study of linear operators of infinite dimensional normed s p aces. Functional analysis originates in the theory of ordinary and partial differential e q uations, a n d i t s t arted t o f o rm a discipline of its own through integral equations in early decades the 20th century. Functional analysis was first u s ed i n b e havior a n alysis b y B . F . S k inner i n 1948. Some of the crucial concepts of functional analysis are the Hahn-Banach theo-rem, the uniform boundedness principle, and the open mapping principle. Metric spaces (a generalization of distance functions) are important in fixed p oint theory. Fixed point theory is the combination of topology, geometry, and analysis. This theory has a great role in finding u n ique s o lutions i n t h e d ifferential an d integral equations theory.

Fixed point theory is productive and progressive work to solve non-linear problems. Fixed point theory was initiated in 1866 by Poincare [1], and he may rightly be considered a pioneer as he gave his first fi xed po int th eorem wi thout it s proof. Afterward, in 1912, Brouwer [2] proved a fixed p oint t heorem o n t he u nit sphere, which confirmed t h e e x istence o f fi xe d po in t, an d it is st at ed as on e of th e early approaches that Kakutani [3] further explored to prove the results on set valued mappings. Fixed point theory is used as a technique of successive approximation, which helps us in three ways:

(i) To guarantee the existence of solutions for non-linear problems.

(ii) To establish the uniqueness of the solution.

(iii) To establish an iterative scheme and conclude that the fixed point is exactly the limit of the iterative sequence.

The appearance of the fixed-point theory that started in the later part of the 19th century was used for successive approximation to search out the existence and detection of a unique solution of differential equations. This methodology is linked with the very prolific mathematicians like Fredholm, Voltera, Liouville, Cauchy, and particularly the works of Picard, search out the existence of a unique differential equation solution.

In the history of mathematics the Banach's fixed point theorem, also known as the contraction mapping theorem (BCP), is an essential break through for the researchers working on metric fixed point theory. It demonstrates the accessibility and uniqueness of fixed points of self-maps of metric spaces and an effective technique for finding those fixed points. It can be expressed as an abstract form of Picard's method of successive approximations. Stefan Banach (1892-1945) is recognized for introducing the theorem in 1922.

A standard application of the theorem is in the existence and uniqueness of solutions to certain ordinary differential equations by using the ingredients used in the Banach contraction principle: a complete space and a suitable integral operator. The renowned theorem guarantees the existence and uniqueness of the integral equations.

This principle occupies a significant part of the field of functional analysis. Afterward, the Banach contraction principle has been extended in various directions. Edelstein [4] gave the first generalization of the Banach contraction condition in 1962 by taking constant $k = 1$ and using distinct points from the space $\mathcal{H}$. In the same year, Rakotch [5] established a contractive condition by replacing the constant k of the contraction with a monotonic decreasing function $k : [0, \infty) \longrightarrow [0, 1]$. Presic [6], Kannan [7], Keeler *et al.* [8] worked on BCP by altering the contraction condition. Fomin [9] and Gupta [10] introduced a rational expression and extended the Banach contraction principle; later on, this result was extended by Dolhare[11].

The idea of $\mathfrak{b}$-metric space was given by Bourbaki [12] and Bakhtin [13]. Czer-wik [14] also gave the axiom of $\mathfrak{b}$-metric space, which was weaker than triangular inequality and formally defined $\mathfrak{b}$ -metric s pace w ith a v iew o f g eneralizing t he Ba-nach contraction theorem.

He introduced a constant k in the triangular inequality, and for the case when $k = 1$, $\mathfrak{b}$-metric space is a metric space. Generally, this concept is weaker than the metric spaces. Czerwik was keen to examine more closely the topological aspects of the spaces.

In 2012, Wardowski [15] presented another well-known contraction, F-contraction. Sagroi *et al.* [16] proved fixed p o int r e sults o n F- c ontraction i n 2 0 13 w i th some applications on integral equations. Alot of work is done in this area; see for ex-amples,[17–24].

F-contraction was further generalized in many ways, for existence an generaliza-tion of F-contraction is (α, F)-contractive mapping. (α, F)-contractive mapping was firstly i ntroduced b y K a mran *e t a l .* [1 9] i n 2 0 16 i n t h e s t ructure o f $\mathfrak{b}$-metric space on single valued mappings. In 2017, this contraction was further extended to multi-valued by Hussain *et al.* [25]. Recently, Sawangsup *et al.* [26] established a new notion of orthogonal F contraction map and proved certain fixed p oint the-orems for the orthogonal complete metric space. Gordji [27], and others introduce the new concept of orthogonality in metric spaces. After that, the fixed-point results in generalized orthogonal metric spaces were demonstrated by Gordji and Habibi [28]. These concepts are further generalized by Beg *et al.* [29] by devel-oping a new idea on the generalized orthogonal F Suzuki contraction map on the orthogonal $\mathfrak{b}$-metric space.

The research in this thesis is in continuation of this studies. In this thesis, the results of Beg *et al.* [29] are further generalized by introducing generalized $\alpha - F$-contraction on the platform of $\mathfrak{b}$-metric spaces. The organization of the thesis is given below:

Chapter 2 covers all the fundamental concepts of functional analysis, which are necessary for the subsequent discussion. The crucial concepts related to metric spaces are explained along with examples. Some important fixed-point results are presented, which provide a base for the main results.

Chapter 3 gives a comprehensive review of the article "Fixed points of orthogonal F-Suzuki contraction mapping on O-Complete $\mathfrak{b}$-metric Spaces using applications" by Beg *et al.* [29]. The theorems are well elaborated, along with examples.

In Chapter 4, the results of Beg *et al.* are further generalized by introducing a new contraction condition, namely generalized $(\alpha - F)$ Suzuki contraction mapping (α-F-$\mathbb{P}_{\triangleleft}$) $\mathcal{S}cm$. It is worth mentioning that the results of Beg *et al.* are a special case of those provided in the present research. One non-trivial example is provided in support of proven theorems. Many existing results are the special case of results given in Chapter 4. Some corollaries authenticate this fact.

Chapter 2

Prelimineries

This chapter covers the fundamental concepts of functional analysis. It introduces the concept of metric space (ms) and $\mathfrak{b}$-metric space($\mathfrak{b}ms$), along with some examples. The chapter also covers different types of mappings, providing appropriate examples to illustrate each concept. Eventually, some classical fixed point results are provided for better understanding of main result.

2.1 Metric Space(ms)

In 1906, M. Frechet introduced the concept of ms, which extenteds the notion of distance to a more general setting. These spaces became an important connection between the fields of topology and real analysis, and helped establish the idea of metric fixed points. Metric spaces provided a framework for addressing many mathematical problems and were instrumental in resolving a number of issues in these fields.

Definition 2.1. Metric Space(ms)
"A Metric Space(ms) is a pair $(\mathcal{H}, \Re)$, where $\mathcal{H}$ is a set and $\Re$ is a metric on $\mathcal{H}$ (or distance function on $\mathcal{H}$) , that is function defined on $\Re : \mathcal{H} \times \mathcal{H} \to \mathbb{R}^+$ such that for all $\mu_1, \mu_2, \mu_3 \in \mathcal{H}$ we have:
(p_1): $\Re$ is real-valued, finite and non-negative, $\Re(\mu, \nu) > 0 \qquad \forall\ \mu, \nu \in \mathcal{H}$,

(p_2): $\Re(\mu_1, \mu_2) = 0$, if and only if $\mu_1 = \mu_2$,

(p_3): $\Re(\mu_1, \mu_2) = \Re(\mu_2, \mu_1)$ (symmetry),

(p_4): $\Re(\mu_1, \mu_2) \leq \Re(\mu_1, \mu_3) + \Re(\mu_3, \mu_2)$, (triangular inequality).

Then $(\mathcal{H}, \Re)$ is called a ms then $\mathcal{H}$ is called underlying set. Its elements are called points." [30]

Example 2.1.1.

Consider $\mathbb{P} = \mathbb{R}^2$, Let $\mu, \nu \in \mathbb{P}$,

where $\mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2) \in \mathbb{R}^2$.

Define $\Re : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ as,

$$\Re((\mu, \nu) = \max\{|\mu_1 - \mu_2|, |\nu_1 - \nu_2|\},$$

then $\Re$ is a metric on $\mathbb{R}^2$ and it is also called box metric. Now check the fourth property since first three are trivial. Let $(\mu_1, \nu_1), (\mu_2, \nu_2) \in \mathbb{R}^2$.

Then,

$$|\mu_1 - \mu_2| \leq |\mu_1 - \mu_3| + |\mu_3 - \mu_2|,$$

and,

$$|\nu_1 - \nu_2| \leq |\nu_1 - \nu_3| + |\nu_3 - \nu_2|,$$

by the triangular inequality.

$$|\mu_1 - \mu_3| + |\mu_3 - \mu_2| \leq \max\{|\mu_1 - \mu_3|, |\nu_1 - \nu_3|\} + \max\{|\mu_3 - \mu_2|, |\nu_3 - \nu_2|\},$$

and similarly for,

$$|\nu_1 - \nu_3| + |\nu_3 - \nu_2|,$$

both,

$$|\mu_1 - \mu_2|, |\nu_1 - \nu_2|,$$

are smaller then or equal to,

$$\Re((\mu_1, \nu_1), (\mu_3, \nu_3)) + \Re((\mu_3, \nu_3), (\mu_2, \nu_2)),$$

so their maximum $\Re((\mu_1, \nu_1), (\mu_2, \nu_2))$ is as well. So we conclude that $\Re$ is a metric on $\mathbb{R}^2$.

Example 2.1.2.

Consider a space of all (bounded and unbounded) sequences of complex numbers and the metric $\Re$ defined by,

$$\Re(\mu, \nu) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\Upsilon_i - \varsigma_i|}{1 + |\Upsilon_i - \varsigma_i|}, \qquad \text{where} \quad \mu = \{\Upsilon_i\} \quad \textit{and} \quad \nu = \{\varsigma_i\}.$$

Axioms (p_1) to (p_3) are trivially satisfied. Now verify the triangle inequality. For this purpose we use the auxiliary function f defined on $\mathbb{R}$.

$$f(c) = \frac{c}{1+c}, \qquad \Rightarrow f'(c) = \frac{1}{(1+c)^2}.$$

which is positive. Hence, f is monotone increasing. Consequently, using the result,

$$|\mu + \nu| \leq |\mu| + |\nu|; \qquad \Rightarrow f|\mu + \nu| \leq f(|\mu| + |\nu|).$$

$$\Rightarrow \frac{|\mu + \nu|}{1 + |\mu + \nu|} \leq \frac{|\mu| + |\nu|}{1 + |\mu| + |\nu|};$$

$$= \frac{|\mu|}{1 + |\mu| + |\nu|} + \frac{|\nu|}{1 + |\mu| + |\nu|};$$

$$\leq \frac{|\mu|}{1 + |\mu|} + \frac{|\nu|}{1 + |\nu|}.$$

In this inequality we let $\mu = \Upsilon_i - \varrho_i \quad \textit{and} \quad \nu = \varrho_i - \varsigma_i, \textit{where} \quad \vartheta = (\varrho_i)$. Then $\mu + \nu = \Upsilon_i - \varsigma_i$ and we have,

$$\frac{|\Upsilon_i - \varsigma_i|}{1 + |\Upsilon_i - \varsigma_i|} \leq \frac{|\Upsilon_i - \varrho_i|}{1 + |\Upsilon_i - \varrho_i|} + \frac{|\varrho_i - \varsigma_i|}{1 + |\varrho_i - \varsigma_i|}.$$

If we multiply both sides by $\frac{1}{2^i}$ and sum over i from 1 to ∞, we obtain, $\Re(\mu,\nu)$ on the left and the sum of $\Re(\mu,\vartheta)$ and $\Re(\vartheta,\nu)$ on the right:

$$\Re(\mu,\nu) \le \Re(\mu,\vartheta) + \Re(\vartheta,\nu).$$

This establishes (p_4) and hence, $\Re$ is a metric.

Example 2.1.3.

Define a metric $\Re$ on C[a,b] by,

$$\Re(\mu,\nu) = \int_a^b |\mu(s) - \nu(s)| d(s).$$

In order to show that $\Re$ is a metric on given space. Define metric

$$\Re(\mu - \nu) = \int_a^b |\mu(s), \nu(s)| d(s), \qquad \forall\ s \in C[a,b], a < b.$$

(p_1): $|\mu(s) - \nu(s)| \ge 0 \Rightarrow \int_c^d |\mu(s) - \nu(s)| d(s) \ge 0$
$\Rightarrow \Re(\mu,\nu) \ge 0,$
(p_2): $\Re(\mu,\nu) = \int_a^b |\mu(s) - \nu(s)| d(s) = 0,$
$\Leftrightarrow |\mu(s) - \nu(s)| = 0 \Leftrightarrow \mu(s) - \nu(s) = 0,$
$\Leftrightarrow \mu(s) = \nu(s) \Leftrightarrow \mu = \nu, \quad \forall, s \in C[a,b],$
(p_3): $\Re(\mu,\nu) = \int_a^b |\mu(s) - \nu(s)| d(s) = \int_a^b |\nu(s) - \mu(s)| d(s) = \Re(\nu,\mu),$
(p_4): let $\mu,\nu,\vartheta \in C[a,b]$ then $|\mu(s) - \nu(s)| = |\mu(s) - \vartheta(s) + \vartheta(s) - \nu(s)|,$
$\Rightarrow |\mu(s) - \nu(s) \le |\mu(s) - \vartheta(s)| + |\vartheta(s) - \nu(s),$
$\Rightarrow \int_a^b |\mu(s) - \nu(s)| d(s) \le \int_a^b |\mu(s) - \vartheta(s)| d(s) + |\vartheta(s) - \nu(s)| d(s),$
$\Rightarrow \Re(\mu,\nu) \le \Re(\mu,\vartheta) + \Re(\vartheta,\nu),$
$\Re$ is a metric on given space.

Definition 2.2. Convergent sequence

"A sequence $\{a_n\}$ in a metric space $\mathcal{H} = (\mathcal{H},\Re)$ is said to converge or to be convergent if there is an $a \in \mathcal{H}$ such that,

$$\lim_{n\to\infty} \Re(a_n, a) = 0,$$

a is called the limit of (a_n) and we write,

$$\lim_{n\to\infty} \Re(a_n, a) = a,$$

or simply, $a_n \to a$." [30]

Definition 2.3. Cauchy Sequence

"A sequence $\{a_n\}$ in a ms $\mathcal{H} = (\mathcal{H}, \Re)$ is said to be Cauchy sequence if every $\epsilon \geq 0$ there is an $\mathbb{N} = \mathbb{N}(\epsilon)$ such that

$$\Re(a_m, a_n) < \epsilon \qquad \forall\ m, n > \mathbb{N}." \quad [30]$$

Example 2.1.4.

Let $(\mathbb{R}, \Re)$ be a usual ms. Consider a sequence $\{a_n\} = \{\frac{n}{n+1}\}$ in $\mathbb{R}$. For every, $\epsilon > 0$, choose $N \in \mathbb{N}$ such that, $\frac{1}{N} < \frac{\epsilon}{2}$. Now if $n, m > N$,

$$\Rightarrow |\frac{n}{n+1} - \frac{m}{m+1}|,$$

$$= |\frac{m+1-n-1}{(n+1)(m+1)}| \leq |\frac{m-n}{mn}| < \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\{a_n\}$ is a Cauchy sequence.

Definition 2.4. Complete Metric Space($\mathcal{C}ms$)

"The ms $\mathcal{H}$ is said to be complete if every Cauchy sequence in $\mathcal{H}$ converges (in $\mathcal{H}$,has a limit which is an element of $\mathcal{H}$)." [30]

Example 2.1.5.

(a): Euclidean space $(\mathbb{R}^n)$ and unitary space $(\mathbb{C}^n)$ are $\mathcal{C}ms$.

(b): The space ℓ^∞ is $\mathcal{C}ms$ with the metric $\Re$ defined as,

$$\Re(\mu, \nu) = \sup_i |\Upsilon_i - \varsigma_i|, \qquad \mu = \{\Upsilon_i\}, \nu = \{\varsigma_i\}.$$

(c): The space l^p is $\mathcal{C}ms$ with the metric define as:

$$\Re(\mu_m, \mu_n) = (\sum_{j=1}^{\infty} |\Upsilon_j^m - \Upsilon_j^n|^p)^{\frac{1}{p}}.$$

2.2 $\mathfrak{b}$-Metric Space ($\mathfrak{b}ms$)

In 1989 Bakhtin gave the first generalization of ms namely $\mathfrak{b}ms$. Later on, Czerwik proved some fixed point results on this platform. Since then, many researchers have worked on $\mathfrak{b}ms$ and achieved impressive results.

Definition 2.5. $\mathfrak{b}$-Metric Space ($\mathfrak{b}ms$)

" Let $\mathcal{H}$ be a non empty set and k be any real number such that $k \geq 1$. A function $\Re_{\mathfrak{b}}$: $\mathcal{H} \times \mathcal{H} \to [0, \infty)$ is called a $\mathfrak{b}ms$, if it satisfies the following properties for all $\mu, \nu, \vartheta \in \mathcal{H}$.

(p_1): $\Re_{\mathfrak{b}}(\mu, \nu) \geq 0$;

(p_2): $\Re_{\mathfrak{b}}(\mu, \nu) = 0$ if and only if $\mu = \nu$;

(p_3): $\Re_{\mathfrak{b}}(\mu, \nu) = \Re_{b}(\nu, \mu)$ for all $\mu, \nu \in \mathcal{H}$;

(p_4): $\Re_{\mathfrak{b}}(\mu, \nu) \leq k[\Re_{\mathfrak{b}}(\mu, \vartheta) + \Re_{\mathfrak{b}}(\vartheta, \nu)]$.

The pair $(\mathcal{H}, \Re_{\mathfrak{b}})$ is called a $\mathfrak{b}ms$. [14]

Remark 2.6.

If k= 1, then $\mathfrak{b}ms$ becomes ms. We can easily conclude that class of $\mathfrak{b}$-metric Spaces is bigger then the class of metric Spaces."

Example 2.2.1.

Let $\mathcal{H} := Ł_p$ of [0,1] be the space of all real functions $\mu(s), s \in [0, 1]$ such that,

$$\int_0^1 |\mu(s)|^p < \infty \quad \text{with } 0 < p < 1.$$

Define $\Re : \mathcal{H} \times \mathcal{H} \to \mathbb{R}^+$,

$$\Re(\mu, \nu) = (\int_0^1 |\mu(s) - \nu(s)|^p d(s))^{\frac{1}{p}}.$$

Then $\Re$ is a $\mathfrak{b}ms$ with coefficient with $\mathfrak{b} = 2^{\frac{1}{p}}$.

Definition 2.7. Convergence of Sequence

"Let $(\mathcal{H}, \Re_{\mathfrak{b}})$ be a $\mathfrak{b}ms$. A sequence $\{a_n\}$ in $\mathcal{H}$ is said to be convergent, if there exist $a \in \mathcal{H}$ such that,

$$\lim_{n\to\infty} \Re_{\mathfrak{b}}(a_n, a_s) = 0,$$

a is called limit of a_n we write,

$$\lim_{n\to\infty} a_n = a \quad \text{or} \quad a_n \to a.\text{"}[31]$$

Definition 2.8. Cauchy Sequence in $\mathfrak{b}ms$

"Let $(\mathcal{H}, \Re_{\mathfrak{b}})$ be a $\mathfrak{b}$-metric Spaces. A sequence $\{a_n\}$ in $\mathcal{H}$ is said to be a Cauchy sequence, if for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that,

$$\Re_{\mathfrak{b}}(a_m, a_n) < \epsilon, \qquad \text{for every } m, n \geq N.\text{"} \quad [31]$$

Definition 2.9. Complete $\mathfrak{b}$-Metric Space ($\mathcal{C}\mathfrak{b}ms$)

"A $\mathfrak{b}$-metric Space $(\mathcal{H}, \Re_{\mathfrak{b}})$ is said to be a complete, if every Cauchy sequence in $\mathcal{H}$ is convergent in $\mathcal{H}$." [31]

It is worth to mention that $\mathfrak{b}$-metric is not a continuous function. Following example depicts this fact:

Example 2.2.2.

Let $\mathcal{H} = \mathbb{N} \cup \{\infty\}$ and $\Re : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, where

$$\Re(p,q) = \begin{cases} 0 & \text{for } p = q. \\ |\frac{1}{p} - \frac{1}{q}| & \text{for } p \text{ and } q \text{ are both even or } p \text{ is even and } q = \infty \text{ or } p = \infty \\ & \text{and } q \text{ is even.} \\ 8 & \text{for } p \text{ and } q \text{ are both odd or } p \text{ is odd and } q = \infty \text{ or } p = \infty \\ & \text{and } q \text{ is odd.} \\ 5 & \text{others.} \end{cases}$$

then $\Re$ is a $\mathfrak{b}$-metric on $\mathcal{H}$ with k= 3. We want to show that $\Re$ is discontinuous at $(\infty, 1) \in \mathcal{H} \times \mathcal{H}$.

Consider a sequence $\{2n\}$ in $\mathcal{H}$ then $2n \to \infty$, but

if we cheese $f(u) = \Re(2u, 1)$,

then

$$|f(2n) - f(\infty)|,$$

$$= |\Re(2n,1) - \Re(\infty,1)|,$$

$$= |5-8| = 3 \quad \Rightarrow f(2n) \to 3.$$

Hence $\Re$ is not continuous.

2.3 Mappings on Metric Spaces

This section is focused on presenting various types of mappings on *ms*. Each definition is accompanied by an appropriate example to help clarify the concept. Fixed point theorems are primarily concerned with identifying conditions on the structure and properties of underlying spaces. They also deals with the properties of self mapping $\mathcal{S}$ on $\mathcal{H}$ in order to increase fixed point outcomes. Thus this section includes the discussion of various conditions on mappings that are useful in this context.

Definition 2.10. Continuous Mapping

"Let $(\mathbb{P}, \Re_1)$ and $(\mathbb{T}, \Re_2)$ be two metric Spaces. A mapping $f : \mathbb{P} \to \mathbb{T}$ is continuous at a point $k_0 \in \mathbb{P}$, if for every $\epsilon > 0$ there is $\delta > 0$, such that,

$$\Re_2(fk, fk_0) < \epsilon \qquad \forall\ k \quad \Re_1(k, k_0) < \delta." \quad [30]$$

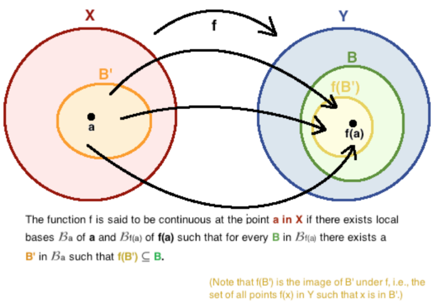

FIGURE 2.1: Continuous Mapping

Definition 2.11. Lipschitzian Mapping

"Let $(\mathcal{H}, \Re)$ be a metric Space. A mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is said to be Lipschitzian mapping if there exist a constant $\mu > 0$ with

$$\Re(\mathbb{P}(k), \mathbb{P}(u)) \leq \mu\Re(k, u) \quad \forall\ k, u \in \mathbb{P}".$$

The Lipschitzian constant for $\mathbb{P}$ is the smallest μ for which condition holds. [32]

Definition 2.12. Contraction Mapping

"let $\mathcal{H} = (\mathcal{H}, \Re)$ is a metric Space. A mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is called a contraction on $\mathcal{H}$ if there is a positive real number $k < 1$ such that, $\forall\ \mu, \nu \in \mathcal{H}$,

$$\Re(\mathbb{P}\mu, \mathbb{P}\nu) \leq k\Re(\mu, \nu)." \quad [30]$$

Example 2.3.1.

Let $\mathcal{H} = [0, 1]$ be a ms and $\Re(\varphi, \Upsilon) = |\varphi - \Upsilon|$.

Then define a mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ by

$$\mathbb{P}(\varphi) = \frac{1}{\varphi + 8}.$$

$$\Re(\mathbb{P}_\varphi, \mathbb{P}_\Upsilon) = |(\frac{1}{\varphi + 8}) - (\frac{1}{\Upsilon + 8})|,$$

$$\begin{aligned}
\Re(\mathbb{P}_\varphi, \mathbb{P}_\Upsilon) &\leq |\frac{\Upsilon + 8 - \varphi - 8}{(\varphi + 8)(\Upsilon + 8)}|, \\
&\leq |\frac{\Upsilon - \varphi}{(\varphi + 8)(\Upsilon + 8)}|, \\
&\leq |\frac{-(\varphi - \Upsilon)}{(\varphi + 8)(\Upsilon + 8)}|, \\
&\leq |\frac{\varphi - \Upsilon}{(\varphi + 8)(\Upsilon + 8)}|, \\
&\leq |\frac{\varphi - \Upsilon}{(8)(8)}|, \\
&\leq \frac{1}{64}|\varphi - \Upsilon|, \\
&\leq \frac{1}{64}\Re(\varphi, \Upsilon),
\end{aligned}$$

then $\mathbb{P}$ is a contraction with $k = \frac{1}{64} < 1$.

Definition 2.13. Contractive mapping

"A mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is said to be a contractive if for $k_1 \neq k_2$, we have,

$$\Re(\mathbb{P}(k_1), \mathbb{P}(k_2)) < \Re(k_1, k_2), \qquad \forall\ k_1, k_2 \in \mathbb{P}". \ [33]$$

2.4 Some Crucial Fixed Point Results

The field of fixed point theory emerged from the important work of Poincare in the last decade of 18th century and first decade of 19th century. Fixed point results become a valuable tool in many areas of mathematics and quantitative sciences, including economics, engineering, and many more. These results have been used to establish the existence of solutions to a wide range of problems, making them an important tool for researchers.

Definition 2.14. Fixed Point

"A fixed point of a mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ of a set $\mathcal{H}$ to itself is $\mu \in \mathcal{H}$ which is mapped on to itself that is,

$$\mathbb{P}(\mu) = \mu.$$

The image $\mathbb{P}(\mu) = \mu$ conincides with μ." [30]

Geometrically, the point of intersection of a real valued function $y = f(x)$ and the line $y = x$ is called fixed point. A function may or may not have a fixed point. Furthermore, if it has then the fixed point may not be unique.

Example 2.4.1.

1 : Define a mapping $f : \mathbb{R} \to \mathbb{R}$ by

$$f(\mu) = \mu^2 - 2,$$

Consider,

$$f(\mu) = \mu.$$

Hence, $\mu = -1, 2$ are fixed points of f.

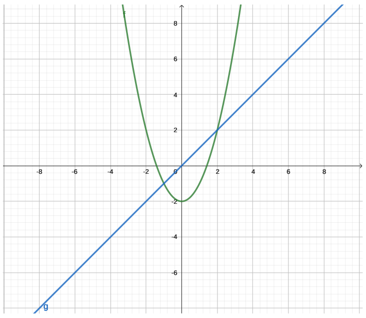

FIGURE 2.2: A graph having two fixed points.

2 : Consider the function

$$f(\mu) = \mu + 1;$$

it is obvious that f has no fixed point.

3 : A mapping $f\colon [0,1] \to [0,1]$ is define by

$$f(\mu) = 4\mu(1-\mu).$$

The fixed points of a function f are simply solution of $f(\mu) = \mu$. Hence fixed points of the function are $\mu = 0$ and $\mu = \frac{3}{4}$.

Geometrically, These are the points where $y = f(\mu)$ and the line $\nu = y$ meet.

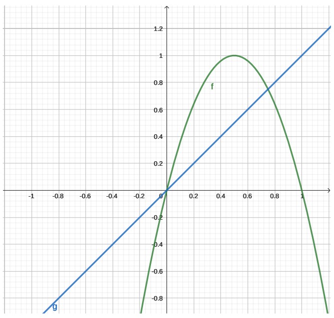

FIGURE 2.3: A graph having two fixed points.

4 : Consider the function

$$f(\mu) = \lfloor \mu \rfloor;$$

has infinitely many fixed point.

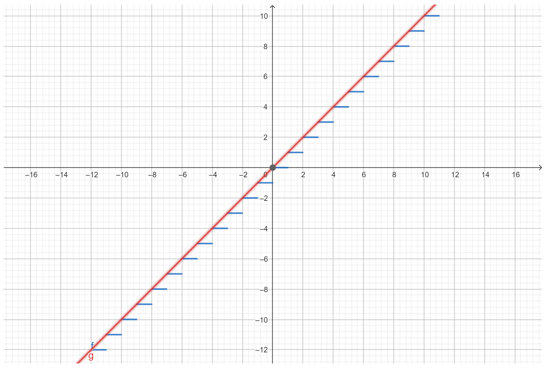

FIGURE 2.4: A graph having infinitely many fixed point.

The Brouwer fixed point theorem is the foundation for more general fixed point theorems that are essential in functional analysis. Brouwer fixed point theorem was one of the early successes of algebraic topology. Brouwer Fixed point theorem was stated and proven in 1912 by the Dutch mathematician L. E. J. Brouwer.

Theorem 2.4.2. Brouwer's Fixed Point Theorem

"Let $\mathbb{P}$ be a closed ball in $\mathbb{R}^n$. Then any continuous mapping $f : \mathbb{P} \to \mathbb{P}$ has atleast one fixed point". [33]

Juliusz Schauder established the theory in 1930 and established it for specific cases which include Banach spaces. His general case assumption was became known in the Scottish book.

Theorem 2.4.3. Schauder's Fixed Point Theorem

"Let $\mathbb{P}$ be a non empty compact convex subset of a Banach space $\mathcal{H}$, and suppose f: $\mathbb{P} \to \mathbb{P}$ is continuous. Then f has atleast one fixed point." [33]

Banach contraction principle theorem is named on the scientist Stefan Banach who firstly stated in 1922. This theorem is also known as contraction mapping theorem. It is considered as most important tool in the theory of ms. It guaranteed the existence and uniqueness of fixed point of self map ms. It can be formulated as an generated as an abstract formulation of Picard's method of successive approximation.

Theorem 2.4.4. Banach Fixed Point Theorem:

"Consider a metric space $\mathcal{H} = (\mathcal{H}, \Re)$, where $\mathcal{H} \neq \phi$. Suppose that $\mathcal{H}$ is complete and let $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ be a contraction on $\mathcal{H}$.Then $\mathbb{P}$ has precisely one fixed point." [30]

Chapter 3

$F_{\lhd\mathfrak{b}}$-Contraction Mapping

In order to demonstrate the fixed point theorems on $\mathcal{C}\mathfrak{b}ms$, Alsulami [34] developed the ideas of generalized F-Suzuki type contraction mapping. Gordji [27] developed the new idea of an orthogonality in ms and established the fixed point results for contraction mappings in ms with this new type of orthogonality.
Inspired by these two ideas Beg et al. [29] presented some fixed point results for F-Suzuki contraction ($\mathcal{F}$-$\mathcal{S}c$) on the platform of O-complete $\mathfrak{b}$-metric space ($\mathcal{OC}\mathfrak{b}ms$). This chapter provides a detailed discussion of this research.

3.1 Basic Concepts

Following is the definition of control function F introduced by Wardowski [15]:

Definition 3.1. F-mappings

Let $\mathfrak{S}$ denote the family of all function $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties.

(F_1) It is strictly increasing.

(F_2) For each sequence $\{a_n\}$ of positive numbers, we have,

$$\lim_{n\to\infty} \mu_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\mu_n) = -\infty.$$

Example 3.1.1.

The following mapping from $\mathbb{R}^+ \to \mathbb{R}$ are examples of F- mapping:

i : $F(\mu) = -\dfrac{1}{\sqrt{\mu}}$; where $\mu > 0$.

ii : $F(\mu) = \ln(\mu)$ where $\mu > 0$.

iii : $F(\mu) = \ln(\mu^2 + \mu)$ where $\mu > 0$.

Definition 3.2. F-Contraction Mapping

A mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$, is called F-contraction if there exist $\tau > 0$ such that, for all $\mu, \nu \in \mathcal{H}$,

$$\Re(\mu, \nu) > 0, \quad \Rightarrow \tau + F(\Re(\mathbb{P}\mu, \mathbb{P}\nu)) \leq F(\Re(\mu, \nu)). \tag{3.1}$$

Remark 3.3.

Obviously, by choosing $F(\mu) = \ln(\mu)$ and $\tau = \ln(\frac{1}{k})$, $k \in [0, 1]$ the conditions of (3.1) becomes the Banach contraction.

Example 3.1.2.

Let $\mathcal{H} = \{a_n : n \in \mathbb{N}\}$ where $a_n = \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ equipped with usual metric,

$$\Re(\mu, \nu) = |\mu - \nu| \quad \forall \ \mu, \nu \in \mathcal{H}.$$

Then $(\mathcal{H}, \Re)$ is complete ms.

Let $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ be defined as

$$\mathbb{P}a_n = \begin{cases} a_n & \text{for } n = 1, \\ a_{n-1} & \text{for otherwise.} \end{cases}$$

Obviously $\mathbb{P}$ is F-contraction but it is not Banach contraction. For $\tau = 1$ and $F(a) = a + \ln a$.

Definition 3.4. Orthogonal set

Let $\mathcal{H}$ be a non-empty set and $\lhd \subseteq \mathcal{H} \times \mathcal{H}$ a binary relation. If $\lhd$ satisfies the following conditions:

$$\exists\ a_0 \in \mathcal{H}\ (\forall\ a \in \mathcal{H}, a \lhd a_0) \text{ or } (\forall\ a \in \mathcal{H}, a_0 \lhd a),$$

then the set $\mathcal{H}$ is called orthogonal set and denoted by $(\mathcal{H}, \lhd)$.

Example 3.1.3.

Let $\mathcal{H}$ be a set of humans in world. Define a binary relation $\Re : \mathcal{H} \to \mathcal{H}$ such that $H_1 \lhd H_2$ if H_1 can donate blood to H_2.

According to this relation if a person whose blood group is O- then $H_0 \lhd H\ \ \forall, H \in \mathcal{H}$. This implies that $(\mathcal{H}, \lhd)$ is orthogonal set. Furthermore, if H_0 is a person with blood group AB+ then $H \lhd H_0\ \ \forall, H \in \mathcal{H}$. Following table depicts the all blood types, donors and receivers of each blood group.

Type	Can give blood to	Can receive blood from
$\mathcal{A}+$	$\mathcal{A}+$, $\mathcal{AB}+$	$\mathcal{A}+$, $\mathcal{A}-$, O+, O−
O+	O−, $\mathcal{A}+$, $\mathcal{B}+$, $\mathcal{AB}+$	O+, O−
$\mathcal{B}+$	$\mathcal{B}+$, $\mathcal{AB}+$	$\mathcal{B}+$, $\mathcal{B}-$, O+, O−
$\mathcal{AB}+$	$\mathcal{AB}+$	Everyone
$\mathcal{A}-$	$\mathcal{A}+$, $\mathcal{A}-$, $\mathcal{AB}+$, $\mathcal{AB}-$	$\mathcal{A}-$, O−
O−	Everyone	O−
$\mathcal{B}-$	$\mathcal{B}+$, $\mathcal{B}-$, $\mathcal{AB}+$, $\mathcal{AB}-$	$\mathcal{B}-$, O−
$\mathcal{AB}-$	$\mathcal{AB}+$, $\mathcal{AB}-$	$\mathcal{AB}-$, $\mathcal{B}-$, O−, $\mathcal{A}-$

Table 3.1: Blood Groups with Donors and Receivers

Definition 3.5. Orthogonal Sequence

Let $(\mathcal{H}, \lhd)$ be an orthogonal set. A sequence $\{a_n\}$ is called an orthogonal sequence if,

$$(\forall\ n \in \mathbb{N}, a_n \lhd a_{n+1}) \text{ or } (\forall\ n \in \mathbb{N}, a_{n+1} \lhd a_n).$$

Definition 3.6. Orthogonal Metric Space

The triple $(\mathcal{H}, \lhd, \Re)$ is called an orthogonal metric space if, $(\mathcal{H}, \lhd)$ is an orthogonal set and $(\mathcal{H}, \Re)$ is a ms.

Definition 3.7. Orthogonal Complete Metric Space ($\mathcal{OC}ms$)

Let $(\mathcal{H}, \lhd, \Re)$ be an orthogonal metric space. Then $\mathcal{H}$ is said to be a $\mathcal{OC}ms$ if every orthogonal Cauchy sequence converges in it.

Remark 3.8. Every complete metric space is O-complete but not conversely.

Following example verifies this fact.

Example 3.1.4.

Let $\mathcal{H} = \{\mu : 0 \leq \mu < 1\}$ and suppose that

$\mu \lhd \nu \Longleftrightarrow \mu \leq \nu \leq \frac{1}{2}$, or $\mu = 0$.

Then $(\mathcal{H}, \lhd)$ is an orthogonal set. In general, $\mathcal{H}$ with the usual ms is not a complete ms but it is in fact orthogonal complete. If $\{a_n\}$ is an arbitrary orthogonal Cauchy sequence of $\mathcal{H}$, then there exists a subsequence $\{a_{np}\}$ of $\{a_p\}$ with $\{a_{np}\} = 0 \ \forall\, n \geq 1$ or there exists a monotonic subsequence $\{a_{np}\}$ of $\{a_p\}$ that $a_{np} \leq \frac{1}{2} \ \ \forall\, n \geq 1$. Since $\{a_{pn}\}$ converges to a point $a \in [0, \frac{1}{2}] \subset \mathcal{H}$. We already know that every Cauchy sequence with a convergent subsequence is convergent. So $\{a_n\}$ is the convergent.

Definition 3.9. Orthogonal Continuous Mapping

Let $(\mathcal{H}, \lhd, \Re)$ be an orthogonal ms. Then map $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is said to be orthogonal continuous in $a \in \mathcal{H}$ if for each sequence $\{a_n\}$ in $\mathcal{H}$ with $\lim_{n\to\infty} a_n = a \Rightarrow \lim_{n\to\infty} \mathbb{P}(a_n) = \mathbb{P}(a)$. $\mathbb{P}$ is called $\lhd$-continuous on $\mathcal{H}$ if $\mathbb{P}$ is $\lhd$-continuous for each $a \in \mathcal{H}$.

Example 3.1.5.

Let $\mathcal{H} = \mathbb{R}$ and suppose that

$$\mu \lhd \nu \text{ if } \mu, \nu \in (n + \frac{1}{3}, n + \frac{2}{3}) \ \ \forall\, n \in \mathbb{Z},$$

or

$$\mu = 0.$$

It is convenient to prove that $(\mathcal{H}, \lhd)$ is an orthogonal set. There is a function f defined: $f : \mathcal{H} \to \mathcal{H}$ by

$$f(\mu) = [\mu] = m \ if \ \mu \in (m + \frac{1}{3}, m + \frac{2}{3}).$$

Then f is $\lhd$-continuous on $\mathcal{H}$. Here we note that if $\{\mu_p\}$ is an arbitrary sequence O in $\mathcal{H}$ then $\{\mu_p\}$ converges to $\mu \in \mathcal{H}$. Then the following possibilities apply:

Case(1)

If $\forall\ p,\ \mu_p = 0$ and $f(\mu_p) = 0 = f(\mu)$.

Case(2)

If $\mu_{p_0} \neq 0$ with some p_0 then $\exists\ q \in \mathbb{Z}$ s.t $\mu_p \in (q + \frac{1}{3}, q + \frac{2}{3}) \quad \forall\ p \geq p_0$. Hence $\mu \in (q + \frac{1}{3}, q + \frac{2}{3})$ and $f(\mu_p) = q = f(\mu)$.

Definition 3.10. Orthogonal Preserving

Let $(\mathcal{H}, \lhd)$ be an orthogonal set. A mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is called $\lhd$-preserving if $\mathbb{P}(p) \lhd \mathbb{P}(q)$ whenever $p \lhd q$.

Definition 3.11. F-Suzuki Contraction mapping ($\mathcal{F}$-$\mathcal{S}cm$)

Let $(\mathcal{H}, \Re)$ be a ms. A map $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is called a general $\mathcal{F}$-$\mathcal{S}cm$ if it exists, $\tau > 0$ such that, $\mu, \nu \in \mathcal{H}$ with $\mu \neq \nu$.

$$\frac{1}{2\mathrm{s}}\Re(\mu, \mathbb{P}\mu) < \Re(\mu, \nu),$$

$$\Rightarrow \tau + F(\Re(\mathbb{P}\mu, \mathbb{P}\nu))$$

$$\leq a_1 F(\Re(\mu, \nu)) + a_2 F(\Re(\mu, \mathbb{P})) + a_3 F(\Re(\nu, \mathbb{P}\nu)),$$

where $a_3 \in [0, 1)$ and $a_1, a_2 \in [0, 1]$ are real numbers with $a_1 + a_2 + a_3 = 1$.

Definition 3.12. Generalized $F_{\lhd\mathfrak{b}}$- Suzuki Contraction

Let $(\mathcal{H}, \lhd, \Re)$ be an orthogonal $\mathfrak{b}ms$ with constant $\mathrm{s} \geq 1$. Mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is called a generalized orthogonal F- Suzuki ($F_{\lhd\mathfrak{b}}$) contraction map on $(\mathcal{H}, \lhd, \Re)$ if there is $F \in \mathfrak{S}$ and $\tau > 0$ such that,

$$\forall\ a, b \in \mathcal{H} \text{ with } a \lhd b\ [\Re(\mathbb{P}a, \mathbb{P}b) > 0, \frac{1}{2\mathrm{s}}\Re(a, \mathbb{P}a) < \Re(a, b)$$

$$\Longrightarrow \tau + F(\Re(\mathbb{P}a, \mathbb{P}b)) \leq a_1 F(\Re(a, b)) + a_2 F(\Re(a, \mathbb{P}a)) + a_3 F(\Re(b, \mathbb{P}b))],$$

where $a_3 \in [0, 1)$ and $a_1, a_2 \in [0, 1]$ are real numbers with $a_1 + a_2 + a_3 = 1$.

If take $a_1 = 1$ and $a_2 = a_3 = 0$; it will gives the following definition.

Definition 3.13. $\mathbb{P}_{\lhd\mathfrak{b}}$-Contraction Mapping

Let $(\mathcal{H}, \lhd, \Re)$ be an orthogonal $\mathfrak{b}ms$ with constant $\mathrm{s} \geq 1$. A mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$

is said to be an orthogonal $\mathcal{F}$-$\mathcal{S}cm$ on $(\mathcal{H}, \lhd, \Re)$ if there are $F \in \mathfrak{S}$ and $\tau > 0$ such that,

$$\forall\ a, b \in \mathcal{H} \text{ with } a \lhd b\ [\Re(\mathbb{P}a, \mathbb{P}b) > 0, \frac{1}{2\mathrm{s}}\Re(a, \mathbb{P}a) < \Re(a, b)$$

$$\Longrightarrow \tau + F(\Re(\mathbb{P}a, \mathbb{P}b)) \leq F(\Re(a, b))].$$

3.2 Fixed Point for $F_{\lhd\mathfrak{b}}$-Contraction Mappings

Theorem 3.2.1.

Consider $(\mathcal{H}, \lhd, \Re)$ an $\mathcal{OC}\mathfrak{b}ms$ with an orthogonal element a_0 and a constant $\mathrm{s} \geq 1$. Suppose that $F \in \mathfrak{S}$; $\tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is a self map satisfying the following axioms:

(i) $\mathbb{P}$ is $\lhd$-preserving,

(ii) $\mathbb{P}$ is generalized $F_{\lhd\mathfrak{b}}$-contraction mapping,

then the sequence $\{\mathbb{P}^n a\}$ converges to a unique fixed point $m \in \mathcal{H}$ of $\mathbb{P}$.

Proof. Since, $(\mathcal{H}, \lhd)$ is an O-set,

$$\exists\ a_0 \in \mathcal{H} : (\forall\ a \in \mathcal{H},\ a \lhd a_0) \text{ or } (\forall\ a \in \mathcal{H},\ a_0 \lhd a).$$

It follows that

$$a_0 \lhd \mathbb{P}a_0 \text{ or } \mathbb{P}a_0 \lhd a_0.$$

Let

$$a_1 = \mathbb{P}a_0,\ a_2 = \mathbb{P}a_1 = \mathbb{P}^2 a_0 ......, a_{n+1} = \mathbb{P}a_n = \mathbb{P}^{n+1}a_0, \quad \forall\ n \in \mathbb{N} \cup \{0\}. \quad (3.2)$$

If $a_n = a_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$, so, then a_n is a fixed point of $\mathbb{P}$. If $a_n \neq a_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, then $\Re(\mathbb{P}a_n, \mathbb{P}a_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. $\Rightarrow \mathbb{P}$ preserves $\lhd$, we have,

$$a_n \lhd a_{n+1} \text{ or } a_{n+1} \lhd a_n, \qquad \forall\ n \in \mathbb{N} \cup \{0\}.$$

This means that $\{a_n\}$ is an orthogonal sequence. Because $\mathbb{P}$ is a generalized $F_{\lhd\mathfrak{b}}$-contraction mapping, we conclude that,

$$\frac{1}{2\mathrm{s}}\Re(a_n, \mathbb{P}a_n) < \Re(a_n, \mathbb{P}a_n), \quad \forall\ n \in \mathbb{N}. \tag{3.3}$$

Thus, in light of this theorem's hypothesis, we have

$$\tau + F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) \leq a_1 F(\Re(a_n, \mathbb{P}a_n)) \ + \ a_2 F(\Re(a_n, \mathbb{P}a_n)) \\ \ + \ a_3 F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)),$$

$$\Rightarrow \tau + (1 - a_3) F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) \leq (a_1 + a_2) F(\Re(a_n, \mathbb{P}a_n)). \tag{3.4}$$

Given that $a_1 + a_2 + a_3 = 1$, the inequality becomes

$$F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) \leq F(\Re(a_n, \mathbb{P}a_n)) - \frac{\tau}{a_1 + a_2} < F(\Re(a_n, \mathbb{P}a_n)).$$

Using (F_1), we determine that

$$\Re(a_{n+1}, \mathbb{P}a_{n+1}) = \Re(\mathbb{P}a_n, \mathbb{P}^2 a_n) < \Re(a_n, \mathbb{P}a_n), \ \ \forall\ n \in \mathbb{N}. \tag{3.5}$$

Thus, $\{\Re(a_n, \mathbb{P}a_n)\}_{n=1}^{\infty}$ is a decreasing sequence of real numbers that is bounded below.

$$\lim_{n\to\infty} \Re(a_n, \mathbb{P}a_n) = \delta = \inf\{\Re(a_n, \mathbb{P}a_n) : \ \forall\ n \in \mathbb{N}\}.$$

To prove $\delta = 0$, Suppose on $\delta > 0$. i.e, for every $\epsilon > 0$ there exists $p \in \mathbb{N}$, such that

$$\Re(a_p, \mathbb{P}a_p) < \delta + \epsilon.$$

$$\Rightarrow F(\Re(a_p, \mathbb{P}a_p)) < F((\delta + \epsilon).$$

However, we have

$$\frac{1}{2\mathrm{s}}\Re(a_p, \mathbb{P}a_p) < \Re(a_p, \mathbb{P}a_p),$$

from (3.3). Given that $\mathbb{P}$ is generalized $F_{\lhd\mathfrak{b}}$-contraction, we acquire,

$$\tau + F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) \leq a_1 F(\Re(a_p, \mathbb{P}a_p)) + a_2 F(\Re(a_p, \mathbb{P}a_p)) + a_3 F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)).$$

$$\Rightarrow \tau + (1-a_3)F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) \leq (a_1+a_2)F(\Re(a_p, \mathbb{P}a_p)).$$

$$\Rightarrow F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) \leq F(\Re(a_p, \mathbb{P}a_p)) - \frac{\tau}{a_1+a_2}. \tag{3.6}$$

Likewise, again by (3.3), gives

$$\frac{1}{2\mathrm{s}}\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p).$$

Due to the fact that $\mathbb{P}$ is generalized $F_{\lhd\mathfrak{b}}$-contraction, we observe that

$$\tau + F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p)) \leq a_1 F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) + a_2 F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) + a_3 F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p)).$$

$$\Rightarrow F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p)) \leq F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) - \frac{\tau}{a_1+a_2}, \tag{3.7}$$

$$\Rightarrow F(\Re(\mathbb{P}^2 a_P, \mathbb{P}^3 a_p)) \leq F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) - \frac{\tau}{a_1+a_2}.$$

$$\leq F(\Re(a_p, \mathbb{P}a_p)0 - \tfrac{2\tau}{a_1+a_2}.$$

Continuing in the same manner:

$$\begin{aligned} F(\Re(\mathbb{P}^n a_p, \mathbb{P}^{n+1} a_p)) &\leq F(\Re(\mathbb{P}^n a_p, \mathbb{P}^{n-1} a_p)) - \frac{\tau}{a_1+a_2}, \\ &\leq F(\Re(\mathbb{P}^{n-1} a_p,\ \mathbb{P}^{n-2} a_p)) - \frac{2\tau}{a_1+a_2}, \\ &\leq F(\Re(\mathbb{P}a_p, a_p)) - \frac{n\tau}{a_1+a_2} \\ &\leq F(\delta+\epsilon) - \frac{n\tau}{a_1+a_2}. \end{aligned}$$

Applying $\lim\limits_{n\to\infty}$ on both sides

$$\lim_{n\to\infty} F(\Re(\mathbb{P}^n a_p, \mathbb{P}^{n+1} a_p)) = -\infty.$$

(F_2) implies $\lim\limits_{n\to\infty} \Re(\mathbb{P}^n a_p, \mathbb{P}^{n+1} a_p) = 0$. Hence, there is $p_1 \in \mathbb{N}$ such that

$$\Re(\mathbb{P}^n a_p, \mathbb{P}^{n+1} a_p) < \delta, \quad \forall\ n \geq p_1,$$

$$\Rightarrow \Re(a_{p+n}, \mathbb{P}a_{p+n}) < \delta, \quad \forall\ n \geq p_1,$$

which contradicts the definition of δ. Hence

$$\lim_{n\to\infty} \Re(a_n, \mathbb{P}a_n) = 0. \tag{3.8}$$

To prove,

$$\lim_{n,p\to\infty} \Re(a_n, a_p) = 0.$$

Assume, on contrary, for $\epsilon > 0$ then exist two sequences of natural number $\{r(n)\}_{n=1}^{\infty}$ and $\{t(n)\}_{n=1}^{\infty}$.

$$r(n) > t(n) > n,\ \Re(a_{r(n)}, a_{t(n)}) \geq \epsilon,$$

$$\Re(a_{r(n)-1}, a_{t(n)}) < \epsilon, \quad \forall\ n \in \mathbb{N}. \tag{3.9}$$

Consider,

$$\begin{aligned} \Re(a_{r(n)}, a_{t(n)}) &\leq \mathrm{s}[\Re(a_{r(n)}, a_{r(n)-1}) + \Re(a_{r(n)-1}, a_{t(n)}] \\ &\leq \mathrm{s}\Re(a_{r(n)}, a_{r(n)-1}) + \mathrm{s}\epsilon \\ &= \mathrm{s}\Re(a_{r(n)-1}, \mathbb{P}a_{r(n)-1}) + \mathrm{s}\epsilon, \qquad \forall\ n \in \mathbb{N}, \end{aligned} \tag{3.10}$$

(3.8) implies that there is $p_2 \in \mathbb{N}$

$$\Re(a_{r(n)}, \mathbb{P}a_{r(n)}) < \epsilon, \quad \forall\ n > p_2. \tag{3.11}$$

(3.11), (3.10) $\Rightarrow$ way

$$\Re(a_{r(n)}, a_{t(n)}) < 2\mathrm{s}\epsilon, \quad \forall\ n > p_2.$$

$$\Rightarrow F(\Re(a_{r(n)}, a_{t(n)})) < F(2\mathrm{s}\epsilon), \quad \forall\ n > p_2. \tag{3.12}$$

Alternatively, one can easily obtain that

$$\frac{1}{2\mathrm{s}}\Re(a_{r(n)}, \mathbb{P}a_{r(n)}) < \frac{\epsilon}{2\mathrm{s}} < \epsilon \leq \Re(a_{r(n)}, a_{t(n)}), \quad \forall\ n > p_2. \tag{3.13}$$

Claim that,

$$\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)}) = \Re(a_{r(n)+1}, a_{t(n)+1}) > 0, \quad \forall\ n \in \mathbb{N}. \tag{3.14}$$

such that, applying contradiction $\exists\ p \geq \mathbb{N}$

$$\Re(a_{r(p)+1}, a_{t(p)+1}) = 0. \tag{3.15}$$

From (3.9), (3.13) and (3.15) it implies that

$$\epsilon \leq \Re(a_{r(p)}, a_{t(p)} \leq \Re(a_{r(p)}, a_{r(p)+1}) + \Re(a_{r(p)+1}, a_{t(p)})$$

$$\leq \Re(a_{r(p)}, a_{r(p)+1}) + \Re(a_{r(p)+1}, a_{t(p)+1}) + \Re(a_{t(p)+1}, a_{t(p)})$$

$$= \Re(a_{r(p)}, \mathbb{P}a_{r(p)}) + \Re(a_{r(p)+1}, a_{t(p)+1}) + \Re(a_{t(p)}, \mathbb{P}a_{t(p)})$$

$$< \tfrac{\epsilon}{2\mathrm{s}} + 0 + \tfrac{\epsilon}{2\mathrm{s}} = \tfrac{\epsilon}{\mathrm{s}},$$

this contradiction leads towards (3.14). Since $\mathbb{P}$ is $\lhd$-preserving, thus we have

$$a_{r(n)} \lhd a_{t(n)} \text{ or } a_{t(n)} \lhd a_{r(n)}.$$

$\mathbb{P}$ is generalized $F_{\lhd \mathfrak{b}}$-contraction, thus for any $n > p_2$.

$$\tau \ + \ F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) \leq a_1 F(\Re(a_{r(n)}, a_{t(n)}))$$

$$+a_2 F(\Re(a_{r(n)}, \mathbb{P}a_{r(n)})) \ + \ a_3 F(\Re(a_{r(n)}, \mathbb{P}a_{t(n)})). \tag{3.16}$$

(3.12) is applied into considered, and (3.16) outcomes.

$\tau \ + \ F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) < a_1 F(2\mathrm{s}\epsilon) \ + \ a_2 F(\Re(a_{r(n)}, \mathbb{P}a_{r(n)}))$

$\quad + \ a_3 F(\Re(a_{t(n)}, \mathbb{P}a_{t(n)})), \quad \forall\ n \in p_2.$

In term of (3.8), we obtain that

$$\lim_{n\to\infty} F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) = -\infty.$$

F_2, ensures that

$$\lim_{n\to\infty} F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) = 0 \Longleftrightarrow \lim_{n\to\infty} F(\Re(a_{r(n)+1}, a_{t(n)+1})) = 0. \tag{3.17}$$

This contradicts the (3.9). Therefore, $\lim_{p,n\to\infty} \Re(a_n, a_p) = 0$;
i.e $\{a_n\}_{n=1}^{\infty}$ where $\mathcal{H}$ is a Cauchy sequence.
$\exists\ m \in \mathcal{H}$ such that due to the $(\mathcal{H}, \Re)$ completeness.

$$\lim_{n\to\infty} \Re(a_n, m) = 0. \tag{3.18}$$

For any n, we say that it belongs to $\mathbb{N}$,

$$\frac{1}{2\mathrm{s}}\Re(a_n, \mathbb{P}a_n) < \Re(a_n, m) \quad \text{or} \quad \frac{1}{2\mathrm{s}}\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n) < \Re(\mathbb{P}a_n, m), \quad \forall\ n \in \mathbb{N}. \tag{3.19}$$

However, if $p \in \mathbb{N}$ exists, in such a way that

$$\frac{1}{2\mathrm{s}}\Re(a_p, \mathbb{P}a_p) \geq \Re(a_p, m),$$

$$\frac{1}{2\mathrm{s}}\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p) \geq \Re(\mathbb{P}a_p, m). \tag{3.20}$$

We have from (3.5) and (F_1),

$$\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p) < \Re(a_p, \mathbb{P}a_p). \tag{3.21}$$

From (3.20) ans (3.21) it is obvious that

$$\begin{aligned}
\Re(a_p, \mathbb{P}a_p) &\leq \mathrm{s}\Re(a_p, m) + \mathrm{s}\Re(m, \mathbb{P}a_p) \\
&\leq \frac{1}{2}\Re(a_p, \mathbb{P}a_p) + \frac{1}{2}\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p). \\
&< \tfrac{1}{2}\Re(a_p, \mathbb{P}a_p) + \tfrac{1}{2}\Re(a_p, \mathbb{P}a_p) = \Re(a_p, \mathbb{P}a_p).
\end{aligned}$$

This is a contradiction. Hence, (3.19) is true. $\mathbb{P}$ is generalized $F_{\lhd\mathfrak{b}}$-contraction, (3.19) provides that, for any $n \in \mathbb{N}$.

$$\tau + F(\Re(\mathbb{P}a_n, \mathbb{P}m)) \leq a_1 F(\Re(a_n, m))$$

$$+\ a_2F(\Re(a_n, \mathbb{P}a_n))\ +\ a_3F(\Re(m, \mathbb{P}m)), \tag{3.22}$$

$$\tau + F(\Re(\mathbb{P}^2a_n, \mathbb{P}m)) \leq a_1F(\Re(\mathbb{P}a_n, m))$$

$$+\ a_2F(\Re(\mathbb{P}a_n, \mathbb{P}^2a_n))\ +\ a_3F(\Re(m, \mathbb{P}m)), \tag{3.23}$$

holds. Due to (F_2), the limits between (3.8) and (3.18) suggests that

$$\lim_{n\to\infty} F(\Re(a_n, m)) = -\infty, \lim_{n\to\infty} F(\Re(a_n, \mathbb{P}a_n) = -\infty.$$

So from (3.22), we determine that

$$\lim_{n\to\infty} F(\Re(\mathbb{P}a_n, \mathbb{P}m)) = -\infty.$$

From (F_2),

$$\lim_{n\to\infty} F(\Re(\mathbb{P}a_n, \mathbb{P}m)) = 0. \tag{3.24}$$

From (3.2), we conclude that

$$\Re(m, \mathbb{P}m) \leq \mathrm{s}[\Re(m, \mathbb{P}a_n) + \Re(\mathbb{P}a_n, \mathbb{P}m)]$$

$$= \mathrm{s}\Re(m, a_{n+1}) + \mathrm{s}\Re(\mathbb{P}a_n, \mathbb{P}m).$$

We determine that $\Re(m, \mathbb{P}m) = 0$ by taking $n \to \infty$ on (3.18) and (3.24). As a result $p = \mathbb{P}p$, where p is a fixed point of $\mathbb{P}$. Consider the second case (3.3). Then from (3.4), we have

$F(\Re(\mathbb{P}^2a_n, \mathbb{P}m)) < \tau\ +\ F(\Re(\mathbb{P}^2a_n, \mathbb{P}m)) \leq a_1F(\Re(\mathbb{P}a_n, m))$

$+\ a_2F(\Re(\mathbb{P}a_n, \mathbb{P}^2a_n))\ +\ a_3F(\Re(m, \mathbb{P}m)),$

$= a_1F(\Re(a_{n+1}, m))\ +\ a_2F(\Re(a_{n+1}, \mathbb{P}a_{n+1}))\ +\ a_3F(\Re(m, \mathbb{P}m)).$

As from (3.8), (3.18), and (F_2):

$$\lim_{n\to\infty} F(\Re(\mathbb{P}^2a_n, \mathbb{P}m)) = -\infty.$$

Likewise, from (F_2) we obtain

$$\lim_{n\to\infty} F(\Re(\mathbb{P}^2a_n, \mathbb{P}m)) = 0. \tag{3.25}$$

Using (3.2), we determine that

$$\Re(m,\mathbb{P}m) \leq s\Re(m,\mathbb{P}^2a_n) + \Re(\mathbb{P}^2a_n,\mathbb{P}m)]$$

$$= s\Re(m,a_{n+2}) + s\Re(\mathbb{P}^2a_n,\mathbb{P}m).$$

$\Rightarrow \Re(m,\mathbb{P}m) = 0$. Hence, m is a fixed point of $\mathbb{P}$. Claim that $\mathbb{P}^nc = c \neq d = \mathbb{P}^nd$ for all $n \in \mathbb{N}$ and that $c, d \in \mathcal{H}$ are two fixed points of $\mathbb{P}$. By using a_0, we have

$$(a_0 \lhd c \text{ and } a_0 \lhd d) \text{ or } (c \lhd a_0 \text{ and } d \lhd a_0).$$

$\mathbb{P}$ is $\lhd$-preserving, so the result is

$$(\mathbb{P}^na_0 \lhd \mathbb{P}^nc \text{ and } \mathbb{P}^na_0 \lhd \mathbb{P}^nd) \text{ or } (\mathbb{P}^nc \lhd \mathbb{P}^na_0 \text{ and } \mathbb{P}^nd \lhd \mathbb{P}^na_0),$$

for each $n \in \mathbb{N}$. Now,

$$\Re(c,d) = \Re(\mathbb{P}^nc,\mathbb{P}^nd) \leq s(\Re(\mathbb{P}^nc,\mathbb{P}^na_0) + \Re(\mathbb{P}^na_0,\mathbb{P}^nd)).$$

As a result $n \to \infty$, we get $\Re(c,d) \leq 0$. So that $c = d$. Therefore $\mathbb{P}$ has a unique fixed point in $\mathcal{H}$. □

Example 3.2.2.

Consider $\mathcal{H} = [0,1] \cup \{2,4\}$, and a mapping $\Re : \mathcal{H} \times \mathcal{H} \to [0,\infty)$ is determined by

$$\Re(c,d) = |c-d|^2, \quad \forall\ c,d \in \mathcal{H}.$$

$\Re$ is $\mathfrak{b}m$ with $s = 2$. If $cd \leq (c \vee d)$ then define the binary relation $\lhd$ on $\mathcal{H}$ as $c \lhd d$, where $c \vee d = c$ or d. Define the mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ by

$$\mathbb{P}(d) = \begin{cases} \ln(1+\frac{d}{2}) & \text{if } d \in [0,1], \\ d-1 & \text{if } d \in \{2,4\}. \end{cases}$$

Suppose $a_1 = \frac{1}{2}, a_2 = \frac{1}{4}, a_3 = \frac{1}{4}$ and $\tau > 0$. Suppose $c = 2, d = 4$, then $cd \leq c$. We obtain

$$\begin{aligned}\Re(\mathbb{P}c, \mathbb{P}d) &= |c - 1 - (d - 1)|^2,\\ &= |c - 1 - d + 1|^2,\\ &= 4.\\ \Re(c, d) &= |c - d|^2,\\ &= 4.\\ \Re(c, \mathbb{P}c) &= |c - (c - 1)|^2,\\ &= 1.\\ \Re(d, \mathbb{P}d) &= |d - (d - 1)|^2,\\ &= 1.\end{aligned}$$

Hence, $\forall, e, f \in \mathcal{H}$ with $e \lhd f$ $[\Re(\mathbb{P}e, \mathbb{P}f) > 0, \frac{1}{2\mathrm{s}}\Re(e, \mathbb{P}e) < \Re(e, f)$ $\Longrightarrow \tau + \Re(\mathbb{P}c, \mathbb{P}d) \geq a_1 F(\Re(c, d)) + a_2 F(\Re(c, \mathbb{P}c)) + a_3 F(\Re(d, \mathbb{P}d))]$.
Thus, $\mathbb{P}$ is not generalized $\mathcal{F}$-$\mathcal{S}cm$, $F(m) = \ln m$, where $m \in (0, \infty)$. If $\tau = \ln 4$, $a_3 \in [0, 1)$ and $a_1, a_2 \in [0, 1]$ are real numbers and $a_1 + a_2 + a_3 = 1$.

$$\forall, e, f \in \mathcal{H} \text{ with } e \lhd f \ [\Re(\mathbb{P}e, \mathbb{P}f) > 0, \frac{1}{2\mathrm{s}}\Re(e, \mathbb{P}e) < \Re(e, f)]$$

$$\Longleftarrow [c, d \in [0, 1] \vee c \in [0, 1] \text{ and } d \in \{2, 4\} \vee c \in \{2, 4\} \text{ and } d \in [0, 1]].$$

Observe that if

$$\begin{aligned}&\forall, e, f \in \mathcal{H} \text{ with } e \lhd f \ [\Re(\mathbb{P}e, \mathbb{P}f) > 0, \frac{1}{2\mathrm{s}}\Re(e, \mathbb{P}e) < \Re(e, f)\\ &\Longrightarrow \tau + F(\Re(\mathbb{P}e, \mathbb{P}f)) \leq a_1 F(\Re(e, f))\\ &+ a_2 F(\Re(e, \mathbb{P}e)) + a_3 F(\Re(f, \mathbb{P}f))],\end{aligned} \tag{3.26}$$

$\mathbb{P}$ is $\lhd$-preserving. Therefore $c \lhd d$. Without sacrificing generality, we can suppose that $cd \leq d$. Then we evaluate the subsequent scenarios:

Case(1):

Suppose that $c, d \in [0,1]$,

$$\Re(\mathbb{P}c, \mathbb{P}d) = |\ln(1+(\frac{c}{2})) - \ln(1+(\frac{d}{2}))|^2 > 0,$$

$$\Re(c, d) = |c-d|^2,$$

$$\Re(c, \mathbb{P}c) = |c - \ln(1+\frac{c}{2})|^2,$$

$$\Re(d, \mathbb{P}d) = |d - \ln(1+\frac{d}{2})|^2.$$

It is clear that

$$c \lhd d, \Re(\mathbb{P}c, \mathbb{P}d) > 0,$$

and

$$\frac{1}{2\mathrm{s}}\Re(c, \mathbb{P}c) < \Re(c, d).$$

This implies (3.25) satisfied.

Case(2):

When $c \in [0,1]$ and $d \in \{2,4\}$, then

$$\Re(\mathbb{P}c, \mathbb{P}d) = |\ln(1+\tfrac{c}{2}) - d + 1|^2,$$
$$\Re(c, d) = |c-d|^2,$$
$$\Re(c, \mathbb{P}c) = |c - \ln(1+\tfrac{c}{2})|^2,$$
$$\Re(d, \mathbb{P}d) = |d-(d-1)|^2 = 1.$$

It is clear that

$$c \lhd d, \ \Re(\mathbb{P}c, \mathbb{P}d) > 0,$$

and

$$\frac{1}{2\mathrm{s}}\Re(c, \mathbb{P}c) < \Re(c, d).$$

$$\tau + F(\Re(\mathbb{P}c, \mathbb{P}d)) \leq a_1 F(\Re(c,d)) + a_2 F(\Re(c, \mathbb{P}c)) + a_3 F(\Re(d, \mathbb{P}d)).$$

Case(3):

When $c \in \{2,4\}$ and $d \in [0,1]$.

This implies (3.25) satisfied.

Similar to **Case(2)**, (3.25) is satisfied.

As all hypothesis of theorem (3.2.1) are satisfied so $\mathbb{P}$ has unique fixed point i.e

$m = 0$.

Theorem 3.2.3.

Consider that $(\mathcal{H}, \lhd, \Re)$ an $\mathcal{OC}\mathfrak{b}ms$ with an orthogonal element a_0 and a constant s ≥ 1. Suppose that $F \in \mathfrak{S}$; $\tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is a self mapping satisfying the following axioms :

(i) $\mathbb{P}$ is $\lhd$- preserving,

(ii) $\mathbb{P}$ is a $F_{\lhd\mathfrak{b}}$-contraction mapping,

then the sequence $\{\mathbb{P}^n a\}$ converges to a unique fixed point $m \in \mathcal{H}$ of $\mathbb{P}$.

Proof. (3.2.3) is concluded by applying $a_1 = 1$ and $a_2 = a_3 = 0$ in (3.2.1). □

Example 3.2.4.

Consider $\mathcal{H} = [0, 1] \cup \{4, 6\}$, and a mapping $\Re : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ is determined by

$$\Re(c, d) = |c - d|^2, \qquad \forall\ c, d \in \mathcal{H}.$$

If $cd \leq (c \vee d)$ then define s $= 2$ the binary relation $\lhd$ on $\mathcal{H}$ as $c \lhd d$, where $c \vee d$ $= c$ or d. Then an $\mathcal{OC}\mathfrak{b}ms$ is $(\mathcal{H}, \Re)$. A mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is define by

$$\mathbb{P}(d) = \begin{cases} \ln(1 + \frac{d}{3}) & \text{if } d \in [0, 1], \\ d - 1 & \text{if } d \in \{4, 6\}. \end{cases}$$

Suppose $a_1 = 1$, and $a_2 = a_3 = 0$ and $\tau > 0$. Suppose $c = 4, d = 6$, then $cd \leq c$.

$$\Re(\mathbb{P}c, \mathbb{P}d) = |c - 1 - (d - 1)|^2,$$

$$= |c - 1 - d + 1|^2,$$

$$= 4.$$

$$\Re(c, d) = |c - d|^2,$$

$$= 4.$$

Hence, $\forall, e, f \in \mathcal{H}$, and $F \in \mathfrak{S}$ and $\tau > 0$, we obtain

$$\tau + F(\Re(\mathbb{P}e, \mathbb{P}f)) > F(\Re(e, f)).$$

Thus, $\mathbb{P}$ is not generalized $\mathcal{F}$-$\mathcal{S}cm$. Consider $F(m) = \ln m$, where $m \in (0, \infty)$ and $\tau = \ln 4$.

$$\forall, e, f \in \mathcal{H} \text{ with } e \lhd f \ [\Re(\mathbb{P}e, \mathbb{P}f) > 0, \frac{1}{2\text{s}}\Re(e, \mathbb{P}) < \Re(e, f)]$$

$$\implies [c, d \in [0,1] \vee c \in [0,1] \text{ and } d \in \{4,6\} \vee c \in \{4,6\} \text{ and } d \in [0,1]].$$

Observe that if

$$e, f \in \mathcal{H} \text{ with } e \lhd f \ [\Re(\mathbb{P}e, \mathbb{P}f) > 0, \frac{1}{2\text{s}}\Re(e, \mathbb{P}e) < \Re(e, f)$$

$$\implies \tau + F(\Re(\mathbb{P}e, \mathbb{P}f)) \leq F(\Re(e, f)). \tag{3.27}$$

Obviously, $\mathbb{P}$ is $\lhd$-preserving. Therefore $c \lhd d$. Without sacrificing generality, we can suppose that $cd \leq d$, then we evaluate the subsequent scenarios:

Case(1):

Suppose that $c, d \in [0,1]$,

$$\Re(\mathbb{P}c, \mathbb{P}d) = |\ln(1 + (\tfrac{c}{3})) - \ln(1 + (\tfrac{d}{3}))|^2 > 0,$$
$$\Re(c, d)) = |c - d|^2,$$
$$\Re(c, \mathbb{P}c) = |c - \ln(1 + (\tfrac{c}{3}))|^2,$$
$$\Re(d, \mathbb{P}d) = |d - \ln(1 + (\tfrac{d}{3}))|^2.$$

It is clear that

$$c \lhd d, \Re(\mathbb{P}c, \mathbb{P}d) > 0,$$

and

$$\frac{1}{2\text{s}}\Re(c, \mathbb{P}c) < \Re(c, d).$$

This implies (3.27) is satisfied.

Case(2):

When $c \in [0,1]$ and $d \in \{4,6\}$, then

$$\Re(\mathbb{P}c, \mathbb{P}d) = |\ln(1 + (\tfrac{c}{3})) - d + 1|^2,$$
$$\Re(c, d) = |c - d|^2,$$
$$\Re(c, \mathbb{P}c) = |c - \ln(1 + (\tfrac{c}{3}))|^2,$$
$$\Re(d, \mathbb{P}d) = 1.$$

It is clear that

$$c \lhd d,\ \Re(\mathbb{P}c, \mathbb{P}d) > 0.$$

and

$$\tfrac{1}{2s}\Re(c, \mathbb{P}c) < \Re(c, d).$$

This implies (3.27) is satisfied.

Case(3):

When $c \in \{4, 6\}$ and $d \in [0, 1]$. Similar to **Case(2)**. As all the hypothesis of (3.2.3) are satisfied, so $\mathbb{P}$ has a unique fixed point i.e $m = 0$.

Corollary 3.2.5.

Consider $(\mathcal{H}, \lhd, \Re)$ an $\mathcal{OC}ms$. Suppose that $F \in \mathfrak{S}$; $\tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is a self mapping satisfying the following axioms:

(i) $\mathbb{P}$ *is* $\lhd$-preserving,

(ii) $\forall, e, f \in \mathcal{H}$ with $e \lhd f$ $[\Re(\mathbb{P}e, \mathbb{P}f) > 0, \frac{1}{2}\Re(e, \mathbb{P}e) < \Re(e, f)$

$\Longrightarrow \tau + F(\Re(\mathbb{P}e, \mathbb{P}f)) \leq a_1 F(\Re(e, f))$,

where $a_3 \in [0, 1)$ and $a_1, a_2 \in [0, 1]$ with $a_1 + a_2 + a_3 = 1$. Then, $\mathbb{P}$ has a unique fixed point that is $m \in \mathcal{H}$.

Proof. Since each ms is a $\mathfrak{b}ms$ with s $= 1$ Hence, proof is analogous to proof of theorem (3.2.1), by taking $a_1 = 1$ and $a_2, a_3 = 0$ in Theorem (3.2.1). □

Corollary 3.2.6.

Consider $(\mathcal{H}, \lhd, \Re)$ an $\mathcal{OC}ms$. Suppose that $F \in \mathfrak{S}; \tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is a self mapping satisfying the following axioms:

(i) $\mathbb{P}$ is $\lhd$-preserving,

(ii) $\forall, e, f \in \mathcal{H}$ with $e \lhd f$ $[\Re(\mathbb{P}e, \mathbb{P}f) > 0, \frac{1}{2}\Re(e, \mathbb{P}e) < \Re(e, f)$

$\Longrightarrow \tau + F(\Re(\mathbb{P}e, \mathbb{P}f)) \leq F(\Re(e, f))]$.

Therefore, $\mathbb{P}$ has a unique fixed point that is $m \in \mathcal{H}$.

Proof. Since, each ms is a $\mathfrak{b}ms$ with s $= 1$ and hence proof is analogous to Theorem (3.2.1) by taking $a_1 = 1$ and $a_2 = a_3 = 0$. □

Theorem 3.2.7.

Consider $(\mathcal{H}, \lhd, \Re)$ an $\mathcal{OC}\mathfrak{b}ms$ with an orthogonal element a_o and a constant s ≥ 1. Suppose that $F \in \mathfrak{S}$; $\tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is a self mapping satisfying the following axioms:

(i) $\mathbb{P}$ is $\lhd$-preserving,

(ii) $\forall\ e, f \in \mathcal{H}$ with $e \lhd f$ $[\Re(\mathbb{P}e, \mathbb{P}f) > 0$

$\Longrightarrow \tau + F(\Re(\mathbb{P}e, \mathbb{P}f)) < a_1 F(\Re(e, f)) + a_2 F(\Re(e, \mathbb{P}e)) + a_3 F(\Re(f, \mathbb{P}f))]$,

where $a_3 \in [0, 1)$ and $a_1, a_2 \in [0, 1]$ with $a_1 + a_2 + a_3 = 1$.

(iii) $\mathbb{P}$ is $\lhd$-continuous.

Then, $\mathbb{P}$ has a unique fixed point $m \in \mathcal{H}$.

Proof. Since, $(\mathcal{H}, \lhd)$ is an O-set,

$$\exists\ a_0 \in \mathcal{H} : (\forall\ a \in \mathcal{H}, a \lhd a_0) \text{ or } (\forall\ a \in \mathcal{H}, a_0 \lhd a).$$

It indicates that either $a_0 \lhd \mathbb{P}a_0$ or $\mathbb{P}a_0 \lhd a_0$. Take

$$a_1 = \mathbb{P}a_0, a_2 = \mathbb{P}a_1 = \mathbb{P}^2 a_0 ....., a_{n+1} = \mathbb{P}a_n = \mathbb{P}^{n+1} a_0, \qquad \forall\ n \in \mathbb{N} \cup \{0\}. \quad (3.28)$$

If $a_n = a_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$, so than a_n is a fixed point of $\mathbb{P}$. If $a_n \neq a_{n+1}$ $\forall\ n \in \mathbb{N}$ then $\Re(\mathbb{P}a_n, \mathbb{P}a_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathbb{P}$ preserves $\lhd$, we have

$$a_n \lhd a_{n+1} \text{ or } a_{n+1} \lhd a_n, \qquad \forall, n \in \mathbb{N} \cup \{0\}.$$

$\Rightarrow \{a_n\}$ is an O-sequence. Using (ii), we have

$$0 < \Re(a_n, \mathbb{P}a_n) = \Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n), \qquad \forall\ n \in \mathbb{N}. \quad (3.29)$$

Hence, we have

$$\tau + F(\Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n)) \leq a_1 F(\Re(a_{n-1}, a_n)) + a_2 F(\Re(a_{n-1}, \mathbb{P}a_{n-1})) + a_3 F(\Re(a_n, \mathbb{P}a_n)),$$

$$= a_1 F(\Re(a_{n-1}, a_n)) + a_2 F(\Re(a_{n-1}, a_n)) + a_3 F(\Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n)).$$

$$\Rightarrow \tau + (1 - a_3) F(\Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n)) \leq (a_1 + a_2) F(\Re(a_{n-1}, a_n)).$$

$a_1 + a_2 + a_3 = 1$, therefore, we obtain

$$F(\Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n)) \leq F(\Re(a_{n-1}, a_n)) - \frac{\tau}{a_1 + a_2} < F(\Re(a_{n-1}, a_n)),$$

(F_1) implies that

$$\Re(a_n, \mathbb{P}a_n) = \Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n) < \Re(a_{n-1}, \mathbb{P}a_{n-1}), \qquad \forall\ n \in \mathbb{N},$$

$\Rightarrow \{\Re(a_n, \mathbb{P}a_n)\}_{n=1}^{\infty}$ is a decreasing sequence that is bounded below, and

$$\lim_{n\to\infty} \Re(a_n, \mathbb{P}a_n) = \delta = \inf\{\Re(a_n, \mathbb{P}a_n) : \ \ \forall\ n \in \mathbb{N}\}.$$

To prove $\delta = 0$. Assume on contrary $\delta > 0$. $\exists\ p \in \mathbb{N}$ such that for every $\epsilon > 0$,

$$\Rightarrow \Re(a_p, \mathbb{P}a_p) < \delta + \epsilon.$$

$$\Rightarrow F(\Re(a_p, \mathbb{P}a_p)) < F(\delta + \epsilon). \tag{3.30}$$

From (3.28), we conclude

$$0 < \Re(a_p, \mathbb{P}a_p) = \Re(a_{p-1}, \mathbb{P}a_p).$$

$$\begin{aligned}\Rightarrow \tau + F(\Re(\mathbb{P}a_{p-1}, \mathbb{P}a_p)) \leq\ & a_1 F(\Re(a_{p-1}, a_p)) + a_2 F(\Re(a_{p-1}, \mathbb{P}a_{p-1})) \\ & + a_3 F(\Re(a_p, \mathbb{P}a_p)), \\ =\ & a_1 F(\Re(a_{p-1}, a_p)) + a_2 F(\Re(a_{p-1}, a_p)) \\ & + a_3 F(\Re(\mathbb{P}a_{p-1}, \mathbb{P}a_p)),\end{aligned}$$

$$\Rightarrow \tau + (1 - a_3) F(\Re(\mathbb{P}a_{p-1}, \mathbb{P}a_p)) < (a_1 + a_2) F(\Re(a_{p-1}, a_p)).$$

Given that $a_1 + a_2 + a_3 = 1$, we get

$$F(\Re(\mathbb{P}a_{p-1}, \mathbb{P}a_p)) \leq F(\Re(a_{p-1}, a_p)) - \frac{\tau}{a_1 + a_2}. \tag{3.31}$$

From (3.29), we have $0 < \Re(\mathbb{P}a_p, \mathbb{P}a_{p+1}) < \Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)$,

and by the assumption of theorem, we obtain

$$\tau + F(\Re(\mathbb{P}a_p, \mathbb{P}a_{p+1})) \leq a_1 F(\Re(a_p, a_{p+1})) \ + \ a_2 F(\Re(a_p, \mathbb{P}a_p))$$
$$+ a_3 F(\Re(a_{p+1}, \mathbb{P}a_{p+1})),$$
$$= a_1 F(\Re(a_p, a_{p+1})) \ + \ a_2 F(\Re(a_p, a_{p+1}))$$
$$+ a_3 F(\Re(\mathbb{P}a_p, \mathbb{P}a_{p+1})),$$

and,

$$\tau + (1 - a_3) F(\Re(\mathbb{P}a_p, \mathbb{P}a_{p+1})) \leq (a_1 + a_2) F(\Re(a_p, \mathbb{P}a_{p+1})).$$

Since, $a_1 + a_2 + a_3 = 1$, we obtain

$$F(\Re(\mathbb{P}a_p, \mathbb{P}a_{p+1})) \leq F(\Re(a_p, \mathbb{P}a_{p+1})) - \frac{\tau}{a_1 + a_2}. \tag{3.32}$$

Now, from (3.30) and carrying out the same procedure as in (3.31) and (3.32), we obtain

$$\begin{aligned}
F(\Re(a_{p+n}, a_{p+n+1})) &= F(\Re(\mathbb{P}a_{p+n-1}, \mathbb{P}a_{p+n})), \\
&\leq F(\Re(a_{p+n-1}, a_{p+n})) - \tfrac{\tau}{a_1+a_2}, \\
&= F(\Re(\mathbb{P}a_{p+n-2}, \mathbb{P}a_{p+n-1}))\tfrac{\tau}{a_1+a_2}, \\
&\leq F(\Re(a_{p+n-2}, a_{p+n-1})) - \tfrac{2\tau}{a_1+a_2}, \\
&= F(\Re(\mathbb{P}a_{p+n-3}, \mathbb{P}a_{p+n-2})) - \tfrac{2\tau}{a_1+a_2}, \\
&\leq F(\Re(a_{p+n-3}, a_{p+n-2})) - \tfrac{3\tau}{a_1+a_2}, \\
&\leq F(\Re(a_{p+1}, a_{p+2})) - \tfrac{(n-1)\tau}{a_1+a_2}, \\
&= F(\Re(\mathbb{P}a_p, \mathbb{P}a_{p+1})) - \tfrac{(n-1)\tau}{a_1+a_2}, \\
&\leq F(\delta + \epsilon) - \tfrac{n\tau}{a_1+a_2}.
\end{aligned}$$

Considering $n \to \infty$, we obtain that

$$\lim_{n\to\infty} F(\Re(a_{p+n}, a_{p+n+1})) = -\infty.$$

$\Rightarrow \lim\limits_{n\to\infty} F(\Re(a_{p+n}, a_{p+n+1})) = 0$. $P_1 \in \mathbb{N}$ exists in such a way

$$\Re(a_{p+n}, a_{p+n+1}) < \delta, \quad \forall\ n \geq P_1,$$

and we obtain from (3.28), and

$$\Rightarrow \Re(a_{p+n}, \mathbb{P}a_{p+n}) < \delta, \quad \forall\ n \geq P_1.$$

Which contradicts the definition of δ.

Hence, $\delta = 0$ and from (3.30), we get

$$\lim_{n\to\infty} \Re(a_n, \mathbb{P}a_n) = 0. \tag{3.33}$$

Claim that,

$$\Rightarrow \lim_{n,p\to\infty} \Re(a_n, a_p) = 0.$$

Assume, on contrary there exists two sequences of natural number $\{r(n)\}_{n=1}^{\infty}$ and $\{t(n)\}_{n=1}^{\infty}$, then

$$r(n) > t(n) > n, \ \Re(a_{r(n)}, a_{t(n)}) \geq \epsilon,$$

$$\Re(a_{r(n)-1}, a_{t(n)}) < \epsilon, \quad \forall\ n \in \mathbb{N}. \tag{3.34}$$

$$\Rightarrow \Re(a_{r(n)}, a_{t(n)}) \leq \mathrm{s}\Re(a_{r(n)}, a_{r(n)-1}) + \Re(a_{r(n)-1}, a_{t(n)})],$$

$$\leq \mathrm{s}\Re(a_{r(n)}, a_{r(n)-1}) + \mathrm{s}\epsilon,$$

$$= \mathrm{s}\Re(a_{r(n)-1}, \mathbb{P}a_{r(n)-1}) + \mathrm{s}\epsilon, \qquad \forall\ n \in \mathbb{N}. \tag{3.35}$$

By (3.33), there are $P_2 \in \mathbb{N}$ in the way that

$$\Re(a_{r(n)}, \mathbb{P}a_{r(n)}) < \epsilon, \quad \forall\ n > P_2. \tag{3.36}$$

(3.35) and (3.36) suggests that

$$\Re(a_{r(n)}, a_{t(n)}) < 2\mathrm{s}\epsilon, \quad \forall\ n > P_2,$$

by (F_2),

$$F(\Re(a_{r(n)}, a_{t(n)}) < F(2\mathrm{s}\epsilon), \quad \forall\ n > P_2, \tag{3.37}$$

using (3.34),

$$\epsilon \leq \Re(a_{r(n)+1}, a_{t(n)+1}) = \Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)}), \quad \forall\ n > P_2.$$

From (3.37),

$$\tau + F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) \leq a_1 F(\Re(a_{r(n)}, a_{t(n)})) \ + \ a_2 F(\Re(a_{r(n)}, \mathbb{P}a_{r(n)}))$$
$$+ \ a_3 F(\Re(a_{t(n)}, \mathbb{P}a_{t(n)})),$$
$$< a_1 F(2s\epsilon) \ + \ a_2 F(\Re(a_{r(n)}, \mathbb{P}a_{r(n)}))$$
$$+ a_3 F(\Re(a_{t(n)}, \mathbb{P}a_{t(n)})), \quad \forall \ n \in \mathbb{N}.$$

From (3.35) and (F_2),

$$\lim_{n\to\infty} F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) = -\infty,$$

by (F_2),

$$\lim_{n\to\infty} F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) = 0 \Leftrightarrow \lim_{n\to\infty} F(\Re(a_{r(n)+1}, a_{t(n)+1})) = 0.$$

Contradiction in relation to (3.34) . Thus, $\lim_{p,n\to\infty} \Re(a_n, a_p) = 0$. There are $m \in \mathcal{H}$ such that due to the completeness of $(\mathcal{H}, \Re)$,

$$\lim_{n\to\infty} \Re(a_n, m) = 0.$$

Since, $\mathbb{P}$ is $\lhd$-continuous,

$$\lim_{n\to\infty} \Re(\mathbb{P}a_n, \mathbb{P}m) = 0.$$

As $\Re(m, \mathbb{P}m) \leq s\Re(m, a_n) + \Re(a_n, \mathbb{P}m)]$, m is a fixed points of $\mathbb{P}$ since $\Re(m, \mathbb{P}m)$ $= 0$. Assuming that $\mathbb{P}^n c = c \neq d = \mathbb{P}^n d \ \ \forall \ n \in \mathbb{N}$, let $c, d \in \mathcal{H}$ be two fixed point of $\mathcal{H}$. By choice of a_0, we get

$$(a_0 \lhd c \text{ and } a_0 \lhd d) \text{ or } (c \lhd a_0 \text{ and } d \lhd a_0).$$

$\mathbb{P}$ is $\lhd$-preserving, we get

$$(\mathbb{P}^n a_0 \lhd \mathbb{P}^n c \text{ and } \mathbb{P}^n a_0 \lhd \mathbb{P}^n d) \text{ or } (\mathbb{P}^n c \lhd \mathbb{P}^n a_0 \text{ and } \mathbb{P}^n d \lhd \mathbb{P}^n a_0), \qquad \forall \ n \in \mathbb{N}.$$

Now,

$$\Re(c, d) = \Re(\mathbb{P}^n c, \mathbb{P}^n d) \leq s(\Re(\mathbb{P}^n c, \mathbb{P}^n a_0) + \Re(\mathbb{P}^n a_0, \mathbb{P}^n d)).$$

We get $\Re(c,d) \leq 0$ as $n \to \infty$. $\Rightarrow c = d$. Therefore, $\mathbb{P}$ has a unique fixed point in $\mathcal{H}$. $\square$

Example 3.2.8.

Suppose $\mathcal{H} = \mathbb{R}$ and the mapping $\Re : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ is determined by

$$\Re(c,d) = \max\{c,d\}^2, \quad \forall, c,d \in \mathcal{H},$$

is a $\mathfrak{b}ms$ with s = 2. If $c, d \in [3, n+4] \quad \forall, n \in \mathbb{N}$ or $c = 0$,

define the binary relation $\lhd$ on $\mathcal{H}$ by $c \lhd d$.

Then the $\mathfrak{b}ms$ $(\mathcal{H}, d)$ is $\mathcal{OC}ms$.

Define the $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ mapping by

$$\mathbb{P}(d) = \begin{cases} 0 & \text{if } d = 0, \\ d+1 & \text{if } d \in [1,2], \\ d+1 & \text{if } d \in (2,\infty). \end{cases}$$

If $c \in \mathcal{H}$ and $\{c_n\}$ is any $\lhd$-sequence in $\mathcal{H}$ that converges to c,

then the following conditions are fulfilled:

Case(1):

$c = 0$ and $\mathbb{P}(c_n) = 0 = \mathbb{P}(c)$ if $c_n = 0 \quad \forall\, n$.

Case(2):

If $c_n \neq 0$ for all n, then $q \in \mathbb{N}$ in order for $c \in [3, q+4]$, and $\mathbb{P}(c_n) = \mathbb{P}(c)$.

$\mathbb{P}$ is thus not continuous on $\mathcal{H}$ but is $\lhd$-continuous on $\mathbb{P}$.

Suppose $a_1 = \frac{1}{2}, a_2 = \frac{1}{4}$, and $a_3 = \frac{1}{4}$. Therefore,

$\Re(\mathbb{P}c, \mathbb{P}d) = 9, \quad \Re(\mathbb{P}1, \mathbb{P}2) = 9, \quad \Re(c,d) = 4, \quad \Re(1,2) = 4, \quad \Re(1, \mathbb{P}1) = 4,$

$\Re(2, \mathbb{P}2) = 9$, $\forall$, $F \in \mathfrak{S}$ and $\tau > 0$ we obtain

$c, d \in \mathcal{H}$ with $c \lhd d$ $[\Re(\mathbb{P}c, \mathbb{P}d) > 0$

$$\Longrightarrow \tau + \Re(\mathbb{P}c, \mathbb{P}d) \geq a_1 F(\Re(c,d)) + a_2 F(\Re(c, \mathbb{P}c)) + a_3 F(\Re(d, \mathbb{P}d))].$$

Consider $F(m) = \ln m$, where $m \in (0, \infty)$. Initially, we note that

$$\forall\, c, d \in \mathcal{H} \text{ with } c \lhd d \ [\Re(\mathbb{P}c, \mathbb{P}d) > 0] \Leftarrow [c = 0 \text{ and } d \in (2, \infty)].$$

Consider that if

$$\forall\ c,d \in \mathcal{H} \text{ with } c \lhd d\ [\Re(\mathbb{P}c,\mathbb{P}d) > 0, \frac{1}{2\mathrm{s}}\Re(c,\mathbb{P}c) < \Re(c,d)$$
$$\Longrightarrow \tau + F(\Re(\mathbb{P}c,\mathbb{P}d)) \leq a_1F(\Re(c,d)) + a_2F(\Re(c,\mathbb{P}c)) + a_3F(\Re(d,\mathbb{P}d))]. \tag{3.38}$$

Obviously, $\mathbb{P}$ is $\lhd$-preserving. Suppose $c \lhd d$ and $c = 0, d \in (2,\infty), \tau = \ln 2$, then
$\Re(\mathbb{P}c,\mathbb{P}d) = (d-1)^2, \Re(c,d) = d^2, \Re(c,\mathbb{P}c) = 0,$ and $\Re(d,\mathbb{P}d) = d^2$. It is obvious that (3.38) is satisfied.

Theorem 3.2.9.

Consider that $(\mathcal{H},\lhd,\Re)$ is an $\mathcal{OCb}ms$ with an orthogonal element a_0 and a constant $\mathrm{s} \geq 1$. Suppose that $\tau > 0$ and $F \in \mathfrak{S}$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is a self mapping satisfying the following axioms:

(i) $\mathbb{P}$ is $\lhd$-preserving $\forall\ e,f \in \mathcal{H}$ with $e \lhd f\ [\mathbb{P}e,\mathbb{P}f) > 0$
$\Longrightarrow \tau + F(\Re(\mathbb{P}e,\mathbb{P}f)) < F(\Re(e,f))]$;

(ii) $\mathbb{P}$ is $\lhd$-continuous,

then, $\mathbb{P}$ has a unique fixed point $m \in \mathcal{H}$.

Proof. Follows by substitution $a_1 = 1$ and $a_2 = a_3 = 0$ in Theorem (3.2.7). □

Theorem 3.2.10.

Consider that $(\mathcal{H},\lhd,\Re)$ is an $\mathcal{OC}ms$. Assume that $\tau > 0$ and $F \in \mathfrak{S}$ exist such that the following conditions are satisfied:

(i) $\mathbb{P}$ is $\lhd$-preserving,

(ii) $\forall\ e,f \in \mathcal{H}$ with $e \lhd f\ [\Re(\mathbb{P}e,\mathbb{P}f) > 0$
$\Longrightarrow \tau + F(\Re(\mathbb{P}e,\mathbb{P}f)) < a_1F(\Re(e,f)) + a_2F(\Re(e,\mathbb{P}e)) + a_3F(\Re(f,\mathbb{P}f))]$,
$a_1 + a_2 + a_3 = 1$ where $a_3 \in [0,1)$ and $a_1,a_2 \in [0,1]$.

(iii) $\mathbb{P}$ is $\lhd$-continuous,

Then, $\mathbb{P}$ has a unique fixed point $m \in \mathcal{H}$.

Proof. Follows by taking $\mathrm{s} = 1$ in Theorem (3.2.7). □

Theorem 3.2.11.

Consider that $(\mathcal{H}, \lhd, \Re)$ is an $\mathcal{OC}ms$. Assume that $\tau > 0$ and $F \in \mathfrak{S}$ exist such that the following conditions are satisfied:

(i) $\mathbb{P}$ is $\lhd$-preserving,

(ii) $\forall\ e, f \in \mathcal{H}$ with $e \lhd f$ $[\Re(\mathbb{P}e, \mathbb{P}f) > 0$
$\Longrightarrow \tau + F(\Re(\mathbb{P}e, \mathbb{P}f)) < F(\Re(e, f))]$,

(iii) $\mathbb{P}$ is $\lhd$-continuous,

then, $\mathbb{P}$ has a unique fixed point $m \in \mathcal{H}$.

Proof. Every ms is a $\mathfrak{b}ms$ with constant s $= 1$, hence, $m \in \mathcal{H}$ is a fixed point of $\mathbb{P}$. In fact, there is another fixed point q such that for any $m, q \in \mathcal{H}$ with $m \lhd q$ $[\Re(\mathbb{P}m, \mathbb{P}q)] > 0$, then $q \in \mathcal{H}$ of $\mathbb{P}$. Since $\tau > 0$, we obtain from our foundation of theorem

$$F(\Re(m, q)) = F(\Re(\mathbb{P}m, \mathbb{P}q)) < \tau + F(\Re(\mathbb{P}m, \mathbb{P}q)) \leq a_1 F(\Re(m, q)),$$

is contradiction. Therefore $\mathbb{P}$ has a unique fixed point. □

Chapter 4

$(\alpha - F - \mathbb{P}_{\lhd})$ Suzuki Contraction Mapping

Beg et al. [29] introduced the orthogonal $\mathcal{F}$-$\mathcal{S}cm$ on $\mathcal{OC}\mathfrak{b}ms$. In this chapter two results are provided by generalizing the contraction of Beg et al. [29]. For this purpose we defined generalized $(\alpha - F - \mathbb{P}_{\lhd})$ Suzuki contractions. Following material is necessary for the proof of main result.

4.1 Some Useful Definitions

Following definitions are taken from [35]

Definition 4.1. Alpha Admissible mapping

A mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is said to be an α- admissible mapping if there exist, a function $\alpha : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ such that, for all $\mu, \nu \in \mathcal{H}$,

$$\alpha(\mu, \nu) \geq 1 \Rightarrow \alpha(\mathbb{P}(\mu), \mathbb{P}(\nu)) \geq 1.$$

Definition 4.2. Alpha- type F- Contraction

Let $(\mathcal{H}, \Re)$ be a metric space. Let D be a non empty subset of $\mathcal{H}$. A mapping $\mathbb{P} : D \to D$ is said to be a α-type F-contraction if there exist $\tau > 0$, and two functions $F \in \mathfrak{F}$, $\alpha : D \times D \to (0, \infty)$ such that, for all $\mu, \nu \in D$ satisfying $\Re(\mathbb{P}\mu, \mathbb{P}\nu) > 0$, the following inequality holds:

$$\tau + \alpha(\mu, \nu) F(\Re(\mathbb{P}\mu, \mathbb{P}\nu)) \leq F(\Re(\mu, \nu)).$$

4.2 Fixed Point Results for $(\alpha - F - \mathbb{P}_{\lhd})$ Suzuki Contraction Mappings

Definition 4.3. $(\alpha - F - \mathbb{P}_{\lhd})$ **Suzuki Contraction**
Let $(\mathcal{H}, \lhd, \Re_{\mathfrak{b}})$ be an orthogonal $\mathfrak{b}ms$ with constant s ≥ 1. A mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is said to be an generalized orthogonal $(\alpha - F - \mathbb{P}_{\lhd})$ Suzuki contraction mapping on $(\mathcal{H}, \lhd, \Re_{\mathfrak{b}})$ if there are $F \in \mathfrak{G}, \alpha : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ and $\tau > 0$ such that,

$$a, b \in \mathcal{H} \text{ with } a \lhd b \ [\Re(\mathbb{P}a, \mathbb{P}b) > 0, \frac{1}{2\mathrm{s}}\Re(a, \mathbb{P}a) < \Re(b, \mathbb{P}b)]$$

$\tau + \alpha(a, b) F(\Re(\mathbb{P}a, \mathbb{P}b)) \leq a_1 F(\Re(a, b)) + a_2 F(\Re(a, \mathbb{P}a))$
$+ a_3 F(\Re(b, \mathbb{P}b)) + a_4 (\frac{F(\Re(a,\mathbb{P}a)) + F(\Re(b,\mathbb{P}b))}{2})$,
with $a_1 + a_2 + a_3 + a_4 < 1$.

Theorem 4.2.1.

Consider $(\mathcal{H}, \lhd, \Re_{\mathfrak{b}})$ an $\mathcal{OC}\mathfrak{b}ms$ with an orthogonal element a_0 and a constant s ≥ 1. Suppose that $F \in \mathfrak{G}$, $\tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is generalized $(\alpha - F - \mathbb{P}_{\lhd})$ Suzuki contraction mapping observing the following axioms:

(i) $\mathbb{P}$ is α-admissible mapping,

(ii) there exist $a_0 \in \mathcal{H}$ such that $\alpha(a_0, \mathbb{P}a_0) \geq 1$;

(iii) $\mathbb{P}$ is $\lhd$-preserving,

then the sequence $\{\mathbb{P}^n a\}$ converges to a unique fixed point $m \in \mathcal{H}$ of $\mathbb{P}$.

Proof. Since, $(\mathcal{H}, \lhd)$ is an O-set,

$$\exists\ a_0 \in \mathcal{H} : (\forall\ a \in \mathcal{H}, a \lhd a_0) \text{ or } (\forall\ a \in \mathcal{H}, a_0 \lhd a).$$

Thus, it implies that $a_0 \lhd \mathbb{P}a_0$ or $\mathbb{P}a_0 \lhd a_0$. Let

$$a_1 = \mathbb{P}a_0, a_2 = \mathbb{P}a_1 = \mathbb{P}^2 a_0....., a_{n+1} = \mathbb{P}a_n = \mathbb{P}^{n+1}a_0, \quad \forall\, n \in \mathbb{N} \cup \{0\}. \tag{4.1}$$

If $a_n = a_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$, then a_n is fixed point of $\mathbb{P}$.
If $a_n \neq a_{n+1} \quad \forall, n \in \mathbb{N} \cup \{0\}$, then $\Re(\mathbb{P}a_n, \mathbb{P}a_{n+1}) > 0$ for all $n \in \mathbb{N}$.
Since $\mathbb{P}$ preserves-$\lhd$, we have

$$a_n \lhd a_{n+1} \text{ or } a_{n+1} \lhd a_n, \quad \forall\, n \in \mathbb{N} \cup \{0\}.$$

This means that $\{a_n\}$ is an orthogonal sequence.
Because $\mathbb{P}$ is a generalized $(\alpha - F - \mathbb{P}_{\lhd})$ Suzuki contraction mapping, we have

$$\frac{1}{2\text{s}}\Re(a_n, \mathbb{P}a_n) < \Re(a_n, \mathbb{P}a_n), \quad \forall\, n \in \mathbb{N}, \tag{4.2}$$

using (i)

$$\alpha(a_0, a_1) = \alpha(a_0, \mathbb{P}a_0) \geq 1.$$

$$\Rightarrow \alpha(a_1, a_2) = \alpha(\mathbb{P}a_0, \mathbb{P}a_1) \geq 1.$$

Inductively, we obtain

$$\alpha(a_n, a_{n+1}) \geq 1, \quad \forall\, n \in \mathbb{N} \cup \{0\}.$$

Thus, in light of the theorem‘s hypothesis, we have

$$\tau + F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) \leq \tau + \alpha(a_n, \mathbb{P}a_n)F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) \leq a_1 F(\Re(a_n, \mathbb{P}a_n)) + a_2 F(\Re(a_n, \mathbb{P}a_n)) + a_3 F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) + a_4\left(\tfrac{F(\Re(a_n, \mathbb{P}a_n)) + F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n))}{2}\right),$$

since,

$$a_1 + a_2 + a_3 + a_4 < 1,$$

$$\tau + (1 - a_3 - \frac{a_4}{2})F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) \leq (a_1 + a_2 + \frac{a_4}{2})F(\Re(a_n, \mathbb{P}a_n)). \tag{4.3}$$

$$\Rightarrow (1 - a_3 - \frac{a_4}{2})F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) \leq (a_1 + a_2 + \frac{a_4}{2})F(\Re(a_n, \mathbb{P}a_n)) - \tau, \tag{4.4}$$

$$F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) \leq (\frac{a_1 + a_2 + \frac{a_4}{2}}{1 - a_3 - \frac{a_4}{2}})F(\Re(a_n, \mathbb{P}a_n)) - \frac{\tau}{1 - a_3 - \frac{a_4}{2}},$$

since $\frac{a_1+a_2+\frac{a_4}{2}}{1-a_3-\frac{a_4}{2}} < 1$, the inequality becomes

$$F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) < F(\Re(a_n, \mathbb{P}a_n)) - \frac{\tau}{1 - a_3 - \frac{a_4}{2}} < F(\Re(a_n, \mathbb{P}a_n)).$$

Using (F_1), we determine that

$$\Re(a_{n+1}, \mathbb{P}a_{n+1}) = \Re(\mathbb{P}a_n, \mathbb{P}^2 a_n) < \Re(a_n, \mathbb{P}a_n) \quad \forall\ n \in \mathbb{N}. \tag{4.5}$$

$\Rightarrow \{\Re(a_n, \mathbb{P}a_n)\}_{n=1}^{\infty}$ is a decreasing sequence of real numbers which is bounded below.

$$\lim_{n\to\infty} \Re(a_n, \mathbb{P}a_n) = \delta = \inf\{\Re(a_n, \mathbb{P}a_n)\ \ \forall\ n \in \mathbb{N}\}.$$

To prove $\delta = 0$, suppose on the contrary, consider that $\delta > 0$.
i.e, for every $\epsilon > 0$ there exist $p \in \mathbb{N}$, such that

$$\Re(a_p, \mathbb{P}a_p) < \delta + \epsilon.$$

$$\Rightarrow F(\Re(a_p, \mathbb{P}a_p)) < F(\delta + \epsilon).$$

However, we have

$$\frac{1}{2\text{s}}\Re(a_p, \mathbb{P}a_p) < \Re(a_p, \mathbb{P}a_p).$$

from (4.1). Given that $\mathbb{P}$ is generalized $(\alpha - F - \mathbb{P}_{\triangleleft})$ Suzuki contraction, we acquire,

$$\tau + F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) \leq \tau + \alpha(a_p, \mathbb{P}a_p)F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p))$$
$$\leq a_1 F(\Re(a_p, \mathbb{P}a_p)) \ + \ a_2 F(\Re(a_p, \mathbb{P}a_p)) \ + \ a_3 F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p))$$
$$+ \ a_4(\tfrac{F(\Re(a_p,\mathbb{P}a_p)) \ + \ F(\Re(\mathbb{P}a_p,\mathbb{P}^2 a_p))}{2}).$$

Which gives

$$\tau + (1 - a_3 - \frac{a_4}{2})F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) \leq (a_1 + a_2 + \frac{a_4}{2})F(\Re(a_p, \mathbb{P}a_p)),$$

$$\Rightarrow (1 - a_3 - \frac{a_4}{2})F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) \leq (a_1 + a_2 + \frac{a_4}{2})F(\Re(a_p, \mathbb{P}a_p)) - \tau,$$

$$F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) \leq (\frac{a_1 + a_2 + \frac{a_4}{2}}{1 - a_3 - \frac{a_4}{2}})F(\Re(a_p, \mathbb{P}a_p)) - \frac{\tau}{1 - a_3 - \frac{a_4}{2}},$$

since,

$$\frac{a_1 + a_2 + \frac{a_4}{2}}{1 - a_3 - \frac{a_4}{2}} < 1,$$

$$F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) < F(\Re(a_p, \mathbb{P}a_p)) - \frac{\tau}{1 - a_3 - \frac{a_4}{2}}. \tag{4.6}$$

Since $\mathbb{P}$ is generalized $(\alpha - F - \mathbb{P}_{\triangleleft})$ Suzuki contraction, we determine

$$\tau + F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p)) \leq \tau + \alpha(\mathbb{P}a_p, \mathbb{P}^2 a_p)F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p))$$
$$\leq a_1 F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) + a_2 F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p))$$
$$+ a_3 F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p)) + a_4(\tfrac{F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) + F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p))}{2}).$$
$$\Rightarrow \tau + F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p)) \leq a_1 F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p))$$
$$+ a_2 F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) + a_3 F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p))$$
$$+ a_4(\tfrac{F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) + F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p))}{2}).$$
$$\tau + (1 - a_3 - \tfrac{a_4}{2})F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p)) \leq (a_1 + a_2 + \tfrac{a_4}{2})F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)).$$
$$\Rightarrow (1 - a_3 - \tfrac{a_4}{2})F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p)) \leq (a_1 + a_2 + \tfrac{a_4}{2})F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) - \tau,$$
$$F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p)) \leq (\tfrac{a_1 + a_2 + \frac{a_4}{2}}{1 - a_3 - \frac{a_4}{2}})F(\mathbb{P}a_p, \mathbb{P}^2 a_p)) - \tfrac{\tau}{1 - a_3 - \frac{a_4}{2}},$$

since

$$\frac{a_1 + a_2 + \frac{a_4}{2}}{1 - a_3 - \frac{a_4}{2}} < 1,$$

$$F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p)) < F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) - \frac{\tau}{1 - a_3 - \frac{a_4}{2}}. \tag{4.7}$$

$a_1 + a_2 + a_3 + a_4 < 1$,

likewise by combining (4.5) and (4.6), we obtain

$$F(\Re(\mathbb{P}^2 a_p, \mathbb{P}^3 a_p)) < F(\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p)) - \frac{\tau}{1 - a_3 - \frac{a_4}{2}},$$

$$< F(\Re(a_p, \mathbb{P}a_p)) - \tfrac{2\tau}{1 - a_3 - \frac{a_4}{2}}.$$

Continuing in the same manner:

$$F(\Re(\mathbb{P}^n a_p, \mathbb{P}^{n+1} a_p)) < F(\Re(\mathbb{P}^n a_p, \mathbb{P}^{n-1} a_p)) - \tfrac{\tau}{1 - a_3 - \frac{a_4}{2}},$$
$$< F(\Re(\mathbb{P}^{n-1}, \mathbb{P}^{n-2} a_p)) - \tfrac{2\tau}{1 - a_3 - \frac{a_4}{2}},$$
$$< F(\Re(\mathbb{P}a_p, a_p)) - \tfrac{n\tau}{1 - a_3 - \frac{a_4}{2}},$$

$$< F(\delta + \epsilon) - \tfrac{n\tau}{1 - a_3 - \frac{a_4}{2}}.$$

Applying $\lim n \to \infty$ on both sides

$$\lim_{n\to\infty} F(\Re(\mathbb{P}^n a_p, \mathbb{P}^{n+1} a_p)) = -\infty.$$

(F_2) implies

$$\lim_{n\to\infty} \Re(\mathbb{P}^n a_p, \mathbb{P}^{n+1} a_p) = 0.$$

Hence, there exists $p_1 \in \mathbb{N}$ such that

$$\Re(\mathbb{P}^n a_p, \mathbb{P}^{n+1} a_p) < \delta, \quad \forall\ n \geq p_1,$$

and from (4.1), we obtain

$$\Re(a_{p+n}, \mathbb{P} a_{p+n}) < \delta, \quad \forall\ n \geq p_1,$$

which contradicts the definition of δ implies. Hence,

$$\lim_{n\to\infty} \Re(a_n, \mathbb{P} a_n) = 0. \tag{4.8}$$

To prove,

$$\lim_{n\to\infty} (\Re(a_n, a_p)) = 0.$$

Assume, on contrary that for $\epsilon > 0$, $\exists$ sequences of natural numbers $\{r(n)\}_{n=1}^{\infty}$ and $\{t(n)\}_{n=1}^{\infty}$ such that

$$r(n) > t(n) > n.$$

$$\Re(a_{r(n)}, a_{t(n)}) \geq \epsilon. \tag{4.9}$$

$$\Re(a_{r(n)-1}, a_{t(n)}) < \epsilon, \quad \forall\ n \in \mathbb{N}. \tag{4.10}$$

The triangular inequality gives us,

$$\Re(a_{r(n)}, a_{t(n)}) \leq \mathrm{s}[\Re(a_{r(n)}, a_{r(n)-1}) + \Re(a_{r(n)-1}, a_{t(n)})],$$

$$\leq \mathrm{s}\Re(a_{r(n)}, a_{r(n)-1}) + \mathrm{s}\epsilon,$$

using (4.7)

$$\lim_{n\to\infty} \Re(a_n, \mathbb{P}a_n) = 0.$$

$\Rightarrow\ \exists\ p_2 \in \mathbb{N}$ such that

$$\Re(a_n, a_{n+1}) < \epsilon, \quad \forall\ n \geq p_2.$$

$$\Re(a_{r(n)}, a_{t(n)}) < \mathrm{s}\Re(a_{r(n)-1}, \mathbb{P}a_{r(n)-1}) + \mathrm{s}\epsilon, \qquad \forall\ n \in \mathbb{N}. \tag{4.11}$$

There $p_2 \in \mathbb{N}$ such that

$$\Rightarrow \Re(a_{r(n)}, a_{t(n)}) < 2\mathrm{s}\epsilon, \quad \forall\ n > p_2.$$

So that from (F_2), we get

$$F(\Re(a_{r(n)}, a_{t(n)}) < F(2\mathrm{s}\epsilon), \quad \forall\ n > p_2.$$

One can easily get,

$$\frac{1}{2\mathrm{s}}\Re(a_{r(n)}, \mathbb{P}a_{r(n)}) < \frac{\epsilon}{2\mathrm{s}} < \epsilon \leq \Re(a_{r(n)}, a_{t(n)}), \quad \forall\ n > p_2.$$

Considering, (4.8) and (4.11) and using the fact that $\mathbb{P}$ is generalized $(\alpha - F - \mathbb{P}_{\triangleleft})$ Suzuki type contraction.

$$\begin{aligned}
\tau\ +\ F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) &\leq \tau\ +\ \alpha(a_{r(n)}, a_{t(n)})F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)}))\\
&\leq a_1F(\Re(a_{r(n)}, a_{t(n)}))\ +\ a_2F(\Re(a_{r(n)}, \mathbb{P}a_{r(n)}))\\
&\quad +\ a_3F(\Re(a_{t(n)}, \mathbb{P}a_{t(n)}))\ +\ a_4(\tfrac{F(\Re(a_{r(n)}, \mathbb{P}a_{r(n)})) + F(\Re(a_{t(n)}, \mathbb{P}a_{t(n)}))}{2}).
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \tau + F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) &\leq a_1F(\Re(a_{r(n)}, a_{t(n)}))\ +\ a_2F(\Re(a_{r(n)}, \mathbb{P}a_{r(n)}))\\
&\quad +\ a_3F(\Re(a_{t(n)}, \mathbb{P}a_{t(n)}))\ +\ a_4(\tfrac{F(\Re(a_{r(n)}, \mathbb{P}a_{r(n)})) + F(\Re(a_{t(n)}, \mathbb{P}a_{t(n)}))}{2}).
\end{aligned}$$

Applying $\lim n \to \infty$, we get

$$\lim_{n\to\infty} F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) = -\infty.$$

From (F_2),

$$\lim_{n\to\infty} \Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)}) = 0.$$

This conflicts with (4.9). Consequently,

$$\lim_{n,p\to\infty} \Re(a_n, a_p) = 0.$$

$\{a_n\}$ is a Cauchy sequence in $\mathcal{H}$.

Due to the completeness of $(\mathcal{H}, \Re)$, there is $m \in \mathcal{H}$ such that

$$\lim_{n\to\infty} \Re(a_n, m) = 0. \tag{4.12}$$

We claim that

$$\frac{1}{2\text{s}}\Re(a_n, \mathbb{P}a_n) < \Re(a_n, m),$$

or

$$\frac{1}{2\text{s}}\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n) < \Re(\mathbb{P}a_n, m), \quad \forall\ n \in \mathbb{N}. \tag{4.13}$$

Suppose on contrary, there exist $m \in \mathbb{N}$ in such a way

$$\frac{1}{2\text{s}}\Re(a_p, \mathbb{P}a_p) \geq \Re(a_p, m), \tag{4.14}$$

$$\frac{1}{2\text{s}}\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p) \geq \Re(\mathbb{P}a_p, m).$$

From (4.5) and (F_1),

$$\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p) \leq \Re(a_p, \mathbb{P}a_p). \tag{4.15}$$

From (4.13) and (4.14)

$$\Re(a_p, \mathbb{P}a_p) \leq \text{s}\Re(a_p, m) + \text{s}\Re(m, \mathbb{P}a_p),$$

$$\begin{aligned} &\leq \tfrac{1}{2}\Re(a_p, \mathbb{P}a_P) + \tfrac{1}{2}\Re(\mathbb{P}a_p, \mathbb{P}^2 a_p), \\ &< \tfrac{1}{2}\Re(a_p, \mathbb{P}a_p) + \tfrac{1}{2}\Re(a_p, \mathbb{P}a_p), \\ &= \Re(a_p, \mathbb{P}a_p). \end{aligned}$$

This is a contradiction. $\mathbb{P}$ is generalized $(\alpha - F - \mathbb{P}_{\triangleleft})$ Suzuki type contraction, hence (4.12) is applicable.

(4.12) outcomes in either for every $n \in \mathbb{N}$,

$$\tau + F(\Re(\mathbb{P}a_n, \mathbb{P}m)) \leq \tau + \alpha(a_n, m)F(\Re(\mathbb{P}a_n, \mathbb{P}m))$$
$$\leq a_1 F(\Re(a_n, m)) + a_2 F(\Re(a_n, \mathbb{P}a_n))$$
$$+ a_3 F(\Re(m, \mathbb{P}m)) + a_4\left(\frac{F(\Re(a_n, \mathbb{P}a_n)) + F(\Re(m, \mathbb{P}m))}{2}\right), \tag{4.16}$$

then

$$\tau + F(\Re(\mathbb{P}a_n, \mathbb{P}m)) \leq a_1 F(\Re(a_n, m)) + a_2 F(\Re(a_n, \mathbb{P}a_n))$$
$$+ a_3 F(\Re(m, \mathbb{P}m)) + a_4\left(\frac{F(\Re(a_n, \mathbb{P}a_n)) + F(\Re(m, \mathbb{P}m))}{2}\right). \tag{4.17}$$

Similarly

$$\tau + F(\Re(\mathbb{P}^2 a_n, \mathbb{P}m)) \leq a_1 F(\Re(\mathbb{P}a_n, m)) + a_2 F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n))$$
$$+ a_3 F(\Re(m, \mathbb{P}m)) + a_4\left(\tfrac{F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) + F(\Re(m, \mathbb{P}m))}{2}\right).$$

Holds as a result of (F_2) as indicated by the limits in (4.7).

$$\lim_{n\to\infty} F(\Re(a_n, m)) = -\infty,$$

$$\lim_{n\to\infty} F(\Re(a_n, \mathbb{P}a_n)) = -\infty.$$

$\Rightarrow$ (4.15),

$$\lim_{n\to\infty} F(\Re(\mathbb{P}a_n, \mathbb{P}m)) = -\infty. \tag{4.18}$$

$$\Rightarrow \lim_{n\to\infty} \Re(\mathbb{P}a_n, \mathbb{P}m) = 0.$$

From (4.1),

$$\Re(m, \mathbb{P}m) \leq \mathrm{s}[\Re(m, \mathbb{P}a_n) + \Re(\mathbb{P}a_n, \mathbb{P}m)],$$

$$= \mathrm{s}\Re(m, a_{n+1}) + \mathrm{s}\Re(\mathbb{P}a_n, \mathbb{P}m).$$

$$\Rightarrow \lim_{n\to\infty} \Re(m, \mathbb{P}m) = 0,$$

$$\Rightarrow \quad m = \mathbb{P}m.$$

Analyze the 2nd case (4.16) immediately. From (4.1),

$F(\Re(\mathbb{P}^2 a_n, \mathbb{P}m)) < \tau + F(\Re(\mathbb{P}^2 a_n, \mathbb{P}m))$

$$\leq a_1 F(\Re(\mathbb{P}a_n, m)) \ + \ a_2 F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) + \ a_3 F(\Re(m, \mathbb{P}m))$$
$$+ \ a_4(\tfrac{F(\Re(\mathbb{P}a_n, \mathbb{P}^2 a_n)) + F(\Re(m, \mathbb{P}m))}{2}),$$
$$= a_1 F(\Re(a_{n+1}, m)) \ + \ a_2 F(\Re(a_{n+1}, \mathbb{P}a_{n+1}))$$
$$+ \ a_3 F(\Re(m, \mathbb{P}m)) \ + \ a_4(\tfrac{F(\Re(a_{n+1}, \mathbb{P}a_{n+1})) + F(\Re(m, \mathbb{P}m))}{2}).$$

From (4.7) and (F_2), then

$$\lim_{n\to\infty} F(\Re(\mathbb{P}^2 a_n, \mathbb{P}m)) = -\infty.$$

$$\Rightarrow \lim_{n\to\infty} \Re(\mathbb{P}^2 a_n, \mathbb{P}m) = 0. \tag{4.19}$$

From the (4.1),

$$\Re(m, \mathbb{P}m) \leq \mathrm{s}[\Re(m, \mathbb{P}^2 a_n) + \Re(\mathbb{P}^2 a_n, \mathbb{P}m)],$$

$$= \mathrm{s}[\Re(m, a_{n+2}) + \Re(\mathbb{P}^2 a_n, \mathbb{P}m)].$$

$$\Rightarrow \Re(m, \mathbb{P}m) = 0,$$

Hence, m is a fixed point of $\mathbb{P}$. Claim that $\mathbb{P}^n c = c \neq d = \mathbb{P}^n d$ for all $n \in \mathbb{N}$ and that $c, d \in \mathcal{H}$ are two fixed points of $\mathbb{P}$.

By using a_0, we have

$$(a_0 \lhd c \text{ and } a_0 \lhd d) \text{ or } (c \lhd a_0 \text{ and } d \lhd a_0).$$

$\mathbb{P}$ is $\lhd$-preserving, so the result is

$$(\mathbb{P}^n a_0 \lhd \mathbb{P}^n c \text{ and } \mathbb{P}^n a_0 \lhd \mathbb{P}^n d) \text{ or } (\mathbb{P}^n c \lhd \mathbb{P}^n a_0 \text{ and } \mathbb{P}^n d \lhd \mathbb{P}^n a_0),$$

for each $n \in \mathbb{N}$. Now,

$$\Re(c, d) = \Re(\mathbb{P}^n c, \mathbb{P}^n d) \leq \mathrm{s}(\Re(\mathbb{P}^n c, \mathbb{P}^n a_0) + \Re(\mathbb{P}^n a_0, \mathbb{P}^n d)).$$

As a result $n \to \infty$, we get $\Re(c, d) \leq 0$. So that $c = d$.

Therefore $\mathbb{P}$ has a unique fixed point in $\mathcal{H}$. □

Main theorem of Beg et al. is restricted case of (4.2.1) as follows:

Corollary 4.2.2.

Consider $(\mathcal{H}, \lhd, \Re_{\mathfrak{b}})$ an $\mathcal{OC}\mathfrak{b}ms$ with an orthogonal element a_0 and a constant $s \geq 1$. Suppose that $F \in \mathfrak{S}$, $\tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is generalized F- Suzuki $(F_{\lhd\mathfrak{b}})$ contraction mapping which is $\lhd$-preserving.

Then the sequence $\{\mathbb{P}^n a\}$ converges to a unique fixed point $m \in \mathcal{H}$ of $\mathbb{P}$.

Proof. Proof of (4.2.2) is obtain by substituting $a_4 = 0$

and $\alpha(a, b) = 1 \quad \forall\, a, b \in \mathcal{H}$ and $a_1 + a_2 + a_3 < 1$ in (4.2.1). □

Corollary 4.2.3.

Consider $(\mathcal{H}, \lhd, \Re)$ an $\mathcal{OC}ms$ with an orthogonal element a_0. Suppose that $F \in \mathfrak{S}$, $\tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is generalized $(\alpha - F - \mathbb{P}_{\lhd})$ Suzuki contraction mapping observing the following axioms:

(i) $\mathbb{P}$ is α-admissible mapping,

(ii) there exist $a_0 \in \mathcal{H}$ such that $\alpha(a_0, \mathbb{P}a_0) \geq 1$;

(iii) $\mathbb{P}$ is $\lhd$-preserving,

Then the sequence $\{\mathbb{P}^n a\}$ converges to a unique fixed point $m \in \mathcal{H}$ of $\mathbb{P}$.

Proof. (4.2.3) is concluded by taking $s = 1$ in (4.2.1). □

Corollary 4.2.4.

Consider $(\mathcal{H}, \lhd, \Re)$ an $\mathcal{OC}ms$ with an orthogonal element a_0. Suppose that $F \in \mathfrak{S}$, $\tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is $\mathbb{P}_{\lhd\mathfrak{b}}$-contraction mapping which is $\lhd$-preserving. Then the sequence $\{\mathbb{P}^n a\}$ converges to a unique fixed point $m \in \mathcal{H}$ of $\mathbb{P}$.

Proof. (4.2.4) is concluded by taking $s = 1$, $a_1 = 1$, $a_2 = a_3 = a_4 = 0$

and $\alpha(a, b) = 1 \quad \forall\, a, b \in \mathcal{H}$ in (4.2.1). □

Example 4.2.5.

Consider $\mathcal{H} = [0, 1] \cup \{2, 4\}$, and a mapping $\Re : \mathcal{H} \times \mathcal{H} \to [0, \infty)$

is determined by

$$\Re(c, d) = |c - d|^2, \quad \forall\, c, d \in \mathcal{H},$$

is a $\mathfrak{b}ms$ with s = 2.

If $cd \leq (c \vee d)$ then define the binary relation $\lhd$ on $\mathcal{H}$ as $c \lhd d$,

where $c \vee d =$ c or d. Then, $\mathcal{OC}\mathfrak{b}ms$ is $(\mathcal{H}, \Re)$.

Determine the mapping $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ by

$$\mathbb{P}(d) = \begin{cases} \ln(1 + \frac{d}{2}) & \text{if } d \in [0, 1], \\ d - 1 & \text{if } d \in \{2, 4\}. \end{cases}$$

Suppose $a_1 = \frac{1}{10}$, $a_2 = \frac{1}{10}$, $a_3 = \frac{1}{10}$, $a_3 = \frac{1}{4}$, and $a_4 = \frac{1}{10}$, and $\tau > 0$.

Consider $\alpha : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ is define as

$$\alpha(c, d) = c + d + 1,$$

Suppose $c = 2, d = 4$, then $\alpha(c, d) = c + d + 1$,

and $\alpha(\mathbb{P}c, \mathbb{P}d) = \ln(1 + \frac{c}{2}) + \ln(1 + \frac{d}{2}) + 1 \geq 1$,

hence $\mathbb{P}$ is α-admissible . $cd \leq c$. We obtain

$$\begin{aligned} \Re(\mathbb{P}c, \mathbb{P}d) &= |c - 1 - (d - 1)|^2, \\ &= |c - 1 - d + 1|^2, \\ &= 4. \\ \Re(c, d) &= |c - d|^2, \\ &= 4. \\ \Re(c, \mathbb{P}c) &= |c - (c - 1)|^2, \\ &= 1. \\ \Re(d, \mathbb{P}d) &= |d - (d - 1)|^2, \\ &= 1. \end{aligned}$$

So we have

$$cd = (2)(4) = 8 \leq c \vee d \Rightarrow c \lhd d,$$

and

$$\Re(\mathbb{P}c, \mathbb{P}d) > 0,$$

and

$$\frac{1}{2s}\Re(c, \mathbb{P}c) = \frac{1}{2s}(1) < 4 = \Re(c, d),$$

$$\Rightarrow \frac{1}{2s}\Re(c, \mathbb{P}c) < \Re(c, d).$$

But Let's take the mapping F which is defined as $F(m) = \ln m$, where $m \in (0, \infty)$. If $\tau = \ln 4$ and $a_1 + a_2 + a_3 + a_4 < 1$.

$\Longrightarrow \tau + F(\Re(\mathbb{P}c, \mathbb{P}d)) \leq \tau \ + \ \alpha(c, d)F(\mathbb{P}c, \mathbb{P}d) \geq a_1 F(\Re(c, d))$
$\quad + \ a_2 F(\Re(c, \mathbb{P}c)) \ + \ a_3 F(\Re(d, \mathbb{P}d)) \ + \ a_4(\frac{F(\Re(c,\mathbb{P}c)) + F(\Re(d,\mathbb{P}d))}{2})]$.

Since $\mathbb{P}$ is not generalized $(\alpha - F - \mathbb{P}_{\lhd})$ Suzuki contraction mapping. Also note that $\mathbb{P}$ is not $\lhd$-preserving in this case because $c \lhd d$ but $\mathbb{P}c \lhd \mathbb{P}d$ does not holds.

Now we consider the following cases.

For $\tau = \ln 4$ s.t $a_1 + a_2 + a_3 + a_4 < 1$.

Case(1):

When $c, d \in [0, 1]$. Then

$$\Re(\mathbb{P}c, \mathbb{P}d) = |\ln(1 + \frac{c}{2}) - (1 + \frac{d}{2})|^2 > 0,$$

$$\Re(c, d) = |c - d|^2,$$
$$\Re(c, \mathbb{P}c) = |c - \ln(1 + \tfrac{c}{2})|^2,$$
$$\Re(d, \mathbb{P}d) = |d - \ln(1 + \tfrac{d}{2})|^2,$$

It is clear that

$$c \lhd d, \Re(\mathbb{P}c, \mathbb{P}d) > 0$$

and

$$\frac{1}{2s}\Re(c, \mathbb{P}c) < \Re(c, d).$$

This implies

$\tau + F(\Re(\mathbb{P}c, \mathbb{P}d)) \leq \tau + \alpha(c, d)F(\mathbb{P}c, \mathbb{P}d)) \leq a_1 F(\Re(c, d))$
$\quad + \ a_2 F(\Re(c, \mathbb{P}c)) \ + a_3 F(\Re(d, \mathbb{P}d)) \ + a_4(\frac{F(\Re(c,\mathbb{P}c)) + F(\Re(d,\mathbb{P}d))}{2})$.

Since $\mathbb{P}$ is generalized $(\alpha - F - \mathbb{P}_{\lhd})$ Suzuki contraction mapping.

Also note that $\mathbb{P}$ is $\lhd$-preserving in this case.

Case(2):

When $c \in [0, 1]$ and $d \in \{2, 4\}$, then

$$\Re(\mathbb{P}c, \mathbb{P}d) = |\ln(1 + \tfrac{c}{2}) - c + 1|^2,$$
$$\Re(c, d) = |c - d|^2,$$
$$\Re(c, \mathbb{P}c) = |c - \ln(1 + \tfrac{c}{2}|^2,$$
$$\Re(d, \mathbb{P}d) = |d - (d-1)|^2 = 1,$$

It is clear that

$$c \lhd d, \Re(\mathbb{P}c, \mathbb{P}d) > 0$$

and

$$\frac{1}{2\text{s}}\Re(c, \mathbb{P}c) < \Re(c, d),$$

$$\Rightarrow \tau + F(\Re(\mathbb{P}c, \mathbb{P}d)) \leq \tau + \alpha(c,d)F(\Re(\mathbb{P}c, \mathbb{P}d)) \leq a_1F(\Re(c,d)) + a_2F(\Re(c, \mathbb{P}c)) + a_3F(\Re(d, \mathbb{P}d)) + a_4(\tfrac{F(\Re(c,\mathbb{P}c))+F(\Re(d,\mathbb{P}d))}{2}).$$

Since $\mathbb{P}$ is generalized $(\alpha - F - \mathbb{P}_{\lhd})$ Suzuki contraction mapping.

Also $\mathbb{P}$ is $\lhd$-preserving in this case.

Case(3):

When $c \in \{2, 4\}$ and $d \in [0, 1]$.

Similar to **Case(2)** As all the hypothesis of theorem are satisfied

so $\mathbb{P}$ has a unique fixed point i.e $m = 0$.

Theorem 4.2.6.

Consider $(\mathcal{H}, \lhd, \Re_{\mathfrak{b}})$ an $\mathcal{OC}\mathfrak{b}ms$ with an element a_0 and a constant s ≥ 1. Suppose that $F \in \mathfrak{S}$; $\tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is generalized $(\alpha - F - \mathbb{P}_{\lhd})$ Suzuki contraction mapping observing the following axioms:

(i) $\mathbb{P}$ is α- admissible mapping,

(ii) there exist $a_0 \in \mathcal{H}$ such that $\alpha(a_0, \mathbb{P}a_0) \geq 1$,

(iii) $\mathbb{P}$ is $\lhd$-preserving,

(iv) $\mathbb{P}$ is $\lhd$-continuous,

then, $\mathbb{P}$ has a unique fixed point $m \in \mathcal{H}$.

Proof. Since, $(\mathcal{H}, \lhd)$ is an O-set,

$$\exists\ a_0 \in \mathcal{H} : (\forall\ a \in \mathcal{H}, a \lhd a_0) \text{ or } (\forall\ a \in \mathcal{H}, a_0 \lhd a).$$

Thus, it implies that $a_0 \lhd \mathbb{P}a_0$ or $\mathbb{P}a_0 \lhd a_0$. Let

$$a_1 = \mathbb{P}a_0, a_2 = \mathbb{P}a_1 = \mathbb{P}^2a_0......, a_{n+1} = \mathbb{P}a_n = \mathbb{P}^{n+1}a_0, \quad \forall\ n \in \mathbb{N} \cup \{0\}. \tag{4.20}$$

If $a_n = a_{n+1}$ for any, $n \in \mathbb{N} \cup \{0\}$, so then a_n is a fixed point of $\mathbb{P}$.

If $a_n \neq a_{n+1} \;\; \forall\, n \in \mathbb{N} \cup \{0\}$, then $\Re(\mathbb{P}a_n, \mathbb{P}a_{n+1}) > 0 \;\; \forall \;\; n \in \mathbb{N}$.

Since $\mathbb{P}$ is $\lhd$-preserving, we have

$$a_n \lhd a_{n+1} \text{ or } a_{n+1} \lhd a_n, \quad \forall\, n \in \mathbb{N} \cup \{0\}.$$

$\Rightarrow \{a_n\}$ is an orthogonal sequence.

As $\mathbb{P}$ map is an α-admissible, then

$$\alpha(a_0, a_1) = \alpha(a_0, \mathbb{P}a_0) \geq 1.$$

$$\Rightarrow (a_0, a_1) = \alpha(\mathbb{P}a_0, \mathbb{P}a_1) \geq 1.$$

Inductively, we obtain

$$\alpha(a_n, a_{n+1}) \geq 1, \quad \forall\, n \in \mathbb{N} \cup \{0\}.$$

Using (iv), we have

$$0 < \Re(a_n, \mathbb{P}a_n) = \Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n), \quad \forall\, n \in \mathbb{N}. \tag{4.21}$$

Hence, we have

$$\begin{aligned}
\tau + F(\Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n)) &\leq \tau + \alpha(a_{n-1}, a_n) F(\Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n)) \\
&\leq a_1 F(\Re(a_{n-1}, a_n)) \;+\; a_2 F(\Re(a_{n-1}, \mathbb{P}a_{n-1})) \;+\; a_3 F(\Re(a_n, \mathbb{P}a_n)) \\
&\quad +\; a_4 (\tfrac{F(\Re(a_{n-1}, \mathbb{P}a_{n-1})) + F(\Re(a_n, \mathbb{P}a_n))}{2}),
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \tau + F(\Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n)) &\leq a_1 F(\Re(a_{n-1}, a_n)) \;+\; a_2 F(\Re(a_{n-1}, a_n)) \\
&\quad +\; a_3 F(\Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n)) \;+\; a_4 (\tfrac{F(\Re(a_{n-1}, a_n)) + F(\Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n))}{2}),
\end{aligned}$$

$a_1 + a_2 + a_3 + a_4 < 1$,

$$\tau + (1 - a_3 - \frac{a_4}{2}) F(\Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n)) \leq (a_1 + a_2 + \frac{a_4}{2}) F(\Re(a_{n-1}, a_n)).$$

$$F(\Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n)) \leq (\frac{a_1 + a_2 + \frac{a_4}{2}}{1 - a_3 - \frac{a_4}{2}}) F(\Re(a_{n-1}, a_n)) - \frac{\tau}{1 - a_3 - \frac{a_4}{2}},$$

since

$$\frac{a_1 + a_2 + \frac{a_4}{2}}{1 - a_3 - \frac{a_4}{2}} < 1,$$

$$F(\Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n)) < F(\Re(a_{n-1}, \mathbb{P}a_n)) - \frac{\tau}{1 - a_3 - \frac{a_4}{2}} < F(\Re(a_{n-1}, a_n)),$$

(F_1) implies that

$$\Re(a_n, \mathbb{P}a_n) = \Re(\mathbb{P}a_{n-1}, \mathbb{P}a_n) < \Re(a_{n-1}, \mathbb{P}a_{n-1}), \quad \forall\ n \in \mathbb{N},$$

$\Rightarrow \{\Re(a_n, \mathbb{P}a_n)\}_{n=1}^{\infty}$ is decreasing sequence that is bounded below, and

$$\lim_{n\to\infty} \Re(a_n, \mathbb{P}a_n) = \delta = \inf\{\Re(a_n, \mathbb{P}a_n) : \quad \forall\ n \in \mathbb{N}\}.$$

To prove $\delta = 0$, assume on contrary, $\delta > 0\ \exists\ p \in \mathbb{N}$ such that for every $\epsilon > 0$,

$$\Re(a_p, \mathbb{P}a_p) < \delta + \epsilon.$$

$$\Rightarrow F(\Re(a_p, \mathbb{P}a_p)) < F(\delta + \epsilon). \tag{4.22}$$

From (4.21), we conclude

$$0 < \Re(a_p, \mathbb{P}a_p) = \Re(a_{p-1}, \mathbb{P}a_p).$$

$$\begin{aligned}\tau\ +\ F(\Re(\mathbb{P}a_{p-1}, \mathbb{P}a_p)) &\le \tau\ +\ \alpha(a_{p-1}, a_p))F(\Re(\mathbb{P}a_{p-1}, \mathbb{P}a_p))\\ &\le a_1F(\Re(a_{p-1}, a_p))\ +\ a_2F(\Re(a_{p-1}, \mathbb{P}a_p))\ +\ a_3F(\Re(a_p, \mathbb{P}a_p))\\ &+\ a_4(\tfrac{F(\Re(a_{p-1}, \mathbb{P}a_{p-1})) + F(\Re(a_p, \mathbb{P}a_p))}{2}),\end{aligned}$$

$$\begin{aligned}\tau + F(\Re(\mathbb{P}a_{p-1}, \mathbb{P}a_p)) &= a_1F(\Re(a_{p-1}, a_p))\ +\ a_2F(\Re(a_{p-1}, a_p))\\ &+\ a_3F(\Re(\mathbb{P}a_{p-1}, \mathbb{P}a_p))\ +\ a_4(\tfrac{F(\Re(a_{p-1}, a_p)) + F(\Re(\mathbb{P}a_{p-1}, \mathbb{P}a_p))}{2}),\end{aligned}$$

$$\Rightarrow \tau + (1 - a_3 - \frac{a_4}{2})F(\Re(\mathbb{P}a_{p-1}, \mathbb{P}a_p)) \le (a_1 + a_2 + \frac{a_4}{2})F(\Re(a_{p-1}, a_p)).$$

$$a_1 + a_2 + a_3 + a_4 < 1,$$

$$F(\Re(\mathbb{P}a_{p-1}, \mathbb{P}a_p)) \le (\frac{a_1 + a_2 + \frac{a_4}{2}}{1 - a_3 - \frac{a_4}{2}})F(\Re(a_{p-1}, a_p)) - \frac{\tau}{1 - a_3 - \frac{a_4}{2}},$$

Since

$$\frac{a_1 + a_2 + \frac{a_4}{2}}{1 - a_3 - \frac{a_4}{2}} < 1,$$

$$F(\Re(\mathbb{P}a_{p-1}, \mathbb{P}a_p)) < F(\Re(a_{p-1}, \mathbb{P}a_p)) - \frac{\tau}{1 - a_3 - \frac{a_4}{2}}. \tag{4.23}$$

From (4.21), we have $0 < \Re(\mathbb{P}a_p, \mathbb{P}a_{p+1}) < \Re(a_p, \mathbb{P}^2 a_p)$,

and by the assumption of theorem, we obtain

$$\tau + F(\Re(\mathbb{P}a_p, \mathbb{P}a_{p+1})) \leq \tau + \alpha(a_p, a_{p+1})F(\Re(a_p, \mathbb{P}a_{p+1}))$$
$$+ \ a_1F(\Re(a_p, a_{p+1})) \ + \ a_2F(\Re(a_p, \mathbb{P}a_p)) \ + \ a_3F(\Re(a_{p+1}, \mathbb{P}a_{p+1}))$$
$$+ \ a_4(\tfrac{F(\Re(a_p,\mathbb{P}a_p))+F(\Re(a_{p+1},\mathbb{P}a_{p+1})}{2}.$$

$$\tau + F(\Re(\mathbb{P}a_p, \mathbb{P}a_{p+1})) = a_1F(\Re(a_p, a_{p+1})) \ + \ a_2F(\Re(a_p, \mathbb{P}a_p))$$
$$+ \ a_3F(\Re(a_{p+1}, \mathbb{P}a_{p+1})) \ + \ a_4(\tfrac{F(\Re(a_p,\mathbb{P}a_p))+F(\Re(a_{p+1},\mathbb{P}a_{p+1})}{2}.$$

$$\Rightarrow \tau + (1 - a_3 - \frac{a_4}{2})F(\Re(\mathbb{P}a_p, \mathbb{P}a_{p+1})) \leq (a_1 + a_2 + \frac{a_4}{2})F(\Re(a_p, a_{p+1})).$$

$a_1 + a_2 + a_3 + a_4 < 1,$

$$F(\Re(\mathbb{P}a_p, \mathbb{P}a_{p+1})) \leq (\frac{a_1 + a_2 + \frac{a_4}{2}}{1 - a_3 - \frac{a_4}{2}})F(\Re(a_p, a_{p+1})) - \frac{\tau}{1 - a_3 - \frac{a_4}{2}},$$

Since

$$\frac{a_1 + a_2 + \frac{a_4}{2}}{1 - a_3 - \frac{a_4}{2}} < 1,$$

$$F(\Re(\mathbb{P}a_p, \mathbb{P}a_{p+1})) < F(\Re(a_p, a_{p+1})) - \frac{\tau}{1 - a_3 - \frac{a_4}{2}}. \tag{4.24}$$

Now,from (4.22) and carrying out the same procedure as in (4.23) and (4.24), we obtain

$$\begin{aligned}
F(\Re(a_{p+n}, a_{p+n+1})) &= F(\Re(\mathbb{P}a_{p+n-1}, \mathbb{P}a_{p+n})),\\
&\leq F(\Re(a_{p+n-1}, a_{p+n})) - \tfrac{\tau}{1-a_3-\frac{a_4}{2}},\\
&= F(\Re(\mathbb{P}a_{p+n-2}, \mathbb{P}a_{p+n-1}))\tfrac{\tau}{1-a_3-\frac{a_4}{2}},\\
&\leq F(\Re(a_{p+n-2}, a_{p+n-1})) - \tfrac{2\tau}{1-a_3-\frac{a_4}{2}},\\
&= F(\Re(\mathbb{P}a_{p+n-3}, \mathbb{P}a_{p+n-2})) - \tfrac{2\tau}{1-a_3-\frac{a_4}{2}},\\
&\leq F(\Re(a_{p+n-3}, a_{p+n-2})) - \tfrac{3\tau}{1-a_3-\frac{a_4}{2}},\\
&\leq F(\Re(a_{p+1}, a_{p+2})) - \tfrac{(n-1)\tau}{1-a_3-\frac{a_4}{2}},\\
&= F(\Re(\mathbb{P}a_p, \mathbb{P}a_{p+1})) - \tfrac{(n-1)\tau}{1-a_3-\frac{a_4}{2}},
\end{aligned}$$

$$\leq F(\delta+\epsilon) - \frac{n\tau}{1-a_3-\frac{a_4}{2}}.$$

$$\Rightarrow \lim_{n\to\infty} F(\Re(a_{p+n}, a_{p+n+1})) = -\infty.$$

$\Rightarrow \lim_{n\to\infty} F(\Re(a_{p+n}, a_{p+n+1})) = 0$, implies $P_1 \in \mathbb{N}$ exists in such a way

$$\Re(a_{p+n}, a_{p+n+1}) < \delta, \quad \forall\, n \geq P_1,$$

and we obtain from (4.21),

$$\Re(a_{p+n}, \mathbb{P}a_{p+n}) < \delta, \quad \forall\, n \geq P_1,$$

which contradicts the definition of δ. Hence, $\delta = 0$ and from (4.22), we get

$$\lim_{n\to\infty} \Re(a_n, \mathbb{P}a_n) = 0. \tag{4.25}$$

To prove

$$\Rightarrow \lim_{n,p\to\infty} \Re(a_n, a_p) = 0.$$

Assume, on contrary that there exists two sequences of natural number $\{r(n)\}_{n=1}^{\infty}$ and $\{t(n)\}_{n=1}^{\infty}$, then

$$r(n) > t(n) > n,\ \Re(a_{r(n)}, a_{t(n)}) \geq \epsilon,$$

$$\Re(a_{r(n)-1}, a_{t(n)}) < \epsilon, \quad \forall\, n \in \mathbb{N}. \tag{4.26}$$

$$\Rightarrow \Re(a_{r(n)}, a_{t(n)}) \leq \mathrm{s}[\Re(a_{r(n)}, a_{r(n)-1}) + \Re(a_{r(n)-1}, a_{t(n)})],$$

$$\leq \mathrm{s}\Re(a_{r(n)}, a_{r(n)-1}) + \mathrm{s}\epsilon,$$

$$= \mathrm{s}\Re(a_{r(n)-1}, \mathbb{P}a_{r(n)-1}) + \mathrm{s}\epsilon, \qquad \forall, n \in \mathbb{N}. \tag{4.27}$$

By (4.25), there are $P_2 \in \mathbb{N}$ in the way that

$$\Re(a_{r(n)}, \mathbb{P}a_{r(n)}) < \epsilon, \quad \forall\, n > P_2. \tag{4.28}$$

(4.26) and (4.27) suggests that

$$\Re(a_{r(n)}, a_{t(n)}) < 2\mathrm{s}\epsilon, \quad \forall\ n > P_2,$$

by (F_2),

$$F(\Re(a_{r(n)}, a_{t(n)}) < F(2\mathrm{s}\epsilon), \quad \forall\ n > P_2, \tag{4.29}$$

using (4.26),

$$\epsilon \leq \Re(a_{r(n)+1}, a_{t(n)+1}) = \Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)}), \quad \forall\ n > P_2.$$

From (4.29),

$$\tau + F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) \leq \tau + \alpha(a_{r(n)}, a_{t(n)})F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)}))$$
$$\leq a_1 F(\Re(a_{r(n)}, a_{t(n)})) \ + \ a_2 F(\Re(a_{r(n)}, \mathbb{P}a_{r(n)})) \ + \ a_3 F(\Re(a_{t(n)}, \mathbb{P}a_{t(n)}))$$
$$+ \ a_4\left(\frac{F(\Re(a_{r(n)}, \mathbb{P}a_{r(n)})) + F(\Re(a_{t(n)}, \mathbb{P}a_{t(n)}))}{2}\right),$$
$$\tau + F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) < a_1 F(\Re(a_{r(n)}, a_{t(n)})) \ + \ a_2 F(\Re(a_{r(n)}, \mathbb{P}a_{r(n)}))$$
$$+ \ a_3 F(\Re(a_{t(n)}, \mathbb{P}a_{t(n)})) \ + \ a_4\left(\frac{F(\Re(a_{r(n)}, \mathbb{P}a_{r(n)})) + F(\Re(a_{t(n)}, \mathbb{P}a_{t(n)}))}{2}\right),$$

From (4.27) and (F_2),

$$\lim_{n\to\infty} F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) = -\infty,$$

by (F_2),

$$\lim_{n\to\infty} F(\Re(\mathbb{P}a_{r(n)}, \mathbb{P}a_{t(n)})) = 0 \Leftrightarrow \lim_{n\to\infty} F(\Re(a_{r(n)+1}, a_{t(n)+1})) = 0,$$

contradiction to (4.26) . By the completeness of $(\mathcal{H}, \Re)$ there exists $m \in \mathcal{H}$ such that

$$\lim_{n\to\infty} \Re(a_n, m) = 0.$$

Since, $\mathbb{P}$ is $\triangleleft$-continuous,

$$\lim_{n\to\infty} \Re(\mathbb{P}a_n, \mathbb{P}m) = 0.$$

As

$$\Re(m, \mathbb{P}m) \leq \mathrm{s}[\Re(m, a_n) + \Re(a_n, \mathbb{P}m)],$$

m is a fixed points of $\mathbb{P}$ since

$$\Rightarrow \Re(m, \mathbb{P}m) = 0.$$

Assuming that $\mathbb{P}^n c = c \neq d = \mathbb{P}^n d \;\; \forall\, n \in \mathbb{N}$,

let $c, d \in \mathcal{H}$ be two fixed point of $\mathcal{H}$. By choice of a_0, we get

$$(a_0 \lhd c \text{ and } a_0 \lhd d) \text{ or } (c \lhd a_0 \text{ and } d \lhd a_0).$$

$\mathbb{P}$ is $\lhd$-preserving, we get

$(\mathbb{P}^n a_0 \lhd \mathbb{P}^n c$ and $\mathbb{P}^n a_0 \lhd \mathbb{P}^n d)$ or $(\mathbb{P}^n c \lhd \mathbb{P}^n a_0$ and $\mathbb{P}^n d \lhd \mathbb{P}^n a_0)$, $\quad \forall\, n \in \mathbb{N}$.

Now,

$$\Re(c, d) = \Re(\mathbb{P}^n c, \mathbb{P}^n d) \leq \mathrm{s}(\Re(\mathbb{P}^n c, \mathbb{P}^n a_0) + \Re(\mathbb{P}^n a_0, \mathbb{P}^n d)).$$

We get $\Re(c, d) \leq 0$ as $n \to \infty$. $\Rightarrow c = d$.

Therefore, $\mathbb{P}$ has a unique fixed point in $\mathcal{H}$. □

Corollary 4.2.7.

Consider $(\mathcal{H}, \lhd, \Re_{\mathfrak{b}})$ an $\mathcal{OC}\mathfrak{b}ms$ with an element a_0 and a constant s ≥ 1. Suppose that $F \in \mathfrak{S}$; $\tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is generalized F- Suzuki $(F_{\lhd\mathfrak{b}})$-contraction mapping observing the following axioms:

(i) $\mathbb{P}$ is $\lhd$-preserving,

(ii) $\mathbb{P}$ is $\lhd$-continuous,

then, $\mathbb{P}$ has a unique fixed point $m \in \mathcal{H}$.

Proof. (4.2.7) is concluded by taking $a_4 = 0$ and $\alpha(a, b) = 1$

$\forall\, a, b \in \mathcal{H}$ and $a_1 + a_2 + a_3 < 1$ in (4.2.6). □

Corollary 4.2.8.

Consider $(\mathcal{H}, \lhd, \Re)$ an $\mathcal{OC}ms$ with an orthogonal element a_0. Suppose that $F \in \mathfrak{S}$; $\tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is generalized $(\alpha - F - \mathbb{P}_{\lhd})$ Suzuki contraction mapping observing the following axioms:

(i) $\mathbb{P}$ is α-admissible mapping,

(ii) there exist $a_0 \in \mathcal{H}$ such that $\alpha(a_0, \mathbb{P}a_0) \geq 1$,

(iii) $\mathbb{P}$ is $\lhd$-preserving,

(iv) $\mathbb{P}$ is $\lhd$-continuous,

then, $\mathbb{P}$ has a unique fixed point $m \in \mathcal{H}$.

Proof. (4.2.8) is concluded by applying s = 1 in Theorem (4.2.6). □

Corollary 4.2.9.

Consider $(\mathcal{H}, \lhd, \Re)$ an $\mathcal{OC}ms$ with an orthogonal element a_0. Suppose that $F \in \mathfrak{S}$; $\tau > 0$ and $\mathbb{P} : \mathcal{H} \to \mathcal{H}$ is $\mathbb{P}_{\lhd\mathfrak{b}}$-contraction mapping observing the following axioms:

(i) $\mathbb{P}$ is $\lhd$-preserving,

(ii) $\mathbb{P}$ is $\lhd$-continuous,

then, $\mathbb{P}$ has a unique fixed point $m \in \mathcal{H}$.

Proof. (4.2.9) is obtain by taking s $= 1$, $a_1 = 1, a_2 = a_3 = a_4 = 0$ and $\alpha(a, b) = 1 \quad \forall\, a, b \in \mathcal{H}$ in Theorem (4.2.6).

Chapter 5

Conclusion

We wrap up our research in the following manners:

- A quick overview of history sets the stage for a brief exploration of fixed point theory.
- Certain mappings are expounded upon to enhance comprehension of contractions.
- Special emphasis is given on F- contraction mapping and its extensions, highlighting its importance along with the examples.
- A section is dedicated to different important concepts related to the metric fixed point theory.
- A comprehensive analysis of the work by Beg et al. [29] is elaborated. This task involves an in-depth examination of orthogonality within a complete $\mathfrak{b}ms$ and the preservation of fixed points for $F_{\triangleleft\mathfrak{b}}$-contraction mapping. These aspects are explored within the framework of a complete $\mathfrak{b}ms$.
- Two fixed point results are established within an $\mathcal{OC}\mathfrak{b}ms$, employing the approach introduced by Beg et al. and using the foundation of a complete $\mathfrak{b}ms$.

Existence of fixed point for an $(\alpha - F - \mathbb{P}_{\triangleleft})$ Suzuki contraction is established.

- $(\alpha - F - \mathbb{P}_{\triangleleft})$ Suzuki contraction mappings is generalization of generalized $F_{\triangleleft\mathfrak{b}}$-contraction introduced in [29].

- One non-trivial example is provided for validating the proven result.

- Several corollaries are presented to exhibit that many existing fixed point results are the special case established in the present research.

- Results proved by Beg et al. [29] are also the restricted case of these theorems.

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