CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Quadruple Fixed Point Results with Matrix Equations in Generalized b-Metric Spaces

by

Sidra Fida

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My this achievement is dedicated to my beloved family. Specially, to my father ***Fida Hussain****,*
mother ***Allah Rakhi****,*
and
brother ***Sarmad Fida****.*

CERTIFICATE OF APPROVAL

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Sidra Fida

Abstract

Hammad and Abdeljawad established some quadruple fixed point and quadruple coincidence point theorems in the setting of generalized metric spaces. In this research study, we extended quadruple fixed point theorems in the framework of generalized b-metric spaces in association with matrices. Some corollaries being the spacial cases of main results are also presented. For the validation of results, some supportive examples are attached. Eventually, an application on the study of unique stationary distribution of Markov process is constructed in the support of the theoretical ideas.

Contents

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Abbreviations

BCP	Banach Contraction Principle
GMS	Generalized Metric Space
GbMS	Generalized b-Metric Space
MMp	Mixed Monotone Property
MQTp	Mixed Quadruple Transcendental Point
MsMP	Mixed s-Monotone Property
PoM	Partially Ordered Metric Space
POCGMS	Partially Ordered Complete Generalized Metric Space
POCGbMS	Partially Ordered Complete Generalized b-Metric Space
QFp	Quadruple Fixed Point
QCp	Quadruple Coincidence Points
QC	Quadruple Comparable Points
TFp	Triple Fixed Point
ZM	Set of all $n \times n$ Matrices over $\mathbb{R}^+$

Symbols

β	Reciprocal of b
d	Metric function
d_b	b-Metric function
$(\mathcal{X}, \mathrm{d})$	Metric space
$(\mathcal{X}, \mathrm{d}_b)$	b-Metric space
υ	Generalized metric function
$(\aleph, \upsilon)$	Generalized metric space
υ_b	Generalized b-metric function
$(\aleph, \upsilon_b)$	Generalized b-metric space
$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^+$	The set of non-negative real numbers
$\mathbb{N}$	The set of natural numbers
$\mathbb{W}$	The set of whole numbers
$M_{n\times n}(\mathbb{R}^+)$	The set of all $n \times n$ matrices over $\mathbb{R}^+$
Ω^4_{n-1}	$4(n-1)$ dimensional unit simplex
O	Zero matrix
I	Identity matrix
I^d	Identity map
$\preceq$	Partial order
$\times$	Cartesian product
$\emptyset$	Empty set
$\neq$	Not equal
$\forall$	For all
$\in$	Belongs to

$\ni$	Such that
$\exists$	There exists
$\Rightarrow$	Implies
$\rightarrow$	Approaches to
$\Longleftrightarrow$	If and only if
lim	Limit
max	Maximum
min	Minimum
∞	Infinity

Chapter 1

Introduction

1.1 Background

"Mathematics is the mother of all sciences", is not only a sentence but there lies a whole universe in it. There are countless applications of mathematics in almost every field of life, for example, in weather prediction, medical science, security concerns, engineering, banking and finance and many others. Mathematics is fractionated into many branches such as arithmetic, algebra, trigonometry, calculus, number theory, probability and statistics and many more. One of the most powerful branch of mathematics is functional analysis. The term " functional" implying "a function whose argument is a function" was introduced by an Italian mathematician Vito Volterra in 19^{th} century and in 1910 it was first used by a French mathematician J.S. Hadamard in his book on that subject. Functional analysis is widely applicable in linear and non-linear analysis, calculus of variation, approximation theory, numerical analysis, integral equations and many others. In non-linear analysis, one of the fundamental tools is metric fixed point theory. In the current research period, one of the most valuable research is finding the solution of differential and integral equation by the way of fixed point theory.

In 1886, the field of fixed point theory was first time deliberated by a French mathematician Poincare [1] and introduced various results on the study of fixed point theorems. A fixed point theorem is a statement which guarantees that under specific conditions there exist one or more points of a mapping $\digamma : \mathcal{X} \to \mathcal{Y}$ such that $\digamma(r) = r$, where $r \in \mathcal{X}$ called fixed point of mapping $\digamma$. In 1910, Brouwer [2] established his fixed point theorem for continuous mappings over Euclidean metric space. Subsequently, Stephan Banach [3] presented his remarkable result in 1922 known as Banach contraction principle (BCP), which was accepted as the fundamental result in fixed point theory. BCP states that a self map $\digamma : \mathcal{X} \to \mathcal{X}$ on a complete metric space $(\mathcal{X}, \mathrm{d})$ has a unique fixed point in $\mathcal{X}$ if it satisfy contraction condition i.e,

$$\mathrm{d}(\digamma(r), \digamma(t) \leq \alpha \mathrm{d}(r,t), \qquad \forall\, r, t \in \mathcal{X},$$

provided that $\alpha \in [0, 1)$.

In 1906, M. Frechet [21] established a new class of spaces termed as metric spaces

under certain conditions satisfied by the mapping over the underlying set. With the passage of time, researchers paid efforts on the generalization of BCP in two different directions (i) by changing the space (ii) working with more properties of contractions mappings. In this regard, a useful result was presented by Edelstein [4] in 1962 by adapting changes in contraction condition.

Perov [5] in 1964 extended classical BCP on spaces endowed with vector-valued metric spaces. Later on, in 1968 Kannan [6] introduced a new version of BCP by way of refinement in the continuity of contraction condition. Subsequently, BCP was generalized by Nadler [7] in 1969 on set-valued mappings by working on multi-valued contraction mappings and opened a new door for researcher to quench the thirst. For more concepts in this regard, we refer to [8–10].

Later on, meddling with the defined properties of metric, authors demonstrated

various types of metric spaces. Working on the idea of non-zero self distance, Matthews [22] introduced the notion of partial metric spaces in 1992. In 1983, by introducing a new parameter in triangular inequality, Bakhtin [23] generate the concept of b-metric space which was remodeled by Czerwik [24] in 1993. Moreover, BCP was also generalized on b-metric space. Afterwards, Akkouchi [25] elaborated his exciting results on the existence of fixed point of mappings in b-metric spaces.

Ever since, many articles have been publicized on numerous type of single-valued and multi-valued operations in frame of b-metric spaces, see [26, 27].

Working on a new track, idea of couple fixed point was first studied by Optoitsev [11] and then in 1987 by Guo and Lakshmikantham [12], later on in the year 2006, Bhaskhar and Lakshmikantham [13] introduced the concept of mixed monotone mappings and worked on coupled fixed point of mappings in a partial ordered metric spaces (PoMs) and the idea was supported by demonstrating an application to the existence of solution of a periodic boundary value problem. Following the path, Berinde and Borcut [14], in 2011, extended the idea of couple fixed point and presented the notion of triple fixed point (TFp) for self mappings and set up remarkable results in PoMs. Subsequently, generalizing the concept of TFp in 2012, Karapinar [15] opened a gateway for researchers in the new direction by proposing the theory of quadruple fixed point (QFp) of mappings and established exciting consequences in this regard [16, 17]. One can have a deeper understanding of the concepts through [18–20].

Perov's [5] extension of BCP on spaces endowed with vector-valued metric spaces gave birth to a new way of research for authors. These spaces named as generalized metric spaces (GMS). For GMS, the notions of convergent sequences, Cauchy sequences, completeness, open subsets, closed subsets and continuous mappings are similar to those for usual metric spaces. In 1884, Gary Gruenhage [28] studied the concept the GMS under weak topological properties in his book on title "Set-Theoretical Topology". Later on, authors established fixed point theorems for fixed point of mappings [29, 30], coupled fixed point results [31, 32], TFp results [33] and QFp results [34, 35] on GMS.

In our research study, a blend of the GMS endowed with vector-valued metric spaces and b-metric spaces is produced and termed as generalized b-metric spaces endowed with vector-valued metric spaces (GbMS) by adapting a new co-efficient in triangular inequality of GMS. Furthermore, motivated by the idea presented in [34], we established coincident points and quadruple fixed points of mappings in the setting of GbMS and introduced some useful theoretical consequences.

1.2 Thesis Contribution

An outline of the forthcoming chapters of our thesis is highlighted below;

(1). **Chapter 2:**

Chapter 2 includes a short review of basic concepts regarding metric space. This chapter is partitioned into four sections. "Matrix Equations" includes the few definitions and important results about matrix conversions. Next section is "Basic Tools" which is about to built a base in metric spaces together with a few examples in this context are also stated. In "Fixed Point of Mappings", a quick review of history of fixed point theorems is illustrated. Eventually, we present few important extensions in metric spaces which will be helpful in the forthcoming chapters.

(2). **Chapter 3:**

In this chapter a detailed review of the work by Hammad *et al*. [34] is articulated. Some definitions and few important consequences on the structure of GMS with the help of examples are demonstrated. Lastly, an applications about stationary distribution of Markov process verifies the obtained results.

(3). **Chapter 4:**

The main theme presented in this chapter is the establishment of coincidence point and quadruple fixed point of mappings in the setting of GbMS, generalized from the idea presented in [34]. Furthermore, we modify few definitions and results of GMS on the structure of GbMS and elaborate by virtue of some examples. Lastly, an application on the study of existence of unique stationary distribution in the frame of GbMS validates our generalized idea.

(4). **Chapter 5:**

Chapter 5 is based on conclusion and future works.

Chapter 2

Preliminaries

Chapter 2 is an introduction to the basics of metric spaces and fixed point theorems. First section covers the main idea of metric spaces along with the notions of convergence, Cauchy sequence, completeness, continuity and contraction mappings. Second section deals with the theory of fixed points. Next section highlights few extensions in metric spaces, important results and examples on b-metric spaces and generalized metric spaces(GMS). Last section includes the concept of matrix convergence in $M_{n\times n}(\mathbb{R}^+)$ with supportive examples.

2.1 Basic Tools

This section is short analysis of few basic concepts and fundamental results in metric spaces along with some examples.

Definition 2.1.1.

"A partially ordered set is a set $\mathcal{X}$ on which there is defined partial ordering, that is, a binary relation which is written as $\preceq$ and satisfies the conditions:

(P 1) $r \preceq r$ for every $r \in \mathcal{X}$; (Reflexive)

(P 2) $r \preceq t$ and $t \preceq r$, then $r = t$; (Anti-symmetric)

(P 3) $r \preceq t$ and $t \preceq q$, then $r \preceq q$. (Transitive)

Partially emphasizes that $\mathcal{X}$ may contain r and t for which neither $r \preceq t$ nor $t \preceq r$ holds. Then r and t are called incomparable elements. In contrast, two elements r and t are called comparable elements if they satisfy $r \preceq t$ or $t \preceq r$ (or both)." [21]

Example 2.1.2.

(i): Let $\mathcal{X}$ be a non empty set and $P(\mathcal{X})$ be the power set of $\mathcal{X}$ i.e contains all subsets of $\mathcal{X}$, define a partial order "$\preceq$" on $P(\mathcal{X})$ as for any $U, V \in P(\mathcal{X})$,

$$U \preceq V \iff U \subseteq V.$$

Then, the pair $(P(\mathcal{X}), \preceq)$ is a partially ordered set.

(ii): Let $\mathcal{Q}$ be the set of all real-valued functions on $[0, 1]$ and "$\preceq$" be the binary relation on $\mathcal{Q}$ defined as;

$$f \preceq g \iff f(r) \leq g(r), \qquad \forall\, r \in [0, 1],$$

for any $f, g \in \mathcal{Q}$. Hence, $(\mathcal{Q}, \preceq)$ is a partially ordered set.

Definition 2.1.3.

"A totally ordered set or chain is a partially ordered set such that every two elements of the set are comparable. In other words, a chain is a partially ordered set that has no incomparable elements." [21]

Remark 1:

Every totally ordered set is a partially ordered set but the converse is not true in general.

Example 2.1.4.

The divisibility relation on the set of natural numbers $\mathbb{N}$ is a partial order and not a total order. However certain subsets of $\mathbb{N}$ with the divisibility relation on them may be totally ordered. For instance, consider the divisibility relation on the subset

$$\mathcal{M} = \{4, 16, 64, 256, 1024\}.$$

As it can be seen, all ordered pairs in the relation are comparable.

M. Frechet [21] introduced the concept of metric spaces in 1906. Later on, it turns into a vast field of research for authors and opens a broader stage for pure as well as applied mathematicians.

Definition 2.1.5.

"A metric space is a pair $(\mathcal{X}, \mathrm{d})$, where $\mathcal{X}$ is a set and d is a metric on $\mathcal{X}$(or distance function on $\mathcal{X}$), that is, a function defined on $\mathcal{X} \times \mathcal{X}$ such that for all $r_1, r_2, r_3 \in \mathcal{X}$, we have:

(M1): d is real-valued, finite and non-negative;

(M2): $\mathrm{d}(r_1, r_2) = 0$ if and only if $r_1 = r_2$;

(M3): $\mathrm{d}(r_1, r_2) = \mathrm{d}(r_2, r_1)$; (Symmetry)

(M4): $\mathrm{d}(r_1, r_3) \leq \mathrm{d}(r_1, r_2) + \mathrm{d}(r_2, r_3)$. (Triangle inequality)

The symbol $\times$ denotes the Cartesian product of two sets. Hence, $\mathcal{X} \times \mathcal{X}$ is the set of all ordered pairs of elements of $\mathcal{X}$." [21]

Example 2.1.6.

Some examples on the idea of metric space are discussed below:

(I) **Real Line $\mathbb{R}$.** Let $\mathcal{X} = \mathbb{R}$ be the set of all real numbers and $\mathrm{d} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be the distance function defined as for all $r_1, r_2 \in \mathcal{X}$

$$\mathrm{d}(r_1, r_2) = |r_1 - r_2|.$$

Then, the metric d satisfies all of the axioms of metric space. Hence, the pair $(\mathcal{X}, \mathrm{d})$ is a metric space known as usual metric space.

(II) **Euclidean Plane** $\mathbb{R}^2$. Let $\mathcal{X} = \mathbb{R}^2$ be the Euclidean space and d be the euclidean metric on $\mathcal{X}$ defined as, for all $r = (r_1, r_2), t = (t_1, t_2) \in \mathcal{X}$

$$\mathrm{d}(r, t) = \sqrt{(r_1 - t_1)^2 + (r_2 - t_2)^2}.$$

Hence, $(\mathbb{R}^2, \mathrm{d})$ is a metric space known as Euclidean metric space.

(III) **Taxicab Metric Space.** Let $\mathcal{X} = \mathbb{R}^2$ be the Cartesian Plane and d be the metric defined in an other way as, for all $r = (r_1, r_2), t = (t_1, t_2) \in \mathcal{X}$

$$\mathrm{d}(r, t) = |r_1 - t_1| + |r_2 - t_2|.$$

Since, one can easily verify that the axioms (M1) - (M4) are satisfied for above defined metric, hence $(\mathbb{R}^2, \mathrm{d})$ is a metric space which is called Taxicab metric space.

Definition 2.1.7.

" A sequence $\{r_n\}$ in a metric space $\mathcal{X} = (\mathcal{X}, \mathrm{d})$ is said to converge or to be convergent if there is an $r \in \mathcal{X}$ such that

$$\lim_{n \to \infty} \mathrm{d}(r_n, r) = 0.$$

r is called limit of $\{r_n\}$ and we write $\lim_{n \to \infty} r_n = r$, or simply, $r_n \to r$." [21]

Definition 2.1.8.

" A sequence $\{r_n\}$ in a metric space $\mathcal{X} = (\mathcal{X}, \mathrm{d})$ is said to be Cauchy Sequence (or fundamental) if for every $\epsilon > 0$ there is an $\mathcal{N} = \mathcal{N}(\epsilon)$ such that

$$\mathrm{d}(r_m, r_n) < \epsilon, \qquad \forall \quad m, n > \mathcal{N}." [21]$$

Example 2.1.9.

Let $\{r_n\}$ be a sequence in $\mathbb{R}$ equipped with usual metric. Suppose that for

$0 < \alpha < 1$, $\{r_n\}$ satisfy the condition $|r_{n+1} - r_n| \leq \alpha^n$ for all $n \in \mathbb{N}$. Then $\{r_n\}$ is a Cauchy sequence in $\mathbb{R}$. Since, for $m, n \in \mathbb{N}$ with $m > n$, we have

$$|r_m - r_n| \leq |r_n - r_{n+1}| + |r_{n+1} - r_{n+2}| + \cdots + |r_{m-1} - r_m|$$

$$\leq \alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}$$
$$= \frac{\alpha^n}{1-\alpha}|1 - \alpha^{m-n}| < \frac{\alpha^n}{1-\alpha}.$$

Since, $0 < \alpha < 1$, $\alpha^n \to 0$ and so given any $\epsilon > 0$, we can choose $\mathcal{N} \in \mathbb{N}$ such that $\frac{\alpha^n}{1-\alpha} < \epsilon$, hence for all $m, n > \mathcal{N}$, we have

$$|r_m - r_n| \leq \frac{\alpha^n}{1-\alpha} < \epsilon.$$

Hence, $\{r_n\}$ is a Cauchy sequence in $\mathbb{R}$.

Remark 2:

Every convergent sequence is a Cauchy sequence in a metric space but converse is not true in general.

Definition 2.1.10.

" A metric space $\mathcal{X} = (\mathcal{X}, \mathrm{d})$ is said to be complete metric space if every Cauchy sequence in $\mathcal{X}$ converges (that is, has a limit which is an element in $\mathcal{X}$)." [21]

Example 2.1.11.

Some examples of complete metric spaces are illustrated bellow.

(1) The **Real line** and the **Complex plane** are complete metric spaces.

(2) **Completeness of $\mathbb{R}^n$ and $\mathbb{C}^n$**: Euclidean Space $\mathbb{R}^n$ and Unitary Space $\mathbb{C}^n$ are complete under the following defined metric

$$\mathrm{d}(r, t) = \left(\sum_{j=1}^{n} (r_j - t_j)^2 \right)^{\frac{1}{2}}$$

where $r = \{r_j\}, t = \{t_j\} \in \mathbb{R}^n$ or $\mathbb{C}^n$.

Let $\{r_m\}$ be a Cauchy sequence in $\mathbb{R}^n$ (or $\mathbb{C}^n$), where $r_m = (r_1^{(m)}, r_2^{(m)}, \cdots, r_n^{(m)})$.

Hence, for each $\epsilon > 0$ there is an $\mathcal{N}(\epsilon)$ such that

$$d(r_m, r_k) = \left(\sum_{j=1}^{n}(r_j^{(m)} - r_j^{(k)})^2\right)^{\frac{1}{2}} < \epsilon, \qquad (m, k > \mathcal{N}(\epsilon)).$$

Squaring, we have for $m, n > \mathcal{N}(\epsilon)$ and $j = 1, 2, \cdots, n$

$$\left(r_j^{(m)} - r_j^{(k)}\right)^2 < \epsilon^2, \qquad \text{and} \qquad |r_j^{(m)} - r_j^{(k)}| < \epsilon.$$

Since, for each j, the sequence $(r_j^{(1)}, r_j^{(2)}, \cdots)$ is a Cauchy sequence in $\mathbb{R}$(or $\mathbb{C}$) and every Cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$) converges i.e for each j, $r_j^{(m)} \to r_j$ as $m \to \infty$. Using n limits, we define $r = (r_1, r_2, \cdots, r_n) \in \mathbb{R}$(or $\mathbb{C}$) such that with $k \to \infty$

$$d(r_m, r) < \epsilon, \qquad (m > \mathcal{N}(\epsilon)).$$

Which proves the completeness of $\mathbb{R}^n$ (or $\mathbb{C}^n$).

Definition 2.1.12.

"Let $\mathcal{X} = (\mathcal{X}, d_1)$ and $\mathcal{Y} = (\mathcal{Y}, d_2)$ be metric spaces. A mapping $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is said to be continuous at a point $r_o \in \mathcal{X}$ if for every $\epsilon > 0$ there is a $\delta > 0$, such that $d_2(\mathcal{F}(r), \mathcal{F}(r_o)) < \epsilon$, for all r satisfying $d_1(r, r_o) < \delta$." [21]

Example 2.1.13.

Let $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ be a mapping on metric space $\mathcal{X} = [0, 1]$ equipped with usual metric defined as;

$$\mathcal{F}(r) = \frac{1}{r+1}, \qquad \forall\, r \in \mathcal{X}.$$

Then, $\mathcal{F}$ is a continuous map on $\mathcal{X}$.

Theorem 2.1.14.

"A mapping $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ of a metric space $(\mathcal{X}, d_1)$ into a metric space $(\mathcal{Y}, d_2)$ is continuous at a point $r \in \mathcal{X}$ if and only if $r_n \to r$ implies $\mathcal{F}(r_n) \to \mathcal{F}(r)$." [21]

Definition 2.1.15.

"Let $\mathcal{X} = (\mathcal{X}, d)$ be a metric space. A mapping $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is called a contraction on $\mathcal{X}$ if there is a positive real number $\alpha < 1$ such that for all $q, r \in \mathcal{X}$,

$$\mathrm{d}(\mathcal{F}(q), \mathcal{F}(r)) \leq \alpha \mathrm{d}(q, r),$$

where, $(\alpha < 1)$.

Geometrically, this means that any points q and r have images that are closer together than those points q and r, more precisely, the ratio $\frac{\mathrm{d}(\mathcal{F}(q),\mathcal{F}(r))}{\mathrm{d}(q,r)}$ does not exceed a constant α which is strictly less than 1." [21]

Example 2.1.16.

Let $\mathcal{X} = [0, 1]$ be a metric space equipped with metric $\mathrm{d}(q, r) = |q - r|$. Define a mapping $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ as

$$\mathcal{F}_q = \frac{q+1}{4}, \qquad \forall\ q \in \mathcal{X}.$$

Then,

$$\begin{aligned}
\mathrm{d}(\mathcal{F}_q, \mathcal{F}_r) &= |\frac{q+1}{4} - \frac{r+1}{4}| \\
&= |\frac{q+1-r-1}{4}| \\
&= |\frac{q-r}{4}| \\
&= \frac{1}{4}|q-r| \\
&= \frac{1}{4}\mathrm{d}(q, r).
\end{aligned}$$

Hence, with contraction constant $\alpha = \frac{1}{4}$, $\mathcal{F}$ is a contraction mapping on $\mathcal{X}$.

Remark 3:

Every contraction mapping is continuous.

2.2 Fixed Points of Mappings

In 19^{th} century, H. Poincare introduced the idea that finding the solution of an equation is equivalent to that of finding the fixed point of parallel mapping. This notion is defined as under;

Definition 2.2.1.

"A fixed point of a mapping $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ on set $\mathcal{X}$ into itself is $r \in \mathcal{X}$ which is mapped onto itself, that is, $\mathcal{F}(r) = r$, the image $\mathcal{F}(r)$ coincides with r." [21]

Geometrically, fixed point of a real-valued function(mapping) is the point of intersection of the line y $= r$ and mapping y $= \mathcal{F}(r)$. One or more than one fixed points of a mapping can exists, moreover, sometimes mappings do not have even a single fixed point.

In 1922, Stephan Banach introduced a fundamental result for the existence of a unique fixed of a mapping.

Theorem 2.2.2.

"Consider a metric space $\mathcal{X} = (\mathcal{X}, \mathrm{d})$, where $\mathcal{X} \neq \emptyset$. Suppose that $\mathcal{X}$ is a complete metric space and let $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ be a contraction mapping on $\mathcal{X}$. Then, $\mathcal{F}$ has precisely one fixed point." [21]

Example 2.2.3.

Some examples of fixed points are:

(a): Let $\mathcal{F} : \mathbb{R} \to [0, 1]$ be a function defined as $\mathcal{F}(r) = \frac{1}{2}\cos^2(r)$. Graphically,

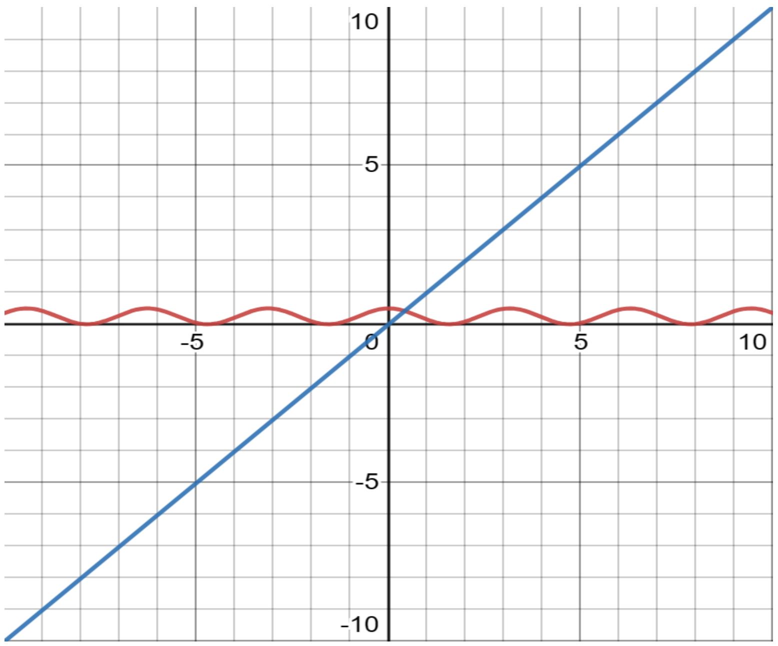

FIGURE 2.1: Unique fixed point

Hence, $\digamma$ has unique fixed point.

(b): The function $\digamma : \mathcal{X} \to \mathbb{R}$, where $\mathcal{X} = \mathbb{R} - \{(2n+1)\frac{\pi}{2}\}$ defined as $\digamma(r) = \tan(r)$ has infinitely many fixed points.

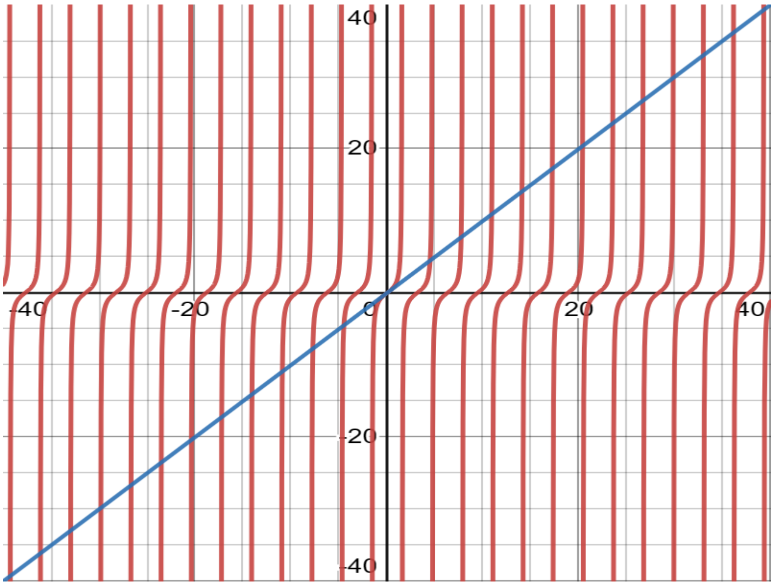

FIGURE 2.2: Infinitely many fixed points

(c): In many of the cases, there dose not exists even a single fixed point of given mapping. One of them is translation of line y $= r$. For example, the function $\digamma(r) = r+1$ has no fixed point. In other words, the graph of the lines $\digamma(r) = r+1$ and $\digamma(r) = r$ has no point of intersection. Graphically, it is presented as;

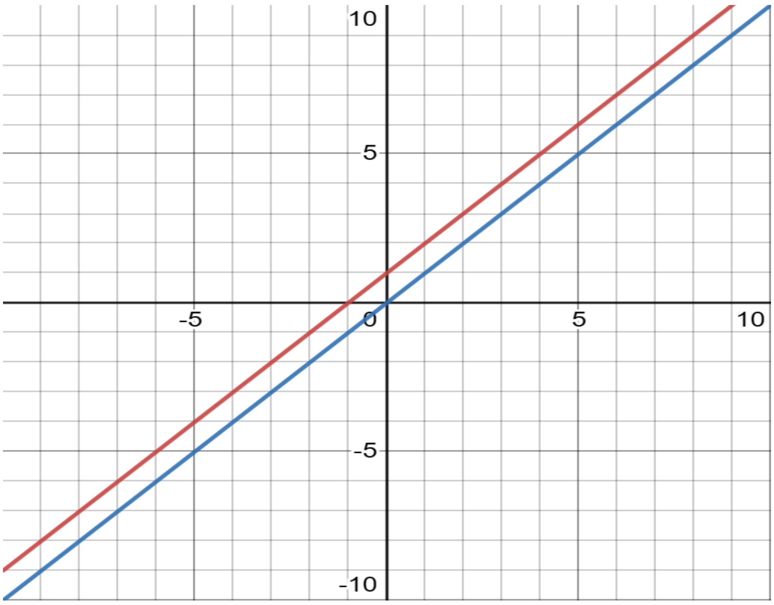

FIGURE 2.3: No fixed point

Bhaskar and Lakshmikantham [13] in 2006 worked on the idea of couple fixed poin [12] in the setting of mixed monotone mappings having mixed monotone property(MMp).

Definition 2.2.4.

"Let $(\mathcal{X}, \preceq)$ be a partially ordered set and $S : \mathcal{X}^2 \to \mathcal{X}$ be a mapping, then S is said to have MMp if for any $r_1, r_2, r_3 \in \mathcal{X}$,

$$r_1^*, r_1^{**} \in \mathcal{X},\ r_1^* \preceq r_1^{**} \Rightarrow S(r_1^*, r_2) \preceq S(r_1^{**}, r_2),$$
$$r_2^*, r_2^{**} \in \mathcal{X},\ r_2^* \preceq r_2^{**} \Rightarrow S(r_1, r_2^*) \succeq S(r_1, r_2^{**}),$$

whenever $S(r_1, r_2)$ is non-decreasing in r_1 and non-increasing in r_2." [13]

Definition 2.2.5.

"An element $(r_1, r_2) \in \mathcal{X}^2$ is called a coupled fixed point of the mapping $S : \mathcal{X}^2 \to \mathcal{X}$ if $S(r_1, r_2) = r_1$ and $S(r_2, r_1) = r_2$." [12]

Definition 2.2.6.

"The two given mappings $S : \mathcal{X}^2 \to \mathcal{X}$ and $s : \mathcal{X} \to \mathcal{X}$ have a common couple fixed point $(r_1, r_2) \in \mathcal{X}^2$ if $S(r_1, r_2) = s(r_1)$ and $S(r_2, r_1) = s(r_2)$." [13]

Example 2.2.7.

Some examples of couple fixed points are stated bellow;

(i): Let $S : \mathbb{R}^2 \to \mathbb{R}$ be a mapping defined as

$$S(r_1, r_2) = r_1 + r_2, \qquad \forall\ (r_1, r_2) \in \mathbb{R}^2.$$

Then, $(0, 0)$ is a coupled fixed point of S.

(ii): Let $S : \mathbb{R}^2 \to \mathbb{R}$ be a mapping defined as

$$S(r, t) = r^3, \qquad \forall\ (r, t) \in \mathbb{R}^2.$$

Hence, $(-1,0)$, $(0,-1)$, $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$ are some coupled fixed points of S.

(iii): Let us define two mappings $S:\mathbb{R}^2\to\mathbb{R}$ and $s:\mathbb{R}\to\mathbb{R}$ such that

$$S(r,r^*)=rr^* \text{ and } s(r)=r^2,\ \forall\, r,r^*\in\mathbb{R},$$

respectively. Then,

$$\mathfrak{f}=\{(r,r^*)\in\mathbb{R}^2 : r=r^*\},$$

is the set of all common couple fixed points of S and s.

Theorem 2.2.8.

"Let $(\mathcal{X},\mathrm{d},\le)$ be a partially ordered complete metric space and $\digamma:\mathcal{X}^2\to\mathcal{X}$ be a continuous mapping having mixed monotone property on $\mathcal{X}$. Assume that there exists a $k\in[0,1)$ with

$$\mathrm{d}(\digamma(r_1,r_2),\digamma(t_1,t_2))\le\frac{k}{2}[\mathrm{d}(r_1,t_1)+\mathrm{d}(r_2,t_2)],\quad \forall\ t_1\le r_1,\ r_2\le t_2.$$

If there exists $r_o,t_o\in\mathcal{X}$ such that $r_o\le\digamma(r_o,t_o)$ and $\digamma(t_o,r_o)\le t_o$, then there exists $(r,t)\in\mathcal{X}^2$ such that $r=\digamma(r,t)$ and $t=\digamma(t,r)$. " [13]

Concept of triple fixed point was presented by Berinde and Borcut [38] and generalized the definition of MMp for triple-valued mappings.

Definition 2.2.9.

" An element $(r_1,r_2,r_3)\in\mathcal{X}^3$ is called a triple fixed point of the mapping $S:\mathcal{X}^3\to\mathcal{X}$ if $S(r_1,r_2,r_3)=r_1$, $S(r_2,r_3,r_1)=r_2$ and $S(r_3,r_1,r_2)=r_3$." [38]

Example 2.2.10.

Some examples of triple fixed points are discussed here:

(i): Let $S:\mathbb{R}\to\mathbb{R}$ be a mapping defined as

$$S(r_1,r_2,r_3)=r_1+r_2+r_3,\quad \forall\ (r_1,r_2,r_3)\in\mathbb{R}^3.$$

Then, $(0,0,0)$ is a triple fixed point of S.

(ii): Let us define a mapping $S:\mathbb{R}^{+3}\to\mathbb{R}$

$$S(r_1,r_2,r_3)=\sqrt{r_1}. \qquad \forall\ (r_1,r_2,r_3)\in\mathbb{R}^{+3}.$$

Then, (0,0,0), (1,0,1), (1,1,0), (0,1,1), (0,0,1), (0,1,0), (1,0,0), (1,1,1) are some triple fixed points of S.

Definition 2.2.11.

"Let $(\mathcal{X},\preceq)$ be a partially ordered set and $S:\mathcal{X}^3\to\mathcal{X}$ be a mapping, then S is

said to have MMp if $S(r_1,r_2,r_3)$ is non-decreasing in r_1 and r_3 and non-increasing in r_2, that is for any $r_1,r_2,r_3\in\mathcal{X}$,

$$r_1^*,r_1^{**}\in\mathcal{X},\ r_1^*\preceq r_1^{**}\Rightarrow S(r_1^*,r_2,r_3)\preceq S(r_1^{**},r_2,r_3),$$
$$r_2^*,r_2^{**}\in\mathcal{X},\ r_2^*\preceq r_2^{**}\Rightarrow S(r_1,r_2^*,r_3)\succeq S(r_1,r_2^{**},r_3),$$
$$r_3^*,r_3^{**}\in\mathcal{X},r_3^*\preceq r_3^{**}\Rightarrow S(r_1,r_2,r_3^*)\preceq S(r_1,r_2,r_3^{**}).\text{"}[38]$$

Theorem 2.2.12.

"Let $(\mathcal{X},\preceq)$ be a partially ordered set and suppose there is a metric d on $\mathcal{X}$ such that $(\mathcal{X},\mathrm{d})$ is a complete metric space. Let $\digamma:\mathcal{X}^3\to\mathcal{X}$ be a continuous mapping having the MMp on $\mathcal{X}$. Assume that there exist the constants $a,b,c\in[0,1)$ with $a+b+c<1$ for which

$$\mathrm{d}(\digamma(r_1,r_2,r_3),\digamma(t_1,t_2,t_3))\leq a\mathrm{d}(r_1,t_1)+b\mathrm{d}(r_2,t_2)+c\mathrm{d}(r_3,t_3),$$

for all $r_1\geq t_1,\ r_2\leq t_2,\ r_3\geq t_3$. If there exists $r_o,t_o,q_o\in\mathcal{X}$ such that

$$r_o\leq\digamma(r_o,t_o,q_o),\ t_o\geq\digamma(t_o,q_o,r_o),\ q_o\leq\digamma(q_o,r_o,t_o).$$

Then, there exist $r,t,q\in\mathcal{X}$ such that $r=\digamma(r,t,q),\ t=\digamma(t,q,r),\ q\leq\digamma(q,r,t)$" [14].

Definitions of quadruple fixed point and MMp for quarter-valued mappings were introduced by Karapinar [16] in 2012.

Definition 2.2.13.

"Let $\mathcal{X} \neq \emptyset$, an element $(r_1, r_2, r_3, r_4) \in \mathcal{X}^4$ is said to be the quadruple fixed point of mapping $S : \mathcal{X}^4 \to \mathcal{X}$ if $S(r_1, r_2, r_3, r_4) = r_1$, $S(r_2, r_3, r_4, r_1) = r_2$, $S(r_3, r_4, r_1, r_2) = r_3$ and $S(r_4, r_1, r_2, r_3) = r_4$." [15]

Definition 2.2.14.

"Let $\mathcal{X} \neq \emptyset$, an element $(r_1, r_2, r_3, r_4) \in \mathcal{X}^4$ is said to be the quadruple coincidence point of mappings $S : \mathcal{X}^4 \to \mathcal{X}$ and $s : \mathcal{X} \to \mathcal{X}$ if $S(r_1, r_2, r_3, r_4) = s(r_1)$, $S(r_2, r_3, r_4, r_1) = s(r_2)$, $S(r_3, r_4, r_1, r_2) = s(r_3)$ and $S(r_4, r_1, r_2, r_3) = s(r_4)$." [15]

Example 2.2.15.

(i): Let $S : \mathbb{R}^4 \to \mathbb{R}$ be a mapping defined as

$$S(r_1, r_2, r_3, r_4) = r_1 + r_2 + r_3 + r_4, \quad \forall\ r_1, r_2, r_3, r_4 \in \mathbb{R}^4.$$

Then, $(0, 0, 0, 0)$ is a unique quadruple fixed point of S.

(ii): Let $S : \mathbb{R}^4 \to \mathbb{R}$ and $s : \mathbb{R} \to \mathbb{R}$ be to mappings

$$S(r_1, r_2, r_3, r_4) = \frac{r_1 r_2 r_3 r_4}{4} \text{ and } s(r) = r^2, \ \forall r_1, r_2, r_3, r_4, r \in \mathbb{R}^4.$$

Then, (2,2,2,2) is a quadruple coincidence point of S and s.

(iii): If we define S and s in such a way that

$$S(r_1, r_2, r_3, r_4) = \frac{r_1 + r_2 + r_3 + r_4}{2} \text{ and } s(r) = r^2, \ \forall r_1, r_2, r_3, r_4, r \in \mathbb{R}^4.$$

Then, in this case (2,2,2,2) is also a unique quadruple coincidence point of S and s.

Note that for $s : \mathcal{X} \to \mathcal{X}$ being identity map concept of quadruple coincidence point is reduces into the notion of quadruple fixed point of mappings.

Definition 2.2.16.

“ Let $S : \mathcal{X}^4 \to \mathcal{X}$ and $s : \mathcal{X} \to \mathcal{X}$ be two mappings, then S and s are called commutative if

$$s(S(r_1, r_2, r_3, r_4)) = S(s(r_1), s(r_2), s(r_3), s(r_4)), \quad \forall\ r_1, r_2, r_3, r_4 \in \mathcal{X}.\text{”}[15]$$

Definition 2.2.17.

“A mapping $S : \mathcal{X}^4 \to \mathcal{X}$ defined on a partial ordered set $(\mathcal{X}, \preceq)$, is said to have MMp if for any $r_1, r_2, r_3, r_4 \in \mathcal{X}$,

$$\begin{aligned}
r_1^*, r_1^{**} \in \mathcal{X},\ r_1^* \preceq r_1^{**} &\Rightarrow\ S(r_1^*, r_2, r_3, r_4) \preceq S(r_1^{**}, r_2, r_3, r_4),\\
r_2^*, r_2^{**} \in \mathcal{X},\ r_2^* \preceq r_2^{**} &\Rightarrow\ S(r_1, r_2^*, r_3, r_4) \succeq S(r_1, r_2^{**}, r_3, r_4),\\
r_3^*, r_3^{**} \in \mathcal{X},\ r_3^* \preceq r_3^{**} &\Rightarrow\ S(r_1, r_2, r_3^*, r_4) \preceq S(r_1, r_2, r_3^{**}, r_4),\\
r_4^*, r_4^{**} \in \mathcal{X},\ r_4^* \preceq r_4^{**} &\Rightarrow\ S(r_1, r_2, r_3, r_4^*) \succeq S(r_1, r_2, r_3, r_4^{**}),
\end{aligned}$$

where, S is non-decreasing in r_1 and r_3 and non-increasing in r_2 and r_4.” [15]

The generalization of above definition for two mappings is given bellow.

Definition 2.2.18.

“Let $S : \mathcal{X}^4 \to \mathcal{X}$ and $s : \mathcal{X} \to \mathcal{X}$ be two mappings defined on a partial ordered set $(\mathcal{X}, \preceq)$. Then S is said to have mixed s-monotone property (MsMP) if for any $r_1, r_2, r_3, r_4 \in \mathcal{X}$,

$$\begin{aligned}
r_1^*, r_1^{**} \in \mathcal{X},\ s(r_1^*) \preceq s(r_1^{**}) &\Rightarrow\ S(r_1^*, r_2, r_3, r_4) \preceq S(r_1^{**}, r_2, r_3, r_4),\\
r_2^*, r_2^{**} \in \mathcal{X},\ s(r_2^*) \preceq s(r_2^{**}) &\Rightarrow\ S(r_1, r_2^*, r_3, r_4) \succeq S(r_1, r_2^{**}, r_3, r_4),\\
r_3^*, r_3^{**} \in \mathcal{X},\ s(r_3^*) \preceq s(r_3^{**}) &\Rightarrow\ S(r_1, r_2, r_3^*, r_4) \preceq S(r_1, r_2, r_3^{**}, r_4),\\
r_4^*, r_4^{**} \in \mathcal{X},\ s(r_4^*) \preceq s(r_4^{**}) &\Rightarrow\ S(r_1, r_2, r_3, r_4^*) \succeq S(r_1, r_2, r_3, r_4^{**}),\text{”}[5]
\end{aligned}$$

Theorem 2.2.19.

“ Let $(\mathcal{X}, \mathrm{d}, \preceq)$ be a partially ordered metric space and $S : \mathcal{X}^4 \to \mathcal{X}$ be a continuous mapping such that S has mixed monotone property. Assume that there exist $\phi \in \Phi$ such that

$$d(S(r_1, r_2, r_3, r_4), S(t_1, t_2, t_3, t_4)) \leq \phi \max\{d(r_1, t_1) + d(r_2, t_2) + d(r_3, t_3) + d(r_4, t_4)\}$$

for all $r_1, r_2, r_3, r_4, t_1, t_2, t_3, t_4 \in \mathcal{X}$ provided that

$$r_1 \preceq t_1, \ r_2 \succeq, r_3 \preceq t_3, r_4 \succeq t_4.$$

Here, Φ is the set of all non-decreasing functions $\phi : [0, +\infty) \to [0, +\infty)$ such that $\lim_{n \to +\infty} \phi^n(t) = 0$, for all $t > 0$. If there exist $r_1^o, r_2^o, r_3^o, r_4^o \in \mathcal{X}$ such that

$$r_1^o \preceq S(r_1^o, r_2^o, r_3^o, r_4^o), \ r_2^o \preceq S(r_2^o, r_3^o, r_4^o, r_1^o)$$
$$r_3^o \preceq S(r_3^o, r_4^o, r_1^o, r_2^o), \ r_4^o \preceq S(r_4^o, r_1^o, r_2^o, r_3^o).$$

Then, S has a quadruple fixed point." [17].

2.3 Some Extensions of Metric Spaces

In the field of abstract spaces, metric space covers a area of research. Authors worked in many directions in ordered to introduce the generalization of metric spaces by manipulating with the existing axioms of metric spaces. In this section, few generalization of metric spaces are articulated, furthermore some examples are also illustrated in this regard.

2.3.1 b-Metric Spaces

Bakhtin [23] in 1989 generalized the notion of metric spaces by introducing a new parameter in triangular inequality and defined the concept of b-metric space. Later on, in 1993 Czerwik [24] modified the idea by using completeness property and generalized BCP.

Definition 2.3.1.

"Let $\mathcal{X}$ be non empty set and $b \geq 1$ be a given real number. A function d_b :

$\mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ is said to be b-metric on $\mathcal{X}$, the pair $(\mathcal{X}, \mathrm{d}_b)$ is called a b-metric space if for all $r_1, r_2, r_3 \in \mathcal{X}$,

(b_1) $\mathrm{d}_b(r_1, r_2) = 0$ if and only if $r_1 = r_2$,

(b_2) $\mathrm{d}_b(r_1, r_2) = \mathrm{d}_b(r_2, r_1)$,

(b_3) $\mathrm{d}_b(r_1, r_3) \leq b\{\mathrm{d}_b(r_1, r_2) + \mathrm{d}_b(r_2, r_3)\}$."[23]

Remark 4:

- If $b = 1$, then the pair $(\mathcal{X}, \mathrm{d})$ becomes metric space.
- Following two inequalities are key points for the proof of existences of triangular inequality in b-metric spaces;

$$(a + b)^t \leq 2^{t-1}(a^t + b^t), \qquad t \geq 1. \tag{2.1}$$

$$\Big(\sum_{i=1}^{\infty} |a_i + b_i|^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{i=1}^{\infty} |a_i|^p\Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{\infty} |b_i|^p\Big)^{\frac{1}{p}}, \quad 1 < p < \infty. \tag{2.2}$$

- The notions of convergent sequences, Cauchy sequences, and completeness hold in the same scenario as in metric spaces.

Example 2.3.2.

To demonstrate the concept of b-metric spaces, some examples are illustrated

bellow.

(1). Let $\mathrm{d}_b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function defined as

$$\mathrm{d}_b(r, t) = |r - t|^2, \qquad \forall\ r, t \in \mathbb{R}.$$

Then, the pair $(\mathbb{R}, \mathrm{d}_b)$ is a b-metric space with $b = 2$.

(2). Let $\mathcal{X} = l_p(\mathbb{R})$ be the space of all sequences in $\mathbb{R}$ such that

$$\sum_{k=1}^{\infty} \{r_k\} < \infty, \qquad \forall\ r = \{r_k\} \subset \mathbb{R},\ p \in (0, 1). \tag{2.3}$$

Let $\mathrm{d}_b : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be a function defined as

$$\mathrm{d}_b(r,t) = (\sum_{k=1}^{\infty} |r_k - t_k|^p)^{\frac{1}{p}}.$$

Since, (b_1) and (b_2) are obvious. Now,

(b_3) : For any $q, r, t \in \mathcal{X}$

$$\begin{aligned}
\mathrm{d}_b(q,r) &= (\sum_{k=1}^{\infty} |q_k - r_k|^p)^{\frac{1}{p}} \\
&= (\sum_{k=1}^{\infty} |q_k - r_k + t_k - t_k|^p)^{\frac{1}{p}} \\
&\leq (\sum_{k=1}^{\infty} 2^{p-1}(|q_k - t_k|^p + |t_k - r_k|^p))^{\frac{1}{p}}, \quad \text{using (2.1)} \\
&\leq 2^{p-1}\{(\sum_{k=1}^{\infty} |q_k - t_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |t_k - r_k|^p)^{\frac{1}{p}}\}, \quad \text{using (2.2)} \\
&= b\{\mathrm{d}_b(q,t) + \mathrm{d}_b(t,r)\}.
\end{aligned}$$

Hence, $(\mathcal{X}, \mathrm{d}_b)$ is b-metric space with $b = 2^{p-1}$.

(3). Let $\mathcal{X} = \{1, 2, 3\}$ and $\mathrm{d}_b(2,0) = \mathrm{d}_b(0,2) = m \geq 2$, $\mathrm{d}_b(0,1) = \mathrm{d}_b(1,0) = \mathrm{d}_b(1,2) = \mathrm{d}_b(2,1) = 1$ and $\mathrm{d}_b(1,1) = \mathrm{d}_b(0,0) = \mathrm{d}_b(2,2) = 0$. Then,

$$\mathrm{d}_b(r_1, r_2) \leq \frac{m}{2}[\mathrm{d}_b(r_1, r_3) + \mathrm{d}_b(r_3, r_2)], \quad r_1, r_2, r_3 \in \mathcal{X}.$$

Hence, $(\mathcal{X}, \mathrm{d}_b)$ is a b-metric space. If $m < 2$, then triangular inequality does not hold.

Theorem 2.3.3.

"Let $(\mathcal{X}, \mathrm{d}_b)$ be a complete b-metric space and $\mathcal{F} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ satisfies

$$\mathrm{d}_b(\mathcal{F}(r_1), \mathcal{F}(r_2)) \leq \phi \mathrm{d}_b(r_1, r_2), \qquad \forall\, r_1, r_2 \in \mathcal{X},$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that $\lim_{n\to\infty} \phi^n(r) = 0$. Then $\mathcal{F}$

has only one fixed point r_o and $\lim_{n\to\infty} \mathrm{d}_b(F^n(r), r_o) = 0$." [24]

2.3.2 Generalized Metric Spaces

Perov's [5] , in 1964, established fixed point results on the spaces endowed with vector valued spaces by generalizing the range of metric function over vector spaces. These spaces were named as generalized metric spaces(GMS). This subsection includes the concept of GMS, few important definitions and result in the frame of GMS.

Definition 2.3.4.

"A mapping $\upsilon : \aleph^2 \to \mathbb{R}^k$ where $(\aleph \neq \emptyset)$ is called a vector-valued metric on $\aleph$, if the conditions bellow are satisfied, for any $r_1, r_2, r_3 \in \aleph$,

($\aleph$1) $\upsilon(r_1, r_2) \succeq \mathbf{0}, \upsilon(r_1, r_2) = \mathbf{0} \Leftrightarrow r_1 = r_2$,

($\aleph$2) $\upsilon(r_1, r_2) = \upsilon(r_2, r_1)$,

($\aleph$3) $\upsilon(r_1, r_2) \preceq \upsilon(r_1, r_3) + \upsilon(r_3, r_2)$.

If $u, v \in \mathbb{R}^k$, where $u = (u_1, u_2, \cdots, u_k)$ and $v = (v_1, v_2, \cdots, v_k)$, then $u \preceq v$ if and only if $u_i \leq v_i$ for $1 \leq i \leq k$. Then, the pair $(\aleph, \upsilon)$ is called a generalized metric space(GMS)." [30]

Remark 5:

- For $k = 1$ in above definition, GMS converts into usual metric space.
- For GMS, the notions of convergent sequences, Cauchy sequences, completeness, open subsets, closed subsets and continuous mappings are similar to those for usual metric spaces.

Example 2.3.5.

Let $\aleph \neq \emptyset$ and $d_1, d_2, d_3, \cdots, d_k$ be the metrics on $\aleph$. Let $\upsilon : \aleph \times \aleph \to \mathbb{R}^k$ be a mapping define as for all $r_1, r_2 \in \aleph$,

$$\upsilon(r_1, r_2) = (d_1(r_1, r_2), d_2(r_1, r_2), \cdots, d_k(r_1, r_2)),$$

then $(\aleph, \upsilon)$ is a generalized metric space.

Theorem 2.3.6.

"Let $(\aleph, \upsilon)$ be a complete generalized metric space and $F : \aleph \to \mathcal{P}_{cl}(\aleph)$ where $(\mathcal{P}_{cl}(\aleph) = \{Y \in \mathcal{P}(\aleph) : Y \text{ is closed}\})$ a multi-valued operator. One suppose that there exists $A, B \in M_{n\times n}(\mathbb{R}^+)$ such that for each $r_1, r_2 \in \aleph$ and all $q \in F(r_1)$, their exists $t \in F(r_2)$ with

$$\upsilon(q,t) \leq A\upsilon(r_1,r_2) + B\upsilon(r_2,q).$$

If A is matrix converges to zero matrix, then F has at least one fixed point." [30]

Definition 2.3.7.

"The mapping $\bar{\upsilon}:\aleph^4 \times \aleph^4 \to \mathbb{R}^k$ defined on a generalized metric space $(\aleph, \upsilon)$ equipped with

$$\bar{\upsilon}((r_1, r_2.r_3, r_4), (t_1, t_2, t_3, t_4)) = \upsilon(r_1, t_1) + \upsilon(r_2, t_2) + \upsilon(r_3, t_3) + \upsilon(r_4, t_4),$$

defines a metric on $\aleph^4$. Moreover for simplicity it will be denoted by υ, for each $r_i, t_i \in \aleph$." [39]

Definition 2.3.8.

"Let $(\aleph, \upsilon, \preceq)$ be a partially ordered set then, $\aleph$ is called regular if the conditions bellow are satisfied:

1. for $n \geq 0$, $r_1^n \preceq r_1$ if a non-decreasing sequence $r_1^n \to r_1$,
2. for $n \geq 0$, $r_2 \leq r_2^n$ if a non-increasing sequence $r_2^n \to r_2$." [40]

2.4 Matrix Equations

Now onward in our thesis, the symbols $M_{n,n}(\mathbb{R}^+), O, I$ represent the set of all $n \times n$ matrices over $\mathbb{R}^+$, zero and identity matrices respectively, and the set $\mathbb{W} =$

$\{0, 1, 2, 3 \cdots\}$. Let ZM be denoted as the set of all $n \times n$ matrices over $\mathbb{R}^+$ such that for all $\Gamma \in M_{n,n}(\mathbb{R}^+)$, $\Gamma^n \to O$ whenever $n \to \infty$.

Definition 2.4.1.

"Let Γ be a matrix in $M_{n,n}(\mathbb{R}^+)$, then Γ is said to be convergent if and only if,

$$\lim_{n\to\infty} \Gamma^n = O.\text{"}[36]$$

Example 2.4.2.

Consider the matrix

$$\Gamma = \begin{pmatrix} r_1 & r_2 \\ r_1 & r_2 \end{pmatrix} \in M_{2,2}(\mathbb{R}^+),$$

such that $r_1 + r_2 < 1$ for some $r_1, r_2 \in \mathbb{R}^+$, then the matrix Γ converges to O.

Example 2.4.3.

Any matrix in $M_{2,2}(\mathbb{R}^+)$ of the form

$$\Gamma = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix},$$

converges to zero matrix provided that $\max\{r_1, r_2\} < 1$.

Example 2.4.4.

Consider the matrix

$$\Psi = \begin{pmatrix} r_1 & r_2 \\ r_1 & r_2 \end{pmatrix}$$

belongs to $M_{2,2}(\mathbb{R}^+)$ with the condition that $r_1 + r_2 \geq 1$, for some $r_1, r_2 \in \mathbb{R}^+$, then the matrix Ψ does not converges in $M_{2,2}(\mathbb{R}^+)$.

Let $\mathbb{R}^k$ be a k dimensional vector space [30]. Let $\mathbf{0} = (0, 0, \cdots, 0)$ and $\mathbf{1} = (1, 1, \cdots, 1)$ be the zero vector and identity vector respectively. Addition and multiplication in $\mathbb{R}^k$ are defined as for any $r, r^* \in \mathbb{R}^k$,

$$r + r^* = (r_1 + r_1^*, r_2 + r_2^*, r_3 + r_3^*, \cdots, r_k + r_k^*) \quad \text{and} \quad r.r^* = (r_1.r_1^*, r_2.r_2^*, \cdots, r_k.r_k^*),$$

where $r = (r_1, r_2, r_3, \cdots, r_k)$ and $r^* = (r_1^*, r_2^*, r_3^*, \cdots, r_k^*)$, In matrix analysis, the following propositions are equivalent;

Lemma 2.4.5.

"Let Γ be a square matrix with entries from $\mathbb{R}^+$, then the following statements are equivalent:

(L1) $\Gamma \to O$;

(L2) $\Gamma^n \to O$ as $n \to \infty$;

(L3) for each $\gamma \in \mathbb{C}$, $|\gamma| < 1$ with $\det(\Gamma - \gamma I) = O$;

(L4) $I - \Gamma$ is a non-singular matrix and
$(I - \Gamma)^{-1} = I + \Gamma + \cdots + \Gamma^n + \cdots$;

(L5) two matrices $\Gamma^n g$ and $g\Gamma^n$ tends to zero as $n \to \infty$, for $g \in \mathbb{R}^k$." [37]

Definition 2.4.6.

"Let $\Gamma = (\Gamma_{ij})$ and $\Upsilon = (\Upsilon_{ij})$ be two matrices in ZM. Then,

$$\Gamma \leq \Upsilon \quad \Longleftrightarrow \quad \Gamma_{ij} \leq \Upsilon_{ij},$$

where $1 \leq i, j \leq n$, and

$$\max\{\Gamma, \Upsilon\} = \Pi = (\Pi_{ij}), \quad \ni \quad \max(\Pi_{ij}) = \max\{\Gamma_{ij}, \Upsilon_{ij}\}.$$

Clearly, if $\Gamma \leq \Upsilon$ then, $\max\{\Gamma, \Upsilon\} = \Upsilon$." [34]

Chapter 3

Some Quadruple Fixed Point Results in GMS Involved with Matrix Equations

In chapter 3, we presented a detailed review of work by Hammad *et al*. [34], in which the contraction condition of mappings and fixed point theorems on the existences of unique quadruple fixed point of mappings over generalized metric space are presented. Also, an application on existence of unique stationary distribution of Markov process in the frame of generalized metric space is discussed.

3.1 Results in Generalized Metric Spaces:

Perov [5] introduced the notion of GMS and generalized BCP on GMS. Later on, authors established exciting results on theory of QFp of mappings under GMS. This section covers some useful definitions on mappings and sequences.

3.1.1 Definitions on Two Mappings

Definition 3.1.1. Let $S : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be two mappings defined on a metric space $(\aleph, \upsilon)$. Then, S and s are said to be compatible if the following conditions holds.

(i) $\lim_{n\to+\infty} \upsilon(s(U_{1234}), S(V_{1234})) = 0,$

where $U_{1234} = S(r_1^n, r_2^n, r_3^n, r_4^n)$ and $V_{1234} = (s(r_1^n), s(r_2^n), s(r_3^n), s(r_4^n)),$

(ii) $\lim_{n\to+\infty} \upsilon(s(U_{2341}), S(V_{2341})) = 0,$

where $U_{2341} = S(r_2^n, r_3^n, r_4^n, r_1^n)$ and $V_{2341} = (s(r_2^n), s(r_3^n), s(r_4^n), s(r_1^n)),$

(iii) $\lim_{n\to+\infty} \upsilon(s(U_{3412}), S(V_{3412})) = 0,$

where $U_{3412} = S(r_3^n, r_4^n, r_1^n, r_2^n)$ and $V_{3412} = (s(r_3^n), s(r_4^n), s(r_1^n), s(r_2^n)),$

(iv) $\lim_{n\to+\infty} \upsilon(s(U_{4123}), S(V_{4123})) = 0,$

where $U_{4123} = S(r_4^n, r_1^n, r_2^n, r_3^n)$ and $V_{4123} = (s(r_4^n), s(r_1^n), s(r_2^n), s(r_3^n)),$

whenever $\{r_1^n\}$, $\{r_2^n\}$, $\{r_3^n\}$ and $\{r_4^n\}$ are sequences in $\aleph$ such that

$$\lim_{n\to+\infty} U_{1234} = \lim_{n\to+\infty} s(r_1^n) = r_1, \quad \lim_{n\to+\infty} U_{2341} = \lim_{n\to+\infty} s(r_2^n) = r_2,$$
$$\lim_{n\to+\infty} U_{3412} = \lim_{n\to+\infty} s(r_3^n) = r_3, \quad \lim_{n\to+\infty} U_{4123} = \lim_{n\to+\infty} s(r_4^n) = r_4,$$

for some $r_1, r_2, r_3, r_4 \in \aleph$.

Definition 3.1.2.

The mappings $S : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ are called reciprocally continuous if for some $r_1, r_2, r_3, r_4 \in \aleph$, following conditions holds.

(i) $\lim_{n\to+\infty} s(U_{1234}) = s(r_1)$ and $\lim_{n\to+\infty} S(V_{1234}) = S(r_1, r_2, r_3, r_4),$

(ii) $\lim_{n\to+\infty} s(U_{2341}) = s(r_2)$ and $\lim_{n\to+\infty} S(V_{2341}) = S(r_2, r_3, r_4, r_1),$

(iii) $\lim_{n\to+\infty} s(U_{3412}) = s(r_3)$ and $\lim_{n\to+\infty} S(V_{3412}) = S(r_3, r_4, r_1, r_2),$

(iv) $\lim_{n\to+\infty} s(U_{4123}) = s(r_4)$ and $\lim_{n\to+\infty} S(V_{4123}) = S(r_4, r_1, r_2, r_3),$

whenever $\{r_1^n\}$, $\{r_2^n\}$, $\{r_3^n\}$, $\{r_4^n\}$ are sequences in $\aleph$ such that

$$\lim_{n\to+\infty} U_{1234} = \lim_{n\to+\infty} s(r_1^n) = r_1, \quad \lim_{n\to+\infty} U_{2341} = \lim_{n\to+\infty} s(r_2^n) = r_2,$$

$$\lim_{n\to+\infty} U_{3412} = \lim_{n\to+\infty} s(r_3^n) = r_3, \quad \lim_{n\to+\infty} U_{4123} = \lim_{n\to+\infty} s(r_4^n) = r_4.$$

Definition 3.1.3.

The mapping $S : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ are called weakly reciprocally continuous if for some $r_1, r_2, r_3, r_4 \in \aleph$, conditions bellow are satisfied.

$$(i) \lim_{n\to+\infty} s(U_{1234}) = s(r_1) \quad \text{or} \quad \lim_{n\to+\infty} S(V_{1234}) = S(r_1, r_2, r_3, r_4),$$

$$(ii) \lim_{n\to+\infty} s(U_{2341}) = s(r_2) \quad \text{or} \quad \lim_{n\to+\infty} S(V_{2341}) = S(r_2, r_3, r_4, r_1),$$

$$(iii) \lim_{n\to+\infty} s(U_{3412}) = s(r_3) \quad \text{or} \quad \lim_{n\to+\infty} S(V_{3412}) = S(r_3, r_4, r_1, r_2),$$

$$(iv) \lim_{n\to+\infty} s(U_{4123}) = s(r_4) \quad \text{or} \quad \lim_{n\to+\infty} S(V_{4123}) = S(r_4, r_1, r_2, r_3),$$

whenever $\{r_1^n\}$, $\{r_2^n\}$, $\{r_3^n\}$, $\{r_4^n\}$ are sequences in $\aleph$ such that

$$\lim_{n\to+\infty} U_{1234} = \lim_{n\to+\infty} s(r_1^n) = r_1, \quad \lim_{n\to+\infty} U_{2341} = \lim_{n\to+\infty} s(r_2^n) = r_2,$$

$$\lim_{n\to+\infty} U_{3412} = \lim_{n\to+\infty} s(r_3^n) = r_3, \quad \lim_{n\to+\infty} U_{4123} = \lim_{n\to+\infty} s(r_4^n) = r_4.$$

3.1.2 Definitions on Mapping of Sequences

Definition 3.1.4.

Let $\xi_n : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be the mappings on a metric space $(\aleph, \upsilon)$, then the sequence $\{\xi_n\}_{n\in\mathbb{W}}$ and s are said to be compatible if the conditions bellow are satisfied.

$$(i) \lim_{n\to+\infty} \upsilon(s(\grave{U}_{1234}), \xi_n(\grave{V}_{1234})) = 0,$$

where $\grave{U}_{1234} = \xi_n(r_1^n, r_2^n, r_3^n, r_4^n)$ and $\grave{V}_{1234} = (s(r_1^n), s(r_2^n), s(r_3^n), s(r_4^n))$,

$$(ii) \lim_{n\to+\infty} \upsilon(s(\grave{U}_{2341}), \xi_n(\grave{V}_{2341})) = 0,$$

where $\grave{U}_{2341} = \xi_n(r_2^n, r_3^n, r_4^n, r_1^n)$ and $\grave{V}_{2341} = (s(r_2^n), s(r_3^n), s(r_4^n), s(r_1^n))$,

(iii) $\lim_{n\to+\infty} \upsilon(s(\grave{U}_{3412}), \xi_n(\grave{V}_{3412})) = 0,$

where $\grave{U}_{3412} = \xi_n(r_3^n, r_4^n, r_1^n, r_2^n)$ and $\grave{V}_{3412} = (s(r_3^n), s(r_4^n), s(r_1^n), s(r_2^n)),$

(iv) $\lim_{n\to+\infty} \upsilon(s(\grave{U}_{4123}), \xi_n(\grave{V}_{4123})) = 0,$

where $\grave{U}_{4123} = \xi_n(r_4^n, r_1^n, r_2^n, r_3^n)$ and $\grave{V}_{4123} = (s(r_4^n), s(r_1^n), s(r_2^n), s(r_3^n)),$

whenever $\{r_1^n\}, \{r_2^n\}, \{r_3^n\}$ and $\{r_4^n\}$ are sequences in $\aleph$ such that

$$\lim_{n\to+\infty} \grave{U}_{1234} = \lim_{n\to+\infty} s(r_1^{n+1}) = r_1, \quad \lim_{n\to+\infty} \grave{U}_{2341} = \lim_{n\to+\infty} s(r_2^{n+1}) = r_2,$$
$$\lim_{n\to+\infty} \grave{U}_{3412} = \lim_{n\to\infty} s(r_3^{n+1}) = r_3, \quad \lim_{n\to+\infty} \grave{U}_{4123} = \lim_{n\to+\infty} s(r_4^{n+1}) = r_4,$$

for some $r_1, r_2, r_3, r_4 \in \aleph$.

Definition 3.1.5.

Let $\xi_n : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be the mappings restricted on a metric space $(\aleph, \upsilon)$, then $\{\xi_n\}_{n\in\mathbb{N}}$ and s are said to be weakly reciprocally continuous if the following conditions hold.

$$(i) \quad \lim_{n\to+\infty} s(\grave{U}_{1234}) = s(r_1), \quad \lim_{n\to+\infty} s(\grave{U}_{2341}) = s(r_2),$$
$$(ii) \quad \lim_{n\to+\infty} s(\grave{U}_{3412}) = s(r_3), \quad \lim_{n\to+\infty} s(\grave{U}_{4123}) = s(r_4),$$

whenever $\{r_1^n\}$, $\{r_2^n\}$, $\{r_3^n\}$ and $\{r_4^n\}$ are sequences in $\aleph$ such that

$$\lim_{n\to+\infty} \grave{U}_{1234} = \lim_{n\to+\infty} s(r_1^{n+1}) = r_1, \quad \lim_{n\to+\infty} \grave{U}_{2341} = \lim_{n\to+\infty} s(r_2^{n+1}) = r_2,$$
$$\lim_{n\to+\infty} \grave{U}_{3412} = \lim_{n\to+\infty} s(r_3^{n+1}) = r_3, \quad \lim_{n\to+\infty} \grave{U}_{4123} = \lim_{n\to+\infty} s(r_4^{n+1}) = r_4,$$

for some r_1, r_2, r_3, r_4 in $\aleph$.

Example 3.1.6.

Let $\aleph = [0, 1]$ be a metric space with metric defined as $\upsilon(r_1, r_2) = |r_1 - r_2|$. Let $\xi_n : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be the mappings defined as

$$\xi_n(r_1, r_2, r_3, r_4) = \frac{1}{2^n} - \frac{r_1 r_2 r_3 r_4}{2} \quad \text{and} \quad s(r_1) = r_1.$$

Let us define four sequences $\{r_1^n\}, \{r_2^n\}, \{r_3^n\}$ and $\{r_4^n\} \in \aleph$ as,

$$r_1^n = \frac{n}{n^2+1},\ r_2^n = \frac{n}{\sqrt{n^2+1}},\ r_3^n = \frac{1}{r_1^n+1} \quad \text{and} \quad r_4^n = \frac{1}{n^3+1}, \forall n \in \mathbb{N}.$$

Then, few simple calculations drives us to the result that

$$\lim_{n\to+\infty} \grave{U}_{1234} = \lim_{n\to+\infty} s(r_1^{n+1}) = r_1 = 0, \quad \lim_{n\to+\infty} \grave{U}_{2341} = \lim_{n\to+\infty} s(r_2^{n+1}) = r_2 = 0,$$
$$\lim_{n\to+\infty} \grave{U}_{3412} = \lim_{n\to+\infty} s(r_3^{n+1}) = r_3 = 0, \quad \lim_{n\to+\infty} \grave{U}_{4123} = \lim_{n\to+\infty} s(r_4^{n+1}) = r_4 = 0,$$

for some $r_1 = r_2 = r_3 = r_4 = 0 \in \aleph$. Also, following conditions are satisfied as

$$\lim_{n\to+\infty} s(\grave{U}_{1234}) = s(r_1) = 0, \quad \lim_{n\to+\infty} s(\grave{U}_{2341}) = s(r_2) = 0,$$
$$\lim_{n\to+\infty} s(\grave{U}_{3412}) = s(r_3) = 0, \quad \lim_{n\to+\infty} s(\grave{U}_{4123}) = s(r_4) = 0,$$
$$\lim_{n\to\infty} \upsilon(s(\grave{U}_{1234}), \xi_n(\grave{V}_{1234})) = 0, \quad \lim_{n\to\infty} \upsilon(s(\grave{U}_{2341}), \xi_n(\grave{V}_{2341})) = 0,$$
$$\lim_{n\to+\infty} \upsilon(s(\grave{U}_{3412}), \xi_n(\grave{V}_{3412})) = 0, \quad \lim_{n\to\infty} \upsilon(s(\grave{U}_{4123}), \xi_n(\grave{V}_{4123})) = 0.$$

Hence, $\{\xi_n\}_{n\in\mathbb{N}}$ and s are compatible and weakly reciprocally continuous.

Definition 3.1.7.

Let $\xi_n : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be the mappings on a partially ordered set $(\aleph, \preceq)$, then $\{\xi_n\}_{n\in\mathbb{W}}$ and s are said to have mixed-s monotone property (MsMP) if for any $r_1, r_2, r_3, r_4, t_1, t_2, t_3, t_4 \in \aleph$,

$$\begin{aligned}
s(r_1) \preceq s(t_1) \quad &\Rightarrow \quad \xi_n(r_1, r_2, r_3, r_4) \preceq \xi_{n+1}(t_1, t_2, t_3, t_4),\\
s(r_2) \succeq s(t_2) \quad &\Rightarrow \quad \xi_n(r_2, r_3, r_4, r_1) \succeq \xi_{n+1}(t_2, t_3, t_4, t_1),\\
s(r_3) \preceq s(t_3) \quad &\Rightarrow \quad \xi_n(r_3, r_4, r_1, r_2) \preceq \xi_{n+1}(t_3, t_4, t_1, t_2),\\
s(r_4) \succeq s(t_4) \quad &\Rightarrow \quad \xi_n(r_4, r_1, r_2, r_3) \succeq \xi_{n+1}(t_4, t_1, t_2, t_3).
\end{aligned}$$

3.2 Main Results

This section deals with the contraction condition of mappings, QFp theorems, corollaries, and supportive examples.

Definition 3.2.1.

Let $\xi_n : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be the mappings, then $\{\xi_i\}_{i\in\mathbb{W}}$ and s are said to satisfy the (O) condition if

$$\begin{aligned} \upsilon(\xi_i(r_1,r_2,r_3,r_4),\xi_j(t_1,t_2,t_3,t_4)) \leq \Gamma[&\upsilon(s(r_1),\xi_i(r_1,r_2,r_3,r_4)) \\ &+ \upsilon(s(t_1),\xi_j(t_1,t_2,t_3,t_4))] \\ &+ \Upsilon[\upsilon(s(r_1),s(t_1))], \end{aligned} \tag{3.1}$$

for some $r_1, r_2, r_3, r_4, t_1, t_2, t_3, t_4, \in \aleph$ provided that $s(r_i) \preceq s(t_i)$ for $1 \leq i \leq 4$, or $s(r_i) \succeq s(t_i)$ for $1 \leq i \leq 4$, $I \neq \Gamma = (\Gamma_{ij})$ and $I \neq \Upsilon = (\Upsilon_{ij}) \in$ ZM satisfy the condition $(\Gamma + \Upsilon)(I - \Gamma)^{-1} \in$ ZM.

Example 3.2.2.

Let $\aleph = [0, 1]$ be a metric space equipped with metric $\upsilon(r_1, r_2) = |r_1 - r_2|$.

(1) Let $\Gamma = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$ and $\Upsilon = \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}$ be two matrices in ZM. Then, it is easy to compute that $(\Gamma + \Upsilon)(I - \Gamma)^{-1} \in$ ZM.

(2) Let $\Gamma = \alpha I$, and $\Upsilon = ((1-\alpha)^3 - \alpha)I \in$ ZM such that $\alpha = \frac{1}{4}, \frac{1}{5}, \frac{1}{7}, \frac{1}{8}$ then we get that $(\Gamma + \Upsilon)(I - \Gamma)^{-1} \in$ ZM.

(3) For $\Gamma = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\Upsilon = \frac{1}{7}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ in ZM, some simple calculations leads us to the result that $(\Gamma + \Upsilon)(I - \Gamma)^{-1} \in$ ZM.

Definition 3.2.3.

Let $\xi_0 : \aleph \to \aleph$ and $s : \aleph \to \aleph$ be two sequences, then ξ_o and s are said to have mixed quadruple transcendence point(MQTp) if $\exists\ r_1^0, r_2^0, r_3^0, r_4^0 \in \aleph$ such that

$$\begin{aligned} &\xi_0(r_1^0,r_2^0,r_3^0,r_4^0) \succeq s(r_1^0), \quad \xi_0(r_2^0,r_3^0,r_4^0,r_1^0) \preceq s(r_2^0) \\ &\xi_0(r_3^0,r_4^0,r_1^0,r_2^0) \succeq s(r_3^0), \quad \text{and} \quad \xi_0(r_4^0,r_1^0,r_2^0,r_3^0) \preceq s(r_4^0), \end{aligned} \tag{3.2}$$

given that ξ_0 and s have non-decreasing transcendence point in r_1^0, r_3^0 and a non-increasing transcendence point in r_2^0, r_4^0.

Lemma 3.2.4.

Let $\xi_i : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be two mappings on a partially ordered complete generalized metric space (POCGMS) $(\aleph, \upsilon, \preceq)$. Suppose that $\{\xi_i\}_{i\in\mathbb{W}}$ have MsMP such that $\xi_i(\aleph^4) \subseteq s(\aleph)$. If ξ_o and s have MQTp, then

1. $\exists$ sequences $\{r_1^n\},\ \{r_2^n\},\ \{r_3^n\}$ and $\{r_4^n\} \in \aleph$ such that

$$s(r_1^n) = \xi_{n-1}(r_1^{n-1}, r_2^{n-1}, r_3^{n-1}, r_4^{n-1}), \quad s(r_2^n) = \xi_{n-1}(r_2^{n-1}, r_3^{n-1}, r_4^{n-1}, r_1^{n-1}),$$
$$s(r_3^n) = \xi_{n-1}(r_3^{n-1}, r_4^{n-1}, r_1^{n-1}, r_2^{n-1}), \text{ and } s(r_4^n) = \xi_{n-1}(r_4^{n-1}, r_1^{n-1}, r_2^{n-1}, r_3^{n-1}).$$

2. $\{s(r_1^n)\},\ \{s(r_3^n)\}$ are non-decreasing sequences and $\{s(r_2^n)\},\ \{s(r_4^n)\}$ are non-increasing sequences.

Proof. (1) Suppose that (3.2) holds for $r_1^o, r_2^o, r_3^o, r_4^o \in \aleph$. Since $\xi_o(\aleph^4) \subseteq s(\aleph)$, we can define $r_1^1, r_2^1, r_3^1, r_4^1 \in \aleph$ such that

$$\begin{aligned} s(r_1^1) =& \xi_o(r_1^o, r_2^o, r_3^o, r_4^o),\ \ s(r_2^1) = \xi_o(r_2^o, r_3^o, r_4^o, r_1^o),\\ s(r_3^1) =& \xi_o(r_3^o, r_4^o, r_1^o, r_2^o),\ \ s(r_4^1), = \xi_o(r_4^o, r_1^o, r_2^o, r_3^o). \end{aligned} \tag{3.3}$$

Since $\xi_o(\aleph^4) \subseteq s(\aleph)$, then $\exists\ \ r_1^2, r_2^2, r_3^2, r_4^2 \in \aleph$ such that

$$\begin{aligned} s(r_1^2) &= \xi_1(r_1^1, r_2^1, r_3^1, r_4^1),\ \ s(r_2^2) = \xi_1(r_2^1, r_3^1, r_4^1, r_1^1),\\ s(r_3^2) &= \xi_1(r_3^1, r_4^1, r_1^1, r_2^1),\ \ s(r_4^2) = \xi_1(r_4^1, r_1^1, r_2^1, r_3^1). \end{aligned}$$

In the similar way, we end with

$$\begin{aligned} s(r_1^n) &= \xi_{n-1}(r_1^{n-1}, r_2^{n-1}, r_3^{n-1}, r_4^{n-1}),\\ s(r_2^n) &= \xi_{n-1}(r_2^{n-1}, r_3^{n-1}, r_4^{n-1}, r_1^{n-1}),\\ s(r_3^n) &= \xi_{n-1}(r_3^{n-1}, r_4^{n-1}, r_1^{n-1}, r_2^{n-1}),\\ s(r_4^n) &= \xi_{n-1}(r_4^{n-1}, r_1^{n-1}, r_2^{n-1}, r_3^{n-1}). \end{aligned} \tag{3.4}$$

(2) By way of mathematical induction, $\forall\ n \geq 0$, it is computed that

$$s(r_1^n) \preceq s(r_1^{n+1}),\ s(r_3^n) \preceq s(r_3^{n+1}),\ s(r_2^n) \succeq s(r_2^{n+1}), \quad \text{and} \quad s(r_4^n) \succeq s(r_4^{n+1}). \tag{3.5}$$

From (3.3), we get that (3.5) holds for $n = 0$, that is

$$s(r_1^o) \preceq s(r_1^1),\ s(r_3^o) \preceq s(r_3^1),\ s(r_2^o) \succeq s(r_2^1) \quad \text{and} \quad s(r_4^o) \succeq s(r_4^1).$$

Hence, (3.4) and (3.5) completes the required result. □

Main theorem of this section is discussed here:

Theorem 3.2.5.

Suppose that all the conditions of the Lemma 3.2.4 holds, assume that $\{\xi_i\}_{i\in\mathbb{W}}$ and s are monotonically decreasing and satisfy (O) condition, furthermore both mappings are compatible and weakly reciprocally continuous provided that s is continuous. If $s(\aleph) \subseteq \aleph$ is complete and regular then, $\{\xi_i\}_{i\in\mathbb{W}}$ and s have a quadruple coincidence point(QCp), whenever $O \neq \Gamma, \Upsilon \in \text{ZM}$.

Proof. Let $\{r_1^n\}, \{r_2^n\}, \{r_3^n\}$ and $\{r_4^n\}$ be the sequences in $\aleph$ constructed by Lemma 3.2.4, then from (3.1) it follows that

$$\begin{aligned}
\upsilon(s(r_1^n), (r_1^{n+1})) &= \upsilon(\xi_{n-1}(r_1^{n-1}, r_2^{n-1}, r_3^{n-1}, r_4^{n-1}), \xi_n(r_1^n, r_2^n, r_3^n, r_4^n)) \\
&\leq \Gamma[\upsilon(s(r_1^{n-1}), \xi_{n-1}(r_1^{n-1}, r_2^{n-1}, r_3^{n-1}, r_4^{n-1})) \\
&\quad + \upsilon(s(r_1^n), \xi_n(r_1^n, r_2^n, r_3^n, r_4^n))] + \Upsilon(s(r_1^{n-1}), s(r_1^n)) \\
&= (\Gamma + \Upsilon)\upsilon(s(r_1^{n-1}), s(r_1^n)) + \Gamma(\upsilon(s(r_1^n), s(r_1^{n+1}))).
\end{aligned}$$

This results in

$$\upsilon(s(r_1^n), s(r_1^{n+1})) \leq (\Gamma + \Upsilon)(I - \Gamma)^{-1}\upsilon(s(r_1^{n-1}), s(r_1^n)), \tag{3.6}$$

similarly, it can be written as

$$\begin{aligned}
\upsilon(s(r_2^n), s(r_2^{n+1})) &\leq (\Gamma + \Upsilon)(I - \Gamma)^{-1}\upsilon(s(r_2^{n-1}), s(r_2^n)), \\
\upsilon(s(r_3^n), s(r_3^{n+1})) &\leq (\Gamma + \Upsilon)(I - \Gamma)^{-1}\upsilon(s(r_3^{n-1}), s(r_3^n)),
\end{aligned} \tag{3.7}$$

and

$$\upsilon(s(r_4^n), s(r_4^{n+1})) \leq (\Gamma + \Upsilon)(I - \Gamma)^{-1}\upsilon(s(r_4^{n-1}), s(r_4^n)). \tag{3.8}$$

Addition of (3.6)-(3.8) provides that

$$\begin{aligned}\lambda_n &= \upsilon(s(r_1^n), s(r_1^{n+1})) + \upsilon(s(r_2^n), s(r_2^{n+1})) + \upsilon(s(r_3^n), s(r_3^{n+1})) + \upsilon(s(r_4^n), s(r_4^{n+1}))\\ &\leq (\Gamma + \Upsilon)(I - \Gamma)^{-1}[\upsilon(s(r_1^{n-1}), s(r_1^n)) + \upsilon(s(r_2^{n-1}), s(r_2^n)) + \upsilon(s(r_3^{n-1}), s(r_3^n))\\ &\quad + \upsilon(s(r_4^{n-1}), s(r_4^n))].\end{aligned}$$

Hence,

$$\lambda_n \leq (\Gamma + \Upsilon)(I - \Gamma)^{-1})\lambda_{n-1}.$$

Let $(\Gamma + \Upsilon)(I - \Gamma)^{-1} = Y$, then for $n \in \mathbb{N}$, we get

$$O \leq \lambda_n \leq Y\lambda_{n-1} \leq Y^2\lambda_{n-2} \leq \cdots \leq Y^n\lambda_o.$$

By using triangular inequality, for $m > 0$, we have,

$$\begin{aligned}&\upsilon(s(r_1^n), s(r_1^{n+m})) + \upsilon(s(r_2^n), s(r_2^{n+m})) + \upsilon(s(r_3^n), s(r_3^{n+m})) + \upsilon(s(r_4^n), s(r_4^{n+m}))\\ &\leq \upsilon(s(r_1^n), s(r_1^{n+1})) + \upsilon(s(r_2^n), s(r_2^{n+1})) + \upsilon(s(r_3^n), s(r_3^{n+1})) + \upsilon(s(r_4^n), s(r_4^{n+1}))\\ &\quad + \upsilon(s(r_1^{n+1}), s(r_1^{n+2})) + \upsilon(s(r_2^{n+1}), s(r_2^{n+2})) + \upsilon(s(r_3^{n+1}), s(r_3^{n+2}))\\ &\quad + \upsilon(s(r_4^{n+1}), s(r_4^{n+2})) + \cdots + \upsilon(s(r_1^{n+m-1}), s(r_1^{n+m})) + \upsilon(s(r_2^{n+m-1}), s(r_2^{n+m}))\\ &\quad + \upsilon(s(r_3^{n+m-1}), s(r_3^{n+m})) + \upsilon(s(r_4^{n+m-1}), s(r_4^{n+m}))\\ &= \lambda_n + \lambda_{n+1} + \cdots + \lambda_{n+m-1}\\ &\leq (Y^n + Y^{n+1} + \cdots + Y^{n+m-1})\lambda_o\\ &= Y^n(I + Y + \cdots + Y^{m-1} + \cdots)\lambda_o\\ &= Y^n(I - Y)^{-1}\lambda_o.\end{aligned}$$

This leads us to the result that

$$\begin{aligned}&\upsilon(s(r_1^n), s(r_1^{n+m})) + \upsilon(s(r_2^n), s(r_2^{n+m})) + \upsilon(s(r_3^n), s(r_3^{n+m})) + \upsilon(s(r_4^n), s(r_4^{n+m}))\\ &\leq [(\Gamma + \Upsilon)(I - \Gamma)^{-1}]^n[I - (\Gamma + \Upsilon)(I - \Gamma)^{-1})]^{-1}\lambda_o.\end{aligned}$$

Taking $\lim_{n\to+\infty}$, provides that

$$\lim_{n\to+\infty}[\upsilon(s(r_1^n), s(r_1^{n+m})) + \upsilon(s(r_2^n), s(r_2^{n+m})) + \upsilon(s(r_3^n), s(r_3^{n+m}))$$

$$+ \upsilon(s(r_4^n), s(r_4^{n+m}))] = 0,$$

$$\Rightarrow \lim_{n\to+\infty}\upsilon(s(r_1^n), s(r_1^{n+m})) = \lim_{n\to+\infty}\upsilon(s(r_2^n), s(r_2^{n+m})) = \lim_{n\to+\infty}\upsilon(s(r_3^n), s(r_3^{n+m}))$$

$$= \lim_{n\to+\infty}\upsilon(s(r_4^n), s(r_4^{n+m})) = 0.$$

This implies that $\{s(r_1^n)\}, \{s(r_2^n)\}, \{s(r_3^n)\}$ and $\{s(r_4^n)\}$ are Cauchy sequences in $\aleph$. Since $s(\aleph)$ is complete, this implies that $\exists$ $(r_1^*, r_2^*, r_3^*, r_4^*) \in \aleph^4$ such that,

$$\lim_{n\to+\infty} s(r_1^n) = s(r_1^*) = r_1, \lim_{n\to+\infty} s(r_2^n) = s(r_2^*) = r_2,$$

$$\lim_{n\to+\infty} s(r_3^n) = s(r_3^*) = r_3, \lim_{n\to+\infty} s(r_4^n) = s(r_4^*) = r_4.$$

Which results in

$$\lim_{n\to+\infty} s(r_1^{n+1}) = \lim_{n\to+\infty}\xi_n(r_1^n, r_2^n, r_3^n, r_4^n), \lim_{n\to+\infty} s(r_2^{n+1}) = \lim_{n\to+\infty}\xi_n(r_2^n, r_3^n, r_4^n, r_1^n),$$

$$\lim_{n\to+\infty} s(r_3^{n+1}) = \lim_{n\to+\infty}\xi_n(r_3^n, r_4^n, r_1^n, r_2^n), \lim_{n\to+\infty} s(r_4^{n+1}) = \lim_{n\to+\infty}\xi_n(r_4^n, r_1^n, r_2^n, r_3^n).$$

Since $\{\xi_i\}_{i\in\mathbb{W}}$ and s are weakly reciprocally continuous and compatible, then we have

$$\lim_{n\to+\infty}\xi_n(s(r_1^n), s(r_2^n), s(r_3^n), s(r_4^n)) = s(r_1),$$

$$\lim_{n\to+\infty}\xi_n(s(r_2^n), s(r_3^n), s(r_4^n), s(r_1^n)) = s(r_2),$$

$$\lim_{n\to+\infty}\xi_n(s(r_3^n), s(r_4^n), s(r_1^n), s(r_2^n)) = s(r_3),$$

$$\lim_{n\to+\infty}\xi_n(s(r_4^n), s(r_1^n), s(r_2^n), s(r_3^n)) = s(r_4).$$

As $\{s(r_1^n)\}, \{s(r_3^n)\}$ are non-decreasing sequences and $\{s(r_2^n)\}, \{s(r_4^n)\}$ are non-

increasing sequences, then from the regularity of $\aleph$, for all $n \geq 0$ we can obtain $s(r_1^n) \preceq r_1, r_2 \preceq s(r_2^n), s(r_3^n) \preceq r_3, r_4 \preceq s(r_4^n)$. Additionally, utilization of (3.1) generates

$$\begin{aligned}
&\upsilon(\xi_i(r_1,r_2,r_3,r_4),\xi_n(s(r_1^n),s(r_2^n),s(r_3^n),s(r_4^n)))\\
&\preceq \Gamma[\upsilon(s(r_1),\xi_i(r_1,r_2,r_3,r_4))+\upsilon(s(s(r_1^n)),\xi_n(r_1^n,r_2^n,r_3^n,r_4^n))]\\
&\quad+\Upsilon[\upsilon(s(r_1),s(s(r_1^n)))].
\end{aligned}$$

Taking $\lim\limits_{n\to+\infty}$, implies that

$$\upsilon(\xi_i(r_1,r_2,r_3,r_4),s(r_1))\preceq\Gamma\upsilon(s(r_1),\xi_i(r_1,r_2,r_3,r_4)),$$

which holds only if

$$\upsilon(\xi_i(r_1,r_2,r_3,r_4),s(r_1))=0,\quad \text{or}\quad \xi_i(r_1,r_2,r_3,r_4)=s(r_1).$$

In similar fashion, it is obtained that

$$\xi_i(r_2,r_3,r_4,r_1)=s(r_2),\ \xi_i(r_3,r_4,r_1,r_2)=s(r_3)\quad\text{and}\quad \xi_i(r_4,r_1,r_2,r_3)=s(r_4).$$

Hence, (r_1,r_2,r_3,r_4) is a QCp of $\{\xi_i\}_{i\in\mathbb{W}}$ and s. □

Next corollary is an extension of Theorem 3.2.5. Let $s=I^d$, where I^d is an identity map.

Corollary 3.2.6.

Let $(\aleph,\upsilon,\preceq)$ is a POCGMS and $\{\xi_i\}_{i\in\mathbb{W}}:\aleph^4\to\aleph$ be a mixed-monotone sequence such that $\{\xi_i\}_{i\in\mathbb{W}}$ and $I^d:\aleph\to\aleph$ satisfy (O) condition, ξ_o and I^d have MQTp and $I^d(\aleph)$ is regular. Then, $\exists$ $(r_1,r_2,r_3,r_4)\in\aleph^4$ such that $\xi_i(r_1,r_2,r_3,r_4)=r_1$, $\xi_i(r_2,r_3,r_4,r_1)=r_2$, $\xi_i(r_3,r_4,r_1,r_2)=r_3$ and $\xi_i(r_4,r_1,r_2,r_3)=r_4$, for $i\in\mathbb{W}$.

By excluding some of the conditions from Corollary 3.2.6, that is, taking Γ as a zero matrix and expanding the distance $\upsilon(r_1,t_1)$, we concludes with an important result.

Corollary 3.2.7.

Let $(\aleph,\upsilon,\preceq)$ be a POCGMS and $\digamma:\aleph^4\to\aleph$ be a mixed monotone mapping, such that

$$\upsilon(F(r_1, r_2, r_3, r_4), F(t_1, t_2, t_3.t_4)) \leq \Upsilon(\upsilon((r_1, r_2, r_3, r_4), (t_1, t_2, t_3, t_4))),$$

where $\Upsilon \in$ ZM. If F has a MQTp, then F has a QFp in $\aleph$.

Definition 3.2.8.

Two points (r_1, r_2, r_3, r_4) and $(t_1, t_2, t_3, t_4) \in \aleph^4$ are said to be quadruple comparable (QC) if and only if

$$\begin{aligned}
&r_1 \preceq t_1,\ r_2 \succeq t_2,\ r_3 \preceq t_3,\ r_4 \succeq t_4 \quad \text{or} \quad r_1 \succeq t_1,\ r_2 \preceq t_2,\ r_3 \succeq t_3,\ r_4 \preceq t_4 \text{ or}\\
&r_1 \preceq t_2,\ r_2 \succeq t_3,\ r_3 \preceq t_4,\ r_4 \succeq t_1 \quad \text{or} \quad r_1 \succeq t_2,\ r_2 \preceq t_3,\ r_3 \succeq t_4,\ r_4 \preceq t_1 \text{ or}\\
&r_1 \preceq t_3,\ r_2 \succeq t_4,\ r_3 \preceq t_1,\ r_4 \succeq t_2 \quad \text{or} \quad r_1 \succeq t_3,\ r_2 \preceq t_4,\ r_3 \succeq t_1,\ r_4 \preceq t_2 \text{ or}\\
&r_1 \preceq t_4,\ r_2 \succeq t_1,\ r_3 \preceq t_2,\ r_4 \succeq t_3 \quad \text{or} \quad r_1 \succeq t_4,\ r_2 \preceq t_1,\ r_3 \succeq t_2,\ r_4 \preceq t_3.
\end{aligned}$$

If we replace (r_1, r_2, r_3, r_4) and (t_1, t_2, t_3, t_4) with $(s(r_1), s(r_2), s(r_3), s(r_4))$ and $(s(t_1), s(t_2), s(t_3), s(t_4))$ in above Definition, then we say that (r_1, r_2, r_3, r_4) a QC with (t_1, t_2, t_3, t_4) with respect to (w.r.t) s.

Theorem 3.2.9.

Let $(\aleph, \upsilon, \preceq)$ be a POCGMS. Assume that $\{\xi_i\}_{i\in\mathbb{W}} : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be the mappings such that $\{\xi_i\}_{i\in\mathbb{W}}$ have QCps and satisfy (O) condition. Moreover, if $\{\xi_i\}_{i\in\mathbb{W}}$ has a QC (w.r.t) s, then there exists a unique common QFp of $\{\xi_i\}_{i\in\mathbb{W}}$ and s.

Proof. By Theorem 3.2.5, we obtained that there exist a non-empty set of QCps of mappings. Let (r_1, r_2, r_3, r_4) and (t_1, t_2, t_3, t_4) be the QC points, that is, if

$$\begin{aligned}
&s(r_1) = \xi_i(r_1, r_2, r_3, r_4), \quad s(r_2) = \xi_i(r_2, r_3, r_4, r_1),\\
&s(r_3) = \xi_i(r_3, r_4, r_1, r_2), \quad \text{and} \quad s(r_4) = \xi_i(r_4, r_1, r_2, r_3),\\
&s(t_1) = \xi_i(t_1, t_2, t_3, t_4), \quad s(t_2) = \xi_i(t_2, t_3, t_4, t_1),\\
&s(t_3) = \xi_i(t_3, t_4, t_1, t_2), \quad \text{and} \quad s(t_4) = \xi_i(t_4, t_1, t_2, t_3),
\end{aligned}$$

then, $s(r_1) = s(t_1),\ s(r_2) = s(t_2),\ s(r_3) = s(t_3),\ s(r_4) = s(t_4)$. Since, QCps are also QC, then from (3.1) we get that,

$$\begin{aligned}\upsilon(s(r_1), s(t_1)) &= \upsilon(\xi_i(r_1, r_2, r_3, r_4), \xi_j(t_1, t_2, t_3, t_4)) \\ &\leq\ \Gamma[\upsilon(s(r_1), \xi_i(r_1, r_2, r_3, r_4)) + \upsilon(s(t_1), \xi_j(t_1, t_2, t_3, t_4))] \\ &\quad + \Upsilon[\upsilon(s(r_1), s(t_1))]\end{aligned}$$

$$\Rightarrow \quad \upsilon(s(r_1), s(t_1)) \leq \Upsilon[\upsilon(s(r_1), s(t_1))].$$

Since, $I \neq \Upsilon \in$ ZM, then $\upsilon(s(r_1), s(t_1)) = \mathbf{0}$, or $s(r_1) = s(t_1)$. Additionally, similar operations generates that $s(r_2) = s(t_2),\ s(r_3) = s(t_3)$ and $s(r_4) = s(t_4)$. Hence $s(r_1) = s(r_2) = s(r_3) = s(r_4) = s(t_1) = s(t_4) = s(t_3) = s(t_4)$. Which drives the uniqueness of QCp $(s(r_1), s(r_2), s(r_3), s(r_4))$ of $\{\xi_i\}_{i\in\mathbb{W}}$ and s. Since $\{\xi_i\}_{i\in\mathbb{W}}$ and s are weakly compatible and coincident points of two compatible mappings are commutable, hence (r_1, r_2, r_3, r_4) is a unique QFp of $\{\xi_i\}_{i\in\mathbb{W}}$ and s. □

3.3 Application

Suppose that $\mathbb{R}^n_+ = \{r_1 = (r_1^1, r_1^2, r_1^3, \cdots, r_1^n) : r_i \geq, i \geq 1\}$ and

$$\Omega^4_{n-1} = \{\rho = (r_1, r_2, r_3, r_4) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \\ \sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} (r_i^1 + r_i^2 + r_i^3 + r_i^4) = 1\},$$

represent a $4(n-1)$ dimensional probability simplex and $\rho \in \Omega^4_{n-1}$ is the probability over respective states. The Markov process is a process which deals with modeling of a randomly changing system over the time. And in this system probability of current state in each period $\ell = 1, 2, 3, ...$ depends only on the probability of previous state. Suppose that for each $\ell = 1, 2, 3, \cdots$, $e_{ij} \geq 0$ shows the probability matrix which is achieved by state i in the next period starting from state j. Then, the preceding probability vector ρ^ℓ and the succeeding probability vector $\rho^{\ell+1}$ in the period ℓ and $\ell + 1$ respectively, written as $\rho_i^{\ell+1} = \sum_j e_{ij}\rho_j^\ell$, for each $j \geq 1$. Let ρ^ℓ be a column vector, then matrix form is achieved by mapping $\rho^{\ell+1} = F\rho^\ell$. In

addition with that for all $e_{ij} \geq 0, \sum_{i=1}^{n} e_{ij} = 1$, required for conditional probability. Finding the stationary distribution for Markov process is equivalent to finding the fixed point of the mapping $\digamma$ that is there exists some $\rho \in \Omega_{n-1}^{4}$ such that $\digamma \rho^{\ell} = \rho^{\ell}$, whenever $\rho^{\ell+1} = \rho^{\ell}$, where the period ρ^{ℓ} is called stationary distribution of Markov process.
Furthermore, for each i, $\pi = \sum_{i=1}^{n} \pi_i$, where $\pi_i = \min_j e_{ij}$.

Now, the main theorem of this section is given bellow.

Theorem 3.3.1.

By the assumption $e_{ij} \geq 0$, there exists a unique stationary distribution for the Markov process.

Proof. Let $\upsilon : \Omega_{n-1}^{4} \times \Omega_{n-1}^{4} \rightarrow \mathbb{R}^2$ be a mapping defined as

$$\begin{aligned}
\upsilon(R,Q) &= \upsilon((r_1,r_2,r_3,r_4),(q_1,q_2,q_3,q_4)) \\
&= \Big(\sum_{i=1}^{n} (|r_1^i - q_1^i| + |r_2^i - q_2^i| + |r_3^i - q_3^i| + |r_4^i - q_4^i|), \\
&\qquad \sum_{i=1}^{n} (|r_1^i - q_1^i| + |r_2^i - q_2^i| + |r_3^i - q_3^i| + |r_4^i - q_4^i|) \Big),
\end{aligned}$$

Where $R = (r_1,r_2,r_3,r_4)$ and $Q = (q_1,q_2,q_3,q_4)$ belongs to Ω_{n-1}^{4}.
Since, $\upsilon(R,Q) \geq (0,0)$ for all R and Q in Ω_{n-1}^{4}. Also, if $\upsilon(R,Q) = (0,0)$, then this implies

$$\begin{aligned}
&\Big(\sum_{i=1}^{n} (|r_1^i - q_1^i| + |r_2^i - q_2^i| + |r_3^i - q_3^i| + |r_4^i - q_4^i|), \\
&\quad \sum_{i=1}^{n} (|r_1^i - q_1^i| + |r_2^i - q_2^i| + |r_3^i - q_3^i| + |r_4^i - q_4^i|) \Big) = (0,0),
\end{aligned}$$

or

$$\begin{aligned}
& |r_1^i - q_1^i| + |r_2^i - q_2^i| + |r_3^i - q_3^i| + |r_4^i - q_4^i| = 0 \\
\Rightarrow \quad & |r_1^i - q_1^i| = |r_2^i - q_2^i| = |r_3^i - q_3^i| = |r_4^i - q_4^i| = 0, \\
\Rightarrow \quad & r_1^i = q_1^i, \ r_2^i = q_2^i, \ r_3^i = q_3^i, \ r_4^i = q_4^i,
\end{aligned}$$

hence, $R = Q$. Conversely, let $R = Q$, then

$$\begin{aligned} & r_1^i = q_1^i,\ r_2^i = q_2^i,\ r_3^i = q_3^i,\ r_4^i = q_4^i, \\ \Rightarrow \quad & |r_1^i - q_1^i| = |r_2^i - q_2^i| = |r_3^i - q_3^i| = |r_4^i - q_4^i| = 0. \end{aligned}$$

Hence,

$$\Big(\sum_{i=1}^{n}(|r_1^i - q_1^i| + |r_2^i - q_2^i| + |r_3^i - q_3^i| + |r_4^i - q_4^i|),$$

$$\sum_{i=1}^{n}(|r_1^i - q_1^i| + |r_2^i - q_2^i| + |r_3^i - q_3^i| + |r_4^i - q_4^i|)\Big) = (0,0)$$

$$\text{or} \quad \upsilon(R,Q) = (0,0).$$

Moreover,

$$\begin{aligned} \upsilon(R,Q) &= \Big(\sum_{i=1}^{n}(|r_1^i - q_1^i| + |r_2^i - q_2^i| + |r_3^i - q_3^i| + |r_4^i - q_4^i|), \\ &\quad \sum_{i=1}^{n}(|r_1^i - q_1^i| + |r_2^i - q_2^i| + |r_3^i - q_3^i| + |r_4^i - q_4^i|)\Big) \\ &= \Big(\sum_{i=1}^{n}(|q_1^i - r_1^i| + |q_2^i - r_2^i| + |q_3^i - r_3^i| + |q_4^i - r_4^i|), \\ &\quad \sum_{i=1}^{n}(|q_1^i - r_1^i| + |q_2^i - r_2^i| + |q_3^i - r_3^i| + |q_4^i - r_4^i|)\Big) \\ &= \upsilon(Q,R). \end{aligned}$$

Now,

$$\begin{aligned} \upsilon(R,Q) =& \Big(\sum_{i-1}^{n}(|r_1^i - q_1^i| + |r_2^i - q_2^i| + |r_3^i - q_3^i| + |r_4^i - q_4^i|), \\ & \sum_{i=1}^{n}(|r_1^i - q_1^i| + |r_2^i - q_2^i| + |r_3^i - q_3^i| + |r_4^i - q_4^i|)\Big) \\ =& \Bigg(\sum_{i=1}^{n}\begin{pmatrix} |(r_1^i - t_1^i) + (t_1^i - q_1^i)| + |(r_2^i - t_2^i) + (t_2^i - q_2^i)| \\ +|(r_3^i - t_3^i) + (t_3^i - q_3^i)| + |(r_4^i - t_4^i) + (t_4^i - q_4^i)| \end{pmatrix}, \\ & \sum_{i=1}^{n}\begin{pmatrix} |(r_1^i - t_1^i) + (t_1^i - q_1^i)| + |(r_2^i - t_2^i) + (t_2^i - q_2^i)| \\ +|(r_3^i - t_3^i) + (t_3^i - q_3^i)| + |(r_4^i - t_4^i) + (t_4^i - q_4^i)| \end{pmatrix}\Bigg) \end{aligned}$$

$$\leq\Big(\sum_{i-1}^{n}\begin{pmatrix}|r_1^i-t_1^i|+|t_1^i-q_1^i|+|r_2^i-t_2^i|+|t_2^i-q_2^i|\\+|r_3^i-t_3^i|+|t_3^i-q_3^i|+|r_4^i-t_4^i|+|t_4^i-q_4^i|\end{pmatrix},$$

$$\sum_{i=1}^{n}\begin{pmatrix}|r_1^i-t_1^i|+|t_1^i-q_1^i|+|r_2^i-t_2^i|+|t_2^i-q_2^i|\\+|r_3^i-t_3^i|+|t_3^i-q_3^i|+|r_4^i-t_4^i|+|t_4^i-q_4^i|\end{pmatrix}\Big)$$

$$=\Big(\sum_{i=1}^{n}(|r_1^i-t_1^i|+|r_2^i-t_2^i|+|r_3^i-t_3^i|+|r_4^i-t_4^i|),$$

$$\sum_{i=1}^{n}(|r_1^i-t_1^i|+|r_2^i-t_2^i|+|r_3^i-t_3^i|+|r_4^i-t_4^i|)\Big)$$

$$+\Big(\sum_{i=1}^{n}(|t_1^i-q_1^i|+|t_2^i-q_2^i|+|t_3^i-q_3^i|+|t_4^i-q_4^i|),$$

$$\sum_{i=1}^{n}(|t_1^i-q_1^i|+|t_2^i-q_2^i|+|t_3^i-q_3^i|+|t_4^i-q_4^i|)\Big)$$

$$=\upsilon(R,T)+\upsilon(T,Q),$$

where $T=(t_1,t_2,t_3,t_4)\in\Omega^4_{n-1}$. Hence, $(\Omega^4_{n-1},\upsilon)$ is a generalized metric space.

Completeness of Ω^4_{n-1} can be easily proved. Now, define partial order on Ω^4_{n-1} as for all $(r_1,r_2,r_3,r_4),(q_1,q_2,q_3,q_4)\in\Omega^4_{n-1}$,

$$(r_1,r_2,r_3,r_4)\preceq(q_1,q_2,q_3,q_4)\iff r_1\succeq q_1,\ r_2\preceq q_2,\ r_3\succeq q_3\quad\text{and}\quad r_4\preceq q_4,$$

hence, $(\Omega^4_{n-1},\upsilon,\preceq)$ is a POCGMS. Let $\digamma:\Omega^4_{n-1}\to\Omega^4_{n-1}$ be a mapping defined as for all $\rho\in\Omega^4_{n-1}$, $\digamma\rho=\ell=\delta_j$ such that for each j, $\delta_j=\sum_{i=1}^{n}e_{ij}\rho_j$. Since,

$$\sum_{j=1}^{n}\delta_j=\sum_{j=1}^{n}\sum_{i=1}^{n}e_{ij}\rho_j=\sum_{i=1}^{n}e_{ij}\sum_{j=1}^{n}(r_1^j+r_2^j+r_3^j+r_4^j)=\sum_{j=1}^{n}(r_1^j+r_2^j+r_3^j+r_4^j)=1,$$

this implies that $\ell\in\Omega^4_{n-1}$ i.e mapping is defined. Now, we have to show that $\digamma$ satisfy the contraction condition, for this, let δ_i be the i^{th} row of δ. Then, for all $(r_1,r_2,r_3,r_4),(q_1,q_2,q_3,q_4)\in\Omega^4_{n-1}$, we get

$$\upsilon(\digamma(r_1,r_2,r_3,r_4),\digamma(q_1,q_2,q_3,q_4))$$

$$
\begin{aligned}
=&\Big(\sum_{i=1}^{n}\big(|\sum_{j=1}^{n}\big(e_{ij}(r_1^j+r_2^j+r_3^j+r_4^j)-e_{ij}(q_1^j+q_2^j+q_3^j+q_4^j)\big)|\big),\\
&\sum_{i=1}^{n}\big(|\sum_{j=1}^{n}\big(e_{ij}(r_1^j+r_2^j+r_3^j+r_4^j)-e_{ij}(q_1^j+q_2^j+q_3^j+q_4^j)\big)|\big)\Big)\\
=&\left(\sum_{i=1}^{n}\begin{pmatrix}|\sum_{j=1}^{n}(e_{ij}-\pi_i)\{(r_1^j+r_2^j+r_3^j+r_4^j)-(q_1^j+q_2^j+q_3^j+q_4^j)\}\\ +\pi_i\{(r_1^j+r_2^j+r_3^j+r_4^j)-(q_1^j+q_2^j+q_3^j+q_4^j)\}|\end{pmatrix},\right.\\
&\left.\sum_{i=1}^{n}\begin{pmatrix}|\sum_{j=1}^{n}(e_{ij}-\pi_i)\{(r_1^j+r_2^j+r_3^j+r_4^j)-(q_1^j+q_2^j+q_3^j+q_4^j)\}\\ +\pi_i\{(r_1^j+r_2^j+r_3^j+r_4^j)-(q_1^j+q_2^j+q_3^j+q_4^j)\}|\end{pmatrix}\right)\\
\leq&\left(\sum_{i=1}^{n}|e_{ij}-\pi_i|\left(|\sum_{j=1}^{n}\{(r_1^j+r_2^j+r_3^j+r_4^j)-(q_1^j+q_2^j+q_3^j+q_4^j)\}|\right)\right.\\
&+\sum_{i=1}^{n}|\sum_{j=1}^{n}\pi_i\{(r_1^j+r_2^j+r_3^j+r_4^j)-(q_1^j+q_2^j+q_3^j+q_4^j)\}|,\\
&\sum_{i=1}^{n}(e_{ij}-\pi_i)\left(|\sum_{j=1}^{n}\{(r_1^j+r_2^j+r_3^j+r_4^j)-(q_1^j+q_2^j+q_3^j+q_4^j)\}|\right)\\
&\left.+\sum_{i=1}^{n}|\sum_{j=1}^{n}\pi_i\{(r_1^j+r_2^j+r_3^j+r_4^j)-(q_1^j+q_2^j+q_3^j+q_4^j)\}|\right)\\
\leq&\begin{pmatrix}\sum_{i=1}^{n}\sum_{j=1}^{n}(|r_1^j-q_1^j|+|r_2^j-q_2^j|+|r_3^j-q_3^j|+|r_4^j-q_4^j|)\times|e_{ij}-\pi_i|\\ \sum_{i=1}^{n}\sum_{j=1}^{n}(|r_1^j-q_1^j|+|r_2^j-q_2^j|+|r_3^j-q_3^j|+|r_4^j-q_4^j|\times|e_{ij}-\pi_i|)\end{pmatrix}\\
=&(I-\pi)\begin{pmatrix}\sum_{j=1}^{n}(|r_1^j-q_1^j|+|r_2^j-q_2^j|+|r_3^j-q_3^j|+|r_4^j-q_4^j|)\\ \sum_{j=1}^{n}(|r_1^j-q_1^j|+|r_2^j-q_2^j|+|r_3^j-q_3^j|+|r_4^j-q_4^j|)\end{pmatrix}\\
=&\Upsilon\upsilon((r_1,r_2,r_3,r_4),(q_1,q_2,q_3,q_4)).
\end{aligned}
$$

Where $(I-\pi)=\Upsilon\in$ ZM, hence all conditions of corollary 3.2.7 are fulfilled. Then, there exists a unique quadruple fixed point of mapping $\digamma$ or in other words a unique stationary distribution of Markov process. Moreover, the sequence $\{\digamma^n\rho^l\}$ converges to a unique stationary distribution for any $\rho^l\in\Omega_{n-1}$. □

Chapter 4

Quadruple Fixed Point Results for GbMS under Matrices

In 1983, the idea of b-metric space was introduced by Bakhtin [23] and generalized by Czerwik [24] in 1993. An extension of BCP on spaces endowed with vector-valued metric was presented by Perov in 1964, called generalized metric spaces. Later on, in 2010, Filip [30] established fixed pint result in generalized metric spaces. Hammad *et al*. [34] worked on the theory of QFp of mappings in the frame me GMS.

In this chapter, we inroduced the definition of generalized b-metric spaces alonwith vector-valued b-metric spaces and extended the results from [34] for quadruple fixed point of mappings in the setting of GbMS. Eventually, our theoretical results are demonstrated by few examples and an application on the existence of unique stationary distribution of Markov process.

4.1 Few Results in GbMS

This section covers some definitions extended on the structure of GbMS along with supportive examples.

Definition 4.1.1.

A mapping $\upsilon_b : \aleph^2 \to \mathbb{R}^k$ where $(\aleph \neq \emptyset)$ is called a vector-valued b-metric on $\aleph$, if for some $b \geq 1$, the conditions bellow are satisfied, that is, for any $r_1, r_2, r_3 \in \aleph$,

$(\aleph_b 1)$ $\upsilon_b(r_1, r_2) \succeq \mathbf{0}, \upsilon_b(r_1, r_2) = \mathbf{0} \Leftrightarrow r_1 = r_2$,

$(\aleph_b 2)$ $\upsilon_b(r_1, r_2) = \upsilon_b(r_2, r_1)$,

$(\aleph_b 3)$ $\upsilon_b(r_1, r_2) \preceq b\{\upsilon_b(r_1, r_3) + \upsilon_b(r_3, r_2)\}$.

If $a, c \in \mathbb{R}^k$, where $a = (a_1, a_2, \cdots, a_k)$ and $c = (c_1, c_2, \cdots, c_k)$, then $a \preceq c$ if and only if $a_i \leq c_i$ for $1 \leq i \leq k$. Then, the pair $(\aleph, \upsilon_b)$ is called a generalized b-metric space (GbMS). Moreover, if $b = 1$ then the pair $(\aleph, \upsilon_b)$ becomes generalized metric space.

Example 4.1.2.

Let $\aleph$ be a non-empty set and $\mathrm{d}_{b_1}, \mathrm{d}_{b_2}, \mathrm{d}_{b_3}, \cdots, \mathrm{d}_{b_k}$ be the b-metrics on $\aleph$ for some $b_1, b_2, b_3, \cdots, b_k \geq 1$ respectively. Let $\upsilon_b : \aleph \times \aleph \to \mathbb{R}^k$ be a mapping defined as for all $r_1, r_2 \in \aleph$,

$$\upsilon_b(r_1, r_2) = (\mathrm{d}_{b_1}(r_1, r_2), \mathrm{d}_{b_2}(r_1, r_2), \cdots, \mathrm{d}_{b_k}(r_1, r_2)),$$

then $(\aleph, \upsilon_b)$ is a generalized b-metric space with $b = \max\{b_1, b_2, b_3, \cdots, b_k\}$. Since, for all $r_1, r_2, r_3 \in \aleph$:

$(\aleph_b 1)$ $\mathrm{d}_{b_i}(r_1, r_2) \succeq 0 \ (1 \leq i \leq k) \quad \Rightarrow \upsilon_b(r_1, r_2) \succeq \mathbf{0}, \quad$ and

$$\upsilon_b(r_1, r_2) = \mathbf{0} \quad \Leftrightarrow \quad \mathrm{d}_{b_i}(r_1, r_2) = 0 \text{ (for each } i) \quad \Leftrightarrow \quad r_1 = r_2,$$

$(\aleph 2)$

$$\begin{aligned}
&\upsilon_b(r_1, r_2) \\
&= (\mathrm{d}_{b_1}(r_1, r_2), \mathrm{d}_{b_2}(r_1, r_2), \cdots, \mathrm{d}_{b_k}(r_1, r_2)) \\
&= (\mathrm{d}_{b_1}(r_2, r_1), \mathrm{d}_{b_2}(r_2, r_1), \cdots, \mathrm{d}_{b_k}(r_2, r_1)) \\
&= \upsilon_b(r_2, r_1),
\end{aligned}$$

(ℵ3)

$$\begin{aligned}
\upsilon_b(r_1, r_3) &= (\mathrm{d}_{b_1}(r_1, r_3), \mathrm{d}_{b_2}(r_1, r_3), \cdots, \mathrm{d}_{b_k}(r_1, r_3)) \\
&\preceq \big(b_1\{\mathrm{d}_{b_1}(r_1, r_2) + \mathrm{d}_{b_1}(r_2, r_3)\}, b_2\{\mathrm{d}_{b_2}(r_1, r_2) + \mathrm{d}_{b_2}(r_2, r_3)\}, \cdots, \\
&\quad b_k\{\mathrm{d}_{b_k}(r_1, r_2) + \mathrm{d}_{b_k}(r_2, r_3)\}\big) \\
&\preceq \big(b\{\mathrm{d}_{b_1}(r_1, r_2) + \mathrm{d}_{b_1}(r_2, r_3)\}, b\{\mathrm{d}_{b_2}(r_1, r_2) + \mathrm{d}_{b_2}(r_2, r_3)\}, \cdots, \\
&\quad b\{\mathrm{d}_{b_k}(r_1, r_2) + \mathrm{d}_{b_k}(r_2, r_3)\}\big) \\
&= b\{\big(\mathrm{d}_{b_1}(r_1, r_2), \mathrm{d}_{b_2}(r_1, r_2), \cdots, \mathrm{d}_{b_k}(r_1, r_2)\big) + \big(\mathrm{d}_{b_1}(r_2, r_3), \mathrm{d}_{b_2}(r_2, r_3) \\
&\quad , \cdots, \mathrm{d}_{b_k}(r_2, r_3)\big)\} \\
&= b\{\upsilon_b(r_1, r_2) + \upsilon_b(r_2, r_3)\}.
\end{aligned}$$

Remark 6:

- (•) For GbMS, the notions of convergent sequences, Cauchy sequences, completeness, open subsets, closed subsets and continuous mappings are similar to those for GMS.
- (•) Definitions of compatibility, reciprocally continuous and weakly reciprocally continuous over two mappings can be extended in the frame of b-metric space in similar fashion as defined in the previous chapter. Forthcoming examples are in support of this theory.

Example 4.1.3.

Let $\aleph = [0, 1]$ be a b-metric space under distance function $\upsilon_b(r_1, r_2) = |r_1 - r_2|^2$, with $b = 2$, and "$\preceq$" be the partial order on $\aleph$. Let $S : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be two mappings defined as

$$S(r_1, r_2, r_3, r_4) = \frac{r_1 r_2 + r_3 r_4}{4} \quad \text{and} \quad s(r_1) = r_1.$$

Consider the sequences $\{r_1^n\}$, $\{r_2^n\}$, $\{r_3^n\}$ and $\{r_4^n\}$ defined by

$$r_1^n = \frac{1}{n^2},\ r_2^n = \frac{1}{n+1},\ r_3^n = \frac{1}{\sqrt{n^2+1}}, \quad \text{and} \quad r_4^n = \frac{1}{n^3}, \quad \forall\, n \in \mathbb{N}.$$

Then, $(\aleph, \upsilon_b)$ is a partially ordered b-metric space. Some simple calculations leads us to the following results

$$\lim_{n\to+\infty} U_{1234} = \lim_{n\to+\infty} s(r_1^n) = 0, \quad \lim_{n\to+\infty} U_{2341} = \lim_{n\to+\infty} s(r_2^n) = 0,$$
$$\lim_{n\to+\infty} U_{3412} = \lim_{n\to+\infty} s(r_3^n) = 0, \quad \lim_{n\to+\infty} U_{4123} = \lim_{n\to+\infty} s(r_4^n) = 0,$$

for some $r_1 = r_2 = r_3 = r_4 = 0 \in \aleph$. Also above defined sequences, functions and metric satisfy the conditions of compatibility, reciprocal continuity and weakly reciprocal continuity. Hence, both mappings S and s are compatible, reciprocally continuous and weakly reciprocally continuous.

Example 4.1.4.

Let $\aleph = [0, 1]$ be a b-metric space with metric defined as $\upsilon_b(r_1, r_2) = |r_1 - r_2|^2$, and $\xi_n : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be the mappings defined as

$$\xi_n(r_1, r_2, r_3, r_4) = \frac{1}{4^n} - \frac{r_1 r_2 r_3 r_4}{4n^2} \quad \text{and} \quad s(r_1) = r_1.$$

Define four sequences $\{r_1^n\}$, $\{r_2^n\}$, $\{r_3^n\}$ and $\{r_4^n\}$ in $\aleph$ as,

$$r_1^n = \frac{n}{n^2+1},\ r_2^n = \frac{n}{\sqrt{n^2+1}},\ r_3^n = \frac{1}{n+1} \quad \text{and} \quad r_4^n = \frac{1}{n^3+1}, \qquad \forall\ n \in \mathbb{N}.$$

Then, some simple steps drives us to the result

$$\lim_{n\to+\infty} \grave{U}_{1234} = \lim_{n\to+\infty} s(r_1^{n+1}) = r_1 = 0, \quad \lim_{n\to+\infty} \grave{U}_{2341} = \lim_{n\to+\infty} s(r_2^{n+1}) = r_2 = 0,$$
$$\lim_{n\to+\infty} \grave{U}_{3412} = \lim_{n\to+\infty} s(r_3^{n+1}) = r_3 = 0, \quad \lim_{n\to+\infty} \grave{U}_{4123} = \lim_{n\to+\infty} s(r_4^{n+1}) = r_4 = 0,$$

for some $r_1 = r_2 = r_3 = r_4 = 0 \in \aleph$. Also

$$\lim_{n\to+\infty} s(\grave{U}_{1234}) = s(r_1) = 0, \quad \lim_{n\to+\infty} s(\grave{U}_{2341}) = s(r_2) = 0,$$
$$\lim_{n\to+\infty} s(\grave{U}_{3412}) = s(r_3) = 0, \quad \lim_{n\to+\infty} s(\grave{U}_{4123}) = s(r_4) = 0,$$

and

$$\lim_{n\to+\infty} \upsilon_b(s(\grave{U}_{1234}), \xi_n(\grave{V}_{1234})) = 0, \quad \lim_{n\to+\infty} \upsilon_b(s(\grave{U}_{2341}), \xi_n(\grave{V}_{2341})) = 0,$$

$$\lim_{n\to+\infty} \upsilon_b(s(\grave{U}_{3412}), \xi_n(\grave{V}_{3412})) = 0, \quad \lim_{n\to+\infty} \upsilon_b(s(\grave{U}_{4123}), \xi_n(\grave{V}_{4123})) = 0.$$

Hence, compatibility and weakly reciprocal continuity of $\{\xi_n\}_{n\in\mathbb{N}}$ and s are proved.

4.2 Main Results

This section deals with contraction condition (B), QFp theorems, corollaries and supportive examples.

Definition 4.2.1.

Let $\xi_n : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be the mappings, then $\{\xi_i\}_{i\in\mathbb{W}}$ and s are said to satisfy the (B) condition if there exist $b \geq 1$ such that

$$\begin{aligned} & b(\upsilon_b(\xi_i(r_1,r_2,r_3,r_4),\xi_j(t_1,t_2,t_3,t_4))) \\ & \preceq \Gamma[\upsilon_b(s(r_1),\xi_i(r_1,r_2,r_3,r_4)) + \upsilon_b(s(t_1),\xi_j(t_1,t_2,t_3,t_4))] \\ & \quad + \Upsilon[\upsilon_b(s(r_1),s(t_1))], \end{aligned} \tag{4.1}$$

for some $r_1, r_2, r_3, r_4, t_1, t_2, t_3, t_4, \in \aleph$ provided that $s(r_i) \preceq s(t_i)$ for $1 \leq i \leq 4$ or $s(r_i) \succeq s(t_i)$ for $1 \leq i \leq 4$, $I \neq \Gamma = (\Gamma_{ij})$ and $I \neq \Upsilon = (\Upsilon_{ij}) \in$ ZM satisfy the condition $(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1} \in$ ZM, where $\beta = \frac{1}{b}$.

Example 4.2.2.

Let $\aleph = [0, 1]$ be a b-metric space equipped with metric

$$\upsilon_b(r_1, r_2) = |r_1 - r_2|^2,$$

with $b = 2$ and $\beta = \frac{1}{2}$.

(1) Let $\Gamma = \begin{pmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{8} \end{pmatrix}$ and $\Upsilon = \begin{pmatrix} 0 & \frac{1}{8} \\ \frac{1}{8} & 0 \end{pmatrix}$ be two matrices in ZM. Then, it is easy to compute that $(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1} \in$ ZM.

(2) Let $\Gamma = \alpha I$, and $\Upsilon = ((1-\gamma)^3 - \alpha)I \in$ ZM such that $\alpha = \frac{1}{4}, \frac{1}{5}, \frac{1}{7}, \frac{1}{8}$ and $\gamma = \alpha\beta$. Then we get that $(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1} \in$ ZM.

(3) For $\Gamma = \frac{1}{7}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\Upsilon = \frac{1}{11}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ in ZM, some simple steps concludes that $(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1} \in$ ZM.

Lemma 4.2.3.

Let $(\aleph, \upsilon_b, \preceq)$ be a partially ordered complete generalized b-metric space (POCGbMS), $\xi_i : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be the mappings such that $\{\xi_i\}_{i\in\mathbb{W}}$ have MsMP and $\xi_i(\aleph^4) \subseteq s(\aleph)$. If ξ_o and s have MQTp, then

1. $\exists$ sequences $\{r_1^n\}$, $\{r_2^n\}$, $\{r_3^n\}$ and $\{r_4^n\} \in \aleph$ such that

$$s(r_1^n) = \xi_{n-1}(r_1^{n-1}, r_2^{n-1}, r_3^{n-1}, r_4^{n-1}), \quad s(r_2^n) = \xi_{n-1}(r_2^{n-1}, r_3^{n-1}, r_4^{n-1}, r_1^{n-1}),$$
$$s(r_3^n) = \xi_{n-1}(r_3^{n-1}, r_4^{n-1}, r_1^{n-1}, r_2^{n-1}), \text{ and } s(r_4^n) = \xi_{n-1}(r_4^{n-1}, r_1^{n-1}, r_2^{n-1}, r_3^{n-1}).$$

2. $\{s(r_1^n)\}$, $\{s(r_3^n)\}$ are non-decreasing sequences and $\{s(r_2^n)\}$, $\{s(r_4^n)\}$ are non-increasing sequences.

Proof of the above stated Lemma can be extracted in same manner from the proof of the Lemma 3.2.4.

Now, the core part of this section is stated bellow.

Theorem 4.2.4.

In addition to the conditions of Lemma 4.2.3, suppose that $\{\xi_i\}_{i\in\mathbb{W}}$ and s are monotonically decreasing and satisfy (B) condition, moreover both mappings are compatible and weakly reciprocally continuous provided that s is continuous. If $s(\aleph) \subseteq \aleph$ is complete and regular then, there exists a quadruple coincidence point (QCp) of $\{\xi_i\}_{i\in\mathbb{W}}$ and s, for any $O \neq \Gamma, \Upsilon \in$ ZM.

Proof. Let $\{r_1^n\}$, $\{r_2^n\}$, $\{r_3^n\}$ and $\{r_4^n\}$ be the sequences in $\aleph$ constructed by Lemma 4.2.3, then from (4.1) it follows that (note that $\beta = \frac{1}{b}$):

$$\begin{aligned}
\upsilon_b(s(r_1^n), (r_1^{n+1})) &= \upsilon_b(\xi_{n-1}(r_1^{n-1}, r_2^{n-1}, r_3^{n-1}, r_4^{n-1}), \xi_n(r_1^n, r_2^n, r_3^n, r_4^n)) \\
&\preceq \beta\Gamma[\upsilon_b(s(r_1^{n-1}), \xi_{n-1}(r_1^{n-1}, r_2^{n-1}, r_3^{n-1}, r_4^{n-1})) \\
&\quad + \upsilon_b(s(r_1^n), \xi_n(r_1^n, r_2^n, r_3^n, r_4^n))] + \beta\Upsilon\upsilon_b(s(r_1^{n-1}), s(r_1^n)) \\
&= \beta(\Gamma + \Upsilon)\upsilon_b(s(r_1^{n-1}), s(r_1^n)) + \beta\Gamma(\upsilon_b(s(r_1^n), s(r_1^{n+1}))).
\end{aligned}$$

This leads us to the result that

$$\upsilon_b(s(r_1^n), s(r_1^{n+1})) \preceq \beta(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1}\upsilon_b(s(r_1^{n-1}), s(r_1^n)), \tag{4.2}$$

similar operations yields,

$$\begin{aligned} &\upsilon_b(s(r_2^n), s(r_2^{n+1})) \preceq \beta(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1}\upsilon_b(s(r_2^{n-1}), s(r_2^n)), \\ &\upsilon_b(s(r_3^n), s(r_3^{n+1})) \preceq \beta(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1}\upsilon_b(s(r_3^{n-1}), s(r_3^n)), \end{aligned} \tag{4.3}$$

and

$$\upsilon_b(s(r_4^n), s(r_4^{n+1})) \preceq \beta(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1}\upsilon_b(s(r_4^{n-1}), s(r_4^n)). \tag{4.4}$$

Addition of (4.2)-(4.4) implies,

$$\begin{aligned} \lambda_n &= \upsilon_b(s(r_1^n), s(r_1^{n+1})) + \upsilon_b(s(r_2^n), s(r_2^{n+1})) + \upsilon_b(s(r_3^n), s(r_3^{n+1})) + \upsilon_b(s(r_4^n), s(r_4^{n+1})) \\ &\preceq \beta(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1}[\upsilon_b(s(r_1^{n-1}), s(r_1^n)) + \upsilon_b(s(r_2^{n-1}), s(r_2^n)) + \upsilon_b(s(r_3^{n-1}), s(r_3^n)) \\ &\quad + \upsilon_b(s(r_4^{n-1}), s(r_4^n))] \\ &= \beta(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1})\lambda_{n-1}. \end{aligned}$$

Take $\beta(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1} = Y$, then for $n \in \mathbb{N}$, we get

$$O \leq \lambda_n \leq Y\lambda_{n-1} \leq Y^2\lambda_{n-2} \leq ... \leq Y^n\lambda_o$$

By way of triangular inequality, for $m > 0$, we have,

$$\begin{aligned} &\upsilon_b(s(r_1^n), s(r_1^{n+m})) + \upsilon_b(s(r_2^n), s(r_2^{n+m})) + \upsilon_b(s(r_3^n), s(r_3^{n+m})) + \upsilon_b(s(r_4^n), s(r_4^{n+m})) \\ &\preceq b[\upsilon_b(s(r_1^n), s(r_1^{n+1})) + \upsilon_b(s(r_2^n), s(r_2^{n+1})) + \upsilon_b(s(r_3^n), s(r_3^{n+1})) + \upsilon_b(s(r_4^n), s(r_4^{n+1}))] \\ &\quad + b^2[\upsilon_b(s(r_1^{n+1}), s(r_1^{n+2})) + \upsilon_b(s(r_2^{n+1}), s(r_2^{n+2})) + \upsilon_b(s(r_3^{n+1}), s(r_3^{n+2})) + \upsilon_b(s(r_4^{n+1}), \\ &\quad s(r_4^{n+2}))] + \cdots + b^m[\upsilon_b(s(r_1^{n+m-1}), s(r_1^{n+m})) + \upsilon_b(s(r_2^{n+m-1}), s(r_2^{n+m})) \\ &\quad + \upsilon_b(s(r_3^{n+m-1}), s(r_3^{n+m})) + \upsilon_b(s(r_4^{n+m-1}), s(r_4^{n+m}))] \\ &= b\lambda_n + b^2\lambda_{n+1} + \cdots + b^m\lambda_{n+m-1} \\ &\preceq (bY^n + b^2Y^{n+1} + \cdots + b^mY^{n+m-1})\lambda_o \\ &= bY^n(I + bY + \cdots + b^{m-1}Y^{m-1} + \cdots)\lambda_o \end{aligned}$$

$$= bY^n(I - bY)^{-1}\lambda_o.$$

Which drives us to the result that

$$\begin{aligned}
&\upsilon_b(s(r_1^n), s(r_1^{n+m})) + \upsilon_b(s(r_2^n), s(r_2^{n+m})) + \upsilon_b(s(r_3^n), s(r_3^{n+m})) + \upsilon_b(s(r_4^n), s(r_4^{n+m}))\\
&\preceq b[\beta(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1}]^n[I - b(\beta(\Gamma + \Upsilon)(I - \beta\Gamma)^{-1})]^{-1}\lambda_o\\
&= \frac{1}{b^{n-1}}[(\Gamma + \Upsilon)(I - \beta\Gamma)]^n[I - (\Gamma + \Upsilon)(I - \beta\Gamma)^{-1}]^{-1}\lambda_o
\end{aligned}$$

Take $\lim\limits_{n\to\infty}$ on both sides implies,

$$\begin{aligned}
&\lim_{n\to+\infty}[\upsilon_b(s(r_1^n), s(r_1^{n+m})) + \upsilon_b(s(r_2^n), s(r_2^{n+m})) + \upsilon_b(s(r_3^n), s(r_3^{n+m}))\\
&\qquad + \upsilon_b(s(r_4^n), s(r_4^{n+m}))] = 0,\\
\Rightarrow\quad &\lim_{n\to+\infty}\upsilon_b(s(r_1^n), s(r_1^{n+m})) = \lim_{n\to+\infty}\upsilon_b(s(r_2^n), s(r_2^{n+m}))\\
&\qquad = \lim_{n\to+\infty}\upsilon_b(s(r_3^n), s(r_3^{n+m})) = \lim_{n\to+\infty}\upsilon_b(s(r_4^n), s(r_4^{n+m})) = 0
\end{aligned}$$

This implies that $\{s(r_1^n)\}$, $\{s(r_2^n)\}$, $\{s(r_3^n)\}$ and $\{s(r_4^n)\}$ are Cauchy sequences in $\aleph$. As $s(\aleph)$ is complete, so $\exists$ $(r_1^*, r_2^*, r_3^*, r_4^*) \in \aleph^4$ such that,

$$\begin{aligned}
&\lim_{n\to+\infty} s(r_1^n) = s(r_1^*) = r_1, \quad \lim_{n\to+\infty} s(r_2^n) = s(r_2^*) = r_2,\\
&\lim_{n\to+\infty} s(r_3^n) = s(r_3^*) = r_3, \quad \lim_{n\to+\infty} s(r_4^n) = s(r_4^*) = r_4.
\end{aligned}$$

which results in

$$\begin{aligned}
&\lim_{n\to+\infty} s(r_1^{n+1}) = \lim_{n\to+\infty}\xi_n(r_1^n, r_2^n, r_3^n, r_4^n),\\
&\lim_{n\to+\infty} s(r_2^{n+1}) = \lim_{n\to+\infty}\xi_n(r_2^n, r_3^n, r_4^n, r_1^n),\\
&\lim_{n\to+\infty} s(r_3^{n+1}) = \lim_{n\to+\infty}\xi_n(r_3^n, r_4^n, r_1^n, r_2^n), \quad \text{and}\\
&\lim_{n\to+\infty} s(r_4^{n+1}) = \lim_{n\to+\infty}\xi_n(r_4^n, r_1^n, r_2^n, r_3^n).
\end{aligned}$$

Since $\{\xi_i\}_{i\in\mathbb{W}}$ and s are weakly reciprocally continuous and compatible, then we have

$$\lim_{n\to+\infty}\xi_n(s(r_1^n), s(r_2^n), s(r_3^n), s(r_4^n)) = s(r_1),$$

$$\lim_{n\to+\infty} \xi_n(s(r_2^n), s(r_3^n), s(r_4^n), s(r_1^n)) = s(r_2),$$

$$\lim_{n\to+\infty} \xi_n(s(r_3^n), s(r_4^n), s(r_1^n), s(r_2^n)) = s(r_3), \quad \text{and}$$

$$\lim_{n\to+\infty} \xi_n(s(r_4^n), s(r_1^n), s(r_2^n), s(r_3^n)) = s(r_4).$$

Since $\{s(r_1^n)\},\ \{s(r_3^n)\}$ are non-decreasing sequences and $\{s(r_2^n)\},\ \{s(r_4^n)\}$ are non-increasing sequences, then from the regularity of $\aleph$, for all $n \geq 0$, we can obtain $s(r_1^n) \preceq r_1,\ r_2 \preceq s(r_2^n),\ s(r_3^n) \preceq r_3,\ r_4 \preceq s(r_4^n)$. Then from (4.1) we get

$$\begin{aligned}
&\upsilon_b(\xi_i(r_1, r_2, r_3, r_4), \xi_n(s(r_1^n), s(r_2^n), s(r_3^n), s(r_4^n)))\\
&\preceq \beta\Gamma[\upsilon_b(s(r_1), \xi_i(r_1, r_2, r_3, r_4)) + \upsilon_b(S(s(r_1^n)), \xi_n(r_1^n, r_2^n, r_3^n, r_4^n))]\\
&\quad + \beta\Upsilon(\upsilon_b(s(r_1), s(s(r_1^n)))),
\end{aligned}$$

Now by applying $\lim\limits_{n\to\infty}$, it is concluded as

$$\upsilon_b(\xi_i(r_1, r_2, r_3, r_4), s(r_1)) \preceq \beta\Gamma\upsilon_b(s(r_1), \xi_i(r_1, r_2, r_3, r_4)),$$

which holds only if

$$\upsilon_b(\xi_i(r_1, r_2, r_3, r_4), s(r_1)) = 0 \quad \text{or} \quad \xi_i(r_1, r_2, r_3, r_4) = s(r_1).$$

Similar operation generates that $\xi_i(r_2, r_3, r_4, r_1) = s(r_2)$, $\xi_i(r_3, r_4, r_1, r_2) = s(r_3)$ and $\xi_i(r_4, r_1, r_2, r_3) = s(r_4)$. Hence, (r_1, r_2, r_3, r_4) is a QCp of $\{\xi_i\}_{i\in\mathbb{W}}$ and s. □

Next result is extended from Theorem 4.2.4 with the addition of $s = I^d$ being identity map.

Corollary 4.2.5.

Let $(\aleph, \upsilon_b, \preceq)$ is a POCGbMS, $\{\xi_i\}_{i\in\mathbb{W}} : \aleph^4 \to \aleph$ be the mixed-monotone mappings such that $\{\xi_i\}_{i\in\mathbb{W}}$ and $I^d : \aleph \to \aleph$ satisfy (B) condition and $I^d(\aleph)$ is regular. If ξ_o and I^d have MQTp, then $\exists\ (r_1, r_2, r_3, r_4) \in \aleph^4$ such that for each $i \in \mathbb{W}$,

$$\xi_i(r_1, r_2, r_3, r_4) = r_1, \quad \xi_i(r_2, r_3, r_4, r_1) = r_2,$$

$$\xi_i(r_3, r_4, r_1, r_2) = r_3 \quad \text{and} \quad \xi_i(r_4, r_1, r_2, r_3) = r_4.$$

By excluding some of the conditions from Corollary 4.2.5, taking Γ as a zero matrix and expanding the distance $\upsilon(r_1, t_1)$ from Definition 2.3.7 in the framework of GbMS, we concludes with an exciting outcome.

Corollary 4.2.6.

Let $(\aleph, \upsilon_b, \preceq)$ be a POCGbMS and $\digamma : \aleph^4 \to \aleph$ be a mixed monotone mapping. If $\digamma$ has a MQTp and satisfy the condition

$$b\upsilon_b(\digamma(r_1, r_2, r_3, r_4), \digamma(t_1, t_2, t_3.t_4)) \preceq \Upsilon(\upsilon_b((r_1, r_2, r_3, r_4), (t_1, t_2, t_3, t_4))),$$

then $\digamma$ has a QFp in $\aleph$.

Next result is on the existence of unique common QFp of mappings.

Theorem 4.2.7.

Let $\{\xi_i\}_{i\in\mathbb{W}} : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be the mappings on a POCGbMS $(\aleph, \upsilon_b, \preceq)$ such that $\{\xi_i\}_{i\in\mathbb{W}}$ and s satisfy (B) condition and have QCps with quadruple comparable (w.r.t) s. Then, $\{\xi_i\}_{i\in\mathbb{W}}$ and s have a unique common QFp.

Proof. From Theorem 4.2.4, we obtain that the set of QCps is non-empty. Now, by proving that (r_1, r_2, r_3, r_4) and (t_1, t_2, t_3, t_4) are QCps, that is, if

$$\begin{aligned}
&s(r_1) = \xi_i(r_1, r_2, r_3, r_4), \quad s(r_2) = \xi_i(r_2, r_3, r_4, r_1),\\
&s(r_3) = \xi_i(r_3, r_4, r_1, r_2), \quad \text{and} \quad s(r_4) = \xi_i(r_4, r_1, r_2, r_3),\\
&s(t_1) = \xi_i(t_1, t_2, t_3, t_4), \quad s(t_2) = \xi_i(t_2, t_3, t_4, t_1),\\
&s(t_3) = \xi_i(t_3, t_4, t_1, t_2), \quad \text{and} \quad s(t_4) = \xi_i(t_4, t_1, t_2, t_3),
\end{aligned}$$

then, $s(r_1) = s(t_1)$, $s(r_2) = s(t_2)$, $s(r_3) = s(t_3)$, $s(r_4) = s(t_4)$. Since QCps are also QC, then from (4.1) we get that,

$$\begin{aligned}
\upsilon_b(s(r_1), s(t_1)) &= \upsilon_b(\xi_i(r_1, r_2, r_3, r_4), \xi_j(t_1, t_2, t_3, t_4))\\
&\preceq \beta\Gamma[\upsilon_b(s(r_1), \xi_i(r_1, r_2, r_3, r_4)) + \upsilon_b(s(t_1), \xi_j(t_1, t_2, t_3, t_4))]\\
&\quad + \beta\Upsilon[\upsilon_b(s(r_1), s(t_1))]\\
\Rightarrow \quad \upsilon_b(s(r_1), s(t_1)) &\preceq \beta\Upsilon[\upsilon_b(s(r_1), s(t_1))].
\end{aligned}$$

Since, $I \neq \Upsilon \in$ ZM, then $\upsilon_b(s(r_1), s(t_1)) = \mathbf{0}$, or $s(r_1) = s(t_1)$. Similar operations generates that $s(r_2) = s(t_2)$, $s(r_3) = s(t_3)$ and $s(r_4) = s(t_4)$. Hence $s(r_1) = s(r_2) = s(r_3) = s(r_4) = s(t_1) = s(t_4) = s(t_3) = s(t_4)$, which shows that $(s(r_1), s(r_2), s(r_3), s(r_4))$ is a unique QCp of $\{\xi_i\}_{i\in\mathbb{W}}$ and s. Since, $\{\xi_i\}_{i\in\mathbb{W}}$ and s are weakly compatible and coincident points of two compatible mappings are commutable, thus it proves that (r_1, r_2, r_3, r_4) is a unique common QFp of $\{\xi_i\}_{i\in\mathbb{W}}$ and s. □

Example 4.2.8.

Let $\aleph = [0, 1]$ be a generalized b-metric space under metric function defined as

$$\upsilon_b(r_1, r_2) = \begin{pmatrix} |r_1 - r_2|^2 \\ |r_1 - r_2|^2 \end{pmatrix},$$

with $b = 2$ and $\beta = \frac{1}{2}$. Let

$$\Gamma = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \Upsilon = \begin{pmatrix} 0 & \frac{1}{64} \\ \frac{1}{64} & 0 \end{pmatrix}$$

be two matrices in ZM. Clearly $(\aleph, \upsilon_b, \leq)$ is a POCGbMS. Let $\xi_i : \aleph^4 \to \aleph$ and $s : \aleph \to \aleph$ be the mappings defined as

$$\xi_i(r_1, r_2, r_3, r_4) = \frac{r_1}{4^i} \quad \text{and} \quad s(r_1) = 4r_1$$

respectively, then

$$\begin{aligned}
\upsilon_b(\xi_i(r_1, r_2, r_3, r_4), \xi_j(t_1, t_2, t_3, t_4)) &= \begin{pmatrix} |\frac{r_1}{4^i} - \frac{t_1}{4^j}|^2 \\ |\frac{r_1}{4^i} - \frac{t_1}{4^j}|^2 \end{pmatrix} \\
&= \frac{1}{16} \begin{pmatrix} |(\frac{5r_1}{4^i} - \frac{r_1}{4^i}) + (\frac{t_1}{4^j} - \frac{5t_1}{4^j})|^2 \\ |(\frac{5r_1}{4^i} - \frac{r_1}{4^i}) + (\frac{t_1}{4^j} - \frac{5t_1}{4^j})|^2 \end{pmatrix} \\
&\preceq \frac{1}{16} \begin{pmatrix} |(4r_1 - \frac{r_1}{4^i}) + (\frac{t_1}{4^j} - 4t_1) + (r_1 - t_1)|^2 \\ |(4r_1 - \frac{r_1}{4^i}) + (\frac{t_1}{4^j} - 4t_1) + (r_1 - t_1)|^2 \end{pmatrix} \\
&\preceq \frac{1}{4} \left(\begin{pmatrix} |4r_1 - \frac{r_1}{4^i}|^2 \\ |4r_1 - \frac{r_1}{4^i}|^2 \end{pmatrix} + \begin{pmatrix} |4t_1 - \frac{t_1}{4^i}|^2 \\ |4t_1 - \frac{t_1}{4^i}|^2 \end{pmatrix} \right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{128}\begin{pmatrix}|4r_1-4t_1|^2\\ |4r_1-4t_1|^2\end{pmatrix}\\
=&\frac{1}{2}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\big(\upsilon_b(s(r_1),\xi_i(r_1,r_2,r_3,r_4))\\
&+\upsilon_b(s(t_1),\xi_j(t_1,t_2,t_3,t_4))\big)\\
&+\frac{1}{2}\begin{pmatrix}0 & \frac{1}{64}\\ \frac{1}{64} & 0\end{pmatrix}\upsilon_b(s(r_1),s(t_1))
\end{aligned}
$$

or

$$
\begin{aligned}
b\upsilon_b(\xi_i(r_1,r_2,r_3,r_4),\xi_j(t_1,t_2,t_3,t_4))&\\
&\preceq \Gamma[\upsilon_b(s(r_1),\xi_i(r_1,r_2,r_3,r_4))+\\
&\quad \upsilon_b(s(t_1),\xi_j(t_1,t_2,t_3,t_4))]\\
&\quad +\Upsilon\upsilon_b(s(r_1),s(t_1)),
\end{aligned}
$$

that is, (B) condition is satisfied. All the conditions of Theorem 4.2.4 are fulfilled, and $(0,0,0,0)$ is a QCp of $\{\xi_i\}_{i\in\mathbb{W}}$ and s, also it is unique quadruple common fixed point of same mappings according to Theorem 4.2.7.

4.3 Application

Suppose that $\mathbb{R}^n_+=\{r_1=(r_1^1,r_1^2,r_1^3,\cdots,r_1^n):r_i\geq,i\geq 1\}$ and

$$
\begin{aligned}
\Omega^4_{n-1}=\{\rho=(r_1,r_2,r_3,r_4)\in\mathbb{R}^n_+\times\mathbb{R}^n_+\times\mathbb{R}^n_+\times\mathbb{R}^n_+ :&\\
\sum_{i=1}^{n}\rho_i=\sum_{i=1}^{n}(r_i^1+r_i^2+r_i^3+r_i^4)=1\},&
\end{aligned}
$$

represent a $4(n-1)$ dimensional probability simplex and $\rho\in\Omega^4_{n-1}$ is the as a probability of each possible state. The Markov process is the process which deals with the modeling of randomly changing process over time. For the probability event, consider the following conditions.

(1): Suppose that for each $\ell = 1, 2, 3, ..$ there exits a scalar $\kappa_i = \frac{1}{b^i}$, where $b^i \geq b^4$ for each i with the condition that at least one of the "$b_i = b^4$".

(2): Let $\kappa_i e_{ij}$ be the probability matrix with the condition that $\sum_{i=1}^{n} \kappa_i e_{ij} = 1$ and for all $e_{ij} \geq 0, \sum_{i=1}^{n} e_{ij} = 1$.

(3): Let ρ^ℓ be a column vector, then to obtain matrix form consider the mapping $\rho^{\ell+1} = F\rho^\ell$, where ρ^ℓ is the prior probability vector and $\rho^{\ell+1}$ is the posterior probability vector.

(4): $\pi_i = \min_j e_{ij}$ for all i and $\pi = \sum_{i=1}^{n} \pi_i$.

(5): $\kappa = \beta^4$ such that $\kappa = \max\{\kappa_i : 1 \leq i \leq n\}$ for all possible periods.

Now, finding the stationary distribution for Markov process is equivalent to finding the fixed point of the mapping F that is there exists some $\rho \in \Omega_{n-1}^4$ such that $F\rho^\ell = \rho^\ell$, whenever $\rho^{\ell+1} = \rho^\ell$ and ρ^ℓ is called stationary distribution of Markov process.
Main theorem of this section is given bellow.

Theorem 4.3.1.

By the hypothesis $0 < \kappa_i \leq 1,\ e_{ij} \geq 0$, there exists a unique stationary distribution for the Markov process.

Proof. Let $\upsilon_b : \Omega_{n-1}^4 \times \Omega_{n-1}^4 \to \mathbb{R}^2$ be a mapping defined as

$$\upsilon_b(R,Q) = \upsilon_b((r_1,r_2,r_3,r_4),(q_1,q_2,q_3,q_4))$$

$$= (\sum_{i=1}^{n}(|r_1^i - q_1^i|^2 + |r_2^i - q_2^i|^2 + |r_3^i - q_3^i|^2 + |r_4^i - q_4^i|^2), \sum_{i=1}^{n}(|r_1^i - q_1^i|^2 + |r_2^i - q_2^i|^2 + |r_3^i - q_3^i|^2 + |r_4^i - q_4^i|^2)),$$

where, $R = (r_1,r_2,r_3,r_4)$ and $Q = (q_1,q_2,q_3,q_4)$ belongs to Ω_{n-1}^4.

Since, $\upsilon_b(R,Q) \succeq (0,0)$ for all R and Q in Ω_{n-1}^4. Also, if $\upsilon_b(R,Q) = (0,0)$, then this implies

$$\Big(\sum_{i=1}^{n}(|r_1^i-q_1^i|^2+|r_2^i-q_2^i|^2+|r_3^i-q_3^i|^2+|r_4^i-q_4^i|^2),$$
$$\sum_{i=1}^{n}(|r_1^i-q_1^i|^2+|r_2^i-q_2^i|^2+|r_3^i-q_3^i|^2+|r_4^i-q_4^i|^2)\Big)=(0,0),$$

or

$$|r_1^i-q_1^i|^2+|r_2^i-q_2^i|^2+|r_3^i-q_3^i|^2+|r_4^i-q_4^i|^2=0$$
$$\Rightarrow |r_1^i-q_1^i|^2=|r_2^i-q_2^i|^2=|r_3^i-q_3^i|^2=|r_4^i-q_4^i|^2=0,$$
$$\text{or}\quad r_1^i=q_1^i,\ r_2^i=q_2^i,\ r_3^i=q_3^i,\ r_4^i=q_4^i.$$

Yields $R=Q$.

Conversely, let $R=Q$, then

$$r_1^i=q_1^i,\ r_2^i=q_2^i,\ r_3^i=q_3^i,\ r_4^i=q_4^i,$$
$$\Rightarrow |r_1^i-q_1^i|^2=|r_2^i-q_2^i|^2=|r_3^i-q_3^i|^2=|r_4^i-q_4^i|^2=0.$$

Hence,

$$\Big(\sum_{i=1}^{n}(|r_1^i-q_1^i|^2+|r_2^i-q_2^i|^2+|r_3^i-q_3^i|^2+|r_4^i-q_4^i|^2),$$
$$\sum_{i=1}^{n}(|r_1^i-q_1^i|^2+|r_2^i-q_2^i|^2+|r_3^i-q_3^i|^2+|r_4^i-q_4^i|^2\Big)=(0,0),$$

or $\quad \upsilon_b(R,Q)=(0,0)$.

Moreover,

$$\begin{aligned}\upsilon_b(R,Q)&=\Big(\sum_{i=1}^{n}(|r_1^i-q_1^i|^2+|r_2^i-q_2^i|^2+|r_3^i-q_3^i|^2+|r_4^i-q_4^i|^2),\\&\quad\sum_{i=1}^{n}(|r_1^i-q_1^i|^2+|r_2^i-q_2^i|^2+|r_3^i-q_3^i|^2+|r_4^i-q_4^i|^2)\Big)\\&=\Big(\sum_{i=1}^{n}(|q_1^i-r_1^i|^2+|q_2^i-r_2^i|^2+|q_3^i-r_3^i|^2+|q_4^i-r_4^i|^2),\\&\quad\sum_{i=1}^{n}(|q_1^i-r_1^i|^2+|q_2^i-r_2^i|^2+|q_3^i-r_3^i|^2+|q_4^i-r_4^i|^2)\Big)\\&=\upsilon_b(Q,R).\end{aligned}$$

Now,

$$
\begin{aligned}
\upsilon_b(R,Q) =& \Big(\sum_{i-1}^{n} (|r_1^i - q_1^i|^2 + |r_2^i - q_2^i|^2 + |r_3^i - q_3^i|^2 + |r_4^i - q_4^i|^2), \\
& \sum_{i=1}^{n} (|r_1^i - q_1^i|^2 + |r_2^i - q_2^i|^2 + |r_3^i - q_3^i|^2 + |r_4^i - q_4^i|^2) \Big) \\
=& \Big(\sum_{i=1}^{n} \begin{pmatrix} |(r_1^i - t_1^i) + (t_1^i - q_1^i)|^2 + |(r_2^i - t_2^i) + (t_2^i - q_2^i)|^2 \\ +|(r_3^i - t_3^i) + (t_3^i - q_3^i)|^2 + |(r_4^i - t_4^i) + (t_4^i - q_4^i)|^2 \end{pmatrix}, \\
& \sum_{i=1}^{n} \begin{pmatrix} |(r_1^i - t_1^i) + (t_1^i - q_1^i)|^2 + |(r_2^i - t_2^i) + (t_2^i - q_2^i)|^2 \\ +|(r_3^i - t_3^i) + (t_3^i - q_3^i)|^2 + |(r_4^i - t_4^i) + (t_4^i - q_4^i)|^2 \end{pmatrix} \Big) \\
\preceq& 2\Big(\sum_{i-1}^{n} \begin{pmatrix} |r_1^i - t_1^i|^2 + |t_1^i - q_1^i|^2 + |r_2^i - t_2^i|^2 + |t_2^i - q_2^i|^2 \\ +|r_3^i - t_3^i|^2 + |t_3^i - q_3^i|^2 + |r_4^i - t_4^i|^2 + |t_4^i - q_4^i|^2 \end{pmatrix}, \\
& \sum_{i=1}^{n} \begin{pmatrix} |r_1^i - t_1^i|^2 + |t_1^i - q_1^i|^2 + |r_2^i - t_2^i|^2 + |t_2^i - q_2^i|^2 \\ +|r_3^i - t_3^i|^2 + |t_3^i - q_3^i|^2 + |r_4^i - t_4^i|^2 + |t_4^i - q_4^i|^2 \end{pmatrix} \Big) \\
=& 2\{ \big(\sum_{i=1}^{n} (|r_1^i - t_1^i|^2 + |r_2^i - t_2^i|^2 + |r_3^i - t_3^i|^2 + |r_4^i - t_4^i|^2), \\
& \sum_{i=1}^{n} (|r_1^i - t_1^i|^2 + |r_2^i - t_2^i|^2 + |r_3^i - t_3^i|^2 + |r_4^i - t_4^i|^2) \big) \\
& + \big(\sum_{i=1}^{n} (|t_1^i - q_1^i|^2 + |t_2^i - q_2^i|^2 + |t_3^i - q_3^i|^2 + |t_4^i - q_4^i|^2), \\
& \sum_{i=1}^{n} (|t_1^i - q_1^i|^2 + |t_2^i - q_2^i|^2 + |t_3^i - q_3^i|^2 + |t_4^i - q_4^i|^2) \big) \} \\
=& 2\{ \upsilon_b(R,T) + \upsilon_b(T,Q) \}
\end{aligned}
$$

where $T = (t_1, t_2, t_3, t_4) \in \Omega_{n-1}^4$. Hence, $(\Omega_{n-1}^4, \upsilon_b)$ is a generalized b-metric space

with $b = 2$. Completeness of Ω_{n-1}^4 can be easily proved. Moreover, define a partial order on Ω_{n-1}^4 as for all $(r_1, r_2, r_3, r_4), (q_1, q_2, q_3, q_4) \in \Omega_{n-1}^4$,

$$(r_1, r_2, r_3, r_4) \preceq (q_1, q_2, q_3, q_4) \iff r_1 \succeq q_1, \; r_2 \preceq q_2, \; r_3 \succeq q_3, \text{ and } r_4 \preceq q_4.$$

Hence, $(\Omega_{n-1}^4, \upsilon_b, \preceq)$ is a POCGbMS.

Let $\digamma : \Omega_{n-1}^4 \to \Omega_{n-1}^4$ be a mapping define as for all $\rho \in \Omega_{n-1}^4$, $\digamma\rho = \delta_j$ such that for each j, $\delta_j = \sum_{i=1}^{n} \kappa_i e_{ij} \rho_j$. Since,

$$\sum_{j=1}^{n} \delta_j = \sum_{j=1}^{n} \sum_{i=1}^{n} \kappa_i e_{ij} \rho_j = \sum_{i=1}^{n} \kappa_i e_{ij} \sum_{j=1}^{n} (r_1^j + r_2^j + r_3^j + r_4^j)$$
$$= \sum_{j=1}^{n} (r_1^j + r_2^j + r_3^j + r_4^j) = 1,$$

this implies that $\delta_j \in \Omega_{n-1}^4$ i.e mapping is defined. Now, we have to show that $\digamma$ satisfy the contraction condition, for this, let δ_i be the i^{th} row of δ. Then, for all $(r_1, r_2, r_3, r_4), (q_1, q_2, q_3, q_4) \in \Omega_{n-1}^4$, we get

$$\begin{aligned}
&\upsilon_b(\digamma(r_1, r_2, r_3, r_4), \digamma(q_1, q_2, q_3, q_4)) \\
=&\Bigg(\sum_{i=1}^{n} \Big(|\sum_{j=1}^{n} \big(\kappa_i e_{ij}(r_1^j + r_2^j + r_3^j + r_4^j) - \kappa_i e_{ij}(q_1^j + q_2^j + q_3^j + q_4^j)\big)|^2\Big), \\
&\quad \sum_{i=1}^{n} \Big(|\sum_{j=1}^{n} \big(\kappa_i e_{ij}(r_1^j + r_2^j + r_3^j + r_4^j) - \kappa_i e_{ij}(q_1^j + q_2^j + q_3^j + q_4^j)\big)|^2\Big)\Bigg) \\
=&\Bigg(\sum_{i=1}^{n} \Bigg(|\sum_{j=1}^{n} (\kappa_i e_{ij} - \kappa\pi_i)\{(r_1^j + r_2^j + r_3^j + r_4^j) - (q_1^j + q_2^j + q_3^j + q_4^j)\} \\
&\quad + \pi_i\{(r_1^j + r_2^j + r_3^j + r_4^j) - (q_1^j + q_2^j + q_3^j + q_4^j)\}|^2\Bigg), \\
&\quad \sum_{i=1}^{n} \Bigg(|\sum_{j=1}^{n} (\kappa_i e_{ij} - \kappa\pi_i)\{(r_1^j + r_2^j + r_3^j + r_4^j) - (q_1^j + q_2^j + q_3^j + q_4^j)\} \\
&\quad + \pi_i\{(r_1^j + r_2^j + r_3^j + r_4^j) - (q_1^j + q_2^j + q_3^j + q_4^j)\}|^2\Bigg)\Bigg) \\
\preceq&2\Bigg(\Bigg(\sum_{i=1}^{n} (\kappa e_{ij} - \kappa\pi_i)^2 |\sum_{j=1}^{n} \{(r_1^j + r_2^j + r_3^j + r_4^j) - (q_1^j + q_2^j + q_3^j + q_4^j)\}|^2 \\
&\quad + \sum_{i=1}^{n} |\sum_{j=1}^{n} \pi_i\{(r_1^j + r_2^j + r_3^j + r_4^j) - (q_1^j + q_2^j + q_3^j + q_4^j)\}|^2\Bigg), \\
&\quad \Bigg(\sum_{i=1}^{n} (\kappa e_{ij} - \kappa\pi_i)^2 |\sum_{j=1}^{n} \{(r_1^j + r_2^j + r_3^j + r_4^j) - (q_1^j + q_2^j + q_3^j + q_4^j)\}|^2 \\
&\quad + \sum_{i=1}^{n} |\sum_{j=1}^{n} \pi_i\{(r_1^j + r_2^j + r_3^j + r_4^j) - (q_1^j + q_2^j + q_3^j + q_4^j)\}|^2\Bigg)\Bigg) \\
\preceq&8\kappa^2 \begin{pmatrix} \sum_{i=1}^{n} \sum_{j=1}^{n} (|r_1^j - q_1^j|^2 + |r_2^j - q_2^j|^2 + |r_3^j - q_3^j|^2 + |r_4^j - q_4^j|^2) \times (e_{ij} - \pi_i)^2, \\ \sum_{i=1}^{n} \sum_{j=1}^{n} (|r_1^j - q_1^j|^2 + |r_2^j - q_2^j|^2 + |r_3^j - q_3^j|^2 + |r_4^j - q_4^j|^2 \times (e_{ij} - \pi_i)^2) \end{pmatrix}
\end{aligned}$$

$$=8\beta^4(I-\pi)^2\begin{pmatrix}\sum_{j=1}^{n}(|r_1^j-q_1^j|^2+|r_2^j-q_2^j|^2+|r_3^j-q_3^j|^2+|r_4^j-q_4^j|^2),\\ \sum_{j=1}^{n}(|r_1^j-q_1^j|^2+|r_2^j-q_2^j|^2+|r_3^j-q_3^j|^2+|r_4^j-q_4^j|^2)\end{pmatrix}$$

$$=\beta\Upsilon\upsilon_b((r_1,r_2,r_3,r_4),(q_1,q_2,q_3,q_4)).$$

Where $(I-\pi)^2=\Upsilon\in$ ZM, hence all conditions of corollary 4.2.6 are fulfilled. Then, there exists a unique quadruple fixed point of mapping $\digamma$ or in other words a unique stationary distribution of Markov process. Moreover, the sequence $\{\digamma^n\rho^l\}$ converges to a unique stationary distribution for any $\rho^l\in\Omega_{n-1}$. □

Remark 7:

- The contraction condition (B) is the generalization of contraction condition (O) presented in [34]. In other words for $b=1$, contraction (O) becomes a spacial case of (B).
- In case of $b=1$, results constructed in [34] turns into a special case of extended ideas obtained in the research study.

Chapter 5

Conclusion and Future Works

5.1 Conclusion

- A detailed review of work by Hammad *et al*. [34] based on study of coincedence points and quadruple fixed point in generalized metric spaces is presented.

- In our research work, we extended the notions of generalized metric spaces in coordination with b-metric spaces and introduced the idea of generalized b-metric spaces.

- Notion of compatible mappings and contraction condition is modified in the setting of GbMS. Examples are also illustrated in this regard.

- Working in the direction presented in [34], results for QFp of mappings are generalized under the umbrella of GbMS.

- Subsequently, an example and an application for unique stationary distribution of Markov process is attached for the validation of obtained theoretical consequences.

5.2 Future Works

In future, one can have a different approach in order to find the quadruple fixed point of mapping.

- An attempt can be done by changing the space e.g by working in the setting of “extended b-metric spaces or double controlled metric spaces”.
- Working with different contraction conditions using more properties of contraction mappings.
- An extension can be made in the articulated theory by extending the notion of quadruple fixed point of mappings.
- Obtained results can be squeezed in the context of couple and triple fixed points.

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