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Weak Partial b-metric Space endowed with a Suzuki-type Multivalued Contraction

by

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I am dedicating this thesis to my beloved parents and my brothers who unconditionally supported me and uplifted me.

CERTIFICATE OF APPROVAL

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Abstract

The idea of H_w-type Suzuki multivalued contraction mapping on weak partial metric spaces is given by Aydi et al. The inspiration behind this idea came from the paper of Beg and Pathak. They introduced almost Hausdroff metric space and generalized Nadler's fixed point theorem for multivalued mappings on weak partial metric spaces. Kanwal et al. presented the concept of weak partial b-metric spaces and the generalization of Nadler's theorem in the setting of weak partial Hausdorff b-metric spaces. This dissertation is the inspiration from both papers. A fixed point theorem using a Suzuki-type multivalued contraction in the context of weak partial b-metric spaces is discussed. To validate the result an example is provided.

Contents

List of Figures

Abbreviations

BCP	Banach Contraction Principle
PMS	Partial Metric Space
WPMS	Weak Partial Metric Space
WPbMS	Weak Partial *b*-metric Space

Symbols

$\mathbb{R}$	Real number
$\mathbb{N}$	Natural number
$\mathbb{C}$	Complex number
$\mathcal{W}$	Weak partial metric
$\mathcal{W}_b$	Weak partial b-metric

Chapter 1

Introduction

1.1 Background

Functional analysis is originated in the early years of twentieth century. In functional analysis, vector spaces and the operators on vector spaces are focused. Functional analysis is a fusion of abstract linear algebra, modern geometry and topology. Now a days functional analysis is very vide subject addressing much of modern analysis. This field deals with the development of vector space and the other abstract spaces. In the dawn of twentieth century, functional analysis has made its own direction through integral equation. Although it was started in solving the differential equations, but it has wide applications to solve many non-linear problems in mathematics.
One of the most important branch of functional analysis is fixed point theory. It is a combination of topology, geometry and analysis. Fixed point theorem has applications in game theory, economics, variational inequalities and many other scientific fields. The concept of fixed point theory was given by Poincare [1] in 1886. Later on, in 1912 Brouwer [2] has provided the solution of equation $\mathcal{F}\mu = \mu$ by proving his fixed point theorem. Brouwer also provided the proofs of fixed point theorem for the n-dimensional counter parts of sphere and square. Afterwards, Kakutani [3] extended these results. He proved the existence of fixed point in

Euclidean n-space for a convex compact set.

In 1922 one of the most rudimentary and valuable result was given by Stephen Banach [4], which is known as Banach Contraction principle. BCP is a very crucial result because it does not only provides the way to find fixed point, but also it gives information about the uniqueness of fixed point. Banach proved his result by stating that a complete metric space U always has a unique fixed point for each self map F which satisfies

$$\text{“}d(Fu, Fv) \leq \alpha d(u, v) \quad \text{for} \quad \text{all} \quad u, v \in U \quad \text{and} \quad \alpha \in [0, 1).\text{”}$$

BCP plays a vital role in solving non-linear problems. The solution of integral and differential equations can be find by using the technique of BCP.

In fixed point theory, the results are being generalized by using BCP in the following two directions.

1. On a mapping the contraction condition is transmuted.
2. On a mapping the under lying space is transmuted.

Edelstein [5] generalized BCP by considering compact space and taking the constant $\alpha = 1$. Later, another contractive condition was introduced by Rakotch [6], in which a monotone decreasing function $\alpha(t)$ is considered and the constant number α is replaced by this function. Presic [7], Kannan [8], Meir et. al [9] worked on BCP by altering the contraction condition. Fomin [10] and Gupta [11] introduced a rational expression and extended Banach contraction principle, later on this result was extended by Dolhare [12].

Nadler [13] has given a new direction for the research in fixed point theory. He changed the underlying space into a space which contains the bounded and closed subsets of a set $(\mathcal{M}, d)$. He generalized Banach contraction principle by changing the single valued contraction mapping to multivalued contraction mapping. This result opened a new door for many researchers who used this new contraction to extend many fixed point results see for example [14], [15], [16]. Later on many authors generated a big revenue in the discipline of fixed point theory by changing the underlying space such as metric like spaces [17], pseudo-metric spaces [18], Partial metric spaces [19], quasi b-metric space [20], [21].

In 1989 Bakhtin [22] presented a new space which is the generalization of metric space and is known as b-metric space. Later on Czerwick [23] used the idea of b-metric space and proved some more results that generalize Banach contraction principle. Afterwards a lot of work has been done in b-matric spaces on multivalued as well as on single valued functions such as [24], [25], [26].

Matthews [27] presented the idea of partial metric space in 1992. Which is another generalization of metric space. Matthews used this new space as underlying space to generalize Banach contraction principle and provided an accurate relationship between the two spaces that is partial metric spaces and quasi-metric spaces [28]. Thereafter, in 1995, O'Neill [29] extended the work of Metthews and introduced the relation between topological aspects of domain theory and partial metric spaces. He amended Metthews's range from non-negative real numbers to the whole real line and this partial metric space was called dualistic partial metric space. Furthermore Oltra et. al [30] put forward Banach contraction principle by considering dualistic partial metric spaces with completeness property.

A lot of work is still being done by mathematicians by considering different contraction conditions on partial metric spaces for instance [31], [32], [33]. In 1999 a generalized notion of partial metric was introduced by Heckmann [34], which is named as weak partial metric space. Heckmann amended the contribution of O'Neill using the range of Matthews, that is nonnegative real numbers.

In [35] Aydi et al. introduced weak partial metric spaces endowed with a Suzuki-type multivalued contraction. Weak partial metric space is the generalization of partial metric space. In this dissertation the work of Aydi et al. has been extended from weak partial metric space to weak partial b-metric space by considering same contraction mapping. The result has been elaborated with an example.

Organization of the rest of dissertation is given as:

• **Chapter** 2:

This chapter is divided into six sections. The first three sections provides the crucial information about metric spaces, b-metric spaces and Partial metric spaces respectively. Some important fixed point theorems are also provided without the proof. Fourth section is about the different Mappings on metric space. Fifth section gives information about fixed point and related ideas. In the last section multivalued mappings are discussed.

• **Chapter** 3:

This chapter is a complete review of the work presented by Aydi et al. [35]. This chapter provides detailed proof for fixed point result on weak partial metric spaces.

• **Chapter** 4:

In this chapter the result reviewed in chapter 3 has been extended from weak partial metric space to weak partial b-metric space. The result of Kanwal et al. [36] on weak partial b-metric space is used and is being applied by considering a Suzuki-type multivalued contraction . This result is verified with an example.

Chapter 2

Literature Review

In this chapter we have discussed some basic definitions, lemmas, theorem and important results from literature. These definitions, lemmas and theorems corresponds to our main result. First section consists of definition of metric space and its examples. Second section is about b-metric space, examples of b-metric space and some fixed point results in it. Third section corresponds to partial metric space and its examples. In fourth and fifth sections fixed point and multivalued mappings have been discussed.

2.1 Metric Space

This section includes some important results about metric spaces.

Definition 2.1. (**Metric Space**) [37]
"A metric space is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is , a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(d1) d is real-valued, finite and nonnegative;

(d2) $d(x, y) = 0$ if and only if $x = y$;

(d3) $d(x, y) = d(y, x)$; symmetric property

(d4) $d(x, y) \leq d(x, z) + d(z, y)$; Triangular inequality

The pair (X, d) is called metric space on X."
In triangular inequality the word triangle is used due to the property of triangle, which is shown in the Figure 2.1

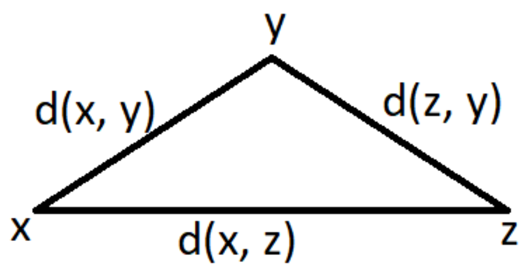

FIGURE 2.1: Triangular inequality in plane.

The structure of metric space is a generalization of the real line. The properties are defined in such a way so that this structure become analogous to real line. In real line the distance between two points is defined as

$$| x - y | .$$

The metric which is defined in above manner is called the usual metric. Some nontrivial examples of metric spaces are given below.

Example 2.1.1. [37]
Let S be the set consisting of all defined and continuous real-valued functions $s_1, s_2, \ldots$, which are functions of an independent real variable u on a given closed interval $I = [a, b]$. A mapping $d : [a, b] \times [a, b] \longrightarrow \mathbb{R}$ by

$$d(s_1, s_2) = \max_{u \in I} | s_1(u) - s_2(u) |,$$

is called metric on a space which is denoted by $C[a, b]$.

Since every point of $C[a, b]$ is a function, so it is also called a function space.

Example 2.1.2. Sequence space l^∞ [37]

Let X be the set of all complex numbers which are bounded sequences, that is every element of X is a sequence which is bounded, that is

$$\xi = (z_1, , z_2, ...) \text{ briefly } \xi = (z_j)$$

such that

$$| z_j | \leq c_x \quad \text{for} \quad \text{all} \quad j = 1, 2, ...$$

where c_x is a real number which may depend on x, but is independent of j.

Define the metric on X as,

$$d(\xi_1, \xi_2) = \sup_{i \in \mathbb{N}} | z_j - y_j |$$

where $\xi_2 = (y_j) \in X$ and $\mathbb{N} = \{1, 2, ...\}$, and sup stands for supremum (least upper bound). This space is denoted by l^∞.

l^∞ is a sequence space because each element of X is a sequence.

Definition 2.1.1. (**Open and Closed Ball in Metric Space**) [37]

"Let $X = (X, d)$ be a metric space. A point $x_0 \in X$ and a real number $r > 0$, we define open ball as

$$B(x_0; r) = \{x \in X \mid d(x, x_0) < r\},$$

and closed ball can be defined as

$$B(x_0; r) = \{x \in X \mid d(x, x_0) \leq r\}."$$

Definition 2.1.2. (**Open Set**) [37]

"A subset M of a metric space X is said to be open if it contains a ball about each of its point which is contained in M."

Definition 2.1.3. (**Closed Set**) [37]

" A subset K of X is said to be closed if its complement (in X) is open, that is,

$K^c = X - K$ is open."

Definition 2.1.4. (**Bounded Set**) [38]

"A set $G \subseteq Y$ is bounded in (Y, d) if there exist $y \in Y$ and $M > 0$ such that $G \subseteq B_d(y, M)$."

Definition 2.1.5. (**Convergence of a sequence**) [39]

"A sequence (p_n) in a metric space X is said to converge if there is a point $p \in X$ with following property:

for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies that

$$d(p_n, p) < \epsilon.$$

This can also be written as

$$\lim_{n \to \infty} d(p_n, p) = 0,$$

or

$$\lim_{n \to \infty} p_n = p$$

We say that (p_n) converges to p or has the limit p. If (p_n) is not convergent, it is said to be divergent."

Example 2.1.3. [39]

• Let $\mathcal{M} = \mathbb{R}$, with metric d defined as

$$d(x_1, x_2) = \mid x_1 - x_2 \mid,$$

then $x_n = \left(\frac{1}{n}\right)$ is convergent sequence. Since

$$\lim_{n \to \infty} d\left(\frac{1}{n}, 0\right) = 0,$$

that is $\frac{1}{n} \to 0$.

• Let $\mathcal{M} = \mathbb{R}$, with metric d defined as

$$d(x_1, x_2) = \mid x_1 - x_2 \mid,$$

then $x_n = n^2$ is a divergent sequence.

Definition 2.1.6. (**Cauchy Sequence**) [37]

"A sequence $\{x_n\}$ in a metric space X is said to be Cauchy, if for every $\epsilon > 0$ there exist a positive integer N such that,

$$d(x_m, x_n) < \epsilon \quad \text{for every} \quad m, n \geqslant N."$$

Theorem 2.1.1. [37]

"Every convergent sequence in metric space is a Cauchy sequence.

Proof.

Let (X, d) be a metric space and x_n be a sequence in X.

Let $x_n \to x$, then for every $\epsilon > 0$, there exist a number $N \in \mathbb{N}$ such that

$$d(x_n, x) < \epsilon \quad \text{for} \quad \text{all} \quad n \geqslant N.$$

$\{x_n\}$ is a Cauchy sequence, by choosing $m, n \in \mathbb{N}$ with $m > n$, we have

$$d(x_m, x) < \frac{\epsilon}{2}, \quad \text{and} \quad d(x_n, x) < \frac{\epsilon}{2}.$$

Now consider,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x) + d(x_n, x) \quad \text{for} \quad \text{all} \quad m, n > \mathbb{N} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\ \Rightarrow d(x_m, x_n) &< \epsilon \quad \text{for} \quad \text{all} \quad m, n > \mathbb{N} \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence." $\square$

Remark 2.1.1. [37]

"If $X = \mathbb{R}$, then every Cauchy sequence is convergent but this is not the case in general.

Cauchy criterion says that a sequence of real or complex numbers converges on $\mathbb{R}$ or in $\mathbb{C}$ if and only if it is a Cauchy sequence. This is the situation for $\mathbb{R}$ and $\mathbb{C}$. But in more general spaces there can be Cauchy sequence which do not converge. In such situation there is lack of the property which is so important so that it deserves a name, called completeness. This consideration leads to the following definition, which was first given by M. Frechet (1906)."

Definition 2.1.7. (**Complete metric space**) [37]

"A metric space (X, d) is said to be complete, if every Cauchy sequence in X has a limit point in X."

Example 2.1.4.

1. Every finite dimensional metric space is complete.
2. The space of continuous functions $C[a, b]$ is a complete metric space.
3. Closed subset of a complete metric space is complete.

2.2 b-Metric Space

The concept of metric space was generalized by Bakhtin [22] and thereafter Czerwick [23] used this idea and has given the concept of b-metric space. It the generalizes the metric space. The contraction mapping principle in b-metric space was proved by Czerwick that generalized the Banach contraction principle [40].

Definition 2.2.1. (**b-Metric Space**) [41]

"Let $X \neq \phi$ be a set and $k \in \mathbb{R}$ such that $k \geq 1$. Consider a function $d : U \times U \longrightarrow [0, \infty)$. The pair (X, d) is called b-metric space if the following conditions are satisfied for all $x, y, z \in X$

(b1) $d(x, y) = 0$ if and only if $x = y$, (b2) $d(x,y) = d(y,x)$,

(b3) $d(x,y) \leq k(d(x,z) + d(z,y))$."

Example 2.2.1. [42]

Let $S = [0,2]$ and $d : S \times S \longrightarrow [0,\infty)$ be defined by

$$d(s,t) = \begin{cases} (s-t)^2, & s,t \in [0,1], \\ \mid \dfrac{1}{s^2} - \dfrac{1}{t^2} \mid, & s,t \in [1,2], \\ \mid s-t \mid, & \text{otherwise.} \end{cases}$$

It can easily seen that d is a b-metric on S with $k = 2$.

Example 2.2.2. [43]

Let

$$\ell^p(\mathbb{R}) = \left\{ (u_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |u_n|^p < \infty \right\}, \qquad 0 < p < 1$$

together with a functional

$$d : \ell^p(\mathbb{R}) \times \ell^p(\mathbb{R}) \longrightarrow \mathbb{R},$$

defined by

$$d(u,v) = (\sum_{n=1}^{\infty} |u_n - v_n|^p)^{\frac{1}{p}},$$

where $u = u_n$ and $v = v_n$ are sequences in $\ell^p(\mathbb{R})$, one can easily verify that $(\ell^p(\mathbb{R}), d)$ is a b-metric space with coefficient $k = 2^{\frac{1}{p}} > 1$.

Definition 2.2.2. (***b*-Cauchy Sequence**)[44]

"Let (X,d) be a b-metric space. Then a sequence (x_n) in X is called Cauchy sequence if and only if for all $\epsilon > 0$ there exist $n(\epsilon) \in \mathbb{N}$ such that for each $m,n > n(\epsilon)$ we have

$$d(x_m, x_n) < \epsilon."$$

Definition 2.2.3. (***b*-Convergent Sequence**) [14]

"Let (X,d) be a b-metric space. Then a sequence (x_n) in X is called convergent sequence if and only if there exist $x \in X$ such that for all $\epsilon > 0$ there exist $n(\epsilon) \in \mathbb{N}$

such that for all $n > n(\epsilon)$ we have

$$d(x_n, x) < \epsilon,$$

in this case we write $\lim_{n\to\propto} x_n = x$."

Definition 2.2.4. (**Complete b-metric Space**) [14]
"A b-metric space is said to be complete if every Cauchy sequence is convergent in it."

2.3 Partial Metric Space

The concept of partial metric was given by Matthews [27] in 1992. The inspiration behind the concept of partial metric space comes from the decipline of computer science. The computer scientists were curious that how an infinite sequence can be computed. They divided an infinite sequence into finite parts and observed that there were many sequences that have non-zero self distances. In this way each finite sequence is actually the partially computed version of an infinite sequence which is totally computed. Partial metric space is the generalization of metric space [45].

Definition 2.3.1. (**Partial metric space**) [35]
"A partial metric on a nonempty set X is a function $p : X \times X \longrightarrow \mathbb{R}^+$ (nonnegative reals) such that for all $x, y, z \in X$:

(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = s(y, y)$,

(p2) $p(x, x) \leq p(x, y)$,

(p3) $p(x, y) = p(y, x)$,

(p4) $p(x, z) \leq p(x, y) + p(y, z) - s(y, y)$

The pair (X, p) is called a partial metric space."

Note 2.3.1.

Partial metric space is the generalization of metric space. In PMS the condition $d(x, x) = 0$ *is replaced by* $d(x, x) \leq d(x, y)$*. That is in PMS self distances are non-zero while in metric spaces self distances are zero.*

Example 2.3.1. [46]

Let W be the collection of all closed interval $[a, b]$ in $\mathbb{R}$, where $a < b$. Let

$$p : W \times W \to \mathbb{R}^+$$

be the mapping defined

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}.$$

Then (W, p) is a partial metric space.

Definition 2.3.2. (**Cauchy Sequence**) [45]

"A sequence $x = (x_n)$ of points in a partial metric space (X, p) is Cauchy if there exists an $a \geq 0$ such that for each $\epsilon > 0$, there exists k such that for all $n, m > k$,

$$\mid p(x_n, x_m) - a \mid < \epsilon.$$

In other words, x is Cauchy if $p(x_n, x_m)$ converges to some a as n and m approach infinity, that is, if

$$\lim_{n,m\to\infty} p(x_n, x_m) = a."$$

Definition 2.3.3. (**Convergent Sequence**) [45]

"A sequence $x = (x_n)$ of points in a partial metric space (X, p) converges to y in X if

$$\lim_{n\to\infty} p(x_n, y) = \lim_{n\to\infty} p(x_n, x_n) = p(y, y).$$

Thus if a sequence converges to a point then the self-distances converge to the self-distance of that point."

Definition 2.3.4. (**Completeness**) [45]

"A partial metric space (X, p) is said to be complete if every Cauchy sequence converges."

2.4 Mappings on Metric Space

Definition 2.4.1. (**Continuous mapping**) [37]

"Let $T : X \to Y$ be a mapping between two metric spaces, T is said to be continuous if for every $x_n \subset X$, we have

$$x_n \to x_0 \in X \Rightarrow T(x_n) \to T(x_0)$$

Alternatively we can say that, a mapping $T : X \to Y$ of a metric space (X, d_1) into a metric space (Y, d_2) is continuous at a point $a_0 \in X$ if for every $\epsilon > 0$, there exists a $\delta > 0$, such that

$$d(Ta_n, Ta_0) < \in \quad \text{whenever} \quad d(a_n, a_0) < \delta."$$

Definition 2.4.2. (**Lipschitzian mapping**) [47]

"Let (X, d) be a metric space. A mapping $F : X \to X$ is said to be Lipschitzian if there exist a constant $\alpha \geqslant 0$ such that

$$d(F(x), F(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$.The smallest number k for which above inequality is true is called Lipschitzian constant."

Example 2.4.1.

Let X be the set of all column vectors in $\mathbb{R}^2$, and let (X, d) metric space, where d is defined as

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Consider a mapping $F : X \longrightarrow X$ defined by $F(u) = Au$, where $u \in X$.

$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $v = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.

$$\begin{aligned}
F(u) &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\
&= \begin{pmatrix} 2x_1 \\ 2y_1 \end{pmatrix} \\
&= 2 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\
&= 2u \\
d(Fu, Fv) &= d(2u, 2v) \\
&= \sqrt{(2x_1 - 2y_1)^2 - (2x_2 - 2y_2)^2} \\
&= 2d(u, v)
\end{aligned}$$

Hence, F is a Lipschitzian mapping.

Definition 2.4.3. (**Contraction**) [47]

"Let (X, d) be a metric space. A mappping $F : X \to X$ is said to be contraction if there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$

$$d(F(x), F(y)) \leq kd(x, y)$$

where k is called contraction constant."

Example 2.4.2.

Let $(\mathcal{M}, d)$ be the usual metric space, where $\mathcal{M} = [0, 1]$. Define $G : \mathcal{M} \to \mathcal{M}$ by

$$G(x) = \frac{1}{c + x} \qquad \text{with} \quad (c > 1)$$

then,

$$\begin{aligned}
d(G(x_1), G(x_2)) &= d(\frac{1}{c + x_1}, \frac{1}{c + x_2}) \\
&=| \frac{1}{c + x_1} - \frac{1}{c + x_2} | \\
&=| \frac{(c + x_2) - (c + x_1)}{(c + x_1)(c + x_2)} |
\end{aligned}$$

$$
\begin{aligned}
&=\mid \frac{x_2 - x_1}{(c+x_1)(c+x_2)} \mid \\
&=\mid x_1 - x_2 \mid \frac{1}{\mid (c+x_1)(c+x_2) \mid} \\
&<\mid x_1 - x_2 \mid \frac{1}{\mid (c+0)(c+0) \mid} \\
&= \frac{1}{c^2} \mid x_1 - x_2 \mid \\
&= kd(x_1, x_2), \qquad \text{where} \qquad k = \frac{1}{c^2}
\end{aligned}
$$

$$\Rightarrow d(G(x_1), G(x_2)) < kd(x_1, x_2)$$

Definition 2.4.4. (**Contractive mapping**) [47]

"Let (X, d) be a metric space and G be a self map, G is called contractive mapping if, for all $x, y \in X$

$$d(G(x), G(y)) < d(x, y), \qquad \text{where} \qquad x \neq y."$$

Example 2.4.3.

Let $(\mathcal{M}, d)$ be the usual metric space, where $\mathcal{M} = \mathbb{R}$. And $G : \mathcal{M} \to \mathcal{M}$ by

$$G(\nu) = \frac{1}{\nu} \quad where \ \ \nu > 1$$

$$
\begin{aligned}
d(G(\nu_1), G(\nu_2)) &= d(\frac{1}{\nu_1}, \frac{1}{\nu_2}) \\
&=\mid \frac{1}{\nu_1} - \frac{1}{\nu_2} \mid \\
&=\mid \frac{\nu_2 - \nu_1}{\nu_1 \nu_2} \mid \\
&=\mid \frac{\nu_1 - \nu_2}{\nu_1 \nu_2} \mid \\
&=\mid \nu_1 - \nu_2 \mid\mid \frac{1}{\nu_1 \nu_2} \mid \\
&<\mid \nu_1 - \nu_2 \mid \\
&= d(\nu_1, \nu_2),
\end{aligned}
$$

$$\Rightarrow d(G(\nu_1), G(\nu_2)) < d(\nu_1, \nu_2)$$

2.5 Fixed Point and Related Ideas

The initiation of fixed point theory belongs to Poincare who has given its concept in nineteenth century. Later on it was used by L. Brouwer who has given his classical result of fixed point theory that is known as "Brouwer's Fixed Point Theorem" in 1912. A fixed point for a function T can be defined as an element taken from the domain of a function and is mapped to itself by the function.
Furthermore, let S and T be the two nonempty subsets of a metric space $(\mathcal{M}, d)$ and

$$\mathcal{F} : S \longrightarrow T$$

be a mapping. For a fixed to be exist it is mandatory that

$$\mathcal{F}(S) \cap S \neq \phi.$$

If above result does not hold, then $d(u, Tu) \neq 0$ that is

$$d(u, Tu) > 0$$

for each $t \in S$.

Definition 2.5.1. (**Fixed Point**) [48]
"Let T be a self mapping on a set X. An element u in X is said to be a fixed point of mapping T if,

$$Tu = u,$$

The set of all points fixed points of T is denoted by Fix(T)."

Geometrically, let $y = f(x)$ be the real valued function then fixed points are the point of intersection of $y = f(x)$ and $y = x$, if the straight line does not intersect the curve then there is no fixed point. The intersection can be seen in the following diagram,

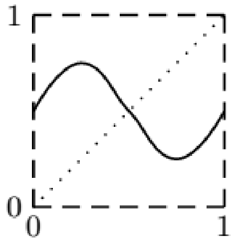

FIGURE 2.2: One Fixed Point.

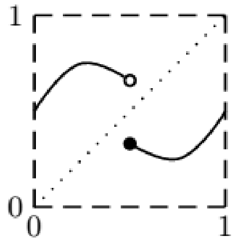

FIGURE 2.3: No Fixed Point.

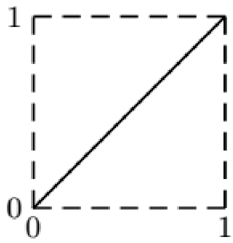

FIGURE 2.4: Infinite Fixed Point.

In 1922, a Polish mathematician Stefan Banach [4] has given a valuable result to fixed point theory, which is later on known as Banach Contraction Principle (BCP). It has wide applications in nonlinear problems of integral and differential equations to check the existence of solution. In computational mathematics it is used to find the proof of convergence of algorithms.

Theorem 2.5.1. [37]
"Consider a metric space $X = (X, d)$, where $X \neq \phi$. Suppose that X is complete and $T : X \longrightarrow X$ be a contraction on X. Then T has precisely one fixed point."

2.6 Multivalued Mapping

Multivalued mapping has a very important role in pure and applied mathematics because of a wide range of applications in real analysis, complex analysis and in optimal control problems [16]. Multivalued function has the same behavior as that of a function. But there could be more than one association of an element of domain X in codomain Y.

Definition 2.6.1. (**Multivalued Mapping**) [15]
"Let X and Y be the two nonempty sets. T is said to be multivalued mapping from X to 2^Y if T is a relation of X to the power set of Y. We denote the multi-valued map by

$$T : X \to 2^Y.$$

That is T is multivalued mapping if and only if for each $x \in X, Tx \subseteq Y$. Unless otherwise stated we always assume Tx is non-empty set for each $x \in X$."

Example 2.6.1.
Let $S = \{l, m, n, o\}$ and $T = \{1, 1.3, 2, 2.3, ..., 10\}$. Define a mapping

$$F : S \to 2^T,$$

F is called multivalued or set valued mapping by

$$F(l) = \{1, 2, 2.3, 3\} \qquad F(m) = \{1.3, 3, 4.3, 5.3\},$$
$$F(n) = \{4, 5, 6\} \qquad F(o) = \{7.3, 8, 9.3, 10\},$$

then F is called a multivalued map.

Remark 2.6.1.
We can see in the above example that $F : S \to F(T)$ is not a function, as in set T there are multiple images of one element of set S.

Example 2.6.2.
Let $X = (-\infty, 1]$ and define a mapping $T : X \longrightarrow 2^{\mathbb{R}}$ by,

$$Tx = 1 + \sqrt{1 - x}$$

is a multivalued mapping which can be seen in the following graph.

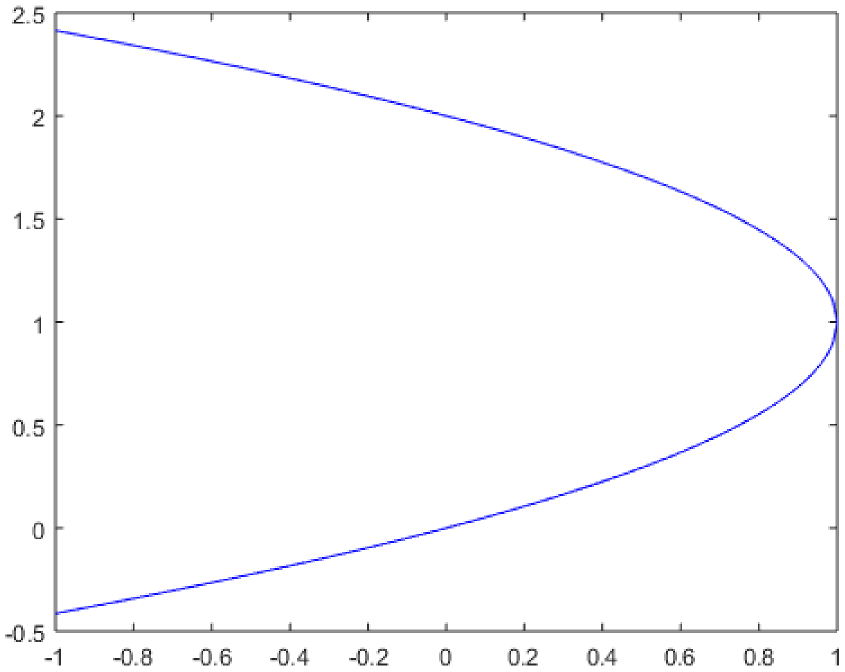

FIGURE 2.5: Multivalued map.

Definition 2.6.2. (**Hausdroff Metric Space**) [49]

"Let (X, d) be a metric space and $CB(X)$ denotes the collection of all nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, define

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

where $d(x, A) = \inf\{d(x, a) : a \in A\}$ is the distance of a point to the set A. It is known that H is a metric on $CB(X)$, called the Hausdorff metric induced by the metric d."

Example 2.6.3.

Let $G = \{1, 2, 3, ..., 8\}$, consider two subsets of G as $S = \{1, 4\}$, $T = \{5, 8\}$.

$$H(S, T) = \max\left\{\sup_{s\in S} d(s, T), \ \sup_{t\in T} d(t, S)\right\} \tag{2.1}$$

Now, to find $H(S,T)$ we proceed as follows

$$\sup_{s\in S} d(s,T) = \sup_{s\in S}\{4,1\}$$

$$\Rightarrow \sup_{s\in S} d(s,T) = 4 \tag{2.2}$$

Also,

$$\sup_{t\in T} d(t,S) = \sup_{t\in T}\{1,4\} \tag{2.3}$$

$$\Rightarrow \sup_{t\in T} d(t,S) = 4$$

From (1.2) and (1.3) using values in (1.1), we have

$$H(S,T) = \max\{4,4\}$$

$$H(S,T) = 4$$

Nadler extended the Banach fixed point theorem for a contraction from a complete metric space into the space of all nonempty closed and bounded subsets of X. He has given the concept of multivalued contractive mappings and proved Banach fixed point theorem for multivalued mappings in complete metric space [50].

Theorem 2.6.1. (**Nadler's Theorem**) [50]
"Let (X,d) be a complete metric space and $T : X \longrightarrow CB(X)$ be such that $H(Tx,Ty) \leq k_0 d(x,y)$, for all $x,y \in X$, and some $k_0 \in [0,1[$, where $CB(x)$ denotes the family of all nonempty closed and bounded subsets of X. Then $Fix(T)$ is nonempty, that is, there exist $x \in X$ such that $x \in Tx$."

Chapter 3

A Fixed Point Result using a Suzuki-type Multivalued Contraction on WPMS

This chapter is the review of the paper of Aydi et al. [35] which addresses a Suzuki-type multivalued contraction on weak partial metric spaces and its applications based on the research work of Beg and Pathak [51]. Beg and Pathak introduced almost partial Hausdroff metric and generalized Nadler's fixed point theorem for multivalued mappings on weak partial metric spaces.

3.1 Basic Definitions

Heckmann introduced the concept of WPMS in 1999. He gave this concept by dropping the nonzero small self distance, which is first axiom of partial metric space [34].

Definition 3.1.1. (**Weak Partial Metric Space**) [35]

Let $\mathcal{M} \neq \phi$ be a set and $\mathcal{W} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ be a function, then for all $\varrho_1, \varrho_2, \varrho_3 \in \mathcal{M}$, (W1) $\mathcal{W}(\varrho_1, \varrho_1) = \mathcal{W}(\varrho_1, \varrho_2)$ if and only if $\varrho_1 = \varrho_2$;

(W2) $\mathcal{W}(\varrho_1, \varrho_1) \leq \mathcal{W}(\varrho_1, \varrho_2)$;

(W3) $\mathcal{W}(\varrho_1, \varrho_2) = \mathcal{W}(\varrho_2, \varrho_1)$;

(W4) $\mathcal{W}(\varrho_1, \varrho_2) \leq \mathcal{W}(\varrho_1, \varrho_3) + \mathcal{W}(\varrho_3, \varrho_2)$,

then $\mathcal{W}$ is called weak partial metric on $\mathcal{M}$. The pair $(\mathcal{M}, \mathcal{W})$ is called a weak partial metric space.

Example 3.1.1. [51]
Consider the set $\mathbb{R}^+$ of all non-negative real numbers. Define a mapping

$$\mathcal{W} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$$

by

$$\mathcal{W}(\varrho_1, \varrho_2) = \frac{1}{4}|\varrho_1 - \varrho_2| + \max\{\varrho_1, \varrho_2\} \qquad \text{for} \qquad \varrho_1, \varrho_2 \in \mathbb{R}^+,$$

then $(\mathbb{R}^+, \mathcal{W})$ is a weak partial metric space.
We can show that $\mathbb{R}^+$ is a weak partial metric space as follows;

(W1) :

$$\text{Let } \mathcal{W}(\varrho_1, \varrho_1) = \mathcal{W}(\varrho_1, \varrho_2)$$
$$\Rightarrow \max\{\varrho_1, \varrho_1\} = \frac{1}{4}|\varrho_1 - \varrho_2| + \max\{\varrho_1, \varrho_2\}$$

Case 1: If $\max\{\varrho_1, \varrho_2\} = \varrho_1$, then

$$\varrho_1 = \frac{1}{4}|\varrho_1 - \varrho_2| + \varrho_1$$
$$\Rightarrow \varrho_1 = \varrho_2$$

Case 2: If $\max\{\varrho_1, \varrho_2\} = \varrho_2$, then

$$\varrho_1 = \frac{1}{4}|\varrho_1 - \varrho_2| + \varrho_2$$

$$\varrho_1 - \varrho_2 = \frac{1}{4}|\varrho_1 - \varrho_2|$$
$$\Rightarrow 4(\varrho_1 - \varrho_2) = |\varrho_1 - \varrho_2|.$$

Which is contradiction, so $\varrho_1 = \varrho_2$.

Conversely let

$$\varrho_1 = \varrho_2,$$

consider

$$\begin{aligned}
\mathcal{W}(\varrho_1, \varrho_2) &= \frac{1}{4}|\varrho_1 - \varrho_2| + \max\{\varrho_1, \varrho_2\} \\
&= \frac{1}{4}|\varrho_1 - \varrho_1| + \max\{\varrho_1, \varrho_1\} \\
&= \max\{\varrho_1, \varrho_1\} \\
&= \varrho_1 \\
&= \mathcal{W}(\varrho_1, \varrho_1) \\
\Rightarrow \mathcal{W}(\varrho_1, \varrho_2) &= \mathcal{W}(\varrho_1, \varrho_1)
\end{aligned}$$

(W2) :

$$\begin{aligned}
\mathcal{W}(\varrho_1, \varrho_1) &= \frac{1}{4}|\varrho_1 - \varrho_1| + \max\{\varrho_1, \varrho_1\} \\
&= \varrho_1 \\
&\leq \frac{1}{4}|\varrho_1 - \varrho_2| + \max\{\varrho_1, \varrho_2\} \\
&= \mathcal{W}(\varrho_1, \varrho_2) \\
\Rightarrow \mathcal{W}(\varrho_1, \varrho_1) &\leq \mathcal{W}(\varrho_1, \varrho_2)
\end{aligned}$$

(W3) : To prove (W3) we proceed as follows

$$\begin{aligned}
\mathcal{W}(\varrho_1, \varrho_2) &= \frac{1}{4}|\varrho_1 - \varrho_2| + \max\{\varrho_1, \varrho_2\} \\
&= \frac{1}{4}|\varrho_2 - \varrho_1| + \max\{\varrho_2, \varrho_1\}
\end{aligned}$$

$$= \mathcal{W}(\varrho_2, \varrho_1)$$
$$\Rightarrow \mathcal{W}(\varrho_1, \varrho_2) = \mathcal{W}(\varrho_2, \varrho_1)$$

(W4) :

$$\begin{aligned}
\mathcal{W}(\varrho_1, \varrho_2) &= \frac{1}{4}|\varrho_1 - \varrho_2| + \max\{\varrho_1, \varrho_2\} \\
&= \frac{1}{4}|\varrho_1 - \varrho_3 + \varrho_3 - \varrho_2| + \max\{\varrho_1, \varrho_2\} \\
&\leq \frac{1}{4}|\varrho_1 - \varrho_3 \,| + \frac{1}{4} \,| \,\varrho_3 - \varrho_2| + \max\{\varrho_1, \varrho_2\} \\
&\leq \frac{1}{4}|\varrho_1 - \varrho_3 \,| + \frac{1}{4} \,| \,\varrho_2 - \varrho_3| + \max\{\varrho_1, \varrho_2\} \\
&+ \max\{\varrho_2, \varrho_3\} + \max\{\varrho_1, \varrho_3\} \\
&= \mathcal{W}(\varrho_1, \varrho_3) + \mathcal{W}(\varrho_2, \varrho_3) \\
\Rightarrow \mathcal{W}(\varrho_1, \varrho_2) &\leq \mathcal{W}(\varrho_1, \varrho_3) + \mathcal{W}(\varrho_2, \varrho_3)
\end{aligned}$$

Since all the conditions are satisfied, so $(\mathbb{R}^+, \mathcal{W})$ is a weak partial metric space.

Remark 3.1.1.
One can easily observe that

(i) If $\mathcal{W}(\varrho_1, \varrho_2) = 0$, then $(W1)$ and $(W2)$ imply that $\varrho_1 = \varrho_2$, but the converse may not need to be true since self distance is non-zero in weak partial metric space.

(ii) First property of partial metric space refers to first property of weak partial metric space that is first property is same for both of partial metric space and weak partial metric space. But the converse may need to be true.

(iii) Also fourth property of partial metric space refers to fourth property of weak partial metric space. But the converse not need to be true.

Remark 3.1.2.
Every weak partial metric $\mathcal{W}$ on $\mathcal{M}$ produces a Hausdroff topology τ_w on $\mathcal{M}$. The family of open w-balls $\{B_w(\mu, \varepsilon) : \mu \in \mathcal{M}, \varepsilon > 0\}$ gives rise to the base of this

topology.

Furthermore the open w-ball can be defined as

$$B_w(\mu, \varepsilon) = \{\nu \in \mathcal{M} : \mathcal{W}(\mu, \nu) < \mathcal{W}(\mu, \mu) + \varepsilon\} \qquad \text{for all} \quad \mu \in \mathcal{M} \quad \text{and} \quad \varepsilon > 0.$$

In order to define a metric on $\mathcal{M}$, for $\mathcal{W}$ to be a weak partial metric space on $\mathcal{M}$, define a function $\mathcal{W}^s : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ by

$$\mathcal{W}^s(\mu, \nu) = \mathcal{W}(\mu, \nu) - \frac{1}{2}\ [\mathcal{W}(\mu, \mu) + \mathcal{W}(\nu, \nu)]$$

Definition 3.1.2. (**Convergent Sequence**) [52]

Let $(\mathcal{M}, \mathcal{W})$ be a weak partial metric spaces. A sequence $\{u_n\}$ in $(\mathcal{M}, \mathcal{W})$ will converge to a point $u \in \mathcal{M}$, with respect to τ_w if

$$\mathcal{W}(u, u) = \lim_{n\to\infty} \mathcal{W}(u, u_n).$$

Definition 3.1.3. (**Cauchy Sequence**)

Let $\{u_n\} \in \mathcal{M}$ be a sequence in a weak partial metric space $(\mathcal{M}, \mathcal{W})$. $\{u_n\}$ will be a Cauchy sequence if $\lim_{n,m\to\infty} \mathcal{W}(u_n, u_m)$ exists and is finite.

Definition 3.1.4. (**Completeness**)

Let $(\mathcal{M}, \mathcal{W})$ be a weak partial metric space. It is will be complete if every Cauchy sequence $\{u_n\}$ in $\mathcal{M}$ converges to a point $u \in \mathcal{M}$ with respect to topology τ_w.

3.2 Fixed Point Results on Weak Partial Metric Spaces

In this section we have discussed some fixed point results on weak partial metric space with respect to the metric defined in weak partial metric spaces.

Lemma 3.2.1. [35]

Let $(\mathcal{M}, \mathcal{W})$ be a weak partial metric space, and $\{u_n\}$ be a sequence in $\mathcal{M}$,

(a) then the sequence $\{u_n\}$ is Cauchy sequence in $(\mathcal{M}, \mathcal{W})$ if and only if this sequence is Cauchy in the metric space $(\mathcal{M}, \mathcal{W}^s)$,

(b) the weak partial metric space $(\mathcal{M}, \mathcal{W})$ is complete if and only if the metric space $(\mathcal{M}, \mathcal{W}^s)$ is complete and vice versa,

(c) a sequence $\{u_n\}$ converge to a point $u \in \mathcal{M}$ if and only if,

$$\lim_{n,m\to\infty} \mathcal{W}(u_n, u_m) = \lim_{n\to\infty} \mathcal{W}(u_n, u) = \mathcal{W}(u, u).$$

Definition 3.2.1. (**Distance between two sets**)

Let $(\mathcal{M}, \mathcal{W})$ be a weak partial metric space and $CB^w(M)$ consists of the family of all bounded and closed subsets of $\mathcal{M}$. One can define the distance between two sets as

$$\psi_w(S, T) = \sup\{\mathcal{W}(s, T) : s \in A\}, \tag{3.1}$$

where

$$\mathcal{W}(s, T) = \inf\{\mathcal{W}(s, t), t \in T\}$$

is the distance between a set and a point.

It is worth mentioning that if $\mathcal{W}(s, T) = 0$, then we have

$$\mathcal{W}^s(s, T) = \inf\{\mathcal{W}^s(s, t), t \in T\}.$$

Proposition 3.2.2. [51]

Let $(\mathcal{M}, \mathcal{W})$ be a weak partial metric space, and let S be a nonempty set in $(\mathcal{M}, \mathcal{W})$, then $s \in \bar{S}$ if and only if

$$\mathcal{W}(s, S) = \mathcal{W}(s, s),$$

where $\bar{S}$ is the closure of S with respect to weak partial metric space $\mathcal{W}$. If S is closed then $\bar{S} = S$.

Proof.

Let $y \in S$ and $s \in \bar{S}$ then for each $\varepsilon > 0$

$$
\begin{aligned}
&B_w(s,\varepsilon)\cap S\neq\emptyset \quad \text{for all} \quad \varepsilon>0\\
&\Leftrightarrow \mathcal{W}(s,y)<\varepsilon+\mathcal{W}(s,s) \quad \text{for all} \quad \varepsilon>0 \;\; \text{and} \;\; \text{some} \;\; y\in S\\
&\Leftrightarrow \mathcal{W}(s,y)-\mathcal{W}(s,s)<\varepsilon\\
&\Leftrightarrow \inf\{\mathcal{W}(s,y)-\mathcal{W}(s,s):y\in S\}=0\\
&\Leftrightarrow \inf\{\mathcal{W}(s,y):y\in S\}-\mathcal{W}(s,s)=0\\
&\Leftrightarrow \inf\{\mathcal{W}(s,y):y\in S\}=\mathcal{W}(s,s)\\
&\Leftrightarrow \mathcal{W}(s,S)=\mathcal{W}(s,s)
\end{aligned}
$$

□

In upcoming discussion the concept of weak partial metric space on the family of bounded and closed subset of $\mathcal{M}$ is discussed. The mapping $\psi_w : CB^w(\mathcal{M})\times CB^w(\mathcal{M})\to[0,\infty)$ is defined in equation (3.1).

Proposition 3.2.3. [51]
Let $(\mathcal{M},\mathcal{W})$ be weak partial metric space, then for all $S,T,U\in CB^w(\mathcal{M})$

(i) $\psi_w(S,S)=\sup\{\mathcal{W}(s,s):s\in S\}$,

(ii) $\psi_w(S,S)\leq\psi_w(S,T)$,

(iii) $\psi_w(S,T)=0$ implies $S\subseteq T$,

(iv) $\psi_w(S,T)\leq\psi_w(S,U)+\psi_w(U,T)$.

Proof.

(i) Let $S\in CB^w(\mathcal{M})$, then for all $s\in S$, we have $\mathcal{W}(s,S)=\mathcal{W}(s,s)$ as $\bar{S}=S$. Therefore $\psi_w(S,S)=\sup\{\mathcal{W}(s,S):s\in S\}=\sup\{\mathcal{W}(s,s):s\in S\}$

(ii) Let $s\in S$,

since,

$$\mathcal{W}(s,s) \leq \mathcal{W}(s,t) \quad \text{for all} \quad t \in T,$$

therefore we have,

$$\mathcal{W}(s,s) \leq \mathcal{W}(s,T) \leq \psi_w(S,T).$$

From (i) it is clear that,

$$\psi_w(S,S) = \sup\{\mathcal{W}(s,s) : s \in S\} \leq \psi_w(S,T),$$

so,

$$\psi_w(S,S) \leq \psi_w(S,T).$$

(iii) Let $\psi_w(S,T) = 0$,

consequently,

$$\mathcal{W}(s,T) = 0 \quad \text{for all} \quad s \in S,$$

from (i) and (ii) we have,

$$\mathcal{W}(s,s) \leq \psi_w(S,T) = 0 \quad \text{for all} \quad s \in S.$$

That is,

$$\mathcal{W}(s,s) = 0 \quad \text{for all} \quad s \in S,$$

so we have,

$$\mathcal{W}(s,T) = \mathcal{W}(s,s) \quad \text{for all} \quad s \in S.$$

Now, using remark (3.1) we have,

$$s \in \bar{T} = T \quad \text{whenever} \quad s \in S,$$

hence,

$$S \subseteq T.$$

(iv) Let $s \in S, t \in T$ and $u \in U$.

Since,

$$\mathcal{W}(s,t) \leq \mathcal{W}(s,u) + \mathcal{W}(u,t),$$

so we have,

$$\mathcal{W}(s,T) \leq \mathcal{W}(s,u) + \mathcal{W}(u,T),$$

as,

$$\mathcal{W}(s,t) \leq \psi_w(S,T),$$

so,

$$\mathcal{W}(s,T) \leq \mathcal{W}(s,u) + \psi_w(U,T).$$

As u is an arbitrary element of U,

therefore,

$$\mathcal{W}(s,T) \leq \mathcal{W}(s,U) + \psi_w(U,T),$$

furthermore, s is also an arbitrary element, so we have

$$\psi_w(S,T) \leq \psi_w(S,U) + \psi_w(U,T).$$

□

Definition 3.2.2.

Let $(\mathcal{M}, \mathcal{W})$ be a weak partial metric space. For $S, T \in CB^w(\mathcal{M})$, define a mapping $H_w : CB^w(\mathcal{M}) \times CB^w(\mathcal{M}) \longrightarrow [0, \infty)$ by

$$H_w(S,T) = \frac{1}{2}\{\psi_w(S,T) + \psi_w(T,S)\},$$

is called H_w **- type Pompeiu-Hausdorff metric space** induced by $\mathcal{W}$.

Proposition 3.2.4. [51]

Let $(\mathcal{M}, \mathcal{W})$ be a weak partial metric space. Then, for all $S, T, U \in CB^w(\mathcal{M})$, we have

(1) $H_w(S,S) \leq H_w(S,T)$;

(2) $H_w(S,T) = H_w(T,S)$;

(3) $H_w(S,T) \leq H_w(S,U) + H_w(U,T)$.

Definition 3.2.3. (**Multivalued Contraction**)
Let $(\mathcal{M}, \mathcal{W})$ be a weak partial metric space and $\mathcal{F} : \mathcal{M} \longrightarrow CB^w(\mathcal{M})$ be a multi-valued mapping. This mapping is said to be H_w - contraction if

(i) there exists k in $(0,1)$ such that

$$H_w(\mathcal{F}x \setminus \{x\}, \mathcal{F}y \setminus \{y\}) \leq k(x,y) \qquad \text{for every} \quad x, y \in \mathcal{M},$$

(ii) for all x in $\mathcal{M}$, y in $\mathcal{F}x$, and $\varepsilon > 0$, there exist z in $\mathcal{F}y$ such that

$$\mathcal{W}(y,z) \leq H_w(\mathcal{F}y, \mathcal{F}x) + \varepsilon.$$

3.3 Fixed Point Result on WPMS

The following fixed point theorem is taken from Beg and Pathak [51]. Aydi et al. [35] generalized this result on Suzuki-type multivalued contraction on weak partial metric spaces.

Theorem 3.3.1.
Let $(\mathcal{M}, \mathcal{W})$ be a complete weak partial metric space, let $\mathcal{T} : \mathcal{M} \longrightarrow CB^w(\mathcal{M})$ be a H_w-type multivalued contraction mapping. This mapping has a fixed point with Lipschitz constant $k < 1$.

To achieve the task of generalization of Theorem (4.1) on H_w - type Suzuki multi-valued contraction on weak partial metric spaces the following mapping is required, which is defined as $\zeta : [0,1) \longrightarrow (0,1]$ is given by

$$\zeta(r) = \begin{cases} 1, & \text{if } 0 \leq r < \dfrac{1}{2} \\ 1 - r, & \text{if } \dfrac{1}{2} \leq r < 1. \end{cases} \tag{3.2}$$

Theorem 3.3.2.

Let $(\mathcal{M}, \mathcal{W})$ be a complete weak partial metric space, and let $T : \mathcal{M} \longrightarrow CB^w(\mathcal{M})$ be a multivalued mapping. Consider $\zeta : [0, 1) \longrightarrow (0, 1]$ as a nonincreasing function which is defined as in (3.2).

Further assume that for $\alpha \in [0, 1)$, T satisfies the following

$$\zeta(\alpha)\mathcal{W}(u_1, Tu_1) \leq \mathcal{W}(u_1, u_2) \Rightarrow H_w(Tu_1 \setminus \{u_1\}, Tu_2 \setminus \{u_2\}) \leq \alpha\mathcal{W}(u_1, u_2) \quad (3.3)$$

for all $u_1, u_2 \in \mathcal{M}$. Suppose also that, for all u_1 in $\mathcal{M}$, and u_2 in Tu_1, and $\beta > 1$, there exist u_3 in Tu_2 such that

$$\mathcal{W}(u_2, u_3) \leq \beta H_w(Tu_2, Tu_3), \quad (3.4)$$

then there is a fixed point of T.

Proof.

Let $\alpha_1 \in (0, 1)$ be such that $0 \leq \alpha \leq \alpha_1 < 1$ and $v_0 \in \mathcal{M}$.

Since Tv_0 is nonempty, it is clear that if $v_0 \in Tv_0$, then there is nothing to prove, as T has a fixed point.

Let $v_0 \notin Tv_0$.

Then there exists $v_1 \in Tv_0$, such that $v_1 \neq v_0$.

Further for $v_1 \notin Tv_1$, there exist $v_2 \in Tv_1$, such that $v_2 \neq v_1$.

As

$$0 \leq \alpha \leq \alpha_1 < 1,$$

so

$$\frac{1}{\sqrt{\alpha_1}} > 1,$$

now using the condition assumed in (3.4), that is

$$\mathcal{W}(v_1, v_2) \leq \frac{1}{\sqrt{\alpha_1}} H_w(Tv_0, Tv_1), \quad (3.5)$$

since,

$$\zeta(r)\mathcal{W}(v_1, Tv_1) \leq \mathcal{W}(v_1, Tv_1) \text{ and } \mathcal{W}(v_1, Tv_1) \leq \mathcal{W}(v_1, v_2),$$

so,

$$\begin{aligned}
&\zeta(r)\mathcal{W}(v_1, Tv_1) \leq \mathcal{W}(v_1, Tv_1) \leq \mathcal{W}(v_1, v_2), \\
\Rightarrow\ &\zeta(r)\mathcal{W}(v_1, Tv_1) \leq \mathcal{W}(v_1, v_2),
\end{aligned}$$

now from equation (3.3) it follows that,

$$H_w(Tv_0 \setminus \{v_0\}, Tv_1 \setminus \{v_1\}) \leq \alpha\mathcal{W}(v_1, v_2),$$

and from equation (3.4),

$$\mathcal{W}(v_1, v_2) \leq \frac{1}{\sqrt{\alpha_1}} H_w(Tv_0, Tv_1),$$

ultimately,

$$\begin{aligned}
\mathcal{W}(v_1, v_2) &\leq \frac{1}{\sqrt{\alpha_1}} H_w(Tv_0, Tv_1) \leq \frac{1}{\sqrt{\alpha_1}} H_w(Tv_0 \setminus \{v_0\}, Tv_1 \setminus \{v_1\}) \\
&\leq \frac{1}{\sqrt{\alpha_1}}.\alpha.\mathcal{W}(v_0, v_1) \leq \sqrt{\alpha_1}.\mathcal{W}(v_0, v_1).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathcal{W}(v_2, v_3) &\leq \frac{1}{\sqrt{\alpha_1}} H_w(Tv_1, Tv_2) \leq \frac{1}{\sqrt{\alpha_1}} H_w(Tv_1 \setminus \{v_1\}, Tv_2 \setminus \{v_2\}) \\
&\leq \frac{1}{\sqrt{\alpha_1}}.\alpha.\mathcal{W}(v_1, v_2) \leq \sqrt{\alpha_1}.\mathcal{W}(v_1, v_2) \\
&\leq \sqrt{\alpha_1}\,[\sqrt{\alpha_1}.\mathcal{W}(v_0, v_1)] \\
\Rightarrow \mathcal{W}(v_2, v_3) &\leq (\sqrt{\alpha_1})^2.\mathcal{W}(v_0, v_1),
\end{aligned}$$

continuing in this way n times,

$$\begin{aligned}
\mathcal{W}(v_n, v_{n+1}) &\leq \frac{1}{\sqrt{\alpha_1}} H_w(Tv_{n-1}, Tv_n) \\
&\leq \frac{1}{\sqrt{\alpha_1}} H_w(Tv_{n-1} \setminus \{v_{n-1}\}, Tv_n \setminus \{v_n\})
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\sqrt{\alpha_1}} H_w(Tv_{n-1} \setminus \{v_{n-1}\}, Tv_n \setminus \{v_n\}) \\
&\leq \frac{1}{\sqrt{\alpha_1}}.\alpha.\mathcal{W}(v_{n-1}, v_n) \leq \sqrt{\alpha_1}.\mathcal{W}(v_{n-1}, v_n) \\
&\leq \sqrt{\alpha_1}\left[(\sqrt{\alpha_1})^{n-1}.\mathcal{W}(v_0, v_1)\right] \\
&\leq (\sqrt{\alpha_1})^n.\mathcal{W}(v_0, v_1) \\
\Rightarrow \mathcal{W}(v_n, v_{n+1}) &\leq (\sqrt{\alpha_1})^n.\mathcal{W}(v_0, v_1). \qquad (3.6)
\end{aligned}
$$

Hence,

$$\lim_{n\to\infty} \mathcal{W}(v_n, v_{n+1}) = 0 \qquad (3.7)$$

Now it is to be proved that $\{v_n\}$ is a Cauchy sequence in $(\mathcal{M}, \mathcal{W}^s)$. For all $m \in N$, where $m > n$,

$$
\begin{aligned}
\mathcal{W}^s(v_n, v_{n+m}) &= \mathcal{W}(v_n, v_{n+m}) - \frac{1}{2}[\mathcal{W}(v_n, v_n) + \mathcal{W}(v_{n+m}, v_{n+m})] \\
&\leq \mathcal{W}(v_n, v_{n+m}) \\
&\leq \mathcal{W}(v_n, v_{n+1}) + \mathcal{W}(v_{n+1}, v_{n+m}) \\
&\leq \mathcal{W}(v_n, v_{n+1}) + \mathcal{W}(v_{n+1}, v_{n+2}) \\
&+ \mathcal{W}(v_{n+2}, v_{n+m}) \\
&\leq \mathcal{W}(v_n, v_{n+1}) + \mathcal{W}(v_{n+1}, v_{n+2}) \\
&+ \mathcal{W}(v_{n+2}, v_{n+3}) + ... + \mathcal{W}(v_{n+m-1}, v_{n+m}) \\
&\leq [(\sqrt{\alpha_1})^n + (\sqrt{\alpha_1})^{n+1}) + (\sqrt{\alpha_1})^{n+2}) + ... \\
&+ (\sqrt{\alpha_1})^{n+m-1})]\mathcal{W}(v_0, v_1) \\
&\leq [(\sqrt{\alpha_1})^n + (\sqrt{\alpha_1})^{n+1}) + (\sqrt{\alpha_1})^{n+2}) + ... \\
&+ (\sqrt{\alpha_1})^{n+m-1}) + (\sqrt{\alpha_1})^{n+m-2}) + ...]\mathcal{W}(v_0, v_1) \\
&\leq (\sqrt{\alpha_1})^n \frac{1}{1 - \sqrt{\alpha_1}} \mathcal{W}(v_0, v_1).
\end{aligned}
$$

$$\lim_{n\to\infty} \mathcal{W}^s(v_n, v_{n+m}) = 0. \qquad (3.8)$$

Hence it is clear that $\{v_n\}$ is a Cauchy sequence in $(\mathcal{M}, \mathcal{W}^s)$. So there exists a $p \in \mathcal{M}$ such that

$$\lim_{n\to\infty} \mathcal{W}(v_n, p) = \lim_{n,m\to\infty} \mathcal{W}(v_n, v_m) = \mathcal{W}(p, p). \tag{3.9}$$

The second property of weak partial metric space gives

$\mathcal{W}(v_n, v_n) \leq \mathcal{W}(v_n, v_{n+1})$ and $\mathcal{W}(v_{n+1}, v_{n+1}) \leqslant \mathcal{W}(v_n, v_{n+1})$

$$\Rightarrow \frac{1}{2}[\mathcal{W}(v_n, v_n) + \mathcal{W}(v_{n+1}, v_{n+1})] \leq \mathcal{W}(v_n, v_{n+1}). \tag{3.10}$$

$$\lim_{n\to\infty} \mathcal{W}(v_n, v_n) = \lim_{n\to\infty} \mathcal{W}(v_{n+1}, v_{n+1}) = \lim_{n\to\infty} \mathcal{W}(v_n, v_{n+1}) = 0, \qquad \text{(from (3.7))} \tag{3.11}$$

since,

$$\lim_{n\to\infty} \mathcal{W}^s(v_n, v_{n+m}) = \lim_{n\to\infty} \mathcal{W}(v_n, v_{n+m}) - \frac{1}{2}\lim_{n\to\infty}[\mathcal{W}(v_n, v_n) + \mathcal{W}(v_{n+m}, v_{n+m})].$$

But from (3.8)

$$\lim_{n\to\infty} \mathcal{W}^s(v_n, v_{n+m}) = 0,$$

using values from (3.11),

$$\Rightarrow \lim_{n\to\infty} \mathcal{W}^s(v_n, v_{n+m}) = 0 = \lim_{n\to\infty} \mathcal{W}(v_n, v_{n+m}). \tag{3.12}$$

Hence,

$$\lim_{n\to\infty} \mathcal{W}(v_n, v_{n+m}) = 0 = \lim_{n\to\infty} \mathcal{W}(v_n, p) = \lim_{n\to\infty} \mathcal{W}(p, p). \tag{3.13}$$

Now, it is to be proved that

$$\mathcal{W}(p, Tu_1) \leq 2\alpha\mathcal{W}(p, u_1) \qquad \text{for all } u_1 \in \mathcal{M} \setminus \{p\} \tag{3.14}$$

As

$$\lim_{n\to\infty} \mathcal{W}(v_n, p) = 0,$$

therefore, there exists $N \in \mathbb{N}$ such that

$$\mathcal{W}(v_n, p) \leq \frac{1}{3}\mathcal{W}(u_1, p) \quad \text{for all } n \geq N. \tag{3.15}$$

Now, from the assumption

$$\begin{aligned}
\zeta(\alpha)\mathcal{W}(v_n, Tv_n) &\leq \mathcal{W}(v_n, Tv_n) \\
&\leq \mathcal{W}(v_n, v_{n+1}) \\
&\leq \mathcal{W}(v_n, p) + \mathcal{W}(p, v_{n+1}) \\
&\leq \frac{1}{3}\mathcal{W}(p, u_1) + \frac{1}{3}\mathcal{W}(p, u_1) \\
&\leq \mathcal{W}(u_1, p) - \frac{1}{3}\mathcal{W}(u_1, p) \\
&\leq \mathcal{W}(u_1, p) - \mathcal{W}(p, v_n) \quad \text{(from (3.15))} \\
\mathcal{W}(p, u_1) &\leq \mathcal{W}(p, v_n) + \mathcal{W}(v_n, u_1) \quad \text{(by triangular inequality)} \\
&= \mathcal{W}(p, v_n) + \mathcal{W}(u_1, v_n) \\
\Rightarrow \mathcal{W}(p, u_1) - \mathcal{W}(p, v_n) &\leq \mathcal{W}(u_1, v_n) \\
\Rightarrow \zeta(\alpha)\mathcal{W}(v_n, Tv_n) &\leq \mathcal{W}(u_1, v_n) \\
\Rightarrow H_w(Tv_n, Tu_1) &\leq \alpha\mathcal{W}(v_n, u_1).
\end{aligned} \tag{3.16}$$

Since,

$$\begin{aligned}
H_w(Tv_n, Tu_1) &= \frac{1}{2}\{\psi_w(Tv_n, Tu_1) + \psi_w(Tu_1, Tv_n)\} \\
\Rightarrow 2H_w(Tv_n, Tu_1) &= \psi_w(Tv_n, Tu_1) + \psi_w(Tu_1, Tv_n) \\
\Rightarrow 2H_w(Tv_n, Tu_1) &\geq \psi_w(Tv_n, Tu_1),
\end{aligned} \tag{3.17}$$

Since $v_{n+1} \in Tv_n$, now using the definition of distance between two sets and the distance between a set and a point,

$$\begin{aligned}
\mathcal{W}(v_{n+1}, Tu_1) &\leq \psi_w(Tv_n, Tu_1) \\
\mathcal{W}(v_{n+1}, Tu_1) &\leq 2H_w(Tv_n, Tu_1) \quad \text{(from (3.17))} \\
\mathcal{W}(v_{n+1}, Tu_1) &\leq 2\alpha\mathcal{W}(v_n, u_1) \quad \text{(from (3.16))} \\
\mathcal{W}(v_{n+1}, Tu_1) &\leq 2\alpha\{\mathcal{W}(v_n, p) + \mathcal{W}(p, u_1)\}
\end{aligned}$$

$$\Rightarrow \lim_{n\to\infty} \mathcal{W}(v_{n+1}, Tu_1) \leq 2\alpha\mathcal{W}(p, u_1).$$

Consider,

$$\begin{aligned} \mathcal{W}(p, Tu_1) &\leq \mathcal{W}(p, v_{n+1}) + \mathcal{W}(v_{n+1}, Tu_1) \\ \mathcal{W}(p, Tu_1) &\leq \lim_{n\to\infty} \mathcal{W}(v_{n+1}, Tu_1) \end{aligned} \tag{3.18}$$

and also,

$$\begin{aligned} \mathcal{W}(v_{n+1}, Tu_1) &\leq \mathcal{W}(v_{n+1}, v_n) + \mathcal{W}(v_n, p) + \mathcal{W}(p, Tu_1) \\ \Rightarrow \lim_{n\to\infty} \mathcal{W}(v_{n+1}, Tu_1) &\leq \mathcal{W}(p, Tu_1) \qquad (3.19) \\ \Rightarrow \lim_{n\to\infty} \mathcal{W}(v_{n+1}, Tu_1) &= \mathcal{W}(p, Tu_1) \qquad \text{(from (3.18) and (3.19))} \\ \Rightarrow \mathcal{W}(p, Tu_1) &\leq 2\alpha\mathcal{W}(p, u_1) \qquad \text{for all } u_1 \in \mathcal{M} \setminus \{p\}. \qquad (3.20) \end{aligned}$$

Now claim that,

$$H_w(Tu_1, Tp) \leq \alpha\mathcal{W}(p, u_1) \quad \text{for all } u_1 \in \mathcal{W}.$$

Above statement clearly hold for $u_1 = p$.

Assume that $u_1 \neq p$, then for every $m \in \mathbb{N}$, there will be a $z_m \in Tu_1$ such that

$$\mathcal{W}(p, z_m) \leq \mathcal{W}(p, Tu_1) + \frac{1}{m}\mathcal{W}(p, u_1). \tag{3.21}$$

Now consider,

$$\begin{aligned} \mathcal{W}(u_1, Tu_1) &\leq \mathcal{W}(u_1, z_m) \\ &\leq \mathcal{W}(u_1, p) + \mathcal{W}(p, z_m) \\ &\leq \mathcal{W}(u_1, p) + \mathcal{W}(p, Tu_1) + \frac{1}{m}\mathcal{W}(u_1, p) \quad \text{(from (3.21))} \qquad (3.22) \\ \mathcal{W}(u_1, Tu_1) &\leq \mathcal{W}(u_1, p) + 2\alpha\mathcal{W}(p, u_1) + \frac{1}{m}\mathcal{W}(u_1, p) \quad \text{(from (3.20) and (3.22))} \\ &= \left[1 + 2\alpha + \frac{1}{m}\right]\mathcal{W}(u_1, p). \end{aligned}$$

$$\Rightarrow \frac{1}{\left[1+2\alpha+\frac{1}{m}\right]}\mathcal{W}(u_1, Tu_1) \le \mathcal{W}(p, u_1).$$

The above inequality implies that

$$H_w(Tu_1, Tp) \le \alpha\mathcal{W}(p, u_1) \quad \text{for all } u_1 \in \mathcal{W}. \tag{3.23}$$

Now the existence of fixed point will be shown.

Consider

$$\begin{aligned}
\mathcal{W}(p, Tp) &= \lim_{n\to\infty} \mathcal{W}(v_{n+1}, Tp) \\
&\le \lim_{n\to\infty} \psi_w(Tv_n, Tp) \\
&\le 2 \lim_{n\to\infty} H_w(Tv_n, Tp) \\
&\le 2\alpha \lim_{n\to\infty} \mathcal{W}(v_n, p) = 0.
\end{aligned}$$

From above calculation it is clear that $\mathcal{W}(p, Tp) = 0 = \mathcal{W}(p, Tp)$.

Since Tp is closed, $p \in \overline{Tp} = Tp$. □

Example 3.3.1. [35]

Let $\mathcal{M} = \{0, \frac{1}{2}, 1\}$. The weak partial metric space $\mathcal{W} : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ can be defined as $\mathcal{W}(0, 0) = 0, \mathcal{W}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{3}, \mathcal{W}(1, 1) = \frac{1}{4}, \mathcal{W}\left(0, \frac{1}{2}\right) = \mathcal{W}\left(\frac{1}{2}, 0\right) = \frac{1}{2}, \mathcal{W}\left(\frac{1}{2}, 1\right) = \mathcal{W}\left(1, \frac{1}{2}\right) = \frac{3}{4}$ and $\mathcal{W}(1, 0) = \mathcal{W}(0, 1) = 1$.

It can easily be seen that $(\mathcal{M}, \mathcal{W})$ is weak partial metric space but is not a partial metric space.

Since

$$\mathcal{W}(1, 0) = 1 \not\le \mathcal{W}\left(\frac{1}{2}, 1\right) + \mathcal{W}\left(\frac{1}{2}, 0\right) - \mathcal{W}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} - \frac{1}{3} = 0.9$$

Let $T : \mathcal{M} \longrightarrow CB^w(\mathcal{M})$ be the mapping defined by $T(0) = T\left(\frac{1}{2}\right) = 0$ and $T(1) = \left\{0, \frac{1}{2}\right\}$. By choosing $\alpha = 0.5$, the definition of ζ gives $\zeta(r) = 1$.

First, the contraction condition of the theorem is to be proved that is,

$$\zeta(\alpha)\mathcal{W}(u_1, Tu_1) \leq \mathcal{W}(u_1, u_2) \Rightarrow H_w(Tu_1 \setminus \{u_1\}, Tu_2 \setminus \{u_2\}) \leq \alpha\mathcal{W}(u_1, u_2)$$

To prove above condition the following cases are to be considered.

Case 1.

Choose $x = 0$,

$$\begin{aligned}
\zeta(\alpha)\mathcal{W}(0, T(0)) &= \mathcal{W}(0, 0) \\
&= 0 \\
&\leq \mathcal{W}(0, u_2) \qquad \text{for all } \ u_1 \in \mathcal{M} \\
\Rightarrow \zeta(\alpha)\mathcal{W}(0, T(0)) &\leq \mathcal{W}(0, u_2).
\end{aligned}$$

Now, for $u_2 = 0$

$$\begin{aligned}
H_w(T(0) \setminus \{0\}, T(0) \setminus \{0\}) &= H_w(\phi, \phi) \\
&= 0 \\
&\leq \alpha\mathcal{W}(0, 0) \\
\Rightarrow H_w(T(0) \setminus \{0\}, T(0) \setminus \{0\}) &\leq \alpha\mathcal{W}(0, 0).
\end{aligned}$$

For $u_2 = \dfrac{1}{2}$, we proceed as follows,

$$\begin{aligned}
H_w\left(T(0) \setminus \{0\}, T\left(\frac{1}{2}\right) \setminus \left\{\frac{1}{2}\right\}\right) &= H_w(\phi, \{0\}) \\
&= 0 \\
&\leq \alpha\mathcal{W}\left(0, \frac{1}{2}\right) \\
\Rightarrow H_w\left(T(0) \setminus \{0\}, T\left(\frac{1}{2}\right) \setminus \left\{\frac{1}{2}\right\}\right) &\leq \alpha\mathcal{W}\left(0, \frac{1}{2}\right).
\end{aligned}$$

For $u_2 = 1$,

$$H_w(T(0) \setminus \{0\}, T(1) \setminus \{1\}) = H_w\left(\phi, \left\{0, \frac{1}{2}\right\}\right)$$

$$
\begin{aligned}
&= 0 \\
&\leq \alpha\mathcal{W}(0,1) \\
\Rightarrow H_w(T(0)\setminus\{0\}, T(1)\setminus\{1\}) &\leq \alpha\mathcal{W}(0,1).
\end{aligned}
$$

Case 2.

At $u_1 = \dfrac{1}{2}$

$$
\begin{aligned}
\zeta(\alpha)\mathcal{W}\left(\frac{1}{2}, T\left(\frac{1}{2}\right)\right) &= \mathcal{W}\left(\frac{1}{2}, 0\right) \\
&= \frac{1}{2} \\
&\leq \mathcal{W}\left(\frac{1}{2}, u_2\right) \qquad \text{for all } \ u_2 \in \mathcal{M}\setminus\left\{\frac{1}{2}\right\} \\
\Rightarrow \zeta(\alpha)\mathcal{W}\left(\frac{1}{2}, T\left(\frac{1}{2}\right)\right) &\leq \mathcal{W}\left(\frac{1}{2}, u_2\right).
\end{aligned}
$$

Now for $u_2 = 0$, the relation becomes

$$
\begin{aligned}
H_w\left(T\left(\frac{1}{2}\right)\setminus\left\{\frac{1}{2}\right\}, T(0)\setminus\{0\}\right) &= H_w(\{0\}, \phi) \\
&= 0 \\
&\leq \alpha\mathcal{W}\left(\frac{1}{2}, 0\right) \\
\Rightarrow H_w\left(T\left(\frac{1}{2}\right)\setminus\left\{\frac{1}{2}\right\}, T(0)\setminus\{0\}\right) &\leq \alpha\mathcal{W}\left(\frac{1}{2}, 0\right).
\end{aligned}
$$

If $u_2 = 1$, then

$$
\begin{aligned}
H_w\left(T\left(\frac{1}{2}\right)\setminus\left\{\frac{1}{2}\right\}, T(1)\setminus\{1\}\right) &= H_w\left(\{0\}, \left\{0, \frac{1}{2}\right\}\right) \\
&= \frac{1}{4} \\
&\leq \alpha\mathcal{W}\left(\frac{1}{2}, 1\right) \\
&= \frac{3}{8} \\
\Rightarrow H_w\left(T\left(\frac{1}{2}\right)\setminus\left\{\frac{1}{2}\right\}, T(1)\setminus\{1\}\right) &\leq \alpha\mathcal{W}\left(\frac{1}{2}, 1\right).
\end{aligned}
$$

Case 3.

At $u_1 = 1$

$$\begin{aligned}
\zeta(\alpha)\mathcal{W}(1, T(1)) &= \mathcal{W}\left(1, \frac{1}{2}\right) \\
&= \frac{3}{4} \\
&\leq \mathcal{W}(1, u_2) \qquad \text{for all} \;\; u_2 \in \mathcal{M}\backslash\{1\} \\
\Rightarrow \zeta(\alpha)\mathcal{W}(1, T(1)) &\leq \mathcal{W}(1, u_2).
\end{aligned}$$

For $u_2 = 0$, the calculation is given as

$$\begin{aligned}
H_w(T(1)\backslash\{1\}, T(0)\backslash\{0\}) &= H_w\left(\left\{0, \frac{1}{2}\right\}, \phi\right) \\
&= 0 \\
&\leq \alpha\mathcal{W}(1, 0) \\
H_w(T(1)\backslash\{1\}, T(0)\backslash\{0\}) &\leq \alpha\mathcal{W}(1, 0).
\end{aligned}$$

If $u_2 = \frac{1}{2}$, then the relation becomes

$$\begin{aligned}
H_w\left(T(1) \setminus \{1\}, T\left(\frac{1}{2}\right) \setminus \left\{\frac{1}{2}\right\}\right) &= H_w\left(\left\{0, \frac{1}{2}\right\}, \{0\}\right) \\
&= \frac{1}{4} \\
&\leq \alpha\mathcal{W}\left(1, \frac{1}{2}\right) \\
&= \frac{3}{8} \\
\Rightarrow H_w\left(T(1) \setminus \{1\}, T\left(\frac{1}{2}\right) \setminus \left\{\frac{1}{2}\right\}\right) &\leq \alpha\mathcal{W}\left(1, \frac{1}{2}\right).
\end{aligned}$$

It is clear from above cases that contraction condition is satisfied. Now, the second condition given in equation (3.4) is to be inquired that is,

$$\mathcal{W}(u_2, u_3) \leq \beta H_w(Tu_2, Tu_3),$$

it will be checked by choosing $\beta = 2$ and by discussing following situations for u_1 and u_2 given below

(i) If $u_1 = 0$ or $u_1 = \frac{1}{2}$, then $u_2 \in T(0)$.
Since $T(0) = T\big(\frac{1}{2}\big) = \{0\}$
This implies that $u_2 = 0$, then there exists $u_3 \in T(u_2)$ such that

$$0 = \mathcal{W}(u_2, u_3) \leq \beta H_w(Tu_2, Tu_3).$$

(ii) If $u_1 = 1$, then $u_2 \in T(1)$.
$\Rightarrow u_2 \in T(1) = \left\{0, \frac{1}{2}\right\}$,
if $u_2 = 0 \Rightarrow u_3 = 0$ and if $u_2 = \frac{1}{2}$, again $u_3 = 0$, so

$$\frac{1}{2} = \mathcal{W}(u_2, u_3) = 2H_w\left(T(1), T\left(\frac{1}{2}\right)\right) = \frac{1}{2}.$$

The function T has a fixed point $u = 0$, since all the conditions of the theorem are satisfied.

Chapter 4

Weak Partial b-metric Space endowed with a Suzuki-type Multivalued Contraction

In 2019, Kanwal et al. [36] presented the concept of weak partial b-metric space. In this article the authors proved a fixed point result, which is the extension of Nadler's theorem. Aydi et al. [35] proved a fixed point result on weak partial metric space endowed with Suzuki type multivalued contraction. This chapter provides a fixed point result which is merging the idea of both.

4.1 Some Important Tools for WPbMS

This section is dedicated to some important results and definition, which are necessary for the proof of upcoming fixed point result.

Definition 4.1.1. (**Weak Partial b-metric Space**) [36]

Let $\mathcal{M}$ be a nonempty set, $k \geq 1$ and $\mathcal{W}_b : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ be a function, then for all $\varrho_1, \varrho_2, \varrho_3 \in \mathcal{M}$,

(W1) $\mathcal{W}_b(\varrho_1, \varrho_1) = \mathcal{W}_b(\varrho_1, \varrho_2)$ if and only if $\varrho_1 = \varrho_2$;

(W2) $\mathcal{W}_b(\varrho_1, \varrho_1) \leq \mathcal{W}_b(\varrho_1, \varrho_2)$;

(W3) $\mathcal{W}_b(\varrho_1, \varrho_2) = \mathcal{W}_b(\varrho_2, \varrho_1)$;

(W4) $\mathcal{W}_b(\varrho_1, \varrho_2) \leq k\{\mathcal{W}_b(\varrho_1, \varrho_3) + \mathcal{W}_b(\varrho_3, \varrho_2)\}$.

Then $\mathcal{W}_b$ is called weak partial b-metric on $\mathcal{M}$. The pair $(\mathcal{M}, \mathcal{W}_b)$ is called a weak partial b-metric space.

It is worth to mention here that one can easily define a b-metric space on $\mathcal{M}$ by using weak partial b-metric space. For this purpose, define a function

$$\mathcal{W}_b^s : \mathcal{M} \times \mathcal{M} \to [0, \infty)$$

by

$$\mathcal{W}_b^s(\mu, \nu) = \mathcal{W}_b(\mu, \nu) - \frac{1}{2}\big[\mathcal{W}_b(\mu, \mu) + \mathcal{W}_b(\nu, \nu)\big],$$

then $(\mathcal{W}_b^s, \mathcal{M})$ is a b-metric space.

Lemma 4.1.1. [36]

Let $(\mathcal{M}, \mathcal{W}_b)$ be a weak partial b-metric space, and $\{u_n\}$ be a sequence in $\mathcal{M}$ then,

(a) the sequence $\{u_n\}$ is Cauchy sequence in $(\mathcal{M}, \mathcal{W}_b)$ if and only if this sequence is Cauchy in the b-metric space $(\mathcal{M}, \mathcal{W}_b^s)$,

(b) the weak partial b-metric space $(\mathcal{M}, \mathcal{W}_b)$ is complete if the b-metric space $(\mathcal{M}, \mathcal{W}_b^s)$ is complete and vice versa,

(c) a sequence $\{u_n\}$ in $(\mathcal{M}, \mathcal{W}_b^s)$ converge to a point $u \in \mathcal{M}$ if and only if,

$$\lim_{n,m\to\infty} \mathcal{W}_b(u_n, u_m) = \lim_{n\to\infty} \mathcal{W}_b(u_n, u) = \mathcal{W}_b(u, u).$$

Definition 4.1.2. [36]

Let $(\mathcal{M}, \mathcal{W}_b)$ be a weak partial b-metric space and $CB^w(M)$ consists of the family of all nonempty, bounded and closed subsets of $\mathcal{M}$. One can define the distance between two sets as

$$\psi_{wb}(S,T) = \sup\{\mathcal{W}_b(s,T) : s \in S\}, \tag{4.1}$$

where

$$\mathcal{W}_b(s,T) = \inf\{\mathcal{W}_b(s,t), t \in T\}$$

is the distance between a set and a point.

It is worth mentioning that if $\mathcal{W}_b(s,T) = 0 \Rightarrow \mathcal{W}_b^s(s,T) = 0$, where $\mathcal{W}_b^s(s,T) = \inf\{\mathcal{W}_b^s(s,t), t \in T\}$.

Remark 4.1.1. [36]

Let $(\mathcal{M}, \mathcal{W}_b)$ be a weak partial b-metric space, and let S be a nonempty set in $\mathcal{M}$, then $s \in \bar{S}$ if and only if

$$\mathcal{W}_b(s,S) = \mathcal{W}_b(s,s),$$

where $\bar{S}$ denotes the closure of S with respect to weak partial b-metric space $\mathcal{W}_b$. If S is closed then $\bar{S} = S$.

Proposition 4.1.2. [36]

Let $(\mathcal{M}, \mathcal{W}_b)$ be weak partial b-metric space, then for all $S, T, U \in CB^w(\mathcal{M})$ and $k \geq 1$

(i) $\psi_{wb}(S,S) = \sup\{\mathcal{W}_b(s,s) : s \in S\}$,

(ii) $\psi_{wb}(S,S) \leq \psi_{wb}(S,T)$,

(iii) $\psi_{wb}(S,T) = 0$ implies $S \subseteq T$,

(iv) $\psi_{wb}(S,T) \leq k\{\psi_{wb}(S,U) + \psi_{wb}(U,T)\}$.

Definition 4.1.3. [36]

Let $(\mathcal{M}, \mathcal{W}_b)$ be a weak partial b-metric space. For $S, T \in CB^{wb}(\mathcal{M})$, define a mapping

$$H_{wb} : CB^{wb}(\mathcal{M}) \times CB^{wb}(\mathcal{M}) \longrightarrow [0,\infty),$$

by

$$H_{wb}(S,T) = \frac{1}{2}\{\psi_{wb}(S,T) + \psi_{wb}(T,S)\},$$

is called H_{wb} **- type Pompeiu-Hausdorff metric space** induced by $\mathcal{W}_b$.

Proposition 4.1.3. [36]

Let $(\mathcal{M}, \mathcal{W}_b)$ be a weak partial b-metric space. Then, for all $S, T, U \in CB^{wb}(\mathcal{M})$,

(1) $H_{wb}(S,S) \leq H_{wb}(S,T)$;

(2) $H_{wb}(S,T) = H_{wb}(T,S)$;

(3) $H_{wb}(S,T) \leq k\{H_{wb}(S,U) + H_w(U,T)\}$.

Proof.

(1) Consider (ii) from Proposition 4.1

$$\psi_{wb}(S,S) \leq \psi_{wb}(S,T),$$

also $H_{wb}(S,S) = \psi_{wb}(S,S)$ and $H_{wb}(S,T) \leq H_{wb}(S,T)$, this implies that $H_{wb}(S,S) = \psi_{wb}(S,S) \leq \psi_{wb}(S,T) \leq H_{wb}(S,T)$. Hence,

$$H_{wb}(S,S) \leq H_{wb}(S,T).$$

(2) Obvious from definition.

(3) Consider H_{wb} - type Pompeiu-Hausdorff metric space

$$H_{wb}(S,T) = \frac{1}{2}\{\psi_{wb}(S,T) + \psi_{wb}(T,S)\},$$

using (iv) from Proposition 4.1 in above equation, it becomes

$$\begin{aligned} H_{wb}(S,T) &\leq \frac{1}{2}\left[k\{\psi_{wb}(S,U) + \psi_{wb}(U,T)\} + k\{\psi_{wb}(T,U) + \psi_{wb}(U,S)\}\right] \\ &= k\left[\frac{1}{2}\{\psi_{wb}(S,U) + \psi_{wb}(U,S\} + \frac{1}{2}\{\psi_{wb}(U,T) + \psi_{wb}(T,U)\}\right] \\ &= k[H_{wb}(S,U) + H_{wb}(U,T)]. \end{aligned}$$

□

Theorem 4.1.4.

Let $(\mathcal{M}, \mathcal{W}_b)$ be a complete weak partial b-metric space with $\mathcal{W}_b$ a continuous function, and let $T : \mathcal{M} \longrightarrow CB^{wb}(\mathcal{M})$ be a multivalued mapping. Consider $\zeta : [0, 1) \longrightarrow (0, 1]$ as a nonincreasing function as defined in (3.2).

Further assume that for $\alpha \in [0, 1)$, T satisfies the following

$$\zeta(\alpha)\mathcal{W}_b(u_1, Tu_1) \leq \mathcal{W}_b(u_1, u_2) \Rightarrow H_{wb}(Tu_1 \setminus \{u_1\}, Tu_2 \setminus \{u_2\}) \leq k\alpha\mathcal{W}_b(u_1, u_2) \tag{4.2}$$

for all $u_1, u_2 \in \mathcal{M}$. Suppose also that, for all u_1 in $\mathcal{M}$, and u_2 in Tu_1, and $\beta > 1$, there exist u_3 in Tu_2 such that

$$\mathcal{W}_b(u_2, u_3) \leq \beta H_{wb}(Tu_2, Tu_3), \tag{4.3}$$

then there is a fixed point of T.

Proof.

Let $\alpha_1 \in (0, 1)$ be such that $0 \leq \alpha \leq \alpha_1 < 1$ and $v_0 \in \mathcal{M}$.

Since Tv_0 is nonempty, it is clear that if $v_0 \in Tv_0$, then there is nothing to prove, as T has a fixed point.

Let $v_0 \notin Tv_0$.

Then there exists $v_1 \in Tv_0$, such that $v_1 \neq v_0$.

Further for $v_1 \notin Tv_1$, there exist $v_2 \in Tv_1$, such that $v_2 \neq v_1$.

As

$$0 \leq \alpha \leq \alpha_1 < 1,$$

so

$$\frac{1}{\sqrt{\alpha_1}} > 1,$$

now using the condition assumed in (4.3), that is

$$\mathcal{W}_b(v_1, v_2) \leq \frac{1}{\sqrt{\alpha_1}} H_{wb}(Tv_0, Tv_1), \tag{4.4}$$

since,

$$\zeta(r)\mathcal{W}_b(v_1, Tv_1) \leq \mathcal{W}_b(v_1, Tv_1) \text{ and } \mathcal{W}_b(v_1, Tv_1) \leq \mathcal{W}_b(v_1, v_2),$$

so we have,

$$\begin{aligned}
&\zeta(r)\mathcal{W}_b(v_1, Tv_1) \leq \mathcal{W}_b(v_1, Tv_1) \leq \mathcal{W}_b(v_1, v_2), \\
\Rightarrow\ &\zeta(r)\mathcal{W}_b(v_1, Tv_1) \leq \mathcal{W}_b(v_1, v_2),
\end{aligned}$$

now from (4.2) it follows that,

$$H_{wb}(Tv_0 \setminus \{v_0\}, Tv_1 \setminus \{v_1\}) \leq \alpha\mathcal{W}_b(v_1, v_2),$$

and from (4.3) we have,

$$\mathcal{W}_b(v_1, v_2) \leq \frac{1}{\sqrt{\alpha_1}} H_{wb}(Tv_0, Tv_1),$$

ultimately the relation becomes,

$$\begin{aligned}
\mathcal{W}_b(v_1, v_2) &\leq \frac{1}{\sqrt{\alpha_1}} H_{wb}(Tv_0, Tv_1) \leq \frac{1}{\sqrt{\alpha_1}} H_{wb}(Tv_0 \setminus \{v_0\}, Tv_1 \setminus \{v_1\}) \\
&\leq \frac{1}{\sqrt{\alpha_1}}.\alpha.\mathcal{W}_b(v_0, v_1) \leq \sqrt{\alpha_1}.\mathcal{W}_b(v_0, v_1).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathcal{W}_b(v_2, v_3) &\leq \frac{1}{\sqrt{\alpha_1}} H_{wb}(Tv_1, Tv_2) \leq \frac{1}{\sqrt{\alpha_1}} H_{wb}(Tv_1 \setminus \{v_1\}, Tv_2 \setminus \{v_2\}) \\
&\leq \frac{1}{\sqrt{\alpha_1}}.\alpha.\mathcal{W}_b(v_1, v_2) \leq \sqrt{\alpha_1}.\mathcal{W}_b(v_1, v_2) \\
&\leq \sqrt{\alpha_1}\,[\sqrt{\alpha_1}.\mathcal{W}_b(v_0, v_1)] \\
\Rightarrow \mathcal{W}_b(v_2, v_3) &\leq (\sqrt{\alpha_1})^2.\mathcal{W}_b(v_0, v_1),
\end{aligned}$$

continuing in this way n times, we have

$$\mathcal{W}_b(v_n, v_{n+1}) \leq \frac{1}{\sqrt{\alpha_1}} H_{wb}(Tv_{n-1}, Tv_n) \leq \frac{1}{\sqrt{\alpha_1}} H_{wb}(Tv_{n-1} \setminus \{v_{n-1}\}, Tv_n \setminus \{v_n\})$$

$$
\begin{aligned}
&\leq \frac{1}{\sqrt{\alpha_1}}.\alpha.\mathcal{W}_b(v_{n-1}, v_n) \leq \sqrt{\alpha_1}.\mathcal{W}_b(v_{n-1}, v_n) \\
&\leq \sqrt{\alpha_1}\left[(\sqrt{\alpha_1})^{n-1}.\mathcal{W}_b(v_0, v_1)\right] \\
&\leq (\sqrt{\alpha_1})^{n}.\mathcal{W}_b(v_0, v_1) \\
\Rightarrow \mathcal{W}_b(v_n, v_{n+1}) &\leq (\sqrt{\alpha_1})^{n}.\mathcal{W}_b(v_0, v_1). \qquad (4.5)
\end{aligned}
$$

Hence,

$$
\lim_{n\to\infty} \mathcal{W}_b(v_n, v_{n+1}) = 0 \qquad (4.6)
$$

Now it is to be proved that $\{v_n\}$ is a Cauchy sequence in $(\mathcal{M}, \mathcal{W}_b^s)$. Suppose $m > n$,

$$
\begin{aligned}
\mathcal{W}_b^s(v_n, v_{n+m}) &= \mathcal{W}_b(v_n, v_{n+m}) - \frac{1}{2}[\mathcal{W}_b(v_n, v_n) + \mathcal{W}_b(v_{n+m}, v_{n+m})] \\
&\leq \mathcal{W}_b(v_n, v_{n+m}) \\
&\leq k\{\mathcal{W}_b(v_n, v_{n+1}) + \mathcal{W}_b(v_{n+1}, v_{n+m})\} \\
&\leq k\{\mathcal{W}_b(v_n, v_{n+1}) + k\{\mathcal{W}_b(v_{n+1}, v_{n+2}) \\
&+ \mathcal{W}_b(v_{n+2}, v_{n+m})\}\} \\
&= k\mathcal{W}_b(v_n, v_{n+1}) + k^2\mathcal{W}_b(v_{n+1}, v_{n+2}) \\
&+ k^2\mathcal{W}(v_{n+2}, v_{n+m}),
\end{aligned}
$$

continuing in this way, we get

$$
\begin{aligned}
&\leq k\mathcal{W}_b(v_n, v_{n+1}) + k^2\mathcal{W}_b(v_{n+1}, v_{n+2}) \\
&+ k^3\mathcal{W}_b(v_{n+2}, v_{n+3}) + ... \\
&+ k^{m+n}\mathcal{W}_b(v_{n+m-1}, v_{n+m}) \\
&\leq [k(\sqrt{\alpha_1})^{n} + k^2(\sqrt{\alpha_1})^{n+1}) + k^3(\sqrt{\alpha_1})^{n+2}) + ... \\
&+ k^m((\sqrt{\alpha_1})^{n+m-1})]\mathcal{W}_b(v_0, v_1) \\
&\leq k(\sqrt{\alpha_1})^{n}[1 + k(\sqrt{\alpha_1}) + k^2(\sqrt{\alpha_1})^2) + ... \\
&+ k^{m-1}((\sqrt{\alpha_1})^{m-1}) + k^{m-2}((\sqrt{\alpha_1})^{m-2}) + ...]\mathcal{W}_b(v_0, v_1)
\end{aligned}
$$

$$\leq k(\sqrt{\alpha_1})^n \frac{1}{1-\sqrt{\alpha_1}} \mathcal{W}_b(v_0, v_1) \quad \text{where } k\sqrt{\alpha_1} < 1$$

So we have

$$\lim_{n\to\infty} \mathcal{W}_b^s(v_n, v_{n+m}) = 0. \tag{4.7}$$

Hence it is clear that $\{v_n\}$ is a Cauchy sequence in $(\mathcal{M}, \mathcal{W}_b^s)$. So there exists a $p \in \mathcal{M}$ such that

$$\lim_{n\to\infty} \mathcal{W}_b(v_n, p) = \lim_{n,m\to\infty} \mathcal{W}_b(v_n, (v_m) = \mathcal{W}_b(p, p). \tag{4.8}$$

Since $\mathcal{W}_b$ is a weak partial b-metric space therefore,

$\mathcal{W}_b(v_n, v_n) \leq \mathcal{W}_b(v_n, v_{n+1})$ and $\mathcal{W}_b(v_{n+1}, v_{n+1}) \leqslant \mathcal{W}_b(v_n, v_{n+1})$

$$\Rightarrow \frac{1}{2}[\mathcal{W}_b(v_n, v_n) + \mathcal{W}_b(v_{n+1}, v_{n+1})] \leq \mathcal{W}_b(v_n, v_{n+1}). \tag{4.9}$$

$$\lim_{n\to\infty} \mathcal{W}_b(v_n, v_n) = \lim_{n\to\infty} \mathcal{W}_b(v_{n+1}, v_{n+1}) = \lim_{n\to\infty} \mathcal{W}_b(v_n, v_{n+1}) = 0 \quad \text{(from (4.6))} \tag{4.10}$$

since ,

$$\lim_{n\to\infty} \mathcal{W}_b^s(v_n, v_{n+m}) = \lim_{n\to\infty} \mathcal{W}_b(v_n, v_{n+m}) - \frac{1}{2} \lim_{n\to\infty} [\mathcal{W}_b(v_n, v_n) + \mathcal{W}_b(v_{n+m}, v_{n+m})].$$

But from (3.7)

$$\lim_{n\to\infty} \mathcal{W}_b^s(v_n, v_{n+m}) = 0,$$

using values from (3.10),

$$\Rightarrow \lim_{n\to\infty} \mathcal{W}_b^s(v_n, v_{n+m}) = 0 = \lim_{n\to\infty} \mathcal{W}(v_n, v_{n+m}). \tag{4.11}$$

Hence,

$$\lim_{n\to\infty} \mathcal{W}_b(v_n, v_{n+m}) = 0 = \lim_{n\to\infty} \mathcal{W}_b(v_n, p) = \lim_{n\to\infty} \mathcal{W}_b(p, p). \tag{4.12}$$

To show

$$\mathcal{W}_b(p, Tu_1) \leq 2k\alpha \mathcal{W}_b(p, u_1) \qquad \text{for all } u_1 \in \mathcal{M} \setminus \{p\} \text{ where, } k\alpha < 1. \tag{4.13}$$

We proceed as follows:

Since $\lim\limits_{n\to\infty} \mathcal{W}_b(v_n, p) = 0$ therefore, there exists $n \geq N$ such that

$$\mathcal{W}_b(v_n, p) \leq \frac{1}{3k}\mathcal{W}_b(u_1, p), \tag{4.14}$$

and

$$\begin{aligned}
\zeta(\alpha)\mathcal{W}_b(v_n, Tv_n) &\leq \mathcal{W}_b(v_n, Tv_n) \\
&\leq \mathcal{W}_b(v_n, v_{n+1}) \\
&\leq k\{\mathcal{W}_b(v_n, p) + \mathcal{W}_b(p, v_{n+1})\}
\end{aligned}$$

Using (4.13),

$$\mathcal{W}_b(v_n, p) \leq \frac{1}{3k}\mathcal{W}_b(u_1, p) \;\text{ and }\; \mathcal{W}_b(p, u_1) \leq \frac{1}{3k}\mathcal{W}_b(u_1, p),$$

$$\begin{aligned}
&\leq k\Big\{\frac{1}{3k}\mathcal{W}_b(p, u_1) + \frac{1}{3k}\mathcal{W}_b(p, u_1)\Big\} \\
&\leq \mathcal{W}_b(u_1, p) - \frac{1}{3}\mathcal{W}_b(u_1, p),
\end{aligned}$$

$$\leq \mathcal{W}_b(u_1, p) - k\mathcal{W}_b(p, v_n) \quad \text{from} \;\; (4.14)$$

consider

$$\begin{aligned}
\mathcal{W}_b(p, u_1) &\leq k\{\mathcal{W}_b(p, v_n) + \mathcal{W}_b(v_n, u_1)\} \\
&= k\{\mathcal{W}_b(p, v_n) + \mathcal{W}_b(u_1, v_n)\} \\
\Rightarrow \mathcal{W}_b(p, u_1) - k\mathcal{W}_b(p, v_n) &\leq k\mathcal{W}_b(u_1, v_n),
\end{aligned}$$

using above value the inequality becomes,

$$\zeta(\alpha)\mathcal{W}_b(v_n, Tv_n) \leq k\mathcal{W}_b(u_1, v_n)$$

$$\Rightarrow H_{wb}(Tv_n, Tu_1) \leq k\alpha\mathcal{W}_b(v_n, u_1). \tag{4.15}$$

Since,

$$\begin{aligned}
H_{wb}(Tv_n, Tu_1) &= \frac{1}{2}\{\psi_{wb}(Tv_n, Tu_1) + \psi_{wb}(Tu_1, Tv_n)\} \\
\Rightarrow 2H_{wb}(Tv_n, Tu_1) &= \psi_{wb}(Tv_n, Tu_1) + \psi_{wb}(Tu_1, Tv_n) \\
\Rightarrow 2H_{wb}(Tv_n, Tu_1) &\geq \psi_{wb}(Tv_n, Tu_1). \qquad (4.16)
\end{aligned}$$

Since $v_{n+1} \in Tv_n$, now using the definition of distance between two sets and the distance between a set and a point,

$$\begin{aligned}
\mathcal{W}_b(v_{n+1}, Tu_1) &\leq \psi_{wb}(Tv_n, Tu_1) \\
\Rightarrow \mathcal{W}_b(v_{n+1}, Tu_1) &\leq 2H_{wb}(Tv_n, Tu_1) \quad \text{(from (4.16))} \qquad (4.17) \\
\mathcal{W}_b(v_{n+1}, Tu_1) &\leq 2k\alpha\mathcal{W}_b(v_n, u_1) \\
\mathcal{W}_b(v_{n+1}, Tu_1) &\leq 2k\alpha\{\mathcal{W}_b(v_n, p) + \mathcal{W}_b(p, u_1)\} \\
\lim_{n\to\infty} \mathcal{W}_b(v_{n+1}, Tu_1) &\leq 2k\alpha\mathcal{W}_b(p, u_1). \qquad (4.18)
\end{aligned}$$

Consider,

$$\begin{aligned}
\mathcal{W}_b(v_{n+1}, Tu_1) &\leq k\{\mathcal{W}_b(v_{n+1}, p) + \mathcal{W}_b(p, Tu_1)\} \\
\lim_{n\to\infty} \mathcal{W}_b(v_{n+1}, Tu_1) &\leq k\mathcal{W}_b(p, Tu_1) \\
&\Rightarrow k \geq \lim_{n\to\infty} \frac{\mathcal{W}_b(v_{n+1}, Tu_1)}{\mathcal{W}_b(p, Tu_1)} \qquad (4.19)
\end{aligned}$$

again,

$$\begin{aligned}
\mathcal{W}_b(v_{n+1}, Tu_1) &\leq k\{\mathcal{W}_b(v_{n+1}, v_n) + \mathcal{W}_b(v_n, Tu_1)\} \\
&\leq k\mathcal{W}_b(v_{n+1}, v_n) + k^2\mathcal{W}_b(v_n, p) + k^2\mathcal{W}_b(p, Tu_1) \\
\Rightarrow \lim_{n\to\infty} \mathcal{W}_b(v_{n+1}, Tu_1) &\leq k^2\mathcal{W}_b(p, Tu_1) \qquad (4.20) \\
\lim_{n\to\infty} \mathcal{W}_b(v_{n+1}, Tu_1) &\leq \lim_{n\to\infty} \left[\frac{\mathcal{W}_b(v_{n+1}, Tu_1)}{\mathcal{W}_b(p, Tu_1)}\right]^2 \mathcal{W}_b(p, Tu_1) \qquad \text{(from (4.19))} \\
\Rightarrow \mathcal{W}_b(p, Tu_1) &\leq \lim_{n\to\infty} \mathcal{W}_b(v_{n+1}, Tu_1) \qquad (4.21)
\end{aligned}$$

(4.18) and (4.21) gives,

$$\mathcal{W}_b(p, Tu_1) \leq 2k\alpha \mathcal{W}_b(p, u_1) \quad \text{for all} \quad u_1 \in \mathcal{M} \setminus \{p\} \quad \text{and} k\alpha < 1. \tag{4.22}$$

Now claim that,

$$H_{wb}(Tu_1, Tp) \leq k\alpha \mathcal{W}_b(p, u_1) \quad \text{for} \quad \text{all} \quad u_1 \in \mathcal{W}_b.$$

Above statement clearly hold for $u_1 = p$.

Assume that $u_1 \neq p$, then for every $m \in \mathbb{N}$, there will be a $z_m \in Tu_1$ such that

$$\mathcal{W}_b(p, z_m) \leq \mathcal{W}_b(p, Tu_1) + \frac{1}{m}\mathcal{W}_b(p, u_1). \tag{4.23}$$

Now consider,

$$\begin{aligned}
\mathcal{W}_b(u_1, Tu_1) &\leq \mathcal{W}_b(u_1, z_m) \\
&\leq k\{\mathcal{W}_b(u_1, p) + \mathcal{W}_b(p, z_m)\} \\
&\leq k\{\mathcal{W}_b(u_1, p) + \mathcal{W}_b(p, Tu_1) + \frac{1}{m}\mathcal{W}_b(u_1, p)\} \\
\mathcal{W}_b(u_1, Tu_1) &\leq k\big[\mathcal{W}_b(u_1, p) + 2k\alpha \mathcal{W}_b(p, u_1) \\
&\quad + \frac{1}{m}\mathcal{W}_b(u_1, p)\big] \text{ (from (4.22) and (4.24))} \\
&= \big[1 + 2k\alpha + \frac{1}{m}\big]\mathcal{W}_b(u_1, p)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \frac{\mathcal{W}_b(u_1, Tu_1)}{\big[1 + 2k\alpha + \frac{1}{m}\big]} \leq \mathcal{W}_b(p, u_1). \\
&\Rightarrow H_{wb}(Tu_1, Tp) \leq k\alpha \mathcal{W}_b(p, u_1) \ \text{ for all } u_1 \in \mathcal{W}_b \text{ and } k\alpha < 1.
\end{aligned}$$

Finally

$$\begin{aligned}
\mathcal{W}_b(p, Tp) &= \lim_{n\to\infty} \mathcal{W}_b(v_{n+1}, Tp) \\
&\leq \lim_{n\to\infty} \psi_{wb}(Tv_n, Tp)
\end{aligned}$$

$$\leq 2 \lim_{n\to\infty} H_{wb}(Tv_n, Tp)$$
$$\leq 2\alpha \lim_{n\to\infty} \mathcal{W}_b(v_n, p) = 0.$$

From above calculation it is clear that $\mathcal{W}_b(p, Tp) = 0 = \mathcal{W}_b(p, Tp)$.
Since Tp is closed, $p \in \overline{Tp} = Tp$. □

Example 4.1.1.
Let $\mathcal{M} = \{0, \frac{1}{2}, 1\}$. The weak partial b-metric space $\mathcal{W}_b : \mathcal{M} \times \mathcal{M} \longrightarrow [0, \infty)$ is defined by

$$\mathcal{W}_b(u_1, u_2) = \frac{1}{2}\left|u_1 - u_2\right|^2 + \frac{1}{2}\max\{u_1, u_2\}.$$

Then $\mathcal{W}_b(0, 0) = 0$,

$$\mathcal{W}_b\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}, \mathcal{W}_b(1, 1) = \frac{1}{2},$$

$$\mathcal{W}_b\left(0, \frac{1}{2}\right) = \mathcal{W}_b\left(\frac{1}{2}, 0\right) = \frac{3}{8},$$

$$\mathcal{W}_b\left(\frac{1}{2}, 1\right) = \mathcal{W}_b\left(1, \frac{1}{2}\right) = \frac{5}{8}$$

and $\mathcal{W}_b(1, 0) = \mathcal{W}_b(0, 1) = 1$.
It can easily be seen that $(\mathcal{M}, \mathcal{W}_b)$ is weak partial b-metric space with $k = 2$.
Let $T : \mathcal{M} \longrightarrow CB^{wb}(\mathcal{M})$ be the mapping defined by $T(0) = \{0\}$,
$T\left(\frac{1}{2}\right) = \left\{0, \frac{1}{2}\right\}$ and $T(1) = \left\{0, \frac{1}{2}, 1\right\}$.
By choosing $\alpha = 0.5$, the definition of ζ that is (3.2) gives $\zeta(\alpha) = 1$.
First, the contraction condition of the theorem is to be proved that is,

$$\zeta(\alpha)\mathcal{W}_b(u_1, Tu_1) \leq \mathcal{W}_b(u_1, u_2) \Rightarrow H_{wb}(Tu_1 \setminus \{u_1\}, Tu_2 \setminus \{u_2\}) \leq k\alpha\mathcal{W}(u_1, u_2)$$

To prove above condition the following cases are to be considered.

Case 1.

Choose $x = 0$,

$$\zeta(\alpha)\mathcal{W}_b(0, T(0)) = \mathcal{W}_b(0, 0)$$

$$
\begin{aligned}
&= 0 \\
&\leq \mathcal{W}_b(0, u_2) \qquad \text{for all} \quad u_1 \in \mathcal{M} \\
\Rightarrow \zeta(\alpha)\mathcal{W}_b(0, T(0)) &\leq \mathcal{W}_b(0, u_2).
\end{aligned}
$$

Now, for $u_2 = 0$

$$
\begin{aligned}
H_{wb}(T(0) \setminus \{0\}, T(0) \setminus \{0\}) &= H_{wb}(\phi, \phi) \\
&= 0 \\
&\leq \alpha \mathcal{W}_b(0, 0) \\
\Rightarrow H_{wb}(T(0) \setminus \{0\}, T(0) \setminus \{0\}) &\leq k\alpha \mathcal{W}_b(0, 0).
\end{aligned}
$$

For $u_2 = \dfrac{1}{2}$

$$
\begin{aligned}
H_{wb}\big(T(0) \setminus \{0\}, T\left(\frac{1}{2}\right) \setminus \left\{\frac{1}{2}\right\}\big) &= H_{wb}(\phi, \{0\}) \\
&= 0 \\
&\leq \alpha \mathcal{W}_b\left(0, \frac{1}{2}\right) \\
\Rightarrow H_{wb}\big(T(0) \setminus \{0\}, T\left(\frac{1}{2}\right) \setminus \left\{\frac{1}{2}\right\}\big) &\leq k\alpha \mathcal{W}_b\left(0, \frac{1}{2}\right).
\end{aligned}
$$

For $u_2 = 1$

$$
\begin{aligned}
H_{wb}(T(0) \setminus \{0\}, T(1) \setminus \{1\}) &= H_{wb}\left(\phi, \left\{0, \frac{1}{2}\right\}\right) \\
&= 0 \\
&\leq \alpha \mathcal{W}_b(0, 1) \\
\Rightarrow H_{wb}(T(0) \setminus \{0\}, T(1) \setminus \{1\}) &\leq k\alpha \mathcal{W}_b(0, 1).
\end{aligned}
$$

Case 2.

The three possibilities can be discussed as

At $u_1 = \dfrac{1}{2}$

$$\begin{aligned}
\zeta(\alpha)\mathcal{W}_b\left(\frac{1}{2},T\left(\frac{1}{2}\right)\right) &= \mathcal{W}_b\left(\frac{1}{2},\left\{0,\frac{1}{2}\right\}\right)\\
&= \inf\left\{\mathcal{W}_b\left(\frac{3}{8},\frac{1}{4}\right)\right\}\\
&= \frac{1}{4}\\
&\leq \mathcal{W}_b\left(\frac{1}{2},u_2\right) \qquad \text{for all } \ u_2\in\mathcal{M}\backslash\left\{\frac{1}{2}\right\}\\
\Rightarrow \zeta(\alpha)\mathcal{W}_b\left(\frac{1}{2},T\left(\frac{1}{2}\right)\right) &\leq \mathcal{W}_b\left(\frac{1}{2},u_2\right).
\end{aligned}$$

Now for $u_2=0$, the relation becomes

$$\begin{aligned}
H_{wb}\left(T\left(\frac{1}{2}\right)\setminus\left\{\frac{1}{2}\right\},T(0)\setminus\{0\}\right) &= H_{wb}(\{0\},\phi)\\
&= 0\\
&\leq k\alpha\mathcal{W}_b\left(\frac{1}{2},0\right)\\
\Rightarrow H_{wb}\left(T\left(\frac{1}{2}\right)\setminus\{\frac{1}{2}\},T(0)\setminus\{0\}\right) &\leq k\alpha\mathcal{W}_b\left(\frac{1}{2},0\right).
\end{aligned}$$

If $u_2=1$, then

$$\begin{aligned}
H_{wb}\left(T\left(\frac{1}{2}\right)\setminus\left\{\frac{1}{2}\right\},T(1)\setminus\{1\}\right) &= H_{wb}\left(\{0\},\left\{0,\frac{1}{2}\right\}\right)\\
&= \frac{3}{16}\\
&\leq k\alpha\mathcal{W}_b\left(\frac{1}{2},1\right)\\
\Rightarrow H_{wb}\left(T\left(\frac{1}{2}\right)\setminus\left\{\frac{1}{2}\right\},T(1)\setminus\{1\}\right) &\leq k\alpha\mathcal{W}_b\left(\frac{1}{2},1\right).
\end{aligned}$$

Case 3.

At $u_1=1$

$$\begin{aligned}
\zeta(\alpha)\mathcal{W}_b(1,T(1)) &= \mathcal{W}_b\left(1,\left\{0,\frac{1}{2},1\right\}\right)\\
&= \mathcal{W}_b(1,1)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \\
&\leq \mathcal{W}_b(1, u_2) \qquad \text{for all} \quad u_2 \in \mathcal{M}\backslash\{1\} \\
\Rightarrow \zeta(\alpha)\mathcal{W}_b(1, T(1)) &\leq \mathcal{W}_b(1, u_2).
\end{aligned}
$$

For $u_2 = 0$

$$
\begin{aligned}
H_{wb}(T(1)\backslash\{1\}, T(0)\backslash\{0\}) &= H_{wb}\left(\left\{0, \frac{1}{2}, 1\right\}, \phi\right) \\
&= 0 \\
&\leq k\alpha\mathcal{W}(1, 0) \\
H_{wb}(T(1)\backslash\{1\}, T(0)\backslash\{0\}) &\leq k\alpha\mathcal{W}(1, 0).
\end{aligned}
$$

If $u_2 = \dfrac{1}{2}$

$$
\begin{aligned}
H_{wb}\left(T(1) \setminus \{1\}, T\left(\frac{1}{2}\right) \setminus \left\{\frac{1}{2}\right\}\right) &= H_{wb}\left(\left\{0, \frac{1}{2}\right\}, \{0\}\right) \\
&= \frac{3}{16} \\
&\leq k\alpha\mathcal{W}_b\left(1, \frac{1}{2}\right) \\
\Rightarrow H_{wb}\left(T(1) \setminus \{1\}, T\left(\frac{1}{2}\right) \setminus \left\{\frac{1}{2}\right\}\right) &\leq k\alpha\mathcal{W}_b\left(1, \frac{1}{2}\right).
\end{aligned}
$$

It is clear from above cases that contraction condition is satisfied. Now, the second condition given in equation (3.4) is to be inquired that is,

$$
\mathcal{W}_b(u_2, u_3) \leq \beta H_{wb}(Tu_2, Tu_3),
$$

it will be checked by choosing $\beta = 2$ and by discussing following situations for u_1 and u_2 given below

(i) If $u_1 = 0$, then $u_2 \in T(0)$.

Since $T(0) = \{0\}$

This implies that $u_2 = 0$, then there exists $u_3 \in T(u_2)$ such that

$$0 = \mathcal{W}_b(u_2, u_3) \leq 2H_w(Tu_2, Tu_3).$$

If $u_1 = \dfrac{1}{2}$, then $u_2 \in T\left(\dfrac{1}{2}\right)$.
Since $T\left(\dfrac{1}{2}\right) = \left\{0, \dfrac{1}{2}\right\}$
This implies that for $u_2 = 0$, then there exists $u_3 \in T(u_2)$ such that

$$0 = \mathcal{W}(u_2, u_3) \leq 2H_{wb}(Tu_2, Tu_3),$$

also, for $u_2 = \dfrac{1}{2}$, there exists $u_3 \in T(u_2)$ that is $T\left(\dfrac{1}{2}\right) = \left\{0, \dfrac{1}{2}\right\}$.
For $u_3 = 0$

$$\begin{aligned}\frac{3}{8} = \mathcal{W}_b\left(\frac{1}{2}, 0\right) &\leq 2H_{wb}\left(T\left(\frac{1}{2}\right), T(0)\right)\\ &= 2\left(\frac{3}{8}\right)\\ &= \frac{3}{4}\end{aligned}$$

For $u_3 = \dfrac{1}{2}$

$$\begin{aligned}\frac{1}{4} = \mathcal{W}_b\left(\frac{1}{2}, \frac{1}{2}\right) &\leq 2H_{wb}\left(T\left(\frac{1}{2}\right), T\left(\frac{1}{2}\right)\right)\\ &= 2\left(\frac{1}{4}\right)\\ &= \frac{1}{2}\\ \Rightarrow \mathcal{W}_b\left(\frac{1}{2}, \frac{1}{2}\right) &\leq 2H_{wb}\left(T\left(\frac{1}{2}\right), T\left(\frac{1}{2}\right)\right)\end{aligned}$$

(ii) If $u_1 = 1$, then $u_2 \in T(1)$.
$\Rightarrow u_2 \in T(1) = \left\{0, \dfrac{1}{2}, 1\right\}$,
if $u_2 = 0 \Rightarrow u_3 = 0$,

the result holds for this situation.
If $u_2 = \dfrac{1}{2}$,
then for $u_3 = 0$, the relation holds.
For $u_2 = \dfrac{1}{2} = u_3$

$$
\begin{aligned}
\mathcal{W}_b\left(\frac{1}{2}, \frac{1}{2}\right) &= \frac{1}{4} \\
&\leq 2H_{wb}\left(T\left(\frac{1}{2}\right), T\left(\frac{1}{2}\right)\right) = \frac{1}{2} \\
\Rightarrow \mathcal{W}_b\left(\frac{1}{2}, \frac{1}{2}\right) &\leq 2H_{wb}\left(T\left(\frac{1}{2}\right), T\left(\frac{1}{2}\right)\right)
\end{aligned}
$$

Now, for $u_2 = \dfrac{1}{2}$ and $u_3 = 1$

$$
\mathcal{W}_b\left(\frac{1}{2}, 1\right) = \frac{5}{8}
$$

$$
\begin{aligned}
\frac{5}{8} &= \mathcal{W}_b\left(\frac{1}{2}, 1\right) \\
&\leq 2H_{wb}\left(T(1), T\left(\frac{1}{2}\right)\right) = \frac{7}{8}
\end{aligned}
$$

From the above discussion it is clear that all the conditions of Theorem (4.1) are satisfied. So, T has fixed points.

Chapter 5

Conclusion

The dissertation is brought to end as follows:

• Some abstract spaces such as metric space, b-metric space and partial metric space are elaborated in the beginning of the dissertation. These elaborations are necessary for the further discussions.

• Some important land marks of the fixed point theory are discussed in systematic manner. To achieve the goal of better understanding a few important results are provided without proof.

• The platform of weak partial metric space is used for this research. Aydi et al. introduced a Suzuki-type multivalued contraction on weak partial metric spaces and provided a fixed point result. The detailed review of the article has been presented in this thesis. The article provides a fixed point theorem established on H_w-type Hausdorff metric space in the context of Suzuki-type multivalued contraction.

• Having inspiration from the paper of Kanwal et al., the result of Aydi et al. is extended on weak partial b-metric spaces over a Suzuki-type multivalued contraction. This result is validated by an example.

• In future one can provide

(i) the application of the result.

(ii) this result can further be extended by using the idea of extended b-metric spaces.

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