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Planar Central Configurations of Restricted Six-Body Problems

by

Hamid Khan

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in the

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To my parents and teachers

CERTIFICATE OF APPROVAL

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Abstract

In this thesis the Central configuration of an isosceles trapezoid for 5 body is discussed. We also analyzed the motion of infinitesimal body m_6 in the gravitational field of 5 massive bodies $m_1 - m_5$. We obtained different equilibrium points ranging between $6 - 8$. Maximum equilibrium points are unstable but couple of stable equilibrium points exist in each case. Interestingly, all stable points are along the y-axis on or off the isosceles trapezoid. There are no equilibrium points off the coordinate axes and along x-axis.

Contents

List of Figures

Abbreviations

2BP	Two-Body Problem
3BP	Three-Body Problem
4BP	Four-Body Problem
CC	Central Configuration
R6BP	Restricted Six-Body Problem
SI	System International
T5BP	Trapezoid Five-Body Problem

Symbols

symbol	name	unit
$\mathbf{F}$	Gravitational Force	Newton
G	Universal Gravitational Constant	$m^3kg^{-1}s^{-2}$
r	Distance	Meter
$\mathbf{P}$	Linear Momentum	$kgms^{-1}$
$\mathbf{L}$	Angular Momentum	kgm^2s^{-1}
m_i	Point Mass	kg

Chapter 1

Introduction

In mechanics the n-body problem is the problem in which the motion of individual has been predicted in the group of celestial bodies related with each other gravitationally. Solution of this problem is the desire and need to understand the motion of sun, moon and the other stars. Newton in 1687 gave the idea of n-body problem [1]. Moreover by using his universal law of gravity Newton solved two body problem but for $n \geq 3$ it has no significant solution. In the last few centuries, astronomers and mathematicians have continued work on the n-body problems. Angel and Gabern [2] worked on 4 and 5 body problems in which they focused on the motion of a small particle nearby Lagrangian point in Jupiter and Sun system. Roy and Steves [3] introduced the restricted 4-body problem which was the intermediate step in the exploitation of three dimensional 4-body problem. Mather and Mcgehee [4] studied the 4-body and found the existence of the solutions of collinear 4-body problem in which they considered four masses on a straight line using inverse square law.

Central configurations (CCs) [5] is a special form of adjustment of point masses attracting each other by Newton's gravity law with the following property the gravitational acceleration vector produced by all others on each mass should point to each other toward the center of mass and proportional to the distance to the center of mass. (CC's) of 3 and 4-body problem is very old and basic problem in celestial mechanics. The configurations in which the Newtonian acceleration on body is taken equal to the position vector of that body multiplied by a constant with respect to the center of mass of that body.

In the study of few body problem the (CC's) is counted in one of the fundamental and important topic. Therefore, in particular few-body problems in central and general configurations have received a lot of attention of researchers over the years. For $n \geq 4$ due to the complexity of problems containing greater number of bodies the study on the (CC's) of n-body problem is limited. In the available literature the main focus for $n \geq 4$ is on the restricted problems. This open a ground for the study of the (CC's) 5-body problem. In the present work we study the (CC's) generalized collinear 4 and 5-body problems. In addition, we also answer the motion of test mass in the five main gravitational fields. We also investigate the position of equilibrium points of test mass or stability of these points.

Several methods are used to study the few body problems. For example, Marchesin [6] studied the restricted rhomboidal 5-body problems and stability of its periodic solutions. Roberts [7] examined the relative equilibria in 5 body problem which contain four bodies $(m_1, m_2, m_3, m_4) = (1, 1, 1, 1)$ taken at the vertices of a quadrilateral (rumbas) with equal masses at the opposite vertices and remaining at the center. Josep et al. [8] studied the (CC's) of $m + 1$ bodies in which one mass is considered large and other masses are taken infinitesimal. Xia [9] used the analytical continuation method to study the (CC's) with small masses and they found exact number of (CC's) for some sets of n masses. Albouy [10] studied the relation between geometrical properties of (CC's) and the masses and proved that in a four body problem in plane if they place equal masses of particles on one diagonal then the convex (CC's) is symmetric with respect to other diagonal.

Chapter 2

Preliminaries

This chapter contains some important definitions, concepts, governing laws which are essential to understand the work presented in next chapters.

2.1 Basic Definitions

2.1.1 Motion [11]

"Motion is the action used to change the location or position of an object with respect to the surroundings over time."

2.1.2 Mechanics [11]

"Mechanics is a branch of physics concerned with motion or change in position of physical objects. It is sometimes further subdivided into:

1. **Kinematics**, which is concerned with the geometry of the motion,
2. **Dynamics**, which is concerned with the physical causes of the motion,
3. **Statics**, which is concerned with conditions under which no motion is apparent."

2.1.3 Scalar [11]

"Various quantities of physics, such as length, mass and time, requires for their specification a single real number (apart from units of measurement which are decided upon in advance). Such quantities are called **Scalars** and the real number is called the magnitude of the quantity."

2.1.4 Vector [11]

"Other quantities of physics, such as displacement, velocity, momentum, force etc require for their specification a direction as well as magnitude. Such quantities are called **Vectors**."

2.1.5 Field [11]

"A field is a physical quantity associated with every point of spacetime. The physical quantity may be either in vector form, scalar form or tensor form."

2.1.6 Scalar Field [11]

"If at every point in a region, a scalar function has a defined value, the region is called a scalar field. i.e.,

$$f : \mathbb{R}^3 \to \mathbb{R},$$

e.g. temperature and pressure fields around the earth."

2.1.7 Vector Field [11]

"If at every point in a region, a vector function has a defined value, the region is called a vector field.

$$V : \mathbb{R}^3 \to \mathbb{R}^3,$$

e.g. tangent vector around a smooth curve."

2.1.8 Conservative Vector Field [11]

"A vector field $\mathbf{V}$ is conservative if and only if there exists a continuously differentiable scalar field f such that $\mathbf{V} = -\nabla f$ or equivalently if and only if

$$\nabla \times \mathbf{V} = \texttt{Curl}\mathbf{V} = \mathbf{0}."$$

2.1.9 Uniform Force Field [11]

"A force field which has constant magnitude and direction is called a uniform or constant force field. If the direction of the field is taken as negative z direction and magnitude is constant $F_0 > 0$, then the force field is given by

$$\mathbf{F} = -F_0\hat{\mathbf{k}}."$$

2.1.10 Central Force [11]

"Suppose that a force acting on a particle of mass m such that
(a) it is always directed from m toward or away from a fixed point O,
(b) its magnitude depends only on the distance r from O.
then we call the force a central force or central force field with O as the center of force. In symbols $\mathbf{F}$ is a central force if and only if

$$\mathbf{F} = f(r)\mathbf{r}_1 = f(r)\tfrac{\mathbf{r}}{r},$$

where $\mathbf{r}_1 = \frac{\mathbf{r}}{r}$ is a unit vector in the direction of $\mathbf{r}$. The central force is one of attraction towards O or repulsion from O according as *f(r)* < 0 or *f(r)* > 0 respectively."

2.1.11 Degree of Freedom [11]

"The number of coordinates required to specify the position of a system of one or more particles is called number of degree of freedom of the system.
Example: A particle moving freely in space requires 3 coordinates, e.g. (x, y, z), to specify its position. Thus the number of degree of freedom is 3."

2.1.12 Center of Mass [11]

"Let $r_1, r_2, ..., r_n$ be the position vector of a system of n particles of masses $m_1, m_2, ...m_n$ respectively. The center of mass or centroid of the system of particles is defined as that point having position vector

$$\hat{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + ... + m_n\mathbf{r}_n}{m_1 + m_2 + ... + m_n} = \frac{1}{\mathbf{M}}\sum_{\nu=1}^{n} m_\nu \mathbf{r}_\nu,$$

where

$$\mathbf{M} = \sum_{\nu=1}^{n} m_\nu,$$

is the total mass of the system."

2.1.13 Center of Gravity [11]

"If a system of particles is in a uniform gravitational field, the center of mass is sometimes called the center of gravity."

2.1.14 Torque [11]

"If a particle with a position vector $\mathbf{r}$ moves in a force field $\mathbf{F}$, we define $\boldsymbol{\tau}$ as torque or moment of the force as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

The magnitude of $\boldsymbol{\tau}$ is

$$\tau = rF \sin \theta.$$

The magnitude of torque is a measure of the turning effect produced on the particle by the force."

2.1.15 Momentum [11]

"The linear momentum $\mathbf{p}$ of an object with mass m and velocity $\mathbf{v}$ is defined as:

$$\mathbf{p} = m\mathbf{v}.$$

Under certain circumstances the linear momentum of a system is conserved. The linear momentum of a particle is related to the net force acting on that object:

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt}.$$

The rate of change of linear momentum of a particle is equal to the net force acting on the object, and is pointed in the direction of the force. If the net force acting on an object is zero, its linear momentum is constant (conservation of linear momentum). The total linear momentum $\mathbf{p}$ of a system of particles is defined as the vector sum of the individual linear momentum.

$$\mathbf{p} = \sum_{1}^{n} \mathbf{p}_i."$$

2.1.16 Point-like Particle [11]

"A point-like particle is an idealization of particles mostly used in different fields of physics. Its defining features is the lacks of spatial extension:being zero-dimensional, it does not take up space. A point-like particle is an appropriate representation of an object whose structure, size and shape is irrelevant in a given context. e.g., from far away, a finite-size mass (object) will look like a point-like particle."

2.1.17 Angular Momentum [11]

"Angular momentum for a point-like particle of mass m with linear momentum $\mathbf{p}$ about a point O, defined by the equation

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

where $\mathbf{r}$ is the vector from the point O to the particle. The torque about the point O acting on the particle is equal to the rate of change of the angular momentum about the point O of the particle i.e.,

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}."$$

2.1.18 Lorentz Transformation [11]

"Lorentz transformation is the relationship between two different coordinate frames that move at a constant velocity and are relative to each other. The name of the transformation comes from a Dutch physicist Hendrik Lorentz. There are two frames of reference, which are"

2.1.18.1 Inertial Frame of Reference

"A frame of reference that remains at rest or moves with constant velocity with respect to other frames of reference is called inertial frame of reference. Actually, an unaccelerated frame of reference is an inertial frame of reference. In this frame of reference a body does not acted upon by external forces. Newton's laws of motion are valid in all inertial frames of reference. All inertial frames of reference are equivalent."

2.1.18.2 Non-Inertial Frame of Reference

"A non-inertial reference frame is a frame of reference that is undergoing acceleration with respect to an inertial frame. While the laws of motion are the same in all inertial frames, in non-inertial frames, they vary from frame to frame depending on the acceleration."

2.1.19 Lagrange Points [11]

"A point in space where a small body with negligible mass under the gravitational influence of two large bodies will remain at rest relative to the larger ones. These points are locations in an orbital arrangement of two large bodies where a third smaller body, affected solely by gravity, is capable of maintaining a stable position relative to the two larger bodies. A lagrange point is also known as a equilibrium point and Liberation point named after a French mathematician and atronomer Joseph-Louis Lagrange. He was first to find these equlibrium points for the earth, sun, and moon system. He found five points out of these three are collinear."

2.1.20 Equilibrium Solution [11]

"The **Equilibrium solution** can guide us through the behavior of the equation that represents the problem without actually solving it. These solutions can be found only if we meet the sufficient condition of all rates equal to zero. If we have two variables then

$$\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = ... = x^{(n)} = y^{(n)} = 0.$$

These solutions may be stable or unstable. The stable solutions regarding in celestial Mechanics helps us find parking spaces where if a satellite or any object placed, it will remain there for ever. These type of places are also found along the Jupiter's orbital path where bodies called trojan are present. These equilibrium points with respect to Celestial Mechanics are also called Lagrange points named after a French mathematician and astronomer Joseph-Louis Lagrange. He was first to find these equilibrium points for the Sun-Earth system. He found that three of these five points were collinear."

2.1.21 Holonomic and non holonomic Constraints [11]

"In classical mechanics, a constraint on a system is a parameter that the system must obey. The limitation on the motion are often called constraints. If the constraints condition can be expressed as an equation,

$$\phi(r_1, r_2, ...r_n, t) = 0,$$

connecting the position vector of the particles and the time, then the constraints are called holonomic, otherwise non-holonomic."

2.1.22 Basin of Attraction [11]

"Newton method is used to find the roots of equations but Arthur Cayley found that if the roots of a function are already know then Newton's method can guide to another problem that is which initial guesses iterate to which roots and the region of these initial guesses is called basins of attraction of the roots."

2.1.23 Galilean Transformation [11]

"In physics, a Galilean transformation is used to transform between the coordinates of two reference frames which differ only by constant relative motion within the constructs of Newtonian physics. Without the translations in space and time the group is the homogeneous Galilean group. Galilean transformations, also called Newtonian transformations, set of equations in classical physics that relate the space and time coordinates of two systems moving at a constant velocity relative to each other."

2.1.24 Celestial Mechanics[11]

"Celestial mechanics is the branch of astronomy that deals with the motions of objects in outer space. Historically, celestial mechanics applies principles of physics (classical mechanics) to astronomical objects, such as stars and planets, to produce ephemeris data. Actually celestial mechanics is the science devoted to the study of the motion of the celestial bodies on the basis of the laws of gravitation. It was founded by Newton and it is the oldest of the chapters of Physical Astronomy."

2.1.25 Kepler's Laws of Planetary Motion [11]

"Kepler's three laws of planetary motion can be described as follows:

1. . Keplers first law states that every planet moves along an ellipse, with the Sun located at a focus of the ellipse. An ellipse is defined as the set of all points such that the sum of the distance from each point to two foci is a constant.

2. Keplers second law states that a planet moves in its ellipse so that the line between it and the Sun placed at a focus sweeps out equal areas in equal times.

3. The cube of the semi major axis of the planetary orbits are proportional to the square of the planets periods of revolution. Mathematically, Kepler's third law can be written as:

   $$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3,$$

   where T is the time period, r is the semi major axis, M_s is the mass of sun and G is the universal gravitational constant."

2.1.26 Procedure for Stability Analysis and Equilibrium Points

We need to follow the following steps to check the stability of equilibrium points [11].

1. "Determine the equilibrium points, $\mathbf{x}^*$, solving $\Omega(\mathbf{x}^*) = \mathbf{0}$.
2. Construct the Jacobian matrix, $J(\mathbf{x}^*) = \frac{\partial \Omega}{\partial \mathbf{x}}$.
3. Compute eigenvalues of $\Omega(\mathbf{x}^*)$: $det|\Omega(\mathbf{x}^*) - \lambda\ I| = 0$.
4. Stability or instability of $\mathbf{x}^*$ based on the real parts of eigenvalues.
5. Point is stable, if all eigenvalues have real parts negative.
6. Unstable, if at least one of the eigenvalues have a real part greater than zero.
7. Otherwise, there is no conclusion, (i.e, require an investigation of higher order terms)."

2.1.27 Newton's Laws of Motion [11]

"The following three laws of motion given by Newton are considered the axioms of mechanics:

1. **First Law of Motion**

   Every particle persists in a state of rest or of uniform motion in a straight line unless acted upon by a force.

2. **Second Law of Motion**

   If $\mathbf{F}$ is the external force acting on a particle of mass m which as a reaction is moving with velocity $\mathbf{v}$, then

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}.$$

If m is independent of time this becomes

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a},$$

where $\mathbf{a}$ is the acceleration of the particle.

3. **Third Law of Motion**

   For every action, there is an equal and opposite reaction."

2.1.28 Newton's Universal Law of Gravitation [11]

"Every particle of matter in the universe attracts every other particle of matter with a force which is directly proportional to the product of the masses and inversely proportional to the square of the distance between them. Hence, for any two particles separated by a distance r, the magnitude of the gravitational force $\mathbf{F}$ is:

$$\mathbf{F} = G\frac{m_1 m_2}{r^3}\mathbf{r}$$

where G is universal gravitational constant. Its numerical value in SI units is $6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$."

2.2 Two Body Problem [12]

"The two-body problem, first studied and resolved by Newton, states: Suppose that t is given at some time the positions and velocities of two heavy bodies moving under their mutual gravitational force, then what should be their location and velocity t at any other time, if the masses are known."
Earth circling around a sun, two stars circling around each other, orbiting a satellite, for instance. Because of the facts below, the 2BP problem is most important.

1. In celestial mechanics, it is the only gravity problem, apart from very limited solutions to the 3BP for which I have a detailed and a suitable solution.

2. A large scope of realistic elliptic motion issues can be viewed as approximately 2BP.

3. In order to provide approximate orbital parameters and forecasts, the two-body solution may be used or deliver as a initial point for producing analytical solutions that are accurate for higher precision orders.

2.3 The Solution to the Two-Body Problem [13]

"Newton's universal gravitational law is the governing law for the two bodies:

$$\mathbf{F} = G\frac{m_1 m_2}{d^3}\mathbf{d}, \tag{2.1}$$

For two masses, m_1 and m_2 are separated by a $\mathbf{d}$ distance, and the universal gravitational constant is G."

The purpose here is to decide if the initial locations and velocities are known, the direction of the particles for some time t. The force of attraction $\mathbf{F}_{12}$ in Figure 2.1 is directed towards m_1 along d, while the force $\mathbf{F}_{21}$ on m_2 is directed in the opposite direction. According to Newton's third law of motion,

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \tag{2.2}$$

From Figure 2.1,

$$\mathbf{F}_{12} = G\frac{m_1 m_2}{d^3}\mathbf{d}. \tag{2.3}$$

Particles under their gravity equations are given by (2.1) and (2.2) respectively, contributing Newton's 2nd law and by equations (2.1) and (2.2).

$$m_1\ddot{\mathbf{d}}_1 = G\frac{m_1 m_2}{d^3}\mathbf{d}, \tag{2.4}$$

and,

$$m_2\ddot{\mathbf{d}}_2 = -G\frac{m_1 m_2}{d^3}\mathbf{d}, \tag{2.5}$$

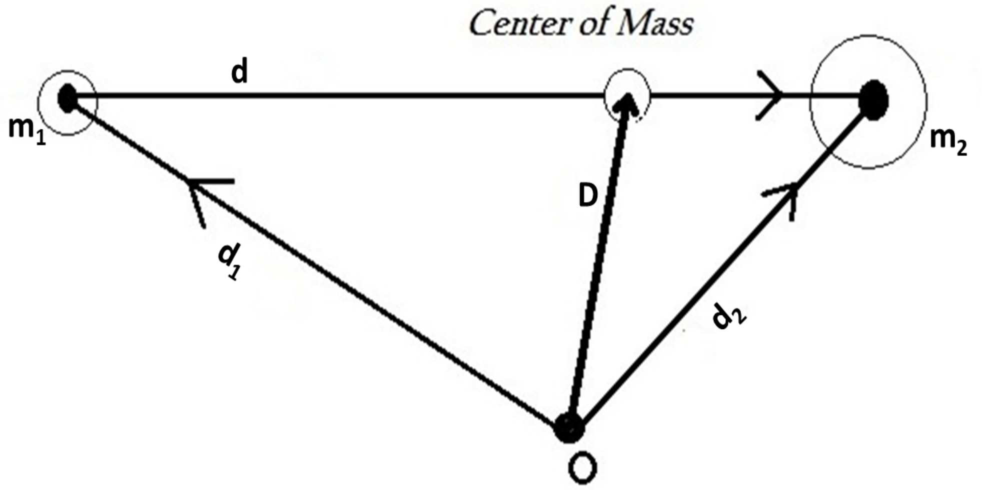

FIGURE 2.1: Center of mass

Where the location vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are from the position point O, as shown in Figure 2.1. When the equations (2.4) and (2.5) are applied, we obtained:

$$m_1\ddot{\mathbf{d}}_1 + m_2\ddot{\mathbf{d}}_2 = \mathbf{0}. \tag{2.6}$$

The integration of the equations above yields:

$$m_1\dot{\mathbf{d}}_1 + m_2\dot{\mathbf{d}}_2 = \mathbf{k}_1, \tag{2.7}$$

The total linear momentum of the system is a constant, i.e., $m_1\mathbf{v}_{m1}+m_2\mathbf{v}_{m_2} = \mathbf{k}_1$. Again integrating equation (2.7) implies that:

$$m_1\mathbf{d}_1 + m_2\mathbf{d}_2 = \mathbf{k}_1 t + \mathbf{k}_2, \tag{2.8}$$

Where $\mathbf{k}_1$ and $\mathbf{k}_2$ represent the constant of integration. Using 2BP's description of the centre of mass, $\mathbf{D}$ is defined as:

$$\begin{aligned}(m_1 + m_2)\mathbf{D} &= m_1\mathbf{d}_1 + m_2\mathbf{d}_2,\\ m_t\mathbf{D} &= m_1\mathbf{d}_1 + m_2\mathbf{d}_2, \end{aligned} \tag{2.9}$$

where $m_t = m_1 + m_2$. We get the derivative of the (2.9) equation and compare it with the (2.7) equation.

$$m_t\dot{\mathbf{D}} = \mathbf{k}_1 \quad \Rightarrow \quad \dot{\mathbf{D}} = \frac{\mathbf{k}_1}{m_t} = constant$$

i.e. linear momentum of the system is constant i.e., $m_1\mathbf{v}_{m1} + m_2\mathbf{v}_{m_2} = \mathbf{k}_1$. Again integrating equation (2.7) implies that:

$$m_1\mathbf{d}_1 + m_2\mathbf{d}_2 = \mathbf{k}_1 t + \mathbf{k}_2, \tag{2.10}$$

Where $\mathbf{k}_1$and $\mathbf{k}_2$ represent the constant of integration. Using 2BP's description of the centre of mass, $\mathbf{D}$ is defined as $\mathbf{D}$:

$$(m_1 + m_2)\mathbf{D} = m_1\mathbf{d}_1 + m_2\mathbf{d}_2,$$

$$m_t\mathbf{D} = m_1\mathbf{d}_1 + m_2\mathbf{d}_2, \tag{2.11}$$

where $m_t = m_1 + m_2$. We get the derivative of the (2.9) equation and compare it with the (2.7) equation.

$$m_t\dot{\mathbf{D}} = \mathbf{k}_1$$

$$\dot{\mathbf{D}} = \frac{\mathbf{k}_1}{m_t} = constant$$

show that $\dot{\mathbf{D}} = \mathbf{v}_c$ is constant.

Subtracting (2.6) from (2.5) from the equations gives:

$$\ddot{\mathbf{d}}_1 - \ddot{\mathbf{d}}_2 = \frac{Gm_2}{d^3}d + \frac{Gm_1}{d^3}\mathbf{d},$$

$$\ddot{\mathbf{d}}_1 - \ddot{\mathbf{d}}_2 = G(m_1 + m_2)\frac{\mathbf{d}}{d^3} \tag{2.12}$$

$$\ddot{\mathbf{d}} = \beta\frac{\mathbf{d}}{d^3},$$

$$\Rightarrow \ddot{\mathbf{d}} - \beta \frac{\mathbf{d}}{d^3} = \mathbf{0}, \tag{2.13}$$

Where a reduced mass is known as $\beta = G(m1 + m2)$ and $\mathbf{d}1 - \mathbf{d}2 = -\mathbf{d}$, is shown in figure 2.1.
Multiplying $\mathbf{r}$ with the (2.12) equation, we get:

$$\mathbf{d} \times \beta\ddot{\mathbf{d}} + \frac{\beta^2}{d^3}\mathbf{d} \times \mathbf{d} = \mathbf{0}$$

$$\Rightarrow \mathbf{d} \times \ddot{\mathbf{d}} = \mathbf{0}, \tag{2.14}$$

integrating above equation yields:

$$\mathbf{d} \times \dot{\mathbf{d}} = \mathbf{H}, \tag{2.15}$$

Where $\mathbf{H}$ is integration constant. We should write the equation (2.12),

$$\Rightarrow \mathbf{d} \times \beta\ddot{\mathbf{d}} = \mathbf{0},$$
$$\Rightarrow \mathbf{d} \times \mathbf{F} = \mathbf{0}, \tag{2.16}$$

where,

$$\mathbf{F} = \beta\ddot{\mathbf{d}}.$$

The description of torque and angular momentum is taken from Chapter 2:

$$T = \mathbf{d} \times \mathbf{F}, \tag{2.17}$$

After comparison equ (2.14) and (2.15), we obtained:

$$T = \mathbf{d} \times \mathbf{F} = \mathbf{0}, \tag{2.18}$$

$$\frac{d\mathbf{L}}{dt} = \mathbf{0}$$

$$\mathbf{L}\text{=conserved.}$$

This means that the angular momentum is conserved.

2.4 Velocity and Acceleration Components of Radial and Transverse

The velocity components along and perpendicular to the radius vector joining m_1 to m_2 are $\dot{d}$ and $d\dot{\theta}$ if the polar co-ordinates d and θ are chosen in this region as shown in fig (2.2), then,

$$\dot{\mathbf{d}} = \dot{d}\mathbf{i} + d\dot{\theta}\mathbf{j}, \tag{2.19}$$

Where the unit vectors $\hat{i}$ and $\hat{j}$ are located along and perpendicular to the vector radius. Thus, by means of equ (2.13) and (2.16),

$$\mathbf{d} \times (\dot{d}\mathbf{i} + d\dot{\theta}\mathbf{j}) = d^2\dot{\theta}\mathbf{k} = L\mathbf{k}, \tag{2.20}$$

Where the unit vector $\mathbf{k}$ is Perpendicular to the orbital plane. Which can be written as,

$$d^2\dot{\theta} = L, \tag{2.21}$$

Since L is a constant shown to be dubble the radius vector definition rate of the field. This is a mathematical version of the second law of Kepler. Now, if we use the scalar product $\dot{\mathbf{d}}$ with the equation (2.11), we obtain equation (2.11) are as under.

$$\dot{\mathbf{d}}.\frac{d^2\mathbf{d}}{dt^2} + \beta\frac{\dot{\mathbf{d}}.\mathbf{d}}{d^3} = 0,$$

after integrated we have get,

$$\frac{1}{2}\dot{\mathbf{d}}.\dot{\mathbf{d}} - \frac{m_1 u}{d} = H,$$

$$\tag{2.22}$$

$$\frac{1}{2}v^2 - \frac{\beta}{d} = H, \tag{2.23}$$

where H is a constant of integration. This is the sort of energy system preservation. The H quantity does not include absolute energy, $\frac{1}{2}\beta^2$ is associated with KE, and $\frac{-\beta}{d}$ is associated with PE of the system's, i.e., the system's total energy is constant. Remember the elements of the acceleration vector from celestial mechanics the radius vector is perpendicular to and along.

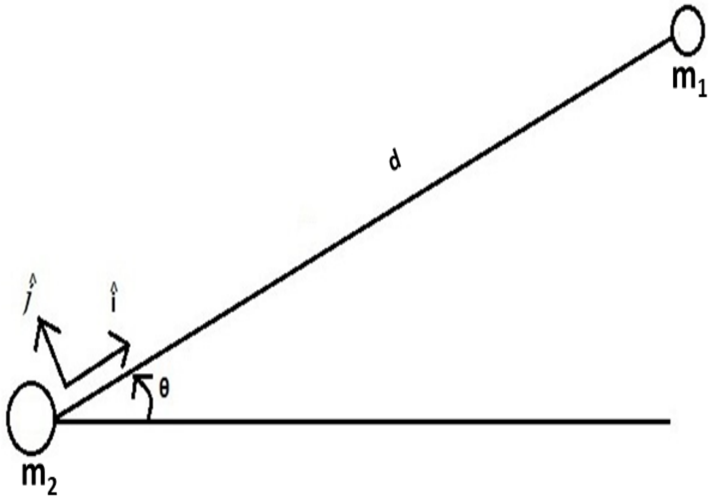

FIGURE 2.2

$$\mathbf{a} = (\ddot{d} - d\dot{\theta}^2)\hat{\imath} + \frac{1}{d}\frac{d}{dt}(d^2\dot{\theta})\hat{\jmath},$$

The above equation is used in (2.11), we obtained,

$$\ddot{d} - d\dot{\theta}^2 = -\frac{\beta}{d^2}, \tag{2.24}$$

$$\frac{1}{d}\frac{d}{dt}(d^2\dot{\theta}) = 0. \tag{2.25}$$

We get the following angular momentum integral after further integrating equation (2.21):

$$d^2\dot{\theta} = L, \tag{2.26}$$

under such type of substitution,

$$u = \frac{1}{d}, \tag{2.27}$$

The exclusion of the time between the equations (2.20) and (2.22) means that:

$$\frac{d^2u}{d\theta^2} + u = \frac{\beta}{L^2}. \tag{2.28}$$

The familiar shape of the equation above is:

$$u = \frac{\beta}{L^2} + B\cos(\theta - \theta_0), \tag{2.29}$$

Where B and θ_0 represent integration constants. In the above equation, substitute $u = \frac{1}{d}$, equation:

$$\frac{1}{d} = \frac{\beta}{L^2} + B\cos(\theta - \theta_0)$$

$$d = \frac{\frac{L^2}{\beta}}{1 + \frac{L^2 B_1}{\beta}\cos(\theta - \theta_0)}.$$

It is possible to write the polar form of the conical equation as:

$$d = \frac{p}{1 + e\cos(\theta - \theta_0)},$$

where

$$p = \frac{L^2}{\beta},$$
$$e = \frac{BL^2}{\beta}. \tag{2.30}$$

The orbit of one celestial body around another is defined by eccentricity e. Thus,

1. The orbit is elliptical if $0 < e < 1$,

2. The orbit is a parabolic If $e = 1$,

3. Similarly the orbit is hyperbolic if $e > 1$.

Therefore, a conic is the solution to the 2BP, including the first law of Kepler as a special case. Mathematically can be defined as,

$$e = \frac{c}{a},$$

$c \Rightarrow$ is the distance from focus to the center and,
$a \Rightarrow$ represent the semi major axis,
$b \Rightarrow$ represent the minor axis,
where,

$$a^2 = b^2 + c^2,$$
$$c^2 = a^2 - b^2.$$

2.5 In the n-Body Problem Equations of Motion

The two body issue deals with much of the crucial work in astrodynamics, but we also need to model the natural world using alternate bodies. Producing 3BP formulas is the next logical step. The n-body problem is a further generalisation of three body problems. In general, it takes a fixed number of integration constants to solve general differential equations of movements in the n-body problem. Consider a basic gravity question in which over time we have constant acceleration, $a(t) = a_0$. We get the velocity, $v(t) = a_0t + v_0$, if we integrate this equation. Again integrating provides, $d(t) = d_0 + v_0t + \frac{1}{2}$ and a_0t^2. Once again integration provides $d(t) = d_0 + v_0t + \frac{1}{2}$ and a_0t^2 respectively. The initial conditions must be known in order to finalise the solution. This example is a straight-forward analytical approach using the initial values, or a function of integration time and constants, called movement integrals. Unfortunately, this isn't always the easy scenario. If initial conditions alone do not provide a solution,The order of differential equations can be reduced by integrals of motion, also referred to as the degrees of freedom of the dynamic system, can be lowered by integrals of motion. Ideally, we should reduce it to order zero if the number of integrals is equal to the order of differential equations. Such integrals are constant behavior of the original conditions, as well as the position and velocity of the motion are constants at any moment, thus the word constants.

We need $6n$ integrals of motion for a complete solution to the n-body problem, a system of $3n$ second order differential equations. Linear momentum conservation provides six, energy one conservation, and total angular momentum conservation three, for a total of ten. There are no rules similar to the first two laws of Kepler to obtain additional constants, so we are left with a $6n-10$ for $n \geq 3$ order structure. These equations defy all attempts at closed-form solutions for n bodies ie., $n \geq 3$. H.Brun, in 1887, proved that there were no other algebraic integrals. We still have only ten known integrals, although Poincaré later generalised Brun 's work. They provide us with insight into the three motions of the body and n-body problems. Conservation of complete linear momentum ensures that there are no outside forces in the system.

First, here we set up the n motion equations of large bodies of mass $m_i(i = 1, 2...n)$ whose radius vectors are $\mathbf{d}i$ from an unexpedited point O, while their mutual radius vectors are d_{ij} where,

$$\mathbf{d}_{ij} = \mathbf{d}_j - \mathbf{d}_i \tag{2.31}$$

From the laws of movement of Newton and the law of gravitation,

$$m_i \ddot{\mathbf{d}}_i = G \sum_{j=1 j \neq i}^{n} \frac{m_i m_j}{d_{ij}^3} \mathbf{d}_{ij}, \qquad (i = 1, ...n). \tag{2.32}$$

Here we notice that $\mathbf{d}_{ij}$ indicates that the vector between m_i and m_j is directed to towards m_i and m_j, thus

$$\mathbf{d}_{ij} = -\mathbf{d}_{ji} \tag{2.33}$$

For the n-body problem, the set of equations (2.27) is the necessary motion equation, G being the universal gravitational constant.

Chapter 3

Planar Central Configurations of Symmetric Five-Body

In this review work of research [14], we find out the central configuration of the set of symmetric planar 5-body problems in which (i) four out of five bodies are on the end of sides(vertices) of a trapezoid and fifth mass take different positions on the symmetry axis both inside and outside the trapezoid. (ii) All the masses are placed in such manner in which (m_1, m_2) and m_4 and similarly (m_2, m_3) and m_5 are take collinear; these two sets of collinear bodies form a triangle with m_2 is the intersection between two sets of masses; now we give the form for expressing the masses ratio and identify the region in which we can choose the positive masses for make the configuration as a center.
Using newtonian Gravitation the equation of motion for n positive masses is

$$m_i \ddot{r}_i = \bigtriangledown_i U_i, \qquad i = 1, ...n, \tag{3.1}$$

where the Newtonian potential

$$U = -G \sum_{i=1}^{n} \sum_{j=1}^{j<i} \frac{m_i m_j}{|r_i - r_j|}$$

r_i and m_i are the position vector and mass of the ith body respectively and G is the constant of gravitation. The n-body system CC(central configuration)is got, if the acceleration of ith body is the common scaler multiple of the position vector of ith body,
i.e.,

$$\ddot{r}_i = \lambda r_i, \qquad i = 1, ...n, \quad \text{where} \quad \lambda \neq 0. \tag{3.2}$$

By Laura/Dziobek equation [24], m_i where $i = 1, ...n$, form non-collinear, planar, CC is

$$f_{ij} = \sum_{k=1,k\neq i,j}^{n} m_k(R_{ik} - R_{jk})\triangle_{ijk} = 0, \tag{3.3}$$

where $\triangle_{ijk} = (r_i - r_j) \wedge (r_i - r_k)$. The $\triangle_{ijk}$ and $R_{ij} =\frac{1}{r_{ij}^3}$ is the area of triangle. The equations (3.3) for five bodies reduces to the given below ten equations

$$f_{12} = m_3(R_{13} - R_{23}) \triangle_{123} + m_4(R_{14} - R_{24}) \triangle_{124} + m_5(R_{15} - R_{25})\triangle_{125} = 0, \tag{3.4}$$

$$f_{13} = m_2(R_{12} - R_{32}) \triangle_{132} + m_4(R_{14} - R_{34}) \triangle_{134} + m_5(R_{15} - R_{35})\triangle_{135} = 0, \tag{3.5}$$

$$f_{14} = m_2(R_{12} - R_{42}) \triangle_{142} + m_3(R_{13} - R_{43}) \triangle_{143} + m_5(R_{15} - R_{45})\triangle_{145} = 0, \tag{3.6}$$

$$f_{15} = m_2(R_{12} - R_{52}) \triangle_{152} + m_3(R_{13} - R_{53}) \triangle_{153} + m_4(R_{14} - R_{54})\triangle_{154} = 0, \tag{3.7}$$

$$f_{23} = m_1(R_{21} - R_{31}) \triangle_{231} + m_4(R_{24} - R_{34}) \triangle_{234} + m_5(R_{25} - R_{35})\triangle_{235} = 0, \tag{3.8}$$

$$f_{24} = m_1(R_{21} - R_{41}) \triangle_{241} + m_3(R_{23} - R_{43}) \triangle_{243} + m_5(R_{25} - R_{45})\triangle_{245} = 0, \tag{3.9}$$

$$f_{25} = m_1(R_{21} - R_{41}) \triangle_{241} + m_3(R_{23} - R_{53}) \triangle_{253} + m_4(R_{24} - R_{54})\triangle_{254} = 0, \tag{3.10}$$

$$f_{34} = m_1(R_{31} - R_{41}) \triangle_{341} + m_2(R_{32} - R_{42}) \triangle_{342} + m_5(R_{35} - R_{45})\triangle_{345} = 0, \tag{3.11}$$

$$f_{35} = m_1(R_{31} - R_{51}) \triangle_{351} + m_2(R_{32} - R_{52}) \triangle_{352} + m_4(R_{34} - R_{54})\triangle_{354} = 0, \tag{3.12}$$

$$f_{45} = m_1(R_{41} - R_{51}) \triangle_4 51 + m_2(R_{42} - R_{52}) \triangle_{452} + m_3(R_{43} - R_{53})\triangle_{453} = 0. \tag{3.13}$$

Here, we recover the central configuration of the isosceles trapezoidal five-body problem and identify the regions in the phase where it is possible to choose positive masses which will make the configuration central. We are motivated by the work of [15] and follow similar ideas to study planar symmetric five-body problem.

3.1 Geometrical Order of Five Body in Isosceles Trapezoid Case having non-zero Equations

We investigate The problems include two types of isosceles trapezoidal five-body problem having four bodies its vertices and fifth body on the symmetry axis which come outside or inside the trapezoid. The third case which is investigated is triangular problem in which two pairs of masses and may or may not equal to one of the four mass is on the perpendicular bisector of that triangle. Suppose $m_1 = m_3$ and $m_4 = m_5$ are symmetrically placed on the isosceles trapezoid vertices and fifth mass is on the axis of symmetry of trapezoid.Four masses form a trapezoid see the figure (3.1)

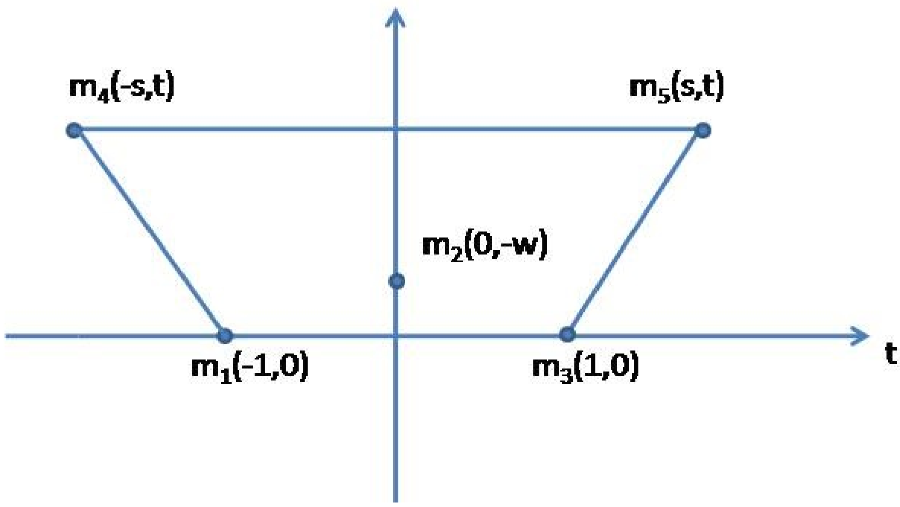

FIGURE 3.1: Trapezoidal five body configuration.

while the mass m_2 is on the line of symmetry. The position vectors of the five masses of 5-body problem have position vectors are: $r_1 = (-1, 0), r_2 = (0, -w), r_3 = (1, 0), r_4 = (-s, t), r_5 = (s, t)$, where $s, t, \in R$. using the values of, we obtain the corresponding values of R_{ij},

$$R_{12} = R_{23} = (1 + w^2)^{\frac{-3}{2}},$$

$$
\begin{aligned}
8R_{13} &= 1, \\
R_{14} = R_{35} &= ((1-t)^2+t^2)^{\frac{-3}{2}}, \\
R_{15} = R_{34} &= ((1+t)^2+t^2)^{\frac{-3}{2}}, \\
R_{24} = R_{25} &= ((t+w)^2+s^2)^{\frac{-3}{2}}, \\
8R_{45} &= t^{-}3.
\end{aligned} \tag{3.14}
$$

By using the following symmetries we find $\triangle_{ijk}$, where $i, j, k = 1, 2, ..., 5$,

$$\triangle_{ijk} = -\triangle_{jik} = -\triangle_{ikj} = -\triangle_{kji},$$

$$\triangle_{ijk} = \triangle_{jki} = \triangle_{kij},$$

$$\triangle_{ijk} = 0, \text{ if } i = j\text{or } i = k\text{or } j = k,$$

and

$$
\begin{aligned}
\triangle_{124} = \triangle_{235} &= (t+w(1-s)), \\
\triangle_{125} = \triangle_{234} &= (t+w(1+t)), \\
\triangle_{145} = \triangle_{345} &= -2t^2, \\
\triangle_{245} &= -2t(t+w), \\
\triangle_{123} &= 2w.
\end{aligned} \tag{3.15}
$$

By the symmetry of the problem we take $m_1 = m_3$ and $m_4 = m_5$ and using the values of R_{ij} and $\triangle_{ijk}$, where $i, j, = 1, 2, ..., 5$, in equations $(3.4 - 3.13)$, we have show that $f_{12} = f_{23}, f_{14} = f_{35}, f_{24} = f_{25}, f_{15} = f_{34}$ and the remaining equations vanish. Consequently, we obtain f_{12}, f_{14}, f_{15}, and f_{24} as the only unique equations:

$$f_{12} = m_1 h_{31} + m_4 h_{33} = 0, \tag{3.16}$$

$$f_{14} = m_1 h_{11} + m_2 h_{12} + m_4 h_{13} = 0, \tag{3.17}$$

$$f_{15} = m_1 h_{21} + m_2 h_{22} + m_4 h_{23} = 0, \tag{3.18}$$

$$f_{24} = m_1 h_{41} + m_4 h_{43} = 0, \tag{3.19}$$

where

$$
\begin{aligned}
h_{11} &= (R_{13} - R_{15})\triangle_{143},\\
h_{12} &= (R_{24} - R_{12})\triangle_{124},\\
h_{13} &= (R_{15} - R_{45})\triangle_{145},\\
h_{21} &= (R_{13} - R_{14})\triangle_{143},\\
h_{22} &= (R_{24} - R_{12})\triangle_{125},\\
h_{23} &= (R_{45} - R_{14})\triangle_{145},\\
h_{31} &= (R_{13} - R_{12})\triangle_{123},\\
h_{33} &= (R_{14} - R_{24})\triangle_{124} + (R_{15} - R_{24})\triangle_{125},\\
h_{41} &= (R_{12} - R_{14})\triangle_{124} + (R_{23} - R_{14})\triangle_{243},\\
h_{43} &= (R_{24} - R_{45})\triangle_{245}.
\end{aligned}
\tag{3.20}
$$

3.2 Main Results

3.2.1 Theorem

Consider a 5-body non-collinear configuration the position vectors $\mathbf{r_i}$ of five masses m_i, where $i = 1, 2, ..., 5$, are:

$$
\begin{aligned}
r_1 &= (-1, 0),\\
r_2 &= (0, -w),\\
r_3 &= (1, 0),\\
r_4 &= (-s, t),\\
r_5 &= (s, t).
\end{aligned}
\tag{3.21}
$$

a. When s = t, there is a continuous family of central configurations determined by the region R and the function $C(t, w) = 0$, given in figure (3.1). There are no central configurations when $w>0$.

a. When $t = 1$ and $s \in (0, 0.64)$ there exists a continuous family of Central Configurations when $w<0$ or when $s>0.64$.

3.2.2 Proof of Theorem

3.2.2.1 Proof of Theorem (a)

Consider five bodies of masses m_i placed at $\mathbf{r_i}$ where $i = 1, 2, ..., 5.$ and imposing the condition $s = t$, the position vectors given in (3.21) of m_i become:

$$\begin{aligned} r_1 &= (-1, 0), \\ r_2 &= (0, -w), \\ r_3 &= (1, 0), \\ r_4 &= (-t, t), \\ r_5 &= (t, t). \end{aligned} \tag{3.22}$$

Write the system of linear homogeneous equations $(3.16 - 3.19)$ for the above arrangement of five masses $(i.e., s = t)$ in matrix form

$$M = \begin{pmatrix} h_{41}1 & 0 & h_{43} \\ h_{31} & 0 & h_{33} \\ h_{21} & h_{22} & h_{23} \\ h_{11} & h_{12} & h_{13} \end{pmatrix}$$

Applying \Gauss Elimination Method", matrix M is reduced to the following matrix:

$$\begin{pmatrix} h_{41} & 0 & h_{43} \\ 0 & h_{22} & -\frac{1}{h_{41}}(h_{21}h_{43} - h_{23}h_{41}) \\ 0 & 0 & -\frac{1}{h_{41}}(h_{21}h_{43} - h_{23}h_{41}) \\ 0 & 0 & 0 \end{pmatrix}$$

The above system takes the form as below,

$$
\begin{aligned}
&m_1 h_{41} + m_4 h_{43} = 0, \\
&m_2 h_{22} - \frac{m_4}{h_{41}}(h_{21} h_{43} - h_{23} h_{41} = 0, \\
&m_4 (h_{31} h_{43} - h_{41} h_{33}) = 0,
\end{aligned}
$$

For the above system to have a non-trivial solution we must have $h_{31}h_{43} - h_{41}h_{33} = 0$. This will be considered as a geometric constraint and will be necessary condition for the existence of trapezoidal central configurations. Setting $\mu_1 = \frac{m_1}{m_4}$, and $\mu_2 = \frac{m_2}{m_4}$, we obtain

$$\mu_1(t, w) = -\frac{h_{43}}{h_{41}} \tag{3.23}$$

$$\mu_2(t, w) = \frac{h_{21}h_{43} - h_{23}h_{41}}{h_{41}h_{22}} = \frac{N_{\mu_2}(t, w)}{D_{\mu_2}(t, w)}. \tag{3.24}$$

Such that $h_{21} \neq 0$ and $h_{41} \neq 0$. Therefore equations (3.23) and (3.24) define central configuration for the trapezoidal 5-body problem for all masses subject to the constraint.

$$C(t, w) = h_{31}h_{43} - h_{41}h_{33} = 0. \tag{3.25}$$

We see from the reduced matrix that (3.17) is not used in deriving μ_1 and μ_2 which gives a second constraint:

$$C^*(t, w) = h_{43}(h_{12}h_{21} - h_{11}h_{22}) + h_{41}(h_{13}h_{22} - h_{12}h_{23}) = 0 \tag{3.26}$$

By using numerical techniques it is confirmed that $C^*(t, w) = 0$ is satisfied everywhere, where $C(t, w) = 0$. The constraint $C^*(t, w) = 0$ has additional solution but that irrelevant as for nontrivial solution both the constraints have to be satisfied. Hence we will only use $C(t, w)$ in our analysis. As only positive solutions define trapezoidal central configuration here therefore we are interested in the regions where all the masses are positive. The analytical proof of these theorems can be seen [16].

3.3 Region where Masses are Positive

To find the central configuration region where μ_1 and μ_2 are both positive and $C(t, w) = 0$, we need to find regions in the tw plane. These region can be seen in figure (3.2) and figure (3.3) .

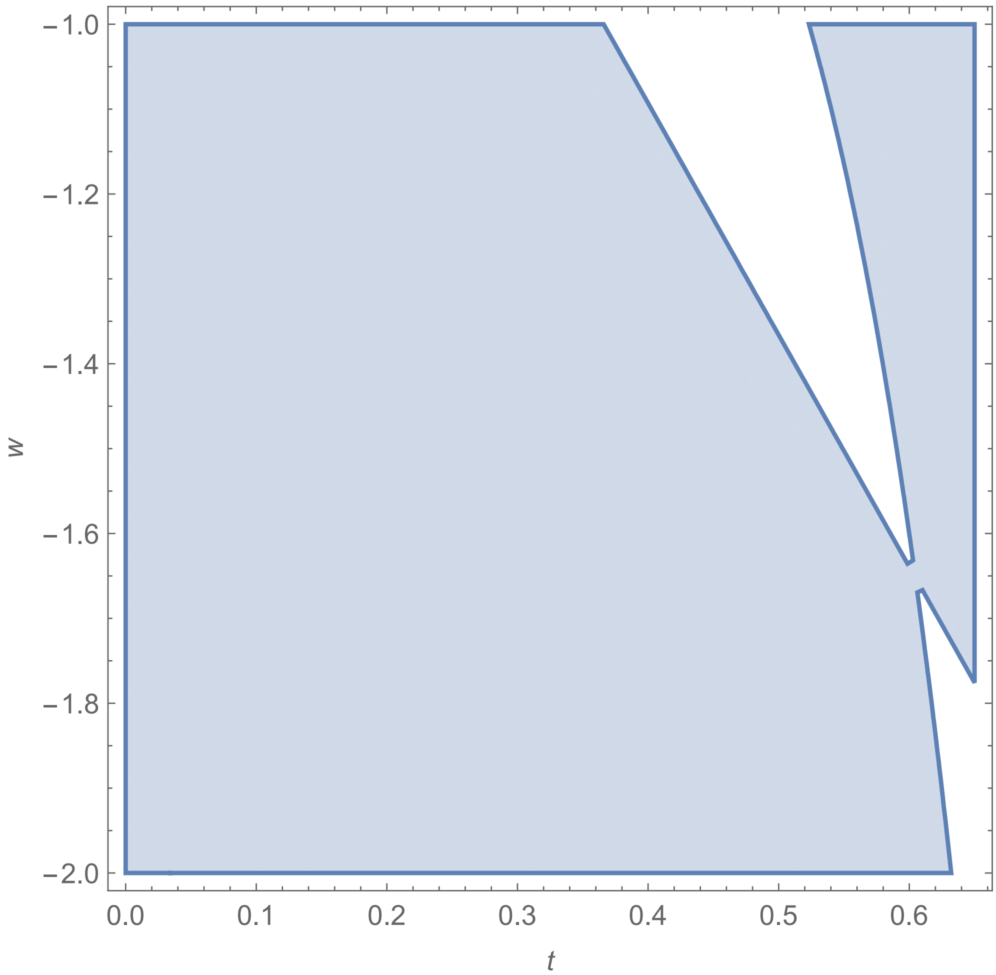

FIGURE 3.2: Central configuration region where both μ_1 and μ_2 are positive.

In this figure t plotted on x-axis and w are in y-axis. This the required region where central configuration exist. As we have discuss in the earlier stage that central configuration exist for positive masses. In this cases the shaded region represent the CC. Moreover if we taking any value form the shaded region the masses i.e., μ_1 and μ_2 are positive. Actually this is the region where masses are positive.

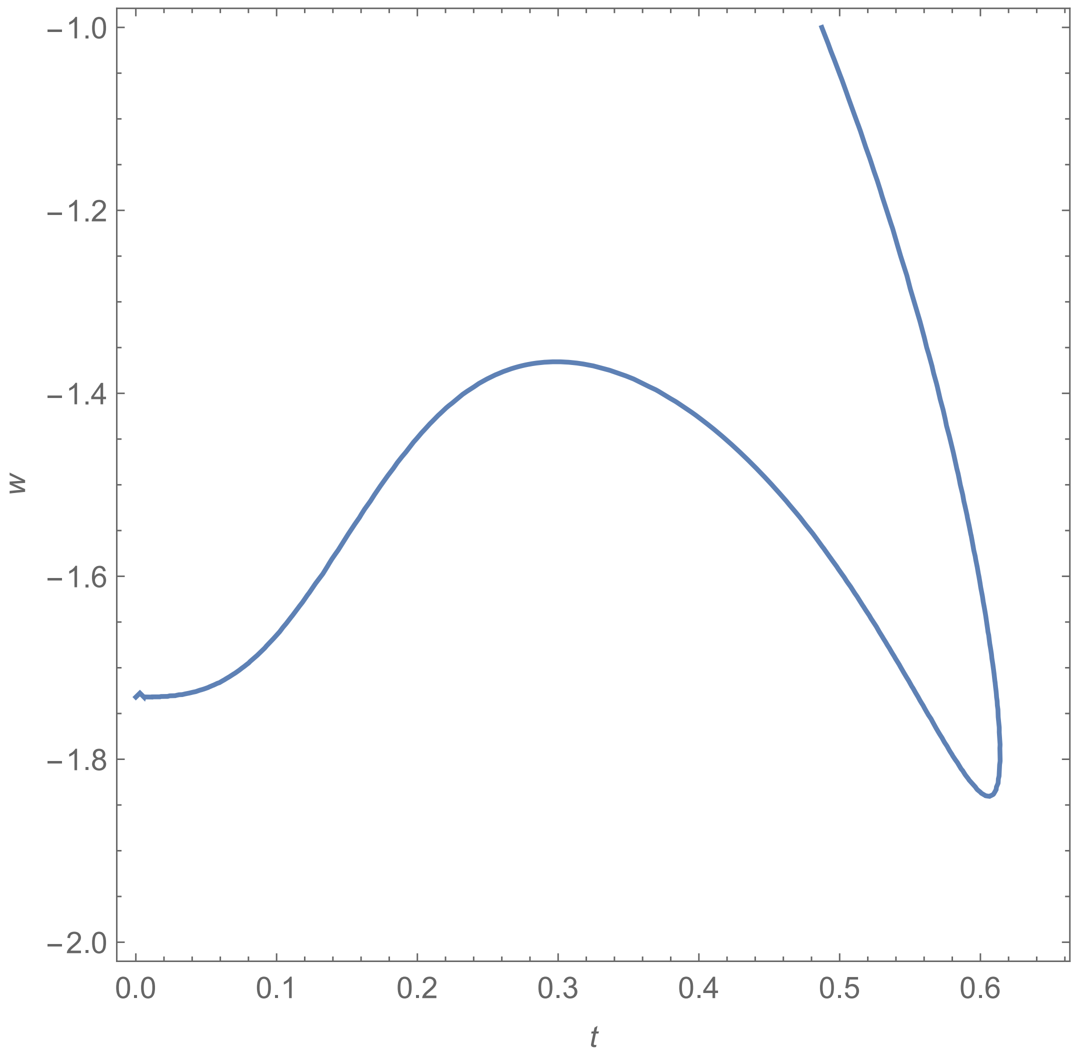

FIGURE 3.3: The line corresponds to $C(t, w) = 0$.

In the above figure we have plot t component on x-axis and w component along y-axis as shown clearly. The central configuration region where both μ_1 and μ_2 are positive, can be found by taking the intersection of the regions found for μ_1 and μ_2. This region is given in Figure (3.2) with the geometric constraint $C(t, w) = 0$. The continuous family of central configurations is shown by intersection of the bold line with the colored region where both μ_1 and μ_2 are positive. In figures (3.2) and (3.3) are merged together. We can see clearly the central configuration region when both figured are merged. The intersection of shaded region and bold line is central configuration region. figure (3.3) just shows increased domain of figure (3.2).

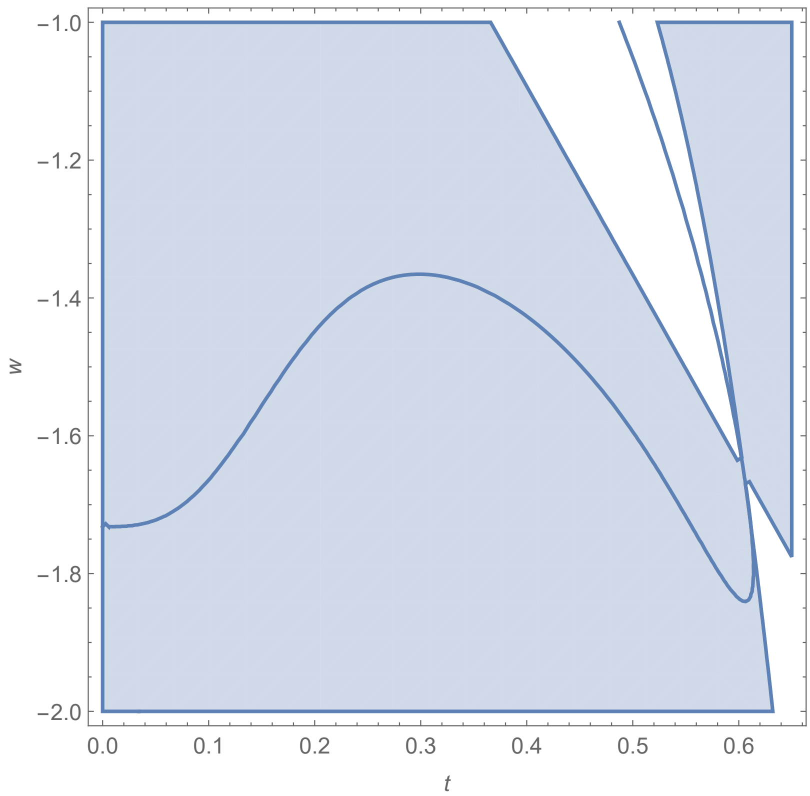

FIGURE 3.4: Figs. 2.1 are merged together

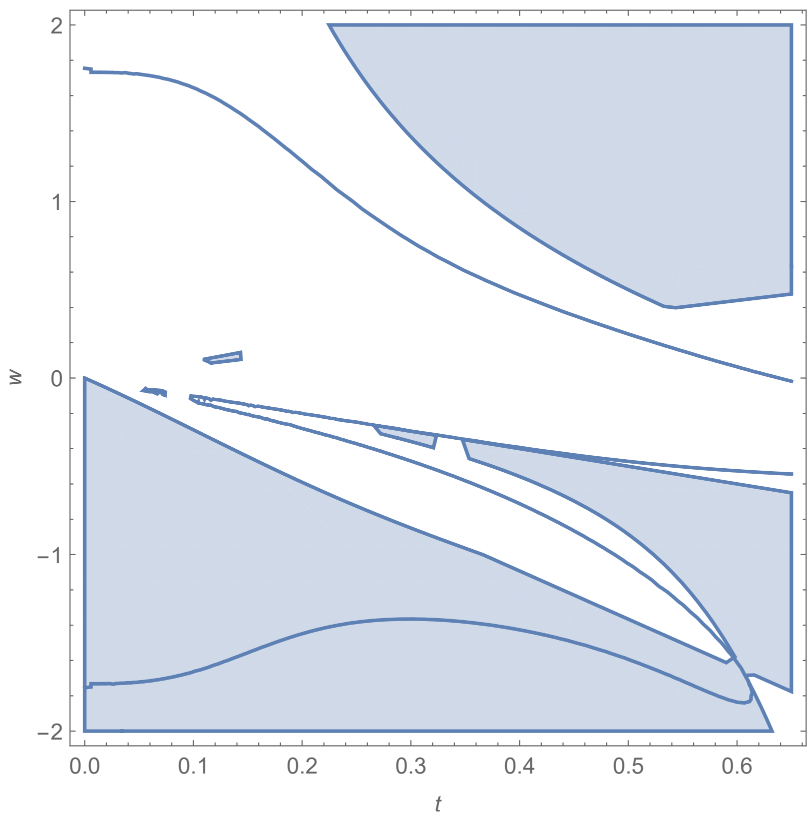

FIGURE 3.5: Fig.(i) larger scale

The CC's region which is the curve in the shaded region of Figure (3.4) is separately shown in Figure (3.5). At all points this curve all mass ratios are positive and the constraints is satisfied.

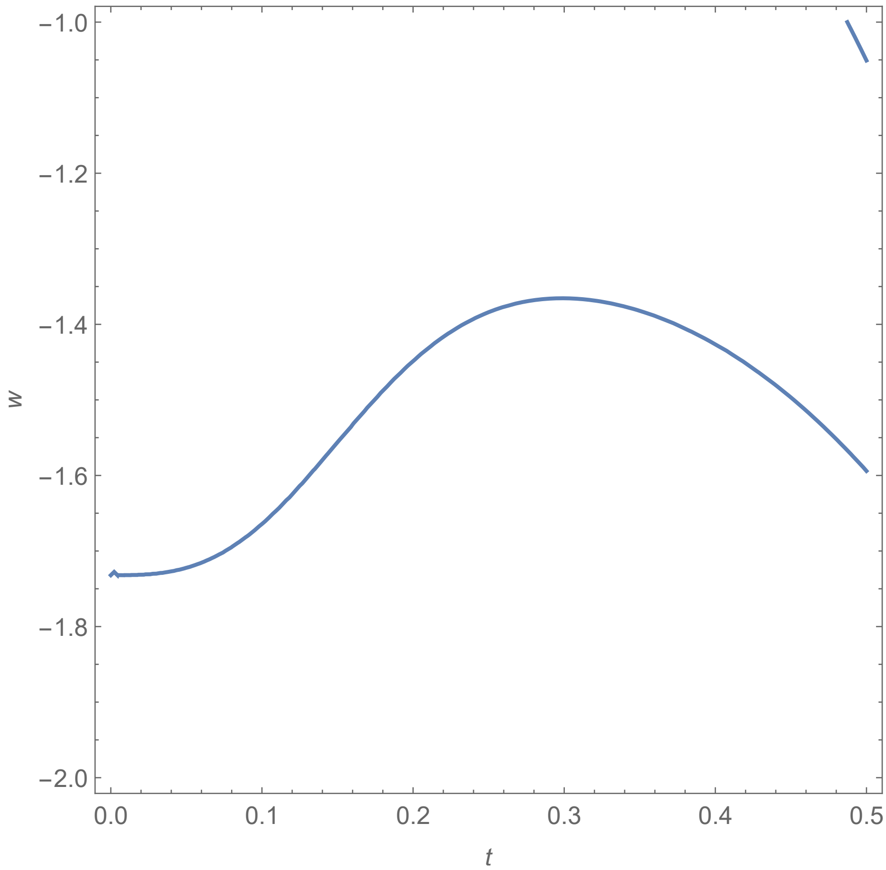

FIGURE 3.6: CCs curve

We have interpolated the CC's curve shown above with 12th degree polynomial with help of Mathematica [17] as

$$\begin{aligned} w = P(t) := & -1.73205 + 7.91088t - 478.155t^2 + 12079.9t^3 - 168059t^4 \\ & + (1.47717 \times 10^6)t^5 - (8.6506 \times 10^6)t^6 + (3.45028 \times 10^7)t^7 \\ & (-9.40066 \times 10^7)t^8 + (1.72083 \times 10^8)t^9 - (2.02264 \times 10^8)t^{10} \\ & + (1.37816 \times 10^8)t^{11} + (-4.1349 \times 10^7)t^{12}. \end{aligned} \tag{3.27}$$

The graph of above polynomial is shown below. And the combined graph of Figure (3.6) and Figure (3.7) is for $t \in (0.04 \;\; 0.5)$ is also shown below.

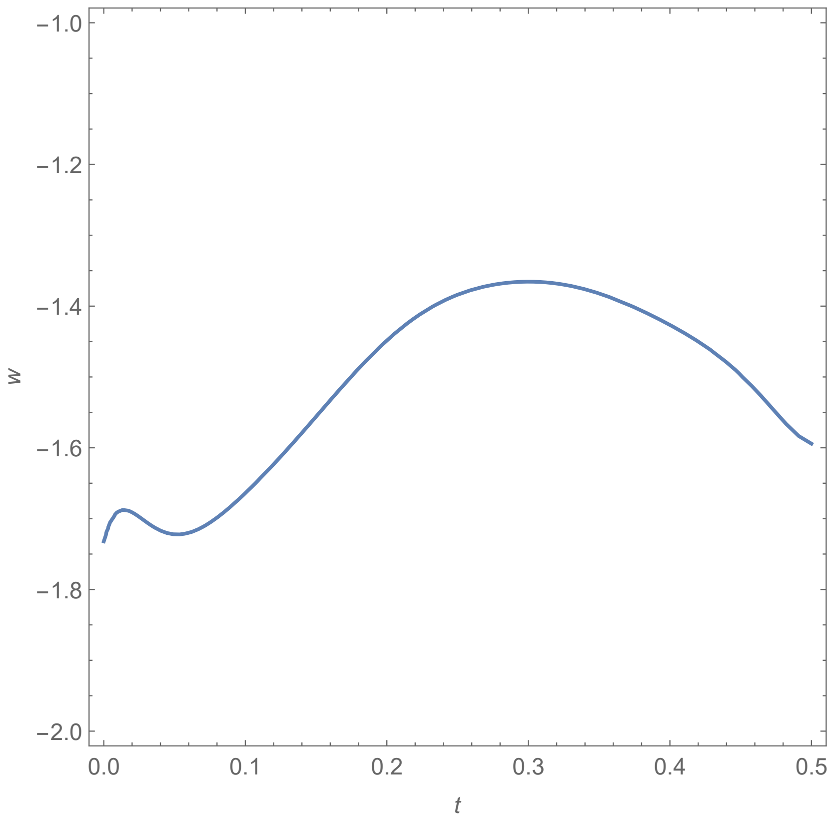

FIGURE 3.7: CCs curve

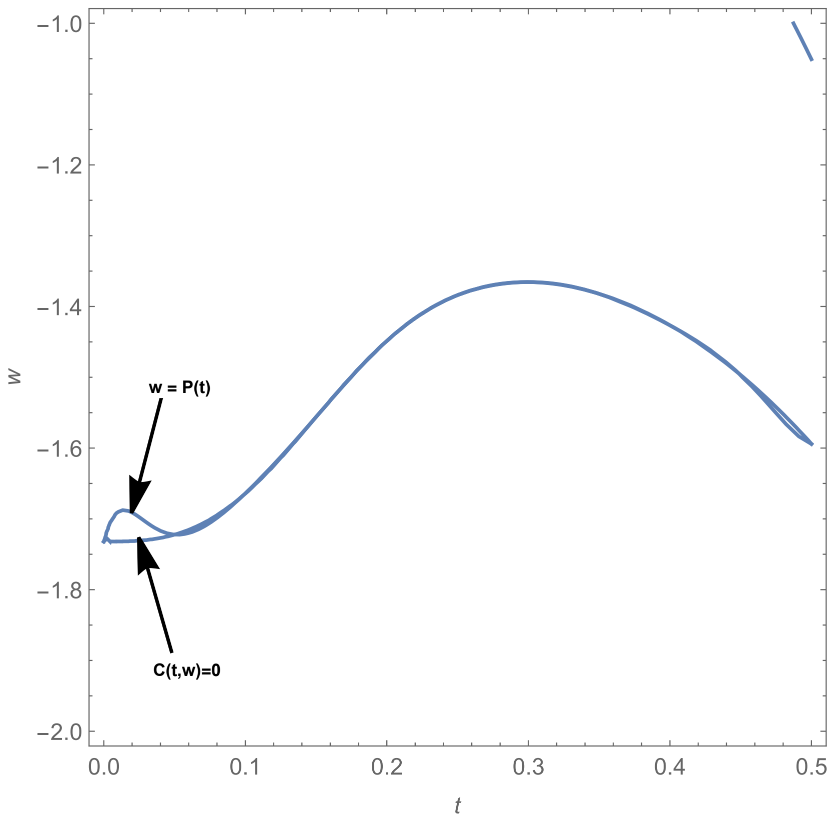

FIGURE 3.8: CCs curve

We have chosen some specific values of $t \in (0.05 \;\; 0.5)$ given in the table (3.1). Using these specific values of t in equation (3.27) we get the corresponding value of w as shown in the table (3.1). For these values of t and w given in table (3.1) the mass ratios defined in equations (3.23) and (3.24) are positive and the constraints given in equation (3.26) is approximately zero. So our CCs holds in $t \in (0.05 \;\; 0.5)$.

Cases	t	w	μ_1	μ_2	$C(t, w)$
Case I	0.1	-1.66404	60.6311	57.9326	1.22125×10^{-14}
Case II	0.2	-1.44846	7.85609	8.56962	-4.7073×10^{-13}
Case III	0.3	-1.36550	2.4098	3.40424	4.1489×10^{-12}
Case IV	0.4	-1.42660	1.08267	2.72339	4.38008×10^{-12}
Case V	0.5	-1.59420	0.77487	3.74963	-4.12704×10^{-11}

Table 3.1: Central Configeration Table for specific values of $t \in (0.05 \;\; 0.5)$

Chapter 4

Dynamics of Sixth Body

4.1 Introduction

In this section, the dynamics of the 6th particle (infinitesimal mass or secondary particle not affecting primary motion) are discussed in the plane, moving in acco rdance with the gravitational field created by the attraction of five primaries mov ing in a trapezoid configuration of planar isosceles as shown in the previous part. We call this problem to restricted six-body problem (R6BP). Equation of motion describe the planer motion of restricted 6th particle mass m_6 written from equation (3.1) will be

$$\ddot{\mathbf{r}}_6 = m_1\frac{\mathbf{r}_1 - \mathbf{r}_6}{|\mathbf{r}_1 - \mathbf{r}_6|^3} + m_2\frac{\mathbf{r}_2 - \mathbf{r}_6}{|\mathbf{r}_2 - \mathbf{r}_6|^3} + m_3\frac{\mathbf{r}_3 - \mathbf{r}_6}{|\mathbf{r}_3 - \mathbf{r}_6|^3} + m_4\frac{\mathbf{r}_4 - \mathbf{r}_6}{|\mathbf{r}_4 - \mathbf{r}_6|^3} + m_5\frac{\mathbf{r}_5 - \mathbf{r}_6}{|\mathbf{r}_5 - \mathbf{r}_6|^3}. \tag{4.1}$$

We are now adding a coordinate system with a uniform angular velocity ω rotating around the centre of mass. In this new rotating frame(non-inertial frame), let (x, y) be the coordinates of m_6. With the following orthogonal system, the equation (4.1) can be transformed from a fixed inertial frame to a rotating coordinate system,

$$\mathbf{e}_1 = e^{iwt} \quad \mathbf{e}_2 = ie^{iwt},$$

where the angular speed is ω and the time is t. The m_6 position vector of the revolving frame is,

$$\mathbf{r}_6 = x(t)\ \mathbf{e}_1 + y(t)\ \mathbf{e}_2, \tag{4.2}$$

Selecting ω (without generality loss) and taking the derivatives of the first and second equation (4.2) yields (4.2),

$$\begin{aligned}\dot{\mathbf{r}}_6 &= [(\dot{x} - y) + i(x + \dot{y})]e^{it},\\ \ddot{\mathbf{r}}_6 &= [(\ddot{x} - 2\dot{y} - x) + i(\ddot{y} + 2\dot{x} - y)]\, e^{it}.\end{aligned} \tag{4.3}$$

Using the (4.2) equation in the (4.3) equation, the m_6 motion equations in the rotating frame in component form are the equations of motion.,

$$\ddot{x} - 2\dot{y} = x - m_1\left(\frac{x+1}{r_{61}^3} + \frac{x-1}{r_{63}^3}\right) - m_4\left(\frac{x+t}{r_{64}^3} + \frac{x-t}{r_{65}^3}\right) - m_2\frac{x}{r_{62}^3}, \tag{4.4}$$

$$\ddot{y} + 2\dot{x} = y - m_1 y\left(\frac{1}{r_{61}^3} + \frac{1}{r_{63}^3}\right) - m_4\left(\frac{y-t}{r_{64}^3} + \frac{y-t}{r_{65}^3}\right) - m_2\frac{y+w}{r_{62}^3}, \tag{4.5}$$

where mutual distances are described as,

$$\begin{aligned} r_{61} &= \sqrt{(x+1)^2 + y^2},\\ r_{62} &= \sqrt{x^2 + (y+w)^2},\\ r_{63} &= \sqrt{(x-t)^2 + (y-t)^2},\\ r_{64} &= \sqrt{(x+t)^2 + (y-t)^2}\\ r_{65} &= \sqrt{(x+t)^2 + (y+t)^2}\end{aligned}$$

The equation of motion of m_6 moving in the plane of primaries can also be written as,

$$U_x = \frac{\partial U}{\partial x} = \ddot{x} - 2\dot{y}, \tag{4.6}$$

$$U_y = \frac{\partial U}{\partial y} = \ddot{y} + 2\dot{x}, \tag{4.7}$$

where the effective potential $U(x, y)$ can pe expressed as,

$$U(x, y) = \frac{x^2 + y^2}{2} + m_1 \left(\frac{1}{r_{61}} + \frac{1}{r_{63}} \right) + m_4 \left(\frac{1}{r_{64}} + \frac{1}{r_{65}} \right) + \frac{m_2}{r_{62}}. \tag{4.8}$$

Comparing the equations (4.4), (4.5) and (4.6), (4.7), we may write equations of motion of m_6 as,

$$U_x(x, y) = x - m_1 \left(\frac{x+1}{r_{61}^3} + \frac{x-1}{r_{63}^3} \right) - m_4 \left(\frac{x+t}{r_{64}^3} + \frac{x-t}{r_{65}^3} \right) - m_2 \frac{x}{r_{62}^3}, \tag{4.9}$$

$$U_y(x, y) = y - m_1 y \left(\frac{1}{r_{61}^3} + \frac{1}{r_{63}^3} \right) - m_4 \left(\frac{y-t}{r_{64}^3} + \frac{y-t}{r_{65}^3} \right) - m_2 \frac{y+w}{r_{62}^3}. \tag{4.10}$$

4.2 Equilibrium Solutions

The equations (4.9) and (4.10) do not have a closed-form analytical solution, since the location of the equilibrium points can be determined by both equations. These are the position in space where there will be zero velocity and acceleration of the infinitesimal mass m_6, i.e., where m_6 appears permanently at rest compared to the main m_1,m_2,m_3, m_4 and m_5 respectively. When placed at an equilibrium point (also called Libration Point / Lagrange Point) a body will apparently remain there. These solutions can only be found if all rates equal to zero have adequate sufficient conditions for all rates,

$$\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0.$$

Finally the equations (4.9) and (4.10) take the form,

$$U_x = x - m_1\left(\frac{x+1}{r_{61}^3} + \frac{x-1}{r_{63}^3}\right) - m_4\left(\frac{x+t}{r_{64}^3} + \frac{x-t}{r_{65}^3}\right) - m_2\frac{x}{r_{62}^3} = 0, \quad (4.11)$$

$$U_y = y - m_1 y\left(\frac{1}{r_{61}^3} + \frac{1}{r_{63}^3}\right) - m_4\left(\frac{y-t}{r_{64}^3} + \frac{y-t}{r_{65}^3}\right) - m_2\frac{y+w}{r_{62}^3} = 0. \quad (4.12)$$

Equations (4.9) and (4.10) are algebraic equations coupled non-linearly. We need to numerically solve these equations or sketch contour plots using Mathematica to determine the zero's (x, y) or equilibrium points. According to table (3.1), the description of balance points for RT6BP is given by,

4.2.1 Case I

For case I: $t = 0.1$, $w = -1.66404$ and $\mu_1 = 60.6311$, $\mu_2 = 57.9326$ (see table (3.1)). Because $\mu_1 = \frac{m_1}{m_4}$ and $\mu_2 = \frac{m_2}{m_4}$.

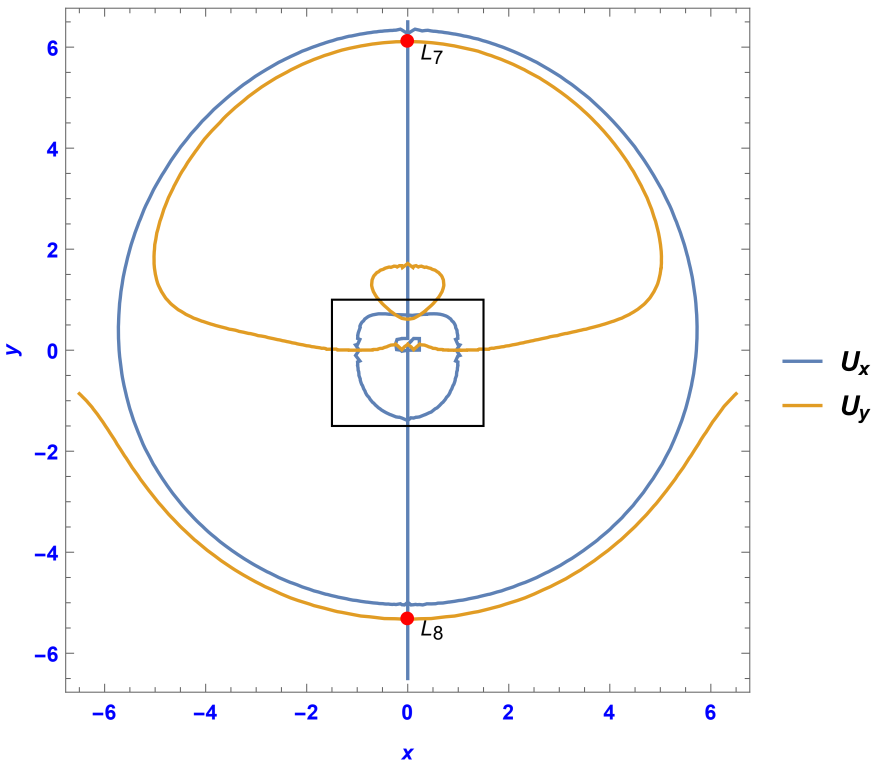

FIGURE 4.1: Contour Plot for Case I

We can suppose (without loss of generality) $m_4 = 1$, then from above expression of μ_1 and μ_2, one can easily get the value of $m_1 = 60.6311$ and $m_2 = 57.9326$. Using all these values in equations (4.11) and (4.12) and drawing the contour plot with help of Mathematica [18] (for Mathematica commands and code please see Appendix) as

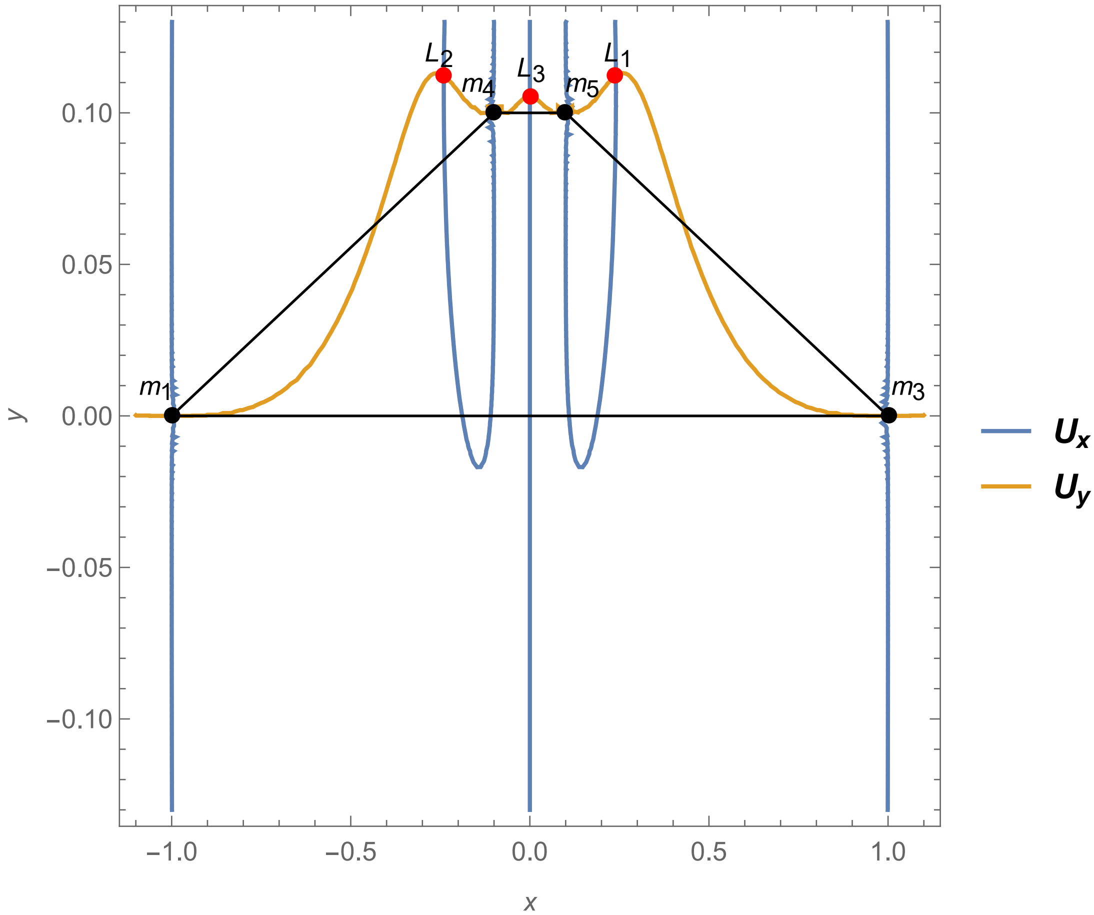

FIGURE 4.2: Contour Plot for Case I

The position of the equilibrium points is defined by the intersections of the non-linear equations $U_x = 0$, (blue) and $U_y = 0$ (orange). The black dots indicate the location of the primary masses and red dots represent the position of stable points for the infinitesimal mass m_6 in the plane of primaries $m_1 - m_5$. (See figures (4.1) to (4.3)). In figure (4.3) the black rectangular cell is zoomed in figure (4.2) and figure (4.3) so that one can easily capture the idea of position of primaries and equilibrium points inside the rectangle. We got eight equilibrium points for this Case I. In the following section we discuss the stability analysis of these equilibrium points.

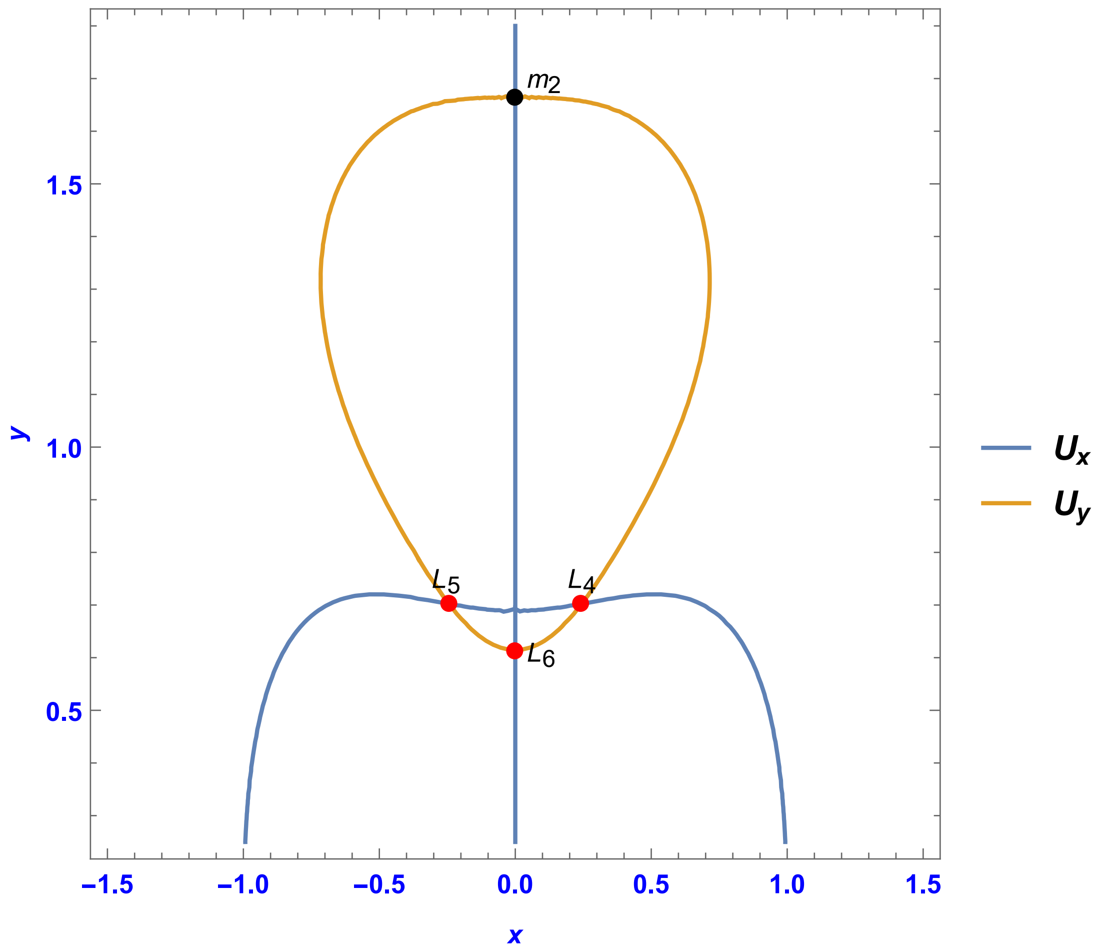

FIGURE 4.3: Contour Plot for Case I

4.2.2 Stability Analysis of Equilibrium Points for Case-I

In order to carry out a proper study of stability we need to examine the behavior of m_6 that is given a small displacement from an equilibrium position. We will follow the standard linearization procedure for linearizing the equation of infinitesimal mass and carrying out stability analysis [19]. Let the location of an equilibrium point in the RRSBP be denoted by (x_0, y_0) and consider a small displacement (X, Y) from the point such that $x_0 + X$ and $y_0 + Y$. Expanding the Taylor's series, then by substituting in equations (4.6) and (4.7), the end result is a set of second order linear differential equations of the from

$$\ddot{X} - 2\dot{Y} = XU_{xx} + YU_{xy}, \tag{4.13}$$

$$\ddot{Y} + 2\dot{X} = XU_{xy} + YU_{yy}, \tag{4.14}$$

where the quantities $U_{xx} = \frac{\partial^2 U}{\partial x^2}$ etc., and these derivatives are all constants since they are evaluated at equilibrium points. The linearized equation matrix form are

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \ddot{X} \\ \ddot{Y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & 0 & -2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ \dot{X} \\ \dot{Y} \end{pmatrix} \tag{4.15}$$

These equations can also be written as in the following matrix form

$$\dot{\boldsymbol{\Psi}} = \mathcal{A}\boldsymbol{\Psi} \tag{4.16}$$

where

$$\boldsymbol{\Psi} = \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \ddot{X} \\ \ddot{Y} \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & 0 & -2 \end{pmatrix}. \tag{4.17}$$

The characteristic polynomial for $\mathcal{A}$ is

$$\lambda^4 + (4 - U_{xx} - U_{yy})\lambda^2 + U_{xx}U_{yy} - U_{xy}^2 = 0. \tag{4.18}$$

Eigenvalues of above equation should be estimated. The number of eigenvalues is equal to the number of state variables. In our case there will be 4 eigenvalues. Eigenvalues are generally complex numbers. If all of four eigenvalues are pure imaginary, then the equilibrium point is stable, otherwise it is unstable. In the following table, case I is investigated:

Equilibrium points	Stability
L_1(0.23976241322412925, 0.1123721126346707)	unstable
L_2(-0.23976241322412925, 0.1123721126346707)	unstable
L_3(0, 0.10570563683271317)	unstable
L_4(0.24416862067148035, 0.7021319890586841)	unstable
L_5(-0.24416862067148035, 0.7021319890586841)	unstable
L_6(0, 0.614711219648052)	stable
L_7(0, 6.109431860113185)	unstable
L_8(0, -5.321006254852686)	stable

TABLE 4.1: Stability Analysis for Case 1: $t = 0.1$, $w = -1.66404$

4.2.3 Case II

For case II: $t = 0.2$, $w = -1.4484606465700836$ and $\mu_1 = 2.4098002600594515$, $\mu_2 = 8.569621485709742$ (see table (3.1)). Taking $m_4 = 1$ and following the same procedure as in Case 1, one can easily get the value of $m_1 = 2.4098002600594515$ and $m_2 = 8.569621485709742$. Using all these values in equations (4.11) and (42) and drawing the contour plot with help of Mathematica [20] (for Mathematica commands and code please see Appendix) as

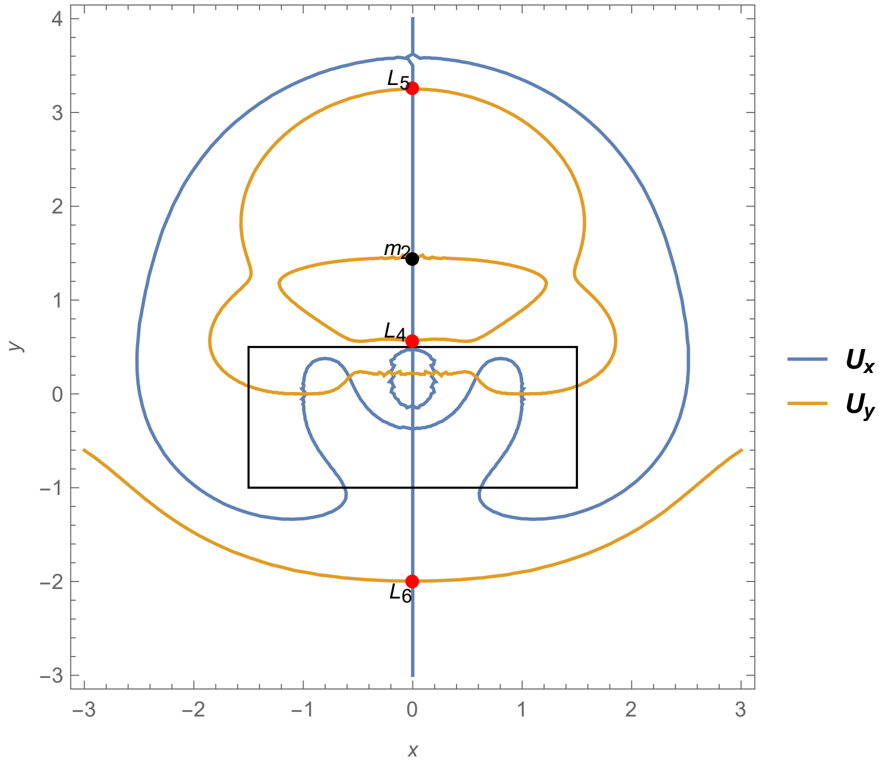

FIGURE 4.4: Contour Plot for Case II

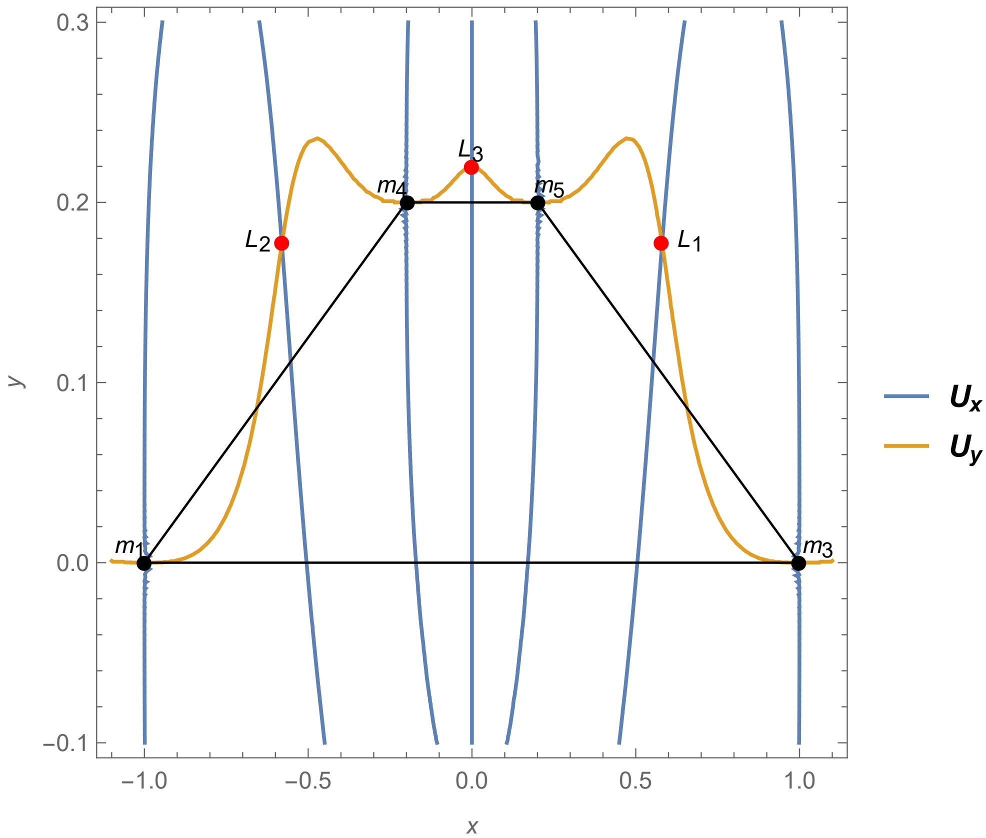

FIGURE 4.5: Contour Plot for Case II

The stability analysis of case II is now discussed in the following table, following the same procedure as in case I.

Equilibrium points	Stability
L_1(0.5804522038141618, 0.17726655699037705)	unstable
L_2(-0.5804522038141618, 0.17726655699037705)	unstable
L_3(0, 0.2199351103848044)	unstable
L_4(0, 0.5711855322922631)	unstable
L_5(0, 3.2505700490381577)	unstable
L_6(0, -1.9962089095551798)	stable

TABLE 4.2: Stability Analysis for Case II: $t = 0.2$, $w = -1.4484606465700836$

4.2.4 Case III

For case III: $t = 0.3$, $w = -1.3654982897601258$ and $\mu_1 = 2.4098002600594515$, $\mu_2 = 3.404243177846762$ (see table (3.1)).

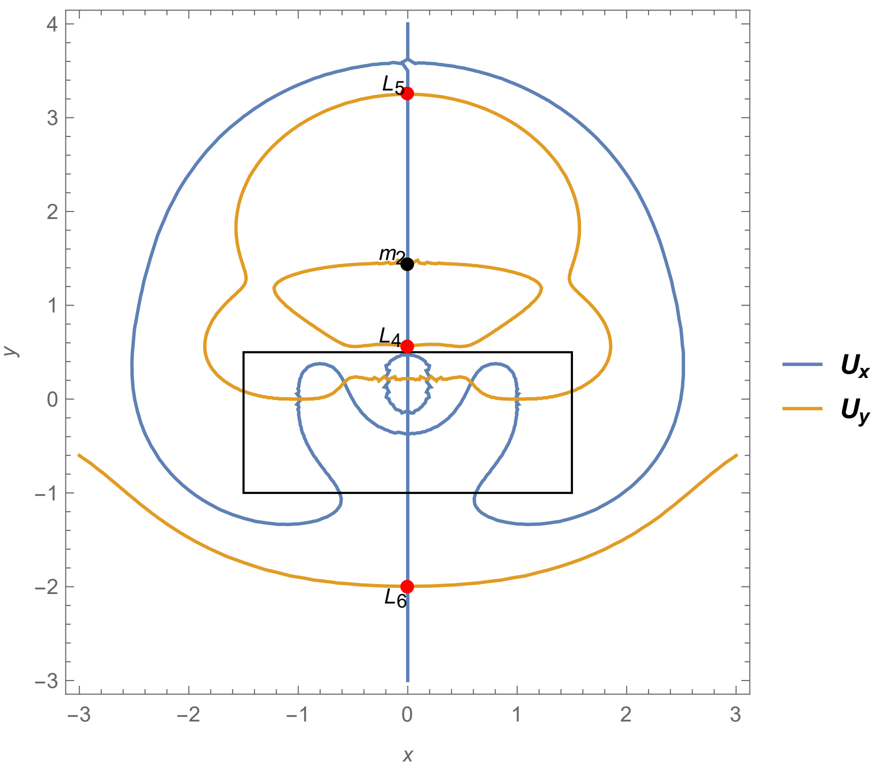

FIGURE 4.6: Contour Plot for Case III

Suppose $m_4 = 1$ and following the same procedure as in Case 1, one can easily get the value of $m_1 = 2.4098002600594515$ and $m_2 = 3.404243177846762$. For the following contour plot (for Mathematica commands and code please see Appendix) using these values in equations (4.11) and (4.12) and drawing the contour plot with help of Mathematica [21] as

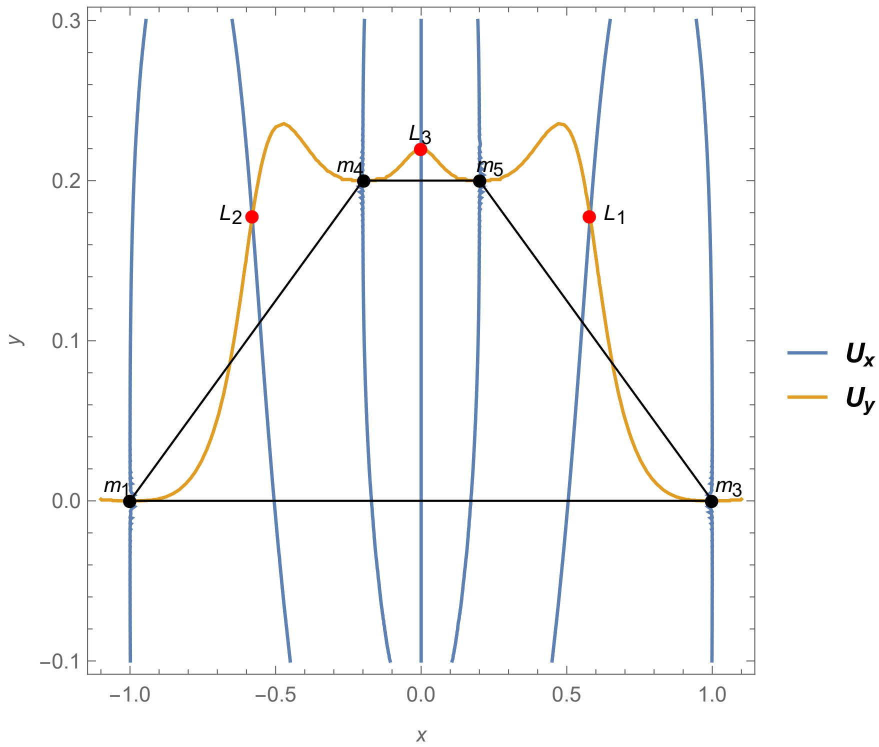

FIGURE 4.7: Contour Plot for Case III

Now following the same procedure as in Case I the stability analysis of Case III is given by:

Equilibrium points	Stability
L_1(0.5804522038141618‘, 0.17726655699037705)	unstable
L_2(-0.6028728540838072‘, 0.18622095274533304)	unstable
L_3(0‘, 0.3293026990503862)	unstable
L_4(0‘, 0.6965947870002095)	unstable
L_5(0‘, -1.7886272297115338)	stable
L_6(0‘, 2.721827016541708)	unstable

TABLE 4.3: Stability Analysis for Case III: $t = 0.3$, $w = -1.3654982897601258$

4.2.5 Case IV

For case IV: $t = 0.4$, $w = -1.4265597661033098$ and $\mu_1 = 1.0826720180786977$, $\mu_2 = 2.7233903520695977$ (see table (3.1)). Suppose $m_4 = 1$ and following the same procedure as in Case 1, one can easily get the value of $m_1 = 1.0826720180786977$ and $m_2 = 2.7233903520695977$ from the expressions of μ_1 and μ_2. For the following contour plot (for Mathematica commands and code please see Appendix) using these values in equations (4.11) and (4.12) and drawing the contour plot with help of Mathematica [22] as

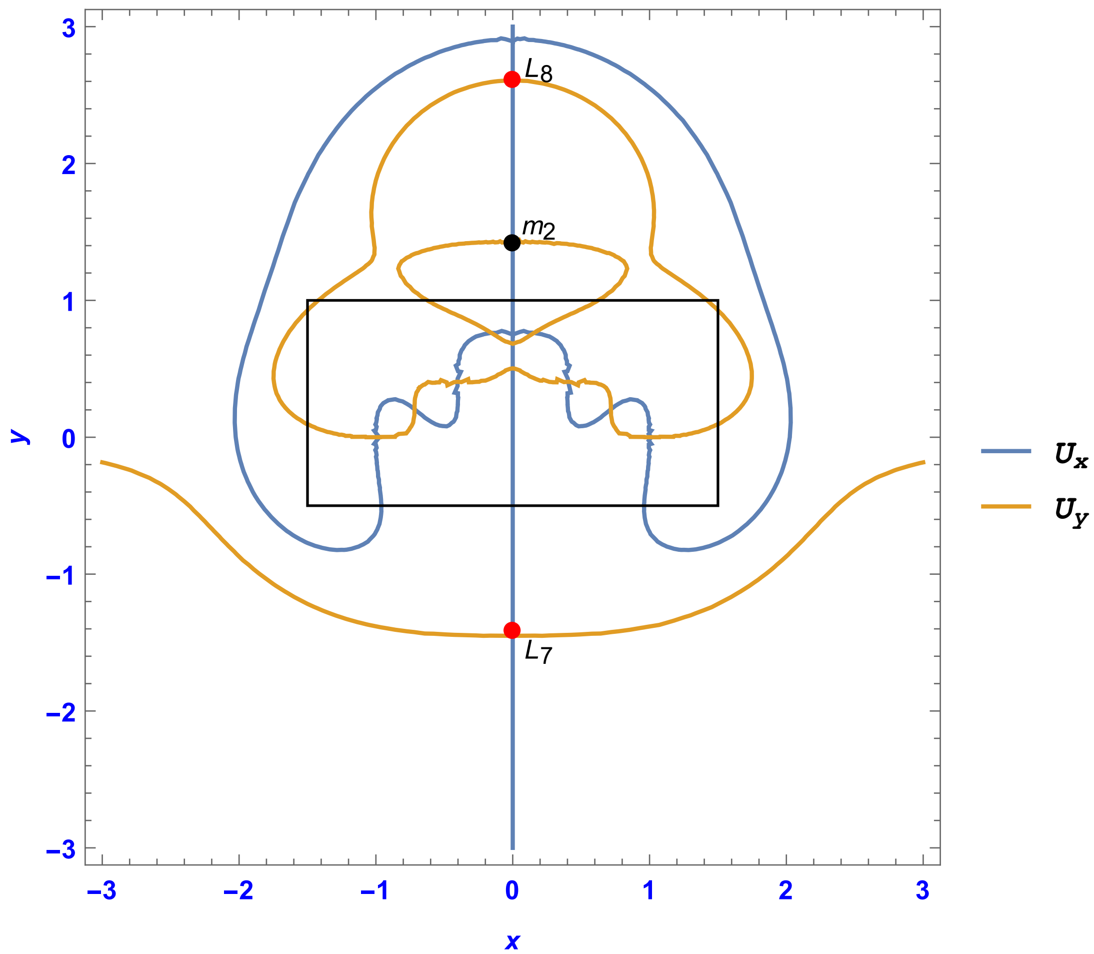

FIGURE 4.8: Contour Plot for Case IV

In the above figure U_x represent the blue and U_y represent the yellow colour.

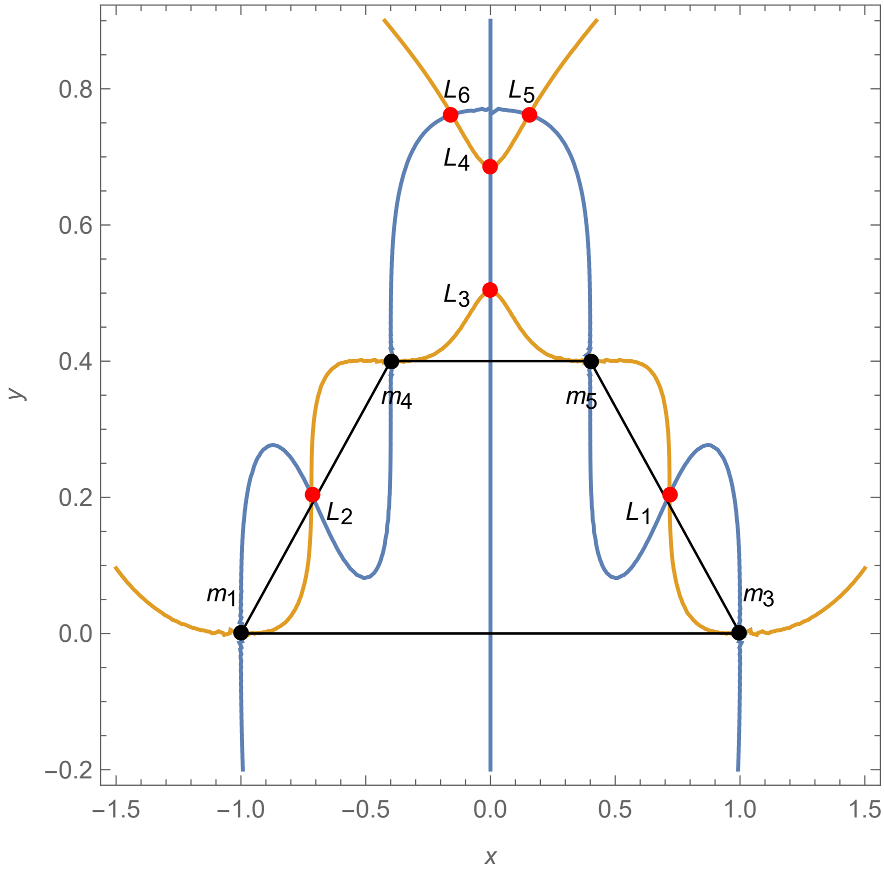

FIGURE 4.9: Contour Plot for Case IV

Now following the same procedure as in Case I the stability analysis of Case IV is given in the below table

Equilibrium points	Stability
L_1(0.7177605877576259, 0.2045771305252147)	unstable
L_2(-0.7177605877576259‘, 0.2045771305252147)	unstable
L_3(0, 0.5027520153490956)	unstable
L_4(0, 0.6854501385873034)	stable
L_5(0.15545298471717345‘, 0.7620251965667376)	unstable
L_6(-0.15545298471717345‘, 0.7620251965667376)	unstable
L_7(0, -1.42)	stable
L_8(0, 2.6065686150198)	unstable

TABLE 4.4: Stability Analysis for Case IV: $t = 0.4$, $w = -1.4265597661033098$

4.2.6 Case V

For case V: $t = 0.5$, $w = -1.5941967693943297$ and $\mu_1 = 0.7748701297751454$, $\mu_2 = 3.749627716706898$ (see table (3.1)). Again we suppose $m_4 = 1$ and following the same procedure as in Case 1, one can easily get the value of $m_1 = 0.7748701297751454$ and $m_2 = 3.749627716706898$ from the expressions of μ_1 and μ_2. In this case contour plot (for Mathematica commands and code please see Appendix) are

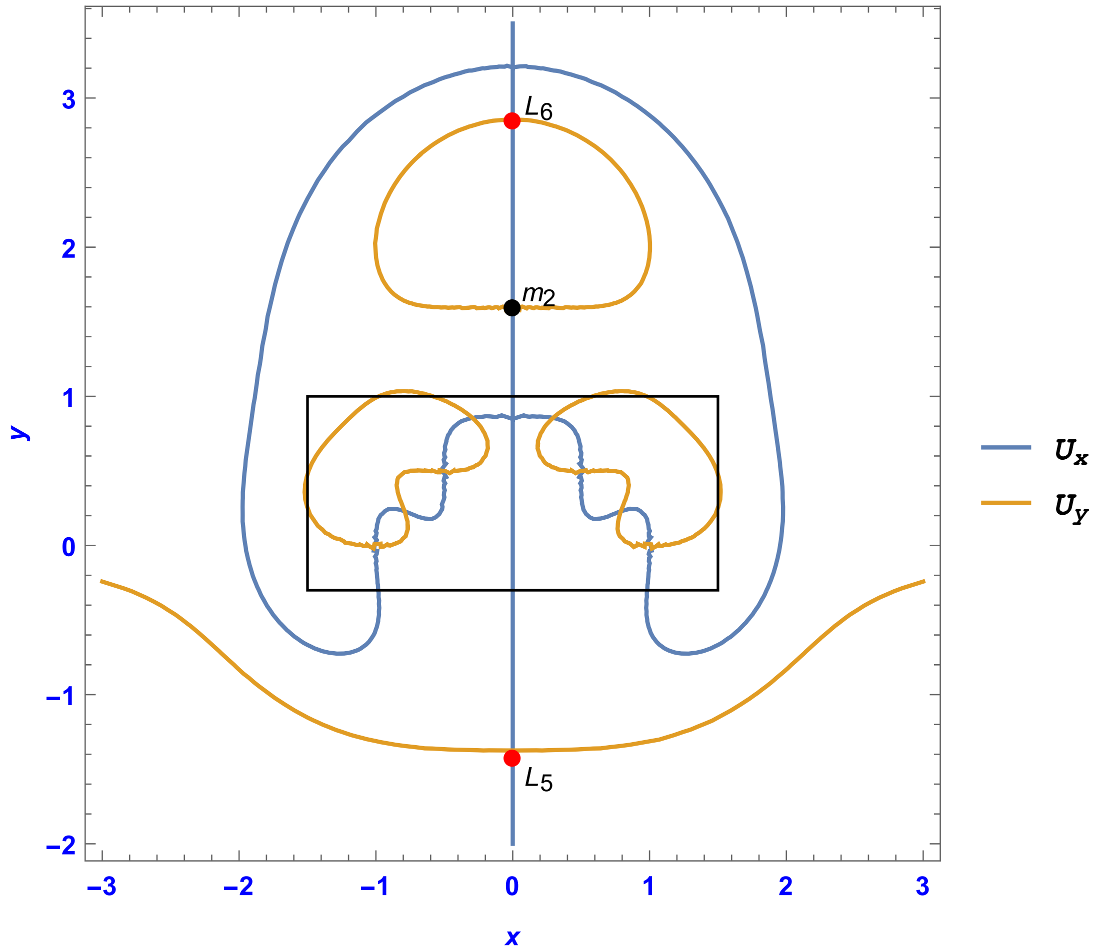

FIGURE 4.10: Contour Plot for Case V

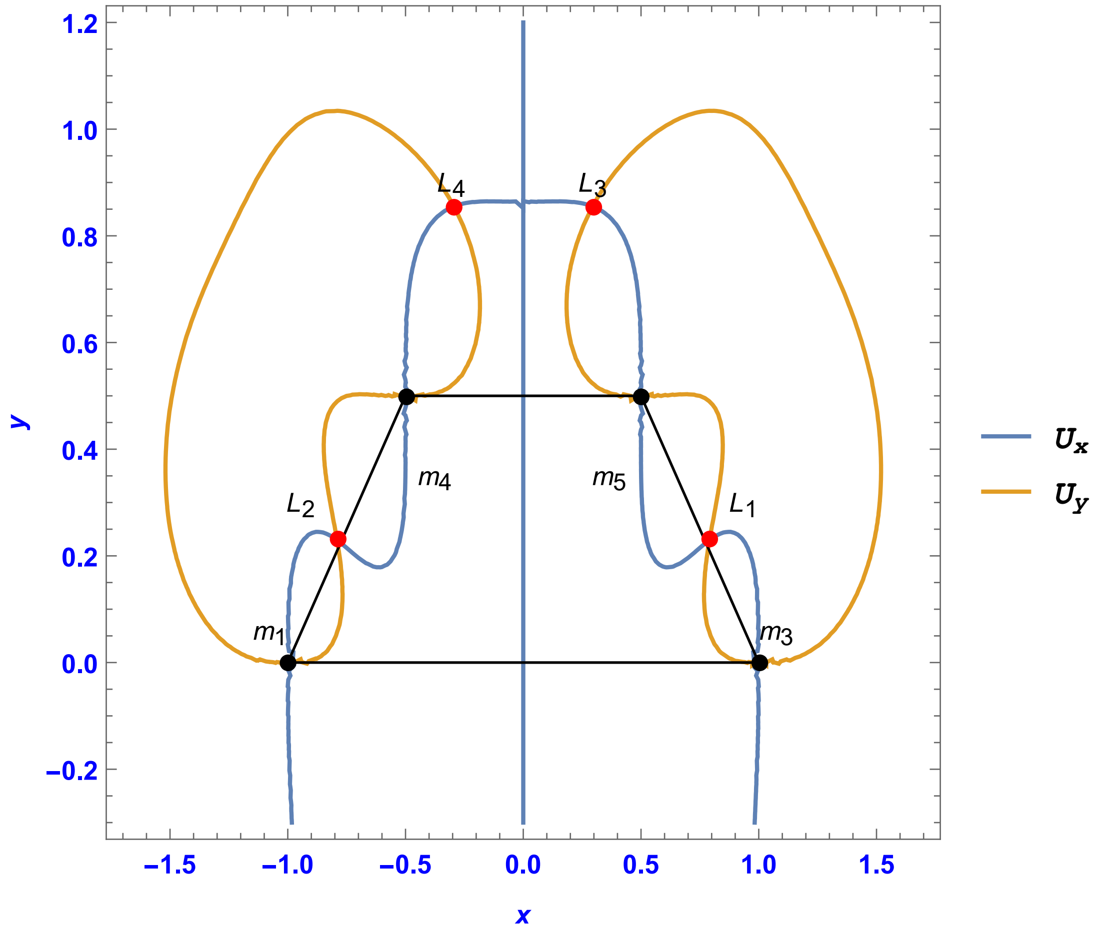

FIGURE 4.11: Contour Plot for Case V

Now following the same procedure as in Case I the stability analysis of Case V is given in the following table

Equilibrium points	Stability
L_1(0.7908694365676083‘, 0.22964236865830492)	unstable
L_2(-0.7908694365676083‘, 0.22964236865830492‘)	unstable
L_3(0.29512268018937554‘, 0.855175610810116)	unstable
L_4(-0.29512268018937554‘, 0.855175610810116)	unstable
L_5(0, -1.42)	stable
L_6(0,0, 2.8552320604711)	unstable

TABLE 4.5: Stability Analysis for Case V: $t = 0.5$, $w = -1.5941967693943297$

Chapter 5

Conclusions

We reviewed the CC's of symmetric isosceles trapezoid for 5BP such that four of the bodies are on the vertices's of an isosceles trapezoid and the fifth mass (m_2) can take various positions on the axis of symmetry both outside and inside the trapezoid. Different cases related to the problem were discussed and it was shown the existence and non existence of CC's related to each case. We picked that CC's curve figure (3.7) and analyses the motion of infinitesimal body m_6 in the gravitational field of five primaries $m_1 - m_5$ for different cases of CC's discussed in table (3.1). We also discussed equilibrium points of five different cases. The stability analysis is also discussed in each case. In Case I-Case V we have got $6, 8, 6, 6$ and 8 equilibrium points. Maximum points are unstable but couple of stable points also exist in each case. Interestingly, all stable points are along the y-axis on or off the isosceles trapezoid. There are no equilibrium points off the axes and along x-axis.

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