CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

MHD Squeezing Flow of Maxwell Nanofluid with Thermal Radiation

by

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A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

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I dedicate my thesis to

my beloved ***family****,* ***sisters*** *specially*

My mother*,*

A determined and aristocratic embodiment who educate me to belief in ALLAH,

believe in hard work and that so much could be done with little.

Muhammad Iqbal (late)

For my father I quote the remarkable words of Hadith,

"A father gives his child nothing better than an education."

CERTIFICATE OF APPROVAL

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Mehmood-ul-Hassan Tahir

Abstract

This thesis investigates the MHD squeezing flow of Maxwell nanofluid with thermal radiation. The proposed problem is modeled as a system of non-linear partial differential equations describing the conservation laws of mass, momentum and energy. The non-linear patrial differential equations are transformed into ordinary differential equations by applying the similarity transformation and are then solved numerically using the shooting technique together with RK4 method by implementing the computational software package MATLAB. The obtained analytical solutions are used to investigate the squeezing phenomena of the nanofluid when the plates are moving apart and when they are coming together. Also, the effect of different parameters such as magnetic field parameter Ha, Eckert number Ec, Deborah number De, and radiation parameter R on the velocity and temperature are analyzed.

Abbreviations

IVP	Initial Value Problem
MHD	Magnetohydrodynamics
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations

Symbols

A_1, A_2, A_3	Dimensionless Constants
C_p	Specific Heat Constant
k	Thermal Conductivity
B	Transverse Magnetic Field
B_0	Uniform Transverse Magnetic Field
H	Initial Position of the Plate
p	Dimensional Pressure
T	Temperature of Fluid
$\triangle T$	Temperature Gradient
T_H	Temperature at Initial Position of Plate
L	Characteristic Length
f	Dimensionless Velocity
h	Distance between the Plates
b	Constant
q_r	Radiative Heat Flux
Pr	Prandtl Number
Ec	Eckert Number
Ha	Hartmann Number
S	Squeezing Parameter
De	Deborah Number
R	Radiation Parameter
u, v	Velocity Components along x and y axis

Greek symbols

α	Thermal Diffusivity
α^*	Absorption Coefficient
η	Dimensionless Variable
ϕ	Volume Fraction of Nanoparticles
θ	Non-Dimensional Temperature
ρ	Density
μ	Dynamic Viscosity
ν	Kinematic Viscosity
σ	Electrical Conductivity
ν	Kinematic Viscosity
σ^*	Stefan-Boltzmann Constant
λ	Dimensionless Suction/Injection Parameter
δ	Dimensionless Length
τ	Shear Stress

Subscripts

f	Fluid
nf	Nanofluid
s	Nanoparticle

Chapter 1

Introduction

1.1 Squeezing Flow

A type of flow in which the material between two parallel plates or artifacts is pushed out or deformed is called the squeezing flow. Stefan [1] first explored in 1874, squeezing flow describes the outward movement of a droplet of liquid, its region of contact with the plate surfaces and the effects of internal and external influences such as temperature, viscoelasticity and heterogeneity of liquids. There are several squeezing flow models for the study of Newtonian and non-Newtonian fluids under various geometries and conditions. Examples of squeezing flow in practical use are numerous across scientific and engineering disciplines including rheometry, welding engineering and material scince. Reynolds [2] investigated the problems for elliptic plates in 1886. In 1972, Archibald [3] explored the squeezed flux across rectangular plates. The researchers have continuously made significant efforts day by day to make simpler and more efficient use of the squeezing flow. The impact of magnetic field in the squeezed flow among infinite parallel plates was identified by Siddique et al. [4]. They observed that for constant value of radiation parameter R and for different magnetic parameter values, velocity rises monotonically. Heat transfer qualities in a squeezing flux among parallel disks have been examined by Duwairi et al. [5]. They noted that as squeezing parameter increases, local coefficient friction reduces and rate of heat conduction enlarges. The unstable two-dimensional axisymmetric squeezing flow among parallel plates was discussed by Khan et al. [6]. They analyzed that the shape element does not influence the velocity of fluid. Domiarry and Aziz [7] provided the approximate analytical solution for the squeeze flow of viscous fluid with blowing or suction among parallel disks. They investigated that the increment in the electrical conductivity does not significantly affect the velocity profile. Due to its various applications, the examinate of heat transmission properties of squeezed viscous fluid flow has acquired extensive attention in several fields of science and engineering. The squeezed flow by a porous surface has been tackled by Mahmood et al. [8]. Analysis shows that with Prandtl number, the intensity of the local Nusselt number rises. Hayat et al. [9] protracted the research of Domiarry and Aziz [7] by implying second grade fluid to evaluate the gripping action of non-Newtonian fluid. The consequences are fulfilled by Mustafa et al. [10]. They observed that the impacts of thermophysical characteristics based on temperature are important for the field of flow. Hamdan and Baron [11] examined the squeezed movement of dirty fluid among side by side disks. Khaled and Vafai [12] studied the impacts of heat transfer and hydrodynamic flux on a horizontal surface. Qayyum et al. [13] achived an empirical analysis of the unstable squeezed flow of viscous Jeffery fluid among parallel disks considering the porosity and squeezed influences on the velocity profiles. It is right to call the modern age the era of science. It is worth mentioning that several papers have been released [14–18] on the mathematical and computational simulation of convective heat transfer process in nanofluids. Mandy [19] examined the heat transfer and mixed convection flow of nanofluids because of an unsteady shrinking sheet. Such simulations have certain advantages over experimental research because of several reasons impacting the nanofluid characteristics. Laterly, the heat transfer and flow properties of nanofluid under various flow structure were studied by Hatami et al. [20, 21]. Pourmehran et al. [22] debated empirical studies of unsteady squeezed nanofluid flow. They demonstrated that the maximum Nusselt number value can be achived by choosing silver as nanoparticles. Regarding complicated geometry, or two phase paradigm for nanofluid simulation, a new aspect has recently been introduced [23, 24]. Modern fields in science and engineering equipped with too many analytical and computational approaches to get valid, estimated results from non-linear equations recently. The methods are differential transformation method (DTM) [25–29], variation of parameter method (VPM) [30, 31], homotopy analysis method (HAM) [32, 33], homotopy perturbation method (HPM) [34], adomian decomposition method (ADM) [35, 36], etc. To enhance heat transfer diverse methods have been investigated in recent years [37].

1.2 Nanofluid

A nanofluid is a substance containing nanoparticles and the base fluid. Such fluids are colloidal clusters of nanoparticles embedded in a base fluid. The nanoparticles used in nanofluids are typically the nanotubes made from metals, oxides, carbides, or carbon. Choi and Eastman [38] were the first to suggest the word nanofluid that describes the fluid in which nanoparticles of poor thermal conductivity are suspended in the base fluid. Zhang et al. [39] developed the heat exchange and nanofluid flow in porous medium in the presence of radiation and magnetic field. They observed a rise in temperature profile with the influence of nanoparticles available in base fluid. Pak and Cho [40] used the titanium dioxide. Radiation impact on nanomaterial flow was explored by Zeeshan et al. [41] and they added the effect of MHD on transportation of titanium dioxide. They analyzed that nanoparticles suspended in base fluid induces a decline in nanofluid velocity. Ibrahim and Shankar [42] investigated the heat transfer and MHD nanofluid flow on a strech surface having the influence of slip conditions and thermal radiation. They found that by enlarging the values of velocity slip parameter, the skin friction coefficent reduces. Malvandi and Ganji [43] analyzed the impact of magnetic fields on the heat exchange inside a circular microchannel of alumina / water nanofluid. Haq et al. [44] acquired the impact of thermal radiation and the slip over the stretching sheet on the MHD nanofluid flow. Govindaraju et al. [45] resolved the issue of magnetohydrodynamic nanofluid flow on entropy production in a slip-velocity stretching layer. Uddin et al. [46] numerically analyzed the issue derived from the influence of slips and blowing on nanofluid bioconvection flow over a horizontal plate in motion. Kameswaran et al. [47] studied the chemical reaction and viscous dissipation effects of expanding or shrinking sheets on nanofluid flow. Matin and Popal [48] analyzed the mass and heat transfer flow in porous channel of a nanofluid with chemical reaction. Pal and Mandalal [49] noticed mixed-convection heat and mass exchange stagnation point flow across stretching surface in a porous channel. Elshehabey and Ahmed et al. [50] examined the effect of mixed nanofluid flow convection with sinusoidal temperature distribution on both vertical walls of the plates utilizing the nanofluid model of Buongiorno. As we glance over, we observe that current modern technology needs more development from an energy-saving perspective in heat transfer. Considering the low thermal conductivity of fluids such as kerosene, ethylene glycol and water, modern science provided encouragement of nanofluid where nano-sized particles are applied to the base fluid to improve the heat transfer capacities of the base fluid.

1.3 Magnetohydrodynamics

A vector field which describes the magnetic effect in relative motion of electrical charges and magnetized materials is known as magnetic field. The effects of magnetic fields are commonly seen in permanent magnets which pull on magnetic materials and attract or repel other magnets. MHD is the study of magnetic properties of electrically conducting fluid. In the magnetic field MHD plays vital role. For example, the charged particles and the conductive fluid both can pass against it quickly but seldom along the path of the magnetic field. There are also several functional variables which depend on fluid stability and containment properties and in magnetic configuration these variables are uniform. In the existence of the magnetic field, we will see how the fluid behaves at various angles. Singh et al. [51] and Seth et al. [52] inspected the impacts of inclined magnetic field on fluid flow. Inspired by the studies above, the present thesis involves the numerical study of squeezed movement of Cu-kerosene and water among two parallel plates with thermal radiation effects.

1.4 Thesis Contribution

The main aim of this research work is to investigate the squeezing flow of Cu-water and Cu-kerosene nanofluid between two parallel plates. The work of Çelik [53] is reviewed by reproducing their results. The problem is moddel as two coupled non-linear PDEs which are converted into a system of coupled ODEs. The numerical technique known as the shooting method has been utilized to solve the system. The impacts of different parameters such as magnetic field parameter Ha, squeezing parameter S, Eckert number Ec, Deborah number De and radiation parameter R on the velocity and temperature are analyzed graphically.

1.5 Thesis Outline

This thesis is further composed of four chapters:

Chapter 2 demonstrates some important definitions, concepts and laws that are helpful in understanding the present work.

Chapter 3 provides the details of numerical analysis of research paper of Çelik [53]. The whole work in this chapter is reproduced by using the shooting method. The non-linear coupled system of the PDEs is transformed into the system of ODEs. The impacts of various parameters are shown and discussed graphically.

Chapter 4 extends the work of Çelik [53] in view of additional impact of thermal radiation. The non-linear coupled system of PDEs is transformed into system of ODEs. The shooting method is utilized to solve the mathematical model. The impacts of different parameters are shown graphically.

Chapter 5 summarizes the overall analysis performed in this thesis and suggests few directions for the further scope of this thesis.

Chapter 2

Preliminaries

This chapter contains on basic definitions, concepts, and governing laws, which will be helpful in the subsequent chapters.

2.1 Important Definitions

Definition 2.1.1. (**Fluid**)
"A fluid is defined as a substance that deforms continuously when acted on by a shearing stress of any magnitude. A shearing stress (force per unit area) is created whenever a tangential force acts on a surface." [54]

Definition 2.1.2. (**Fluid Mechanics**)
"Fluid mechanics is that branch of science which deals with the behavior of the fluid (liquids or gases) at rest as well as in motion." [55]

Definition 2.1.3. (**Fluid Statics**)
"The study of fluid at rest is called fluid statics." [55]

Definition 2.1.4. (**Fluid Dynamics**)
"The study of fluid if the pressure forces are also considered for the fluids in motion, that branch of science is called fluid dynamics." [55]

Definition 2.1.5. (**Viscosity**)

"Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. Mathematically,

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}},$$

where μ is viscosity coefficient, τ is shear stress and $\frac{\partial u}{\partial y}$ represents the velocity gradient." [55]

Definition 2.1.6. (**Kinematic Viscosity**)

"It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by symbol ν called **'nu'**. Mathematically,

$$\nu = \frac{\mu}{\rho}.\text{"} \ [55]$$

Definition 2.1.7. (**Ideal Fluid**)

"A fluid which is incompressible and has no viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity." [55]

Definition 2.1.8. (**Real Fluid**)

"A fluid which possesses viscosity, is known as a real fluid. In actual practice, all the fluids are real fluids." [55]

Definition 2.1.9. (**Newtonian Fluid**)

"A real fluid in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as a Newtonian fluid." [55]

Definition 2.1.10. (**Non-Newtonian Fluid**)

"A real fluid in which the shear stress is not directly proportional to the rate of shear strain (or velocity gradient), is known as a Non-Newtonian fluid." [55]

Definition 2.1.11. (**Magnetohydrodynamics**)

"Magnetohydrodynamics is concerned with the mutual interaction of fluid flow and magnetic fields. The fluids in question must be electrically conducting and non-magnetic, which limits us to liquid metals, hot ionised gases (plasmas) and strong electrolytes." [56]

2.2 Types of Flows

Definition 2.2.1. (**Flow**)
"It is the deformation of the material under the influence of different forces. If the deformation increases continuously without any limit then the process is known as flow." [57]

Definition 2.2.2. (**Uniform Flow**)
"When the velocity of flow does not change either in magnitude or in direction at any point in a flowing fluid, for a given time, it is said to be a uniform flow. In other words, it is the flow of a fluid in which each particle moves along its line of flow with constant speed and the cross section of each stream tube remains unchanged." [57]

Definition 2.2.3. (**Non-uniform Flow**)
"When the velocity of flow changes at different points in a flowing fluid, for a given time, the flow is said to be a non-uniform flow." [57]

Definition 2.2.4. (**Compressible Flow**)
"Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid, Mathematically,

$$\rho \neq b,$$

where b is constant." [55]

Definition 2.2.5. (**Incompressible Flow**)
"Incompressible flow is that type of flow in which the density is constant for the fluid. Liquids are generally incompressible while gases are compressible, Mathematically,

$$\rho = b,$$

where b is constant." [55]

Definition 2.2.6. (**Steady Flow**)

"If the flow characteristics such as depth of flow, velocity of flow, rate of flow at any point in open channel flow do not change with respect to time, the flow is said to be steady flow. Mathematically,

$$\frac{\partial Q}{\partial t} = 0,$$

where Q is any fluid property." [55]

Definition 2.2.7. (**Unsteady Flow**)

"If at any point in open channel flow, the velocity of flow, depth of flow or rate of flow changes with respect to time, the flow is said to be unsteady. Mathematically,

$$\frac{\partial Q}{\partial t} \neq 0,$$

where Q is any fluid property." [55]

Definition 2.2.8. (**Internal Flow**)

"If the flows is completely bounded by a solid surfaces are called internal or duct flows." [57]

Definition 2.2.9. (**External Flow**)

"Flows over bodies immersed in an unbounded fluid are said to be an external flow." [57]

2.3 Fundamental Equations of Flow

2.3.1 Continuity Equation

"The conservation of mass of fluid entering and leaving the control volume, the resulting mass balance is called the equation of continuity. This equation reflects the fact that mass is conserved. Mathematically it can be write as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\ V) = 0.$$

For steady case rate of time will be constant, so continuity equation becomes

$$\nabla.(\rho\ V) = 0.$$

In the case of incompressible flow, density does not variate so continuity equation can be re-written as:

$$\nabla.V = 0.$$

where, V is velocity of fluid." [57]

2.3.2 Momentum Equation

"The net force F acting on a fluid particle is equal to the time rate of change of linear momentum. Consider the mass in a system denoted by control surface of infinitesimally small dimensions dx, dy and dz. The mass of the system is steady. Newtons second law can be composed as:

$$m\frac{DV}{Dt} = F.$$

$$\rho\frac{DV}{Dt} = \nabla.\tau + \rho b,$$

where ρb, is the net body force, $\nabla.\tau$ is the surface forces and τ is the Cauchy stress tensor." [57]

2.3.3 Energy Equation

"Energy can be transferred to or from a closed system by heat or work, and the conservation of energy principle requires that the net energy transfer to or from a system during a process be equal to the change in energy content of the system.

Control volume involves energy transfer via mass flow also, and the conservation of energy principle, also called the energy balance, is expressed as:

$$E_{in} - E_{out} = \frac{dE}{dt}.$$

where E_{in} and E_{out} are the total rates of energy transfer into and out of the control volume respectively, and $\frac{dE}{dt}$ is the rate of change of energy within the control volume boundaries." [57]

2.4 Modes of Heat Transfer and Related Properties

Definition 2.4.1. (**Heat Transfer**)
"Heat transfer is a branch of engineering that deals with the transfer of thermal energy from one point to another within a medium or from one medium to another due to the occurrence of a temperature difference." [58]

Definition 2.4.2. (**Conduction**)
"The transfer of heat within a medium due to a diffusion process is called conduction." [58]

Definition 2.4.3. (**Convection**)
"Convection heat transfer is usually defined as energy transport effected by the motion of a fluid. The convection heat transfer between two dissimilar media is governed by Newton's law of cooling." [58]

Definition 2.4.4. (**Thermal radiation**)
"Thermal radiation is defined as radiant (electromagnetic) energy emitted by a medium and is solely to the temperature of the medium." [58]

2.5 Dimensionless Numbers

Definition 2.5.1. (**Prandtl Number**)

"It is the ratio between the momentum diffusivity ν and thermal diffusivity α. Mathematically, it can be defined as

$$Pr = \frac{\nu}{\alpha} = \frac{\frac{\mu}{\rho}}{\frac{k}{C_p \rho}} = \frac{\mu C_p}{k}$$

where μ represents the dynamic viscosity, Cp denotes the specific heat and k stands for thermal conductivity. The relative thickness of thermal and momentum boundary layer is controlled by Prandtl number. For small Pr, heat distributed rapidly corresponds to the momentum." [57]

Definition 2.5.2. (**Eckert Number**)

"It is the dimensionless number used in continuum mechanics. It describes the relation between flows and the boundary layer enthalpy difference and it is used for characterized heat dissipation. Mathematically,

$$Ec = \frac{u^2}{C_p \nabla T}\text{"} \ [57]$$

Definition 2.5.3. (**Hartmann Number**)

"Hartmann number Ha is the ratio of electromagnetic force to the viscous force. It is frequently encountered in fluid flows through magnetic fields. It is defined by:

$$Ha = BL\sqrt{\frac{\sigma}{\mu}}$$

where B is the magnetic field intensity, L is the characteristic length scale, σ is the electrical conductivity and μ is the dynamic viscosity." [57]

Definition 2.5.4. (**Deborah Number**)

"It is defined as the ratio of relaxation time to deformation time, It is denoted by γ. Mathematically,

$$\gamma = \frac{\lambda U}{2x}.\text{"} \ [59]$$

2.6 Solution Methodology

"In a shooting method, the missing (unspecified) initial condition at the initial point of the interval is assumed, and the differential equation is then integrated numerically as an initial value problem to the terminal point. The accuracy of the assumed missing initial condition is then checked by comparing the calculated value of the dependent variable at the terminal point with its given value there. If a difference exists, another value of the missing initial condition must be assumed and the process is repeated. This process is continued until the agreement between the calculated and the given condition at the terminal point is within the specified degree of accuracy. For this type of iterative approach, one naturally inquires whether or not there is systematic way of finding each succeeding (assumed) value of the missing initial condition. In this method, the differential equation is kept in its nonlinear form and the missing slope is found systematically by Newton's method. This method provides quadratic convergence of the iteration and is far better than the usual cut-and-try methods. Consider the second-order differential equation.

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \tag{2.1}$$

subject to the boundary conditions

$$y(0) = 0, \quad y(L) = A. \tag{2.2}$$

First, *Eq.* (2.1) is written in terms of a system of two first-order differential equations:

$$\frac{dy}{dx} = u, \quad \frac{du}{dx} = f(x, y, u) \tag{2.3}$$

We denote the missing initial slope by

$$\frac{dy(0)}{dx} = s, \quad \text{or} \quad u(0) = s \tag{2.4}$$

The problem is to find 's' such that the solution of *Eq.* (2.3), subject to the initial conditions (2.2) and (2.4), satisfies the boundary condition at the second point, (2.2). In other words, if the solutions of the initial value problem are denoted by $y(x, s)$ and $u(x, s)$, one searches for the value of s such that

$$y(L, s) - A = \phi(s) = 0 \tag{2.5}$$

For the Newton's method, the iteration formula for s is given by

$$s^{(n+1)} = s^{(n)} - \frac{\phi(s^{(n)})}{\frac{d\phi(s^{(n)})}{ds}} \tag{2.6}$$

or

$$s^{(n+1)} = s^{(n)} - \frac{y(L, s^n) - A}{\frac{\partial y(L, s^n)}{\partial s}} \tag{2.7}$$

To find the derivative of y with respect to s, *Eqs.* (2.3), (2.2), and (2.4) are differentiated with respect to s, and we get

$$\frac{dY}{dx} = U, \ \frac{dU}{dx} = \frac{\partial f}{\partial y}Y + \frac{\partial f}{\partial u}U \tag{2.8}$$

and

$$Y(0) = 0, \ U(0) = 1 \tag{2.9}$$

where

$$Y = \frac{\partial y}{\partial s}, \quad U = \frac{\partial u}{\partial s} \tag{2.10}$$

The solution of *Eq.* (2.3), subject to the boundary conditions (2.2), can therefore be obtained by the following steps.

1. Assume a value of s for the missing initial slope, (2.4). Let us denote this approximative value of s by $s^{(1)}$.

2. Integrate *Eq.* (2.3), subject to the boundary conditions (2.2) and (2.4), as an initial value problem from $x = 0$ to $x = L$.

3. Integrate *Eq.* (2.8), subject to the boundary conditions (2.9), as an initial value problem from $x = 0$ to $x = L$.

4. Substituting the values of $y(L, s^{(1)})$ and $Y(L, s^{(1)})$ into *Eq.* (2.7), we get

$$s^{(2)} = s^{(1)} - \frac{[y(L, s^{(1)}) - A]}{Y(L, s^{(1)})} \tag{2.11}$$

the next approximation of the missing initial slope $s^{(2)}$ is obtained.

5. Repeat steps 2-4 until the value of s is within the specified degree of accuracy." [60]

Chapter 3

Numerical Study of Squeezing Flow of Nanofluids of Cu-Water and Kerosene between Two Parallel Plates

3.1 Introduction

This chapter provides the review study of Çelik [53]. In this chapter, consideration has been given to the numerical analysis of incompressible, viscous nanofluid flow and heat transmission among two parallel plates exhibiting a squeezing movement. Meanwhile the governing non-linear PDEs are transformed into a system of ODEs by utilizing the similarity transformations. By using shooting technique implementing MATLAB, the solution of ODEs is obtained. At the end of this chapter the numerical solution for various parameters such as magnetic field parameter Ha, squeezing parameter S, Eckert number Ec, Deborah number De, nanoparticle volume fraction ϕ and radiation parameter R is discussed for the dimensionless velocity and temperature. Investigation of obtained numerical results is given through graphs.

3.2 Problem Formulation

In this chapter, the numerical investigation of the viscous, incompressible, squeezing nanofluid flow and transmission of heat among parallel plates has been taken into account as shown in Figure 3.1. The coordinate system is selected as, the x-axis is along the plate and y-axis is normal to it. In addition, the fluid is flowing in the presence of a magnetic field B and all the body forces are ignored. The distance between the plates is $h(t) = H(1 - \alpha t)^{\frac{1}{2}}$, where H is the initial position of the plate at time $t = 0$ and α is characteristic parameter.

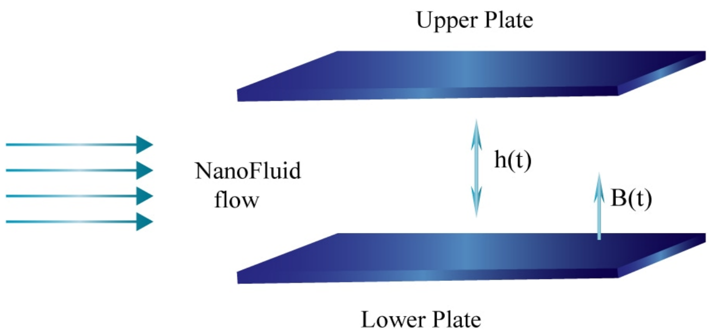

FIGURE 3.1: Physical model of the problem

3.2.1 The Governing Equations

The flow is described by the continuity equation, momentum equation, and energy equation as:

- Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

- x-Momentum Equation:

$$\rho_{nf}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu_{nf}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \sigma_{nf}B^2(t)u, \tag{3.2}$$

- y-Momentum Equation:

$$\rho_{nf}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu_{nf}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{3.3}$$

- Energy Equation:

$$\begin{aligned}&\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{k_{nf}}{(\rho C_p)_{nf}}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{\mu_{nf}}{(\rho C_p)_{nf}}\\&\left(4\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right),\end{aligned} \tag{3.4}$$

Here u and v represents the velocity components along x and y direction respectively. T denotes the temperature of the nanofluid, p denotes the pressure, ρ represents the density, C_p is specific heat, μ denotes the dynamic viscosity, and κ denotes the thermal conductivity.

Dimensional Boundary Conditions

The imposed boundry conditions for the problem are given by:

$$\left.\begin{aligned}&u=0,\ v=\frac{dh}{dt},\ T=T_H \text{ at } y\ =\ h(t),\\&\frac{\partial u}{\partial y}=0,\ v=0,\ \frac{\partial T}{\partial y}=\ 0 \text{ at } y\ =0.\end{aligned}\right\} \tag{3.5}$$

The following similarity transformation has been introduced for transforming the mathematical model Eq. (3.1) - Eq. (3.4) into the dimensionless form [53]:

$$\left.\begin{aligned}&u=\frac{\alpha x}{2(1-\alpha t)}f'(\eta),\quad v=\frac{-\alpha H}{2\sqrt{1-\alpha t}}f(\eta),\\&\theta=\frac{T}{T_H},\quad \eta=\frac{y}{H\sqrt{1-\alpha t}}.\end{aligned}\right\} \tag{3.6}$$

Now using Eq. (3.6) into Eq. (3.1). For this we differentiate 'u' and 'v' w.r.t 'x' and 'y' respectively.

$$\begin{aligned}\frac{\partial u}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{\alpha x}{2(1-\alpha t)}f'(\eta)\right),\\\frac{\partial u}{\partial x}&=\frac{\alpha}{2(1-\alpha t)}f'(\eta).\end{aligned} \tag{3.7}$$

$$\begin{aligned}\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-\alpha H}{2\sqrt{1-\alpha t}}f(\eta)\right),\\ \frac{\partial v}{\partial y} &= -\frac{\alpha}{2(1-\alpha t)}f'(\eta).\end{aligned} \tag{3.8}$$

Using Eq. (3.7) and Eq. (3.8) in Eq. (3.1) to satisfy continuity equation,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\alpha}{2(1-\alpha t)}f'(\eta)-\frac{\alpha}{2(1-\alpha t)}f'(\eta)=0. \tag{3.9}$$

Now we include the procedure for the conversion of Eq. (3.2) and Eq. (3.3) into dimensionless form.

We have to convert momentum equations into single ODE to utilize similarity transformation. Since 'v' does not depend on x, so the derivative of v approaches to zero and second derivative of 'u' also approaches to zero.

In order to use Eq. (3.6) into Eq.(3.2) and Eq.(3.3), we differentiate above equation w.r.t 't',

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left(f'(\eta)\frac{\alpha x}{2(1-\alpha t)}\right),\\ \frac{\partial u}{\partial t} &= \frac{\alpha x}{2}\left(f'(\eta)\frac{\alpha}{(1-\alpha t)^2}+\frac{f''(\eta)}{(1-\alpha t)}\frac{\alpha y}{2H(1-\alpha t)^{\frac{3}{2}}}\right),\\ \frac{\partial u}{\partial t} &= \frac{\alpha x}{2}\left(f'(\eta)\frac{\alpha}{(1-\alpha t)^2}+\frac{f''(\eta)\alpha}{2(1-\alpha t)^2}\frac{y}{H\sqrt{1-\alpha t}}\right),\\ \frac{\partial u}{\partial t} &= \frac{\alpha^2 x}{2(1-\alpha t)}\left(f'(\eta)+\eta\frac{f''(\eta)}{2}\right).\end{aligned} \tag{3.10}$$

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{-\alpha H}{2\sqrt{1-\alpha t}}f(\eta)\right),\\ \frac{\partial v}{\partial t} &= \frac{-\alpha H}{2}\left(\frac{\alpha}{2(1-\alpha t)^{\frac{3}{2}}}f(\eta)+\frac{f'(\eta)}{\sqrt{1-\alpha t}}\frac{\alpha y}{2H(1-\alpha t)^{\frac{3}{2}}}\right),\\ \frac{\partial v}{\partial t} &= \frac{-\alpha^2 H}{4(1-\alpha t)^{\frac{3}{2}}}\left(f(\eta)+f'(\eta)\,\eta\right).\end{aligned} \tag{3.11}$$

Similarly, we differentiate the above equation w.r.t 'x', we have

$$\frac{\partial u}{\partial x}=\frac{\alpha}{2(1-\alpha t)}f'(\eta). \tag{3.12}$$

$$\frac{\partial^2 u}{\partial x^2} = 0. \tag{3.13}$$

$$\frac{\partial v}{\partial x} = 0. \tag{3.14}$$

$$\frac{\partial^2 v}{\partial x^2} = 0. \tag{3.15}$$

Next, we differentiate the above equation w.r.t 'y', we have

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left(f'(\eta)\frac{\alpha x}{2(1-\alpha t)}\right), \\
\frac{\partial u}{\partial y} &= \frac{\alpha x}{2(1-\alpha t)} f''(\eta)\frac{\partial \eta}{\partial y}, \\
\frac{\partial u}{\partial y} &= \frac{\alpha x}{2(1-\alpha t)} f''(\eta)\frac{1}{H\sqrt{1-\alpha t}}, \\
\frac{\partial u}{\partial y} &= \frac{\alpha x}{2H(1-\alpha t)^{\frac{3}{2}}} f''(\eta).
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\alpha x}{2H(1-\alpha t)^{\frac{3}{2}}} f''(\eta)\right), \\
\frac{\partial^2 u}{\partial y^2} &= \left(\frac{\alpha x}{2H(1-\alpha t)^{\frac{3}{2}}} f'''(\eta)\frac{\partial \eta}{\partial y}\right), \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\alpha x}{2H^2(1-\alpha t)^2} f'''(\eta).
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
\frac{\partial v}{\partial y} &= \frac{-\alpha H}{2\sqrt{1-\alpha t}} f'(\eta)\frac{\partial \eta}{\partial y}, \\
\frac{\partial v}{\partial y} &= \frac{-\alpha}{2(1-\alpha t)} f'(\eta).
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
\frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{-\alpha}{2(1-\alpha t)} f'(\eta)\right), \\
\frac{\partial^2 v}{\partial y^2} &= \frac{-\alpha}{2(1-\alpha t)} f''(\eta)\frac{\partial \eta}{\partial y}, \\
\frac{\partial^2 v}{\partial y^2} &= \frac{-\alpha}{2(1-\alpha t)} f''(\eta)\frac{1}{H\sqrt{1-\alpha t}}, \\
\frac{\partial^2 v}{\partial y^2} &= \frac{-\alpha}{2H(1-\alpha t)^{\frac{3}{2}}} f''(\eta).
\end{aligned} \tag{3.19}$$

Substituting these values from Eq. (3.10) to Eq. (3.19) into Eq. (3.2) and Eq. (3.3), Eq. (3.2)⇒

$$\rho_{nf}\left[\frac{\alpha^2 x}{4(1-\alpha t)^2}\left(2f^{'}(\eta)+f^{''}(\eta)\ (\eta)+(f^{'})^2(\eta)-f^{''}(\eta)\ f(\eta)\right)\right]$$
$$=-\frac{\partial p}{\partial x}+\mu_{nf}\frac{\alpha x}{2H^2(1-\alpha t)^2}f^{'''}(\eta)-\sigma_{nf}B^2(t)u. \tag{3.20}$$

Eq. (3.3)⇒

$$\rho_{nf}\left(\frac{-\alpha^2 H}{4(1-\alpha t)^{\frac{3}{2}}}\left(f(\eta)+f^{'}(\eta)\ \eta\right)+(0)+\frac{-\alpha H}{2\sqrt{1-\alpha t}}f(\eta)-\frac{\alpha}{2(1-\alpha t)}f^{'}(\eta)\right)$$
$$=-\frac{\partial p}{\partial y}+\mu_{nf}\left((0)+\frac{-\alpha}{2H(1-\alpha t)^{\frac{3}{2}}}f^{''}(\eta)\right),$$
$$\rho_{nf}\left(\frac{-\alpha^2 H}{4(1-\alpha t)^{\frac{3}{2}}}\left(f(\eta)+f^{'}\ \eta-f(\eta)f^{\prime}(\eta)\right)\right)=-\frac{\partial p}{\partial y}-\mu_{nf}$$
$$\left(\frac{\alpha}{2H(1-\alpha t)^{\frac{3}{2}}}f^{''}(\eta)\right). \tag{3.21}$$

Now differentiating Eq. (3.21) w.r.t 'x' and Eq. (3.20) w.r.t 'y',

Eq. (3.21) ⇒

$$\frac{\partial^2 p}{\partial x \partial y}=0. \tag{3.22}$$

Eq. (3.20)⇒

$$\rho_{nf}\frac{\alpha^2 x}{4(1-\alpha t)^2}\left(2f^{''}(\eta)\frac{1}{H\sqrt{1-\alpha t}}+f^{'''}(\eta)(\eta)\frac{1}{H\sqrt{1-\alpha t}}+f^{''}(\eta)\frac{1}{H\sqrt{1-\alpha t}}\right.$$
$$\left.+2f^{'}(\eta)f^{''}(\eta)\frac{1}{H\sqrt{1-\alpha t}}-f^{'''}(\eta)f(\eta)\frac{1}{H\sqrt{1-\alpha t}}-f^{''}(\eta)f^{'}(\eta)\frac{1}{H\sqrt{1-\alpha t}}\right)$$
$$=\frac{\partial^2 p}{\partial x \partial y}+\mu_{nf}\frac{\alpha x}{2H^2(1-\alpha t)^2}f^{''''}(\eta)\frac{1}{H\sqrt{1-\alpha t}}$$
$$-\sigma_{nf}B_0(t)\frac{\alpha x}{2H(1-\alpha t)^{\frac{5}{2}}}f^{''}(\eta). \tag{3.23}$$

Putting Eq. (3.22) in Eq. (3.23), we have,

$$\rho_{nf}\left(\frac{\alpha}{2}\left(f^{'''}\eta+3f^{''}+f^{''}f^{'}-f^{'''}f^{'}\right)\right)=\mu_{nf}\frac{1}{H^2}f^{''''}-\sigma_{nf}B_0^2(t)f^{''},$$

multiplying by H^2 and dividing by μ_{nf}, we have

$$f^{''''} - \frac{\rho_{nf}}{\mu_{nf}}\left(\frac{\alpha H^2}{2}\left(f^{'''}\eta + 3f^{''} + f^{''}f^{'} - f^{'''}f\right)\right) - \frac{\sigma_{nf}}{\mu_{nf}}B_0^2(t)f^{''} = 0. \tag{3.24}$$

$$\therefore \frac{\rho_{nf}}{\mu_{nf}} = \frac{(1-\Phi)^{2.5}}{\mu_f}\left(\Phi\rho_s + (1-\Phi)\rho_f\right),$$

multiplying and dividing by ρ_f

$$\frac{\rho_{nf}}{\mu_{nf}} = \frac{(1-\Phi)^{2.5}}{\nu_f}\left(\Phi\frac{\rho_s}{\rho_f} + (1-\Phi)\right).$$

Eq (3.24) becomes

$$f'''' - SA_1(1-\Phi)^{2.5}\left(f'''\eta + 3f'' + f''f' - f'''f\right) - (Ha)^2 f'' = 0.$$

Now we have to convert energy equation into ODE to utilize similarity transformation, in Eq. (3.4) T does not depend on 'x', so the derivative of T approaches to zero.

In order to use Eq. (3.6) into Eq. (3.4), we differentiate 'T' w.r.t 't',

$$\begin{aligned}\frac{\partial T}{\partial t} &= T_H\theta^{'}\frac{\partial \eta}{\partial t},\\ \frac{\partial T}{\partial t} &= T_H\theta^{'}\frac{\alpha\eta}{2(1-\alpha t)}.\end{aligned} \tag{3.25}$$

Similarly, differentiating 'T' w.r.t 'x' and 'y', we have,

$$\begin{aligned}\frac{\partial T}{\partial x} &= \frac{\partial}{\partial y}\left(\theta T_H\right),\\ \frac{\partial T}{\partial x} &= 0.\end{aligned} \tag{3.26}$$

$$\frac{\partial^2 T}{\partial x^2} = 0. \tag{3.27}$$

$$\begin{aligned}\frac{\partial T}{\partial y} &= \frac{\partial}{\partial y}\left(\theta T_H\right),\\ \frac{\partial T}{\partial y} &= \frac{\theta^{'}T_H}{H\sqrt{1-\alpha t}}.\end{aligned} \tag{3.28}$$

$$
\begin{aligned}
\frac{\partial^2 T}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\theta' T_H}{H\sqrt{1-\alpha t}}\right), \\
\frac{\partial^2 T}{\partial y^2} &= \frac{\theta'' T_H}{H^2(1-\alpha t)}.
\end{aligned} \tag{3.29}
$$

Similarly, we have,

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left(f'(\eta)\frac{\alpha x}{2(1-\alpha t)}\right), \\
\frac{\partial u}{\partial y} &= \frac{\alpha x}{2(1-\alpha t)} f''(\eta)\frac{\partial \eta}{\partial y}, \\
\frac{\partial u}{\partial y} &= \frac{\alpha x}{2(1-\alpha t)} f''(\eta)\frac{1}{H\sqrt{1-\alpha t}}, \\
\frac{\partial u}{\partial y} &= \frac{\alpha x}{2H(1-\alpha t)^{\frac{3}{2}}} f''(\eta).
\end{aligned} \tag{3.30}
$$

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\alpha x}{2(1-\alpha t)} f'(\eta)\right), \\
\frac{\partial u}{\partial x} &= \frac{\alpha}{2(1-\alpha t)} f'(\eta).
\end{aligned} \tag{3.31}
$$

Substituting values from Eq. (3.25) to Eq. (3.31) in Eq. (3.4), we have,

$$
\begin{aligned}
&\frac{\alpha\eta T_H\theta'}{2(1-\alpha t)} - \frac{\alpha H}{2\sqrt{1-\alpha t}} f(\eta)\frac{\theta' T_H}{H\sqrt{1-\alpha t}} = \frac{k_{nf}}{(\rho C_p)_{nf}}\left(\frac{\theta'' T_H}{H^2(1-\alpha t)}\right) \\
&+ \frac{\mu_{nf}}{(\rho C_p)_{nf}}\left[4\left(\frac{\alpha}{2(1-\alpha t)} f'(\eta)\right)^2 + \frac{\alpha^2 x^2}{4H^2(1-\alpha t)^3}(f'')^2\right], \\
&\frac{\alpha T_H}{2(1-\alpha t)}\left(\theta'\eta - \theta' f\right) = \frac{1}{(1-\alpha t)}\left(\frac{k_{nf}}{(\rho C_p)_{nf}}\frac{\theta'' T_H}{H^2}\right) \\
&+ \frac{\mu_{nf}}{(\rho C_p)_{nf}}\frac{\alpha^2}{(1-\alpha t)^2}\left((f')^2 + \frac{x^2}{4H^2(1-\alpha t)}(f'')^2\right), \\
&\theta'' + \left(\theta' f - \theta'\eta\right)\frac{\alpha T_H}{2}\frac{(\rho C_p)_{nf}}{k_{nf}}\frac{H^2}{T_H} + \frac{\mu_{nf}}{k_{nf}}\frac{H^2}{T_H}\frac{\alpha^2}{(1-\alpha t)} \\
&\left((f')^2 + \frac{x^2}{4H^2(1-\alpha t)}(f'')^2\right) = 0, \\
&\theta'' + \left(\theta' f - \theta'\eta\right)\frac{\alpha H^2}{2}\frac{(1-\Phi)(\rho C_p)_f + \Phi(\rho C_p)_s}{A_3} k_f + \frac{\mu_f}{(1-\Phi)^{2.5}A_3 k_f}\frac{H^2}{T_H} \\
&\frac{\alpha^2}{(1-\alpha t)}\left((f')^2 + \frac{x^2}{4H^2(1-\alpha t)}(f'')^2\right) = 0.
\end{aligned}
$$

$$\theta'' + \frac{(\rho C_p)_f}{(\rho C_p)_f}\frac{\mu_f}{\rho_f}\frac{\rho_f}{\mu_f}\left(\theta' f - \theta'\eta\right)\frac{\alpha}{2H^2}\frac{(1-\Phi)(\rho C_p)_f + \Phi(\rho C_p)_s}{A_3}k_f$$
$$+ \frac{(\rho C_p)_f}{(\rho C_p)_f}\frac{\mu_f}{\rho_f}\frac{\rho_f}{\mu_f}\frac{\mu_f}{(1-\Phi)^{2.5}A_3k_f}\frac{H^2}{T_H}\frac{\alpha^2}{(1-\alpha t)}\left((f')^2 + \frac{x^2}{4H^2(1-\alpha t)}(f'')^2\right) = 0,$$
$$\theta'' + \left(\theta' f - \theta'\eta\right)\frac{A_2}{A_3}PrS + \frac{PrEc}{A_3(1-\Phi)^{2.5}}\left((f'')^2 + 4(\delta)^2(f')^2\right) = 0. \tag{3.32}$$

Now for converting the associated boundary conditions into the dimensionless form, the following steps have been taken:

$$u = 0 \ at \ y = h(t), \quad \frac{\alpha x}{2(1-\alpha t)}f'(\eta) = 0,$$
$$\frac{\alpha x}{2(1-\alpha t)} \neq 0,$$
$$f'(\eta) = 0, \ \text{when} \ \eta \to 1, \ f'(1) = 0.$$
$$v = \frac{dh}{dt} \ at \ y = h(t), \ v = \frac{-\alpha H}{2\sqrt{1-\alpha t}}f(\eta),$$
$$\frac{dh}{dt} = \frac{-\alpha H}{2\sqrt{1-\alpha t}},$$
$$\frac{-\alpha H}{2\sqrt{1-\alpha t}} = \frac{-\alpha H}{2\sqrt{1-\alpha t}}f(\eta),$$
$$f(\eta) = 1, \ \text{when} \ \eta \to 1, \ f(1) = 1.$$
$$T = T_H \ at \ y = h(t), \ \theta T_H = T_H,$$
$$\text{when} \ \eta \to 1, \ \theta(1) = 1.$$
$$\frac{\partial u}{\partial y} = 0 \ at \ y = 0, \ \text{so} \ \frac{\alpha x}{2H(1-\alpha t)^{\frac{3}{2}}}f''(\eta) = 0,$$
$$\frac{\alpha x}{2H(1-\alpha t)^{\frac{3}{2}}} \neq 0, \ f''(\eta) = 0,$$
$$\text{when} \ \eta \to 0, \ f''(0) = 0.$$
$$v = 0 \ at \ y = 0, \ \frac{-\alpha H}{2\sqrt{1-\alpha t}}f(\eta) = 0,$$
$$\frac{-\alpha H}{2\sqrt{1-\alpha t}} \neq 0, \ f(\eta) = 0,$$
$$\text{when} \ \eta \to 0, \ f(0) = 0.$$
$$\frac{\partial T}{\partial y} = 0 \ at \ y = 0, \ \frac{\partial T}{\partial y} = \theta' T_H\frac{1}{H\sqrt{1-\alpha t}}, \ T_H\frac{1}{H\sqrt{1-\alpha t}} \neq 0,$$
$$\theta'(\eta) = 0, \ \text{when} \ \eta \to 0, \ \theta'(0) = 0.$$

The final dimensionless form of the governing model is:

$$f'''' - SA_1(1-\Phi)^{2.5}\left(f'''\eta + 3f'' + f''f' - f'''f\right) - (Ha)^2 f'' = 0. \quad (3.33)$$

$$\theta'' + \left(\theta' f - \theta'\eta\right)\frac{A_2}{A_3}PrS + \frac{PrEc}{A_3(1-\Phi)^{2.5}}\left((f'')^2 + 4(\delta)^2(f')^2\right) = 0. \quad (3.34)$$

The associated boundary conditions of Eq. (3.33) and Eq. (3.34) shown as:

$$\left.\begin{aligned} &f(0)=0, f''(0)=0, \theta'(0)=0,\\ &f(1)=1, f'(1)=0, \theta(1)=1. \end{aligned}\right\} \quad (3.35)$$

Followinng parameters are used in Eq. (3.33) and Eq. (3.34):

$$A_1 = (1-\Phi) + \Phi\frac{\rho_s}{\rho_f},\ A_2 = (1-\Phi) + \Phi\frac{(\rho C_p)_s}{(\rho C_p)_f},\ A_3 = \frac{k_{nf}}{k_f},\ Pr = \frac{\mu_f(\rho C_p)_f}{\rho_f k_f},$$
$$\delta = \frac{H(1-\alpha t)^{\frac{1}{2}}}{x},\ Ec = \frac{\rho_f}{T_H(\rho C_p)_f}\left(\frac{\alpha x}{2(1-\alpha t)}\right)^2,\ Ha = \sqrt{\frac{\sigma_{nf}}{\mu_{nf}}}HB_0, S = \frac{\alpha H^2}{2v_f}.$$

3.3 Solution Methodology

For solving Eq. (3.33) with associated boundary conditions (3.35), we used shooting method. First of all we convert the fourth order ODE into the system of first order ODEs. In order to solve IVP, we used the RK4 method and assume the missing initial conditions. Now for Eq. (3.33), we used following notations:

$$f = f_1,\quad f' = f_1' = f_2,\quad f'' = f_2' = f_3,\ f''' = f_3' = f_4,\quad f'''' = f_4'. \quad (3.36)$$

The resulting initial value problem takes the form:

$$f_1' = f_2; \qquad f_1(0) = 0, \quad (3.37)$$

$$f_2' = f_3; \qquad f_2(0) = r, \quad (3.38)$$

$$f_3' = f_4; \qquad f_3(0) = 0, \quad (3.39)$$

$$f_4' = A_1S(1-\Phi)^{2.5}\left(f'''\eta + 3f'' + f''f' - f'''f\right) + (Ha)^2 f''; \qquad f_4(0) = s. \quad (3.40)$$

Missing conditions 'r' and 's' are assumed to satisfy the following relations:

$$f_1(1, r, s) - 1 = 0, \tag{3.41}$$

$$f_2(1, r, s) = 0. \tag{3.42}$$

We use Newton's method for two equations with two variables to solve Eq. (3.41) and Eq. (3.42),

$$\begin{bmatrix} r^{(n+1)} \\ s^{(n+1)} \end{bmatrix} = \begin{bmatrix} r^{(n)} \\ s^{(n)} \end{bmatrix} - \left(\begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial s} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial s} \end{bmatrix} \right)^{-1}_{(r=r^{(n)}, s=s^{(n)})} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{3.43}$$

where $n = 0, 1, 2, ...$

We utilize following notations:

$$\begin{aligned} \frac{\partial f_1}{\partial r} &= f_5, & \frac{\partial f_2}{\partial r} &= f_6, \\ \frac{\partial f_3}{\partial r} &= f_7, & \frac{\partial f_4}{\partial r} &= f_8, \\ \frac{\partial f_1}{\partial s} &= f_9, & \frac{\partial f_2}{\partial s} &= f_{10}, \\ \frac{\partial f_3}{\partial s} &= f_{11}, & \frac{\partial f_4}{\partial s} &= f_{12}. \end{aligned} \tag{3.44}$$

Using above notations in Eq. (3.43),

$$\begin{bmatrix} r^{(n+1)} \\ s^{(n+1)} \end{bmatrix} = \begin{bmatrix} r^{(n)} \\ s^{(n)} \end{bmatrix} - \left(\begin{bmatrix} f_5 & f_9 \\ f_6 & f_{10} \end{bmatrix} \right)^{-1}_{(r=r^{(n)}, s=s^{(n)})} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Differentiating Eq. (3.37) to Eq. (3.40) and using notations Eq. (3.44), we have first order ODEs,

$$f_1' = f_2; \qquad f_1(0) = 0, \tag{3.45}$$

$$f_2' = f_3; \qquad f_2(0) = r, \tag{3.46}$$

$$f_3' = f_4; \qquad f_3(0) = 0, \tag{3.47}$$

$$f_4' = A_1 S(1 - \Phi)^{2.5} (3f_3 + f_4\eta + f_2 f_3 - f_1 f_4) + (Ha)^2 f_3; \qquad f_4(0) = s, \tag{3.48}$$

$$f_5' = f_6; \qquad f_5(0) = 0, \tag{3.49}$$

$$f_6' = f_7; \qquad f_6(0) = 1, \tag{3.50}$$

$$f_7' = f_8; \qquad f_7(0) = 0, \quad (3.51)$$

$$f_8' = A_1 S(1-\Phi)^{2.5}\left(3f_7 + f_8\eta + (f_2f_7 + f_3f_6) - (f_1f_8 + f_4f_5)\right) + (Ha)^2 f_7;$$
$$f_8(0) = 0, \quad (3.52)$$

$$f_9' = f_{10}; \qquad f_9(0) = 0, \quad (3.53)$$

$$f_{10}' = f_{11}; \qquad f_{10}(0) = 0, \quad (3.54)$$

$$f_{11}' = f_{12}; \qquad f_{11}(0) = 0, \quad (3.55)$$

$$f_{12}' = A_1 S(1-\Phi)^{2.5}\left(3f_{11} + f_{12}\eta + (f_2F_{11} + f_3f_{10}) - (f_1f_{12} + f_4f_9)\right) + (Ha)^2 f_{11};$$
$$f_{12}(0) = 1. \quad (3.56)$$

RK4 method is used to solve the IVP and the initial values are choosen arbitrarily. During the execution of iterations these initial guesses will be updated by the Newton's method and the process will be continued until the following criteria is met.

$$max(|f_1(1) - 1|, |f_2(1) - 0|) < \epsilon.$$

Throughout this work, ϵ has been taken as 10^{-6} unless otherwise mentioned. Now for Eq. (3.34), following notations are used:

$$\begin{aligned} \theta &= z_1, \\ \theta' &= z_1' = z_2, \\ \theta'' &= z_1'' = z_2'. \end{aligned} \quad (3.57)$$

Initial value problem takes the form:

$$z_1' = z_2; \qquad z_1(0) = \xi, \quad (3.58)$$

$$z_2' = -PrS\frac{A_2}{A_3}(z_2f_1 - z_2\eta) - \frac{PrEc}{A_3(1-\Phi)^{2.5}}\left((f_3)^2 + 4(\delta)^2(f_2)^2\right);$$
$$z_2(0) = 0. \quad (3.59)$$

Missing condition 'ξ' assumed to satisfy the following relation :

$$z_1(1,\xi) - 1 \;= 0.$$

We use Newton's method for Eq. (3.60),

$$\xi_{n+1} = \xi_n - \frac{Z(\xi)}{Z'(\xi)}, \tag{3.60}$$

$$\text{where} \quad Z(\xi) = z_1(1, \xi) - 1.$$

Following notations are used:

$$\frac{\partial z_1}{\partial \xi} = z_3, \qquad \frac{\partial z_2}{\partial \xi} = z_4.$$

Differentiating Eq. (3.58) and Eq. (3.59) w.r.t 'ξ',we have,

$$z_3' = z_4; \qquad z_3(0) = 1. \tag{3.61}$$

$$z_4' = -PrS\frac{A_2}{A_3}\left(z_4 f_1 - z_4\eta\right); \qquad z_4(0) = 0. \tag{3.62}$$

The RK4 method is used to solve IVP for some suitable choice ξ. The missing condition ξ is updated by the Newton's method and process will be continued until the following criteria is met.

$$|z_1(1) - 1| < \epsilon.$$

Throughout this work, ϵ is taken as 10^{-6} unless otherwise mentioned.

3.4 Results and Discussion

This section's aim is to examine the numerical results demonstrated in the graphical form. Computations are conducted for different values of the effect of squeezeing parameter S, nanoparticle volume fraction ϕ, Hartmann number Ha, Eckert number Ec and impact of these parameters on velocity and temperature are debated.

Figure 3.2 illustrates the behaviour of velocity $f'(\eta)$ of Cu-water and Cu-kerosene for different values of squeezing parameter S. The negative and positive squeeze numbers impact velocity differently. The squeezing number physically explains how the plate moves. It can be seen clearly from Figure 3.2 that velocity of the fluid $f'(\eta)$ close to the lower plate surface periodically declines with a rising value of S, and this value increases as we step away from the surface of lower plate. Figure 3.3 demonstrates the influence of temperature $\theta(\eta)$ of Cu-water and Cu-kerosene for different values of squeezing parameter S. The positive and negative squeeze numbers have different effects on temperature. When $S<0$ thermal boundry layer thickness increases. Therefore, nanofluid temperature rises. It is clear from Figure 3.3 that nanofluid flow temperature decreases as S increases. If the squeezing parameter increases, the friction among the nanoparticles and boundry surface of the plates reduces due to decreased kinematic viscosity and speed. Consequently, nanofluid temperatue decreases.

Figure 3.4 and 3.5 demonstrate the impact of the volume fraction of nanoparticles on fluid velocity for negative and positive S. When $S < 0$, further collisions arise among nanoparticles and particles at the boundry surface of plates as the volume fraction of nanoparticles rises. For this reason, the velocity of fluid diminish close to the boundry and rises near the central region. For $S > 0$, if the volume fraction of the nanoparticles arises, the boundry layer decreases close to the plate, while the boundry layer increases far away from the plate surface.

For $S < 0$ and $S > 0$, Figure 3.6 and 3.7 demonstrate the impact of nanoparticles fraction on nanofluid temperatures. The thickness of thermal boundry layer increases by introducing nanoparticles to the base fluid. It is a sign that as the fluid moves, the nanoparticles drain the heat from boundry layer. Due to their high thermal conductivity, heat transfer may increase if nanoparticles are added to the base fluid, so temperature may decrease.

Figure 3.8 and 3.9 show the influence of magnetic field parameter on velocity distribution for $S<0$ and $S>0$. This is because, enlarging the value of Hartmann number, the Lorentz force, which resists the flow, induces a reduction in the maximum velocity due to the retarding effect of the magnetic force and thus reduces the thickness of the boundry momentum sheet. It is found that the values of Ha gradually decreases the velocity profile. In addition, it means two plates are quite similar to one another while $S < 0$, the scenario then produces adverse pressure gradient along with retarding Lorentz force. If these forces operate over a long period of time, a divergence point may occur and the back flow may occur. The explanation for $S > 0$ is very special. As two plates pass back, an open space exist and fluid moves at high speed in that region. So, there will be no violation of the conservation law of mass flow.

Figure 3.10 and 3.11 demonstrate the impact of magnetic field parameter on temperature of nanofluid for positive and negative S. Figure 3.10 and 3.11 demonstrate that as the Hartmann number increases the temperature of nanofluid declines. Impact becomes more pronounced for $S < 0$ and the Cu-kerosene temperature is the greater than the Cu-water nanofluid. For $S > 0$ the result is the positive. This is worthy of note for $S > 0$ we observe a temperature profile crossover at $\eta = 0.82$. But, the impact is trivial after that area.

The influence of Eckert number Ec on nanofluid temperature for both positive and negative S is shown in Figure 3.12 and 3.13. It is seen from Figure 3.12 and 3.13 as the rise in Eckert number, the thermal boundry layer declines and the temperature of nanofluid among two parallel plates rises due to a rise in the viscous dissipation effect.

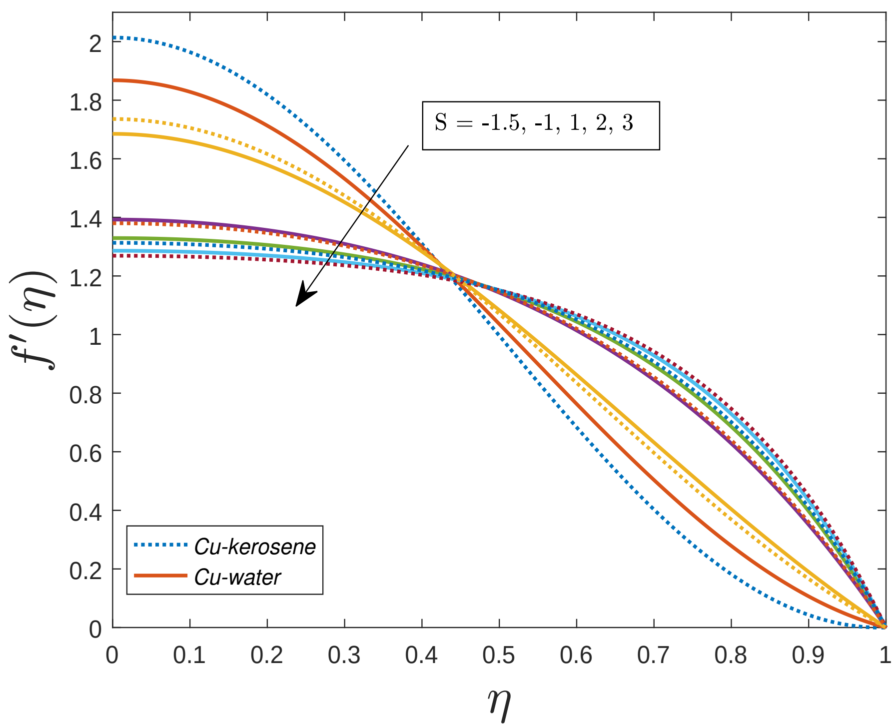

FIGURE 3.2: Impacts of S on $f'(\eta)$.

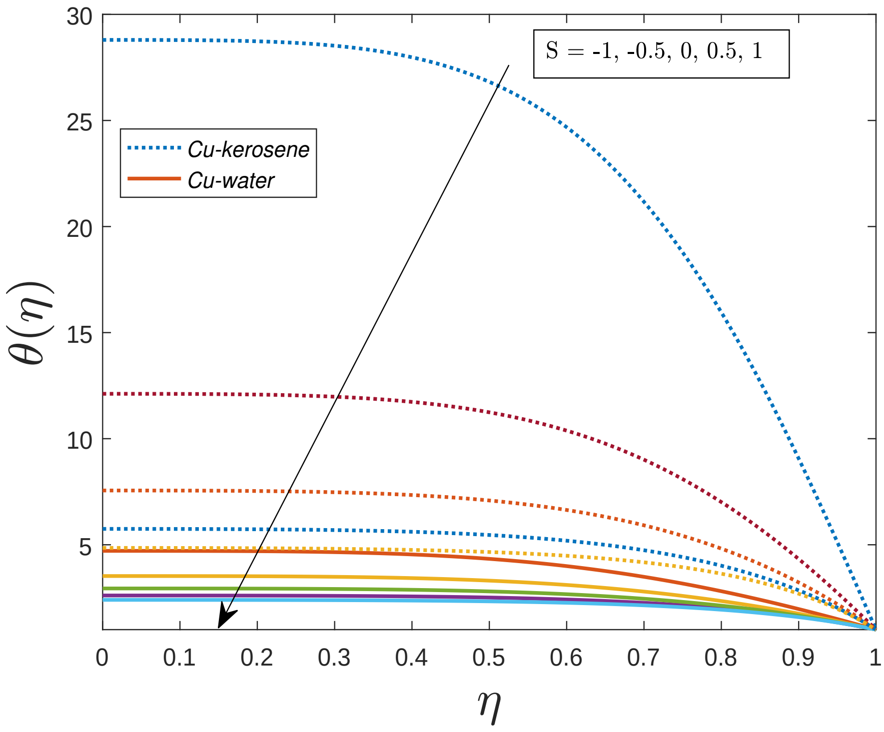

FIGURE 3.3: Impacts of S on $\theta(\eta)$.

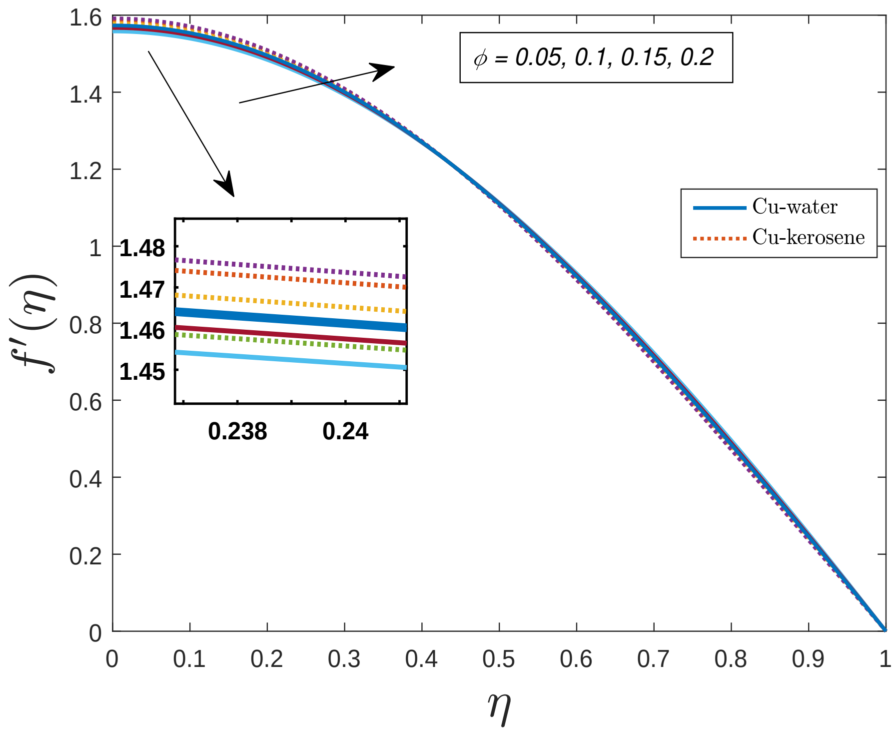

FIGURE 3.4: Impacts of ϕ on $f'(\eta)$ when $S < 0$.

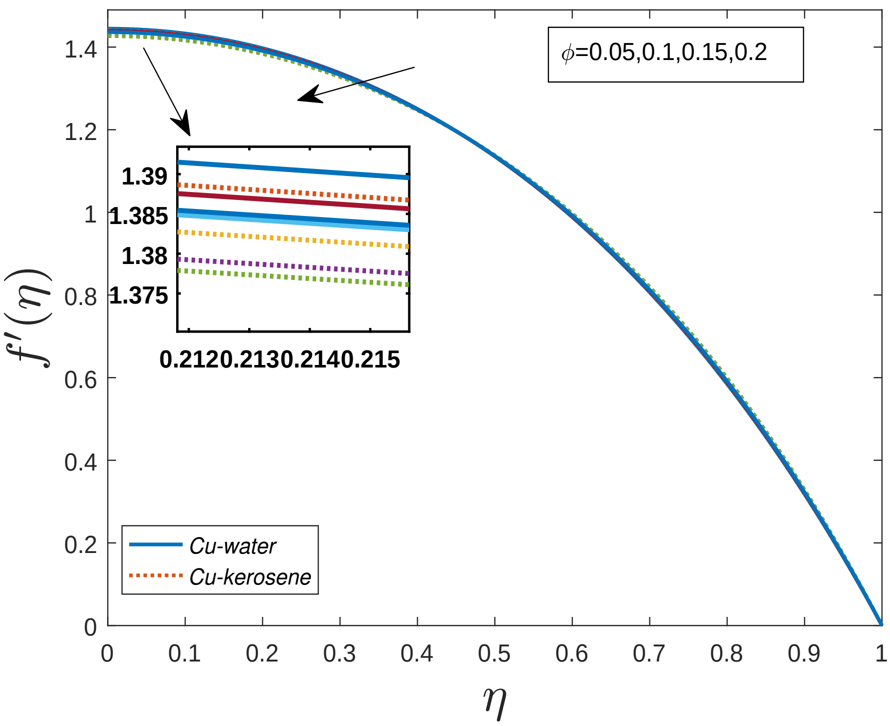

FIGURE 3.5: Impacts of ϕ on $f'(\eta)$ when $S > 0$.

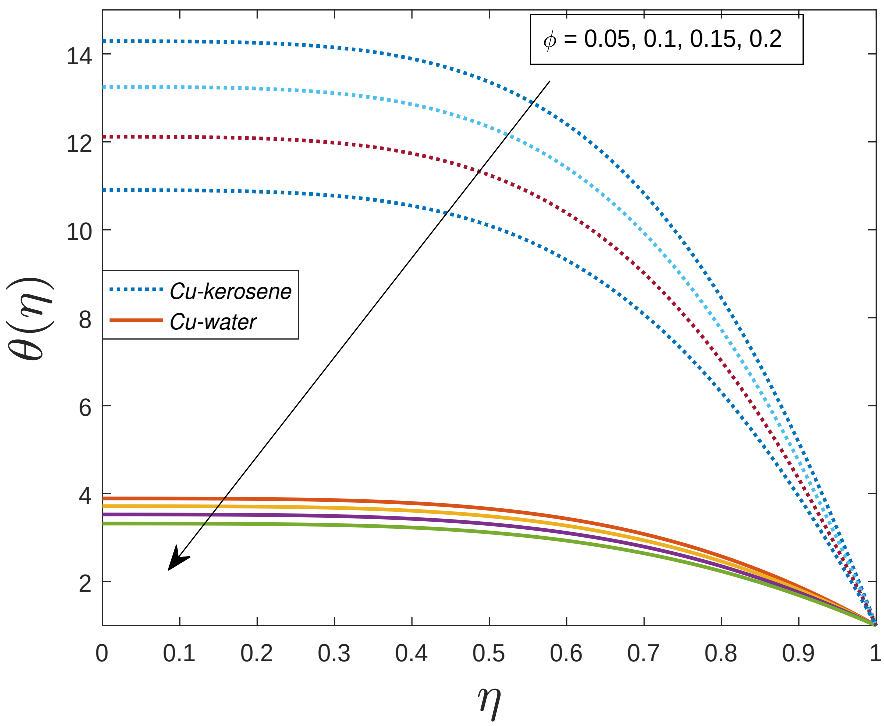

FIGURE 3.6: Impacts of ϕ on $\theta(\eta)$ when $S < 0$.

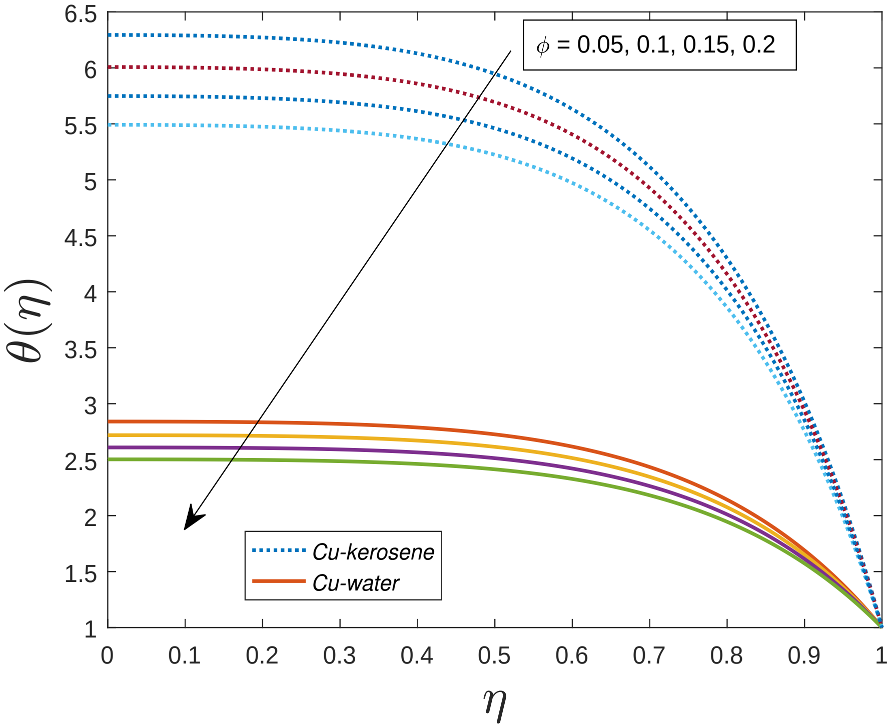

FIGURE 3.7: Impacts of ϕ on $\theta(\eta)$ when $S > 0$.

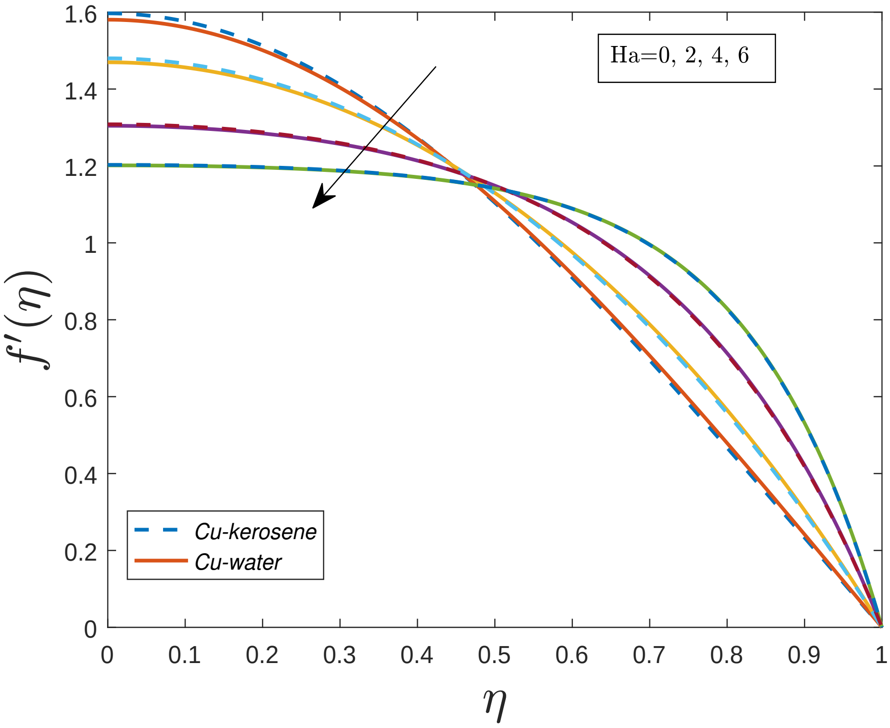

FIGURE 3.8: Impacts of Ha on $f'(\eta)$ when $S < 0$.

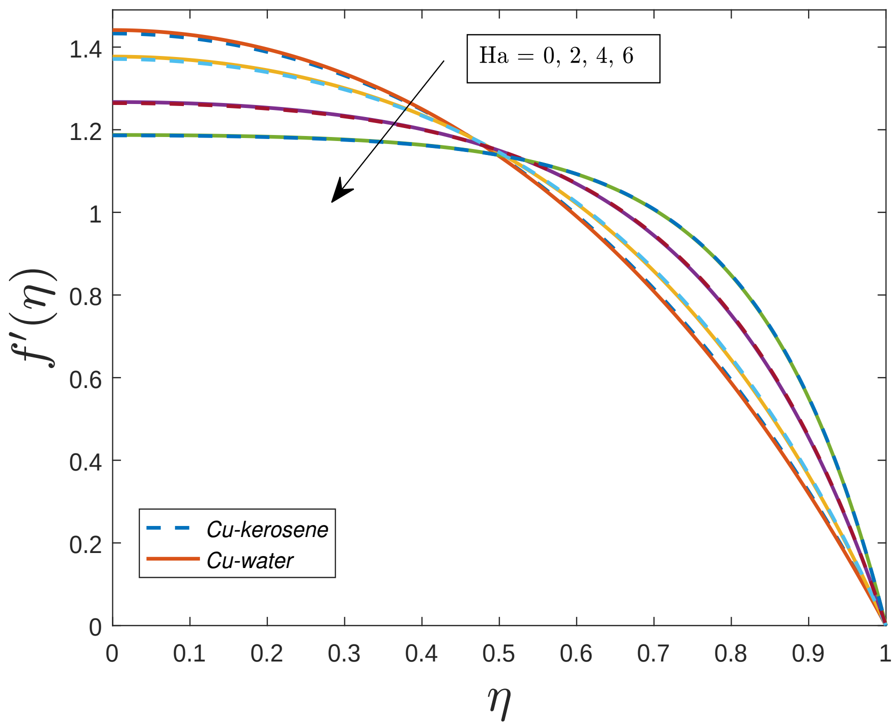

FIGURE 3.9: Impacts of Ha on $f'(\eta)$ when $S > 0$.

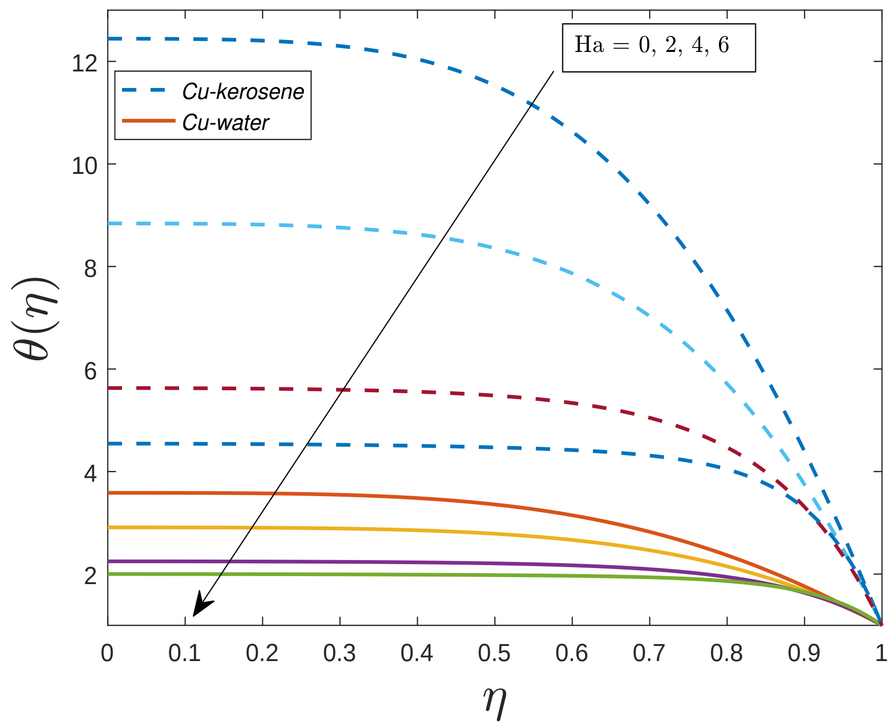

FIGURE 3.10: Impacts of Ha on $\theta(\eta)$ when $S < 0$.

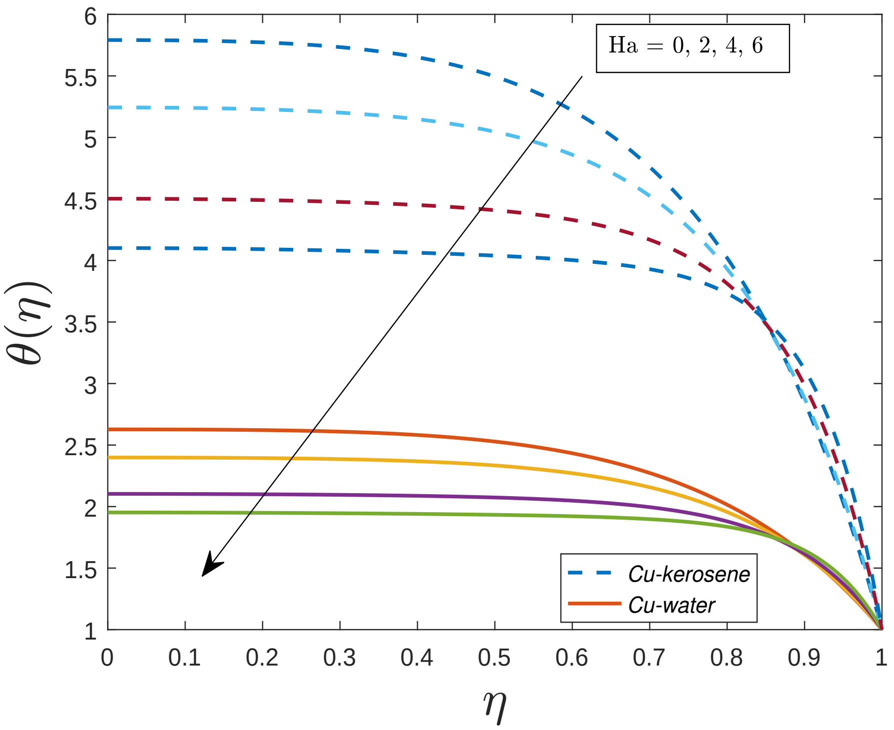

FIGURE 3.11: Impacts of Ha on $\theta(\eta)$ when $S > 0$.

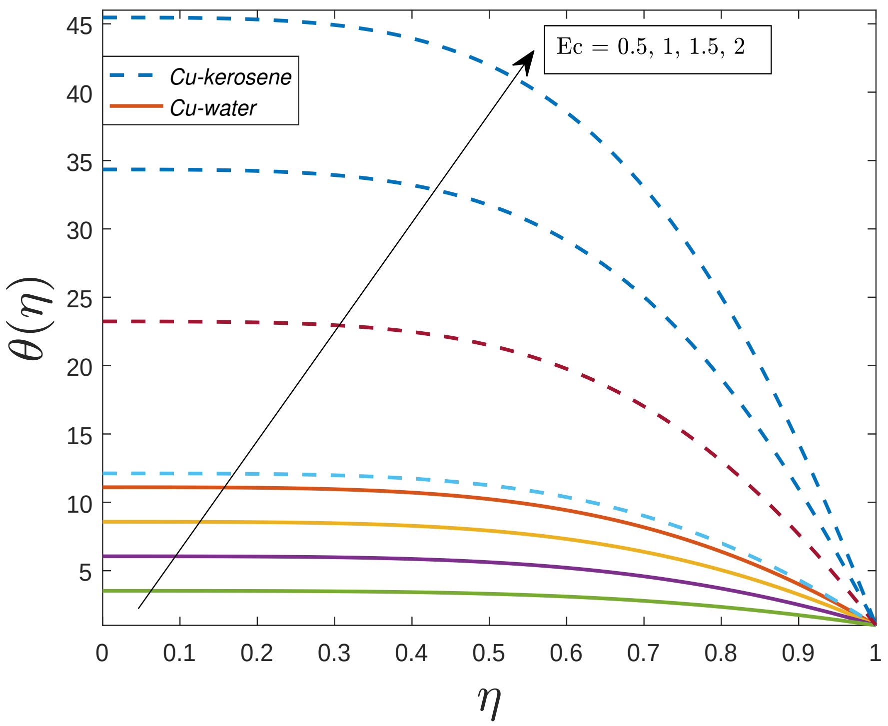

FIGURE 3.12: Impacts of Ec on $\theta(\eta)$ when $S < 0$.

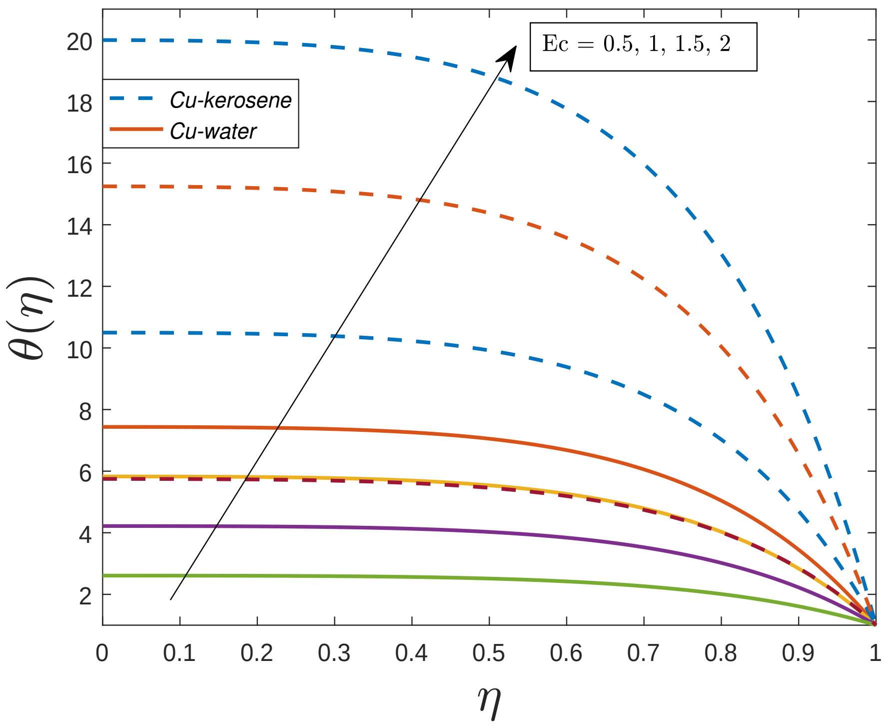

FIGURE 3.13: Impacts of Ec on $\theta(\eta)$ when $S > 0$.

Chapter 4

Numerical Analysis of Thermal Radiation in Maxwell Nanofluid

4.1 Introduction

The work of Çelik [53] presented in the preceding chapter has been extended in this chapter. Additionally, the governing non-linear PDEs of momentum and temperature are converted into system of ODEs by utilizing the similarity transformation. The solution of ordinary differential equations is obtained by using shooting technique by implementing the computational software package MATLAB. At the end of this chapter the numerical solution for various parameters is discussed for dimensionless velocity and temperature. Investigation of obtained numerical results is given through the graphs.

4.2 Mathematical Modeling

In this chapter, the numerical investigation of the viscous, incompressible, squeezing nanofluid flow and transmission of heat among parallel plates has been taken into account as shown in Figure 3.1. The coordinate system is selected as, the x-axis is along the plate and y-axis is normal to it. A uniform magnetic field

$B = B_0(1-\alpha t)^{\frac{-1}{2}}$ is applied and all body forces are ignored where B_0 is uniform transverse magnetic field and α is characteristic parameter. The distance between the plates is $h(t) = H(1-\alpha t)^{\frac{1}{2}}$, where H is the initial position of the plate.

4.2.1 The Governing Equations

The governing equations describes the flow as:

- Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.1}$$

- x-Momentum Equation:

$$\rho_{nf}\left[\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \lambda\left(u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x \partial y}\right)\right]$$
$$= -\frac{\partial p}{\partial x} + \mu_{nf}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \sigma_{nf}B^2(t)\left(u + \lambda v\frac{\partial u}{\partial y}\right), \tag{4.2}$$

- y-Momentum Equation:

$$\rho_{nf}\left[\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \lambda\left(u^2\frac{\partial^2 v}{\partial x^2} + v^2\frac{\partial^2 v}{\partial y^2} + 2uv\frac{\partial^2 v}{\partial x \partial y}\right)\right]$$
$$= -\frac{\partial p}{\partial y} + \mu_{nf}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \sigma_{nf}B^2(t)\left(u\frac{\partial v}{\partial x}\right), \tag{4.3}$$

- Energy Equation:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k_{nf}}{(\rho C_p)_{nf}}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{\mu_{nf}}{(\rho C_p)_{nf}}$$
$$\left(2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right) + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right) - \frac{1}{(\rho C_p)}\frac{\partial q_r}{\partial y}, \tag{4.4}$$

Here u and v denotes the velocity components along x and y direction respectively. T denotes the temperatutr of nanofluid, p denotes the pressure, ρ denotes the density, μ denotes the dynamic viscosit, C_p denotes the specific heat, and k denotes the thermal conductivity.

The radiative heat flux q_r can be written as

$$q_r = -\frac{4\sigma^*}{3\alpha^*}\frac{\partial T^4}{\partial y}.$$

Here σ^* denotes the Stefan-Boltzmann constant and the cooeficent of Rosseland mean absorption is α^*.

Dimensional Boundary Conditions

The imposed boundry conditions for the problem are given by:

$$\left.\begin{aligned} &u = 0,\ v = \frac{dh}{dt},\ T = T_H \text{ at } y\ =\ h(t), \\ &\frac{\partial u}{\partial y} = 0,\ v = 0,\ \frac{\partial T}{\partial y}\ =\ 0 \text{ at } y\ = 0. \end{aligned}\right\} \tag{4.5}$$

For transforming mathematical model Eq. (4.1) - Eq. (4.4) into the dimensionless form, the following similarity transformation has been introduced [53]:

$$\left.\begin{aligned} &u = \frac{\alpha x}{2(1-\alpha t)} f'(\eta),\quad v = \frac{-\alpha H}{2\sqrt{1-\alpha t}} f(\eta), \\ &\theta = \frac{T}{T_H},\quad \eta = \frac{y}{H\sqrt{1-\alpha t}}. \end{aligned}\right\} \tag{4.6}$$

Now using Eq. (4.6) into Eq. (4.1), we differentiate above equation w.r.t 'x' and 'y'.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\alpha x}{2(1-\alpha t} f'(\eta)\right), \\ \frac{\partial u}{\partial x} &= \frac{\alpha}{2(1-\alpha t)} f'(\eta). \end{aligned} \tag{4.7}$$

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-\alpha H}{2\sqrt{1-\alpha t}} f(\eta)\right), \\ \frac{\partial v}{\partial y} &= \frac{-\alpha H}{2\sqrt{1-\alpha t}} f'(\eta)\frac{\partial \eta}{\partial y}, \\ \frac{\partial v}{\partial y} &= -\frac{\alpha}{2(1-\alpha t)} f'(\eta). \end{aligned} \tag{4.8}$$

Using Eq. (4.7) and Eq. (4.8) in Eq. (4.1) to satisfy continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\alpha}{2(1-\alpha t)} f'(\eta) - \frac{\alpha}{2(1-\alpha t)} f'(\eta) = 0. \tag{4.9}$$

Now we include the procedure for the conversion of Eq. (4.2) and Eq. (4.3) into dimensionless form.

We have to convert momentum equations into single ODE to utilize similarity transformation. Since 'v' does not depend on x, so the derivative of v approaches to zero and second derivative of 'u' also approaches to zero.

In order to use Eq. (4.6) into Eq. (4.2) and Eq. (4.3), we differentiate above equation w.r.t 't', we have

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left(f'(\eta)\frac{\alpha x}{2(1-\alpha t)}\right), \\
\frac{\partial u}{\partial t} &= \frac{\alpha x}{2}\left(f'(\eta)\frac{\alpha}{(1-\alpha t)^2} + \frac{f''(\eta)}{(1-\alpha t)}\frac{\alpha y}{2H(1-\alpha t)^{\frac{3}{2}}}\right), \\
\frac{\partial u}{\partial t} &= \frac{\alpha x}{2}\left(f'(\eta)\frac{\alpha}{(1-\alpha t)^2} + \frac{f''(\eta)\alpha}{2(1-\alpha t)^2}\frac{y}{H\sqrt{1-\alpha t}}\right), \\
\frac{\partial u}{\partial t} &= \frac{\alpha^2 x}{2(1-\alpha t)}\left(f'(\eta) + \eta\frac{f''(\eta)}{2}\right).
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{-\alpha H}{2\sqrt{1-\alpha t}} f(\eta)\right), \\
\frac{\partial v}{\partial t} &= \frac{-\alpha H}{2}\left(\frac{\alpha}{2(1-\alpha t)^{\frac{3}{2}}} f(\eta) + \frac{(f'(\eta)}{\sqrt{1-\alpha t}}\frac{\alpha y}{2H(1-\alpha t)^{\frac{3}{2}}}\right), \\
\frac{\partial v}{\partial t} &= \frac{-\alpha^2 H}{4(1-\alpha t)^{\frac{3}{2}}}\left(f(\eta) + (f'(\eta)\ \eta\right).
\end{aligned} \tag{4.11}$$

Similarly, we differentiate above equation w.r.t 'x', we have,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\alpha x}{2(1-\alpha t} f^{'}(\eta)\right), \\
\frac{\partial u}{\partial x} &= \frac{\alpha}{2(1-\alpha t)} f^{'}(\eta).
\end{aligned} \tag{4.12}$$

$$\frac{\partial^2 u}{\partial x^2} = 0. \tag{4.13}$$

$$\frac{\partial v}{\partial x} = 0. \tag{4.14}$$

$$\frac{\partial^2 v}{\partial x^2} = 0. \tag{4.15}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\alpha}{2H(1-\alpha t)^{\frac{3}{2}}} f^{''}(\eta). \tag{4.16}$$

Similarly, differentiating above equation w.r.t 'y', we have

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left((f'(\eta)\frac{\alpha x}{2(1-\alpha t)}\right), \\
\frac{\partial u}{\partial y} &= \frac{\alpha x}{2(1-\alpha t)} f^{''}(\eta)\frac{\partial \eta}{\partial y}, \\
\frac{\partial u}{\partial y} &= \frac{\alpha x}{2(1-\alpha t)} f^{''}(\eta)\frac{1}{H\sqrt{1-\alpha t}}, \\
\frac{\partial u}{\partial y} &= \frac{\alpha x}{2H(1-\alpha t)^{\frac{3}{2}}} f^{''}(\eta).
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\alpha x}{2H(1-\alpha t)^{\frac{3}{2}}} f''(\eta)\right), \\
\frac{\partial^2 u}{\partial y^2} &= \left(\frac{\alpha x}{2H(1-\alpha t)^{\frac{3}{2}}} f'''(\eta)\frac{\partial \eta}{\partial y}\right), \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\alpha x}{2H^2(1-\alpha t)^2} f'''(\eta).
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-\alpha H}{2\sqrt{1-\alpha t}} f(\eta)\right), \\
\frac{\partial v}{\partial y} &= \frac{-\alpha H}{2\sqrt{1-\alpha t}} f^{'}(\eta)\frac{1}{H\sqrt{1-\alpha t}}, \\
\frac{\partial v}{\partial y} &= \frac{-\alpha}{2(1-\alpha t)} f^{'}(\eta).
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
\frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{-\alpha}{2(1-\alpha t)} f^{'}(\eta)\right), \\
\frac{\partial^2 v}{\partial y^2} &= \frac{-\alpha}{2(1-\alpha t)} f^{''}(\eta)\frac{1}{H\sqrt{1-\alpha t}}, \\
\frac{\partial^2 v}{\partial y^2} &= \frac{-\alpha}{2H(1-\alpha t)^{\frac{3}{2}}} f^{''}(\eta).
\end{aligned} \tag{4.20}$$

Substituting these values from Eq. (4.10) to Eq. (4.20) into Eq. (4.2) and Eq. (4.3), we have,

Eq. (4.2)⇒

$$\rho_{nf}\left[\frac{\alpha^2 x}{4(1-\alpha t)^2}\left(2f^{'}(\eta)+f^{''}(\eta)\ (\eta)+(f^{'})^2(\eta)-f^{''}(\eta)\ f(\eta)\right)\right.$$
$$\left.+\lambda\left(\frac{\alpha^3 x}{8(1-\alpha t)^3}\left(f^2(\eta)f'''(\eta)-2f(\eta)f'(\eta)f''(\eta)\right)\right)\right]$$
$$=-\frac{\partial p}{\partial x}+\mu_{nf}\frac{\alpha x}{2H^2(1-\alpha t)^2}f^{'''}(\eta)-\sigma_{nf}B^2(t)\left(u+\lambda v\frac{\partial u}{\partial y}\right). \quad (4.21)$$

Eq. (4.3)⇒

$$\rho_{nf}\left[\frac{-\alpha^2 H}{4(1-\alpha t)^{\frac{3}{2}}}\left(f(\eta)+f^{'}\ \eta-f(\eta)f'(\eta)\right)+\lambda\frac{\alpha^3 x}{4(1-\alpha t)}\right.$$
$$\left.\left(\frac{1}{2}f^2(\eta)f''(\eta)-f(\eta)f'(\eta)f''(\eta)\right)\right]=-\frac{\partial p}{\partial y}-\mu_{nf}$$
$$\frac{\alpha}{2H(1-\alpha t)^{\frac{3}{2}}}f^{''}(\eta)+0. \quad (4.22)$$

Now differentiating Eq. (4.22) w.r.t 'x' and Eq. (4.21) w.r.t 'y',

Eq. (4.22) ⇒

$$\frac{\partial^2 p}{\partial x\partial y}=0. \quad (4.23)$$

Eq. (4.21)⇒

$$\rho_{nf}\left[\frac{\alpha^2 x}{4(1-\alpha t)^2}\left(2f^{''}(\eta)\frac{1}{H\sqrt{1-\alpha t}}+f'''(\eta)(\eta)\frac{1}{H\sqrt{1-\alpha t}}+f^{''}(\eta)\frac{1}{H\sqrt{1-\alpha t}}\right.\right.$$
$$\left.+2f^{'}(\eta)f^{''}(\eta)\frac{1}{H\sqrt{1-\alpha t}}-f^{'''}(\eta)f(\eta)\frac{1}{H\sqrt{1-\alpha t}}-f^{''}(\eta)f^{'}(\eta)\frac{1}{H\sqrt{1-\alpha t}}\right)$$
$$+\lambda\frac{\alpha^3 x}{8(1-\alpha t)^3}\left(f^2(\eta)f''''(\eta)\frac{1}{H\sqrt{1-\alpha t}}-2f'(\eta)f'(\eta)f''(\eta)\frac{1}{H\sqrt{1-\alpha t}}\right.$$
$$\left.\left.-2f(\eta)f''(\eta)f''(\eta)\frac{\partial\eta}{\partial y}\right)\right]=\frac{\partial^2 p}{\partial x\partial y}+\mu_{nf}\frac{\alpha x}{2H^2(1-\alpha t)^2}f^{''''}(\eta)\frac{1}{H\sqrt{1-\alpha t}}$$
$$-\sigma_{nf}B_0^2(t)\frac{\alpha x}{2H(1-\alpha t)^{\frac{5}{2}}}\left[f^{''}(\eta)-\frac{De}{2}\left(f'(\eta)f''(\eta)+f'''(\eta)f(\eta)\right)\right]. \quad (4.24)$$

Putting Eq. (4.23) in Eq. (4.24), we have,

$$\rho_{nf}\frac{\alpha}{2}\left[\left(f^{'''}\eta+3f^{''}-f^{'''}f^{'}+f^{''}f^{'}\right)+\lambda\frac{\alpha}{2(1-\alpha t)}\left(f^2f''''-2(f')^2f''-2f(f'')^2\right)\right]$$

$$= \mu_{nf}\frac{1}{H^2}f'''' - \sigma_{nf}B_0^2(t)\left[f'' - \frac{De}{2}\Big(f'f'' + f'''f\Big)\right]. \tag{4.25}$$

Multiplying by H^2

$$\rho_{nf}\frac{\alpha H^2}{2}\left[\Big(f'''\eta + 3f'' - f'''f' + f''f'\Big) + \lambda\frac{\alpha}{2(1-\alpha t)}\Big(f^2f'''' - 2(f')^2f'' - 2f(f'')^2\Big)\right] = \mu_{nf}f'''' - \sigma_{nf}B_0^2(t)H^2\left[f'' - \frac{De}{2}\Big(f'f'' + f'''f\Big)\right],$$

$$f'''' - \frac{\rho_{nf}}{\mu_{nf}}\frac{\alpha H^2}{2}\left[(f'''\eta + 3f'' + f''f' - f''f') + \lambda\frac{\alpha}{2(1-\alpha t)}\Big(f^2f'''' - 2(f')^2f'' - 2f(f'')^2\Big)\right] - \frac{\sigma_{nf}}{\mu_{nf}}B_0^2(t)H^2\left[f'' - \frac{De}{2}\Big(f'f'' + f'''f\Big)\right] = 0. \tag{4.26}$$

$$\therefore \frac{\rho_{nf}}{\mu_{nf}} = \frac{(1-\Phi)^{2.5}}{\mu_f}\left(\Phi\rho_s + (1-\Phi)\rho_f\right),$$

multiplying and dividing by ρ_f

$$\frac{\rho_{nf}}{\mu_{nf}} = \frac{(1-\Phi)^{2.5}}{\nu_f}\left(\Phi\frac{\rho_s}{\rho_f} + (1-\Phi)\right).$$

Eq. (4.26) becomes

$$f'''' - SA_1(1-\Phi)^{2.5}\left[\Big(f'''\eta + 3f'' + f''f' - f'''f\Big) + \frac{De}{2}\Big(f^2f'''' - 2(f')^2f'' - 2f(f'')^2\Big)\right] - (Ha)^2\left[f'' - \frac{De}{2}\Big(f'f'' + f'''f\Big)\right] = 0. \tag{4.27}$$

Now we have to convert energy equation into ODE to utilize similarity transformation, in Eq. (4.4). T does not depend on 'x', so the derivative of T approaches to zero.

By using Eq. (4.6) into Eq. (4.4), we differentiate above equation w.r.t 'T', we have

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial}{\partial t}(\theta T_H)\\ \frac{\partial T}{\partial t} &= T_H\theta'\frac{\partial\eta}{\partial t},\\ \frac{\partial T}{\partial t} &= T_H\theta'\frac{\alpha\eta}{2(1-\alpha t)}.\end{aligned} \tag{4.28}$$

Similarly, differentiating above equation w.r.t 'x' and 'y', we have

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left(\theta T_H\right) = 0. \tag{4.29}$$

$$\frac{\partial^2 T}{\partial x^2} = 0. \tag{4.30}$$

$$\begin{aligned}\frac{\partial T}{\partial y} &= \frac{\partial}{\partial y}\left(\theta T_H\right),\\ \frac{\partial T}{\partial y} &= \frac{\theta' T_H}{H\sqrt{1-\alpha t}}.\end{aligned} \tag{4.31}$$

$$\begin{aligned}\frac{\partial^2 T}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\theta' T_H}{H\sqrt{1-\alpha t}}\right),\\ \frac{\partial^2 T}{\partial y^2} &= \frac{\theta'' T_H}{H^2(1-\alpha t)}.\end{aligned} \tag{4.32}$$

Similarly, we have,

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left((f'(\eta)\frac{\alpha x}{2(1-\alpha t)}\right),\\ \frac{\partial u}{\partial y} &= \frac{\alpha x}{2(1-\alpha t)}f''(\eta)\frac{\partial \eta}{\partial y},\\ \frac{\partial u}{\partial y} &= \frac{\alpha x}{2H(1-\alpha t)^{\frac{3}{2}}}f''(\eta).\end{aligned} \tag{4.33}$$

$$\frac{\partial v}{\partial x} = 0. \tag{4.34}$$

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(f'(\eta)\frac{\alpha x}{2(1-\alpha t}\right),\\ \frac{\partial u}{\partial x} &= \frac{\alpha}{2(1-\alpha t)}f'(\eta).\end{aligned} \tag{4.35}$$

$$\begin{aligned}\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-\alpha H}{2\sqrt{1-\alpha t}}f(\eta)\right)\\ \frac{\partial v}{\partial y} &= \frac{-\alpha}{2(1-\alpha t)}f'(\eta),\end{aligned} \tag{4.36}$$

Next, we differentiate 'q_r' w.r.t 'y'

$$\frac{\partial q_r}{\partial y} = \frac{-16\sigma^* T^3}{3\alpha^*}\frac{\partial^2 T}{\partial y^2}. \tag{4.37}$$

Putting values from Eq. (4.28) to Eq. (4.37) in Eq. (4.4), we have,

$$\begin{aligned}
&\frac{\alpha\eta T_H\theta'}{2(1-\alpha t)} - \frac{\alpha H}{2\sqrt{1-\alpha t}}f(\eta)\frac{\theta' T_H}{H\sqrt{1-\alpha t}} = \frac{k_{nf}}{(\rho C_p)_{nf}}\left(\frac{\theta'' T_H}{H^2(1-\alpha t)}\right)\\
&+\frac{\mu_{nf}}{(\rho C_p)_{nf}}\left[2\left(\left(\frac{\alpha}{2(1-\alpha t)}f'(\eta)\right)^2 + \left(\frac{-\alpha}{2(1-\alpha t)}f'(\eta)\right)^2\right) + \frac{\alpha^2 x^2}{4H^2(1-\alpha t)^3}(f'')^2\right]\\
&-\frac{1}{(\rho C_p)_{nf}}\left(-\frac{16\sigma^* T^3}{3\alpha^*}\frac{T_H\theta''}{H^2(1-\alpha t)}\right),\\
&\frac{\alpha\eta T_H\theta'}{2(1-\alpha t)} - \frac{\alpha H}{2\sqrt{1-\alpha t}}f(\eta)\frac{\theta' T_H}{H\sqrt{1-\alpha t}} = \frac{T_H}{H^2(1-\alpha t)}\left[\frac{k_{nf}}{(\rho C_p)_{nf}}\theta''\right.\\
&\left.+\frac{\mu_{nf}}{(\rho C_p)_{nf}}\left(2\frac{\alpha^2 H^2 (f')^2}{T_H 2(1-\alpha t)} + \frac{\alpha^2 x^2 (f'')^2}{T_H 4(1-\alpha t)^2}\right)\right] + \frac{16}{3}\frac{\sigma^* T^3}{\alpha^*(\rho C_p)_{nf}}\theta'',
\end{aligned}$$

multiplying and dividing by $(\rho C_p)_f$ and $\frac{\mu_f}{\rho_f}$,

$$\begin{aligned}
&\theta'' + \frac{\mu_{nf}}{(\rho C_p)_{nf}}\frac{(\rho C_p)_{nf}}{k_{nf}}\left(2\frac{\alpha^2 H^2 (f')^2}{T_H 2(1-\alpha t)} + \frac{\alpha^2 x^2 (f'')^2}{T_H 4(1-\alpha t)^2}\right) + \frac{4}{3}\frac{4\sigma^* T^3}{\alpha^*(\rho C_p)_{nf}}\theta''\\
&\frac{(\rho C_p)_{nf}}{k_{nf}} - \frac{\alpha H^2}{2}\theta'\eta\frac{(\rho C_p)_{nf}}{k_{nf}} + \frac{\alpha H^2}{2}f\theta'\frac{(\rho C_p)_{nf}}{k_{nf}} = 0,\\
&(1+\frac{4}{3}R)\theta'' + \left(\theta' f - \theta'\eta\right)\frac{A_2}{A_3}PrS + \frac{PrEc}{A_3(1-\Phi)^{2.5}}\left((f'')^2 + 4(\delta)^2(f')^2\right) = 0.
\end{aligned} \tag{4.38}$$

Now for converting the associated boundary conditions into the dimensionless form, the following steps have been taken

$$\begin{aligned}
&u = 0 \; at \; y = h(t), \quad \frac{\alpha x}{2(1-\alpha t)}f'(\eta) = 0,\\
&\quad\frac{\alpha x}{2(1-\alpha t)} \neq 0,\\
&f'(\eta) = 0, \text{ when } \eta \to 1, \; f'(1) = 0.\\
&v = \frac{dh}{dt} \; at \; y = h(t), \; v = \frac{-\alpha H}{2\sqrt{1-\alpha t}}f(\eta),\\
&\quad\frac{dh}{dt} = \frac{-\alpha H}{2\sqrt{1-\alpha t}},
\end{aligned}$$

$$\frac{-\alpha H}{2\sqrt{1-\alpha t}} = \frac{-\alpha H}{2\sqrt{1-\alpha t}} f(\eta),$$

$$f(\eta) = 1,\ \text{when}\ \eta \to 1,\ f(1) = 1.$$

$$T = T_H\ at\ y = h(t),\ \theta T_H = T_H,$$

$$\text{when}\ \eta \to 1,\ \theta(1) = 1.$$

$$\frac{\partial u}{\partial y} = 0\ at\ y = 0,$$

$$\frac{\alpha x}{2H(1-\alpha t)^{\frac{3}{2}}} f''(\eta) = 0,$$

$$\frac{\alpha x}{2H(1-\alpha t)^{\frac{3}{2}}} \neq 0,\ f''(\eta) = 0,$$

$$\text{when}\ \eta \to 0,\ f''(0) = 0.$$

$$v = 0\ at\ y = 0,\ \frac{-\alpha H}{2\sqrt{1-\alpha t}} f(\eta) = 0,$$

$$\frac{-\alpha H}{2\sqrt{1-\alpha t}} \neq 0,\ f(\eta) = 0,$$

$$\text{when}\ \eta \to 0,\ f(0) = 0.$$

$$\frac{\partial T}{\partial y} = 0\ at\ y = 0,\ \frac{\partial T}{\partial y} = \theta' T_H \frac{1}{H\sqrt{1-\alpha t}},$$

$$T_H \frac{1}{H\sqrt{1-\alpha t}} \neq 0,$$

$$\theta'(\eta) = 0,\ \text{when}\ \eta \to 0,\ \theta'(0) = 0.$$

The final dimensionless form of the governing model is:

$$f'''' - SA_1(1-\Phi)^{2.5}\left[\left(f'''\eta + 3f'' + f''f' - f'''f\right) + \frac{De}{2}\left(f^2 f'''' - 2(f')^2 f'' - 2f(f'')^2\right)\right] - (Ha)^2\left[f'' - \frac{De}{2}\left(f'f'' + f'''f\right)\right] = 0. \tag{4.39}$$

$$(1 + \frac{4}{3}R)\theta'' + \left(\theta' f - \theta'\eta\right)\frac{A_2}{A_3} PrS + \frac{PrEc}{A_3(1-\Phi)^{2.5}}\left((f'')^2 + 4(\delta)^2(f')^2\right) = 0. \tag{4.40}$$

The associated boundary conditions of Eq. (4.39) and Eq. (4.40) shown as:

$$\left.\begin{array}{l} f(0) = 0,\ f''(0) = 0,\ \theta'(0) = 0, \\ f(1) = 1,\ f'(1) = 0,\ \theta(1) = 1. \end{array}\right\} \tag{4.41}$$

Following parameters are used in Eq. (4.39) and Eq. (4.40):

$$A_1 = (1-\Phi) + \Phi\frac{\rho_s}{\rho_f},\ A_2 = (1-\Phi) + \Phi\frac{(\rho C_p)_s}{(\rho C_p)_f},\ A_3 = \frac{k_{nf}}{k_f},$$
$$Pr = \frac{\mu_f(\rho C_p)_f}{\rho_f k_f},\ \delta = \frac{H(1-\alpha t)^{\frac{1}{2}}}{x},\ Ec = \frac{\rho_f}{T_H(\rho C_p)_f}\left(\frac{\alpha x}{2(1-\alpha t)}\right)^2,$$
$$Ha = \sqrt{\frac{\sigma_f}{\mu_f}}HB_0,\ De = \frac{\lambda\alpha}{1-\alpha t},\ R = \frac{4\sigma^* T^3}{\alpha^*(\rho C_p)_{nf}},$$

4.3 Solution Methodology

To solve the Eq. (4.39) with corresponding boundary conditions (4.41), we used shooting method. First of all we convert the higher order ODE into the first order ODEs. We used the RK4 method to solve IVP and assume the missing initial conditions. Now for Eq. (4.39), we utilize following notations:

$$\begin{aligned} f &= f_1, f' = f_1' = f_2, \\ f'' &= f_2' = f_3, \\ f''' &= f_3' = f_4, \\ f'''' &= f_4'. \end{aligned} \tag{4.42}$$

The resulting initial value problem takes the form:

$$f_1' = f_2; \qquad f_1(0) = 0, \tag{4.43}$$
$$f_2' = f_3; \qquad f_2(0) = r, \tag{4.44}$$
$$f_3' = f_4; \qquad f_3(0) = 0, \tag{4.45}$$
$$f_4' = \frac{1}{\left(1 - SA_1(1-\Phi)^{2.5}\frac{De}{2}f_1^2\right)}\Bigg[A_1S(1-\Phi)^{2.5}\bigg((3f_3 + f_4\eta + f_3f_2 - f_4f_1) - \frac{De}{2}\left(2f_2^2f_3 + 2f_1f_3^2\right)\bigg) + (Ha)^2\bigg(f_3 - \frac{De}{2}\left(f_2f_3 + f_4f_1\right)\bigg)\Bigg];$$
$$f_4(0) = s. \tag{4.46}$$

Missing conditions 'r' and 's' are assumed to satisfy the following relations:

$$f_1(1, r, s) - 1 = 0, \tag{4.47}$$

$$f_2(1, r, s) = 0. \tag{4.48}$$

We use Newton's method for two equations with two variables to solve Eq. (4.47) and Eq. (4.48),

$$\begin{bmatrix} r^{(n+1)} \\ s^{(n+1)} \end{bmatrix} = \begin{bmatrix} r^{(n)} \\ s^{(n)} \end{bmatrix} - \left(\begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial s} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial s} \end{bmatrix} \right)^{-1}_{(r=r^{(n)}, s=s^{(n)})} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{4.49}$$

where $n = 0, 1, 2, ...$

We utilize following notations:

$$\frac{\partial f_1}{\partial r} = f_5, \quad \frac{\partial f_2}{\partial r} = f_6, \frac{\partial f_3}{\partial r} = f_7, \quad \frac{\partial f_4}{\partial r} = f_8,$$
$$\frac{\partial f_1}{\partial s} = f_9, \quad \frac{\partial f_2}{\partial s} = f_{10}, \frac{\partial f_3}{\partial s} = f_{11}, \quad \frac{\partial f_4}{\partial s} = f_{12}.$$

Using above notations in Eq. (4.49),

$$\begin{bmatrix} r^{(n+1)} \\ s^{(n+1)} \end{bmatrix} = \begin{bmatrix} r^{(n)} \\ s^{(n)} \end{bmatrix} - \left(\begin{bmatrix} f_5 & f_9 \\ f_6 & f_{10} \end{bmatrix} \right)^{-1}_{(r=r^{(n)}, s=s^{(n)})} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Differentiating Eq. (4.43) to Eq. (4.46) and using above notations, we have first order ODEs,

$$f_1' = f_2; \qquad f_1(0) = 0, \tag{4.50}$$

$$f_2' = f_3; \qquad f_2(0) = r, \tag{4.51}$$

$$f_3' = f_4; \qquad f_3(0) = 0, \tag{4.52}$$

$$f_4' = \frac{A_1 S(1-\Phi)^{2.5}}{(1 - SA_1(1-\Phi)^{2.5} S \frac{De}{2} f_1^2)} \Bigg[\Bigg((3f_3 + f_4\eta + f_3 f_2 - f_4 f_1) - \frac{De}{2} (2f_2^2 f_3 + 2f_1 f_3^2) \Bigg) + (Ha)^2 \Bigg(f_3 - \frac{De}{2} (f_2 f_3 + f_4 f_1) \Bigg) \Bigg];$$
$$f_4(0) = s, \tag{4.53}$$

$$f_5' = f_6; \qquad f_5(0) = 0, \tag{4.54}$$

$$f_6' = f_7; \qquad f_6(0) = 1, \tag{4.55}$$

$$f_7' = f_8; \qquad f_7(0) = 0, \quad (4.56)$$

$$f_8' = \frac{SA_1(1-\phi)^{2.5}}{((1-A_1(1-\phi)^{2.5}S\frac{De}{2})f_1^2)^2}\Bigg[(1-A_1(1-\phi)^{2.5}S\frac{De}{2})f_1^2\Bigg((f_8\eta+3f_7 + f_7f_2+f_3f_6-f_4f_5-f_1f_8)-\frac{De}{2}\left(4f_6f_3f_2+2f_2^2f_7+2f_5f_3^2+4f_1f_7f_3\right) + (Ha)^2\Bigg(f_7-\frac{De}{2}\left(f_2f_7+f_3f_6+f_4f_5+f_1f_8\right)\Bigg)\Bigg)-\Bigg((f_4\eta+3f_3+f_3f_2 - f_4f_1)-\frac{De}{2}\left(2f_2^2f_3+2f_1f_3^2\right)+(Ha)^2\Bigg(f_3-\frac{De}{2}\left(f_2f_3+f_4f_1\right)\Bigg)\Bigg)(2f_5f_1)\Bigg];$$

$$f_8(0) = 0, \quad (4.57)$$

$$f_9' = f_{10}; \qquad f_9(0) = 0, \quad (4.58)$$

$$f_{10}' = f_{11}; \qquad f_{10}(0) = 0, \quad (4.59)$$

$$f_{11}' = f_{12}; \qquad f_{11}(0) = 0, \quad (4.60)$$

$$f_{12}' = \frac{SA_1(1-\phi)^{2.5}}{((1-A_1(1-\phi)^{2.5}S\frac{De}{2})f_1^2)^2}\Bigg[(1-A_1(1-\phi)^{2.5}S\frac{De}{2})f_1^2\Bigg((f_{12}\eta+3f_{11} + f_3f_{10}+f_2f_{11}-f_4f_9-f_1f_{12})-\frac{De}{2}\left(4f_{10}f_3f_2+2f_2^2f_{11}+2f_9f_3^2+4f_1f_{11}f_3\right) + (Ha)^2\Bigg(f_{11}-\frac{De}{2}\left(f_2f_{11}+f_3f_{10}+f_4f_9+f_{12}f_1\right)\Bigg)\Bigg)-\Bigg((f_4\eta+3f_3+f_3f_2 - f_4f_1)-\frac{De}{2}\left(2f_2^2f_3+2f_1f_3^2\right)+(Ha)^2\Bigg(f_3-\frac{De}{2}\left(f_2f_3+f_4f_1\right)\Bigg)(2f_9f_1)\Bigg)\Bigg];$$

$$f_{12}(0) = 1. \quad (4.61)$$

In order to solve IVP, we used RK4 method, the initial values are chosen arbitrarily. During the execution of iterations these initial guesses will be updated by the Newton's method and the process will be continued until the following criteria is met,

$$max(|f_1(1)-1|, |f_2(1)-0|) < \epsilon.$$

Throughout this work, ϵ has been taken as 10^{-6} unless otherwise mentioned. Now for Eq. (4.40), following notations are used:

$$\begin{aligned}
\theta &= z_1, \\
\theta' &= z_1' = z_2, \\
\theta'' &= z_1'' = z_2'.
\end{aligned} \quad (4.62)$$

Initial value problem takes the form:

$$z_1' = z_2; \qquad z_1(0) = \xi, \tag{4.63}$$

$$z_2' = -\frac{1}{(1+\frac{4}{3}R)}\left[PrS\frac{A_2}{A_3}(z_2f_1 - z_2\eta) + \frac{PrEc}{A_3(1-\Phi)^{2.5}}\left((f_3)^2 + 4(\delta)^2(f_2)^2\right)\right]; \qquad z_2(0) = 0. \tag{4.64}$$

Missing condition 'ξ' assumed to satisfy the following relation:

$$z_1(1,\xi) - 1 = 0. \tag{4.65}$$

We use Newton's method for Eq. (4.65),

$$\xi_{n+1} = \xi_n - \frac{Z(\xi)}{Z'(\xi)}, \tag{4.66}$$

$$\text{where} \quad Z(\xi) = z_1(1,\xi) - 1.$$

Following notations are used:

$$\frac{\partial z_1}{\partial \xi} = z_3, \qquad \frac{\partial z_2}{\partial \xi} = z_4.$$

Differentiating Eq. (4.63) and Eq. (4.64) w.r.t 'ξ',we have,

$$z_3' = z_4; \qquad z_3(0) = 1. \tag{4.67}$$

$$z_4' = -\frac{1}{(1+\frac{4}{3}R)}\left[PrS\frac{A_2}{A_3}(z_4f_1 - z_4\eta)\right]; \qquad z_4(0) = 0. \tag{4.68}$$

The RK4 method is used to solve IVP for some suitable choice ξ. The missing condition ξ is updated by the Newton's method and process will be continued until the following criteria is met.

$$|z_1(1) - 1| < \epsilon.$$

Throughout this work, ϵ is taken as 10^{-6} unless otherwise mentioned.

4.4 Results and Discussion

The purpose of this section is to analyze the numerical results demonstrated in the graphical form. Computations are carried out for different values and impact of these parameters on velocity and temperature are discussed. Figure 4.1 exhibits the effect of the squeezing parameter S on the velocity of nanofluid. This indicates by increasing S the velocity profile declines for $0 \leq \eta \leq 0.42$ but for $\eta > 0.42$ this begins to rise. This can be clarified physically as the reduction in velocity of fluid close to the wall area leads a rise in velocity gradient there. If the rate of mass flow is sustained conservatively, the fluid velocity decline close to the wall region must be compensated with rising velocity closer the central field. That's why we noticed a break point and backflow happens at $\eta = 0.42$

Figure 4.2 demonstrates the behaviour of S on $\theta(\eta)$. Figure 4.2 illustrates that $\theta(\eta)$ decreases with increasing S. The rise in S thus indicates a reduction in kinematic viscosity and a decline in velocity where the plates travel separately. Correspondingly, the friction among the boundry surface of the plate and the nanoparticles decreases and thus the nanofluid temperature decreases.

Figure 4.3 and 4.4 demonstrate the impact of ϕ on $f'(\eta)$ for both positive and negative S. For $S < 0$, in the range of $0 \leq \eta \leq 0.436$, the velocity rises and fall in the range of $0.436 \leq \eta \leq 1$. This could be the case described as when the volume fraction of the nanoparticle enlarges, more collisions among nanoparticles and particles at the surface of boundry layer occur. Velocity falls near the boundry and rises above the central field. For $S > 0$, when nanoparticles volume fraction rises, boundry layer reduces closer to the plates, but it increases away from the surface.

Figure 4.5 and 4.6 demonstrate the impacts of nanoparticle volume fraction on the temperature of nanofluid for both positive and negative S. The incorporation of nanoparticles into the base fluid increases the thickness of thermal boundry layer. This is indication that when the fluid flows, the nanoparticles take away the heat from the boundry layer. It is obvious that when nanoparticles are added to the base fluid, because of their high thermal conductivity, heat transfer will increase and thus temperature profile will reduce.

The influence of the magnetic field on velocity distribution has been shown in Figure 4.7 and 4.8. For $S < 0$, the flow velocity declines as the Hartmann number Ha increases in the interval $\eta \leq 0.5$. This decreases the fluid's motion in the boundry layer area because the restricting force produced by magnetic field. Nonetheless, the opposite effect arises because of the same explanations for mass flow for $\eta > 0.5$. Moreover, if $S < 0$, that means two plates are nearest to one another, then the circumstances creates an adverse pressure gradient together with a retarding Lorentz force. A separation point occurs, when these forces act for a long period and backflow may occur. For positive S, nanofluid velocity also decreases, a empty area exists as two plates move apart and in that region fluid moves at high speed such that no violation occur for law of conservation of mass flow.

For both $S < 0$ and $S > 0$, Figure 4.9 and 4.10 demonstrate the effect of magnetic field parameter on temperature of nanofluid. From the Figure 4.9 and 4.10, it can be seen as the Hartmann number rises there is a diminish in nanofluid temperature . Nanofluid temperature increases when $\eta > 0.86$ and it decreases when $\eta < 0.86$ for positive S.

Impacts of Eckert number Ec on the distribution of nanofluid temperature for positive and negative S are given in Figure 4.11 and 4.12. It is shown that by increasing Eckert number, the nanofluid temparature among the plates rises and the thermal boundry layer tends to reduce due to the increased viscous dissipation effect.

Figure 4.13 and 4.14 demonstrate the impact of Deborah number on velocity of fluid for negative and positive S. The velocity of fluid declines as Deborah number rises, can be seen from figures. Figure 4.15 and 4.16 shows the influence of radiation parameter on temperature of fluid for $S < 0$ and $S > 0$. It is cleared in figures when radiation parameter increases the temperature of the fluid rises significantly. This is because the rising radiation parameter values reduce the thickness of the boundry layer and increase the heat transfer rate with a chemical impact on the melting surface.

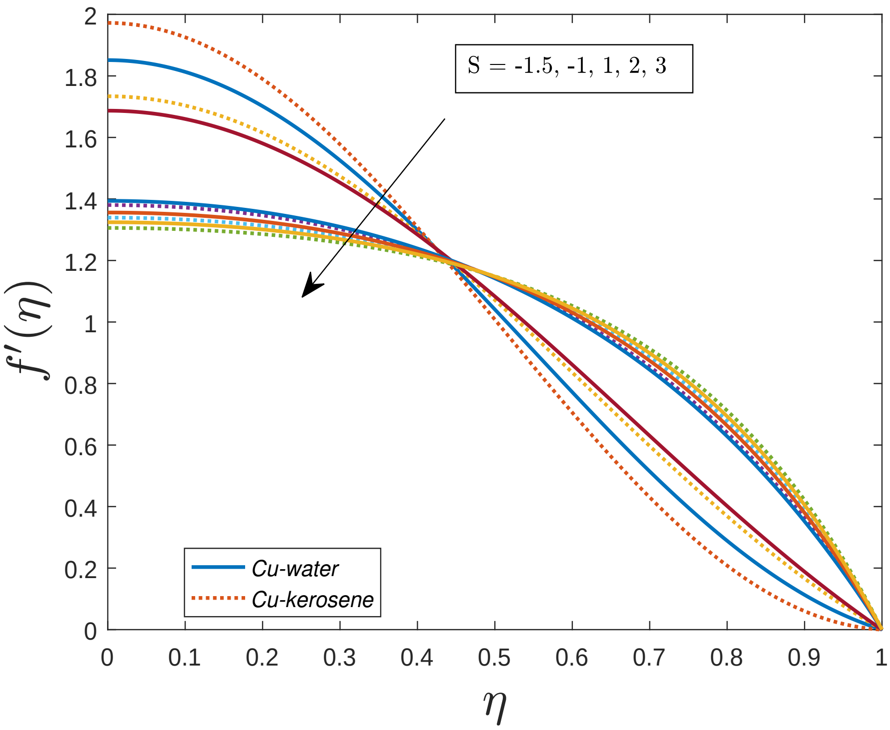

FIGURE 4.1: Impacts of S on $f'(\eta)$.

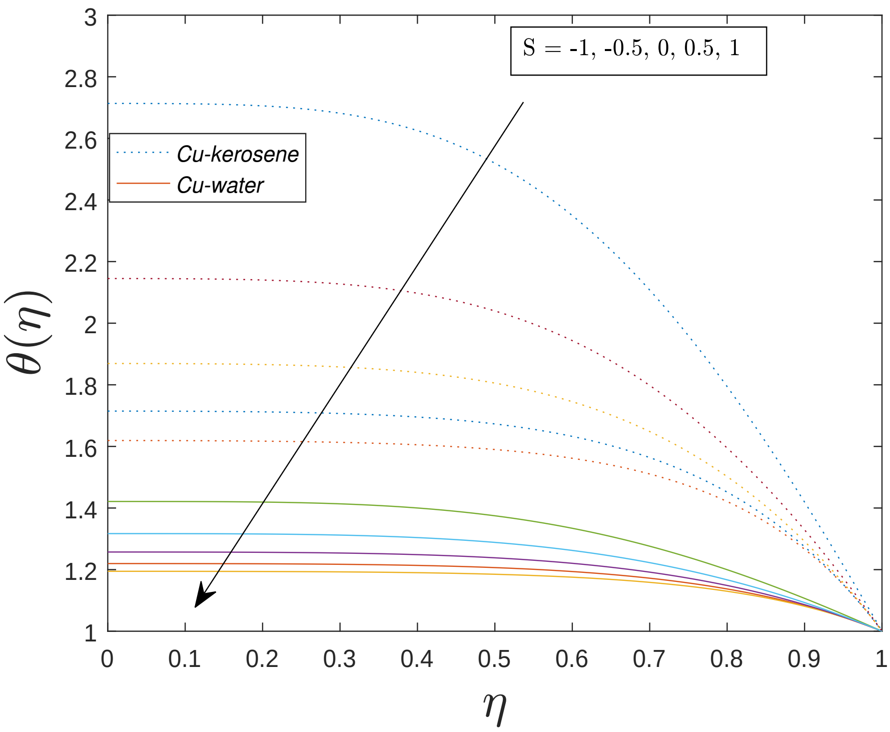

FIGURE 4.2: Impacts of S on $\theta(\eta)$.

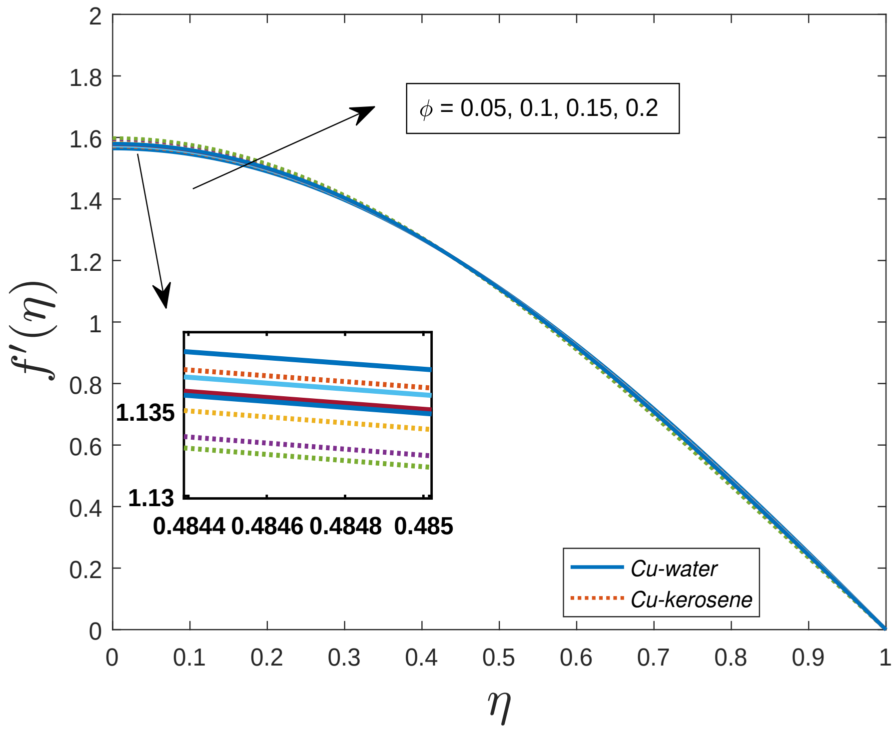

FIGURE 4.3: Impacts of ϕ on $f'(\eta)$ when $S < 0$.

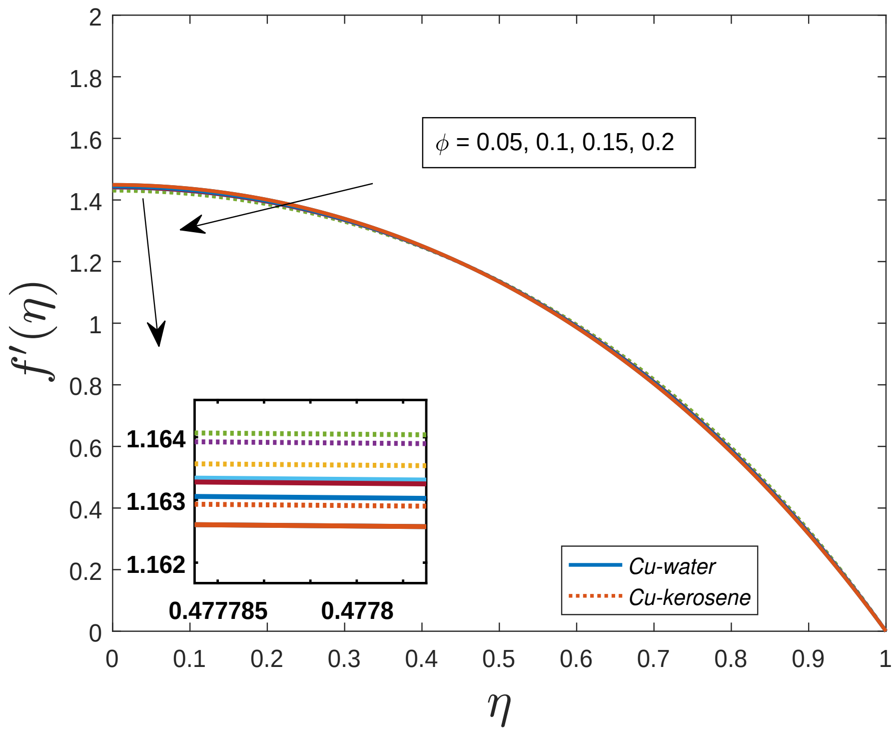

FIGURE 4.4: Impacts of ϕ on $f'(\eta)$ when $S > 0$.

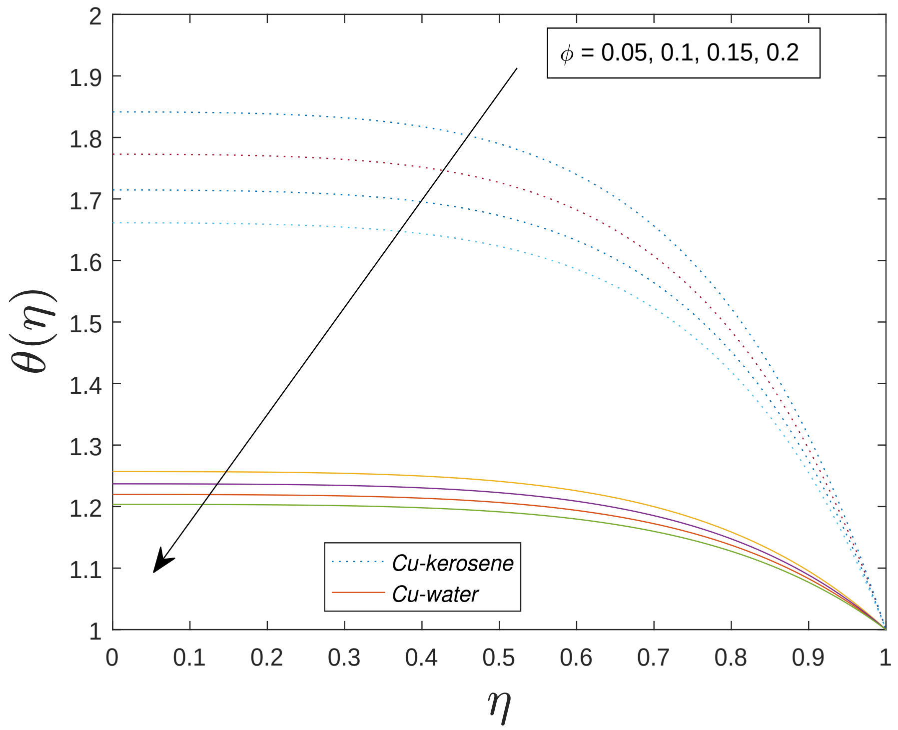

FIGURE 4.5: Impacts of ϕ on $\theta(\eta)$ when $S < 0$.

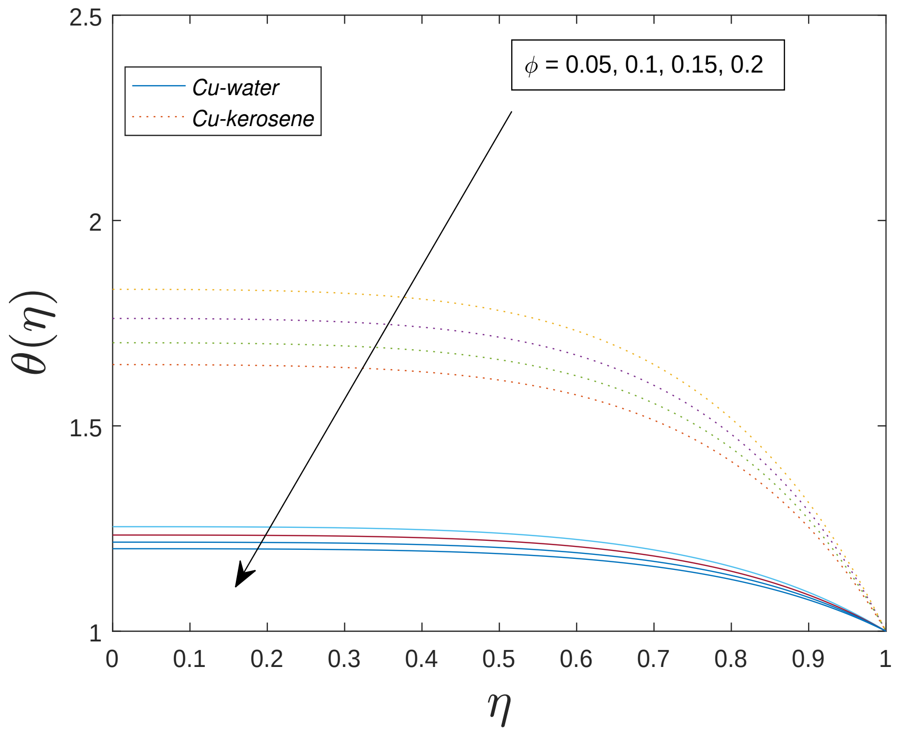

FIGURE 4.6: Impacts of ϕ on $\theta(\eta)$ when $S > 0$.

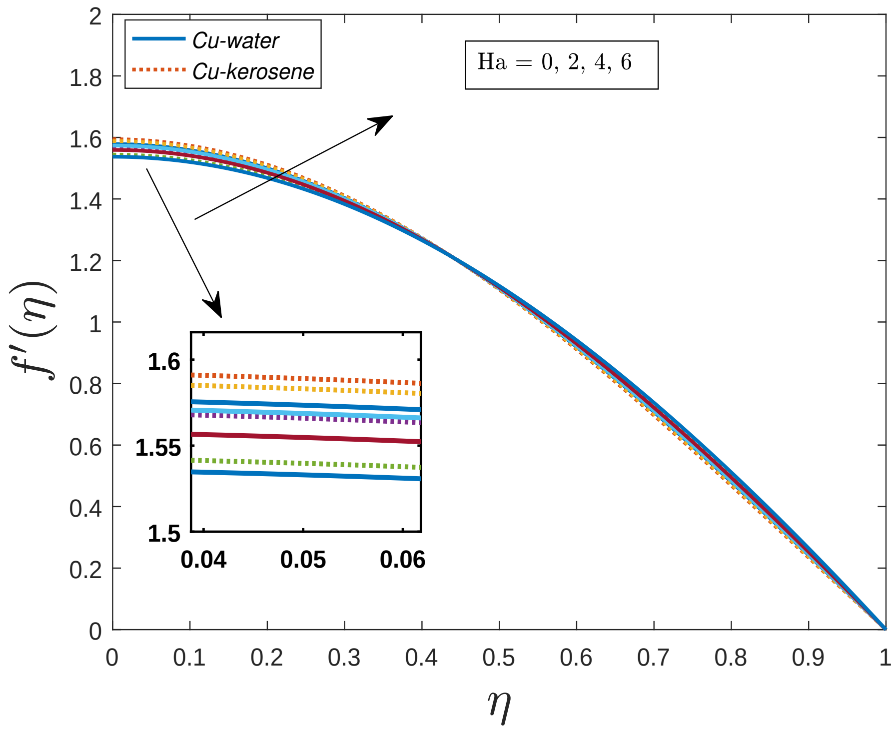

FIGURE 4.7: Impacts of Ha on $f'(\eta)$ when $S < 0$.

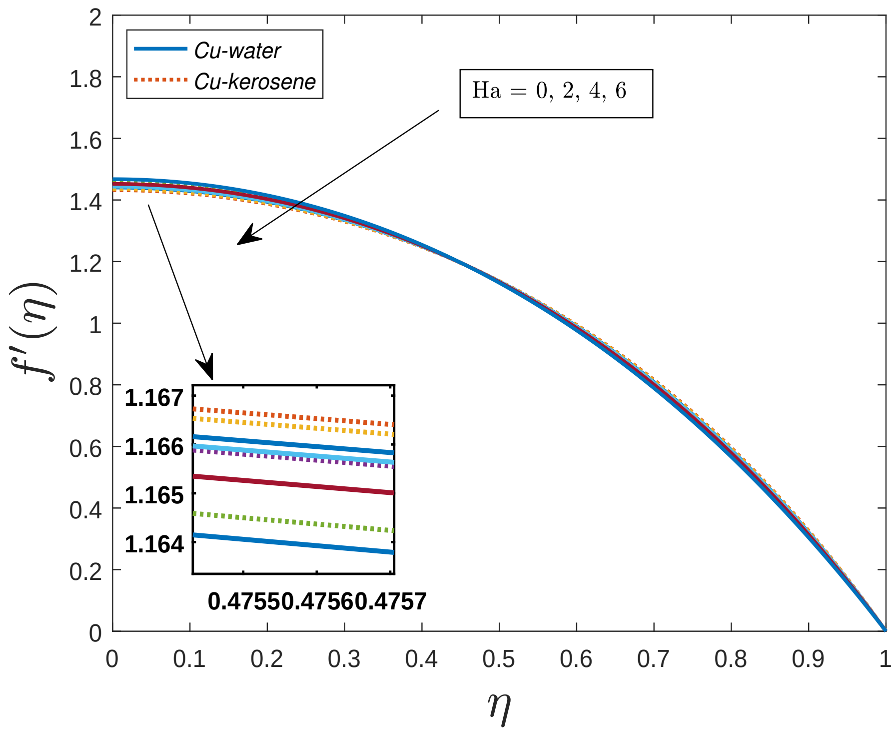

FIGURE 4.8: Impacts of Ha on $f'(\eta)$ when $S > 0$.

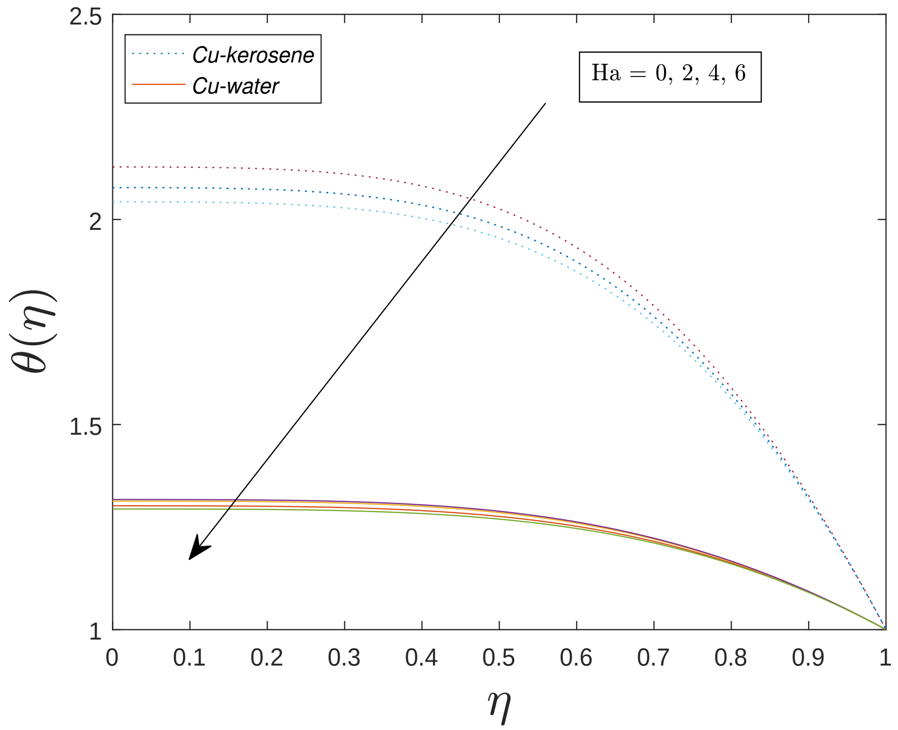

FIGURE 4.9: Impacts of Ha on $\theta(\eta)$ when $S < 0$.

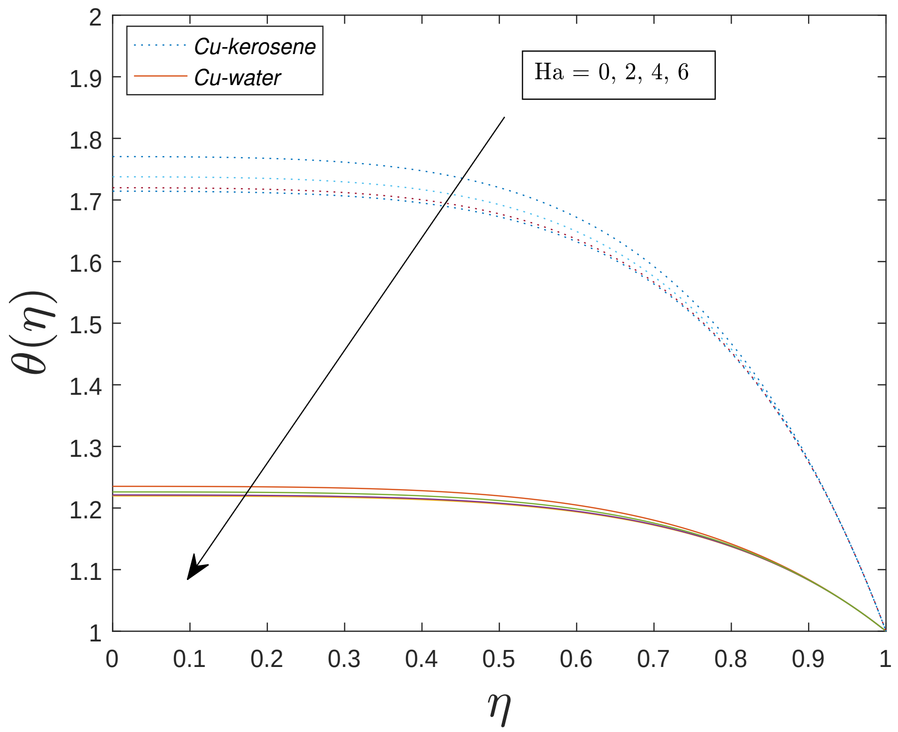

FIGURE 4.10: Impacts of Ha on $\theta(\eta)$ when $S > 0$.

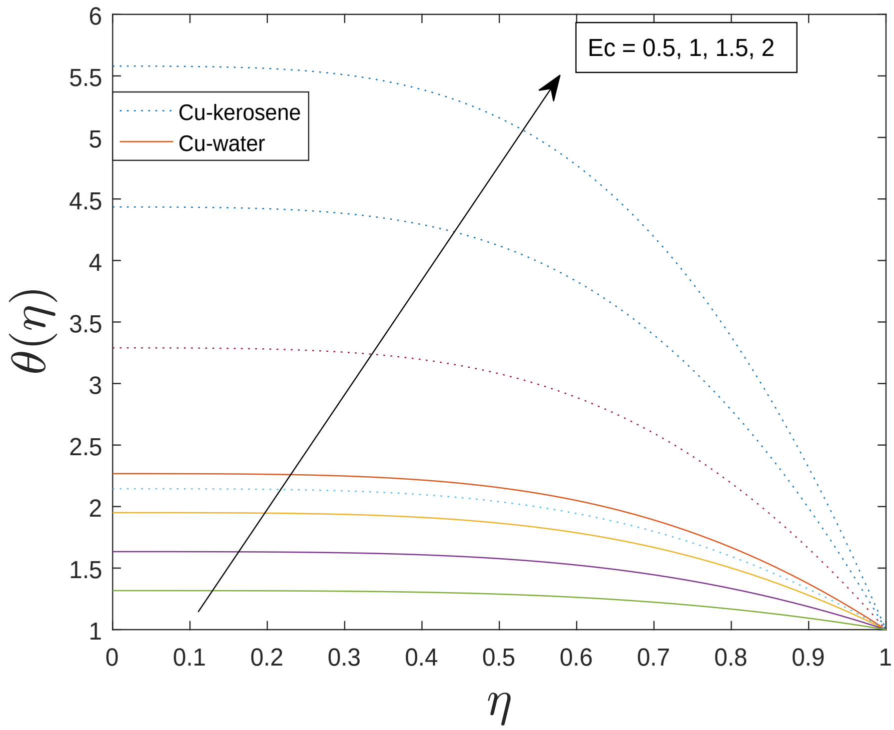

FIGURE 4.11: Impacts of Ec on $\theta(\eta)$. when $S < 0$.

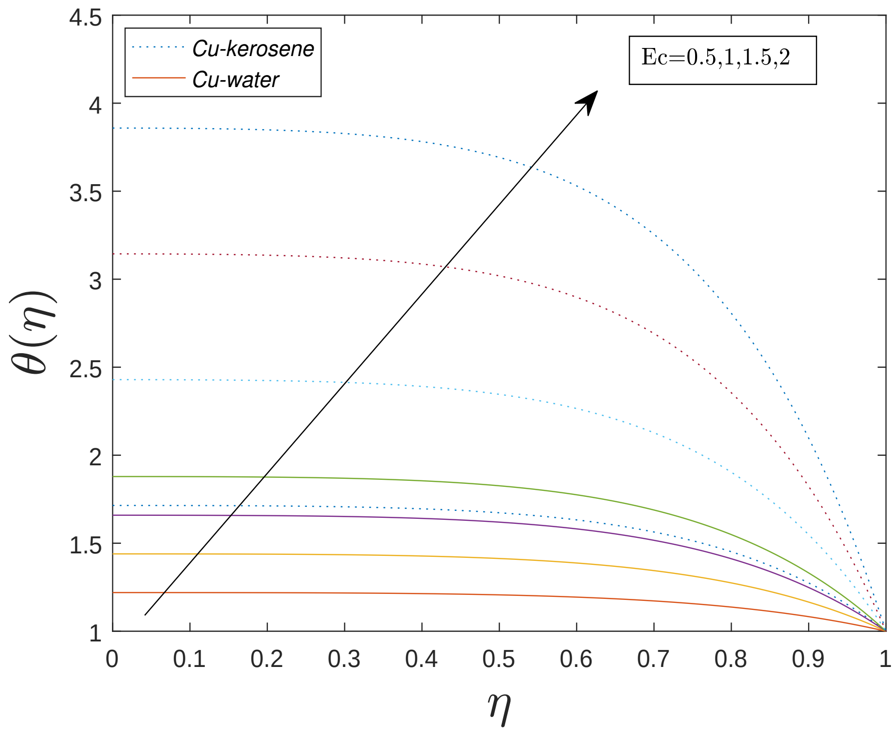

FIGURE 4.12: Impacts of Ec on $\theta(\eta)$ when $S > 0$.

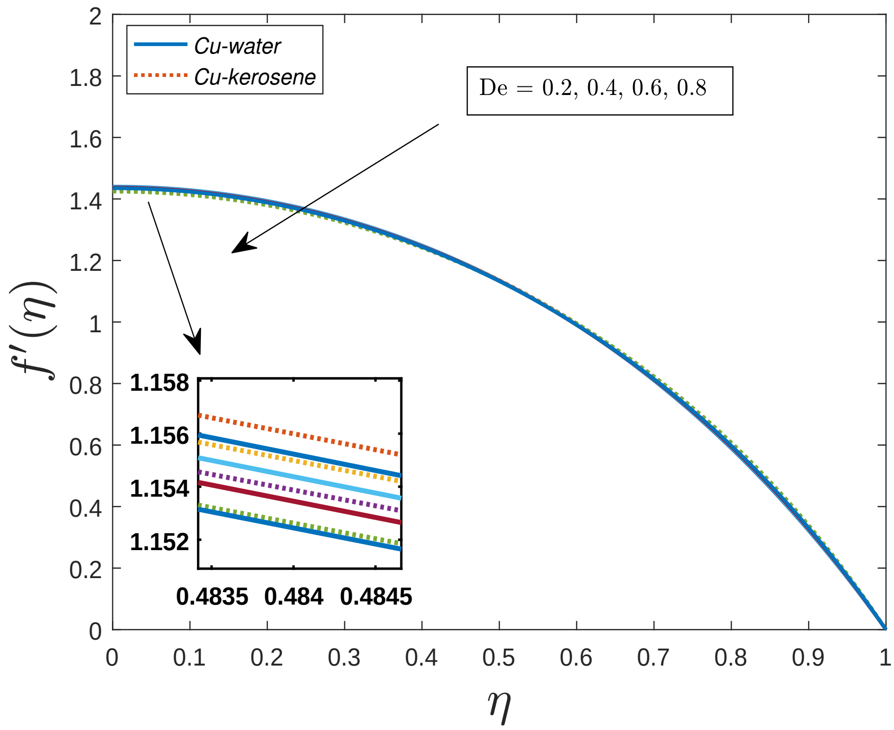

FIGURE 4.13: Impacts of De on $f'(\eta)$ when $S < 0$.

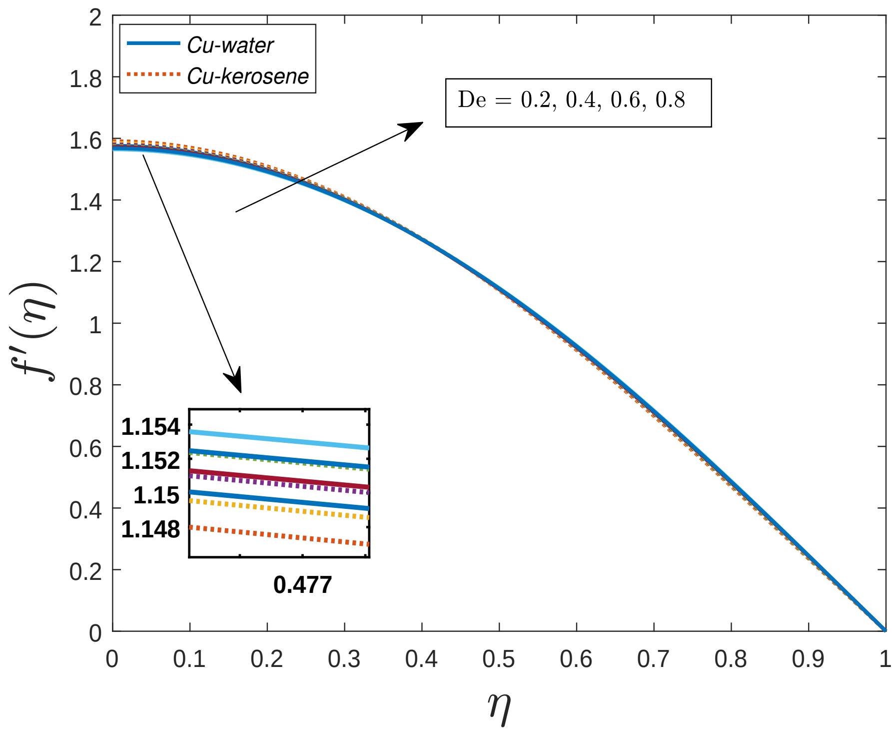

FIGURE 4.14: Impacts of De on $f'(\eta)$ when $S > 0$.

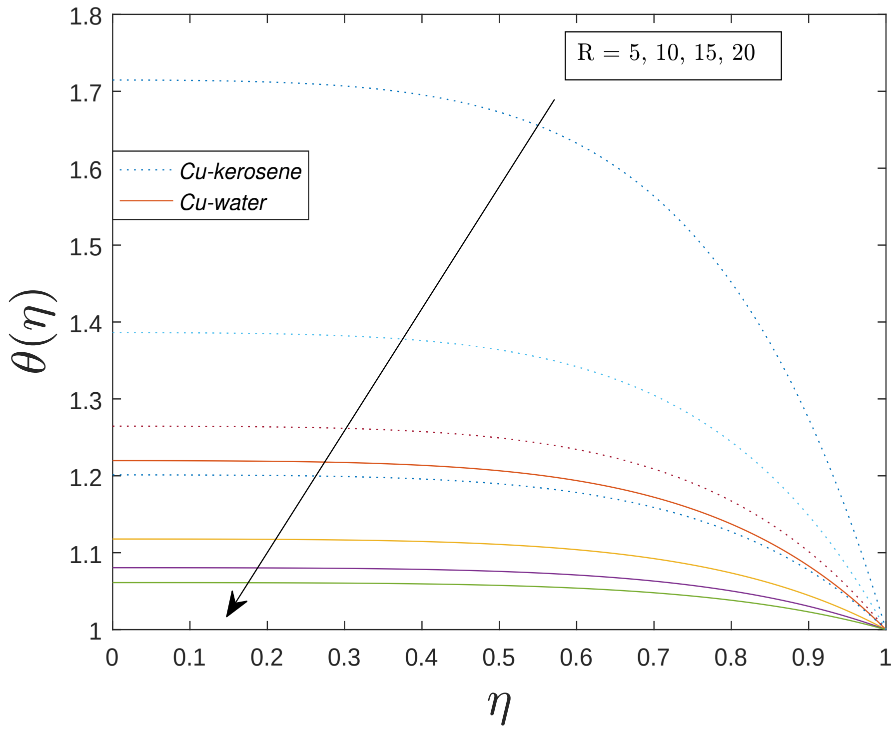

FIGURE 4.15: Impacts of R on $\theta(\eta)$ when $S < 0$.

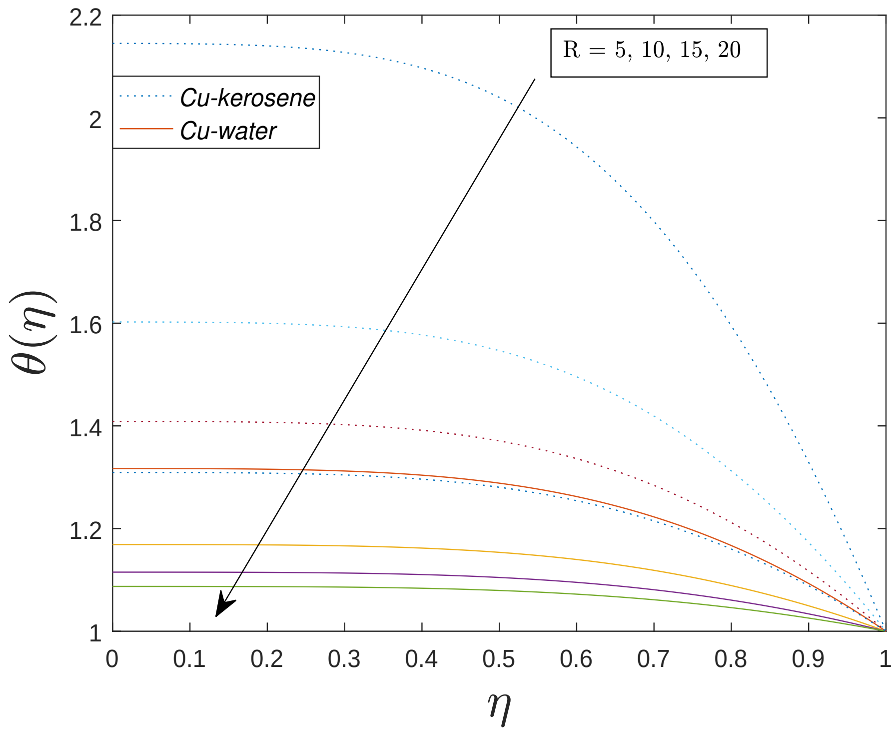

FIGURE 4.16: Impacts of R on $\theta(\eta)$ when $S > 0$.

Chapter 5

Conclusion

In this thesis, the numerical analysis of MHD squeezed movement of Cu-kerosene and water among two parallel plates has been considered and analyzed under the impact of thermal radiation. The governing non-linear PDEs of momentum and energy are transformed into ODEs by utilizing a proper transformation of similarities. The numerical technique known as the shooting method has been utilized to solve the system. Impact of different physical parameters such as squeezing parameter S, Magnetic field parameter Ha, Eckert number Ec, Deborah number De, and radiation parameter R on the velocity and temperature is discussed graphically.

From the present study that has been numerically performed the following worthy points can be concluded:

- With an increase in the nanoparticle volume fraction, the temperature of the fluid decreases.
- Because of strong magnetic parameter, the velocity profile diminish.
- The temperature profile reduces by rising the Hartmann number.
- With the rising Eckert number temperature profile tends to increase.

- With the rise in Deborah number the velocity field decreases.
- Increasing the value of radiation parameter results in an increase in the temperature profile.

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