CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Cattaneo-Christov Double Diffusions Model and Activation Energy for Entropy Analysis in a Hydromagnetic Nanofluid Flow

by

Abdul Rehman

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

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*I dedicate my dissertation work to my **family** and dignified **teachers**. A special feeling of gratitude to my loving parents who have supported me in my studies.*

CERTIFICATE OF APPROVAL

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Abstract

In this thesis, a numerical investigation of Cattaneo-Christov double diffusion model and activation energy in a porous medium on boundary layer flow of a thermally radiating nanofluid past a nonlinear stretching sheet has been carried out. The temperature and concentration distributions are associated with Cattaneo-Christove double diffusion. The diffusion of chemically reactive specie is investigated with Arrhenius activation energy. The governing partial differential equations are modified to a system of ordinary differential equations using some appropriate similarity transformation. The resulting system of the ordinary differential equations has been solved numerically by the shooting method. The influence of physical parameters such as permeability parameter, nonlinear stretching sheet parameter, velocity slip, unsteadiness parameter magnetic field parameter, Eckert number, Prandtl number, thermophoresis parameter, Brownain motion parameter, Biot number, chemical reaction parameter, activation energy parameter, mass relaxation parameter and temperature ratio parameter on the velocity profile, temperature distribution and concentration profile, are analyzed through graphs. The effects of different parameters on skin friction coefficient, Nusselt number, and Sherwood number have been discussed. Furthermore, an analysis of the system's entropy generation is provided. The entropy generation via the Bejan number significantly affects the applied magnetic field, Brinkman number, thermal radiation, Biot number and diffusion parameter. The results of the present analysis show that chemical reactions, mass relaxation and Brownian motion lead to a decrease in the concentration profile of the nanofluid. It has been noted that the temperature distribution is decreased by the thermal relaxation parameter. However, the thermal relaxation parameter and activation energy parameters increase the concentration distribution.

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Abbreviations

BVP	Boundary value problem
IVPs	Initial value problems
MHD	Magnetohydrodynamics
ODEs	Ordineary differential equations
PDEs	Partial differential equations
RK	Runge-Kutta

Symbols

ν	Kinematic viscosity
τ	Stress tensor
k	Thermal conductivity
α_{nf}	Thermal diffisuitivity
σ	Electrical conductivity
u	x-component of fluid velocity
v	y-component of fluid velocity
B_0	Magnetic field constant
a	Stretching constant
T_w	Temperature of the wall
T_∞	Ambient temperature of the nanofluid
T	Temperature
ν_{nf}	Kinematic viscosity of the base fluid
μ_{nf}	Viscosity of the nanofluid
q_r	Radiative heat flux
D_T	thermophoretic diffusion coefficient
D_B	Brownian diffusion coefficient
σ^*	Stefan Boltzmann constant
k^*	Absorption coefficient
ψ	Stream function
ζ	Similarity variable
C_f	Skin friction coefficient
Nu_x	Local Nusselt number

Sh_x	Local Sherwood number
Re_x	Local Reynolds number
R_d	Thermal radiation parameter
n	Stretching parameter
M	Magnetic parameter
K	Permeability parameter
Ec	Eckert number
Pr	Prandtl number
λ	the reciprocal
Nb	Brownain motion parameter
Nt	Thermophoresis parameter
γ	velocity slip parameter
γ_1	Chemical reaction rate parameter
ρ_{nf}	Density of the nanofluid
σ_f	Electrical conductivity of the base fluid
σ_s	Electrical conductivity of the nanoparticle
k_{nf}	Thermal conductivity of the nanofluid
f	Dimensionless velocity
θ	Dimensionless temperature
ϕ	Dimensionless concentration
C_∞	Ambient concentration
C	Concentration
C_w	Nanoparticles concentration at the stretching surface
E	is the non-dimensional Arrhenius activation energy
L_t	is thermal relaxation parameter
L_c	is mass relaxation
θ_w	is relative temperature ratio parameter

Chapter 1

Introduction

The fluid state of a substance is defined as the free movement of atoms and molecules of that substance without any definite shape. In liquids, the atoms never lose their molecular structural vitality, hence assuming the shape of anything in which they are kept. However in gases, atoms become completely detached. Flow of fluids has different types of aspects, compressible and incompressible, steady and unsteady, viscous and inviscid, uniform and non-uniform, rotational and irrational [1]. The term nanofluid refers to a fluid in which nanoparticles, floating in the base fluid, create a colloidal solution. The basic fluids used in nanofluids are water, ethylene glycol, oil etc. The nanoparticles employed in them are typically comprised of metals, oxides, carbides, or carbon nanotubes.

Choi [2] was the first to use the concept nanofluid. In general, nanofluids frequently have a volume proportion of nanoparticles up to 5%, which accelerates the rate of efficient heat transmission. This is why nanofluids have a tremendous impact on modern engineering and technology. The nanometer-sized substances have particular chemical and physical characteristics. Due to their small size which is almost comparable to that of liquid molecules nanoparticles may move through microchannels without being clogged. The thermal conductivity of the base fluid is observed to increase by 15–40% when nanoparticles are present. An increase in heat conductivity of this magnitude cannot be simply attributable to the additional nanoparticles' higher thermal conductivity.

Other characteristics of the nanofluids, such as volume concentration, particle agglomeration, thermophoresis, Brownian motion, surface area, particle shape, etc., should be taken into account for the nanofluids to perform more effectively. Due to numerous remarkable uses in a variety of industrial and scientific fields, including the production of rubber and plastic sheets, geothermal energy extraction, fibreglass, hot rolling, etc. The fluid dynamics community has given the study of boundary layer nanofluid flow a great deal of attention. To investigate the rotating flow of nanofluids brought on by an exponentially stretched layer, Mushtaq et al. [3] carried out a numerical analysis.

Magnetohydrodynamic (MHD) convective flow of nanofluids over a stretching sheet in a porous media considering heat generation/absorption was discussed by Reddy and Chamkha [4]. In a convective heat transfering hydromagnetic nanofluid flow, Shit et al. [5] described the mechanism of entropy formation. Geisha et al. [6] recently published a Kirchhoff's Voltage Law (KVL) based Hall influenced two-phase transient nanofluid flow over a stretching sheet. MHD mixed convection stagnation point flow and heat transfer of nanofluid over an inclined stretching sheet with chemical reaction and thermal radiation was investigated by Gupta et al. [7].
Mokhtar et al. [8] conducted a new study on the impact of feedback control and an internal heat source on the commencement of Rayleigh-Benard convection in a horizontal nanofluid layer when Soret and Dufour effects are present. Joshi et al. [9] described the Effect of particle shape and slip mechanism on buoyancy induced convective heat transport with nanofluids. Because of this fact, the problems with fluid flow across a stretching surface have become one of the primary areas of study nowadays, due to a variety of applications in various scientific problems and industries. A few practical applications of this type of research include the manufacturing of rubber and plastic sheets, the creation of fibreglass, melt-spinning, the cooling of metallic plates, etc.

Sakiadis [10], who studied the hydromagnetic flow through a solid surface moving at a uniform velocity, was the first who attempted to break a new ground in this area. Crane [11] explored the behaviour of a two-dimensional MHD flow caused by a continuously deforming surface. He predicted that the surface would deform linearly as one moved away from the origin. Later, Gupta and Gupta [12] expanded the Crane's work and examined the boundary layer flow, mass, and heat transfer past a stretching sheet while taking the effect of suction or blowing into account. For a continuous surface that is stretching exponentially, Magyari and Keller [13] outlined the features of mass and transmission of heat in the boundary layer. Other pertinent research studies were conducted by Vajravelu [14]; Cortell [15] studied how viscous dissipation via a nonlinear stretching affected the thermal boundary layer.

Khan and Pop, [16] investigated the steady boundary layer flow of nanofluids and heat transfer properties over a stretching surface by considering the volume fraction of nanoparticles. They mentioned that both the local Nusselt number and the local Sherwood number strongly depend on the outcomes of thermophoretic parameters, Brownian motion and Lewis number. Makinde and Aziz [17] have examined the convective boundary layer flow and heat transfer of nanofluids past a linearly stretching sheet by considering the effects of thermophoresis and Brownian motion.

Rana and Bhargava, [18] investigated numerically the laminar boundary fluid flow which results from the non-linear stretching of a flat surface in a nanofluid. The literature stated above only applies to steady flows. However, from a practical standpoint, the flow created is not steady when there is a sharp rise or fall in the surface temperature. Furthermore, the induction of an erratic fluid flow is also caused by impulsive stretching of the surface. A model for unstable power-law fluid flow over a stretching surface was investigated by Andersson et al. [19] Three-dimensional MHD flow that is unstable with Joule heating and viscous dissipation was examined by Hayat et al. [20].

References [21] to [22] provide more outstanding studies illuminating flow behaviour under varying conditions. All the research articles mentioned above do not include the condition of slip velocity, since the writers believe that there is no relative velocity between the fluid molecules and the surface. The existence of a slip velocity between the solid-fluid interfaces has been confirmed by numerous theoretical and computational experiments. The linear proportionality between the slip velocity and the shear stress at the wall was investigated by Navier [23]. Imtiaz et al. [24] performed the boundary layer flow of nanofluid caused by an exponentially stretching sheet withinside the presence of thermal radiation and viscous dissipation.

Seth and Mishra [25] explored Navier's slip boundary conditions for unsteady hydromagnetic nanofluid flow over a non-linear stretching sheet. The flows in porous media are very important in the fuel cell technologies, geothermal energy and oil recovery, material processing, drying processes, trickle bed chromatography and in many others. The combined influence of heat and mass transfer in the boundary layer flows of nanofluids saturated in porous media in the presence of magnetic field is an efficient method to enhance thermal performance.

In this direction, Chamkha et al. [26] investigated the convection flow from an inclined plate saturated in a porous medium with varying porosity as a result of solar radiation. Double-diffusion natural convection boundary layer nanofluid flow across a vertical plate embedded in a porous media was described by Khan and Aziz [27].

Oyelakin et al. [28] looked into the effects of heat radiation and the slip boundary condition on unsteady Casson nanofluid over a stretched sheet. It is a widely recognized fact that the number of irreversibilities that take place during any thermal process is determined by the generation of entropy. The heating and cooling processes are extremely important in various energy and electrical equipment in numerous industrial and engineering sectors. Because of this, it is crucial to optimize entropy production to stop any irreversibility loss that can impair the functionality of a specific system.

The Bejan number Be, which is a measure of thermal irreversibility about the total loss of heat due to fluid frictional effects, was first described by Bejan [29, 30] because of its relevance. Abolbashari et al. [31] performed an inspection for the generation of entropy in the situation of an unstable magnetohydrodynamic nanofluid flow past an accelerating permeable stretching sheet. Qing et al. [32] studied the entropy generation on the magnetohydrodynamic flow of nanofluids over a porous permeable linear stretching or shrinking surface with the aid of using the Casson fluid model. They have taken into consideration the trouble wherein the stretching surface is kept at a steady surface temperature.

Butt et al. [33] numerically investigated the entropy generation of a viscous fluid over a stretching cylinder embedded in a porous medium withinside the presence of magnetic field. However, their work is limited in how it takes into account the effect of thermal radiation and the physical properties of nanoparticles. Seth et al. [34] numerically investigated the entropy generation of dissipative float of carbon nanotubes in rotating frame with Darcy-Forchheimer porous medium.

Fourier [35] and Fick [36] were the first persons to describe the phenomenon of heat and mass transfer, respectively. According to them, temperature and concentration distributions have parabolic equations. Cattaneo [37], later on, added the thermal relaxation term to modify the Fourier's law of heat conduction and as a result discussed the heat transfer with finite speed in thermal waves. A new model was introduced by Christov [38] in which the time derivative is replaced by Oldroyd's upper-convected derivative to achieve the materialinvariant formulation.

Tibullo and Zampoli. [39] studied the uniqueness of the Cattaneo-Christov model of heat flux for the incompressible fluids. Hayat et al. [40] explored the three dimensional boundary-layer flow of viscous nanofluid over a bidirectional linearly stretching sheet in the presence of Cattaneo-Christov double diffusion. Malik et al. [41] explored numerically analysis of Cattaneo-Christov double-diffusion model for Sisko fluid flow with velocity slip. M.Azam et al. [42] examined the Effects of Cattaneo-Christov heat flux and nonlinear thermal radiation on MHD Maxwell nanofluid with Arrhenius activation energy.

1.1 Thesis Contribution

In the present study, which is motivated by the aforementioned literature review, we aim to describe the characteristics of unsteady nanofluid flow with Cattaneo-Christov double diffusions model and activation energy over a nonlinearly stretching sheet immersed in a porous medium. The effects of Brownian motion, thermophoresis, thermal relaxation parameter, mass relaxation, chemical reaction, activation energy parameter and the fraction volume of nanoparticles have also been discussed. An analysis of system's entropy generation is the conculding goal of the present work.

1.2 Thesis Layout

This dissertation is divided into the following four chapters.

Chapter 2 demonstrates some important definitions, laws and concepts which are useful in understanding upcoming work.

Chapter 3 provides the details of numerical analysis of the research paper by Seth et al. [43]. An appropriate similarity transformation is used for the conversion of PDEs into ODEs and obtained the numerical results by solving the system of ODEs with the help of the shooting method.

Chapter 4 extends the work of [43] by considering the Cattaneo-Christov double diffusions and activation energy.

Chapter 5 provides a summary of the research conducted for this dissertation .

The Bibliography includes all the references that were used in this research:

Chapter 2

Preliminaries

This chapter contains some basic definitions and governing laws, which will be helpful in the subsequent chapters.

2.1 Important Definitions

Definition 2.1.1 (Fluid)

"A substance exists in three primary phases. Solid, Liquid and Gas (at very high temperatures, it also exists as plasma). A substance in the liquid or gas phase is referred to as a fluid. Distinction between a solid and fluid is made on the basis of substances ability to resist an applied shear or (tangential) stress that tends to change its shape." [44]

Definition 2.1.2 (Entropy)

"Thermal energy per unit temperature which is unavailable for doing useful work is known as entropy of a system. Every system has the energy to do useful work. But during this useful work some energy is lost in the form of heat due to friction and other factors. This loss of energy is known as the entropy of system." [45]

Definition 2.1.2 (Magnetohydrodynamics)

"Magnetohydrodynamics(MHD) is concerned with the mutual interaction of fluid flow and magnetic fields. The fluids in question must be electrically conducting and non-magnetic, which limits us to liquid metals, hot ionised gases (plasmas) and strong electrolytes." [46]

Definition 2.1.3 (Fluid Mechanics)

"Fluid mechanics is the branch of science which deals with the behavior of the fluids (liquids or gases) at rest as well as in motion. Thus this branch of science deals with the static, kinematics and dynamic aspects of fluids" [47]

Definition 2.1.4 (Fluid Dynamics)

"The study of fluid if the pressure forces are also considered for the fluids in motion, that branch of science is called fluid dynamics." [47]

Definition 2.1.5 (Fluid Statics)

"The study of fluid at rest is called fluid statics." [47]

Definition 2.1.6 (Viscosity)

"Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. Mathematically,

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}},$$

where μ is viscosity coefficient, τ is shear stress and $\frac{\partial u}{\partial y}$ represents the velocity gradient." [47]

Definition 2.1.7 (Kinematic Viscosity)

"It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by symbol ν called **'nu'**. Mathematically,

$$\nu = \frac{\mu}{\rho}.\text{"}\ [47]$$

Definition 2.1.8 (Thermal Conductivity)

"The Fourier heat conduction law states that the heat flow is proportional to the temperature gradient. The coefficient of proportionality is a material parameter known as the thermal conductivity which may be a function of a number of variables." [48]

Definition 2.1.9 (Thermal Diffusivity)

"The rate at which heat diffuses by conducting through a material depends on the thermal diffusivity and can be defined as,

$$\alpha = \frac{k}{\rho C_p},$$

where α is the thermal diffusivity, k is the thermal conductivity, ρ is the density and C_p is the specifc heat at constant pressure." [49]

2.2 Types of Flow

Definition 2.2.1 (Laminar and Turbulent Flow)

"Fluid partical follows a smooth trajectory, the flow is then said to be laminar. Further increases in speed may lead to instability that eventually produces a more random type of flow that is called turbulent." [50]

Definition 2.2.2 (Rotational Flow)

"Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis." [47]

Definition 2.2.3 (Irrotational Flow)

"Irrotational flow is that type of flow in which the fluid particles while flowing along stream-lines, do not rotate about their own axis then this type of flow is called irrotational flow." [47]

Definition 2.2.4 (Compressible Flow)

"Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid, Mathematically,

$$\rho \neq k,$$

where k is constant." [47]

Definition 2.2.5 (Incompressible Flow)

"Incompressible flow is that type of flow in which the density is constant for the fluid. Liquids are generally incompressible while gases are compressible, Mathematically,

$$\rho = k,$$

where k is constant." [47]

Definition 2.2.6 (Steady Flow)

"If the flow characteristics such as depth of flow, velocity of flow, rate of flow at any point in open channel flow do not change with respect to time, the flow is said to be steady flow. Mathematically,

$$\frac{\partial Q}{\partial t} = 0,$$

where Q is any fluid property." [47]

Definition 2.2.7 (Unsteady Flow)

"If at any point in open channel flow, the velocity of flow, depth of flow or rate of flow changes with respect to time, the flow is said to be unsteady. Mathematically,

$$\frac{\partial Q}{\partial t} \neq 0,$$

where Q is any fluid property." [47]

Definition 2.2.8 (Internal Flow)

"Flows completely bounded by a solid surfaces are called internal or duct flows." [51]

Definition 2.2.9 (External Flow)

"Flows over bodies immersed in an unbounded fluid are said to be an external flow." [51]

2.3 Classification of Fluids

2.3.1 (Types of Fluid)

"The fluids may be classified into the following five types:

1. Ideal fluid,
2. Real fluid,
3. Newtonian fluid,
4. Non-Newtonian fluid." [47]

Definition 2.3.1 (Ideal Fluid)

"A fluid, which is incompressible and has no viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity." [47]

Definition 2.3.2 (Real Fluid)

"A fluid, which possesses viscosity, is known as a real fluid. In actual practice, all the fluids are real fluids." [47]

Definition 2.3.3 (Newtonian Fluid)

"A real fluid, in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as a Newtonian fluid." [47]

Definition 2.3.4 (Non-Newtonian Fluid)

"A real fluid in which the shear stress is not directly proportional to the rate of shear strain (or velocity gradient), is known as a non-Newtonian fluid." [47]

2.4 Modes of Heat Transfer

Definition 2.4.1 (Heat Transfer)

"Heat transfer is a branch of engineering that deals with the transfer of thermal energy from one point to another within a medium or from one medium to another due to the occurrence of a temperature difference." [52]

2.4.2 (Modes of Heat Transfer) "There are three modes of heat transfer namely conduction, convection and radiation.

1. Conduction
2. Convection
3. Radiation."[52]

Definition 2.4.3 (Conduction)

"The transfer of heat within a medium due to a diffusion process is called conduction." [48]

Definition 2.4.4 (Convection)

"Convection heat transfer is usually defined as energy transport effected by the motion of a fluid. Newtons law of cooling governs the convection heat transfer between two different media." [48]

Definition 2.4.5 (Thermal Radiation)

"Thermal radiation is defined as radiant (electromagnetic) energy emitted by a medium and is sole to the temperature of the medium. Sometimes radiant energy is taken to be transported by electromagnetic waves while at other times it is supposed to be transported by particle like photons." [48]

2.5 Dimensionless Numbers

Definition 2.5.1 (Bejan Number)

"The ratio of heat transfer irreversibility to the total entropy(that is heat transfer irreversibility, fluid friction irreversibility and diffusion irreversibility) of the system is called the Bejan number. It is represented by Be and is given by

$$Be = \frac{N_{HT}}{N_{HT} + N_{MT} + N_{PM}}.$$

Where N_s is the system's overall entropy generation. N_{HT} Entropy factors, N_{MT} is the entropy factor due to mass transfer, N_{PM} is the fluid friction entropy factor." [53]

Definition 2.5.2 (Brinkman number)

"The Brinkman number related to heat conduction from a wall to a flowing viscous fluid. Mathematically,

$$Br = \frac{\mu u^2}{k(T_w - T_0)} = Pr.Ec$$

where μ is the dynamic viscosity,u is flow velocity,k is the thermal conductivity, $T_0$is the bluk fluid temperature,T_W is the wall temperature,Pr is prandtl number,Ec is the Eckert number." [54]

Definition 2.5.3 (Nusselt Number)

"The hot surface is cooled by a cold fluid stream. The heat from the hot surface, which is maintained at a constant temperature, is diffused through a boundary layer and convected away by the cold stream. Mathematically,

$$Nu = \frac{qL}{k}$$

where q stands for the convection heat transfer, L for the characteristic length and k stands for thermal conductivity.” [52]

Definition 2.5.4 (Eckert Number)

“It is the dimensionless number used in continuum mechanics. It describes the relation between flows and the boundary layer enthalpy difference and it is used for characterized heat dissipation. Mathematically,

$$Ec = \frac{u^2}{C_p \nabla T}$$

where C_p denotes the specific heat.” [51]

Definition 2.5.5 (Prandtl Number)

“It is the ratio between the momentum diffusivity ν and thermal diffusivity α. Mathematically, it can be defined as

$$Pr = \frac{\nu}{\alpha} = \frac{\frac{\mu}{\rho}}{\frac{k}{C_p\rho}} = \frac{\mu C_p}{k}$$

where μ represents the dynamic viscosity, Cp denotes the specific heat and k stands for thermal conductivity. The relative thickness of thermal and momentum boundary layer is controlled by Prandtl number. For small Pr, heat distributed rapidly corresponds to the momentum.” [51]

Definition 2.5.6 (Sherwood Number)

“It is the nondimensional quantity which show the ratio of the mass transport by convection to the transfer of mass by diffusion. Mathematically:

$$Sh = \frac{kL}{D}$$

here L is characteristics length, D is the mass diffusivity and k is the mass transfer coeffcient.” [55]

Definition 2.5.7 (Reynolds Number)

“It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. Mathematically,

$$Re = \frac{VL}{\nu},$$

where U denotes the free stream velocity, L is the characteristic length and ν stands for kinematic viscosity." [47]

2.5.8 (Thermophoresis Parameter Nt)

"In a temperature gradient, small particles are pushed towards the lower temperature because of the asymmetry of molecular impact." [55]

2.5.9 (Biot Number)

"Biot number expresses the ratio of the heat flow transferred by convection on a body surface to the heat flow transferred by conduction in a body.The criterion was first introduced by French physicist, Jean-Baptiste Biot.
Mathematically it can be expressed as

$$Bi = \frac{h_h L}{k},$$

where h_h is heat transfer coefficient, L denotes the characteristic length and k is the thermal conductivity." [55]

Definition 2.5.10 (Skin Friction Coefficient)

"The skin friction coefficient can be defined as

$$C_f = \frac{2\tau_w}{\rho w_\infty^2}$$

where τ_w denotes the wall shear stress, the velocity of free fluid flow is denoted by w_∞ and ρ is the density." [56]

2.6 Governing Laws

Definition 2.6.1 (Continuity Equation)

"The principle of conservation of mass can be stated as the time rate of change of mass in a fixed volume is equal to the net rate of flow of mass across the surface. The mathematical statement of the principle results in the following equation, known as the continuity (of mass) equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0. \tag{2.1}$$

where ρ is the density (kg/m^3) of the medium, v the velocity vector (m/s), and ∇ is the nabla or del operator.
For steady-state conditions, the continuity equation (2.1) becomes

$$\nabla.(\rho \mathbf{v}) = 0. \tag{2.2}$$

When the density changes following a fluid particle are negligible, the continuum is termed incompressible. The continuity equation (2.2) then becomes

$$\nabla.\mathbf{v} = 0. \tag{2.3}$$

which is often referred to as the incompressibility condition or incompressibility constraint."[48]

Definition 2.6.2 (Momentum Equation)
"The principle of conservation of linear momentum (or Newton's Second Law of motion) states that the time rate of change of linear momentum of a given set of particles is equal to the vector sum of all the external forces acting on the particles of the set, provided Newton's Third Law of action and reaction governs the internal forces. Newton's Second Law can be written as

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla.[(\rho \mathbf{v})\mathbf{v}] = \nabla.\sigma + \rho \mathbf{f}. \tag{2.4}$$

Where is the tensor (or dyadic) product of two vectors, σ is the Cauchy stress tensor (N/m^2) and f is the body force vector, measured per unit mass and normally taken to be the gravity vector. Equation (2.1) describes the motion of a continuous medium, and in fluid mechanics they are also known as the Navier equations. The form of the momentum equation shown in (2.4) is the conservation (divergence) form that is most often utilized for compressible flows. This equation may be simplified to a form more commonly used with incompressible flows. Expanding the first two derivatives and collecting terms

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\nabla.\mathbf{v}\right)+\mathbf{v}\left(\frac{\partial \rho}{\partial t}+\nabla.\rho\mathbf{v}\right)=\nabla.\sigma+\rho\mathbf{f}. \quad (2.5)$$

The second term in parentheses is the continuity equation (2.1) and neglecting this term allows (2.5) to reduce to the non-conservation (advective) form

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\nabla.\mathbf{v}\right)=\nabla.\sigma+\rho\mathbf{f}.\text{''} \ [48] \quad (2.6)$$

Definition 2.6.3 (Energy Equation)

"The law of conservation of energy (or the First Law of Thermodynamics) states that the time rate of change of the total energy is equal to the sum of the rate of work done by applied forces and the change of heat content per unit time. In the general case, the First Law of Thermodynamics can be expressed in conservation form as

$$\frac{\partial \rho e^t}{\partial t}+\nabla.\rho\mathbf{v}e^t=-\nabla.\mathbf{q}+\nabla.(\sigma.\mathbf{v})+Q+\rho\mathbf{f}.\mathbf{v} \quad (2.7)$$

where $e^t = e + 1/2\mathbf{v}.\mathbf{v}$ is the total energy (J/m^3), e is the internal energy, $\mathbf{q}$ is the heat flux vector (W/m^2) and Q is the internal heat generation (W/m^3). The total energy equation (2.7) is useful for high speed compressible flows where the kinetic energy is significant. For incompressible flows, an internal energy equation is more appropriate and can be derived from (2.7) with use of the momentum equation(2.4) Taking the dot product of the velocity vector with the momentum equation produces an equation for the kinetic energy this equation is subtracted from the total energy equation (2.7) to produce the conservation (divergence) form of the internal energy equation

$$\frac{\partial \rho e}{\partial t}+\nabla.\rho\mathbf{v}e=-\nabla.\mathbf{q}+Q+\phi \quad (2.8)$$

where ϕ is the dissipation function that is defined by

$$\phi=\sigma:\nabla\mathbf{v} \quad (2.9)$$

In Eq.(2.9) $\nabla\mathbf{v}$ is the velocity gradient tensor. " [48]

2.7 Solution Methodology

To elaborate the shooting method, consider the following nonlinear boundary value problem.

$$2V'''(x) + V(x)V'' = 0. \tag{2.10}$$

along with boundary conditions

$$V(0) = 0, \quad V'(0) = 0, \quad V'(m) = 1. \tag{2.11}$$

To reduce the order of the above boundary value problem, introduce the following notations.

$$V = g_1, \quad V' = g_1' = g_2, \quad V'' = g_2' = g_3, \quad V''' = g_3'. \tag{2.12}$$

As a result, (2.10)and (2.11) converted into the system of first order ODEs.

$$g_1' = g_2, \qquad g_1(0) = 0. \tag{2.13}$$

$$g_2' = g_3, \qquad g_2(0) = 0. \tag{2.14}$$

$$g_3' = \frac{-1}{2} g_3, \qquad g_3(0) = k. \tag{2.15}$$

where k is the missing initial condition which will be guessed. The above IVP will be numerically solved by the RK-4 method. The missing condition k is to be chosen such that.

$$g_2(m, k) = 1. \tag{2.16}$$

For convenience, now onward, $g_2\big(m, k\big)$ will be denoted by $g_2(k)$. Let us further denote $g_2(k) - 1$ by $H(k)$, so that

$$H(k) = 0. \tag{2.17}$$

The above equation can be solved by using Newton's method, which has the following iterative formula.

$$k_{n+1} = k_n - \frac{H(k_n)}{\frac{\partial H(k_n)}{\partial k}},$$
$$k_{n+1} = k_n - \frac{g_2(k_n) - 1}{\frac{\partial g_2(k_n)}{\partial k}}. \tag{2.18}$$

To find $\frac{\partial g_2(k_n)}{\partial k}$, introduce the following notations.

$$\frac{\partial g_1}{\partial k} = g_4, \quad \frac{\partial g_2}{\partial k} = g_5, \quad \frac{\partial g_3}{\partial k} = g_6. \tag{2.19}$$

As a result of these new notations, the Newton's iterative scheme, will then get the following form.

$$k_{n+1} = k_n - \frac{g_2(k_n) - 1}{g_5(k_n)}. \tag{2.20}$$

Now differentiating the system of two first order ODEs (2.13)-(2.15) with respect to k, we get another system of ODEs, as follows

$$g_4' = g_5, \qquad g_4(0) = 0. \tag{2.21}$$
$$g_5' = g_6, \qquad g_5(0) = 0. \tag{2.22}$$
$$g_6' = \frac{-1}{2}\Big[g_1 g_6 + g_3 g_4\Big], \qquad g_6(0) = 1. \tag{2.23}$$

Writing all the six ODEs (2.13), (2.14), (2.15), (2.21), (2.22), and (2.23) together, we have the following initial value problem.

$$g_1' = g_2, \qquad g_1(0) = 0.$$
$$g_2' = g_3, \qquad g_2(0) = 0.$$
$$g_3' = \frac{-1}{2} g_3, \qquad g_3(0) = k$$
$$g_4' = g_5, \qquad g_4(0) = 0.$$
$$g_5' = g_6, \qquad g_5(0) = 0.$$

$$g_6' = \frac{-1}{2}\left[g_1 g_6 + g_3 g_4\right], \qquad g_6(0) = 1.$$

The above system together will be solved numerically by Runge-Kutta method of order four. The missing condition will be updated by the Newton's formula in (2.20). The stopping criteria for the Newton's technique is set as,

$$|g_2(k) - 1| < \epsilon^*,$$

where $\epsilon^* > 0$ is an arbitrarily small positive number. [57]

Chapter 3

Entropy Generation in Hydromagnetic Nanofluid Flow Over a Non-linear Stretching with Navier's Velocity Slip and Convective Heat Transfer

3.1 Introduction

In the ongoing chapter, we examine the formation of entropy in a hydromagnetic nanofluid flowing past a nonlinearly stretching sheet encased in a porous medium under the impact of thermal radiation, a magnetic field, and slip velocity [43].

By converting the governing PDEs into ODEs with the appropriate similarity transformation, the shooting method is used to obtain the numerical results. The computational software MATLAB is used for numerical computation. Graphical representations are also provided to explain the effect of evolving parameters. Tables and graphs are used to investigate the numerical results produced.

3.2 Mathematical Modeling

We have considered a $2D$ unsteady flow of an electro-conducting, viscous, incompressible, optically thick, and thermally radiating nanofluid past a horizontal nonlinear stretching sheet embedded in a porous medium. The x axis is aligned along the stretching sheet, and the flow is constrained in the domain $y \geq 0$, where y denotes the coordinate axis selected in the direction normal to the sheet. The application of a force while maintaining the origin fixed causes the sheet to continuously elongate with a time dependent nonlinear velocity, $u_w = \frac{ax^n}{(1-\lambda t)}$.

The sheet is being convectively heated with temperature T_f, which is fluid temperature at the bottom of the sheet. In the direction perpendicular to the sheet, a homogeneous transverse magnetic field of intensity B_0 is imposed. The model's configuration and a physical drawing of the coordinate system are shown in Figure 3.1.

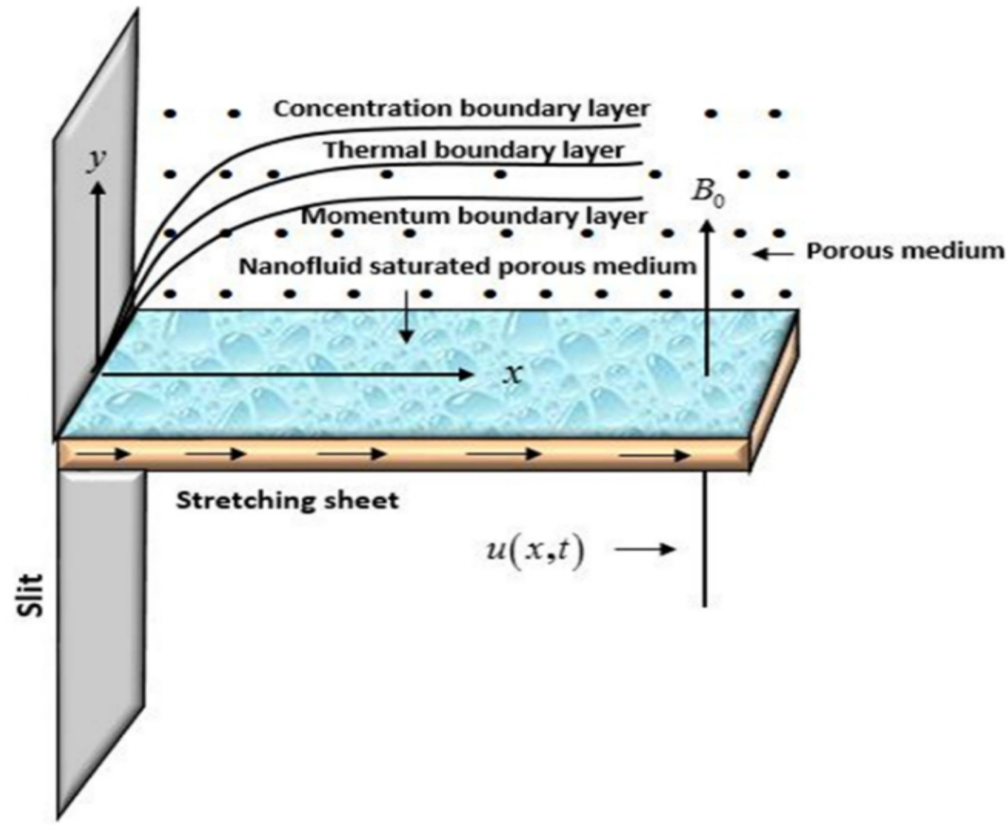

FIGURE 3.1: Problem schematic diagram.

The set of equations describing the flow is as follows.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3.1}$$

$$\rho_{nf}\left(u\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu_{nf}\left(\frac{\partial^2 u}{\partial y^2}\right)-\frac{\partial p}{\partial x}-\sigma B_0^2 u-\frac{\mu_{nf}}{k_p}u, \tag{3.2}$$

$$\frac{\partial p}{\partial y}=0, \tag{3.3}$$

$$\begin{aligned}\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_{nf}\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho_{nf}c_p}\Big(\frac{\partial q_r}{\partial y}\Big)+\frac{v_{nf}}{c_p}\Big(\frac{\partial u}{\partial y}\Big)^2\\ +\frac{\sigma B_0^2}{\rho_{nf}c_p}u^2+\frac{v_{nf}}{k_p c_p}u^2\\ +\tau\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right],\end{aligned} \tag{3.4}$$

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D_B\frac{\partial^2 C}{\partial y^2}+\frac{D_T}{T_\infty}\Big(\frac{\partial^2 T}{\partial y^2}\Big). \tag{3.5}$$

The associated BCs have been taken as

$$\left.\begin{aligned} &u=u_w+u_{slip}=\frac{ax^n}{1-\lambda t}+N\frac{\partial u}{\partial y}, \quad v=0,\\ &k_{nf}\frac{\partial T}{\partial y}=h_f(T_f-T), \quad C=C_w, \quad at \quad y=0,\\ &u\to 0, \quad T\to T_\infty, \quad C\to C_\infty, \quad as \quad y\to\infty. \end{aligned}\right\} \tag{3.6}$$

Expand T^4 about T_∞ by the Taylor series as follows.

$$T^4=T_\infty^4+4T_\infty^3(T-T_\infty)+6T_\infty^2(T-T_\infty)^2+...$$

Ignoring the highest order terms, we have

$$\begin{aligned}T^4&=T_\infty^4+4T_\infty^3(T-T_\infty)\\ &=4T_\infty^3 T-3T_\infty^4.\end{aligned}$$

The heat flux of radiation can be used as a function of temperature according to Rosseland approximation for radiation [58], as follows.

$$q_r=-\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}=-\frac{4\sigma^*}{3k^*}4\frac{\partial T^3}{\partial y}=-\frac{16\sigma^*}{3k^*}T^3\frac{\partial T}{\partial y}. \tag{3.7}$$

Here Stefan-Boltzman constant is σ^* and the absorption coefficient is k^*.

Using (3.7) in (3.4) we have

$$\begin{aligned}\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}&=\alpha_{nf}\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho_{nf}c_p}\frac{\partial}{\partial y}\left(-\frac{16\sigma^*}{3k^*}T^3\frac{\partial T}{\partial y}\right)+\frac{v_{nf}}{c_p}\left(\frac{\partial u}{\partial y}\right)^2\\&\quad+\frac{\sigma B_0^2}{\rho_{nf}c_p}u^2+\frac{v_{nf}}{k_pc_p}u^2+\tau\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right]\\&=\frac{\partial}{\partial y}\left[\left(\alpha_{nf}+\frac{16\sigma^*T^3}{3k^*\rho_{nf}c_p}\right)\frac{\partial T}{\partial y}\right]+\frac{v_{nf}}{c_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\sigma B_0^2}{\rho_{nf}c_p}u^2\\&\quad+\frac{v_{nf}}{k_pc_p}u^2+\tau\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right].\end{aligned}\tag{3.8}$$

Equation (3.3) indicates p is a fixed function of x, and in the perpendicular motion, p unchanged.

$$\frac{\partial U_\infty}{\partial t}+U_\infty\frac{\partial U_\infty}{\partial x}=-\frac{1}{\rho_{nf}}\left(\frac{\partial p}{\partial x}\right),\tag{3.9}$$

where U_∞ denotes the flow motion away from the boundary layer.

$$\begin{aligned}&u=U_\infty\to 0\\&\frac{\partial U_\infty}{\partial t}+U_\infty\frac{\partial U_\infty}{\partial x}=-\frac{1}{\rho_{nf}}\left(\frac{\partial p}{\partial x}\right),\\\Rightarrow\quad&\left(\frac{\partial p}{\partial x}\right)=0\end{aligned}$$

$$\rho_{nf}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu_{nf}\frac{\partial^2 u}{\partial y^2}-\sigma B_0^2u-\frac{\mu_{nf}}{k_p}u.\tag{3.10}$$

For the conversion of the equations (3.1), (3.5), (3.8) and (3.10) into a system of ODEs, the following similarity transformation was used.

$$\left.\begin{aligned}&\zeta=y\sqrt{\frac{u_w(x,t)}{v_{nf}x}},\\&\psi=\sqrt{v_{nf}xu_w(x,t)}f(\zeta)\\&\phi(\zeta)=\frac{C-C_\infty}{C_w-C_\infty},\theta(\zeta)=\frac{T-T_\infty}{T_f-T_\infty},\\&T=T_\infty\Big(1+(\theta_w-1)\theta(\zeta)\Big),\end{aligned}\right\}\tag{3.11}$$

where $\theta_w = \frac{T_f}{T_\infty}$. The detailed procedure for the conversion of (3.1)-(3.5) into the dimensionless form has been discussed in the upcoming discussion

$$\begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ &= \frac{\partial}{\partial y}\left(\sqrt{v_{nf} x u_w} f(\zeta)\right) \\ &= \sqrt{v_{nf} x u_w} f'(\zeta) \frac{\partial \zeta}{\partial y} \\ &= \sqrt{v_{nf} x u_w} f'(\zeta) \sqrt{\frac{u_w}{v_{nf} x}} \\ &= u_w f'(\zeta) \\ &= \frac{a x^n}{1-\lambda t} f'(\zeta) \qquad (3.12) \\ &\qquad \left(\because u_w = \frac{a x^n}{1-\lambda t}\right) \end{aligned}$$

Now differentiating both sides of (3.12) w.r.t. to x,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{a x^n}{1-\lambda t} f'(\zeta)\right) \\ &= \frac{n a x^{n-1}}{1-\lambda t} f'(\zeta) + \left(\frac{a x^n}{1-\lambda t} f''(\zeta)\right) \frac{\partial \zeta}{\partial x} \\ &= \frac{n a x^{n-1}}{1-\lambda t} f'(\zeta) + \left(\frac{a x^n}{1-\lambda t} f''(\zeta)\right) y \sqrt{\frac{a}{(1-\lambda t) v_{nf}}} \left(\frac{n-1}{2}\right) x^{\frac{n-3}{2}} \\ &= \frac{n a x^{n-1}}{1-\lambda t} f'(\zeta) + \left(\frac{a x^n}{1-\lambda t} f''(\zeta)\right) y \sqrt{\frac{a x^n}{(1-\lambda t) x v_{nf}}} \left(\frac{n-1}{2}\right) x^{-1} \\ &= \frac{a x^{n-1}}{1-\lambda t}\left(n f'(\zeta) + \left(\frac{n-1}{2}\right) \zeta f''(\zeta)\right). \qquad (3.13) \\ v &= -\frac{\partial \psi}{\partial x} \\ &= \frac{\partial}{\partial x}\left(-\sqrt{v_{nf} x u_w(x,t)} f(\zeta)\right) \qquad \left(\because u_w = \frac{a x^n}{1-\lambda t}\right) \\ &= -\sqrt{\frac{v_{nf} a}{1-\lambda t}}\left[\frac{\partial}{\partial x}\left(x^{\frac{n+1}{2}} f(\zeta)\right)\right] \\ &= -\sqrt{\frac{v_{nf} a}{1-\lambda t}}\left(\frac{n+1}{2} x^{\frac{n-1}{2}} f(\zeta) + x^{\frac{n+1}{2}} f'(\zeta) \frac{\partial \zeta}{\partial x}\right) \\ &= -\sqrt{\frac{v_{nf} a}{1-\lambda t}}\left[\frac{n+1}{2} x^{\frac{n-1}{2}} f(\zeta) + x^{\frac{n+1}{2}} f'(\zeta) \frac{n-1}{2} \zeta \frac{1}{x}\right] \\ &= -\sqrt{\frac{v_{nf} a}{1-\lambda t}}\left[\frac{n+1}{2} x^{\frac{n-1}{2}} f(\zeta) + x^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right) \zeta f'(\zeta)\right] \end{aligned}$$

$$
\begin{aligned}
&= -\sqrt{\frac{v_{nf}a}{1-\lambda t}}(x)^{\frac{n-1}{2}}\left[\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right] \\
&= -\sqrt{\frac{ax^{n-1}v_{nf}}{(1-\lambda t)}}\left(\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right) \\
&= -\sqrt{\frac{ax^{n-1}v_{nf}}{(1-\lambda t)}}\left(\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right) \\
&= -\sqrt{\frac{(ax^{n-1})v_{nf}}{(1-\lambda t)}}\left(\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right). \qquad (3.14)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial v}{\partial y} &= -\frac{\partial}{\partial y}\left(\sqrt{\frac{ax^{n-1}v_{nf}}{(1-\lambda t)}}\left(\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)\right) \\
&= -\sqrt{\frac{ax^{n-1}v_{nf}}{(1-\lambda t)}}\frac{\partial}{\partial y}\left(\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right) \\
&= -\sqrt{\frac{ax^{n-1}v_{nf}}{(1-\lambda t)}}\left(\frac{n+1}{2}f'(\zeta)\frac{\partial \zeta}{\partial y}+\left(\frac{n-1}{2}\right)\zeta f''(\zeta)\frac{\partial \zeta}{\partial y}+\left(\frac{n-1}{2}\right)f'(\zeta)\frac{\partial \zeta}{\partial y}\right) \\
&= -\sqrt{\frac{ax^{n-1}v_{nf}}{(1-\lambda t)}}\left(\frac{n+1}{2}f'(\zeta)\sqrt{\frac{ax^{n}}{(1-\lambda t)v_{nf}x}}\right. \\
&\quad \left.+\left(\frac{n-1}{2}\right)\zeta f''(\zeta)\sqrt{\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}}+\left(\frac{n-1}{2}\right)f'(\zeta)\sqrt{\frac{ax^{n}}{(1-\lambda t)v_{nf}x}}\right) \\
&= -\frac{ax^{n-1}}{(1-\lambda t)}\left(nf'(\zeta)+\left(\frac{n-1}{2}\right)\zeta f''(\zeta)\right). \qquad (3.15)
\end{aligned}
$$

Equation (3.1) is easily satisfied by using (3.13) and (3.15), as follows

$$
\begin{aligned}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= \frac{ax^{n-1}}{(1-\lambda t)}\left(nf'(\zeta)+\left(\frac{n-1}{2}\right)\zeta f''(\zeta)-nf'(\zeta)-\left(\frac{n-1}{2}\right)\zeta f''(\zeta)\right) \\
&= 0
\end{aligned}
$$

Now, we include below the procedure for the conversion of the momentum equation (3.10) into the dimessionless form.

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{ax^{n}}{(1-\lambda t)}f'(\zeta)\right) \\
&= ax^{n}\frac{\lambda}{(1-\lambda t)^2}f'(\zeta)+\frac{ax^{n}}{(1-\lambda t)}f''(\zeta)\frac{\partial \zeta}{\partial t} \\
&= ax^{n}\frac{\lambda}{(1-\lambda t)^2}f'(\zeta)+\frac{ax^{n}}{(1-\lambda t)}f''(\zeta)y\sqrt{\frac{ax^{n}}{v_{nf}x}}\frac{\partial}{\partial t}\left(1-\lambda t\right)^{\frac{-1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= ax^n\frac{\lambda}{(1-\lambda t)^2}f'(\zeta) + \frac{ax^n}{(1-\lambda t)}f''(\zeta)y\sqrt{\frac{ax^n}{\upsilon_{nf}x}}\frac{\lambda}{2}\Big(1-\lambda t\Big)^{\frac{-3}{2}}\\
&= ax^n\frac{\lambda}{(1-\lambda t)^2}f'(\zeta) + \frac{ax^n}{(1-\lambda t)^2}f''(\zeta)y\sqrt{\frac{ax^n}{(1-\lambda t)\upsilon_{nf}x}}\frac{\lambda}{2}\\
&= \frac{ax^n\lambda}{(1-\lambda t)^2}\Big(f'(\zeta) + \frac{\zeta}{2}f''(\zeta)\Big). \qquad (3.16)
\end{aligned}
$$

$$
\begin{aligned}
u\frac{\partial u}{\partial x} &= \frac{ax^n}{(1-\lambda t)}f'(\zeta)\Bigg(\frac{ax^{n-1}}{1-\lambda t}\Big(nf'(\zeta) + (\frac{n-1}{2})f''(\zeta)\Big)\Bigg)\\
&= \Big(\frac{ax^n}{1-\lambda t}\Big)^2\frac{1}{x}\Big(nf'^2(\zeta) + \frac{n-1}{2}f''(\zeta)f'(\zeta)\Big). \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\Big(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\Big)\\
&= \frac{ax^n}{(1-\lambda t)}\frac{\partial}{\partial y}\Big(f'(\zeta)\Big)\\
&= \frac{ax^n}{(1-\lambda t)}f''(\zeta)\frac{\partial \zeta}{\partial y}\\
&= \frac{ax^n}{(1-\lambda t)}f''(\zeta)\sqrt{\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}}. \qquad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
v\frac{\partial u}{\partial y} &= -\sqrt{\frac{ax^{n-1}\upsilon_{nf}}{(1-\lambda t)}}\Big(\frac{n+1}{2}f(\zeta) + (\frac{n-1}{2})\zeta f'(\zeta)\Big)\\
&\quad \frac{ax^n}{(1-\lambda t)}f''(\zeta)\sqrt{\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}}\\
&= -\Big(\frac{ax^n}{(1-\lambda t)}\Big)^2\frac{1}{x}\Big(\frac{n+1}{2}f(\zeta)f''(\zeta) + \frac{n-1}{2}\zeta f'(\zeta)f''(\zeta)\Big). \qquad (3.19)
\end{aligned}
$$

$$
\frac{\partial^2 u}{\partial y^2} = \Big(\frac{ax^n}{(1-\lambda t)}\Big)^2\frac{1}{\upsilon_{nf}x}f'''(\zeta). \qquad (3.20)
$$

Using (3.16), (3.17) and (3.19) in the left side of (3.10), it becomes:

$$
\begin{aligned}
\rho_{nf}\Big(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\Big) &= \rho_{nf}\Bigg(\frac{ax^n\lambda}{(1-\lambda t)^2}\Big(f'(\zeta) + \frac{\zeta}{2}f''(\zeta)\Big)\Bigg)\\
&+ \rho_{nf}\Bigg(\Big(\frac{ax^n}{1-\lambda t}\Big)^2\frac{1}{x}\Big(nf'^2(\zeta) + \frac{n-1}{2}f''(\zeta)f'(\zeta)\Big)\Bigg)\\
&+ \rho_{nf}\Bigg(-\Big(\frac{ax^n}{(1-\lambda t)}\Big)^2\frac{1}{x}\Big(\frac{n+1}{2}f(\zeta)f''(\zeta) + \frac{n-1}{2}\zeta f'(\zeta)f''(\zeta)\Big)\Bigg)\\
&= \rho_{nf}\frac{ax^n\lambda}{(1-\lambda t)^2}f'(\zeta) + \rho_{nf}\frac{ax^n\lambda}{(1-\lambda t)^2}\frac{\zeta}{2}f''(\zeta) + \rho_{nf}\Bigg(\Big(\frac{ax^n}{1-\lambda t}\Big)^2\frac{1}{x}nf'^2(\zeta)\\
&+ \rho_{nf}\Bigg(\Big(\frac{ax^n}{1-\lambda t}\Big)^2\frac{1}{x}\frac{n-1}{2}\zeta f'(\zeta)f''(\zeta)\Bigg)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&-\rho_{nf}\left(\left(\frac{ax^n}{1-\lambda t}\right)^2\frac{1}{x}\frac{n+1}{2}f(\zeta)f''(\zeta)\right.\\
&-\rho_{nf}\left(\left(\frac{ax^n}{1-\lambda t}\right)^2\frac{1}{x}\frac{n-1}{2}\zeta f'(\zeta)f''(\zeta)\right)\Bigg).\\
&=\rho_{nf}\frac{ax^n}{(1-\lambda t)^2}\Bigg(\lambda f'(\zeta)+\lambda\frac{\zeta}{2}f''(\zeta)\\
&+ax^{n-1}nf'^2(\zeta)-\left(ax^{n-1}\right)\frac{n+1}{2}f(\zeta)f''(\zeta)\Bigg).
\end{aligned}
\tag{3.21}
$$

Using (3.12) and (3.20), in the right side of (3.10), it becomes:

$$
\begin{aligned}
\mu_{nf}\frac{\partial^2 u}{\partial y^2}-\sigma B_0^2 u-\frac{\mu_{nf}}{k_p}u=\mu_{nf}&\left(\frac{ax^n}{(1-\lambda t)}\right)^2\frac{1}{\upsilon_{nf}x}f'''(\zeta)\\
&-\sigma B_0^2\frac{ax^n}{1-\lambda t}f'(\zeta)-\frac{\mu_{nf}}{k_p}\frac{ax^n}{1-\lambda t}f'(\zeta).
\end{aligned}
\tag{3.22}
$$

Using (3.21) and (3.22) in equation (3.10), we obtain

$$
\begin{aligned}
&\rho_{nf}\frac{ax^n}{(1-\lambda t)^2}\left(\lambda f'(\zeta)+\lambda\frac{\zeta}{2}f''(\zeta)+ax^{n-1}nf'^2(\zeta)-ax^{n-1}\frac{n+1}{2}f(\zeta)f''(\zeta)\right)\\
&=\mu_{nf}\left(\frac{ax^n}{(1-\lambda t)}\right)^2\frac{1}{\upsilon_{nf}x}f'''(\zeta)-\sigma B_0^2\frac{ax^n}{1-\lambda t}f'(\zeta)-\frac{\mu_{nf}}{k_p}\frac{ax^n}{1-\lambda t}f'(\zeta).
\end{aligned}
\tag{3.23}
$$

Multiplying each term of (3.23) by $\frac{\upsilon_{nf}(1-\lambda t)^2 x}{(ax^n)^2\mu_{nf}}$, we get

$$
\begin{aligned}
&\rho_{nf}\frac{\upsilon_{nf}}{\mu_{nf}}\left(\frac{\lambda}{ax^{n-1}}\right)\left(f'(\zeta)+\frac{\zeta}{2}f''(\zeta)\right)+\rho_{nf}\frac{\upsilon_{nf}}{\mu_{nf}}\\
&\quad\left(nf'^2(\zeta)-\frac{n+1}{2}f(\zeta)f''(\zeta)\right)\\
&=f'''(\zeta)-\left(\frac{\sigma B_0^2 x(1-\lambda t)}{ax^n}\right)\frac{\upsilon_{nf}}{\mu_{nf}}f'(\zeta)-\left(\frac{\upsilon_{nf}x(1-\lambda t)}{k_p ax^n}\right)f'(\zeta).
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
\Rightarrow\quad & f'''(\zeta)-\frac{\sigma x}{\rho_{nf}u_w}B_0^2 f'(\zeta)-\frac{\upsilon_{nf}x}{k_p u_w}f'(\zeta)-nf'^2(\zeta)\\
&-\frac{\lambda}{ax^{n-1}}\left(f'(\zeta)+\frac{\zeta}{2}f''(\zeta)\right)+\frac{n+1}{2}f(\zeta)f''(\zeta)=0.\\
\Rightarrow\quad & f'''(\zeta)+\frac{n+1}{2}f(\zeta)f''(\zeta)-nf'^2(\zeta)-(M+K)\,f'(\zeta)\\
&-A\left((f'(\zeta)+\frac{\zeta}{2}f''(\zeta)\right)=0.
\end{aligned}
\tag{3.25}
$$

Now, for the conversion of energy equation (3.8), the following derivatives are required:

- $\theta(\zeta) = \dfrac{T - T_\infty}{T_f - T_\infty}$.

$$\Rightarrow \quad T = T_\infty\Big(1 + (\theta_w - 1)\theta(\zeta)\Big)$$
$$= T_\infty + T_\infty(\theta_w - 1)\theta(\zeta).$$

- $$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\big(T_\infty + T_\infty(\theta_w - 1)\theta(\zeta)\big)$$
$$= T_\infty(\theta_w - 1)\theta'(\zeta)\frac{\partial \zeta}{\partial t}$$
$$= T_\infty(\theta_w - 1)\theta'(\zeta) y\sqrt{\frac{ax^n}{\upsilon_{nf} x}}\frac{\partial}{\partial t}\Big(1 - \lambda t\Big)^{\frac{-1}{2}}$$
$$= T_\infty(\theta_w - 1)\theta'(\zeta) y\sqrt{\frac{ax^{n-1}}{\upsilon_{nf}}}\big(\frac{\lambda}{2}\big)\Big(1 - \lambda t\Big)^{\frac{-3}{2}}$$
$$= T_\infty(\theta_w - 1)\theta'(\zeta)\frac{\lambda}{(1 - \lambda t)}\frac{\zeta}{2}. \tag{3.26}$$

- $$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\big(T_\infty + T_\infty(\theta_w - 1)\theta(\zeta)\big)$$
$$= T_\infty(\theta_w - 1)\theta'(\zeta)\frac{\partial \zeta}{\partial x}$$
$$= T_\infty(\theta_w - 1)\theta'(\zeta)\left(\frac{n-1}{2}\right)\frac{\zeta}{x}. \tag{3.27}$$

- $$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\big(T_\infty + T_\infty(\theta_w - 1)\theta(\zeta)\big)$$
$$= T_\infty(\theta_w - 1)\theta'(\zeta)\frac{\partial \zeta}{\partial y}$$
$$= T_\infty(\theta_w - 1)\theta'(\zeta)\sqrt{\frac{ax^{n-1}}{\upsilon_{nf}(1 - \lambda t)}}$$
$$= T_\infty(\theta_w - 1)\theta'(\zeta)\sqrt{\frac{ax^{n-1}}{\upsilon_{nf}(1 - \lambda t)}}. \tag{3.28}$$

- $$\frac{\partial^2 T}{\partial y^2} = \frac{\partial}{\partial y}\left[T_\infty(\theta_w - 1)\theta'(\zeta)\sqrt{\frac{ax^{n-1}}{\upsilon_{nf}(1 - \lambda t)}}\ \right]$$
$$= \sqrt{\frac{ax^{n-1}}{\upsilon_{nf}(1 - \lambda t)}}\frac{\partial}{\partial y}\Big(T_\infty(\theta_w - 1)\theta'(\zeta)\Big)$$
$$= T_\infty(\theta_w - 1)\theta''(\zeta)\frac{ax^{n-1}}{\upsilon_{nf}(1 - \lambda t)}. \tag{3.29}$$

- $$u\frac{\partial T}{\partial x} = \frac{ax^n}{(1 - \lambda t)}\left(T_\infty(\theta_w - 1)\theta'(\zeta)\left(\frac{n-1}{2}\right)\frac{\zeta}{x}\right)$$

$$= \frac{ax^{n-1}}{(1-\lambda t)} f'(\zeta)\left(T_\infty(\theta_w - 1)\theta'(\zeta)\left(\frac{n-1}{2}\right)\zeta\right). \tag{3.30}$$

- $$v\frac{\partial T}{\partial y} = -\sqrt{\frac{ax^n v_{nf}}{(1-\lambda t)x}}\left(\frac{n+1}{2}f(\zeta)\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)$$

$$T_\infty(\theta_w - 1)\theta'(\zeta)\sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}}$$

$$= -\frac{ax^{n-1}}{(1-\lambda t)}\left(\frac{n+1}{2}f(\zeta) + \left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)T_\infty(\theta_w - 1)\theta'(\zeta). \tag{3.31}$$

Using (3.26), (3.30) and (3.31) in left the side of (3.8), we obtain

$$\begin{aligned}
\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= T_\infty(\theta_w - 1)\theta'(\zeta)\frac{\lambda}{2(1-\lambda t)}\zeta \\
&+ \frac{ax^{n-1}}{(1-\lambda t)} f'(\zeta)\left(T_\infty(\theta_w - 1)\theta'(\zeta)\left(\frac{n-1}{2}\right)\zeta\right) \\
&- \frac{ax^{n-1}}{(1-\lambda t)}\left(\frac{n+1}{2}f(\zeta) + \left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)T_\infty(\theta_w - 1)\theta'(\zeta) \\
&= \frac{T_\infty(\theta_w - 1)\theta'(\zeta)}{(1-\lambda t)}\left(\frac{\zeta}{2}\lambda + ax^{n-1}\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right. \\
&\left. - ax^{n-1}\left(\frac{n-1}{2}\right)\zeta f'(\zeta) - ax^{n-1}\left(\frac{n+1}{2}f(\zeta)\right)\right) \\
&= \frac{T_\infty(\theta_w - 1)\theta'(\zeta)}{(1-\lambda t)}\left(\lambda\frac{\zeta}{2} - ax^{n-1}\left(\frac{n+1}{2}f(\zeta)\right)\right).
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
\frac{\partial q_r}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{16\sigma^*}{3k^*}T^3\frac{\partial T}{\partial y}\right) \\
&= -\frac{16\sigma^*}{3k^*}\frac{\partial}{\partial y}\left(T^3\frac{\partial T}{\partial y}\right) \\
&= -\frac{16\sigma^*}{3k^*}\left[3T^2\left(\frac{\partial T}{\partial y}\right)^2 + T^3\frac{\partial^2 T}{\partial y^2}\right] \\
&= -\frac{16\sigma^*}{3k^*}\left(3T_\infty^2\left(1 + (\theta_w - 1)\theta(\zeta)\right)^2\left(\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}\right)T_\infty^2(\theta_w - 1)^2\theta'^2(\zeta)\right) \\
&- \frac{16\sigma^*}{3k^*}\left(T_\infty^3\left(1 + (\theta_w - 1)\theta(\zeta)\right)^3\left(\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}\right)T_\infty(\theta_w - 1)\theta''(\zeta)\right).
\end{aligned} \tag{3.33}$$

- $$\alpha_{nf}\frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho_{nf}c_p}\frac{\partial}{\partial y}\left(-\frac{16\sigma^*}{3k^*}T^3\frac{\partial T}{\partial y}\right) = \alpha_{nf}T_\infty(\theta_w - 1)$$

$$\theta''(\zeta)\left(\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}\right) + \frac{16\sigma^*}{\rho_{nf}c_p 3k^*}$$

$$\left(\left(3T_\infty^2\left(1 + (\theta_w - 1)\theta(\zeta)\right)^2\left(\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}\right)T_\infty^2(\theta_w - 1)^2\theta'^2(\zeta)\right)\right.$$

$$\left. + \left(T_\infty^3\left(1 + (\theta_w - 1)\theta(\zeta)\right)^3\left(\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}\right)T_\infty(\theta_w - 1)\theta''(\zeta)\right)\right)$$

$$\alpha_{nf}\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho_{np}c_p}\frac{\partial}{\partial y}\bigg(-\frac{16\sigma^*}{3k^*}T^3\frac{\partial T}{\partial y}\bigg)$$
$$=\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}T_\infty(\theta_w-1)\alpha_{nf}\Bigg(\theta''(\zeta)+\frac{16\sigma^*}{\rho_{np}c_p\alpha_{nf}}$$
$$\Bigg(3T_\infty^3\left(1+(\theta_w-1)\theta(\zeta)\right)^2(\theta_w-1)\theta'^2(\zeta)$$
$$+T_\infty^3\left(1+(\theta_w-1)\theta(\zeta)\right)^3\theta''(\zeta)\Bigg)\Bigg). \tag{3.34}$$

- $$\phi(\zeta)=\frac{C-C_\infty}{C_w-C_\infty}$$
$$\Rightarrow\quad C=\phi(\zeta)\big(C_w-C_\infty\big)+C_\infty \tag{3.35}$$
- $$\frac{\partial C}{\partial y}=\frac{\partial}{\partial y}\big(\phi(\zeta)\big(C_w-C_\infty\big)+C_\infty\big)$$
$$=\big(C_w-C_\infty\big)\phi'(\zeta)\frac{\partial\zeta}{\partial y}$$
$$=\big(C_w-C_\infty\big)\phi'(\zeta)\sqrt{\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}}. \tag{3.36}$$
- $$\frac{\partial T}{\partial y}\frac{\partial C}{\partial y}=T_\infty(\theta_w-1)\theta'(\zeta)\sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}}\big(C_w-C_\infty\big)\phi'(\zeta)\sqrt{\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}}$$
$$=T_\infty(\theta_w-1)\theta'(\zeta)\big(C_w-C_\infty\big)\phi'(\zeta)\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}. \tag{3.37}$$
- $$\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{ax^n}{(1-\lambda t)}\right)^2 f''^2(\zeta)\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}. \tag{3.38}$$
- $$\frac{v_{nf}}{c_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\sigma B_0^2}{\rho_{nf}c_p}u^2+\frac{v_{nf}}{k_pc_p}u^2=\frac{v_{nf}}{c_p}\left(\frac{ax^n}{(1-\lambda t)}f''(\zeta)\right)^2$$
$$\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}+\frac{\sigma B_0^2}{\rho_{nf}c_p}$$
$$\left(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\right)^2+\frac{v_{nf}}{k_pc_p}\left(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\right)^2. \tag{3.39}$$
- $$\tau\left(D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right)=\tau\Bigg(D_BT_\infty(\theta_w-1)\theta'(\zeta)\big(C_w-C_\infty\big)$$
$$\phi'(\zeta)\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}+\frac{D_T}{T_\infty}\left(T_\infty(\theta_w-1)\theta'(\zeta)\right)^2\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}\Bigg). \tag{3.40}$$

Using (3.34), (3.39) and (3.40) in right the side of (3.8), we get

$$\alpha_{nf}\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho_{nf}c_p}\frac{\partial}{\partial y}\bigg(-\frac{16\sigma^*}{3k^*}T^3\frac{\partial T}{\partial y}\bigg)+\frac{v_{nf}}{c_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\sigma B_0^2}{\rho_{nf}c_p}u^2+\frac{v_{nf}}{k_pc_p}u^2$$
$$+\tau\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right]=\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}T_\infty(\theta_w-1)\alpha_{nf}\Bigg(\theta''(\zeta)+\frac{16\sigma^*}{\rho_{np}c_p\alpha_{nf}}$$

$$\left(3T_\infty^3\left(1+(\theta_w-1)\theta(\zeta)\right)^2(\theta_w-1)\theta'^2(\zeta)+T_\infty^3\left(1+(\theta_w-1)\theta(\zeta)\right)^3\theta''(\zeta)\right)\Bigg)$$
$$+\frac{\upsilon_{nf}}{c_p}\left(\frac{ax^n}{(1-\lambda t)}f''(\zeta)\right)^2\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}+\frac{\sigma B_0^2}{\rho_{nf}c_p}\left(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\right)^2$$
$$+\frac{\upsilon_{nf}}{k_pc_p}\left(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\right)^2\tau\Bigg(D_BT_\infty(\theta_w-1)\theta'(\zeta)\big(C_w-C_\infty\big)\phi'(\zeta)\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}$$
$$+\frac{D_T}{T_\infty}\left(T_\infty(\theta_w-1)\theta'(\zeta)\right)^2\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}\Bigg). \tag{3.41}$$

Using (3.32) and (3.41) in equation (3.8), we obtain dimensionless form of Energy equation

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_{nf}\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho_{nf}c_p}\frac{\partial}{\partial y}\left(-\frac{16\sigma^*}{3k^*}T^3\frac{\partial T}{\partial y}\right)$$
$$+\frac{\upsilon_{nf}}{c_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\sigma B_0^2}{\rho_{nf}c_p}u^2$$
$$+\frac{\upsilon_{nf}}{k_pc_p}u^2+\tau\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right].$$
$$\Rightarrow\ \frac{T_\infty(\theta_w-1)\theta'(\zeta)}{(1-\lambda t)}\left(\frac{\zeta}{2}\lambda-ax^{n-1}\left(\frac{n+1}{2}f(\zeta)\right)\right)=\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}T_\infty(\theta_w-1)\alpha_{nf}$$
$$\left(\theta''(\zeta)+\frac{16\sigma^*}{\rho_{nf}c_p3k^*\alpha_{nf}}\left(3T_\infty^3\left(1+(\theta_w-1)\theta(\zeta)\right)^2(\theta_w-1)\theta'^2(\zeta)\right.\right.$$
$$\left.\left.+T_\infty^3\left(1+(\theta_w-1)\theta(\zeta)\right)^3\theta''(\zeta)\right)\right)+\frac{\upsilon_{nf}}{c_p}\left(\frac{ax^n}{(1-\lambda t)}f''(\zeta)\right)^2\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}$$
$$+\frac{\sigma B_0^2}{\rho_{nf}c_p}\left(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\right)^2+\frac{\upsilon_{nf}}{k_pc_p}\left(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\right)^2$$
$$\tau\Bigg(D_BT_\infty(\theta_w-1)\theta'(\zeta)\big(C_w-C_\infty\big)\phi'(\zeta)\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}$$
$$+\frac{D_T}{T_\infty}\left(T_\infty(\theta_w-1)\theta'(\zeta)\right)^2\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}\Bigg).$$
$$\Rightarrow\ \frac{T_\infty(\theta_w-1)\theta'(\zeta)}{(1-\lambda t)}\left(\frac{\zeta}{2}\lambda-ax^{n-1}\left(\frac{n+1}{2}f(\zeta)\right)\right)=\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}T_\infty(\theta_w-1)\alpha_{nf}$$
$$\left(\theta''(\zeta)+\frac{16\sigma^*T_\infty^3}{k_{nf}3k^*}\left(3\left(1+(\theta_w-1)\theta(\zeta)\right)^2(\theta_w-1)\theta'^2(\zeta)\right.\right.$$
$$\left.\left.+\left(1+(\theta_w-1)\theta(\zeta)\right)^3\theta''(\zeta)\right)\right)+\frac{\sigma B_0^2}{\rho_{nf}c_p}\left(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\right)^2$$
$$+\frac{\upsilon_{nf}}{k_pc_p}\left(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\right)^2\tau\Bigg(D_BT_\infty(\theta_w-1)\theta'(\zeta)\big(C_w-C_\infty\big)$$
$$\phi'(\zeta)\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}+\frac{D_T}{T_\infty}\left(T_\infty(\theta_w-1)\theta'(\zeta)\right)^2\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}\Bigg). \tag{3.42}$$

Multiplying by $\left(\frac{(1-\lambda t)v_{nf}}{ax^{n-1}T_\infty(\theta_w-1)\alpha_{nf}}\right)$ on both sides of (3.42), we get

$$\frac{v_{nf}}{\alpha_{nf}}\theta'(\zeta)\left(\frac{\zeta}{2}\frac{\lambda}{ax^{n-1}}-\frac{n+1}{2}f(\zeta)\right)=\theta''(\zeta)+R_d\Bigg(3\left(1+(\theta_w-1)\theta(\zeta)\right)^2(\theta_w-1)$$
$$\theta'^2(\zeta)+\left(1+(\theta_w-1)\theta(\zeta)\right)^3\theta''(\zeta)\Bigg)+\frac{v_{nf}}{\alpha_{nf}}\left(\frac{ax^n}{(1-\lambda t)}\right)^2\frac{1}{(T_f-T_\infty)c_p}f''^2(\zeta)$$
$$+\frac{x\sigma(1-\lambda t)}{\rho_{nf}ax^n}B_0^2\frac{v_{nf}}{\alpha_{nf}}\left(\frac{ax^n}{(1-\lambda t)}\right)^2\frac{1}{(T_f-T_\infty)c_p}f'^2(\zeta)$$
$$+\frac{\tau D_B(C_W-C_\infty)}{v_{nf}}\frac{v_{nf}}{\alpha_{nf}}\phi'(\zeta)\theta'(\zeta)+\frac{\tau D_T(T_f-T_\infty)}{T_\infty v_{nf}}\frac{v_{nf}}{\alpha_{nf}}\theta'^2(\zeta)$$
$$+\frac{v_{nf}x(1-\lambda t)}{k_pax^n}\frac{v_{nf}}{\alpha_{nf}}\left(\frac{ax^n}{(1-\lambda t)}\right)^2\frac{1}{(T_f-T_\infty)c_p}f'^2(\zeta).$$
$$\Rightarrow\ Pr\left(\frac{\zeta}{2}A-\frac{n+1}{2}f(\zeta)\right)=\left[\left(1+R_d\left(1+(\theta_w-1)\theta\right)^3\right)\theta'\right]'$$
$$+Pr\frac{u_w^2}{(T_f-T_\infty)c_p}f''^2(\zeta)+\frac{\sigma x}{\rho_{nf}u_w}B_0^2Pr\frac{u_w^2}{(T_f-T_\infty)c_p}f'^2(\zeta)$$
$$+Nb\phi'(\zeta)\theta'(\zeta)Pr$$
$$+Nt\theta'^2(\zeta)Pr+\frac{v_{nf}x}{k_pu_w}Pr\frac{u_w^2}{(T_f-T_\infty)c_p}f'^2(\zeta)$$
$$\Rightarrow\ \left[\left(1+R_d\left(1+(\theta_w-1)\theta\right)^3\right)\theta'\right]'+Pr\left(-\frac{\zeta}{2}A+\frac{n+1}{2}f(\zeta)\right)+Pr.Ecf''^2(\zeta)$$
$$+Mf'^2(\zeta)Pr.Ec+Nb\phi'(\zeta)\theta'(\zeta)Pr+Nt\theta'^2(\zeta)Pr+Kf'^2(\zeta)Ec.Pr=0.$$

The dimensionless form of (3.8) is given below:

$$\left[\left(1+R_d\left(1+(\theta_w-1)\theta\right)^3\right)\theta'\right]'+Pr\left(\frac{n+1}{2}f-\frac{\zeta}{2}A+Nb\phi'+Nt\theta'\right)\theta'$$
$$+Br\left[f''^2+\left(M+K\right)f'^2\right]=0. \tag{3.43}$$

Now, we use the following procedure to convert (3.5) into the dimensionless form:

- $C=\phi(\zeta)\big(C_w-C_\infty\big)+C_\infty.$
- $$\frac{\partial C}{\partial t}=\frac{\partial}{\partial t}\Big[\phi(\zeta)\big(C_w-C_\infty\big)+C_\infty\Big]$$
$$=\big(C_w-C_\infty\big)\phi'(\zeta)\frac{\partial\zeta}{\partial t}$$
$$=\big(C_w-C_\infty\big)\phi'(\zeta)\frac{\zeta}{2}\frac{\lambda}{(1-\lambda t)}. \tag{3.44}$$

- $$\frac{\partial C}{\partial x} = \frac{\partial}{\partial x}\left[\phi(\zeta)(C_w - C_\infty) + C_\infty\right]$$
$$= (C_w - C_\infty)\phi'(\zeta)\frac{\partial \zeta}{\partial x}$$
$$= (C_w - C_\infty)\phi'(\zeta)\left(\frac{n-1}{2}\right)\frac{\zeta}{x}. \tag{3.45}$$
- $$u\frac{\partial C}{\partial x} = \frac{ax^n}{(1-\lambda t)}f'(\zeta)(C_w - C_\infty)\phi'(\zeta)\left(\frac{n-1}{2}\right)\frac{\zeta}{x}$$
$$= \frac{ax^{n-1}}{(1-\lambda t)}f'(\zeta)(C_w - C_\infty)\phi'(\zeta)\left(\frac{n-1}{2}\right)\zeta. \tag{3.46}$$
- $$\frac{\partial C}{\partial y} = \frac{\partial}{\partial y}\left[\phi(\zeta)(C_w - C_\infty) + C_\infty\right]$$
$$= (C_w - C_\infty)\phi'(\zeta)\frac{\partial \zeta}{\partial y}$$
$$= (C_w - C_\infty)\phi'(\zeta)\sqrt{\frac{ax^n}{v_{nf}x(1-\lambda t)}}. \tag{3.47}$$
- $$\frac{\partial^2 C}{\partial y^2} = \frac{\partial}{\partial y}\left((C_w - C_\infty)\phi'(\zeta)\sqrt{\frac{ax^n}{v_{nf}x(1-\lambda t)}}\right)$$
$$= (C_w - C_\infty)\phi''(\zeta)\sqrt{\frac{ax^n}{v_{nf}x(1-\lambda t)}}\frac{\partial \zeta}{\partial y}$$
$$= (C_w - C_\infty)\phi''(\zeta)\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}. \tag{3.48}$$
- $$v\frac{\partial C}{\partial y} = -\sqrt{\frac{ax^n v_{nf}}{(1-\lambda t)x}}\left(\frac{n+1}{2}f(\zeta) + \left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)$$
$$(C_w - C_\infty)\phi'(\zeta)\sqrt{\frac{ax^n}{v_{nf}x(1-\lambda t)}}$$
$$= -\frac{ax^{n-1}}{(1-\lambda t)}\left(\frac{n+1}{2}f(\zeta) + \left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)(C_w - C_\infty)\phi'(\zeta). \tag{3.49}$$

Adding (3.44),(3.46) and (3.49) we have

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = (C_w - C_\infty)\phi'(\zeta)\frac{\zeta}{2}\frac{\lambda}{(1-\lambda t)} + \frac{ax^{n-1}}{(1-\lambda t)}f'(\zeta)(C_w - C_\infty)$$
$$\phi'(\zeta)\left(\frac{n-1}{2}\right)\zeta - \frac{ax^{n-1}}{(1-\lambda t)}\left(\frac{n+1}{2}f(\zeta) + \left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)(C_w - C_\infty)\phi'(\zeta)$$
$$= \frac{(C_w - C_\infty)}{(1-\lambda t)}\phi'(\zeta)\left(\frac{\zeta}{2}\lambda + ax^{n-1}f'(\zeta)\left(\frac{n-1}{2}\right)\zeta - ax^{n-1}\left(\frac{n+1}{2}\right)f(\zeta)\right.$$
$$\left. - ax^{n-1}f'(\zeta)\left(\frac{n-1}{2}\right)\zeta\right)$$

$$= \frac{(C_w - C_\infty)}{(1-\lambda t)}\phi'(\zeta)\left(\frac{\zeta}{2} - ax^{n-1}\left(\frac{n+1}{2}\right)f(\zeta)\right). \tag{3.50}$$

Using (3.29)and (3.48) the right side of (3.5), gets the following form:

$$\begin{aligned} D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} &= D_B\big(C_w - C_\infty\big)\phi''(\zeta)\frac{ax^{n-1}}{\upsilon_{nf}(1-\lambda t)} \\ &+ \frac{D_T}{T_\infty}(\theta_w - 1)\theta''(\zeta)\frac{ax^{n-1}}{\upsilon_{nf}(1-\lambda t)} \\ &= \frac{ax^{n-1}}{\upsilon_{nf}(1-\lambda t)}\left(D_B\big(C_w - C_\infty\big)\phi''(\zeta) + \frac{D_T}{T_\infty}T_\infty(\theta_w - 1)\theta''(\zeta)\right). \end{aligned} \tag{3.51}$$

Putting the values of (3.50) and (3.51) in (3.5), gets the following form:

$$\begin{aligned} &\frac{(C_w - C_\infty)}{(1-\lambda t)}\phi'(\zeta)\left(\frac{\lambda\zeta}{2} - ax^{n-1}\left(\frac{n+1}{2}\right)f(\zeta)\right) \\ &= \frac{ax^{n-1}}{\upsilon_{nf}(1-\lambda t)}\left(D_B\big(C_w - C_\infty\big)\phi''(\zeta) + \frac{D_T}{T_\infty}T_\infty(\theta_w - 1)\theta''(\zeta)\right). \end{aligned} \tag{3.52}$$

Multiplying $\left(\frac{(1-\lambda t)\upsilon_{nf}}{ax^{n-1}D_B\left(C_w - C_\infty\right)}\right)$ on both sides of (3.52), we get

$$\begin{aligned} &\frac{\lambda}{ax^{n-1}}\frac{\upsilon_{nf}}{D_B}\frac{\zeta}{2}\phi'(\zeta) - \frac{\upsilon_{nf}}{D_B}\left(\frac{n+1}{2}\right)f(\zeta)\phi'(\zeta) = \phi''(\zeta) + \frac{D_T(T_f - T_\infty)}{T_\infty D_B\big(C_w - C_\infty\big)}\theta''(\zeta). \\ \Rightarrow\ &\phi''(\zeta) + \frac{D_T(T_f - T_\infty)}{T_\infty D_B\big(C_w - C_\infty\big)}\theta''(\zeta) + \frac{\upsilon_{nf}}{D_B}\left(\frac{n+1}{2}\right)f(\zeta)\phi'(\zeta) \\ &- \frac{\lambda}{ax^{n-1}}\frac{\upsilon_{nf}}{D_B}\frac{\zeta}{2}\phi'(\zeta) = 0. \\ \Rightarrow\ &\phi''(\zeta) + \frac{Nt(\upsilon_{nf})\tau}{Nb(\upsilon_{vf})\tau}\theta''(\zeta) + Sc\left(\frac{n+1}{2}\right)f(\zeta)\phi'(\zeta) - A(Sc)\frac{\zeta}{2}\phi'(\zeta) = 0. \\ \Rightarrow\ &\phi'' + \frac{Nt}{Nb}\theta'' + Sc\left(\frac{n+1}{2}f - \frac{A}{2}\zeta\right)\phi' = 0. \end{aligned} \tag{3.53}$$

The corresponding BCs are transformed into the non-dimensional form through the following procedure.

$$u = u_w + u_{slip} = \frac{ax^n}{(1-\lambda t)} + N\frac{\partial u}{\partial y}, \qquad at \quad y = 0.$$

$$\Rightarrow\ \frac{ax^n}{(1-\lambda t)}f'(\zeta) = \frac{ax^n}{(1-\lambda t)} + N\frac{ax^n}{(1-\lambda t)}\sqrt{\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}}f''(\zeta), \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad f'(\zeta) = 1 + N\sqrt{\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}}f''(\zeta), \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad f'(\zeta) = 1 + N\left(\frac{a}{v_{nf}}\right)^{\frac{1}{2}}\left(\frac{x^{n-1}}{(1-\lambda t)}\right)^{\frac{1}{2}}f''(\zeta), \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad f'(\zeta) = 1 + N_1\left(\frac{a}{v_{nf}}\right)^{\frac{1}{2}}, \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad f'(\zeta) = 1 + \gamma f''(\zeta), \qquad at \quad \zeta = 0.$$

$$v(x,t) = 0, \qquad at \quad y = 0.$$

$$\Rightarrow \quad \frac{ax^{n-1}v_{nf}}{(1-\lambda t)}\left(\frac{n+1}{2}f(\zeta) + \left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right) = 0, \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad -\frac{ax^{n-1}v_{nf}}{(1-\lambda t)}\left(\frac{n+1}{2}f(\zeta)\right) = 0, \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad f(\zeta) = 0, \qquad at \quad \zeta = 0.$$

$$-k_{nf}\frac{\partial T}{\partial y} = h_f\big(T_f - T\big), \qquad at \quad y = 0.$$

$$\Rightarrow \quad \frac{\partial T}{\partial y} = -\frac{h_f}{k_{nf}}\big(T_f - T\big), \qquad at \quad y = 0.$$

$$\Rightarrow \quad T_\infty(\theta_w - 1)\theta'(\zeta)\sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}} = -\frac{h_f}{k_{nf}}\big(T_f - T\big), \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad T_\infty(\theta_w - 1)\theta'(\zeta) = -\frac{h_f}{k_{nf}}\sqrt{\frac{v_{nf}(1-\lambda t)}{ax^{n-1}}}\big(T_f - T\big), \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad T_\infty(\theta_w - 1)\theta'(\zeta) = -\frac{h_f}{k_{nf}}\sqrt{\frac{v_{nf}(1-\lambda t)}{ax^{n-1}}}\big(T_f - T\big), \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad T_\infty(\theta_w - 1)\theta'(\zeta) = -\frac{h_f}{k_{nf}}\sqrt{\frac{v_{nf}(1-\lambda t)}{ax^{n-1}}}\big(T_f - T\big), \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad \big(T_f - T_\infty\big)\theta'(\zeta) = -\frac{h_f}{k_{nf}}\sqrt{\frac{v_{nf}x(1-\lambda t)}{ax^{n}}}\big(T_f - T\big), \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad \big(T_f - T_\infty\big)\theta'(\zeta) = -\frac{h_f}{k_{nf}}\sqrt{\frac{xv_{nf}}{u_w}}\big(T_f - T\big), \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad \big(T_f - T_\infty\big)\theta'(\zeta) = -Bi\big(T_f - T\big), \qquad \text{at} \quad \zeta = 0.$$

$$\Rightarrow \quad \theta'(\zeta) = -Bi\frac{\big(T_f - T\big)}{\big(T_f - T_\infty\big)}, \qquad \text{at} \quad \zeta = 0.$$

$$\Rightarrow \quad \theta'(\zeta) = -Bi\left(\frac{\theta_w T_\infty - T_\infty - T_\infty\big(\theta_w - 1\big)\theta(\zeta)}{T_\infty\big(\theta_w - 1\big)}\right), \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad \theta'(\zeta) = -Bi\left(\frac{T_\infty\left((\theta_w - 1) - (\theta_w - 1)\theta(\zeta)\right)}{T_\infty\big(\theta_w - 1\big)}\right), \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad \theta'(\zeta) = -Bi\left(\frac{(\theta_w - 1)\big(1 - \theta(\zeta)\big)}{(\theta_w - 1)}\right), \qquad at \quad \zeta = 0.$$

$$\Rightarrow \quad \theta'(\zeta) = -Bi\Big(1-\theta(\zeta)\Big), \qquad at \ \ \zeta = 0.$$

$$C = C_w, \qquad at \ \ y = 0.$$

$$\Rightarrow \quad \phi(\zeta)\big(C_w - C_\infty\big) + C_\infty = C_w, \qquad at \ \ \zeta = 0.$$

$$\Rightarrow \quad \phi(\zeta)\big(C_w - C_\infty\big) = C_w - C_\infty, \qquad at \ \ \zeta = 0.$$

$$\Rightarrow \quad \phi(\zeta) = 1, \qquad at \ \ \zeta = 0.$$

$$u \to 0, \qquad as \ \ y \to \infty.$$

$$\Rightarrow \quad \frac{ax^n}{(1-\lambda t)} f'(\zeta) \to 0, \qquad as \ \ \zeta \to \infty.$$

$$\Rightarrow \quad f'(\zeta) \to 0, \qquad as \ \ \zeta \to \infty.$$

$$T \to T_\infty, \qquad as \ \ y \to \infty.$$

$$\Rightarrow \quad T_\infty + T_\infty\big(\theta_w - 1\big)\theta(\zeta) \to T_\infty, \qquad as \ \zeta \to \infty.$$

$$\Rightarrow \quad T_\infty\big(\theta_w - 1\big)\theta(\zeta) \to 0, \qquad as \ \ \zeta \to \infty.$$

$$\Rightarrow \quad \big(\theta_w - 1\big)\theta(\zeta) \to 0, \qquad as \ \ \zeta \to \infty.$$

$$\Rightarrow \quad \theta(\zeta) \to 0, \qquad as \ \ \zeta \to \infty.$$

$$C \to C_\infty, \qquad as \ \ y \to \infty.$$

$$\Rightarrow \quad C_\infty + \phi(\zeta)\big(C_w - C_\infty\big) \to C_\infty, \qquad as \ \ \zeta \to \infty.$$

$$\Rightarrow \quad \phi(\zeta)\big(C_w - C_\infty\big) \to 0, \qquad as \ \ \zeta \to \infty.$$

$$\Rightarrow \quad \phi(\zeta) \to 0, \qquad as \ \ \zeta \to \infty.$$

The final dimensionless form of the governing model is

$$f''' + \frac{n+1}{2} f f'' - n f'^2 - (M+K)\, f' - A\left(f' + \frac{\zeta}{2} f''\right) = 0, \tag{3.54}$$

$$\left[\left(1 + R_d\big(1 + (\theta_w - 1)\theta\big)^3\right)\theta'\right]' + Pr\left(\frac{n+1}{2} f - \frac{\zeta}{2} A + Nb\phi' + Nt\theta'\right)\theta'$$
$$+ Br\left[f''^2 + \Big(M + K\Big) f'^2\right] = 0, \tag{3.55}$$

$$\phi'' + \frac{Nt}{Nb}\theta'' + Sc\left(\frac{n+1}{2} f - \frac{A}{2}\zeta\right)\phi' = 0. \tag{3.56}$$

The associated BCs (3.6) in the dimensionless form are

$$\left.\begin{aligned}
&f' = 1 + \gamma f'', \quad f(\zeta) = 0\\
&\theta' = -Bi(1-\theta(\zeta)), \quad \phi(\zeta) = 1 \;\; at \; \zeta = 0\\
&f' \to 0, \;\; \theta(\zeta) \to 0, \;\; \phi(\zeta) \to 0 \;\; as \;\; \zeta \to \infty
\end{aligned}\right\} \tag{3.57}$$

Different dimensionless parameters used in (3.54)-(3.57) are formulated as follows.

$$\left.\begin{aligned}
&Nb = \frac{\tau D_B(C_w - C_\infty)}{\upsilon_{nf}}, K = \frac{\upsilon_{nf} x}{k_p u_w}, A = \frac{\lambda}{a x^{n-1}},\\
&Pr = \frac{\upsilon_{nf}}{\alpha_{nf}}, R_d = \frac{16\sigma^* T_\infty^3}{3k^* k_{nf}}, Nt = \frac{\tau D_T}{T_\infty}\frac{(T_f - T_\infty)}{\upsilon_{nf}},\\
&Nb = \frac{\tau D_B(C_w - C_\infty)}{\upsilon_{nf}}, Ec = \frac{u_w^2}{c_p(T_f - T_\infty)},\\
&Br = \left(\frac{u_w^2}{c_p(T_f - T_\infty)}\right)\frac{\upsilon_{nf}}{\alpha_{nf}}, \gamma = N_1\left(\frac{a}{\upsilon_{nf}}\right)^{\frac{1}{2}}, Bi = \frac{h_f}{k_{nf}}\sqrt{\frac{\upsilon_{nf} x}{u_w}}.
\end{aligned}\right\} \tag{3.58}$$

3.3 Physical Quantities of Interest

Currently, our research interests include the skin friction coefficient Cf_x, the local Nusselt number Nu_x, and the Sherwood number Sh_x. The following is a description of such physical quantities. Skin coefficient friction has a mathematical form that is

$$Cf_x = \frac{\tau_w}{\rho_{nf} u_w^2(x,t)}. \tag{3.59}$$

Nusselt number is defined as

$$Nu_x = \frac{x q_w}{k_{nf}(T_f - T_\infty)}. \tag{3.60}$$

The mathematical formulation of the Sherwood number is

$$Sh_x = \frac{x q_m}{D_B(C_w - C_\infty)}. \tag{3.61}$$

Here τ_w represents the surface shear stress, q_w the wall heat flux and q_m the mass flow from the surface. These quantities are formulated as:

$$\tau_w = \mu_{nf}\left(\frac{\partial u}{\partial y}\right)_{y=0}, \tag{3.62}$$

$$q_w = -k_{nf}\left(\frac{\partial T}{\partial y}\right)_{y=0} + \left(q_r\right)_w, \tag{3.63}$$

$$q_m = -D_B\left(\frac{\partial C}{\partial y}\right)_{y=0}. \tag{3.64}$$

The following steps will be utilized in order to get the dimensionless form of Cf_x.

$$\frac{\partial u}{\partial y} = \frac{ax^n}{(1-\lambda t)} f''(\zeta)\sqrt{\frac{ax^n}{(1-\lambda t)\upsilon_{nf} x}}.$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{ax^n}{(1-\lambda t)} f''(0)\sqrt{\frac{ax^n}{(1-\lambda t)\upsilon_{nf} x}}.$$

$$\tau_w = \mu_{nf}\frac{ax^n}{(1-\lambda t)} f''(0)\sqrt{\frac{ax^n}{(1-\lambda t)\upsilon_{nf} x}}.$$

The skin friction can be written by:

$$\begin{aligned}
Cf_x &= \frac{\tau_w}{\rho_{nf} u_w^2(x,t)} \\
&= \frac{1}{\rho_{nf} u_w^2(x,t)}\left(\mu_{nf}\frac{ax^n}{(1-\lambda t)} f''(0)\sqrt{\frac{ax^n}{(1-\lambda t)\upsilon_{nf} x}}\right) \\
&= \frac{\mu_{nf}}{\rho_{nf}}\frac{(1-\lambda t)^2 ax^n}{(ax^n)^2(1-\lambda t)}\sqrt{\frac{ax^n}{(1-\lambda t)\upsilon_{nf} x}} f''(0) \\
&= \upsilon_{nf}\frac{(1-\lambda t)}{ax^n}\left(\frac{ax^n}{(1-\lambda t)\upsilon_{nf} x}\right)^{\frac{1}{2}} f''(0) \\
&= \left(\frac{\upsilon_{nf}(1-\lambda t)}{ax^n x}\right)^{\frac{1}{2}} f''(0) \\
&= \left(\frac{\upsilon_{nf}}{u_w(x,t)x}\right)^{\frac{1}{2}} f''(0).
\end{aligned}$$

$$\Rightarrow \quad Cf_x\left(\frac{u_w(x,t)x}{\upsilon_{nf}}\right)^{\frac{1}{2}} = f''(0).$$

$$\Rightarrow \quad Cf_x Re_x^{\frac{1}{2}} = f''(0). \tag{3.65}$$

The following steps will be utilized in order to get the dimensionless form of Nu_x.

$$
\begin{aligned}
\frac{\partial T}{\partial y} &= \big(T_\infty(\theta_w - 1)\theta'(\zeta)\big)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}}. \\
\Rightarrow \left(\frac{\partial T}{\partial y}\right)_{y=0} &= \big(T_\infty(\theta_w - 1)\theta'(0)\big)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}}. \\
q_w &= -k_{nf}\left(\frac{\partial T}{\partial y}\right)_{y=0} + \big(q_r\big)_w \\
&= -k_{nf}\big(T_\infty(\theta_w - 1)\theta'(0)\big)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}} - \frac{16\sigma^* T^3}{3k^*}\frac{\partial T}{\partial y} \\
&= -k_{nf}\big(T_\infty(\theta_w - 1)\theta'(0)\big)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}} \\
&\quad - \frac{16\sigma^* T^3}{3k^*}\big(T_\infty(\theta_w - 1)\theta'(0)\big)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}} \\
&= -k_{nf}\big(T_\infty(\theta_w - 1)\theta'(0)\big)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}}\left(1 + \frac{16\sigma^* T^3}{3k^* k_{nf}}\right) \\
&= -k_{nf}\big(T_\infty(\theta_w - 1)\theta'(0)\big)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}} \\
&\quad \left(1 + \frac{16\sigma^*}{3k^* k_{nf}} T_\infty^3\big(1 + (\theta_w - 1)\theta(0)\big)^3\right) \\
&= -k_{nf}\big(T_\infty(\theta_w - 1)\theta'(0)\big)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}}\left(1 + R_d\big(1 + (\theta_w - 1)\theta(0)\big)^3\right).
\end{aligned}
$$

The Nusselt number can be written by:

$$
\begin{aligned}
Nu_x &= \frac{xq_w}{k_{nf}(T_f - T_\infty)} \\
&= \frac{-xk_{nf}\big(T_\infty(\theta_w - 1)\theta'(0)\big)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}}\left(1 + R_d\big(1 + (\theta_w - 1)\theta(0)\big)^3\right)}{k_{nf}(T_f - T_\infty)} \\
&= \frac{x\big(T_\infty(\theta_w - 1)\theta'(0)\big)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}}\left(1 + R_d\big(1 + (\theta_w - 1)\theta(0)\big)^3\right)}{T_\infty(\theta_w - 1)} \\
&= -x\theta'(0)\big)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}}\left(1 + R_d\big(1 + (\theta_w - 1)\theta(0)\big)^3\right) \\
&= -\theta'(0)\big)\sqrt{\frac{ax^n x}{(1-\lambda t)v_{nf}}}\left(1 + R_d\big(1 + (\theta_w - 1)\theta(0)\big)^3\right)
\end{aligned}
$$

$$= -\theta'(0))\sqrt{\frac{u_w(x,t)x}{v_{nf}}}\left(1 + R_d\big(1 + (\theta_w - 1)\theta(0)\big)^3\right).$$

$$\Rightarrow \quad Nu_x\left(\frac{u_w(x,t)x}{v_{nf}}\right)^{\frac{-1}{2}} = -\theta'(0)\left(1 + R_d\big(1 + (\theta_w - 1)\theta(0)\big)^3\right).$$

$$\Rightarrow \quad Nu_x Re_x^{\frac{-1}{2}} = -\theta'(0)\left(1 + R_d\big(1 + (\theta_w - 1)\theta(0)\big)^3\right). \tag{3.66}$$

The following steps will be utilized in order to get the dimensionless form of Sh_x.

$$\frac{\partial C}{\partial y} = \left(\big(C_w - C_\infty\big)\phi'(\zeta)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}}\right)$$

$$\Rightarrow \quad \left(\frac{\partial C}{\partial y}\right)_{y=0} = \left(\big(C_w - C_\infty\big)\phi'(0)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}}\right).$$

$$q_m = -D_B\left(\frac{\partial C}{\partial y}\right)_{y=0}$$

$$= -D_B\left(\big(C_w - C_\infty\big)\phi'(0)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}}\right).$$

The Sherwood number can be written by:

$$Sh_x = \frac{xq_m}{D_B(C_w - C_\infty)}$$

$$= \frac{-xD_B\left(\big(C_w - C_\infty\big)\phi'(0)\sqrt{\frac{ax^n}{(1-\lambda t)v_{nf}x}}\right)}{D_B(C_w - C_\infty)}$$

$$= -\phi'(0)\sqrt{\frac{ax^n x}{(1-\lambda t)v_{nf}}}.$$

$$= -\phi'(0)\sqrt{\frac{u_w(x,t)x}{v_{nf}}}.$$

$$\Rightarrow \quad Sh_x\left(\frac{u_w(x,t)x}{v_{nf}}\right)^{\frac{-1}{2}} = -\phi'(0).$$

$$\Rightarrow \quad Sh_x Re_x^{\frac{-1}{2}} = -\phi'(0). \tag{3.67}$$

Here $Re_x = \frac{u_w(x,t)x}{v_{nf}}$ represents the Reynolds number.

3.4 Entropy Generation Formulations

Entropy generation is a way of quantifying how irreversible thermal energy is there in a system

$$E_G = \frac{k_{nf}}{T_{\infty 2}}\left[1+\frac{16\sigma^* T_\infty^3}{3k^* k_{nf}}\right]\left(\frac{\partial T}{\partial y}\right)^2 + \frac{D_B}{C_\infty}\left(\frac{\partial C}{\partial y}\right)^2 + \frac{D_B}{T_\infty}\left(\frac{\partial C}{\partial y}\right)\left(\frac{\partial T}{\partial y}\right)$$
$$+ \frac{\mu_{nf}}{T_\infty}\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\mu_{nf}}{k_{np}T_\infty} + \frac{\sigma B_0^2}{T_\infty}\right)u^2 \tag{3.68}$$

Using (3.28), (3.37) and (3.38) in (3.68), we get

$$E_G = \frac{k_{nf}}{T_\infty^2}\left[1+\frac{16\sigma^* T_\infty^3}{3k^* k_{nf}}\right]\left(T_\infty(\theta_w - 1)\theta'(\zeta)\sqrt{\frac{ax^{n-1}}{\upsilon_{nf}(1-\lambda t)}}\right)^2$$
$$+ \frac{D_B}{C_\infty}\left((C_w - C_\infty)\phi'(\zeta)\sqrt{\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}}\right)^2 + \frac{D_B}{T_\infty}\left(T_\infty(\theta_w - 1)\theta'(\zeta)\right.$$
$$\left.(C_w - C_\infty)\phi'(\zeta)\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}\right) + \frac{\mu_{nf}}{T_\infty}\left(\frac{ax^n}{(1-\lambda t)}f''(\zeta)\sqrt{\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}}\right)^2$$
$$+ \frac{(ax^n)^2}{(1-\lambda t)^2}f'^2(\zeta)\left(\frac{\mu_{nf}}{k_{np}T_\infty} + \frac{\sigma B_0^2}{T_\infty}\right)$$
$$= \frac{k_{nf}}{T_\infty^2}\left(1+R_d\right)T_\infty^2\left(\frac{T_f - T_\infty}{T_\infty}\right)^2\left(\frac{\zeta}{y}\right)^2\theta'^2 + \frac{D_B}{C_\infty}((C_w - C_\infty)^2\left(\frac{\zeta}{y}\right)^2\phi'^2$$
$$+ \frac{D_B}{T_\infty}T_\infty\left(\frac{T_f - T_\infty}{T_\infty}\right)((C_w - C_\infty)\left(\frac{\zeta}{y}\right)^2\phi'\theta' + \frac{\rho_{nf}\upsilon_{nf}}{T_\infty}u_w^2\left(\frac{\zeta}{y}\right)^2 f''^2$$
$$+ \frac{u_w^2}{T_\infty}f'^2\left(\frac{\rho_{nf}\upsilon_{nf}}{k_p} + \sigma B_0^2\right). \tag{3.69}$$
$$\left(\because \frac{\zeta}{y} = \sqrt{\frac{ax^{n-1}}{\upsilon_{nf}(1-\lambda t)}}\right)$$

Multiplying both sides of (3.69) by $\frac{T_\infty^2\left(\frac{\zeta}{y}\right)^2}{k_{nf}(T_f - T_\infty)^2}$ we get

$$E_G\left[\frac{T_\infty^2\left(\frac{\zeta}{y}\right)^2}{k_{nf}(T_f - T_\infty)^2}\right] = (1+R_d)\theta'^2 + \frac{D_B}{k_{nf}}((C_w - C_\infty)\frac{(C_w - C_\infty)}{C_\infty}\left(\frac{T_f}{T_f - T_\infty}\right)^2$$
$$\phi'^2 + \frac{D_B}{k_{nf}}((C_w - C_\infty)\left(\frac{T_\infty}{T_f - T_\infty}\right)\phi'\theta' + \frac{\upsilon_{nf}}{\alpha_{nf}}\frac{u_w^2}{c_p((T_f - T_\infty}\left(\frac{T_\infty}{T_f - T_\infty}\right)f''^2$$
$$+ \frac{\upsilon_{nf}}{\alpha_{nf}}\frac{u_w^2}{c_p((T_f - T_\infty}\left(\frac{T_\infty}{T_f - T_\infty}\left(\frac{(1-\lambda t)x\upsilon_{nf}}{k_p ax^n} + \frac{\sigma B_0^2(1-\lambda t)x}{ax^n\rho_{nf}}\right)f'^2.\right.$$

After simplifying the above expression, we get the dimensionless form the entropy generation as follows:

$$Ns = \big(1 + R_d\big)\theta'^2 + \lambda\bigg[\frac{\epsilon}{\Omega^2}\phi'^2 + \frac{1}{\Omega}\theta'\phi'\bigg] + \frac{Br}{\Omega}\bigg[f''^2 + \big(M + K\big)\bigg], \tag{3.70}$$

Here N_s is referred to as the system's overall entropy generation

$$\left.\begin{aligned} &N_s = \frac{T_\infty^2\left(\frac{\varsigma}{y}\right)^2}{k_{nf}(T_f - T_\infty)^2}, \quad N_{HT} = \big(1 + R_d\big)\theta'^2, \\ &N_{MT} = \lambda\bigg[\frac{\epsilon}{\Omega}^2\phi'^2 + \frac{1}{\Omega}\theta'\phi'\bigg],\ N_{PM} = \frac{Br}{\Omega}\bigg[f''^2 + \big(M + K\big)\bigg], \\ &\Omega = \frac{T_f - T_\infty}{T_\infty}, \quad \epsilon = \frac{C_w - C_\infty}{C_\infty}, \quad \lambda = \frac{D_B}{k_{nf}}((C_w - C_\infty). \end{aligned}\right\} \tag{3.71}$$

where N_{HT} the entropy factors resulting from frictional heating, such as thermal radiation, N_{MT} the entropy factor due to mass transfer, N_{PM} is the fluid friction entropy factor. Ω is the non-dimensional temperature difference parameter, ϵ is the non-dimensional volume difference parameter, λ the mass transfer parameter.

3.5 Bejan Number

In this specific circumstance, the assessment of the Bejan number Be is quite possibly the main assignment to concentrate on both the fluid flow mechanism and heat transfer. The Bejan number [29] is defined as

$$Be = N_s = \frac{N_{HT}}{N_s} = \frac{N_{HT}}{N_{HT} + N_{MT} + N_{PM}}. \tag{3.72}$$

3.6 Solution Methodology

The system of nonlinear ODEs (3.54)-(3.56) along with the boundary conditions (3.57) will be modified into the first order ODEs. The shooting method will be utilized to solve the first-order system of ODEs with the boundary conditions

(3.57). Equation (3.54)-(3.56) can be written in the following form.

$$f''' = nf'^2 - \frac{n+1}{2}ff'' + (M+K)f' + A\left(f' + \frac{\zeta}{2}f''\right), \tag{3.73}$$

$$\theta'' = \frac{-1}{1+3R_d\big(1+(\theta_w-1)\theta\big)^3}\Big[3R_d\big(1+(\theta_w-1)\theta\big)^2(\theta_w-1)\theta'^2 + Pr\left(\frac{n+1}{2}f - \frac{\zeta}{2}A + Nb\phi' + Nt\theta'\right)\theta' + Br(f'' + f'^2(M+K))\Big], \tag{3.74}$$

$$\phi'' = -\frac{Nt}{Nb}\theta'' - Sc\left(\frac{n+1}{2}f - \frac{A}{2}\zeta\right)\phi'. \tag{3.75}$$

Initially, the momentum equation (3.73) has been solved independently and then the computed of f, will be utilized in coupled equations equations (3.74) and (3.75). Following notations have been considered for further procedure:

$$f = Z_1, \qquad f' = Z_1' = Z_2, \qquad f'' = Z_1'' = Z_2' = Z_3, \qquad f''' = Z_3'.$$

By using the above notations in (3.73), the successive ODEs are obtained:

$$\begin{aligned}
&Z_1' = Z_2, && Z_1(0) = 0.\\
&Z_2' = Z_3, && Z_2(0) = 1 + \gamma s.\\
&Z_3' = nZ_2^2 + (M+K)Z_2 + A(Z_2 + \frac{\zeta}{2}Z_3) - \frac{n+1}{2}Z_1Z_3, && Z_3(0) = s.
\end{aligned}$$

Here s is the missing initial condition. The above IVP will be numerically solved by the RK technique of order four. The domain of our problem is considered to be bounded i.e. $[0, \zeta_\infty]$, where ζ_∞ has been a positive number and for which the variation in the solution is negligible after $\zeta = \zeta_\infty$. Furthermore, s is assumed as the missing condition for the solution of (3.73) such that:

$$Z_2(\zeta_\infty, s) = 0. \tag{3.76}$$

To solve the above algebraic equation (3.76), we apply the Newton's method which has the following iterative scheme:

$$s_{k+1} = s_k - \frac{Z_2(\zeta_\infty, s)}{\frac{\partial}{\partial s}Z_2(\zeta_\infty, s)}.$$

To incorporate Newton's method, we further utilize the following notions:

$$\frac{\partial Z_1}{\partial s} = Z_4, \qquad \frac{\partial Z_2}{\partial s} = Z_5, \qquad \frac{\partial Z_3}{\partial s} = Z_6.$$

Now, differentiating the system of three first order ODEs with respect to s, we get another system of ODEs is as follows:

$$Z_4' = Z_5, \qquad Z_4(0) = 0.$$

$$Z_5' = Z_6, \qquad Z_5(0) = \gamma.$$

$$Z_6' = 2nZ_2Z_5 + \big(M + k\big)Z_5 + A\big(Z_5 + \frac{\zeta}{2}Z_6\big) - \frac{n+1}{2}\big(Z_1Z_6 + Z_3Z_4\big), \quad Z_6(0) = 1.$$

The stopping criteria for Newton's technique are set as

$$|Z_2(\zeta_\infty, s)| < \epsilon^*,$$

where $\epsilon^* > 0$ is an arbitrarily small positive number. From now onward ϵ has been taken as 10^{-10}.

Now, the coupled equations (3.74) and (3.75) will be treated similarly by taking f, f' and f'' as known functions. The initial missing condition $\theta(0)$ and $\phi(0)$ can be represented by p and q respectively. The following notations have been further considered

$$\left.\begin{array}{l} \theta = Y_1,\ \theta' = Y_1' = Y_2,\ \theta'' = Y_2'\ \phi = Y_3,\ \phi' = Y_3' = Y_4, \\ \phi'' = Y_4',\ Y_5 = \dfrac{\partial\theta}{\partial p},\ Y_6 = \dfrac{\partial\theta'}{\partial p},\ Y_7 = \dfrac{\partial\phi}{\partial p},\ Y_8 = \dfrac{\partial\phi'}{\partial p} \\ Y_9 = \dfrac{\partial\theta}{\partial q},\ Y_{10} = \dfrac{\partial\theta'}{\partial q},\ Y_{11} = \dfrac{\partial\phi}{\partial q},\ Y_{12} = \dfrac{\partial\phi'}{\partial q}. \end{array}\right\}$$

By using the above notations in equations (3.74) and (3.75), the following ODEs are obtained:

$$Y_1' = Y_2, \qquad Y_1(0) = p.$$

$$Y_2' = \frac{1}{-1-3R_d\big(1+(\theta_w-1)Y_1\big)^3}\Big[3R_d\big(1+(\theta_w-1)Y_1\big)^2(\theta_w-1)Y_2^2 + Pr$$
$$\Big(\frac{n+1}{2}f - \frac{\zeta}{2}A + NbY_4 + NtY_2\Big)Y_2 + Br(f'' + f'^2(M+K))\Big],\ Y_2(0) = -Bi(1-p).$$

$$Y_3' = Y_4, \qquad Y_3(0) = 1.$$

$$Y_4' = -Sc\Big(\frac{n+1}{2}f - \frac{A}{2}\zeta\Big)Y_4 - \frac{Nt}{Nb}\Bigg(\frac{1}{-1-3R_d\big(1+(\theta_w-1)Y_1\big)^3}$$
$$\Big[3R_d\big(1+(\theta_w-1)Y_1\big)^2(\theta_w-1)Y_2^2 + Pr\Big(\frac{n+1}{2}f - \frac{\zeta}{2}A$$
$$+ NbY_4 + NtY_2\Big)Y_2 + Br(f'' + f'^2(M+K))\Big]\Bigg), \qquad Y_4(0) = q.$$

$$Y_5' = Y_6, \qquad Y_5(0) = 1.$$

$$Y_6' = \frac{3R_d(1+(\theta_w-1)Y_1)^2(\theta_w-1)Y_5}{\big(1+3R_d(1+(\theta_w-1)Y_1)^3\big)^2}\Big[3R_d\big(1+(\theta_w-1)Y_1\big)^2(\theta_w-1)Y_2^2$$
$$+ Pr\Big(\frac{n+1}{2}f - \frac{\zeta}{2}A + NbY_4 + NtY_2\Big)Y_2 + Br(f'' + f'^2(M+K))\Big]$$
$$+ \frac{1}{\big(-1-3R_d\big(1+(\theta_w-1)Y_1\big)^3\big)}\Big[6R_d\big(1+(\theta_w-1)Y_1\big)(\theta_w-1)^2Y_5Y_2^2$$
$$+ 6R_d\big(1+(\theta_w-1)Y_1\big)(\theta_w-1)^2Y_6Y_2 + Pr\Big(\frac{n+1}{2}f - \frac{\zeta}{2}A$$
$$+ NbY_4 + NtY_2\Big)Y_6 + Pr\big(NbY_8 + NtY_6\big)Y_2\Big], \qquad Y_6(0) = Bi.$$

$$Y_7' = Y_8, \qquad Y_7(0) = 0.$$

$$Y_8 = -Sc(\frac{n+1}{2}f - \frac{A}{2})Y_8 - \frac{Nt}{Nb}$$
$$\Bigg[\frac{3R_d(1+(\theta_w-1)Y_1)^2(\theta_w-1)Y_5}{\big(1+3R_d(1+(\theta_w-1)Y_1)^3\big)^2}\Big[3R_d\big(1+(\theta_w-1)Y_1\big)^2(\theta_w-1)Y_2^2$$
$$+ Pr\Big(\frac{n+1}{2}f - \frac{\zeta}{2}A + NbY_4 + NtY_2\Big)Y_2 + Br(f'' + f'^2(M+K))\Big]$$
$$+ \frac{1}{\big(-1-3R_d\big(1+(\theta_w-1)Y_1\big)^3\big)}\Big[6R_d\big(1+(\theta_w-1)Y_1\big)(\theta_w-1)^2Y_5Y_2^2$$
$$+ 6R_d\big(1+(\theta_w-1)Y_1\big)(\theta_w-1)^2Y_6Y_2 + Pr\Big(\frac{n+1}{2}f - \frac{\zeta}{2}A$$
$$+ NbY_4 + NtY_2\Big)Y_6 + Pr\big(NbY_8 + NtY_6\big)Y_2\Big]\Bigg], \qquad Y_8(0) = 0.$$

$$Y_9' = Y_{10}, \qquad Y_9(0) = 0.$$

$$Y_{10}' = \frac{3R_d(1+(\theta_w-1)Y_1)^2(\theta_w-1)Y_9}{\big(1+3R_d(1+(\theta_w-1)Y_1)^3\big)^2}\Big[3R_d\big(1+(\theta_w-1)Y_1\big)^2(\theta_w-1)Y_2^2$$

$$+ Pr\left(\frac{n+1}{2}f - \frac{\zeta}{2}A + NbY_4 + NtY_2\right)Y_2 + Br(f'' + f'^2(M+K))\Bigg]$$

$$+ \frac{1}{\left(-1 - 3R_d\big(1 + (\theta_w - 1)Y_1\big)^3\right)}\Bigg[6R_d\big(1 + (\theta_w - 1)Y_1\big)(\theta_w - 1)^2 Y_9 Y_2^2$$

$$+ 6R_d\big(1 + (\theta_w - 1)Y_1\big)(\theta_w - 1)^2 Y_{10} Y_2 + Pr\Bigg(\frac{n+1}{2}f$$

$$- \frac{\zeta}{2}A + NbY_4 + NtY_2\Bigg)Y_{10} + Pr\big(NbY_{12} + NtY_{10}\big)Y_2\Bigg], \qquad Y_{10}(0) = 0.$$

$$Y'_{11} = Y_{12}, \qquad Y_{11}(0) = 0.$$

$$Y'_{12} = Sc\left(\frac{n+1}{2}f - \frac{A}{2}\right)Y_{12} - \frac{Nt}{Nb}\Bigg[\frac{3R_d(1 + (\theta_w - 1)Y_1)^2(\theta_w - 1)Y_9}{\big(1 + 3R_d(1 + (\theta_w - 1)Y_1)^3\big)^2}$$

$$\Bigg[3R_d\big(1 + (\theta_w - 1)Y_1\big)^2(\theta_w - 1)Y_2^2$$

$$+ Pr\left(\frac{n+1}{2}f - \frac{\zeta}{2}A + NbY_4 + NtY_2\right)Y_2 + Br(f'' + f'^2(M+K))\Bigg]$$

$$+ \frac{1}{\left(-1 - 3R_d\big(1 + (\theta_w - 1)Y_1\big)^3\right)}\Bigg[6R_d\big(1 + (\theta_w - 1)Y_1\big)(\theta_w - 1)^2 Y_9 Y_2^2$$

$$+ 6R_d\big(1 + (\theta_w - 1)Y_1\big)(\theta_w - 1)^2 Y_{10} Y_2 + Pr\left(\frac{n+1}{2}f - \frac{\zeta}{2}A + NbY_4 + NtY_2\right)$$

$$Y_{10} + Pr\big(NbY_{12} + NtY_{10}\big)Y_2\Bigg]\Bigg], \qquad Y_{12}(0) = 0.$$

To solve the above initial value problem, we use the RK4 method for which the missing conditions are chosen as:

$$(Y_1(p,q))_{\zeta=\zeta_\infty} = 0, \qquad (Y_3(p,q))_{\zeta=\zeta_\infty} = 0. \tag{3.77}$$

The above set of equations can be solved by using Newton's method with the following iterative formula

$$\begin{bmatrix} p \\ q \end{bmatrix}_{k+1} = \begin{bmatrix} p \\ q \end{bmatrix}_k - \left(\begin{bmatrix} \frac{\partial Y_1}{\partial p} & \frac{\partial Y_1}{\partial q} \\ \frac{\partial Y_3}{\partial p} & \frac{\partial Y_3}{\partial q} \end{bmatrix}^{-1} \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix}\right)_k. \tag{3.78}$$

$$\Rightarrow \begin{bmatrix} p \\ q \end{bmatrix}_{k+1} = \begin{bmatrix} p \\ q \end{bmatrix}_k - \left(\begin{bmatrix} Y_5 & Y_9 \\ Y_7 & Y_{11} \end{bmatrix}^{-1} \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix}\right)_k. \tag{3.79}$$

The iterative process is repeated until the following stopping criteria is met:

$$max\Big(\big|Y_1(\zeta_\infty)\big|, \big|Y_3(\zeta_\infty)\big|\Big) < \epsilon^*.$$

3.7 The Outcomes with Discussion

Analyzing the effects of different parameters on the velocity, temperature, and concentration distributions is the main objective of this part. The influence of different factors such as Schmidt number Sc, Brownian parameter Nb, thermophoresis parameter Nt, Prandtl number Pr, Eckert number Ec, velocity slip factor γ, Biot number Bi, stretching index n, permeability parameter K, unsteadiness parameter A, magnetic field parameter M, is observed graphically. Numerical results of the skin friction coefficient $Cf_xRe_x^{\frac{1}{2}}$, Nusselt number $Nu_xRe_x^{\frac{-1}{2}}$ and Sherwood number $Sh_xRe_x^{\frac{-1}{2}}$, for the distinct values of some fixed parameters are shown in Tables 3.1-3.3. In Tables 3.1-3.3, the missing conditions are taken from the intervals represented by I_f, I_θ and I_ϕ. In Table 3.1, increasing the values of the magnetic parameter M, stretching index parameter n, permeability parameter K, and unsteadiness parameter A would increase the skin friction coefficient but when the value of velocity slip parameter γ increases, the skin friction coefficient decreases.

In Table 3.2, the effects of different parameters on Nusselt number demonstrate a progressive decline in the Nusselt number by raising the numerical values of various parameters like thermophoresis parameter Nt, unsteadiness parameter A and Eckert parameter, Ec. But as Prandtl and Biot parameter values rise, Nusselt number value increases, as well. From Table 3.3, it can be observed that by raising the values of the Schmidt number, the Brownian parameter, the Sherwood number increases, whereas it decreases for the rising the values of the unsteadiness parameter.

The transverse magnetic field effect on the field of velocity is sketched in Figure 3.2. It is illustrated that the velocity diminishes while M increases. The existence of a transverse magnetic field ultimately leads to a drag force called Lorentz force that will cause a retardation in the distribution of velocity. Figure 3.3 indicates the impact of the porous permeability parameter on the field of velocity and it is noticed that the fluid velocity reduces by raising the values of K. Figure 3.4 shows how n affects the velocity profile. The velocity profile $f'(\zeta)$ declines with increasing values of n. Figure 3.5 exhibits the influence of the velocity slip parameter on the velocity profile. It is determined that the dimensionless velocity profile declines with increasing values of γ. As the velocity slip parameter increases, the slip velocity will increase and the fluid velocity decline. This is due to the fact that while the slip condition occurs, the speed of the stretching sheet is not the same as the velocity of the flow close to the sheet.

From Figure 3.6 it can be observed that the velocity profile falls as the values of the unsteadiness parameter A increases. Figure 3.7 illustrates the consequences of the Prandtl number Pr on the temperature profile. It is found that the temperature distribution declines as Pr values rise. Figure 3.8 shows the relationship between Nt and the temperature distribution. By increasing the values of Nt, the temperature profile $\theta(\zeta)$ increases. Physically, increasing the values of Nt pulls the nanoparticles from hotter to cooler regions. As a result, the overall temperature of the nanofluid increases. Figure 3.9 shows that the temperature distribution grows as Ec values are increased. When Ec values rise, the dissipation also rises. The internal energy of the fluid also increases as a consequence of this rise in dissipation. The fluid's temperature distribution is also improved by this change in internal energy. The Biot number Bi affects the temperature distribution as seen in Figure 3.10. When the value of the Biot number Bi escalates, the temperature distribution $\theta(\zeta)$ will also rise. Greater Biot number values result in more dynamic heat generation on the sheet as described in this trend. Thus the thickness of the thermal boundary layer increases.

Figure 3.11 represents the concentration distribution $\phi(\zeta)$ for different values of the Schmidt number parameter Sc. It is simple to see that as the value of Sc rises, the concentration distribution changes into a decreasing function. The effects of the Brownian motion parameter Nb on the concentration distribution $\phi(\zeta)$ are explained in Figure 3.12. It has been noted that the concentration profile diminishes

as Nb is enhanced. Nb produces a reduction in the concentration distribution.
Figure 3.13 investigates the influence of the unsteadiness parameter on the concentration profile. By expanding A, the concentration distribution is found to increase.
It is obvious from Figure 3.14 that the Bejan number exhibits an increasing behaviour when there is an applied magnetic field. The effect of heat transfer entropy increases as we move up from the surface. Heat transmission causes this entropy effect to fully dominate when it is far from the surface. This is due to the larger frictional effect that arises from an increase in M, which raises the fluid temperature and improves the Bejan number as seen in Figure 3.14.
In Figure 3.15, the effects of thermal radiation parameter R_d on the Bejan number is discussed. It is observed that an enhancement in R_d diminishes the Bejan number.
Figure 3.16 reflects the behaviour of Bejan number Be for various values of Brinkman number Br. Due to an increment in Br, the Bejan number is decreased. Figure 3.17 determines the effect of the Biot number on Bejan number. Enhancing the Bi, increases the Bejan number.

TABLE 3.1: Result of skin friction coefficient.

M	n	γ	K	A	$-f''(0)$	I_f
0.1	2	0.1	0.5	0.1	1.306365	[-1.3,-0.9]
0.3					1.353518	[-1.3,-1.1]
0.5					1.398425	[-1.4,-1.1]
0.5	0	0.1	0.5	0.1	0.997508	[-1.1,-0.9]
	1				1.225453	[-1.3,-0.1]
0.5	2	0	0.5	0.1	1.702257	[-1.7,-1.4]
		0.5			0.843713	[-0.8,-0.7]
0.5	2	0.1	0.1	0.1	1.306365	[-1.3,-1.1]
			0.3		1.353518	[-1.3,-1.1]
0.5	2	0.1	0.5	0.15	1.406414	[-1.4,-1.2]
				0.2	1.414362	[-1.4,-1.2]

TABLE 3.2: Result of the Nusselt number.
$R_d = 0.5$, $n = 2$, $M = 0.5$, $\gamma = 0.1$

Pr	Nt	Ec	A	Bi	$-\boldsymbol{\theta}'(\mathbf{0})$	I_θ
1	0.2	0.1	0.1	0.21	0.110354	[-1,7]
3					0.128700	[-3,4]
5					0.132731	[-3,5]
1	0.5	0.1	0.1	0.21	0.115938	[1,5]
	0.7				0.114137	[-5,3]
1	0.2	0.3	0.1	0.21	0.078176	[-0.5,8]
		0.5			0.029995	[1,3]
1	0.2	0.1	0.15	0.21	0.117745	[-2,3]
			0.2		0.104656	[1,3]
1	0.2	0.1	0.1	0.4	0.154147	[-1.3,3]
				0.6	0.167387	[0,3]

TABLE 3.3: Result of the Sherwood number.
$R_d = 0.5$, $n = 2$, $M = 0.5$, $\gamma = 0.1$

Sc	Nb	A	$-\boldsymbol{\phi}'(\mathbf{0})$	I_ϕ
1	0.2	0.1	0.478728	[1,3]
1.5			0.664326	[1,3]
2.0	0.2	0.1	0.836305	[1,3]
1.5	0.5	0.1	0.708828	[0,3]
	1.0		0.724192	[1,4]
1.5	0.2	0.15	0.638447	[1,8]
		0.2	0.587264	[1,8]

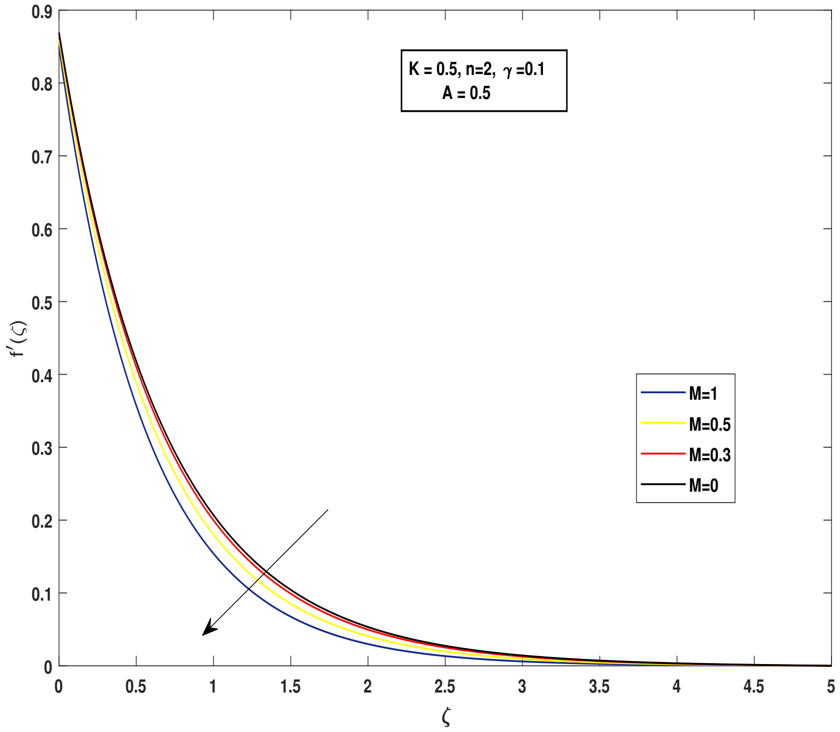

FIGURE 3.2: Impact of M on the velocity profile .

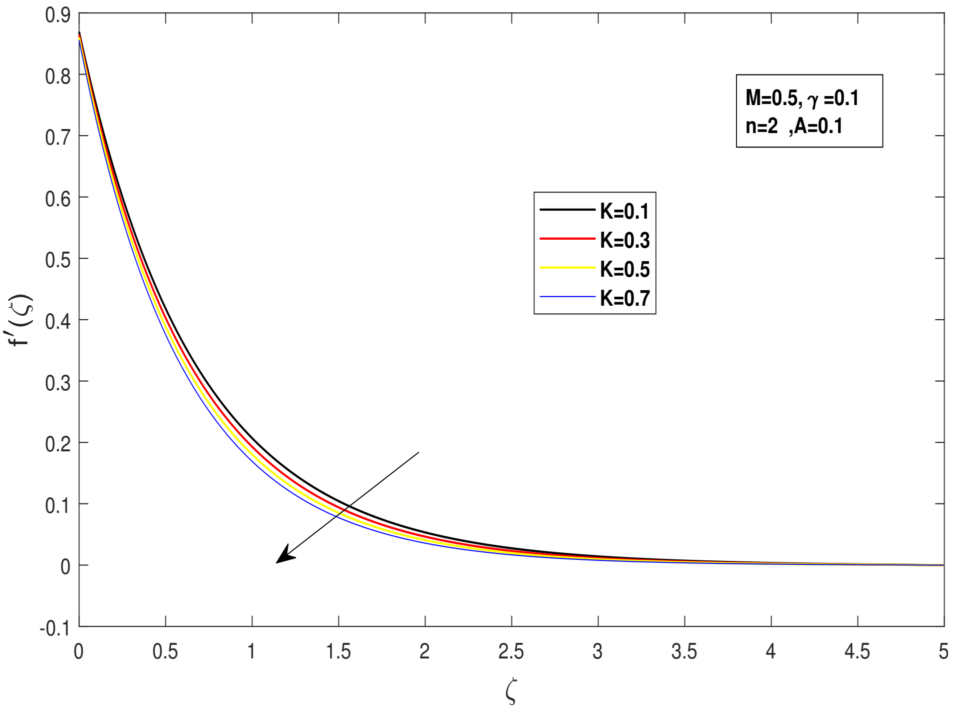

FIGURE 3.3: Impact of K on the velocity profile

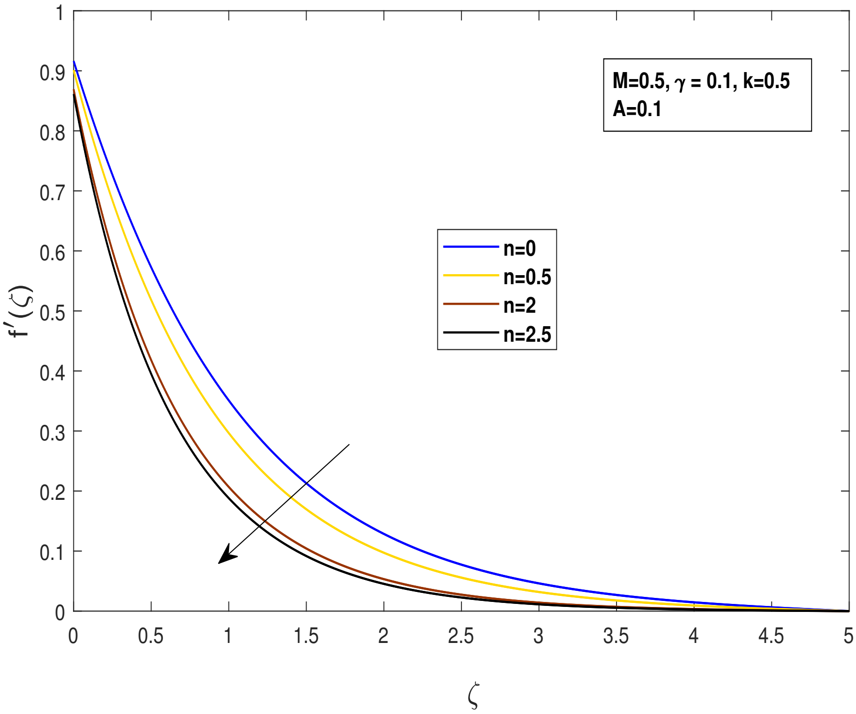

FIGURE 3.4: Impact of n on the $f'(\zeta)$.

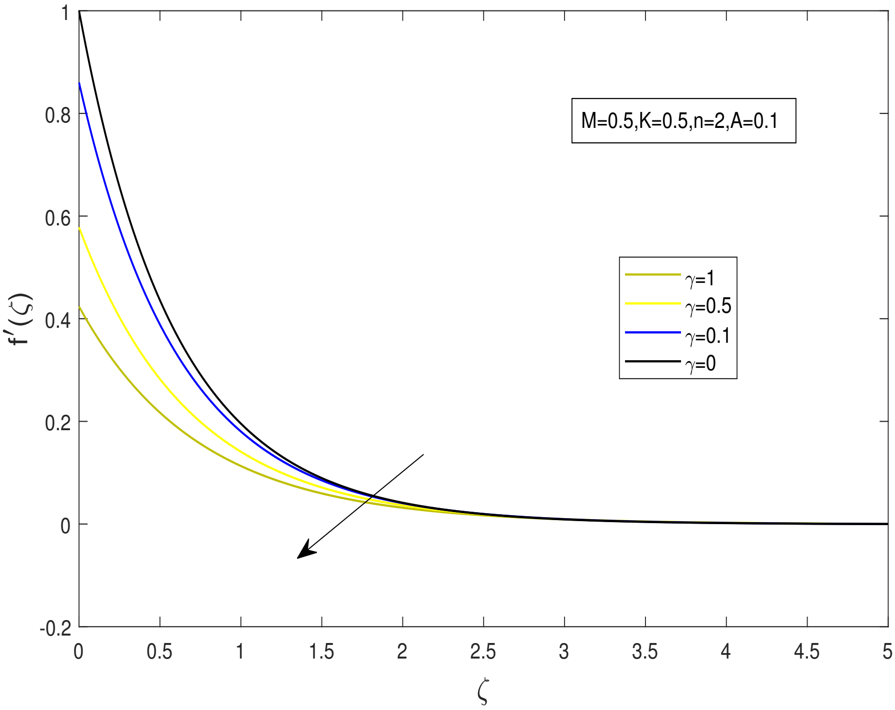

FIGURE 3.5: Impact of γ on the velocity profile.

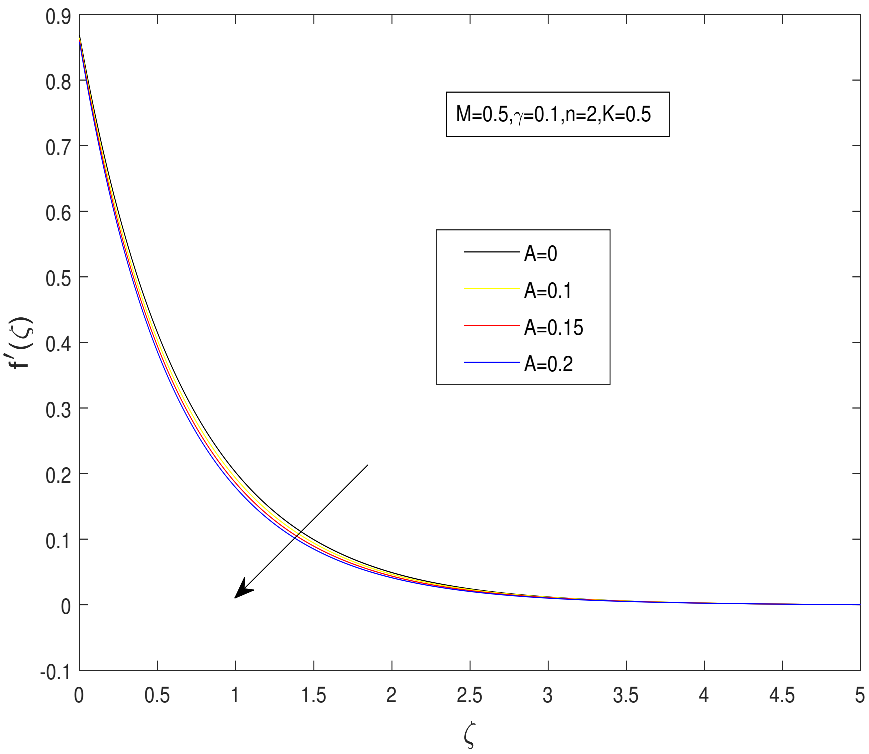

FIGURE 3.6: Impact of A on the $f'(\zeta)$.

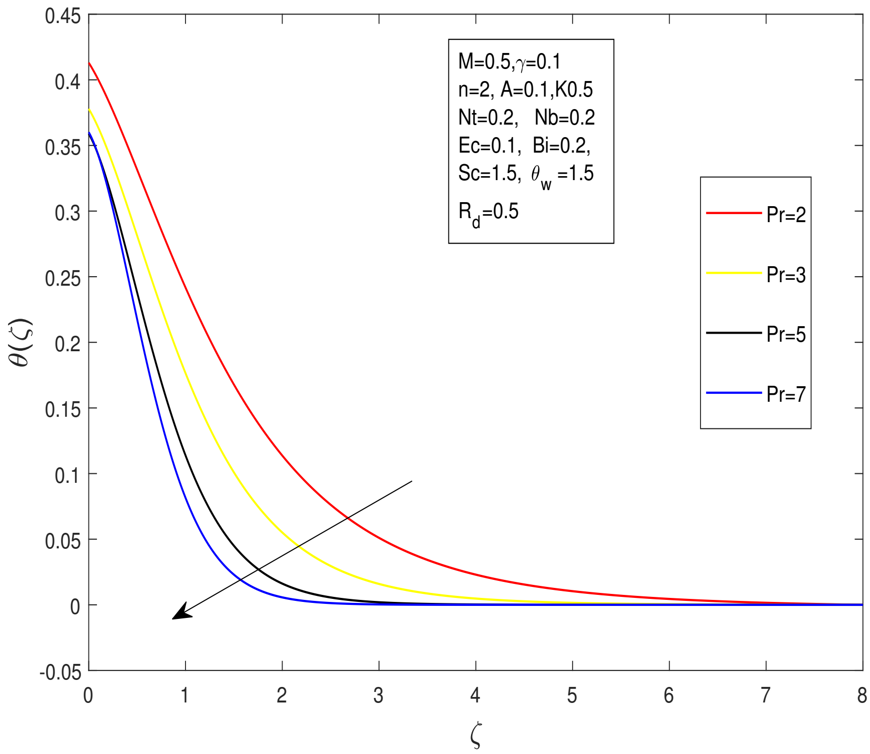

FIGURE 3.7: Impact of Pr on the temperature distribution.

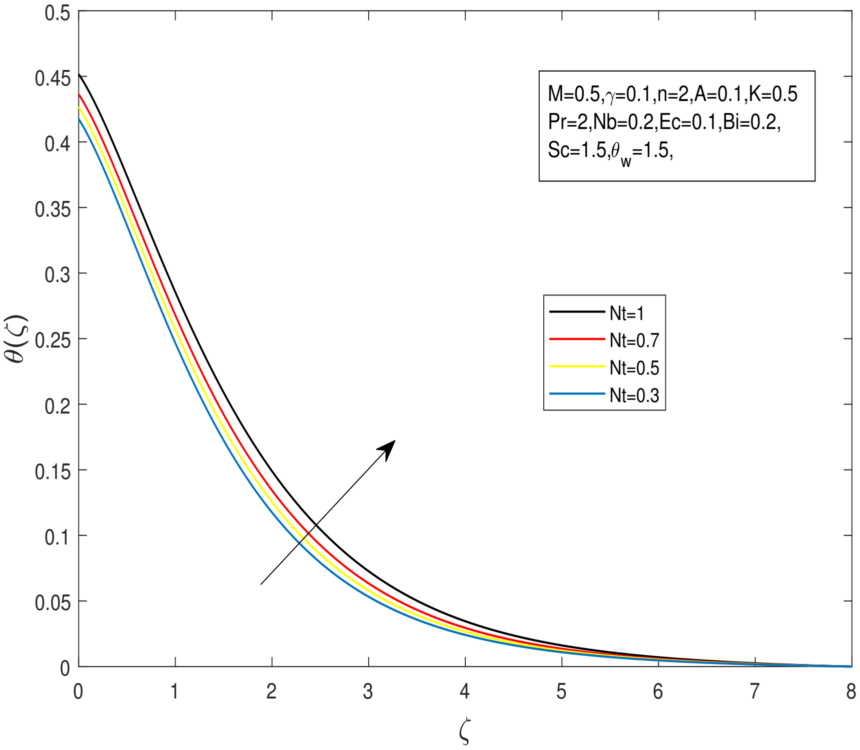

FIGURE 3.8: Impact of Nt on the temperature distribution.

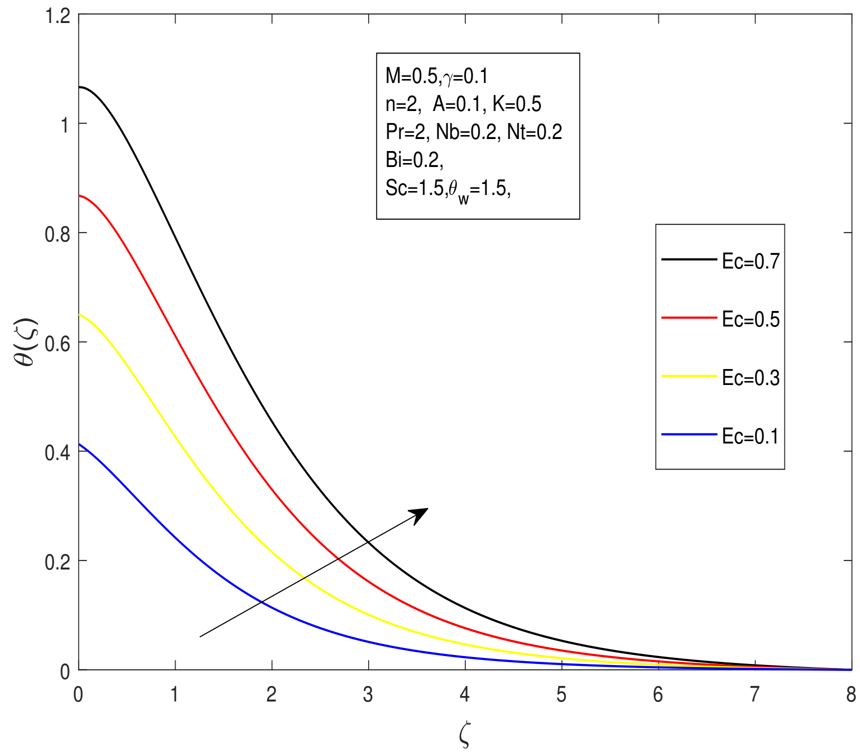

FIGURE 3.9: Influence of Ec on the temperature distribution.

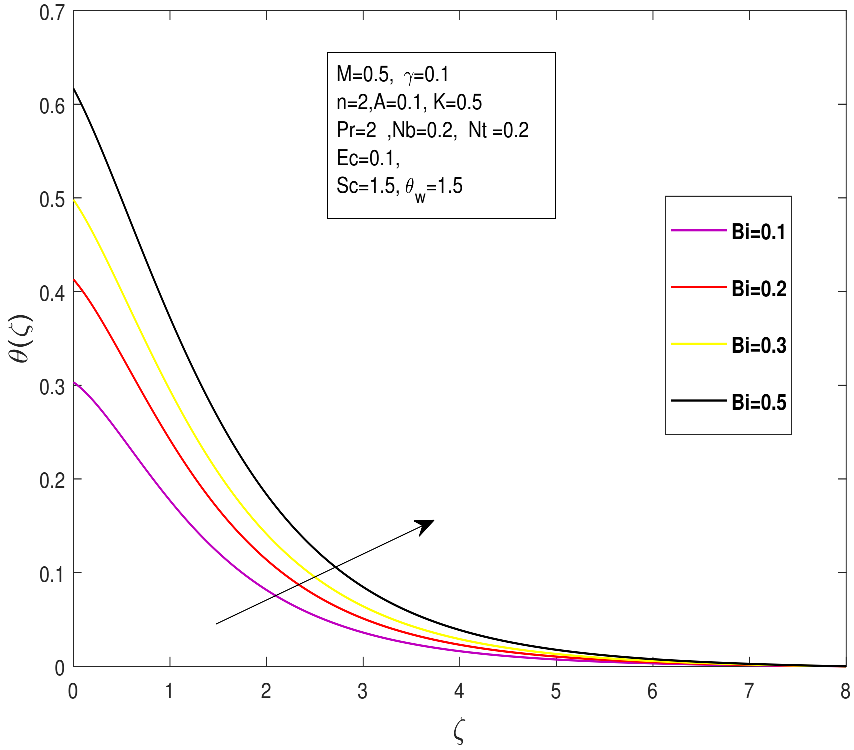

FIGURE 3.10: Impact of Bi on the $\theta(\zeta)$.

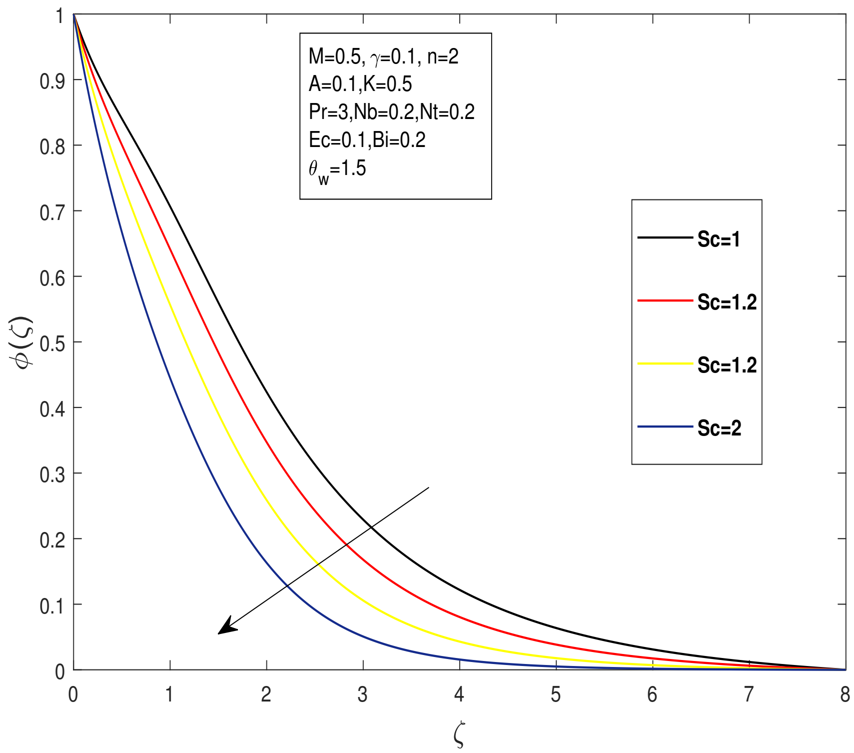

FIGURE 3.11: Influence of Sc concentration distribution.

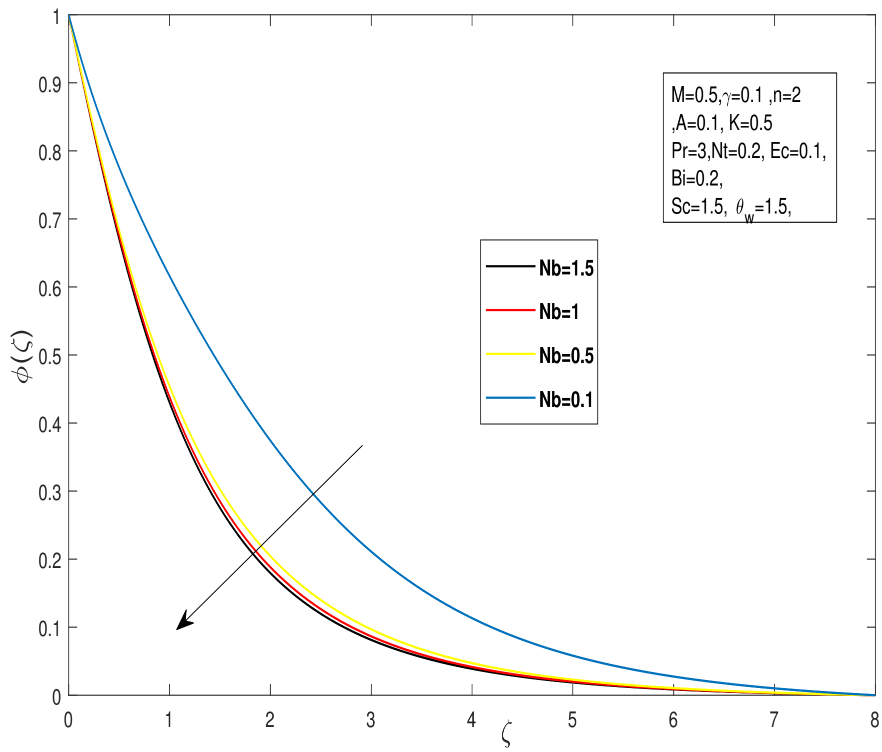

FIGURE 3.12: Impact of Nb on the $\phi(\zeta)$.

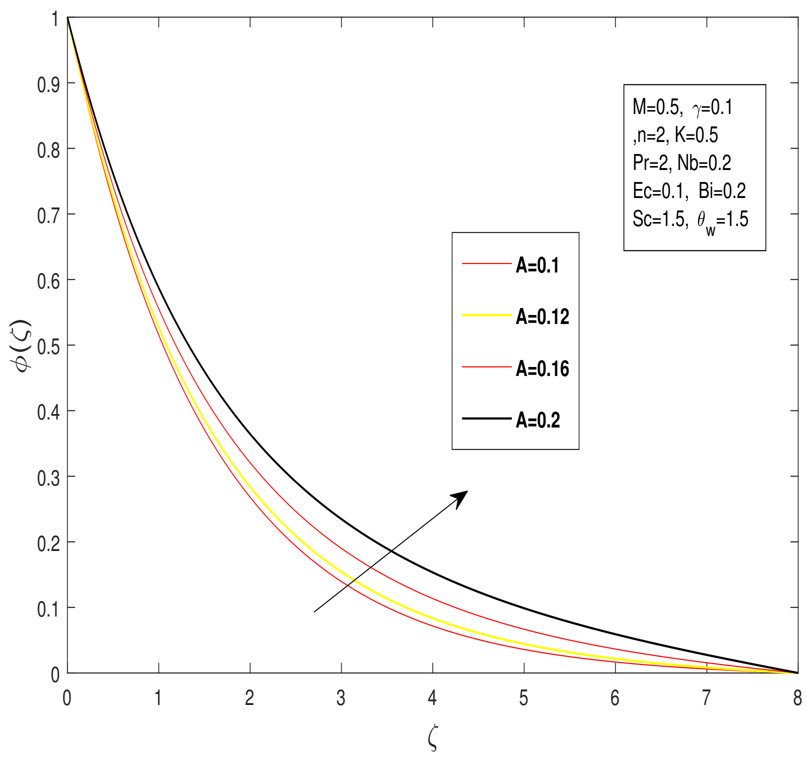

FIGURE 3.13: Influence of A on the concentration distribution.

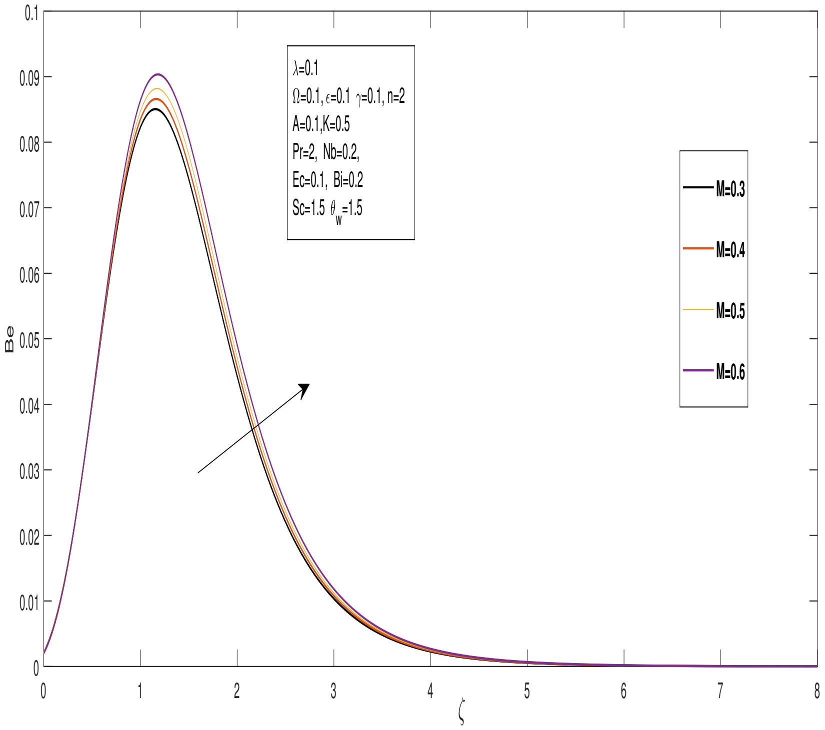

FIGURE 3.14: Impact of M on the Bejan number.

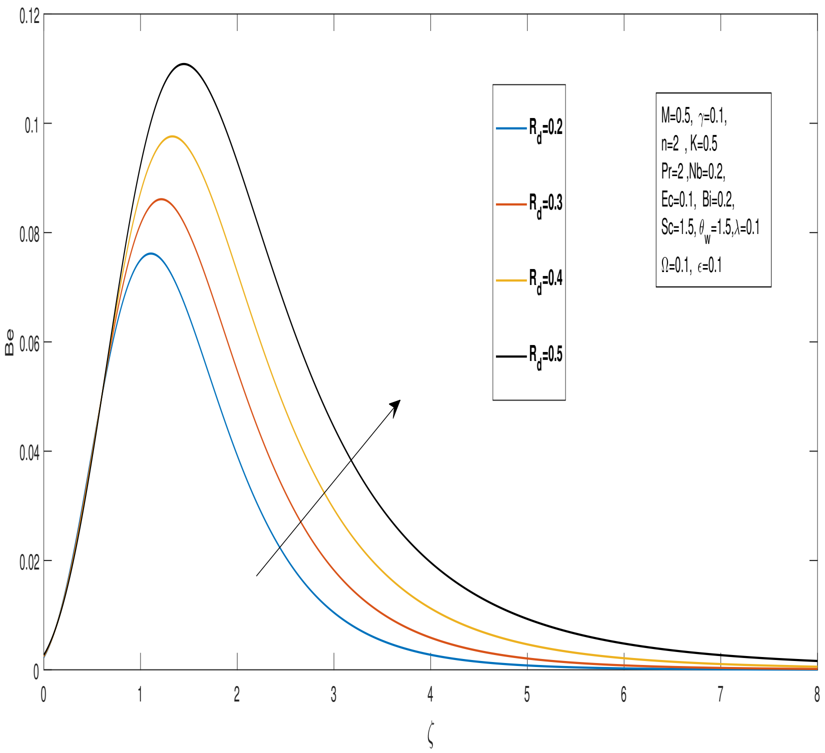

FIGURE 3.15: Impact of R_d on the Bejan number.

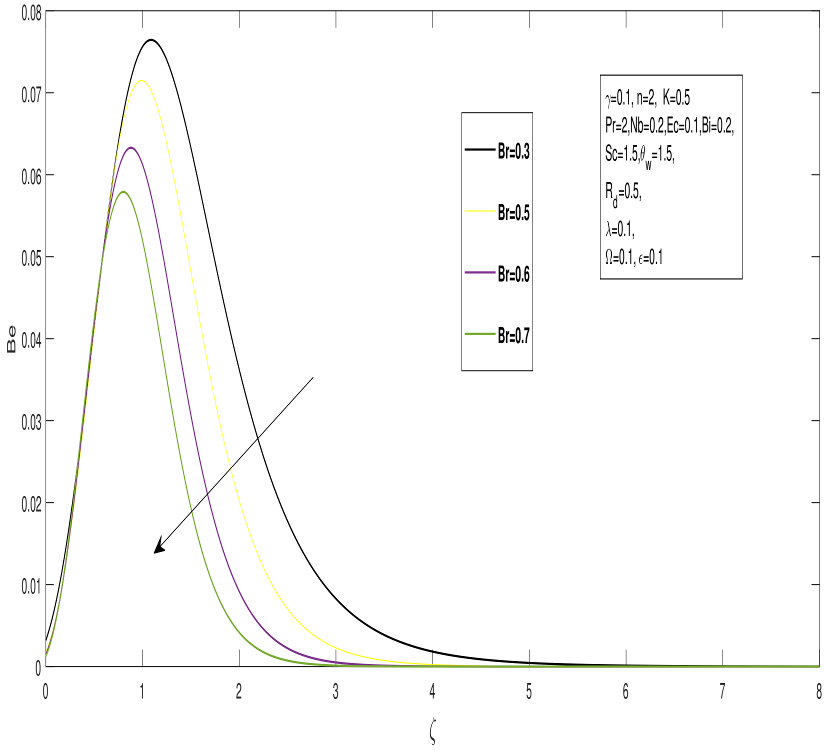

FIGURE 3.16: Impact of Br on the Bejan number.

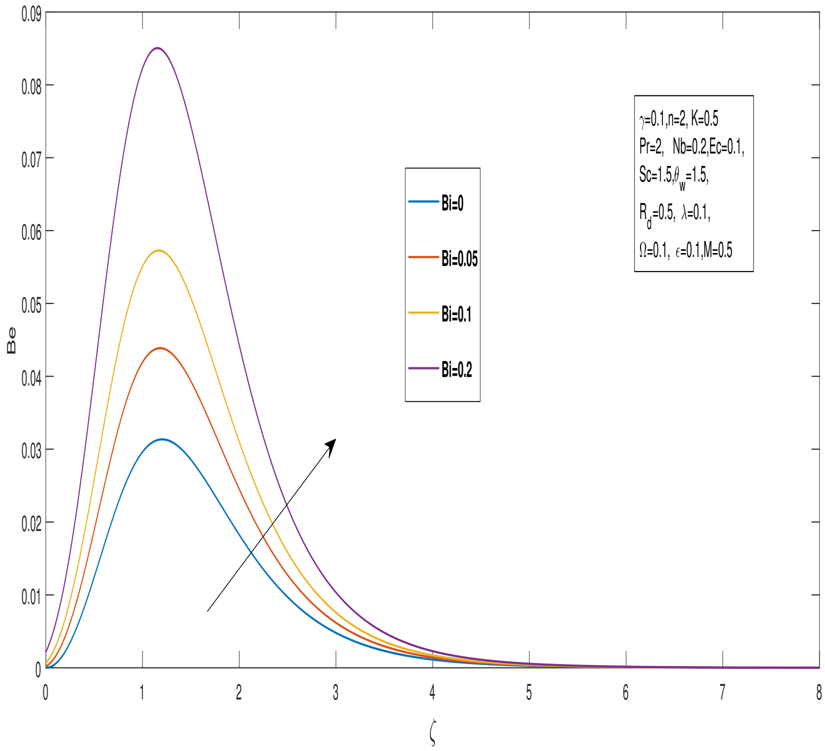

FIGURE 3.17: Impact of Bi on the Bejan number.

Chapter 4

Cattaneo-Christov Double Diffusions model and Activation Energy for Entropy Analysis in Hydromagnetic Nanofluid Flow

4.1 Introduction

The objective of this chapter is to extend the work of Seth et al. [43] discussed in the previous chapter. Flow model of Seth et al. has been extended by considering Cattaneo-Christov Double Diffusions model and Activation energy.

By converting the governing PDEs into ODEs with the help of an appropriate similarity transformation, the shooting method is used to obtain the numerical results.

The computational software MATLAB is used for numerical computation. Graphical representations are also provided to explain the effect of the evolving parameters. Tables and graphs are used to investigate the numerical results produced.

4.2 Mathematical Modeling

The set of equations describing the flow is as follows

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{4.1}$$

$$\rho_{nf}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\mu_{nf}\frac{\partial^2 u}{\partial y^2}-\sigma B_0^2 u-\frac{\mu_{nf}}{k_p}u, \tag{4.2}$$

$$\begin{aligned}&\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+\lambda_T\Bigg[u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x}+v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y}+u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y}+v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x}+2uv\frac{\partial^2 T}{\partial x\partial y}\\&+u^2\frac{\partial^2 T}{\partial x^2}+v^2\frac{\partial^2 T}{\partial y^2}\Bigg]=\alpha_{nf}\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho_{nf}c_p}\frac{\partial}{\partial y}\left(-\frac{16\sigma^*}{3k^*}T^3\frac{\partial T}{\partial y}\right)+\frac{\upsilon_{nf}}{c_p}\left(\frac{\partial u}{\partial y}\right)^2\\&+\frac{\sigma B_0^2}{\rho_{nf}c_p}u^2+\frac{\upsilon_{nf}}{k_p c_p}u^2+\tau\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right],\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+\lambda_C\Bigg[u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x}+v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y}+u\frac{\partial v}{\partial x}\frac{\partial C}{\partial y}+v\frac{\partial u}{\partial y}\frac{\partial C}{\partial x}\\&+2uv\frac{\partial^2 C}{\partial x\partial y}+u^2\frac{\partial^2 C}{\partial x^2}\\&+v^2\frac{\partial^2 C}{\partial y^2}\Bigg]=D_B\frac{\partial^2 C}{\partial y^2}+\frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial y^2}\right)-k_r^2(C-C_\infty)\left[\frac{T}{T_\infty}\right]^m\exp\left[\frac{-E_a}{kT_\infty}\right].\end{aligned} \tag{4.4}$$

The associated BCs have been taken as

$$\left.\begin{aligned}&u=u_w+u_{slip}=\frac{ax^n}{1-\lambda t}+N\frac{\partial u}{\partial y},\;\; v=0,\\&k_{nf}\frac{\partial T}{\partial y}=h_f(T_f-T),\quad C=C_w,\quad at\quad y=0,\\&u\to 0,\quad T\to T_\infty,\quad C\to C_\infty,\quad as\;\; y\to\infty.\end{aligned}\right\} \tag{4.5}$$

Following similarity, the transformation has been used to convert PDEs (4.1)-(4.4) into a system of ODEs.

$$\left.\begin{aligned}&\zeta=y\sqrt{\frac{u_w(x,t)}{\upsilon_{nf}x}},\\&\theta(\zeta)=\frac{T-T_\infty}{T_f-T_\infty},\\&\phi(\zeta)=\frac{C-C_\infty}{C_w-C_\infty},\\&\theta(\zeta)=\frac{T-T_\infty}{T_f-T_\infty}.\end{aligned}\right\} \tag{4.6}$$

The detailed procedure for the conversion of continuity equation (4.1) and momentum equation (4.2), has been already discussed in Chapter 3. Now, we include below the procedure for the conversion of (4.3), into the dimensionless form. The already worked out derivatives (3.12)-(3.15), (3.16)-(3.19), (3.26), (3.27), (3.29) and (3.30) in Chapter 3, will be directly used here.

- $$\begin{aligned} u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} &= \left(\frac{ax^n}{1-\lambda t}\right)^2\frac{1}{x}\left(nf'^2(\zeta)+(\frac{n-1}{2})\zeta f''(\zeta)f'(\zeta)\right)T_\infty(\theta_w-1)\theta'(\zeta) \\ &\quad \left(\frac{n-1}{2}\right)T_\infty(\theta_w-1)\theta'(\zeta)\left(\frac{n-1}{2}\right)\frac{\zeta}{x} \\ &= \left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2 T_\infty(\theta_w-1)\theta'(\zeta)\left(\left(\frac{n-1}{2}\right)nf'^2(\zeta)\zeta \right. \\ &\quad \left. +\zeta^2\left(\frac{n-1}{2}\right)^2 f'(\zeta)f''(\zeta)\right). \end{aligned} \tag{4.7}$$
- $$\begin{aligned} v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y} &= -\frac{ax^{n-1}}{(1-\lambda t)}\left(nf'(\zeta)+\left(\frac{n-1}{2}\right)\zeta f''(\zeta)\right) \\ &\quad \left[-\frac{ax^{n-1}}{(1-\lambda t)}\left(\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)T_\infty(\theta_w-1)\theta'(\zeta)\right] \\ &= \left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2 T_\infty(\theta_w-1)\theta'(\zeta)\left[n\left(\frac{n+1}{2}\right)f(\zeta)f'(\zeta)+n\left(\frac{n-1}{2}\right)\zeta\right. \\ &\quad \left. f'^2(\zeta)+\left(\frac{n^2-1}{4}\right)\zeta ff''(\zeta)+\left(\frac{n-1}{2}\right)^2\zeta^2 f''(\zeta)f'(\zeta)\right] \end{aligned} \tag{4.8}$$
- $$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{\partial}{\partial x}\left(-\sqrt{\frac{ax^{n-1}\upsilon_{nf}}{(1-\lambda t)}}\left(\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)\right) \\ &= -\sqrt{\frac{ax^{n-1}\upsilon_{nf}}{(1-\lambda t)}}\frac{\partial}{\partial x}\left(\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right) \\ &\quad +\left(\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)\frac{\partial}{\partial x}\left(\sqrt{\frac{ax^{n-1}\upsilon_{nf}}{(1-\lambda t)}}\right) \\ &= -\sqrt{\frac{ax^{n-1}\upsilon_{nf}}{(1-\lambda t)}}\left[\left(\frac{n^2-1}{4}\right)\frac{\zeta}{x}f'(\zeta)+\left(\frac{n-1}{2}\right)^2\frac{\zeta^2}{x}f''(\zeta)+\left(\frac{n-1}{2}\right)^2\frac{\zeta}{x}f'(\zeta)\right] \\ &\quad -\left[\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right]\sqrt{\frac{ax^{n-1}\upsilon_{nf}}{(1-\lambda t)}}\left(\frac{n-1}{2}\right)\frac{1}{x}. \end{aligned} \tag{4.9}$$
- $$\begin{aligned} u\frac{\partial T}{\partial y}\frac{\partial v}{\partial x} &= \frac{ax^n}{(1-\lambda t)\upsilon_{nf}}f'(\zeta)\big(T_\infty(\theta_w-1)\theta'(\zeta)\sqrt{\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}} \\ &\quad \left(-\sqrt{\frac{ax^{n-1}\upsilon_{nf}}{(1-\lambda t)}}\left[\left(\frac{n^2-1}{4}\right)\frac{\zeta}{x}f'(\zeta)+\left(\frac{n-1}{2}\right)^2\frac{\zeta^2}{x}f''(\zeta)\right.\right. \end{aligned}$$

$$+\left(\frac{n-1}{2}\right)^2\frac{\zeta}{x}f'(\zeta)\Bigg]-\left[\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right]$$
$$\sqrt{\frac{ax^{n-1}v_{nf}}{(1-\lambda t)}}\left(\frac{n-1}{2}\right)\frac{1}{x}\Bigg)$$
$$=-\left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2T_\infty(\theta_w-1)\theta'(\zeta)\Bigg[\left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)+\left(\frac{n-1}{2}\right)^2$$
$$\zeta^2f'(\zeta)f''(\zeta)+f'^2(\zeta)\left(\frac{n-1}{2}\right)^2\zeta$$
$$+\left(\frac{n^2-1}{4}\right)f(\zeta)f'(\zeta)+\left(\frac{n-1}{2}\right)^2(\zeta)f'^2(\zeta)\Bigg]. \tag{4.10}$$

- $$v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x}=-\left(\frac{ax^n}{(1-\lambda t)}\right)^2\frac{1}{x}\Bigg(\frac{n+1}{2}f(\zeta)f''(\zeta)$$
$$+\frac{n-1}{2}\zeta f'(\zeta)f''(\zeta)\Bigg)T_\infty(\theta_w-1)\theta'(\zeta)\left(\frac{n-1}{2}\right)\frac{\zeta}{x}$$
$$=-\left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2T_\infty(\theta_w-1)\theta'(\zeta)\Bigg[\left(\frac{n^2-1}{4}\right)\zeta f(\zeta)f''(\zeta)$$
$$+\left(\frac{n-1}{2}\right)^2\zeta^2f''(\zeta)f'(\zeta)\Bigg]. \tag{4.11}$$

- $$\frac{\partial^2T}{\partial x\partial y}=\frac{\partial}{\partial x}\left(T_\infty(\theta_w-1)\theta'(\zeta)\sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}}\right)$$
$$=\sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}}(T_\infty(\theta_w-1)\theta''(\zeta)\zeta\left(\frac{n-1}{2}\right)\frac{1}{x}$$
$$+\sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}}(T_\infty(\theta_w-1)\theta'(\zeta)\left(\frac{n-1}{2}\right)\frac{1}{x}$$
$$=\sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}}(T_\infty(\theta_w-1)\left(\frac{n-1}{2}\right)\frac{1}{x}\Big[\zeta\theta''(\zeta)+\theta'(\zeta)\Big]. \tag{4.12}$$

- $$2uv\frac{\partial^2T}{\partial x\partial y}=2\left(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\right)\left(-\sqrt{\frac{ax^nv_{nf}}{(1-\lambda t)x}}\left(\frac{n+1}{2}f(\zeta)+\left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)\right)$$
$$\sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}}(T_\infty(\theta_w-1)\left(\frac{n-1}{2}\right)\frac{1}{x}\Big[\zeta\theta''(\zeta)+\theta'(\zeta)\Big]$$
$$=-2\left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2T_\infty(\theta_w-1)\Bigg[\left(\frac{n^2-1}{4}\right)f'(\zeta)f(\zeta)\zeta\theta''(\zeta)$$
$$+\left(\frac{n^2-1}{4}\right)f'(\zeta)f(\zeta)\theta'(\zeta)$$
$$+\left(\frac{n-1}{2}\right)^2\zeta^2\theta''(\zeta)f'^2(\zeta)+\left(\frac{n-1}{2}\right)^2\zeta\theta'(\zeta)f'^2(\zeta)\Bigg]. \tag{4.13}$$

- $$\frac{\partial^2T}{\partial x^2}=\frac{\partial}{\partial x}\left(T_\infty(\theta_w-1)\theta'(\zeta)\left(\frac{n-1}{2}\right)\frac{\zeta}{x}\right)$$

$$= T_\infty(\theta_w - 1)\frac{\partial}{\partial x}\left(\theta'(\zeta)\left(\frac{n-1}{2}\right)\frac{\zeta}{x}\right)$$

$$= T_\infty(\theta_w - 1)\left(\theta''(\zeta)\zeta^2\left(\frac{n-1}{2}\right)^2\frac{1}{x^2}\right.$$
$$\left. - \theta'(\zeta)\zeta\frac{1}{x^2}\left(\frac{n-1}{2}\right) + \theta'(\zeta)\zeta\frac{1}{x^2}\left(\frac{n-1}{2}\right)^2\right). \tag{4.14}$$

- $$u^2\frac{\partial^2 T}{\partial x^2} = \frac{(ax^n)^2}{(1-\lambda t)^2}f'^2(\zeta)T_\infty(\theta_w - 1)\left(\theta''(\zeta)\zeta^2\left(\frac{n-1}{2}\right)^2\frac{1}{x^2}\right.$$
$$\left. - \theta'(\zeta)\zeta\frac{1}{x^2}\left(\frac{n-1}{2}\right) + \theta'(\zeta)\zeta\frac{1}{x^2}\left(\frac{n-1}{2}\right)^2\right)$$
$$= \left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2 T_\infty(\theta_w - 1)\left[\theta''(\zeta)\left(\frac{n-1}{2}\right)^2\zeta^2 f'^2\right.$$
$$\left. - \left(\frac{n-1}{2}\right)\theta'(\zeta)\zeta f'^2(\zeta) + \theta'(\zeta)\zeta\left(\frac{n-1}{2}\right)^2 f'^2(\zeta)\right]. \tag{4.15}$$

- $$v^2 = \left(-\sqrt{\frac{ax^n v_{nf}}{(1-\lambda t)x}}\left(\frac{n+1}{2}f(\zeta) + \left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)\right)^2$$
$$= \frac{ax^{n-1}v_{nf}}{(1-\lambda t)}\left[\left(\frac{n+1}{2}\right)^2 f^2(\zeta) + \left(\frac{n-1}{2}\right)^2\zeta^2 f'^2(\zeta)\right.$$
$$\left. + 2\left(\frac{n^2-1}{4}\right)\zeta f'(\zeta)f(\zeta)\right]. \tag{4.16}$$

- $$v^2\frac{\partial^2 T}{\partial y^2} = \left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2 T_\infty(\theta_w - 1)\left[\left(\frac{n+1}{2}\right)^2 f^2(\zeta)\theta''(\zeta)\right.$$
$$\left. + \theta''(\zeta)\zeta^2\left(\frac{n-1}{2}\right)^2 f'^2(\zeta) + 2\left(\frac{n^2-1}{4}\right)\zeta f'(\zeta)f(\zeta)\theta''(\zeta)\right]. \tag{4.17}$$

By adding (4.7), (4.8), (4.10), (4.11), (4.13), (4.15) and (4.17), we get

$$\left[u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} + v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y} + u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y} + v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} + 2uv\frac{\partial^2 T}{\partial x\partial y} + u^2\frac{\partial^2 T}{\partial x^2} + v^2\frac{\partial^2 T}{\partial y^2}\right]$$
$$= \left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2 T_\infty(\theta_w - 1)\left[\left(\frac{n-1}{2}\right)nf'^2(\zeta)\zeta\theta'(\zeta)\right.$$
$$+ \zeta^2\left(\frac{n-1}{2}\right)^2 f'(\zeta)f''(\zeta)\theta'(\zeta) + n\left(\frac{n+1}{2}\right)f(\zeta)f'(\zeta)\theta'(\zeta)$$
$$+ n\left(\frac{n-1}{2}\right)\zeta f'^2(\zeta)\theta'(\zeta) + \left(\frac{n^2-1}{4}\right)\zeta f(\zeta)f''(\zeta)\theta'(\zeta)$$
$$+ \left(\frac{n-1}{2}\right)^2\zeta^2 f''(\zeta)f'(\zeta)\theta'(\zeta) - \left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\theta'(\zeta)$$
$$- \left(\frac{n-1}{2}\right)^2\zeta^2 f'(\zeta)f''(\zeta)\theta'(\zeta) - \left(\frac{n-1}{2}\right)^2 f'^2(\zeta)\zeta\theta'(\zeta) - \left(\frac{n^2-1}{4}\right)$$

$$
\begin{aligned}
& f(\zeta)f'(\zeta)\theta'(\zeta) - \left(\frac{n-1}{2}\right)^2 \zeta f'^2(\zeta)\theta'(\zeta) - \left(\frac{n^2-1}{4}\right)\zeta f(\zeta)f''(\zeta)\theta'(\zeta) \\
& - \theta'(\zeta)\left(\frac{n-1}{2}\right)^2 \zeta^2 f''(\zeta)f'(\zeta) - 2\left(\frac{n^2-1}{4}\right) f'(\zeta)f(\zeta)\zeta\theta''(\zeta) \\
& - 2\left(\frac{n^2-1}{4}\right) f'(\zeta)f(\zeta)\theta'(\zeta) - 2\left(\frac{n-1}{2}\right)^2 \zeta^2\theta''(\zeta)f'^2(\zeta) \\
& - 2\left(\frac{n-1}{2}\right)^2 \zeta\theta'(\zeta)f'^2(\zeta) + \theta''(\zeta)\left(\frac{n-1}{2}\right)^2 \zeta^2 f'^2(\zeta) - \left(\frac{n-1}{2}\right) \\
& \theta'(\zeta)\zeta f'^2(\zeta) + \theta'(\zeta)\zeta\left(\frac{n-1}{2}\right)^2 f'^2(\zeta) + \left(\frac{n+1}{2}\right)^2 f^2(\zeta)\theta''(\zeta) \\
& + 2\left(\frac{n^2-1}{4}\right)\zeta f'(\zeta)f(\zeta)\theta''(\zeta) + \theta''(\zeta)\zeta^2\left(\frac{n-1}{2}\right)^2 f'^2(\zeta)\Bigg] \\
= & \left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2 T_\infty(\theta_w - 1)\Bigg[2\left(\frac{n-1}{2}\right) n f'^2(\zeta)\zeta\theta'(\zeta) \\
& + n\left(\frac{n+1}{2}\right) f(\zeta)f'(\zeta)\theta'(\zeta) - \left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\theta'(\zeta) \\
& - 3\left(\frac{n^2-1}{4}\right) f(\zeta)f'(\zeta)\theta'(\zeta) - 3\left(\frac{n-1}{2}\right)^2 \zeta\theta'(\zeta)f'^2(\zeta) \\
& - \left(\frac{n-1}{2}\right)\theta'(\zeta)\zeta f'^2(\zeta) + \left(\frac{n+1}{2}\right)^2 f^2(\zeta)\theta''(\zeta)\Bigg].
\end{aligned} \tag{4.18}
$$

Using (3.32) and (4.18) in the left side of (4.3), we obtain dimensionless form of energy equation

$$
\begin{aligned}
& \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \lambda_T\Bigg[u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} + v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y} + u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y} + v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} \\
& + 2uv\frac{\partial^2 T}{\partial x\partial y} + u^2\frac{\partial^2 T}{\partial x^2} + v^2\frac{\partial^2 T}{\partial y^2}\Bigg] \\
= & \frac{T_\infty(\theta_w - 1)\theta'(\zeta)}{(1-\lambda t)}\left(\frac{\zeta}{2} - ax^{n-1}\left(\frac{n+1}{2}f(\zeta)\right)\right) + \lambda_T\left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2 T_\infty(\theta_w - 1) \\
& \Bigg[2\left(\frac{n-1}{2}\right) n f'^2(\zeta)\zeta\theta'(\zeta) + n\left(\frac{n+1}{2}\right) f(\zeta)f'(\zeta)\theta'(\zeta) \\
& - \left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\theta'(\zeta) - 3\left(\frac{n^2-1}{4}\right) f(\zeta)f'(\zeta)\theta'(\zeta) - 3\left(\frac{n-1}{2}\right)^2 \zeta\theta'(\zeta) \\
& - \left(\frac{n-1}{2}\right)\theta'(\zeta)\zeta f'^2(\zeta) + \left(\frac{n+1}{2}\right)^2 f^2(\zeta)\theta''(\zeta)\Bigg].
\end{aligned} \tag{4.19}
$$

The conversion of the Right-hand side of (4.3) into dimensionless form is already discussed in chapter 3. from (3.43)

$$\begin{aligned}
&\alpha_{nf}\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho_{np}c_p}\frac{\partial}{\partial y}\bigg(-\frac{16\sigma^*}{3k^*}T^3\frac{\partial T}{\partial y}\bigg)+\frac{\upsilon_{nf}}{c_p}\bigg(\frac{\partial u}{\partial y}\bigg)^2\\
&+\frac{\sigma B_0^2}{\rho_{nf}c_p}u^2+\frac{\upsilon_{nf}}{k_pc_p}u^2+\tau\bigg[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\bigg(\frac{\partial T}{\partial y}\bigg)^2\bigg]=\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}T_\infty(\theta_w-1)\\
&\alpha_{nf}\bigg(\theta''(\zeta)+\frac{16\sigma^*T_\infty^3}{k_{nf}3k^*}\bigg(3\left(1+(\theta_w-1)\theta(\zeta)\right)^2(\theta_w-1)\theta'^2(\zeta)\\
&+\left(1+(\theta_w-1)\theta(\zeta)\right)^3\theta''(\zeta)\bigg)\bigg)\frac{\sigma B_0^2}{\rho_{nf}c_p}\bigg(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\bigg)^2+\frac{\upsilon_{nf}}{k_pc_p}\bigg(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\bigg)^2\\
&\tau\bigg(D_BT_\infty(\theta_w-1)\theta'(\zeta)\big(C_w-C_\infty\big)\phi'(\zeta)\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}\\
&+\frac{D_T}{T_\infty}\bigg(T_\infty(\theta_w-1)\theta'(\zeta)\bigg)^2\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}\bigg).
\end{aligned}\tag{4.20}$$

Puting the (4.19) and (4.20) in (4.3), we get

$$\begin{aligned}
&\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+\lambda_T\bigg[u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x}+v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y}+u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y}+v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x}\\
&+2uv\frac{\partial^2 T}{\partial x\partial y}+u^2\frac{\partial^2 T}{\partial x^2}+v^2\frac{\partial^2 T}{\partial y^2}\bigg]=\alpha_{nf}\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho_{nf}c_p}\frac{\partial}{\partial y}\bigg(-\frac{16\sigma^*}{3k^*}T^3\frac{\partial T}{\partial y}\bigg)\\
&+\frac{\upsilon_{nf}}{c_p}\bigg(\frac{\partial u}{\partial y}\bigg)^2+\frac{\sigma B_0^2}{\rho_{nf}c_p}u^2+\frac{\upsilon_{nf}}{k_pc_p}u^2+\tau\bigg[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\bigg(\frac{\partial T}{\partial y}\bigg)^2\bigg].\\
\Rightarrow\ &\frac{T_\infty(\theta_w-1)\theta'(\zeta)}{(1-\lambda t)}\bigg(\frac{\zeta}{2}-ax^{n-1}\bigg(\frac{n+1}{2}f(\zeta)\bigg)\bigg)+\lambda_T\bigg(\frac{ax^{n-1}}{(1-\lambda t)}\bigg)^2\\
&T_\infty(\theta_w-1)\bigg[(n-1)\,nf'^2(\zeta)\zeta\theta'(\zeta)+n\left(\frac{n+1}{2}\right)f(\zeta)f'(\zeta)\theta'(\zeta)\\
&-\left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\theta'(\zeta)-3\left(\frac{n^2-1}{4}\right)f(\zeta)f'(\zeta)\theta'(\zeta)\\
&-3\left(\frac{n-1}{2}\right)^2\zeta\theta'(\zeta)f'^2(\zeta)-\left(\frac{n-1}{2}\right)\theta'(\zeta)\zeta f'^2(\zeta)\\
&+\left(\frac{n+1}{2}\right)^2f^2(\zeta)\theta''(\zeta)\bigg]=\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}T_\infty(\theta_w-1)\alpha_{nf}\\
&\quad\bigg(\theta''(\zeta)+\frac{16\sigma^*T_\infty^3}{k_{nf}3k^*}\bigg(3\left(1+(\theta_w-1)\theta(\zeta)\right)^2(\theta_w-1)\theta'^2\\
&\quad(\zeta)+\left(1+(\theta_w-1)\theta(\zeta)\right)^3\theta''(\zeta)\bigg)\bigg)\frac{\sigma B_0^2}{\rho_{nf}c_p}\bigg(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\bigg)^2+\frac{\upsilon_{nf}}{k_pc_p}
\end{aligned}$$

$$\left(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\right)^2\tau\Bigg(D_BT_\infty(\theta_w-1)\theta'(\zeta)\big(C_w-C_\infty\big)\phi'(\zeta)\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}\Bigg). \quad (4.21)$$

Multiplying $\left(\frac{(1-\lambda t)v_{nf}}{ax^{n-1}T_\infty(\theta_w-1)\alpha_{nf}}\right)$ on both sides of the (4.21), we get

$$\begin{aligned}
&\frac{v_{nf}}{\alpha_{nf}}\theta'(\zeta)\Bigg(\frac{\zeta}{2}\frac{\lambda}{ax^{n-1}}-\frac{n+1}{2}f(\zeta)\Bigg)+\lambda_T\Bigg(\frac{ax^{n-1}}{(1-\lambda t)}\Bigg)\frac{v_{nf}}{\alpha_{nf}}\Bigg[(n-1)\,nf'^2(\zeta)\\
&\zeta\theta'(\zeta)+n\left(\frac{n+1}{2}\right)f(\zeta)f'(\zeta)\theta'(\zeta)-\left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\theta'(\zeta)\\
&-3\left(\frac{n^2-1}{4}\right)f(\zeta)f'(\zeta)\theta'(\zeta)-3\left(\frac{n-1}{2}\right)^2\zeta\theta'(\zeta)f'^2(\zeta)\\
&-\left(\frac{n-1}{2}\right)\theta'(\zeta)\zeta f'^2(\zeta)+\left(\frac{n+1}{2}\right)^2f^2(\zeta)\theta''(\zeta)\Bigg]=\theta''(\zeta)\\
&+R_d\Big(3\left(1+(\theta_w-1)\theta(\zeta)\right)^2(\theta_w-1)\theta'^2(\zeta)+\left(1+(\theta_w-1)\theta(\zeta)\right)^3\theta''(\zeta)\Big)\\
&+\frac{v_{nf}}{\alpha_{nf}}\left(\frac{ax^n}{(1-\lambda t)}\right)^2\frac{1}{(T_f-T_\infty)c_p}f''^2(\zeta)+\frac{x\sigma(1-\lambda t)}{\rho_{nf}ax^n}B_0^2\\
&\frac{v_{nf}}{\alpha_{nf}}\left(\frac{ax^n}{(1-\lambda t)}\right)^2\frac{1}{(T_f-T_\infty)c_p}f'^2(\zeta)+\frac{\tau D_B(C_W-C_\infty)}{v_{nf}}\frac{v_{nf}}{\alpha_{nf}}\phi'(\zeta)\theta'(\zeta)\\
&+\frac{\tau D_T(T_f-T_\infty)}{T_\infty v_{nf}}\frac{v_{nf}}{\alpha_{nf}}\theta'^2(\zeta)+\frac{v_{nf}x(1-\lambda t)}{k_pax^n}\frac{v_{nf}}{\alpha_{nf}}\left(\frac{ax^n}{(1-\lambda t)}\right)^2\frac{f'^2(\zeta)}{(T_f-T_\infty)c_p}.\\
\Rightarrow\quad &Pr\Bigg(\frac{\zeta}{2}A-\frac{n+1}{2}f(\zeta)\Bigg)\theta'(\zeta)+\lambda_T\Bigg(\frac{ax^{n-1}}{(1-\lambda t)}\Bigg)Pr\Bigg[(n-1)\,nf'^2(\zeta)\zeta\theta'(\zeta)\\
&+n\left(\frac{n+1}{2}\right)f(\zeta)f'(\zeta)\theta'(\zeta)-\left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\theta'(\zeta)-3\left(\frac{n^2-1}{4}\right)f(\zeta)\\
&f'(\zeta)\theta'(\zeta)-3\left(\frac{n-1}{2}\right)^2\zeta\theta'(\zeta)f'^2(\zeta)-\left(\frac{n-1}{2}\right)\theta'(\zeta)\zeta f'^2(\zeta)\\
&+\left(\frac{n+1}{2}\right)^2f^2(\zeta)\theta''(\zeta)\Bigg]=\Bigg[\bigg(1+R_d\big(1+(\theta_w-1)\theta(\zeta)\big)^3\bigg)\theta'(\zeta)\Bigg]'\\
&+Pr\frac{u_w^2}{(T_f-T_\infty)c_p}f''^2(\zeta)+\frac{\sigma x}{\rho_{nf}u_w}B_0^2Pr\frac{u_w^2}{(T_f-T_\infty)c_p}f'^2(\zeta)\\
&+Nb\phi'(\zeta)\theta'(\zeta)Pr+Nt\theta'^2(\zeta)Pr+\frac{v_{nf}x}{k_pu_w}Pr\frac{u_w^2}{(T_f-T_\infty)c_p}f'^2(\zeta).\\
\Rightarrow\quad &\Bigg[\bigg(1+R_d\big(1+(\theta_w-1)\theta(\zeta)\big)^3\bigg)\theta'(\zeta)\Bigg]'-Pr\lambda_T\Bigg(\frac{ax^{n-1}}{(1-\lambda t)}\Bigg)\left(\frac{n+1}{2}\right)^2f^2(\zeta)\\
&\theta''(\zeta)+Pr\Bigg(-\frac{\zeta}{2}A+\frac{n+1}{2}f(\zeta)\Bigg)\theta'(\zeta)+Pr.Ecf''^2(\zeta)-Pr\lambda_T\Bigg(\frac{ax^{n-1}}{(1-\lambda t)}\Bigg)\\
&\Bigg[(n-1)\,nf'^2(\zeta)\zeta\theta'(\zeta)+n\left(\frac{n+1}{2}\right)f(\zeta)f'(\zeta)\theta'(\zeta)-\left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\theta'(\zeta)
\end{aligned}$$

$$- 3\left(\frac{n^2-1}{4}\right) f(\zeta)f'(\zeta)\theta'(\zeta) - 3\left(\frac{n-1}{2}\right)^2 \zeta\theta'(\zeta)f'^2(\zeta)$$
$$- \left(\frac{n-1}{2}\right)\theta'(\zeta)\zeta f'^2(\zeta)\Bigg] = 0.$$

The dimensionless form of (4.3) has been given below

$$\begin{aligned}
&\left[\left(1 + R_d\big(1 + (\theta_w - 1)\theta(\zeta)\big)^3\right)\theta'(\zeta)\right]' - PrL_t\left(\frac{n+1}{2}\right)^2 f^2(\zeta)\theta''(\zeta) \\
&- PrL_t\Bigg[(n-1)\, nf'^2(\zeta)\zeta\theta'(\zeta) + n\left(\frac{n+1}{2}\right) f(\zeta)f'(\zeta)\theta'(\zeta) \\
&- \left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\theta'(\zeta) - 3\left(\frac{n^2-1}{4}\right) f(\zeta)f'(\zeta)\theta'(\zeta) \\
&- 3\left(\frac{n-1}{2}\right)^2 \zeta\theta'(\zeta)f'^2(\zeta) - \left(\frac{n-1}{2}\right)\theta'(\zeta)\zeta f'^2(\zeta)\Bigg] \\
&+ Pr\left(\frac{n+1}{2}f - \frac{\zeta}{2}A + Nb\phi' + Nt\theta'\right)\theta' \\
&+ Br\left[f''^2(\zeta) + \Big(M + K\Big)f'^2(\zeta)\right] = 0.
\end{aligned} \tag{4.22}$$

Now we include below the procedure for the conversion of (4.4) into dimensionless form. The already worked out derivatives (3.36)-(3.40) and (3.45)-(3.50) in Chapter 3, will be directly used here

- $$\begin{aligned}
u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x} &= \left(\frac{ax^n}{1-\lambda t}\right)^2 \frac{1}{x}\left(nf'^2(\zeta) + (\frac{n-1}{2})'(\zeta)\right)(C_w - C_\infty)\phi'(\zeta)\left(\frac{n-1}{2}\right)\frac{\zeta}{x} \\
&= \left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2 (C_w - C_\infty)\phi'(\zeta)\Bigg(\left(\frac{n-1}{2}\right) nf'^2(\zeta)\zeta \\
&\quad + \zeta^2\left(\frac{n-1}{2}\right)^2 f'(\zeta)f''(\zeta)\Bigg).
\end{aligned} \tag{4.23}$$
- $$\begin{aligned}
v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y} &= -\frac{ax^{n-1}}{(1-\lambda t)}\left(nf'(\zeta) + \left(\frac{n-1}{2}\right)\zeta f''(\zeta)\right)\Bigg[-\frac{ax^{n-1}}{(1-\lambda t)}\left(\frac{n+1}{2}f(\zeta)\right. \\
&\quad \left. + \left(\frac{n-1}{2}\right)\zeta f'(\zeta)\right)(C_w - C_\infty)\phi'(\zeta)\Bigg] \\
&= \left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2 (C_w - C_\infty)\phi'(\zeta)\Bigg[n\left(\frac{n+1}{2}\right) f(\zeta)f'(\zeta) + n\left(\frac{n-1}{2}\right) \\
&\quad \zeta f'^2(\zeta) + \left(\frac{n^2-1}{4}\right)\zeta f f''(\zeta) + \left(\frac{n-1}{2}\right)^2 \zeta^2 f''(\zeta)f'(\zeta)\Bigg].
\end{aligned} \tag{4.24}$$
- $$\frac{\partial C}{\partial y}\frac{\partial v}{\partial x} = \frac{ax^n}{(1-\lambda t)v_{nf}} f'(\zeta)(C_w - C_\infty)\phi'(\zeta)\sqrt{\frac{ax^{n-1}}{(1-\lambda t)v_{nf}}}$$

$$\Big(-\sqrt{\frac{ax^{n-1}v_{nf}}{(1-\lambda t)}}\Big[\left(\frac{n^2-1}{4}\right)\frac{\zeta}{x}f'(\zeta) + \left(\frac{n-1}{2}\right)^2\frac{\zeta^2}{x}f''(\zeta) + \left(\frac{n-1}{2}\right)^2$$

$$\frac{\zeta}{x}f'(\zeta)\Big] - \Big[\frac{n+1}{2}f(\zeta) + \left(\frac{n-1}{2}\right)\zeta f'(\zeta)\Big]\sqrt{\frac{ax^{n-1}v_{nf}}{(1-\lambda t)}}\left(\frac{n-1}{2}\right)\frac{1}{x}\Big)$$

$$= -\left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2 (C_w - C_\infty)\phi'(\zeta)\Big[\left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)$$

$$+ \left(\frac{n-1}{2}\right)^2 \zeta^2 f'(\zeta)f''(\zeta) + f'^2(\zeta)\left(\frac{n-1}{2}\right)^2\zeta$$

$$+ \left(\frac{n^2-1}{4}\right) f(\zeta)f'(\zeta) + \left(\frac{n-1}{2}\right)^2(\zeta)f'^2(\zeta)\Big]. \tag{4.25}$$

- $$v\frac{\partial u}{\partial y}\frac{\partial C}{\partial x} = -\left(\frac{ax^n}{(1-\lambda t)}\right)^2\frac{1}{x}\Big(\frac{n+1}{2}f(\zeta)f''(\zeta)$$

$$+ \frac{n-1}{2}\zeta f'(\zeta)f''(\zeta)\Big)(C_w - C_\infty)\phi'(\zeta)\left(\frac{n-1}{2}\right)\frac{\zeta}{x}$$

$$= -\left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2 (C_w - C_\infty)\phi'(\zeta)\Big[\left(\frac{n^2-1}{4}\right)\zeta f(\zeta)f''(\zeta)$$

$$+ \left(\frac{n-1}{2}\right)^2\zeta^2 f''(\zeta)f'(\zeta)\Big]. \tag{4.26}$$

- $$\frac{\partial^2 C}{\partial x\partial y} = \frac{\partial}{\partial x}(C_w - C_\infty)\phi'(\zeta)\sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}}$$

$$= \sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}}(C_w - C_\infty)\phi''(\zeta)\left(\frac{n-1}{2}\right)\frac{1}{x}$$

$$+ \sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}}(C_w - C_\infty)\phi'(\zeta)\left(\frac{n-1}{2}\right)\frac{1}{x}$$

$$= \sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}}(C_w - C_\infty)\left(\frac{n-1}{2}\right)\frac{1}{x}\Big[\zeta\phi''(\zeta) + \phi'(\zeta)\Big]. \tag{4.27}$$

- $$2uv\frac{\partial^2 C}{\partial x\partial y} = 2\left(\frac{ax^n}{(1-\lambda t)}f'(\zeta)\right)\Big(-\sqrt{\frac{ax^n v_{nf}}{(1-\lambda t)x}}\Big(\frac{n+1}{2}f(\zeta)$$

$$+ \left(\frac{n-1}{2}\right)\zeta f'(\zeta)\Big)\sqrt{\frac{ax^{n-1}}{v_{nf}(1-\lambda t)}}(C_w - C_\infty)$$

$$\left(\frac{n-1}{2}\right)\frac{1}{x}\Big[\zeta\phi''(\zeta) + \phi'(\zeta)\Big]$$

$$= -2\left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2 (C_w - C_\infty)\Big[\left(\frac{n^2-1}{4}\right) f'(\zeta)f(\zeta)\zeta\phi''(\zeta)$$

$$+ \left(\frac{n^2-1}{4}\right) f'(\zeta)f(\zeta)\phi'(\zeta) + \left(\frac{n-1}{2}\right)^2\zeta^2\phi''(\zeta)f'^2(\zeta)$$

$$+ \left(\frac{n-1}{2}\right)^2\zeta f'^2(\zeta)\phi'(\zeta)\Big]. \tag{4.28}$$

- $$\frac{\partial^2 C}{\partial x^2} = \frac{\partial}{\partial x}\left((C_w - C_\infty)\phi'(\zeta)\left(\frac{n-1}{2}\right)\frac{\zeta}{x}\right)$$
$$= (C_w - C_\infty)\frac{\partial}{\partial x}\left(\phi'(\zeta)\left(\frac{n-1}{2}\right)\frac{\zeta}{x}\right)$$
$$= (C_w - C_\infty)\left(\phi''(\zeta)\zeta^2\left(\frac{n-1}{2}\right)^2\frac{1}{x^2}\right.$$
$$\left. - \phi'(\zeta)\zeta\frac{1}{x^2}\left(\frac{n-1}{2}\right) + \phi'(\zeta)\zeta\frac{1}{x^2}\left(\frac{n-1}{2}\right)^2\right. . \tag{4.29}$$

- $$u^2\frac{\partial^2 C}{\partial x^2} = \frac{ax^{2n}}{(1-\lambda t)^2}f'^2(\zeta)(C_w - C_\infty)\left(\phi''(\zeta)\zeta^2\left(\frac{n-1}{2}\right)^2\frac{1}{x^2}\right.$$
$$\left. - \phi'(\zeta)\zeta\frac{1}{x^2}\left(\frac{n-1}{2}\right) + \phi'(\zeta)\zeta\frac{1}{x^2}\left(\frac{n-1}{2}\right)^2\right)$$
$$= \left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2(C_w - C_\infty)\left[\phi''(\zeta)\left(\frac{n-1}{2}\right)^2\zeta^2 f'^2\right.$$
$$\left. - \left(\frac{n-1}{2}\right)\phi'(\zeta)\zeta f'^2(\zeta) + \phi'(\zeta)\zeta\left(\frac{n-1}{2}\right)^2 f'^2(\zeta)\right]. \tag{4.30}$$

- $$v^2\frac{\partial^2 C}{\partial y^2} = \left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2(C_w - C_\infty)\left[\left(\frac{n+1}{2}\right)^2 f^2(\zeta)\phi''(\zeta)\right.$$
$$\left. + \phi''(\zeta)\zeta^2\left(\frac{n-1}{2}\right)^2 f'^2(\zeta) + 2\left(\frac{n^2-1}{4}\right)\zeta f'(\zeta)f(\zeta)\theta''(\zeta)\right]. \tag{4.31}$$

Adding (4.23), (4.24), (4.25), (4.26), (4.28), (4.30), and (4.31), we get

$$\left[u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x} + v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y} + u\frac{\partial v}{\partial x}\frac{\partial C}{\partial y} + v\frac{\partial u}{\partial y}\frac{\partial C}{\partial x} + 2uv\frac{\partial^2 C}{\partial x\partial y} + u^2\frac{\partial^2 C}{\partial x^2} + v^2\frac{\partial^2 C}{\partial y^2}\right]$$
$$= \left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2(C_w - C_\infty)\left[\left(\frac{n-1}{2}\right)nf'^2(\zeta)\zeta\phi'(\zeta)\right.$$
$$+ \zeta^2\left(\frac{n-1}{2}\right)^2 f'(\zeta)f''(\zeta)\phi'(\zeta)$$
$$+ n\left(\frac{n+1}{2}\right)f(\zeta)f'(\zeta)\phi'(\zeta) + n\left(\frac{n-1}{2}\right)\zeta f'^2(\zeta)\phi'(\zeta) + \left(\frac{n^2-1}{4}\right)\zeta$$
$$f(\zeta)f''(\zeta)\phi'(\zeta) + \left(\frac{n-1}{2}\right)^2\zeta^2 f''(\zeta)f'(\zeta)\phi'(\zeta) - \left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\phi'(\zeta)$$
$$- \left(\frac{n-1}{2}\right)^2\zeta^2 f'(\zeta)f''(\zeta)\phi'(\zeta) - \left(\frac{n-1}{2}\right)^2 f'^2(\zeta)\zeta\phi'(\zeta)$$
$$- \left(\frac{n^2-1}{4}\right)f(\zeta)f'(\zeta)\phi'(\zeta) - \left(\frac{n-1}{2}\right)^2\zeta f'^2(\zeta)\phi'(\zeta)$$
$$- \left(\frac{n^2-1}{4}\right)\zeta f(\zeta)f''(\zeta)\phi'(\zeta) - \phi'(\zeta)\left(\frac{n-1}{2}\right)^2\zeta^2 f''(\zeta)f'(\zeta)$$

$$
\begin{aligned}
&-2\left(\frac{n^2-1}{4}\right)f'(\zeta)f(\zeta)\zeta\phi''(\zeta)-2\left(\frac{n^2-1}{4}\right)f'(\zeta)f(\zeta)\phi'(\zeta)\\
&-2\left(\frac{n-1}{2}\right)^2\zeta^2\phi''(\zeta)f'^2(\zeta)-2\left(\frac{n-1}{2}\right)^2\zeta\phi'(\zeta)f'^2(\zeta)\\
&+\phi''(\zeta)\left(\frac{n-1}{2}\right)^2\zeta^2f'^2(\zeta)-\left(\frac{n-1}{2}\right)\phi'(\zeta)\zeta f'^2(\zeta)\\
&+\phi'(\zeta)\zeta\left(\frac{n-1}{2}\right)^2f'^2(\zeta)+\left(\frac{n+1}{2}\right)^2f^2(\zeta)\phi''(\zeta)+2\left(\frac{n^2-1}{4}\right)\zeta\\
&f'(\zeta)f(\zeta)\phi''(\zeta)+\phi''(\zeta)\zeta^2\left(\frac{n-1}{2}\right)^2f'^2(\zeta)\Bigg]\\
&=\left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2(C_w-C_\infty)\Bigg[(n-1)\,nf'^2(\zeta)\zeta\phi'(\zeta)\\
&+n\left(\frac{n+1}{2}\right)f(\zeta)f'(\zeta)\phi'(\zeta)-\left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\phi'(\zeta)\\
&-3\left(\frac{n^2-1}{4}\right)f(\zeta)f'(\zeta)\phi'(\zeta)-3\left(\frac{n-1}{2}\right)^2\zeta\phi'(\zeta)f'^2(\zeta)\\
&-\left(\frac{n-1}{2}\right)\phi'(\zeta)\zeta f'^2(\zeta)+\left(\frac{n+1}{2}\right)^2f^2(\zeta)\phi''(\zeta)\Bigg].
\end{aligned}
\tag{4.32}
$$

To convert the right side of (4.4), we need only the following conversion and the rest of the conversions already discussed in chapter 3.

- $$
\begin{aligned}
&-k_r^2(C-C_\infty)\left[\frac{T}{T_\infty}\right]^m\exp\left[\frac{-E_a}{kT_\infty}\right]\\
&=-k_r^2(C-C_\infty)\left[\frac{T_\infty\big(1+(\theta_w-1)\theta(\zeta)\big)}{T_\infty}\right]^m\exp\left[\frac{-E_a}{kT_\infty\big(1+(\theta_w-1)\theta(\zeta)\big)}\right]\\
&=-k_r^2(C-C_\infty)\Big[\big(1+(\theta_w-1)\theta(\zeta)\big)\Big]^m\exp\left[\frac{-E_a}{kT_\infty\big(1+(\theta_w-1)\theta(\zeta)\big)}\right].
\end{aligned}
\tag{4.33}
$$

Using the (3.50) and (4.32) in the left side of (4.4), we get

$$
\begin{aligned}
&\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+\lambda_C\Bigg[u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x}+v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y}+u\frac{\partial v}{\partial x}\frac{\partial C}{\partial y}+v\frac{\partial u}{\partial y}\frac{\partial C}{\partial x}+2uv\frac{\partial^2 C}{\partial x\partial y}\\
&+u^2\frac{\partial^2 C}{\partial x^2}+v^2\frac{\partial^2 C}{\partial y^2}\Bigg]=\frac{(C_w-C_\infty)\phi'(\zeta)}{(1-\lambda t)}\left(\frac{\zeta}{2}-ax^{n-1}\left(\frac{n+1}{2}f(\zeta)\right)\right)\\
&+\lambda_c\left(\frac{ax^{n-1}}{(1-\lambda t)}\right)^2(C_w-C_\infty)\Bigg[2\left(\frac{n-1}{2}\right)nf'^2(\zeta)\zeta\phi'(\zeta)\\
&+n\left(\frac{n+1}{2}\right)f(\zeta)f'(\zeta)\phi'(\zeta)-\left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\phi'(\zeta)
\end{aligned}
$$

$$- 3\left(\frac{n^2-1}{4}\right) f(\zeta)f'(\zeta)\phi'(\zeta) - 3\left(\frac{n-1}{2}\right)^2 \zeta\phi'(\zeta)f'^2(\zeta)$$
$$- \left(\frac{n-1}{2}\right) \phi'(\zeta)\zeta f'^2(\zeta) + \left(\frac{n+1}{2}\right)^2 f^2(\zeta)\phi''(\zeta)\Bigg]. \tag{4.34}$$

Using the (3.51) and (4.33) in the right side of (4.4), we get

$$D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\Big(\frac{\partial^2 T}{\partial y^2}\Big) - k_r^2(C - C_\infty)\left[\frac{T}{T_\infty}\right]^m \exp\left[\frac{-E_a}{kT_\infty}\right]$$
$$= \frac{ax^{n-1}}{v_{nf}(1-\lambda t)}\bigg(D_B\big(C_w - C_\infty\big)\phi''(\zeta) + \frac{D_T}{T_\infty}T_\infty(\theta_w - 1)\theta''(\zeta)\bigg)$$
$$- k_r^2(C - C_\infty)\Big[\big(1 + (\theta_w - 1)\theta(\zeta)\big)\Big]^m$$
$$\exp\left[\frac{-E_a}{kT_\infty\big(1 + (\theta_w - 1)\theta(\zeta)\big)}\right]. \tag{4.35}$$

Using the equations (4.34) and (4.35) in equation (4.4), we have

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + \lambda_C\Bigg[u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x} + v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y} + u\frac{\partial v}{\partial x}\frac{\partial C}{\partial y} + v\frac{\partial u}{\partial y}\frac{\partial C}{\partial x}$$
$$+ 2uv\frac{\partial^2 C}{\partial x\partial y} + u^2\frac{\partial^2 C}{\partial x^2} + v^2\frac{\partial^2 C}{\partial y^2}\Bigg] = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\Big(\frac{\partial^2 T}{\partial y^2}\Big)$$
$$- k_r^2(C - C_\infty)\left[\frac{T}{T_\infty}\right]^m \exp\left[\frac{-E_a}{kT_\infty}\right].$$
$$\Rightarrow \quad \frac{(C_w - C_\infty)\phi'(\zeta)}{(1-\lambda t)}\bigg(\frac{\zeta}{2} - ax^{n-1}\Big(\frac{n+1}{2}f(\zeta)\Big)\bigg) + \lambda_c\bigg(\frac{ax^{n-1}}{(1-\lambda t)}\bigg)^2(C_w - C_\infty)$$
$$\Bigg[2\left(\frac{n-1}{2}\right) nf'^2(\zeta)\zeta\phi'(\zeta) + n\left(\frac{n+1}{2}\right) f(\zeta)f'(\zeta)\phi'(\zeta)$$
$$- \left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\phi'(\zeta) - 3\left(\frac{n^2-1}{4}\right) f(\zeta)f'(\zeta)\phi'(\zeta)$$
$$- 3\left(\frac{n-1}{2}\right)^2 \zeta\phi'(\zeta)f'^2(\zeta) - \left(\frac{n-1}{2}\right)\phi'(\zeta)\zeta f'^2(\zeta)$$
$$+ \left(\frac{n+1}{2}\right)^2 f^2(\zeta)\phi''(\zeta)\Bigg]$$
$$= \frac{ax^{n-1}}{v_{nf}(1-\lambda t)}\bigg(D_B\big(C_w - C_\infty\big)\phi''(\zeta) + \frac{D_T}{T_\infty}T_\infty(\theta_w - 1)\theta''(\zeta)\bigg) - k_r^2\phi(\zeta)$$
$$(C_w - C_\infty)\Big[\big(1 + (\theta_w - 1)\theta(\zeta)\big)\Big]^m \exp\left[\frac{-E_a}{kT_\infty\big(1 + (\theta_w - 1)\theta(\zeta)\big)}\right]. \tag{4.36}$$

Multiplying by $\left(\frac{(1-\lambda t)v_{nf}}{ax^{n-1}D_B\left(C_w-C_\infty\right)}\right)$ on both sides of (4.36), we obtain

$$
\begin{aligned}
&\frac{\lambda}{ax^{n-1}}\frac{v_{nf}}{D_B}\frac{\zeta}{2}\phi'(\zeta)-\frac{v_{nf}}{D_B}\left(\frac{n+1}{2}\right)f(\zeta)\phi'(\zeta)+\lambda_c\left(\frac{ax^{n-1}}{(1-\lambda t)}\right)\frac{v_{nf}}{D_B}\\
&\Bigg[2\left(\frac{n-1}{2}\right)nf'^2(\zeta)\zeta\phi'(\zeta)+n\left(\frac{n+1}{2}\right)f(\zeta)f'(\zeta)\phi'(\zeta)\\
&-\left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\phi'(\zeta)-3\left(\frac{n^2-1}{4}\right)f(\zeta)f'(\zeta)\phi'(\zeta)\\
&-3\left(\frac{n-1}{2}\right)^2\zeta\phi'(\zeta)f'^2(\zeta)-\left(\frac{n-1}{2}\right)\phi'(\zeta)\zeta f'^2(\zeta)+\left(\frac{n+1}{2}\right)^2f^2(\zeta)\phi''(\zeta)\Bigg]\\
&=\phi''(\zeta)+\frac{D_T(T_f-T_\infty)}{T_\infty D_B\big(C_w-C_\infty\big)}\theta''(\zeta)-k_r^2\phi(\zeta)\frac{v_{nf}}{D_B}\frac{(1-\lambda)}{ax^{n-1}}\Big[\big(1+(\theta_w-1)\theta(\zeta)\big)\Big]^m\\
&\quad\exp\left[\frac{-E_a}{kT_\infty\big(1+(\theta_w-1)\theta(\zeta)\big)}\right].\\
\Rightarrow\quad &Sc\frac{\zeta}{2}\phi'(\zeta)A-Sc\left(\frac{n+1}{2}\right)f(\zeta)\phi'(\zeta)+\lambda_c\left(\frac{ax^{n-1}}{(1-\lambda t)}\right))Sc\\
&\quad\Bigg[2\left(\frac{n-1}{2}\right)nf'^2(\zeta)\zeta\phi'(\zeta)+n\left(\frac{n+1}{2}\right)f(\zeta)f'(\zeta)\phi'(\zeta)\\
&\quad-\left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\phi'(\zeta)-3\left(\frac{n^2-1}{4}\right)f(\zeta)f'(\zeta)\phi'(\zeta)-\left(\frac{n-1}{2}\right)\phi'(\zeta)\zeta\\
&f'^2(\zeta)-3\left(\frac{n-1}{2}\right)^2\zeta\phi'(\zeta)f'^2(\zeta)+\left(\frac{n+1}{2}\right)^2f^2(\zeta)\phi''(\zeta)\Bigg]\\
&=\phi''(\zeta)+\frac{Nt(v_{nf})\tau}{Nb(v_{nf})\tau}\theta''(\zeta)-Sc\ \gamma_1\phi(\zeta)\big(1+(\theta_w-1)\theta(\zeta)\big)^m\\
&\quad\exp\left[\frac{-E}{\big(1+(\theta_w-1)\theta(\zeta)\big)}\right].\\
\Rightarrow\quad &\phi''(\zeta)-L_cSc\left(\frac{n+1}{2}\right)^2f^2(\zeta)\phi''(\zeta)+\frac{Nt}{Nb}\theta''(\zeta)+Sc\left(\frac{n+1}{2}f-\frac{A}{2}\zeta\right)\phi'(\zeta)\\
&-L_cSc\Bigg[2\left(\frac{n-1}{2}\right)nf'^2(\zeta)\zeta\phi'(\zeta)+n\left(\frac{n+1}{2}\right)f(\zeta)f'(\zeta)\phi'(\zeta)\\
&-\left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\phi'(\zeta)-3\left(\frac{n^2-1}{4}\right)f(\zeta)f'(\zeta)\phi'(\zeta)\\
&-3\left(\frac{n-1}{2}\right)^2\zeta\phi'(\zeta)f'^2(\zeta)-\left(\frac{n-1}{2}\right)\phi'(\zeta)\zeta f'^2(\zeta)\Bigg]\\
&-Sc\ \gamma_1\ \phi(\zeta)\big(1+(\theta_w-1)\theta(\zeta)\big)^m\\
&\exp\left[\frac{-E}{\big(1+(\theta_w-1)\theta(\zeta)\big)}\right]=0. \qquad (4.37)
\end{aligned}
$$

The final dimensionless form of the governing model are

$$
\begin{aligned}
& f'''(\zeta) + \frac{n+1}{2} f(\zeta) f''(\zeta) - n f'^2(\zeta) - (M+K)\, f'(\zeta) \\
& - A\left(f'(\zeta) + \frac{\zeta}{2} f''(\zeta)\right) = 0, \qquad (4.38) \\
& \left[\left(1 + R_d\big(1 + (\theta_w - 1)\theta(\zeta)\big)^3\right)\theta'(\zeta)\right]' - PrL_t\left(\frac{n+1}{2}\right)^2 f^2(\zeta)\theta''(\zeta) \\
& - PrL_t\Bigg[(n-1)\, n f'^2(\zeta)\zeta\theta'(\zeta) + n\left(\frac{n+1}{2}\right) f(\zeta) f'(\zeta)\theta'(\zeta) \\
& - \left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\theta'(\zeta) - 3\left(\frac{n^2-1}{4}\right) f(\zeta) f'(\zeta)\theta'(\zeta) \\
& - 3\left(\frac{n-1}{2}\right)^2 \zeta\theta'(\zeta) f'^2(\zeta) - \left(\frac{n-1}{2}\right)\theta'(\zeta)\zeta f'^2(\zeta)\Bigg] \\
& + Pr\left(\frac{n+1}{2} f - \frac{\zeta}{2} A + Nb\phi' + Nt\theta'\right)\theta' + Br\left[f''^2 + \Big(M+K\Big) f'^2\right] = 0, \qquad (4.39) \\
& \phi''(\zeta) - L_c Sc\left(\frac{n+1}{2}\right)^2 f^2(\zeta)\phi''(\zeta) + \frac{Nt}{Nb}\theta'' + Sc\left(\frac{n+1}{2} f - \frac{A}{2}\zeta\right)\phi' \\
& - L_c Sc\Bigg[2\left(\frac{n-1}{2}\right) n f'^2(\zeta)\zeta\phi'(\zeta) + n\left(\frac{n+1}{2}\right) f(\zeta) f'(\zeta)\phi'(\zeta) \\
& - \left(\frac{n^2-1}{4}\right)\zeta f'^2(\zeta)\phi'(\zeta) - 3\left(\frac{n^2-1}{4}\right) f(\zeta) f'(\zeta)\phi'(\zeta) \\
& - 3\left(\frac{n-1}{2}\right)^2 \zeta\phi'(\zeta) f'^2(\zeta) - \left(\frac{n-1}{2}\right)\phi'(\zeta)\zeta f'^2(\zeta)\Bigg] \\
& - Sc\,\gamma_1\,\phi(\zeta)\big(1 + (\theta_w - 1)\theta(\zeta)\big)^m \exp\left[\frac{-E}{\big(1 + (\theta_w - 1)\theta(\zeta)\big)}\right] = 0. \qquad (4.40)
\end{aligned}
$$

The associated BCs (4.5) in the dimensionless form are:

$$
\left.
\begin{aligned}
& f' = 1 + \gamma f'', \quad f(\zeta) = 0 \\
& \theta' = -Bi(1 - \theta(\zeta)), \quad \phi(\zeta) = 1 \;\; at\; \zeta = 0 \\
& f' \to 0, \quad \theta(\zeta) \to 0, \quad \phi(\zeta) \to 0 \;\; as\;\; \zeta \to \infty
\end{aligned}
\right\} \qquad (4.41)
$$

Most of the following parameters have been taken into account from (3.59). The following list of dimensionless parameter involved four new parameters which are arisen due Cattaneo-Christov Double Diffusions model and Activation energy that we have used for the extension of Seth et al. [43] Different dimensionless parameters used in (4.38)-(4.40). E represent the activation energy parameter, γ_1 denote chemical reaction parameter, L_t dimensionless thermal relaxation parameter and L_c dimensionless mass relaxation.

$$\left.\begin{aligned}
&M = \frac{\sigma x}{\rho_{nf} u_w} B_0, \qquad K = \frac{\upsilon_{nf} x}{k_p u_w}, \qquad A = \frac{\lambda}{a x^{n-1}},\\
&Pr = \frac{\upsilon_{nf}}{\alpha_{nf}}, \quad R_d = \frac{16\sigma^* T_\infty^*}{3k^* k_{nf}}, \qquad Br = Pr.Ec,\\
&Nt = \frac{\tau D_T}{T_\infty}\frac{(T_f - T_\infty)}{\upsilon_{nf}}, \qquad Nb = \frac{\tau D_B (C_w - C_\infty)}{\upsilon_{nf}},\\
&Ec = \frac{u_w^2}{c_p (T_f - T_\infty)}, \qquad Bi = \frac{h_f}{k_{nf}}\sqrt{\frac{\upsilon_{nf} x}{u_w}}, \qquad \gamma_1 = k_r^2 \frac{(1-\lambda t)}{a x^{n-1}},\\
&E = \frac{E_a}{k T_\infty}, \qquad L_c = \lambda_c \left(\frac{a x^{n-1}}{(1-\lambda t)}\right) = \lambda_c \left(\frac{u_w}{x}\right),\\
&L_t = \lambda_T \left(\frac{a x^{n-1}}{(1-\lambda t)}\right) = \lambda_T \left(\frac{u_w}{x}\right).
\end{aligned}\right\}$$

4.3 Transformation of Physical Quantities

$$Cf_x = \frac{\tau_w}{\rho_{nf} u_w^2 (x,t)}, \tag{4.42}$$

$$Nu_x = \frac{x q_w}{k_{nf} (T_f - T_\infty)}, \tag{4.43}$$

$$Sh_x = \frac{x q_m}{D_B (C_w - C_\infty)}. \tag{4.44}$$

The complete discussion to get the expression for the dimensionless form of Skin Friction Coefficient, Nusselt Number and Sherwood Number corresponding to (4.42)-(4.44) have been discussed in chapter 3. The dimensionless form of physical quantities are given below

$$\left.\begin{aligned}
&Cf_x Re_x^{\frac{1}{2}} = f''(0),\\
&Nu_x Re_x^{\frac{-1}{2}} = -\theta'(0)\Big(1 + R_d \big(1 + (\theta_w - 1)\theta(0)\big)^3\Big),\\
&Sh_x Re_x^{\frac{-1}{2}} = -\phi'(0).
\end{aligned}\right\}$$

4.4 Entropy Generation Formulations

The rate of entropy generation E_G for Cattaneo Christov Double Diffusion Model and activation energy has been discussed briefly in [59, 60]. The dimensional form of E_G is given below:

$$E_G = \frac{k_{nf}}{T_{\infty 2}}\left[1+\frac{16\sigma^* T_\infty^3}{3k^* k_{nf}}\right]\left(\frac{\partial T}{\partial y}\right)^2 + \frac{RD}{C_\infty}\left(\frac{\partial C}{\partial y}\right)^2 + \frac{RD}{T_\infty}\left(\frac{\partial C}{\partial y}\right)\left(\frac{\partial T}{\partial y}\right) + \frac{\mu_{nf}}{T_\infty}\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\mu_{nf}}{k_{np}T_\infty}+\frac{\sigma B_0^2}{T_\infty}\right)u^2 \tag{4.45}$$

Using (3.28), (3.37), and (3.38) in (4.45), we get

$$\begin{aligned} E_G =& \frac{k_{nf}}{T_\infty^2}\left[1+\frac{16\sigma^* T_\infty^3}{3k^* k_{nf}}\right]\left(T_\infty(\theta_w-1)\theta'(\zeta)\sqrt{\frac{ax^{n-1}}{\upsilon_{nf}(1-\lambda t)}}\right)^2 \\ &+ \frac{RD}{C_\infty}\left((C_w-C_\infty)\phi'(\zeta)\sqrt{\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}}\right)^2 + \frac{RD}{T_\infty}\Bigg(T_\infty(\theta_w-1)\theta'(\zeta) \\ &(C_w-C_\infty)\phi'(\zeta)\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}\Bigg) + \frac{\mu_{nf}}{T_\infty}\left(\frac{ax^n}{(1-\lambda t)}f''(\zeta)\sqrt{\frac{ax^{n-1}}{(1-\lambda t)\upsilon_{nf}}}\right)^2 \\ &+ \frac{(ax^n)^2}{(1-\lambda t)^2}f'^2(\zeta)\left(\frac{\mu_{nf}}{k_{np}T_\infty}+\frac{\sigma B_0^2}{T_\infty}\right) \\ =& \frac{k_{nf}}{T_\infty^2}\Big(1+R_d\Big)T_\infty^2\left(\frac{T_f-T_\infty}{T_\infty}\right)^2\left(\frac{\zeta}{y}\right)^2\theta'^2 + \frac{RD}{C_\infty}((C_w-C_\infty)^2\left(\frac{\zeta}{y}\right)^2\phi'^2 \\ &+ \frac{RD}{T_\infty}T_\infty\left(\frac{T_f-T_\infty}{T_\infty}\right)((C_w-C_\infty)\left(\frac{\zeta}{y}\right)^2\phi'\theta' + \frac{\rho_{nf}\upsilon_{nf}}{T_\infty}u_w^2\left(\frac{\zeta}{y}\right)^2 f''^2 \\ &+ \frac{u_w^2}{T_\infty}f'^2\left(\frac{\rho_{nf}\upsilon_{nf}}{k_p}+\sigma B_0^2\right). \end{aligned} \tag{4.46}$$

Multiplying both sides of (4.46) by $\frac{T_\infty^2\left(\frac{\zeta}{y}\right)^2}{k_{nf}(T_f-T_\infty)^2}$ we get

$$\begin{aligned} E_G\left[\frac{T_\infty^2\left(\frac{\zeta}{y}\right)^2}{k_{nf}(T_f-T_\infty)^2}\right] =& \Big(1+R_d\Big)\theta'^2 + \frac{RD}{k_{nf}}(C_w-C_\infty)\frac{(C_w-C_\infty)}{C_\infty}\left(\frac{T_f}{T_f-T_\infty}\right)^2\phi'^2 \\ &+ \frac{RD}{k_{nf}}(C_w-C_\infty)\left(\frac{T_\infty}{T_f-T_\infty}\right)\phi'\theta' + \frac{\upsilon_{nf}}{\alpha_{nf}}\frac{u_w^2}{c_p(T_f-T_\infty)}\left(\frac{T_\infty}{T_f-T_\infty}\right)f''^2 \\ &+ \frac{\upsilon_{nf}}{\alpha_{nf}}\frac{u_w^2}{c_p(T_f-T_\infty)}\left(\frac{T_\infty}{T_f-T_\infty}\left(\frac{(1-\lambda t)x\upsilon_{nf}}{k_pax^n}+\frac{\sigma B_0^2(1-\lambda t)x}{ax^n\rho_{nf}}\right)f'^2\right. \end{aligned}$$

After simplifying the above expression, we get the dimensionless form

$$Ns = \big(1 + R_d\big)\theta'^2 + L\left[\frac{\epsilon}{\Omega^2}\phi'^2 + \frac{1}{\Omega}\theta'\phi'\right] + \frac{Br}{\Omega}\left[f''^2 + \big(M + K\big)\right]. \tag{4.47}$$

Bejan number Be is defined as

$$Be = \frac{N_{HT}}{N_s} = \frac{N_{HT}}{N_{HT} + N_{MT} + N_{PM}}. \tag{4.48}$$

In the above equations the dimensionless quantities are formulated as:

$$\left.\begin{aligned} &N_s = \frac{T_\infty^2\left(\frac{\zeta}{y}\right)^2}{k_{nf}(T_f - T_\infty)^2},\ N_{HT} = \big(1 + R_d\big)\theta'^2, \Omega = \frac{T_f - T_\infty}{T_\infty},\\ &N_{MT} = \lambda\left[\frac{\epsilon}{\Omega}^2\phi'^2 + \frac{1}{\Omega}\theta'\phi'\right], N_{PM} = \frac{Br}{\Omega}\left[f''^2 + \big(M + K\big)\right],\\ &\epsilon = \frac{C_w - C_\infty}{C_\infty}, \lambda = \frac{D_B}{k_{nf}}((C_w - C_\infty), L = \frac{RD}{k_{nf}}(C_w - C_\infty) \end{aligned}\right\} \tag{4.49}$$

4.5 Solution Methodology

The system of nonlinear ODEs (4.39), (4.40) along with the boundary conditions (4.41) will be modified into the first order ODEs. The shooting method will be utilized to solve the first-order system of ODEs with the boundary conditions (4.41). Equation (4.39), (4.40) can be written in the following form.

$$\begin{aligned} \theta'' = &\frac{1}{-1 - 3R_d\big(1 + (\theta_w - 1)\theta\big)^3 + PrL_t\left(\frac{n+1}{2}\right)^2 f^2}\Big[3R_d\big(1 + (\theta_w - 1)\theta\big)^2(\theta_w - 1)\theta'^2\\ &- PrL_t\Bigg[(n-1)\,nf'^2\zeta\theta' + n\left(\frac{n+1}{2}\right)ff'\theta' - \left(\frac{n^2-1}{4}\right)\zeta f'^2\theta'\\ &- 3\left(\frac{n^2-1}{4}\right)ff'\theta' - 3\left(\frac{n-1}{2}\right)^2\zeta\theta' f'^2 - \left(\frac{n-1}{2}\right)\theta'\zeta f'^2\Bigg]\\ &+ Pr\left(Nb\phi' + Nt\theta' + \frac{n+1}{2}f - \frac{\zeta}{2}A\right)\theta' + Br(f'' + f'^2(M + K))\Big], \end{aligned} \tag{4.50}$$

$$\begin{aligned} \phi'' = &\frac{1}{-1 + ScL_c\left(\frac{n+1}{2}\right)^2 f^2}\Bigg[Sc\left(\frac{n+1}{2}f - \frac{A}{2}\zeta\right)\phi' - L_cSc\Bigg((n-1)\,nf'^2\zeta\phi'\\ &+ n\left(\frac{n+1}{2}\right)ff'\phi' - \left(\frac{n^2-1}{4}\right)\zeta f'^2\phi' - 3\left(\frac{n^2-1}{4}\right)ff'\phi' \end{aligned}$$

$$- 3\left(\frac{n-1}{2}\right)^2 \zeta\phi' f'^2 - \left(\frac{n-1}{2}\right)\phi'\zeta f'^2\right) + \frac{Nt}{Nb}\theta''$$
$$- Sc\,\gamma_1\,\phi\big(1+(\theta_w-1)\theta\big)^m \exp\left(\frac{-E}{\big(1+(\theta_w-1)\theta\big)}\right)\Bigg]. \tag{4.51}$$

Now, the coupled equations (4.50) and (4.51) will be treated similarly by considering f, f' and f'' as a known functions. The initial missing condition $\theta(0)$ and $\phi(0)$ can be represented by r and w respectively. The following notations have been further considered.

$$\left.\begin{aligned} &\theta = z_1,\ \theta' = z_1' = z_2,\ \theta'' = z_2'\ \phi = z_3,\ \phi' = z_3' = z_4,\\ &\phi'' = z_4',\ z_5 = \frac{\partial\theta}{\partial r},\ z_6 = \frac{\partial\theta'}{\partial r},\ z_7 = \frac{\partial\phi}{\partial r},\ z_8 = \frac{\partial\phi'}{\partial r}\\ &z_9 = \frac{\partial\theta}{\partial w},\ z_{10} = \frac{\partial\theta'}{\partial w},\ z_{11} = \frac{\partial\phi}{\partial w},\ z_{12} = \frac{\partial\phi'}{\partial w}. \end{aligned}\right\} \tag{4.52}$$

By using the above notations in equations (4.50)-(4.51), the following ODEs are obtained:

$$z_1' = z_2, \qquad z_1 = r.$$
$$z_2' = \frac{1}{-1-3R_d\big(1+(\theta_w-1)z_1\big)^3 + PrL_t\left(\frac{n+1}{2}\right)^2 f^2}\Bigg(3R_d\big(1+(\theta_w-1)z_1\big)^2$$
$$(\theta_w-1)z_2^2 - PrL_t\Bigg[(n-1)\,nf'^2\zeta z_2 + n\left(\frac{n+1}{2}\right)ff'z_2$$
$$-\left(\frac{n^2-1}{4}\right)\zeta f'^2 z_2 - 3\left(\frac{n^2-1}{4}\right)ff'z_2 - 3\left(\frac{n-1}{2}\right)^2\zeta z_2 f'^2$$
$$-\left(\frac{n-1}{2}\right)z_2\zeta f'^2\Bigg] + Pr\left(Nbz_4 + Ntz_2 + \frac{n+1}{2}f - \frac{\zeta}{2}A\right)z_2$$
$$+ Br\Big(f'' + f'^2(M+K)\Big)\Bigg), \qquad z_2(0) = -Bi(1-r).$$
$$z_3' = z_4, \qquad z_3(0) = 1.$$
$$z_4' = \frac{1}{-1+ScL_c\left(\frac{n+1}{2}\right)^2 f^2}\Bigg(Sc\left(\frac{n+1}{2}f - \frac{A}{2}\zeta\right)z_4 - L_cSc\Bigg[(n-1)\,nf'^2\zeta z_4$$
$$+ n\left(\frac{n+1}{2}\right)ff'z_4 - \left(\frac{n^2-1}{4}\right)\zeta f'^2 z_4 - 3\left(\frac{n^2-1}{4}\right)ff'z_4$$
$$- 3\left(\frac{n-1}{2}\right)^2\zeta z_4 f'^2 - \left(\frac{n-1}{2}\right)z_4\zeta f'^2\Bigg] + \frac{Nt}{Nb}z_2' - Sc\,\gamma_1\,z_3$$

$$\big(1+(\theta_w-1)z_1\big)^m \exp\left(\frac{-E}{\big(1+(\theta_w-1)z_1\big)}\right)\Bigg), \qquad z_4(0)=w.$$

$$z_5'=z_6, \qquad z_5(0)=1.$$

$$z_6'=\frac{3R_d(1+(\theta_w-1)z_1)^2(\theta_w-1)z_5}{\big(1+3R_d(1+(\theta_w-1)z_1)^3-PrL_t\left(\frac{n+1}{2}\right)^2f^2\big)^2}\Bigg[3R_d\big(1+(\theta_w-1)z_1\big)^2$$
$$(\theta_w-1)z_2^2+Pr\Bigg(\frac{n+1}{2}f-\frac{\zeta}{2}A-PrL_t\Bigg[(n-1)\,nf'^2\zeta z_2+n\left(\frac{n+1}{2}\right)ff_2'$$
$$-\left(\frac{n^2-1}{4}\right)\zeta f'^2z_2-3\left(\frac{n^2-1}{4}\right)ff'z_2-3\left(\frac{n-1}{2}\right)^2\zeta z_2f'^2$$
$$-\left(\frac{n-1}{2}\right)z_2\zeta f'^2+\Big(Nbz_4+Ntz_2\Big)z_2+Br(f''+f'^2(M+K))\Bigg]$$
$$+\frac{1}{\Big(-1-3R_d\big(1+(\theta_w-1)z_1\big)^3+PrL_t\left(\frac{n+1}{2}\right)^2f^2}\Bigg[6R_d\big(1+(\theta_w-1)z_1\big)$$
$$(\theta_w-1)^2z_5z_2^2+6R_d\big(1+(\theta_w-1)z_1\big)(\theta_w-1)^2z_6z_2+Pr\Bigg(\frac{n+1}{2}f$$
$$-\frac{\zeta}{2}A+Nbz_4+Ntz_2\Bigg)z_6+Pr\big(Nbz_8+Ntz_6\big)z_2-PrL_t\Bigg((n-1)\,nf'^2\zeta$$
$$+n\left(\frac{n+1}{2}\right)ff'-\left(\frac{n^2-1}{4}\right)\zeta f'^2-3\left(\frac{n^2-1}{4}\right)ff'-3\left(\frac{n-1}{2}\right)^2\zeta f'^2$$
$$-\left(\frac{n-1}{2}\right)\zeta f'^2\Bigg)z_6\Bigg], \qquad z_6(0)=Bi.$$

$$z_7'=z_8, \qquad z_7(0)=0.$$

$$z_8'=\frac{1}{-1+ScL_c\left(\frac{n+1}{2}\right)^2f^2}\Bigg[Sc\Bigg(\frac{n+1}{2}f-\frac{A}{2}\zeta\Bigg)z_8-L_cSc\Bigg[(n-1)\,nf'^2\zeta z_8$$
$$+n\left(\frac{n+1}{2}\right)ff'z_8-\left(\frac{n^2-1}{4}\right)\zeta f'^2z_8-3\left(\frac{n^2-1}{4}\right)ff'z_8$$
$$-3\left(\frac{n-1}{2}\right)^2\zeta z_8f'^2-\left(\frac{n-1}{2}\right)z_8\zeta f'^2\Bigg]+\frac{Nt}{Nb}z_6'$$
$$-Sc\Bigg(\gamma_1\ z_7\big(1+(\theta_w-1)z_1\big)^m\exp\left(\frac{-E}{\big(1+(\theta_w-1)z_1\big)}\right)+\gamma_1m\big(1+(\theta_w-1)z_1\big)^{m-1}$$
$$(\theta_w-1)z_5z_3\exp\left(\frac{-E}{\big(1+(\theta_w-1)z_1\big)}\right)+\gamma_1\ z_3\big(1+(\theta_w-1)z_1\big)^m$$
$$\exp\left(\frac{-E}{\big(1+(\theta_w-1)z_1\big)}\right)\left(\frac{E(\theta_w-1)z_5}{\big(1+(\theta_w-1)z_1\big)^2}\right)\Bigg)\Bigg], \qquad z_8(0)=0.$$

$$z_9'=z_{10}, \qquad z_9(0)=0.$$

$$z_{10}'=\frac{3R_d(1+(\theta_w-1)z_9)^2(\theta_w-1)z_{10}}{\big(1+3R_d(1+(\theta_w-1)z_1)^3-PrL_t\left(\frac{n+1}{2}\right)^2f^2\big)^2}\Bigg[3R_d\big(1+(\theta_w-1)z_1\big)^2$$
$$(\theta_w-1)z_2^2+Pr\Bigg(\frac{n+1}{2}f-\frac{\zeta}{2}A-PrL_t\Bigg[(n-1)\,nf'^2\zeta z_2+n\left(\frac{n+1}{2}\right)ff_2'$$

$$
\begin{aligned}
&-\left(\frac{n^2-1}{4}\right)\zeta f'^2 z_2 - 3\left(\frac{n^2-1}{4}\right) f f' z_2 - 3\left(\frac{n-1}{2}\right)^2 \zeta z_2 f'^2 \\
&-\left(\frac{n-1}{2}\right) z_2 \zeta f'^2 + \Big(Nb z_4 + Nt z_2\Big) z_2 + Br(f'' + f'^2(M+K))\Big] \\
&+ \frac{1}{\big(-1 - 3R_d\big(1+(\theta_w-1)z_1\big)^3 + PrL_t\left(\frac{n+1}{2}\right)^2 f^2} \Big[6R_d\big(1+(\theta_w-1)z_1\big) \\
&(\theta_w-1)^2 z_9 z_2^2 + 6R_d\big(1+(\theta_w-1)z_1\big)(\theta_w-1)^2 z_{10} z_2 \\
&+ Pr\Big(\frac{n+1}{2} f - \frac{\zeta}{2} A + Nb z_4 + Nt z_2\Big) z_{10} + Pr\big(Nb z_{12} + Nt z\big) z_{10} \\
&- PrL_t\Big(\,(n-1)\, n f'^2 \zeta + n\left(\frac{n+1}{2}\right) f f' - \left(\frac{n^2-1}{4}\right)\zeta f'^2 - 3\left(\frac{n^2-1}{4}\right) \\
&f f' - 3\left(\frac{n-1}{2}\right)^2 \zeta f'^2 - \left(\frac{n-1}{2}\right)\zeta f'^2\Big) z_{10}\Big], \qquad z_{10}(0) = 0.
\end{aligned}
$$

$$
z_{11}' = z_{12}, \qquad z_{11}(0) = 0.
$$

$$
\begin{aligned}
z_{12}' = &\frac{1}{-1 + ScL_c\left(\frac{n+1}{2}\right)^2 f^2}\Big[Sc\Big(\frac{n+1}{2} f - \frac{A}{2}\zeta\Big) z_{12} - L_c Sc\Big[\,(n-1)\, n f'^2 \zeta z_{12} \\
&+ n\left(\frac{n+1}{2}\right) f f' z_{12} - \left(\frac{n^2-1}{4}\right)\zeta f'^2 z_{12} - 3\left(\frac{n^2-1}{4}\right) f f' z_{12} \\
&- 3\left(\frac{n-1}{2}\right)^2 \zeta z_{12} f'^2 - \left(\frac{n-1}{2}\right) z_{12}\zeta f'^2\Big] + \frac{Nt}{Nb} z_{10}' \\
&- Sc\Big(\gamma_1\; z_{11}\big(1+(\theta_w-1)z_1\big)^m \exp\left(\frac{-E}{\big(1+(\theta_w-1)z_1\big)}\right) \\
&+ \gamma_1\; m\big(1+(\theta_w-1)z_1\big)^{m-1}(\theta_w-1) z_9 z_3 \exp\left(\frac{-E}{\big(1+(\theta_w-1)z_1\big)}\right) + \gamma_1\; z_3 \\
&\big(1+(\theta_w-1)z_1\big)^m \exp\left(\frac{-E}{\big(1+(\theta_w-1)z_1\big)}\right)\left(\frac{E(\theta_w-1)z_9}{\big(1+(\theta_w-1)z_1\big)^2}\right)\Big)\Big], \qquad z_{12}(0) = 1.
\end{aligned}
$$

To solve the above initial value problem, we use the RK4 method for which the missing conditions are chosen as:

$$
(Y_1(r,w))_{\zeta=\zeta_\infty} = 0, \qquad (Y_3(r,w))_{\zeta=\zeta_\infty} = 0. \tag{4.53}
$$

The above set of equations can be solved by using Newton's method with the following iterative formula

$$
\begin{bmatrix} r \\ w \end{bmatrix}_{k+1} = \begin{bmatrix} r \\ w \end{bmatrix}_k - \left(\begin{bmatrix} \frac{\partial z_1}{\partial r} & \frac{\partial z_1}{\partial w} \\ \frac{\partial z_3}{\partial r} & \frac{\partial z_3}{\partial w} \end{bmatrix}^{-1} \begin{bmatrix} z_1 \\ z_3 \end{bmatrix}\right)_k. \tag{4.54}
$$

$$\Rightarrow \begin{bmatrix} r \\ w \end{bmatrix}_{k+1} = \begin{bmatrix} r \\ w \end{bmatrix}_{k} - \left(\begin{bmatrix} z_5 & z_9 \\ z_7 & z_{11} \end{bmatrix}^{-1} \begin{bmatrix} z_1 \\ z_3 \end{bmatrix} \right)_k . \tag{4.55}$$

The iterative process is repeated until the following stopping criteria is met:

$$max\Big(\big|z_1(\zeta_\infty)\big|, \big|z_3(\zeta_\infty)\big| \Big) < \epsilon^*.$$

where $\epsilon^* > 0$ is a small positive real number.

4.6 Result and Discussion

The main objective of this part is to examine the effects of different dimensionless parameters on the temperature and concentration distribution of the flow.The transformed ordinary differential equations (4.50) and (4.51) along with the boundary conditions (4.41) are numerically solved by using the shooting method. By assuming different values for distinct physical parameters, the numerical solutions of Nusselt and Sherwood numbers have been presented through tables. The missing conditions $\theta(0)$ and $\phi(0)$ can be chosen from the intervals represented by I_θ and I_ϕ respectively in Table 4.1 and Table 4.2 . In Table 4.1, increasing the values of the thermophoresis parameter Nt, unsteadiness parameter A and Eckert parameter Ec would decrease the Nusselt number but when the value of Prandtl, Biot and mass relaxation parameter increase, the Nusselt number increases. From Table 4.2, it can be observed that by raising the values of the Schmidt number, the Brownian motion parameter, temperature ratio parameter, chemical reaction parameter and mass relaxation parameter, the Sherwood number increases, whereas it decreases for the rising values of the unsteadiness parameter and activation energy parameter.

Impact of Prandtl Number Pr

Figure 4.1 represents the temperature profile for different values of the Prandtl Number parameter Pr. It is simple to see that as the value of Pr rises, the temperature profile changes into a decreasing function. In Figure 4.2, by increasing

Pr, concentration distribution increases in the lower half of the surface whereas it decreases in the upper half.

Impact of Thermophoresis Parameter Nt

Figure 4.3 shows the relationship between Nt and the temperature distribution. By increasing the values of Nt, the temperature profile $\theta(\zeta)$ increases. Physically, increasing the values of Nt pulls the nanoparticles from hotter to cooler regions. As a result, the overall temperature of the nanofluid increases. Figure 4.4 represents the concentration distribution $\phi(\zeta)$ for different values of the thermophoresis parameter Nt. When Nt values rise, the concentration distribution also rises.

Influence of Eckert number Ec

Figure 4.5 shows that the temperature distribution grows as Ec values increase. When Ec values rise, the dissipation rises. The internal energy of the fluid also increases as a result of this increase in dissipation. The fluid's temperature distribution is also improved by this change in the internal energy. In Figure 4.6, by increasing Ec, the concentration profile reduces in the lower half of the surface whereas it enhances in the upper half.

Effect of Biot number parameter Bi

The Biot number Bi affects the temperature distribution as seen in Figure 4.7. When the value of the Biot number Bi escalates, the temperature distribution $\theta(\zeta)$ will also rise. Greater Biot number values result in more dynamic heat generation on the sheet as described in this trend. Thus the thickness of the thermal boundary layer increases. In Figure 4.8, by increasing the Biot number parameter, the concentration profile can be seen to be increasing.

Implications of the of thermal relaxation parameter L_t

Figure 4.9 illustrates the consequences of the thermal relaxation time parameter on the temperature profile. It is found that the temperature distribution declines as L_t values rise. Physically, less heat is transferred from the sheet to the fluid when the thermal relaxation time parameter is improved. $\theta(\zeta)$ is a decreasing function of the thermal relaxation parameter L_t. Figure 4.10 demonstrates that as the thermal relaxation parameter L_t values rise, the concentration profile rises as well.

Implications of the Schmidt Number Sc

Figure 4.11 represents the concentration distribution $\phi(\zeta)$ for different values of the Schmidt number parameter Sc. It is simple to see that as the value of Sc rises, the concentration distribution changes into a decreasing function.

Effect of Brownian Motion Parameter Nb

The effect of Brownian motion parameter Nb on the concentration profile $\phi(\zeta)$ is presented in Figure 4.8. Increasing the values of the Brownian motion parameter reduces the concentration profile.

Effect of unsteadiness parameter A

Figure 4.13 investigates the influence of the unsteadiness parameter on the concentration profile. By expanding A, the concentration distribution is found to increase.

Implications of the Activation Energy parameter E

Figure 4.14 shows how the concentration profile $\phi(\zeta)$ is affected by activation energy parameter E. The distribution of concentration rises with an increase in the activation energy value. This phenomenon is caused by higher activation energy at a given temperature, which accelerates the chemical reaction and as a result, increases the concentration profile.

Implications of the of chemical reaction rate parameter γ_1

The concentration profile is plotted in Figure 4.15 for various chemical reaction rate parameter γ_1 values. Increasing the values of chemical reaction rate parameter reduces the concentration profile. This phenomenon is noted because a greater value of γ_1 corresponds to a faster rate of chemical reaction, which slows diffusion and lowers the concentration profile.

Impact of mass relaxation parameter T_c

Figure 4.16 shows the Impact of the mass relaxation parameter T_c on concentration distribution $\phi(\zeta)$. An increase T_c in decreases the concentration profile $\phi(\zeta)$. The reason for such a decrease is that the fluid takes more time to diffuse. Higher values of mass relaxation parameter T_c correspond to a situation when the material starts behaving as a solid and a reduction in the concentration profile is observed.

Effect of different parameters on Bejan number

Figure 4.17 demonstrates that as the thermal radiation parameter R_d is raised, the Bejan number Be increases. From Figure 4.18, it is clear that when the Brinkman number Br rises, the value of the Bejan number Be decreases.

Figure 4.19 determines the effect of the Biot number on Bejan number. Enhancing the Bi, increases the Bejan number.

Figure 4.20 demonstrates the impact of the activation energy parameter E on the Bejan number parameter Be. As E is increased, the Bejan number decreases.

Figure 4.21 reflects the behaviour of Bejan number Be for various values of L. Due to an increment in L, the Bejan number is decreased. It is obvious from Figure 4.22 that the Bejan number exhibits an increasing behaviour when there is an applied magnetic field.

TABLE 4.1: Variation of the Nusselt number.
$R_d = 0.5,\ L_c = 0.1, n = 2,\ M = 0.5,\ \gamma = 0.1$

Pr	Nt	Ec	A	Bi	L_t	$-\boldsymbol{\theta'(0)}$	I_θ
1	0.2	0.1	0.1	0.2	0.1	0.99994	[1,3]
2						0.11878	[1,3]
3						0.12582	[1,3]
	0.5					0.09765	[1,5]
	0.7					0.09607	[1,3]
		0.3				0.06568	[1,8]
		0.5				0.03433	[1,3]
			0.15			0.09441	[1,8]
			0.2			0.08777	[1,3]
				0.3		0.12053	[1,3]
				0.5		0.14203	[1,5]
					0.3	0.10209	[1,3]
					0.5	0.10454	[1,3]

TABLE 4.2: Variation of the Sherwood number.
$R_d = 0.5$, $L_t = 0.1$, $n = 2$, $M = 0.5$, $\gamma = 0.1$

Sc	Nb	A	L_c	γ_1	θ_w	E	$-\phi'(0)$	I_ϕ
1	0.2	0.1	0.1	0.2	1.5	1	0.6049	[1,3]
1.5							0.8215	[1,3]
2							1.0097	[1,3]
1.5	0.3						0.8355	[1,5]
	0.5						0.8475	[1,3]
		0.15					0.8034	[1,8]
		0.2					0.7967	[1,3]
			0.3				0.8696	[1,8]
			0.5				0.9235	[1,3]
				0.3			0.8780	[1,3]
				0.5			0.9776	[1,5]
					1.0		0.7867	[1,3]
					2.0		0.8681	[1,3]
						2	0.7487	[1,3]
						3	0.7139	[1,3]

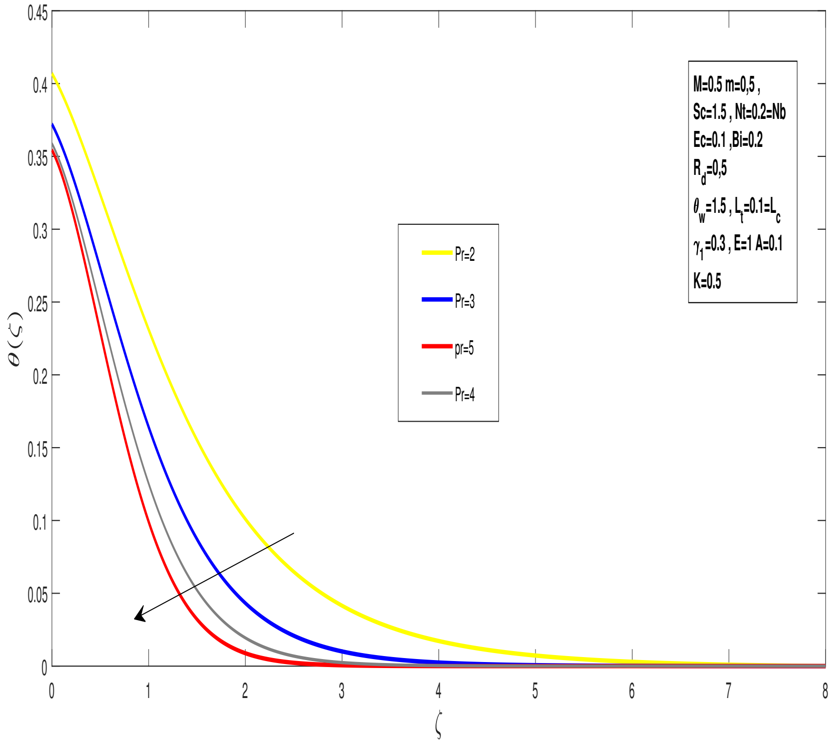

FIGURE 4.1: Impact of Pr on the $\theta(\zeta)$.

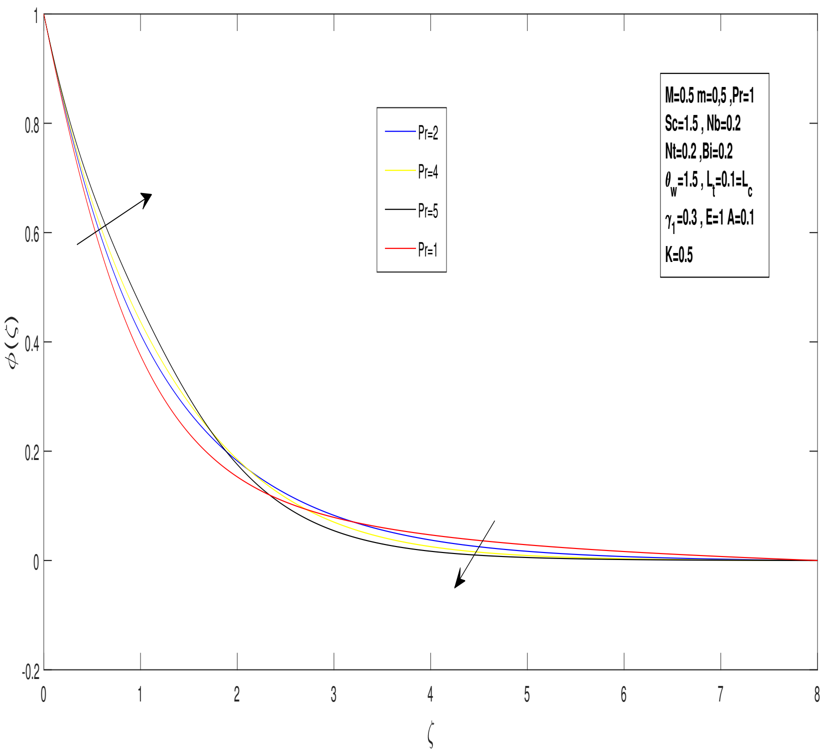

FIGURE 4.2: Effect of Pr on the $\phi(\zeta)$.

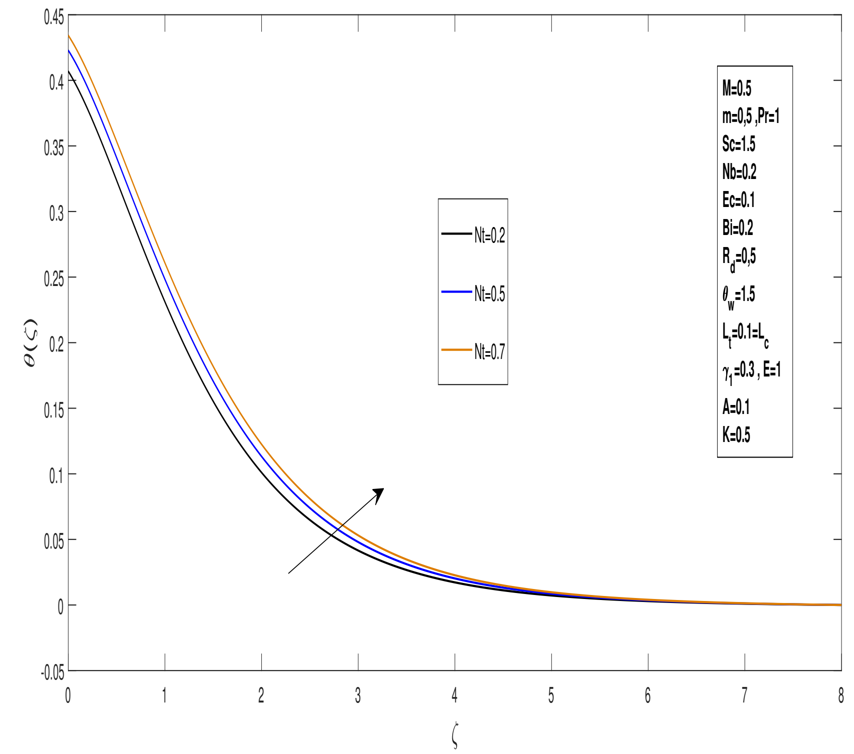

FIGURE 4.3: influence of Nt on the temperature distribution .

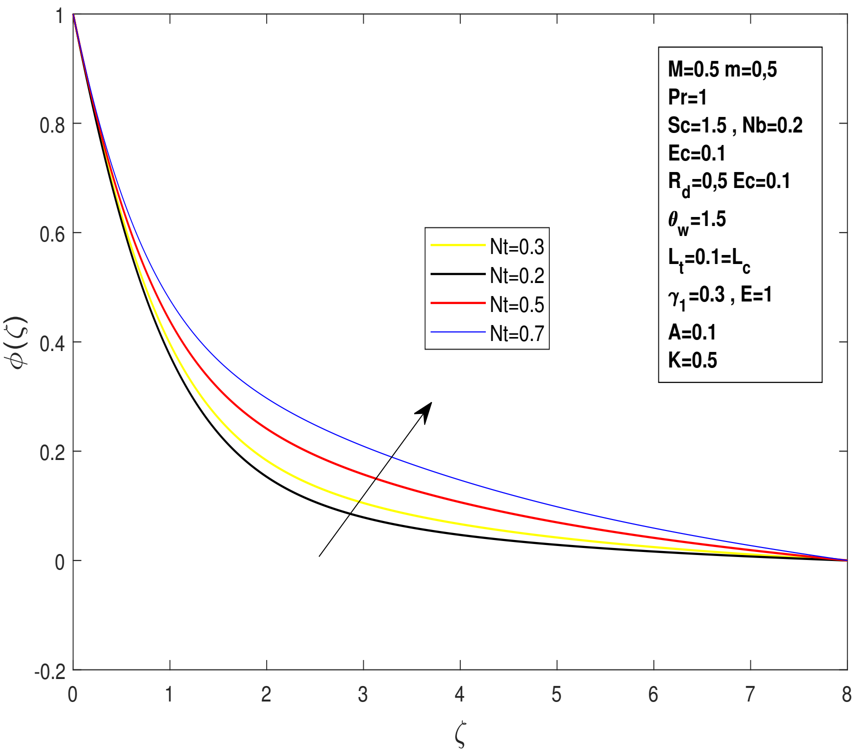

FIGURE 4.4: influence of Nt on the concentration distribution.

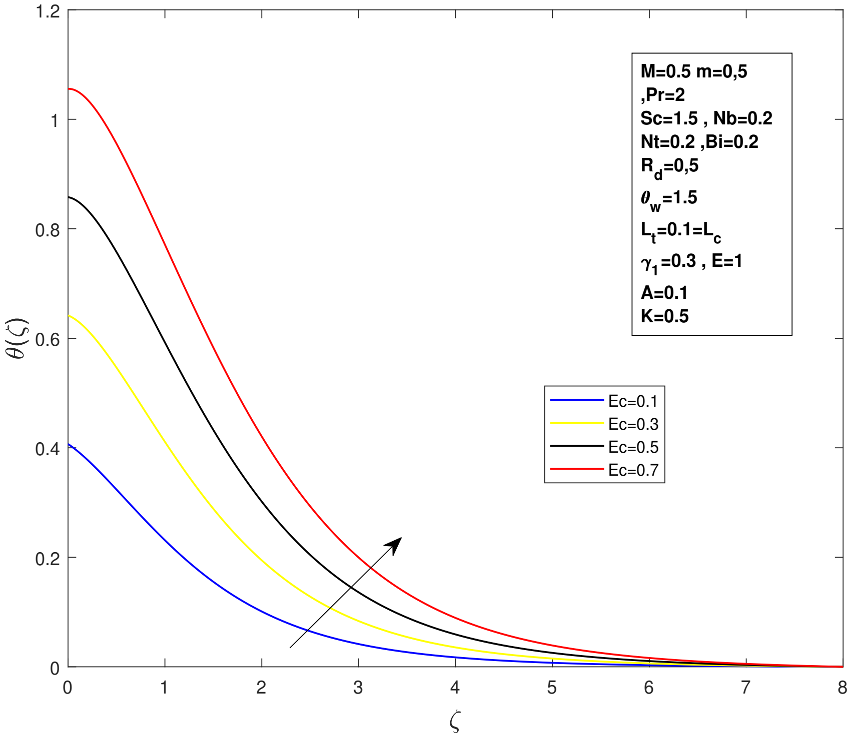

FIGURE 4.5: Effect of Ec on the temperature distribution .

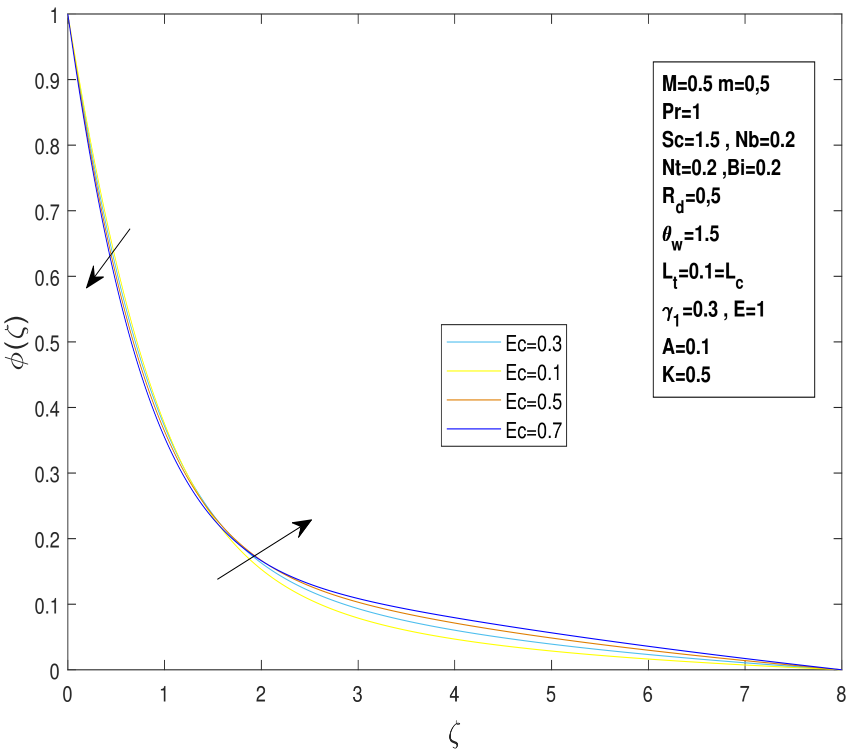

FIGURE 4.6: Impact of Ec on the concentration distribution.

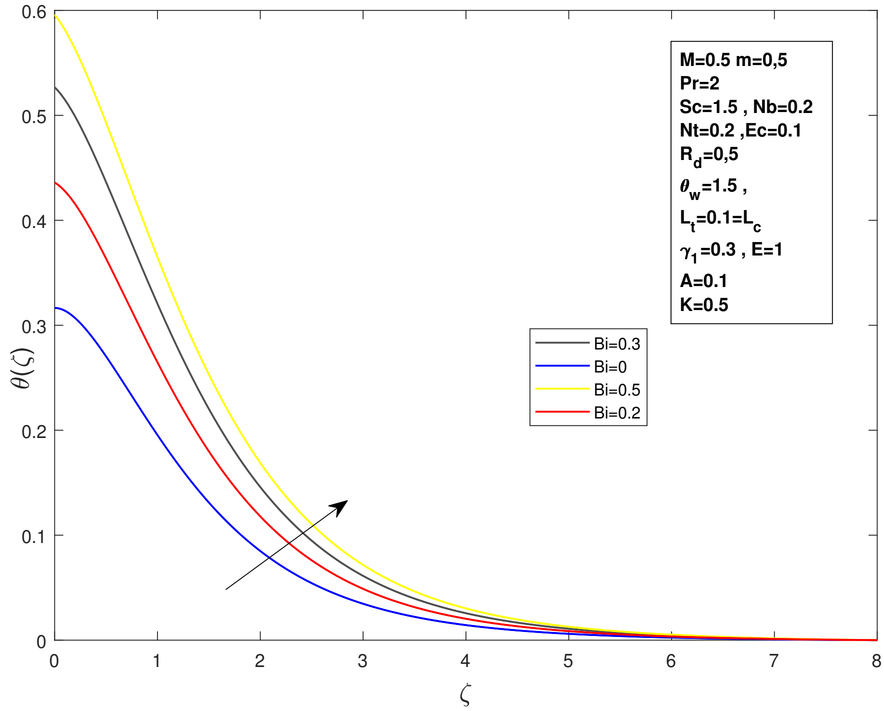

FIGURE 4.7: Impact of Bi on the $\theta(\zeta)$.

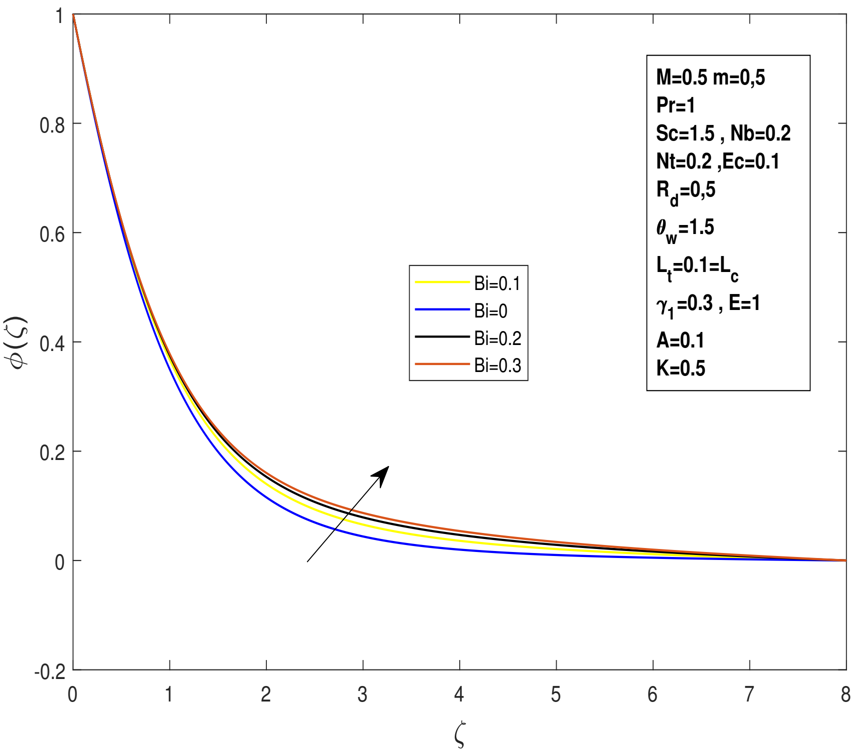

FIGURE 4.8: Impact of Bi on the $\phi(\zeta)$.

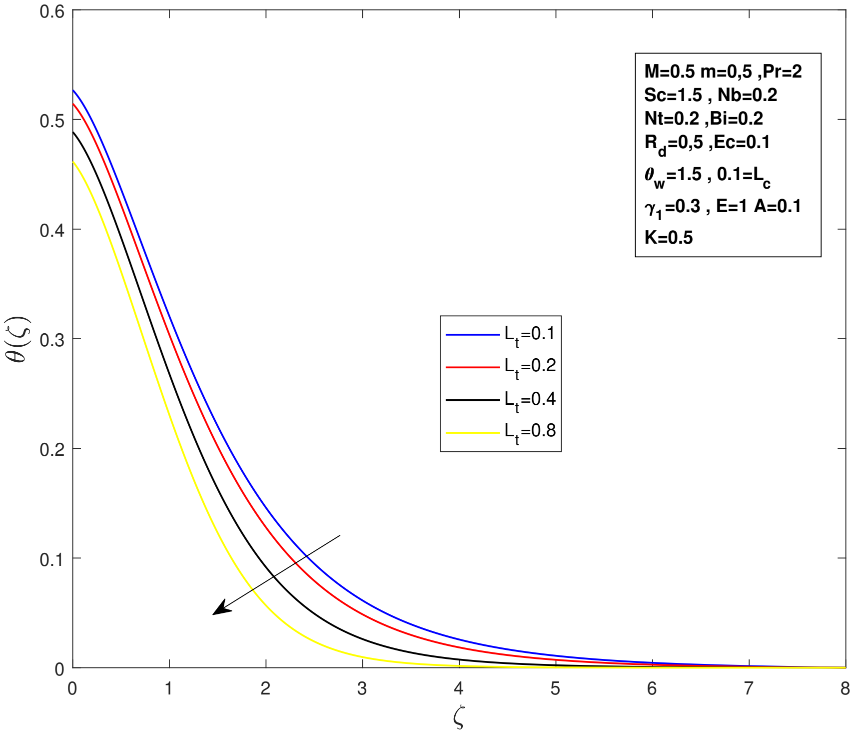

FIGURE 4.9: Effect of L_t on the temperature distribution

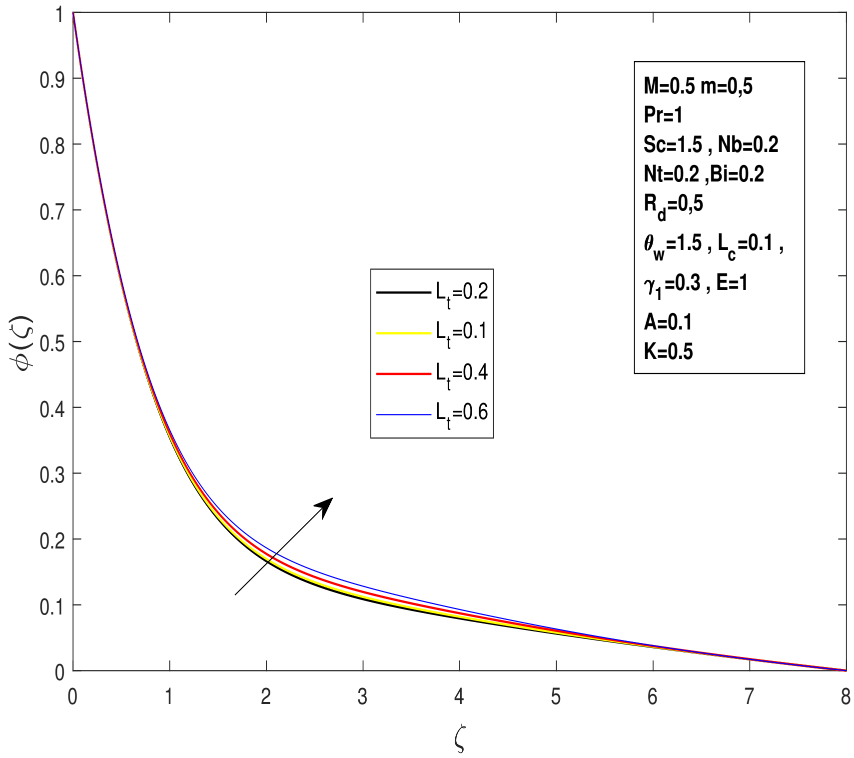

FIGURE 4.10: Impact of L_t on the concentration distribution.

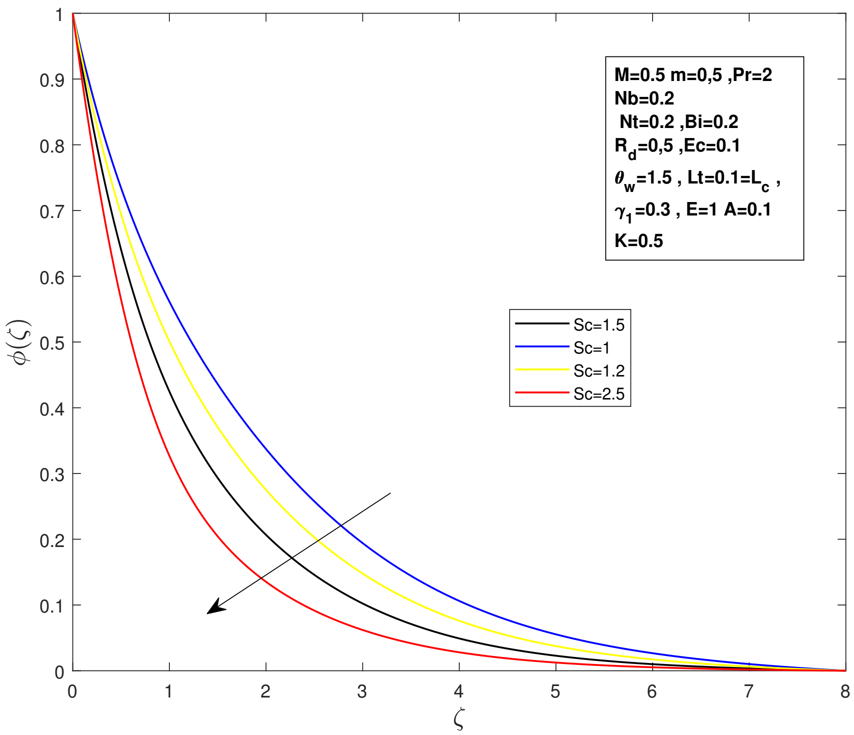

FIGURE 4.11: Impact of Sc on the $\phi(\zeta)$.

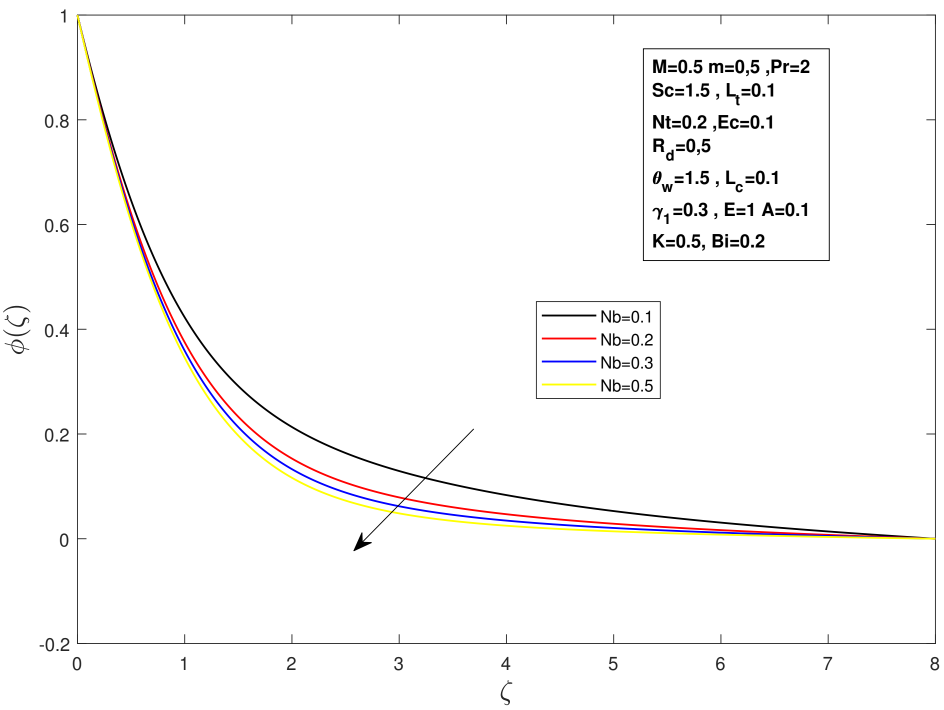

FIGURE 4.12: Impact of Nb on the $\phi(\zeta)$.

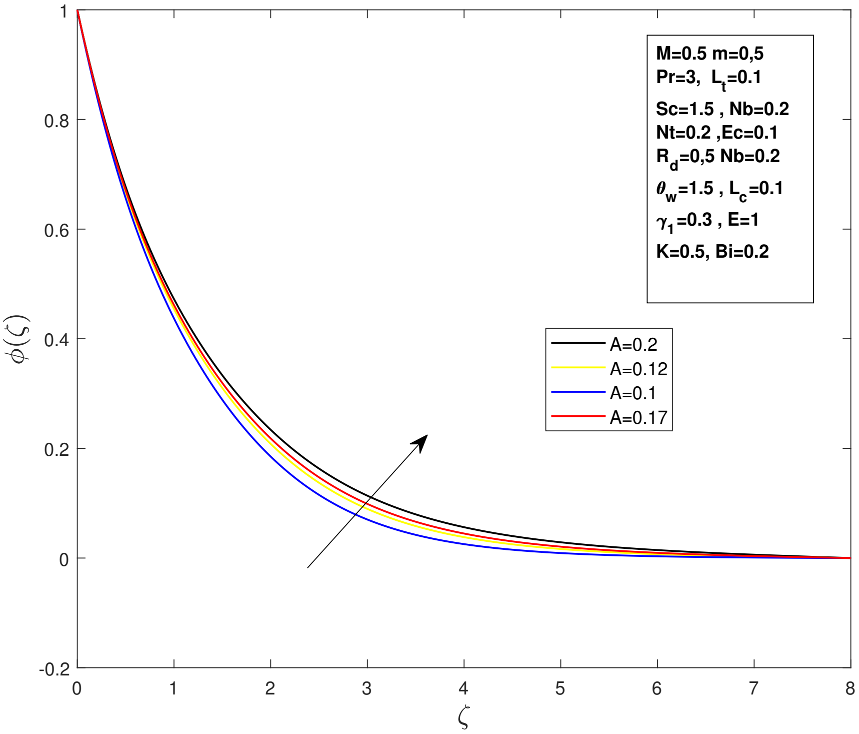

FIGURE 4.13: Impact of A on the $\phi(\zeta)$.

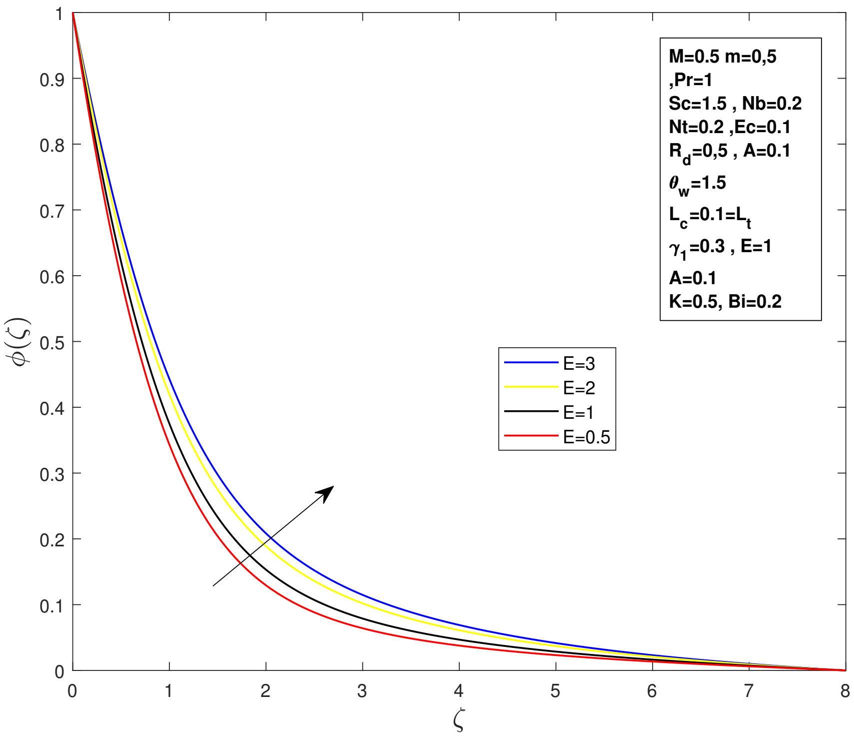

FIGURE 4.14: Impact of E on the $\phi(\zeta)$.

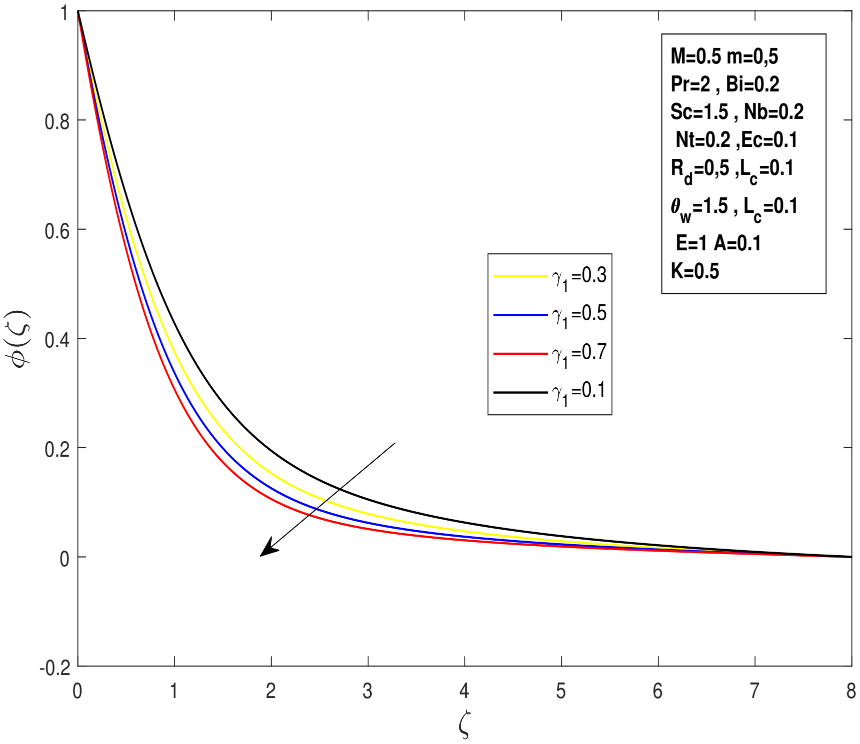

FIGURE 4.15: Effect of γ_1 on the $\phi(\zeta)$.

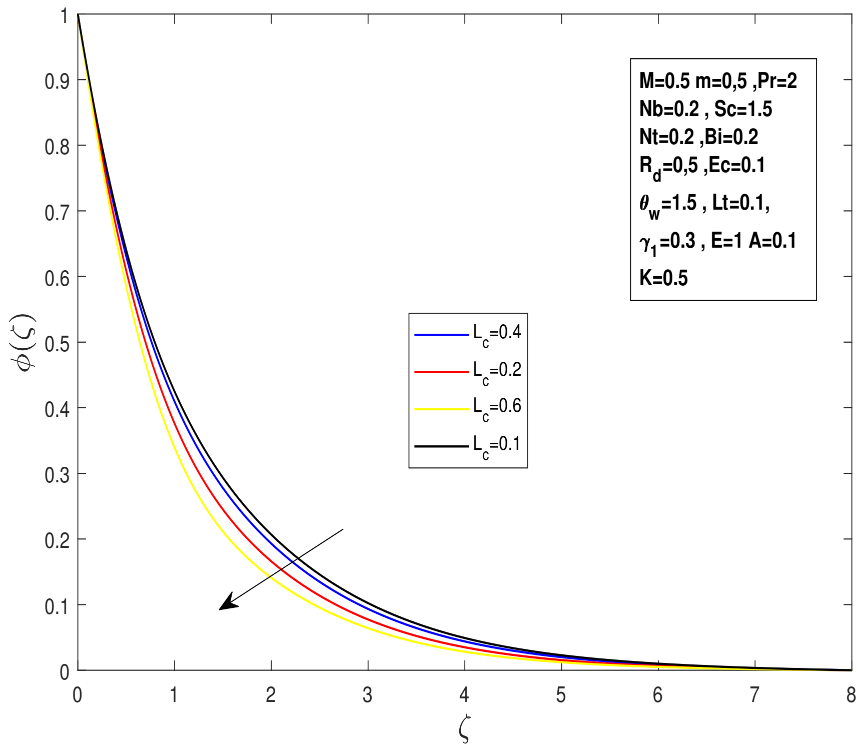

FIGURE 4.16: Impact of L_c on the $\phi(\zeta)$.

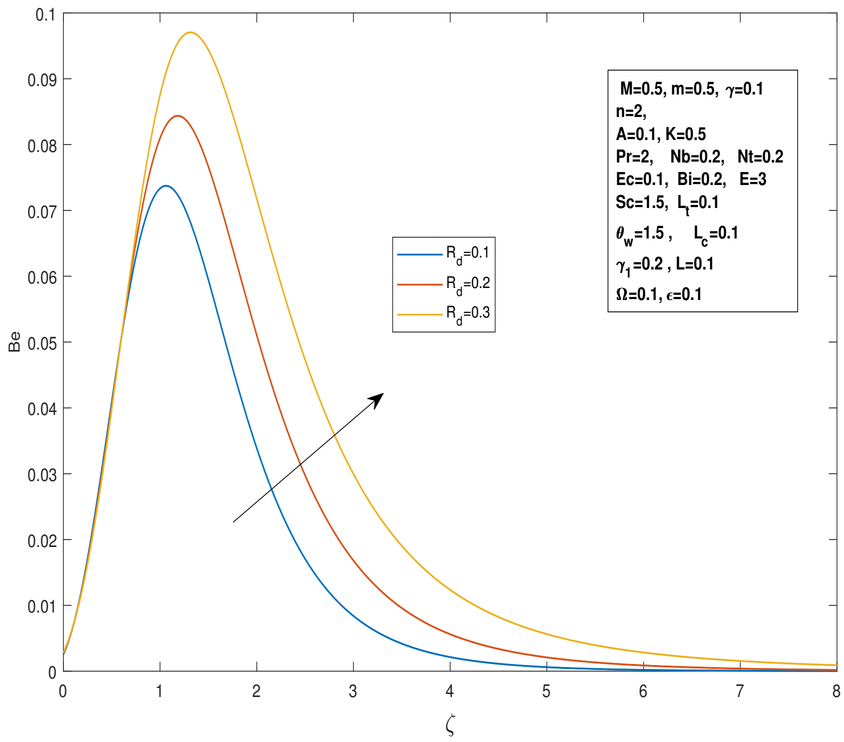

FIGURE 4.17: Impact of R_d on the Be.

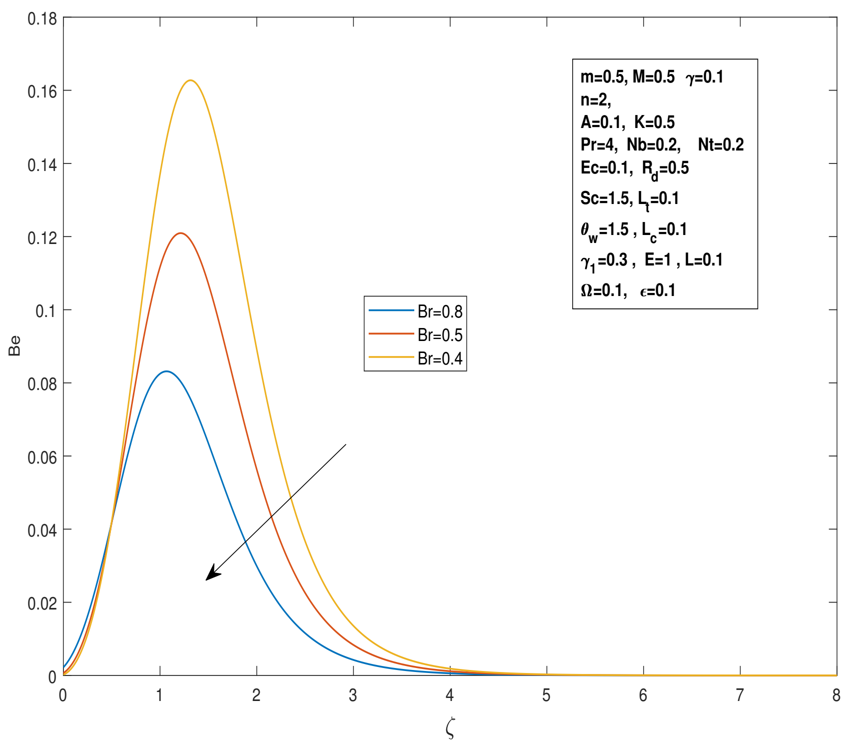

FIGURE 4.18: Impact of Br on the Be.

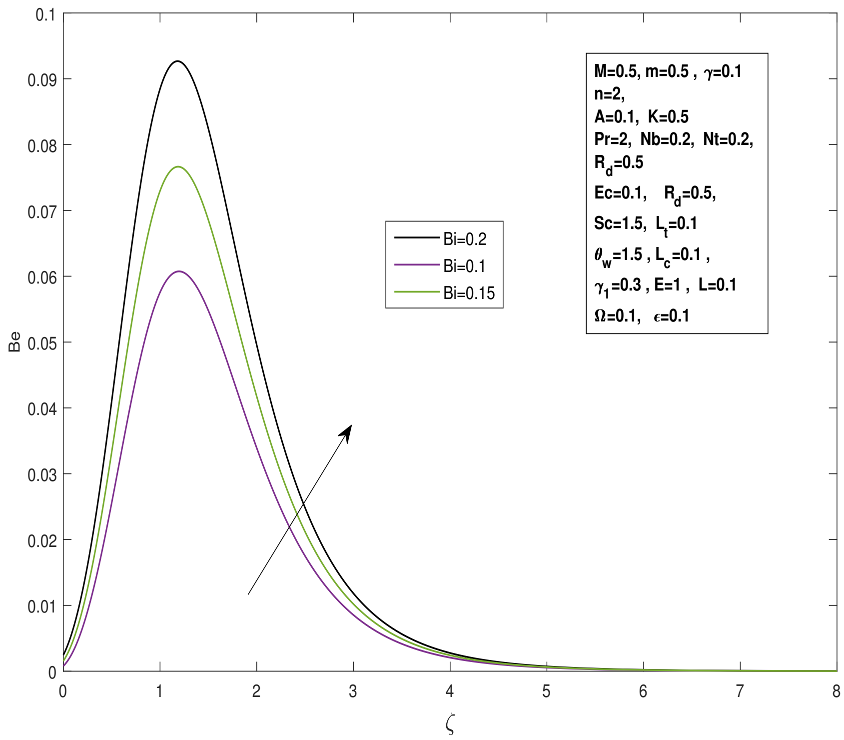

FIGURE 4.19: Impact of Bi on the Be.

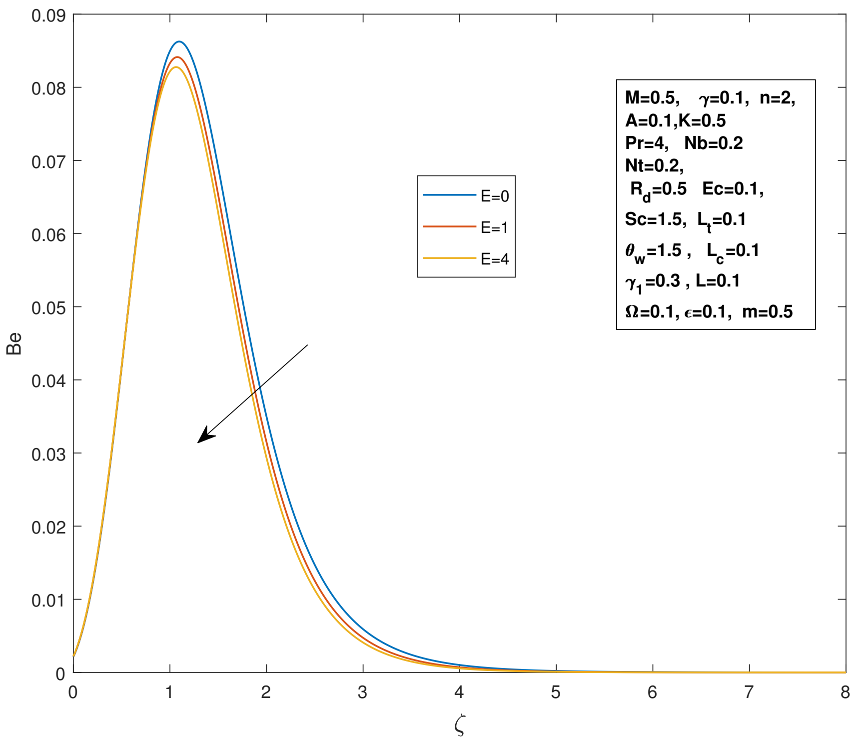

FIGURE 4.20: Impact of E on the Be.

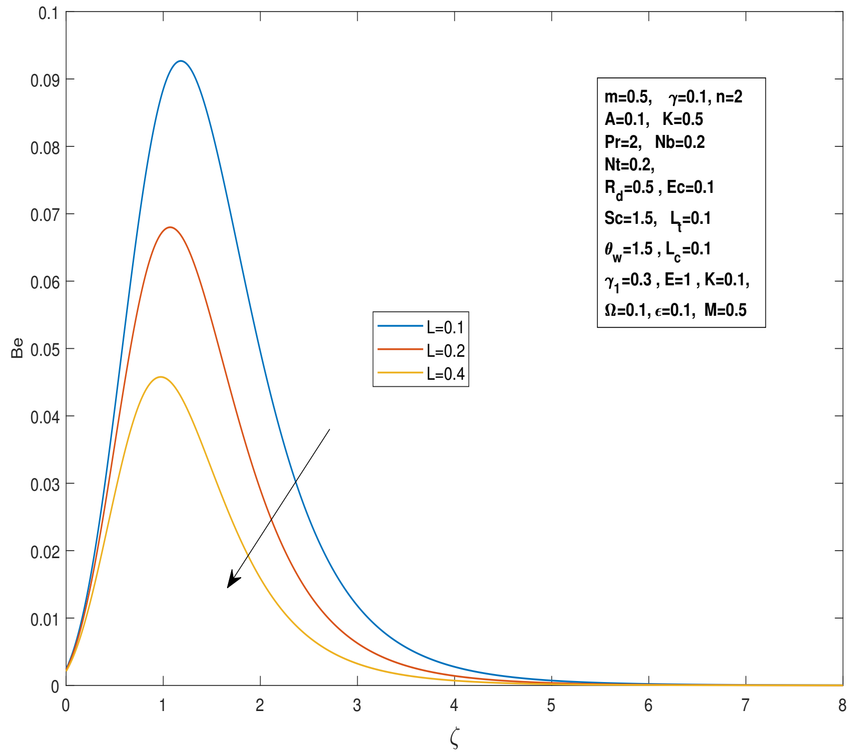

FIGURE 4.21: Impact of L on the Be.

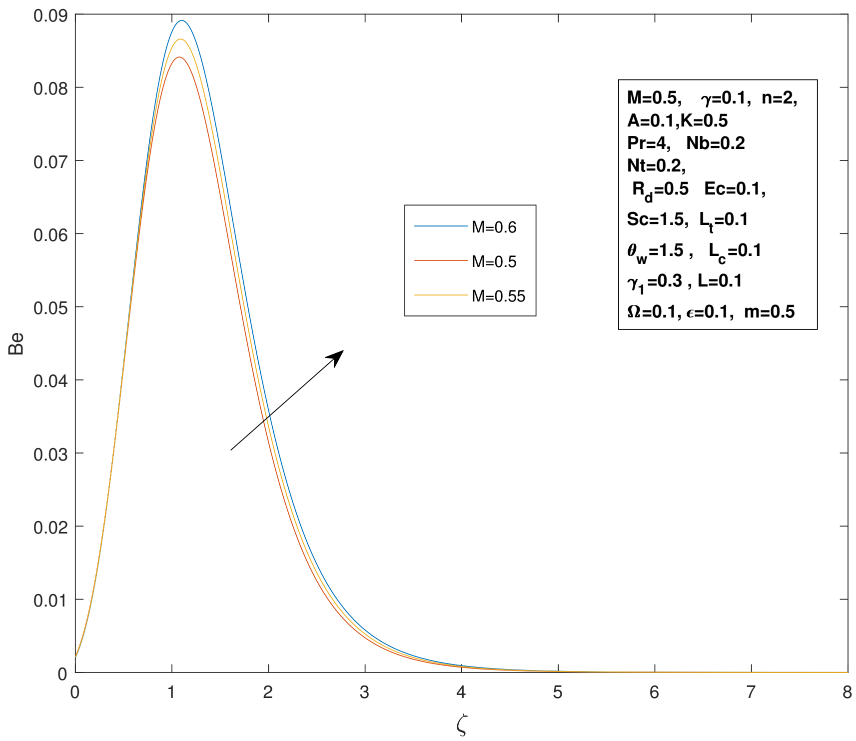

FIGURE 4.22: Impact of M on the Be.

Chapter 5

Conclusion

In this thesis, the work of Seth et al. [43] is examined and extended with the Cattaneo-Christov double diffusions and activation energy. First of all, momentum, energy and concentration equations are converted into the ODEs by using some suitable transformations. Numerical solutions to the modified ODEs have been discovered using the shooting technique. The results for velocity, temperature, and concentration profiles are shown in graphs. Tables display the results of Nusselt, skin friction and the Sherwood number. The following is a summary of the present research accomplishments:

- The velocity profile diminishes as M rises.
- The velocity profile declines with an increase in K.
- The velocity distribution decreases as the value of γ increases.
- It is shown that velocity profiles progressively diminish as the value of A rises.
- The velocity profile is reduced as the value of n is increased.
- The growing Prandtl number decreases the temperature profile but augments the concentration profile near wall and decreases the profile toward the stream.

- The distributions of temperature and concentration are observed to be increasing as a result of the rising values of the thermophoresis parameter.
- Schmidt number and Brownian motion are both found to have a negative impact on the nanoparticle concentration, although increasing the unsteadiness parameter increases the concentration.
- By increasing the values of Eckert number, a rise in the temperature distribution is found.
- When the thermal relaxation parameter increases, the fluid temperature and associated thermal boundary layer thickness diminish, and contradicting behaviour is shown in the concentration distribution.
- As the mass relaxation parameter T_c value rise, concentration distribution $\phi(\zeta)$ declines.
- As activation energy parameter$E$ increases, the concentration profile does as well.
- It has been noted that an increase in the temperature ratio parameter θ_w reduces the concentration profiles.
- Each parameter, M, Br, R_d, and Bi, individually increase the irreversibility of thermal energy.
- Each parameter Br, E and L, individually decrease the irreversibility of thermal energy.
- The growing influence of the magnetic field, porous permeability, unsteadiness parameter, and stretching index is observed to have a positive impact on the local skin friction coefficient while the slip parameter is thought to have a negative impact.
- We find that the Nusselt number tends to decrease when the Eckert number, thermophoresis parameter, and unsteadiness parameter values rise . But as Prandtl, Biot and the mass relaxation parameter values rise, Nusselt number value increases as well.

- It can be concluded that the Sherwood number tends to increase with rising values of the Schmidt number, Brownian motion parameter, temperature ratio parameter, chemical reaction parameter and mass relaxation parameter whereas the value of this physical decreases by increasing the values of the unsteadiness parameter and the activation energy parameter.

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