CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

MHD Squeezing Nanofluid Flow between Two Parallel Plates with Cattaneo-Christov Double Diffusion and Thermal Radiation

by

Asma Batool

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
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*I dedicate my dissertation work to my **family** and dignified **teachers**. A special feeling of gratitude to my loving parents who have supported me in my studies.*

CERTIFICATE OF APPROVAL

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Abstract

A mathematical model has been presented for analysing the flow of MHD squeezing nanofluid between two parallel plates with Cattaneo-Christov Double Diffusion and thermal radiation. One of the plates is fixed and the other is kept stretched. The proposed problem is modeled as a system of non-linear partial differential equations describing the conservation laws of mass, momentum and energy. The non-linear partial differential equations are transformed into ordinary differential equations by applying the similarity transformation and are then solved numerically using the shooting technique together with RK4 method by implementing the computational software package MATLAB. The obtained analytical solutions are used to investigate the squeezing phenomena of the nanofluids between two parallel plates. Also, the effect of different parameters such as prandtl number Pr, Levis number Le, radiation parameter Rd and thermophoretic parameter Nt on the velocity temperature and concentration are analysed.

Contents

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Abbreviations

IVPs	Initial value problems
BVPs	Boundary value problems
MHD	Magnetohydrodynamics
ODEs	Ordineary differential equations
PDEs	Partial differential equations
RK	Runge-Kutta

Symbols

μ	Viscosity
ρ	Density
ν	Kinematic viscosity
τ	Stress tensor
k	Thermal conductivity
α	Thermal diffisuitivity
σ	Electrical conductivity
u	x-component of fluid velocity
v	y-component of fluid velocity
B_0	Magnetic field constant
a	Stretching constant
T_0	Temperature of the wall
T_2	Ambient temperature of the nanofluid
T	Temperature
ρ_f	Density of the fluid
μ_f	Viscosity of the fluid
ν_f	Kinematic viscosity of the base fluid
ρ_{nf}	Density of the nanofluid
μ_{nf}	Viscosity of the nanofluid
q_r	Radiative heat flux
q	Heat generation constant
q_w	Heat flux
q_m	Mass flux

σ^*	Stefan Boltzmann constant
k^*	Absorption coefficient
ψ	Stream function
C_f	Skin friction coefficient
Nu	Nusselt number
Nu_x	Local Nusselt number
Sh	Sherwood number
Sh_x	Local Sherwood number
Re	Reynolds number
Re_x	Local Reynolds number
ϕ	Nanoparticle volume fraction
Rd	Thermal radiation parameter
n	Stretching parameter
M	Magnetic parameter
Pr	Prandtl number
Q	Heat generation parameter
Nb	Brownian motion parameter
Nt	Thermophoresis parameter
$\wedge_T$	Thermal relaxation term
$\wedge_C$	Mass relaxation term
L_e	Lewis number
ρ_f	Density of the pure fluid
ρ_s	Density of nanoparticle
μ_f	Viscosity of the base fluid
$(\rho C_p)_f$	Heat capacitance of base fluid
$(\rho C_p)_s$	Heat capacitance of nanoparticle
σ_s	Electrical conductivity of the nanoparticle
k_f	Thermal conductivity of the base fluid
k_s	Thermal conductivity of the nanoparticle
f	Dimensionless velocity
θ	Dimensionless temperature

ϕ	Dimensionless concentration
C_∞	Ambient concentration
C	Concentration
C_0	Nanoparticles concentration at the stretching surface

Chapter 1

Introduction

The efficiency of heat transfer speration is dependant on the function of thermal conduction of operating liquid, such as water, oil and ethyl glycol. If a little portion of nanoparticles (such as Cu, Ag, TiO_2 and Al_2O_3) is immersed into a conventional fluid, a new category of fluids is obtained which is called nanofluids. Nanofluids paved a new pathway to innovations in the improvement of the characteristics of heat transfer. There is wide variety of nanoparticles which are categorised according to their size, shape, thermal and electrical conductivity and heat transfer abilities. They are made up of metals, carbides and oxides. Some are named as nanofibers, nanowires, nanotubes and nanosheets [1]. Nanofluid has various applications in industrial devices, heat exchanger [2], drug delivery, medicines, car radiators, cooling of heat exchanging equipments, transformer oil cooling, electronic cooling [3, 4]. The diameter of the suspended nanoparticles varies between 1 to 100nm. There appears a histrionic boost in the thermophysical characteristics of the conventional liquid when nanoparticles are suspended in it.

On account of the point mentioned above, Choi et al. [5] introduced solid nanoparticles into the operating conventional fluid with the target of forming a new class of fluids that will have high heat conductivity in contrast to usual conventional fluid. They designated the combination of nanoparticles with nanofluid. The combination of Cu nanofluid and deionized water afterwards is analyzed by Xuan and Li [6]. They mentioned that the thermal conductivity of the water-based Cu fluid is higher as compared with that of the deionized water in a ratio which is approximately ranging from 1.24 to 1.78. Moreover, Choi et al. [7] figured that a small inclusion of rigid nanoparticles into conventional heat spread liquid raises the thermal conductivity of conventional liquid by a percentage of 200.

In 2006, Buongiorno [8] presented a detailed discussion related to convective transmit of system in nanofluid. He encountered the fact that Brownian diffusion and thermophoresis are the basic implement for the improvement of heat transmission. He deduced that immense fluctuations of temperature near the boundary layer zone result in noticeable reduction in the viscosity which as a consequence accelerate a rise in the coefficient of heat transmission.

Tiwari and Das [9] in 2007, further devised a model to examine nanofluid and heat transmission within a lid-driven cavity and analyzed the role of nanoparticle on volume fraction. They emphasized on the prime role of nanoparticle volume fraction for evaluating the impact of nanoparticles in the fluid flow and rate of heat transfer. Yang et al. [10] mentioned that, the thermal conductivity of nanofluid relies highly on nanoparticles volume fraction and their different properties such as diameter and shape.

Khan and Pop [11] were the first to perform an experiment depicting the response of nanofluid flow over a stretchable sheet with the help of Buongiorno's configuration. They came out with a conclusion that the rate of heat transfer is reduced with an increase in the Brownain diffusion and thermophoresis parameters. With time, Rana and Bhargava [12] added slight modifications in Khan and Pop's original experiment. They focused on the steady viscous nanofluid flow by applying finite element method (FEM) over a nonlinear stretchable sheet . Their findings indicated that an increase in the Brownain motion and thermophoresis parameters cause an improvement in the thermal boundary layer thickness. Also, Hamad and Ferdows [13] followed the model of Tiwari and Das. They addressed the similarity solution of viscous boundary layer flow of nanofluid over a nonlinear stretchable surface. Soon it was made clear that in the presence of nanoparticles in base fluid is capable of bringing about change in the pattern and behaviour of fluid flow based on the impacts of nanoparticle and nonlinear stretching sheet parameter. The impact of radiation and variable wall temperature on nanofluid flow past a nonlinear stretchable surface was investigated by Hady et al. [14]. Their point of view is, nanofluid temperature is reduced with a rise in radiation parameters. Due to partial condition effect, Das [15] again performed the same technique by taking account of the specified surface temperature. His main finding was that when there is increase in the nonlinear stretching sheet parameter and slip parameter causes a fall in the nanofluid velocity and a rise in the thickness of the boundary layer. Khan et al. [16] observed a three dimensional nanofluid flow past a nonlinear stretching sheet depicting the 4th and 5th order Runge-Kutta methods. Malvandi et al. [17] demonstrated a stagnation point nanofluid flow past a nonlinear stretching sheet with suction/injection. They demonstrate that when there ia a increment in suction parameter then heat transfer rate rises and decreases with the increased blowing parameter. Khan and Shehzad [18] worked on the thermophoresis's effect and Brownain motion on third grade nanofluid and rate of heat transmission past an oscillatory dynamic sheet.

Many authors [19–24] have contributed generously to the vastness of study of electrically conducting nanofluids covering the fields of engineering and technological process such as the plasma studies, MHD pumps, MHD generators and bearings. Noteable considerations also include either the viscous dissipation, heat radiation or generation of heat responses on the boundary layer flow of nanofluid. The features of heat transfer rate embedded in porous medium. This method is commonly used in oil reservoirs and geothermal engineering. Ahmad et al. [25] analyzed the behaviour of MHD viscous flow over an exponentially stretching surface with effect of radiation in a porous medium. In the presence of thermal radiation through a porous medium over a linear stretching sheet, Williamson fluid film flow and transfer of heat were examined by Shah et al. [26]. In their study, they noted that an increment in the porosity parameter decreases thin films flow and that the Lorentz force affects the flow of liquid film. Research regarding MHD boundary layer flow of nanofluids in a porous medium was also put forth by Zeeshan et al. [27]. Pal and Mandal [28] demonstrated the impact of heat radiation and heat generation on convective nanofluid flow through a stagnation point in a porous medium. Hybrid approach to numerically dissect the effects of viscous dissipation on MHD boundary layer nanofluid flow over a nonlinear stretching sheet saturated in a porous medium was triumphantly used by Bhargawa and Chandra [29]. Haroun et al. [30] devised the technique of spectral relaxation for examine of the influence of viscous dissipation, chemical reaction and radiation on MHD nanofluid flow in a porous medium and found that velocity field is decreased with a rise in porosity parameter. when porosity parameter is increased, it also increases the temperature distribution. On the same theme, MHD nanofluid flow and rate of heat transfer between porous medium and stretching sheet was examined by Geng et al. [31]. Further adding to the list, Patel [32] throughly studied homotopy analysis, the influence of nonlinear thermal diffusion, heat generation and cross-diffusion on an electrically conducting Casson fluid saturated in a porous medium. His conclusion is that with a decrease in the value of magnetic field, skin friction can be minimized.

The chemical reactions can further be classified as heterogeneous and homogenous processes. In the case of the strong compound system, the reaction is heterogenous. In most of the cases of chemical reaction processes, the concentration rate depends upon the species itself as discussed by Magyari and Chamkha et al. [33]. Chamkha and Rashad [34] talked about the impact of chemical reaction on MHD flow in the presence of heat generation or absorption of uniform vertical permeable surface. Das [35] explained the impact of chemical reaction with radiation on the heat and mass exchange along the MHD flow.

1.1 Thesis Contributions

The present survey is focused on the numerical analysis of MHD squeezing nanofluid flow with inclined magnetic field, Cattaneo-Christov Double Diffusion, thermophoresis diffusion, Brownain motion and thermal radiation. The proposed nonlinear PDEs are converted into system of ODEs by applying similarity transformations. Further, for finding the numerical results of nonlinear ODEs and the shooting method is utilized for numerical solution. The numerically obtained results are computed by using MATLAB. The impact of significant parameters on velocity distribution $f(\kappa)$, temperature distribution $\theta(\kappa)$ and concentration distribution $\phi(\kappa)$, skin friction coefficient C_f, local Nusselt number Nu_x and local Sherwood number Sh_x have been discussed in graphs and tables.

1.2 Layout of Thesis

A brief overview of the contents of the thesis is provided below.

Chapter 2 includes some basic definitions and terminologies, which are useful to understand the concepts discussed later on.

Chapter 3 provides the proposed numerical study of MHD squeezing nanofluid flow and thermal radiation between two parallel plates. The numerical results of the governing flow equations are reproduced by the shooting method.

Chapter 4 extends the proposed model flow discussed in Chapter 3 by including the impacts Cattaneo-Christov Double Diffusion and thermal radiation.

Chapter 5 provides the concluding remarks of the thesis.

References used in the thesis are mentioned in **Bibliography**.

Chapter 2

Preliminaries

This chapter contains some basic definitions and governing laws, which will be helpful in the subsequent chapters.

2.1 Important Definitions

Definition 2.1.1 (Fluid)
"A substance exists in three primary phases. Solid, Liquid and Gas (at very high temperatures, it also exists as plasma). A substance in the liquid or gas phase is referred to as a fluid. Distinction between a solid and fluid is made on the basis of substances ability to resist an applied shear or (tangential) stress that tends to change its shape." [36]

Definition 2.1.2 (Magnetohydrodynamics)
"Magnetohydrodynamics(MHD) is concerned with the mutual interaction of fluid flow and magnetic fields. The fluids in question must be electrically conducting and non-magnetic, which limits us to liquid metals, hot ionised gases (plasmas) and strong electrolytes." [37]

Definition 2.1.3 (Fluid Mechanics)
"Fluid mechanics is the branch of science which deals with the behavior of the fluids (liquids or gases) at rest as well as in motion. Thus this branch of science

deals with the static, kinematics and dynamic aspects of fluids" [38]

Definition 2.1.4 (Fluid Dynamics)

"The study of fluid if the pressure forces are also considered for the fluids in motion, that branch of science is called fluid dynamics." [38]

Definition 2.1.5 (Fluid Statics)

"The study of fluid at rest is called fluid statics." [38]

Definition 2.1.6 (Viscosity)

"Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. Mathematically,

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}},$$

where μ is viscosity coefficient, τ is shear stress and $\frac{\partial u}{\partial y}$ represents the velocity gradient." [38]

Definition 2.1.7 (Kinematic Viscosity)

"It is defined as the ratio between the dynamic viscosity and density of fluid. It is denoted by symbol ν called **'nu'**. Mathematically,

$$\nu = \frac{\mu}{\rho}.\text{" [38]}$$

Definition 2.1.8 (Thermal Conductivity)

"The Fourier heat conduction law states that the heat flow is proportional to the temperature gradient. The coefficient of proportionality is a material parameter known as the thermal conductivity which may be a function of a number of variables." [39]

Definition 2.1.9 (Thermal Diffusivity)

"The rate at which heat diffuses by conducting through a material depends on the thermal diffusivity and can be defined as,

$$\alpha = \frac{k}{\rho C_p},$$

where α is the thermal diffusivity, k is the thermal conductivity, ρ is the density and C_p is the specifc heat at constant pressure." [40]

2.2 Types of Flow

Definition 2.2.1 (Laminar and Turbulent Flow)

"Fluid partical follows a smooth trajectory, the flow is then said to be laminar. Further increases in speed may lead to instability that eventually produces a more random type of flow that is called turbulent." [41]

Definition 2.2.2 (Rotational Flow)

"Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis." [38]

Definition 2.2.3 (Irrotational Flow)

"Irrotational flow is that type of flow in which the fluid particles while flowing along stream-lines, do not rotate about their own axis then this type of flow is called irrotational flow." [38]

Definition 2.2.4 (Compressible Flow)

"Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid, Mathematically,

$$\rho \neq k,$$

where k is constant." [38]

Definition 2.2.5 (Incompressible Flow)

"Incompressible flow is that type of flow in which the density is constant for the fluid. Liquids are generally incompressible while gases are compressible, Mathematically,

$$\rho = k,$$

where k is constant." [38]

Definition 2.2.6 (Steady Flow)

"If the flow characteristics such as depth of flow, velocity of flow, rate of flow at any point in open channel flow do not change with respect to time, the flow is said to be steady flow. Mathematically,

$$\frac{\partial Q}{\partial t} = 0,$$

where Q is any fluid property." [38]

Definition 2.2.7 (Unsteady Flow)

"If at any point in open channel flow, the velocity of flow, depth of flow or rate of flow changes with respect to time, the flow is said to be unsteady. Mathematically,

$$\frac{\partial Q}{\partial t} \neq 0,$$

where Q is any fluid property." [38]

Definition 2.2.8 (Internal Flow)

"Flows completely bounded by a solid surfaces are called internal or duct flows." [42]

Definition 2.2.9 (External Flow)

"Flows over bodies immersed in an unbounded fluid are said to be an external flow." [42]

2.3 Classification of Fluids

2.3.1 (Types of Fluid)

"The fluids may be classified into the following five types:

1. Ideal fluid,
2. Real fluid,
3. Newtonian fluid,
4. Non-Newtonian fluid." [38]

Definition 2.3.1 (Ideal Fluid)

"A fluid, which is incompressible and has no viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity." [38]

Definition 2.3.2 (Real Fluid)

"A fluid, which possesses viscosity, is known as a real fluid. In actual practice, all the fluids are real fluids." [38]

Definition 2.3.3 (Newtonian Fluid)

"A real fluid, in which the shear stress is directly proportional to the rate of shear strain (or velocity gradient), is known as a Newtonian fluid." [38]

Definition 2.3.4 (Non-Newtonian Fluid)

"A real fluid in which the shear stress is not directly proportional to the rate of shear strain (or velocity gradient), is known as a non-Newtonian fluid." [38]

2.4 Modes of Heat Transfer

Definition 2.4.1 (Heat Transfer)

"Heat transfer is a branch of engineering that deals with the transfer of thermal energy from one point to another within a medium or from one medium to another due to the occurrence of a temperature difference." [43]

2.4.2 (Modes of Heat Transfer)

"There are three modes of heat transfer namely conduction, convection and radiation.

1. Conduction
2. Convection
3. Radiation."[43]

Definition 2.4.3 (Conduction)

"The transfer of heat within a medium due to a diffusion process is called conduction." [39]

Definition 2.4.4 (Convection)

"Convection heat transfer is usually defined as energy transport effected by the motion of a fluid. Newtons law of cooling governs the convection heat transfer between two different media." [39]

Definition 2.4.5 (Thermal Radiation)

"Thermal radiation is defined as radiant (electromagnetic) energy emitted by a medium and is sole to the temperature of the medium. Sometimes radiant energy is taken to be transported by electromagnetic waves while at other times it is supposed to be transported by particle like photons." [39]

2.5 Dimensionless Numbers

Definition 2.5.1 (Nusselt Number)

"The hot surface is cooled by a cold fluid stream. The heat from the hot surface, which is maintained at a constant temperature, is diffused through a boundary layer and convected away by the cold stream. Mathematically,

$$Nu = \frac{qL}{k}$$

where q stands for the convection heat transfer, L for the characteristic length and k stands for thermal conductivity." [43]

Definition 2.5.2 (Lewis Number)

"The Lewis number can be defined as the ratio of thermal diffusivity to molecular diffusivity. It characterizes the mutual relation of heat and mass transfers in various materials. Mathematically

$$Le = \frac{\lambda}{\rho\ D_m C_p}$$

where λ is the thermal conductivity, Dm the molecular diffusivity, and C_p the specific heat capacity at constant pressure." [42]

Definition 2.5.3 (Prandtl Number)

"It is the ratio between the momentum diffusivity ν and thermal diffusivity α. Mathematically, it can be defined as

$$Pr = \frac{\nu}{\alpha} = \frac{\frac{\mu}{\rho}}{\frac{k}{C_p\rho}} = \frac{\mu C_p}{k}$$

where μ represents the dynamic viscosity, Cp denotes the specific heat and k stands for thermal conductivity. The relative thickness of thermal and momentum boundary layer is controlled by Prandtl number. For small Pr, heat distributed rapidly corresponds to the momentum." [42]

Definition 2.5.4 (Sherwood Number)

"It is the nondimensional quantity which show the ratio of the mass transport by convection to the transfer of mass by diffusion. Mathematically:

$$Sh = \frac{kL}{D}$$

here L is characteristics length, D is the mass diffusivity and k is the mass transfer coeffcient." [44]

Definition 2.5.5 (Reynolds Number)

"It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. Mathematically,

$$Re = \frac{VL}{\nu},$$

where U denotes the free stream velocity, L is the characteristic length and ν stands for kinematic viscosity." [38]

2.5.6 (Thermophoresis Parameter Nt)

"In a temperature gradient, small particles are pushed towards the lower temperature because of the asymmetry of molecular impact." [44]

Definition 2.5.7 (Skin Friction Coefficient)

"The skin friction coefficient can be defined as

$$C_f = \frac{2\tau_w}{\rho w_\infty^2}$$

where τ_w denotes the wall shear stress, the velocity of free fluid flow is denoted by w_∞ and ρ is the density." [45]

2.6 Governing Laws

Definition 2.6.1 (Continuity Equation)

"The principle of conservation of mass can be stated as the time rate of change of mass in a fixed volume is equal to the net rate of flow of mass across the surface. The mathematical statement of the principle results in the following equation, known as the continuity (of mass) equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0. \tag{2.1}$$

where ρ is the density (kg/m^3) of the medium, v the velocity vector (m/s), and ∇ is the nabla or del operator.

For steady-state conditions, the continuity equation (2.1) becomes

$$\nabla.(\rho \mathbf{v}) = 0. \tag{2.2}$$

When the density changes following a fluid particle are negligible, the continuum is termed incompressible. The continuity equation (2.2) then becomes

$$\nabla.\mathbf{v} = 0. \tag{2.3}$$

which is often referred to as the incompressibility condition or incompressibility constraint."[39]

Definition 2.6.2 (Momentum Equation)

"The principle of conservation of linear momentum (or Newton's Second Law of motion) states that the time rate of change of linear momentum of a given set of particles is equal to the vector sum of all the external forces acting on the particles of the set, provided Newton's Third Law of action and reaction governs the internal forces. Newton's Second Law can be written as

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla.[(\rho \mathbf{v})\mathbf{v}] = \nabla.\sigma + \rho \mathbf{f}. \tag{2.4}$$

Where is the tensor (or dyadic) product of two vectors, σ is the Cauchy stress tensor (N/m^2) and f is the body force vector, measured per unit mass and normally taken to be the gravity vector. Equation (2.1) describes the motion of a continuous medium, and in fluid mechanics they are also known as the Navier equations. The form of the momentum equation shown in (2.4) is the conservation (divergence) form that is most often utilized for compressible flows. This equation may be simplified to a form more commonly used with incompressible flows. Expanding the first two derivatives and collecting terms

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\nabla.\mathbf{v}\right)+\mathbf{v}\left(\frac{\partial \rho}{\partial t}+\nabla.\rho\mathbf{v}\right)=\nabla.\sigma+\rho\mathbf{f}. \tag{2.5}$$

The second term in parentheses is the continuity equation (2.1) and neglecting this term allows (2.5) to reduce to the non-conservation (advective) form

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\nabla.\mathbf{v}\right)=\nabla.\sigma+\rho\mathbf{f}.\text{” [39]} \tag{2.6}$$

Definition 2.6.3 (Energy Equation)
“The law of conservation of energy (or the First Law of Thermodynamics) states that the time rate of change of the total energy is equal to the sum of the rate of work done by applied forces and the change of heat content per unit time. In the general case, the First Law of Thermodynamics can be expressed in conservation form as

$$\frac{\partial \rho e^t}{\partial t}+\nabla.\rho\mathbf{v}e^t=-\nabla.\mathbf{q}+\nabla.(\sigma.\mathbf{v})+Q+\rho\mathbf{f}.\mathbf{v} \tag{2.7}$$

where $e^t = e + 1/2\mathbf{v}.\mathbf{v}$ is the total energy (J/m^3), e is the internal energy, $\mathbf{q}$ is the heat flux vector (W/m^2) and Q is the internal heat generation (W/m^3). The total energy equation (2.7) is useful for high speed compressible flows where the kinetic energy is significant. For incompressible flows, an internal energy equation is more appropriate and can be derived from (2.7) with use of the momentum equation(2.4) Taking the dot product of the velocity vector with the momentum equation produces an equation for the kinetic energy this equation is subtracted from the total energy equation (2.7) to produce the conservation (divergence) form of the internal energy equation

$$\frac{\partial \rho e}{\partial t} + \nabla.\rho \mathbf{v} e = -\nabla.\mathbf{q} + Q + \phi \tag{2.8}$$

where ϕ is the dissipation function that is defined by

$$\phi = \sigma : \nabla \mathbf{v} \tag{2.9}$$

In Eq.(2.9) $\nabla \mathbf{v}$ is the velocity gradient tensor. " [39]

2.7 Shooting Method

To elaborate the shooting method, consider the following nonlinear BVP.

$$\left.\begin{aligned} &M''(x) = 2M'(x) + M(x) \\ &M(0) = 0, \qquad N(H) = J. \end{aligned}\right\} \tag{2.10}$$

To reduce the order of the above boundary value problem, introduce the following notations.

$$M = W_1 \qquad M' = W_1' = W_2 \qquad M'' = W_2'. \tag{2.11}$$

As a result,

$$W_1' = X_2, \qquad\qquad W_1(0) = 0, \tag{2.12}$$

$$W_2' = X_1 X_2 + 2X_1^2 \qquad\qquad W_2(0) = p \tag{2.13}$$

where p is the missing initial condition. The missing condition p is to be chosen such that

$$W_1(H, p) = J. \tag{2.14}$$

Now onward $W_1(H,p)$ will be denoted by $W_1(p)$. Let us further denote $W_1(p) - J$ by $N(p)$, so that

$$N(p) = 0. \tag{2.15}$$

The above equation can be solved by using Newton's method with the following iterative formula

$$\begin{aligned} p_{n+1} &= p_n - \frac{Np_n}{\frac{\partial Np_n}{\partial p}}, \\ p_{n+1} &= p_n - \frac{W_1p_n - J}{\frac{\partial W_1p_n}{\partial p}}. \end{aligned} \tag{2.16}$$

To find $\frac{\partial W_1p_n}{\partial p}$, introduce the following notations

$$\frac{\partial W_1}{\partial p} = W_3, \quad \frac{\partial W_2}{\partial p} = W_4. \tag{2.17}$$

As a result of these new notations, the Newton's iterative scheme, will then get the form

$$p_{n+1} = p_n - \frac{W_1p_n - J}{W_3p_n}. \tag{2.18}$$

Now differentiating the system of two first order ODEs (2.12)-(2.13) with respect to p, we get another system of ODEs, as follows

$$W_3' = W_4, \qquad W_3(0) = 0, \tag{2.19}$$

$$W_4' = 2W_4 + W_3. \qquad W_4(0) = 1. \tag{2.20}$$

Writing all the four ODEs (2.12), (2.13), (2.19) and (2.20) together, we have the following initial value problem

$$\begin{aligned} W_1' &= W_2, & W_1(0) &= 0, \\ W_2' &= 2W_2 + W_1, & W_2(0) &= p, \\ W_3' &= W_4, & W_3(0) &= 0, \\ W_4' &= 2W_4 + W_3. & W_4(0) &= 1. \end{aligned}$$

The above system together will be numerically solved by Runge-Kutta technique of order four. The stopping criteria for the Newton's technique is set as,

$$\mid W_1(p) - J \mid < \epsilon.$$

Here $\epsilon > 0$ is small positive real number.

Chapter 3

MHD Squeezing Nanofluid Flow between Two Parallel Plates and Thermal Radiation

3.1 Introduction

In this chapter unsteady, two-dimensional and symmetric flow for viscous incompressible fluid among two plates kept parallel is discussed under the influence of MHD and thermal radiations. The lower plate is on the horizontal $x - axis$ and the $y - axis$ is at the perpendicular position to the lower plate which is fixed. The relevant nonlinear PDEs are converted to a system of non-dimensional ODEs with the help of some appropriate transformations. For solving the ODEs, the shooting technique is applied with the help of MATLAB code. The numerical results of various parameters are elaborated at the end of this chapter for the dimensionless velocity $f(\kappa)$ profile, temperature distribution $\theta(\kappa)$ and concentration profile $\phi(\kappa)$. The reproduced findings of the current study which is a detailed review of the work presented by Muhammad et al. [46] are given through tables and graphs.

3.2 Mathematical Modeling

A $2D$ MHD flow for nanofluid flow has been investigated among two parallel plates, one kept fixed at $y = 0$ and the other at a variable distance $y = h(t)$. The lower plate is on the horizontal $x-axis$ and the $y-axis$ is at the perpendicular position to the lower plate which is fixed. In addition, the fluid is flowing subjected to the magnetic field B_0. The maintenance of a constant temperature between both plates has also been assumed. The upper plate which is placed at $y = h(t)$, has passive auxiliary conditions and the particles are constantly, uninterruptedly and stably distributed on the lower plate which is kept at $y = 0$. Here particles are scattered uniformly. Figure 3.1 represents the fluid flow geometry.

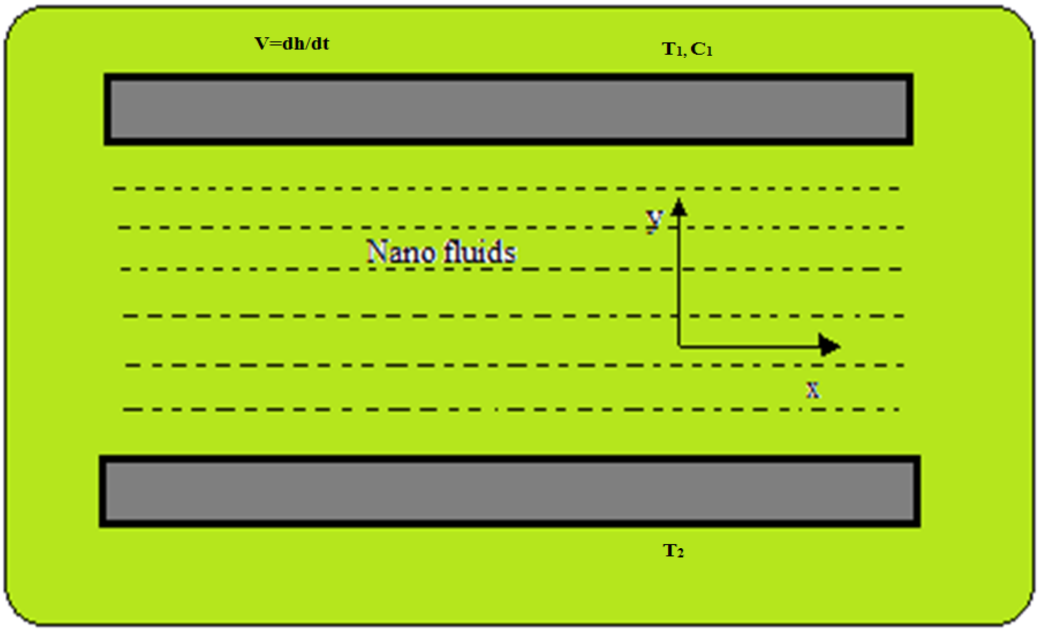

FIGURE 3.1: Physical model of fluid flow geometry.

The set of equations describing the flow are articulated [46] as follows.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \sigma B_o^2 u(t), \tag{3.2}$$

$$\rho_{nf}\left(\frac{\partial v}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{3.3}$$

$$\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\hat{\alpha}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$
$$+\tau\left[D_B\left\{\frac{\partial C}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}\right\}+\left(\frac{D_T}{T_0}\right)\left\{\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2\right\}\right]$$
$$-\frac{1}{(\rho * c_p)_f}\frac{\partial q_{rd}}{\partial y}, \tag{3.4}$$

$$\left(\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}\right)=D_B\left\{(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right\}+\frac{D_T}{T_0}\left\{(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right\}. \tag{3.5}$$

For lower and upper walls, the necessary BCs have been taken as [46]:

$$\left.\begin{array}{ll} v=0, u=0, T=T_2, D_B\left(\frac{\partial C}{\partial y}\right)+\frac{D_T}{T_0}\left(\frac{\partial T}{\partial y}\right)=0 & at \quad y=0 \\ v=\dfrac{dh}{dt}, u=0, C=C_1, T=T_1 & at \quad y=h(t). \end{array}\right\} \tag{3.6}$$

In equations (3.1) - (3.5), u and v are the velocities in x and y direction respectively, T is taken as temperature at the plates and C represents volumetric fraction of the nano particles. Moreover, ρ represents the density of the nanofluid, μ represents viscosity, D_B is taken as Brownian diffusion and D_T represents the thermophoretic coefficient.

In equation (3.4), the radiative heat flux is

$$q_r=-\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}.$$

Here σ^* and k^* are the Stefan-Boltzman constant and the absorption coefficient respectively. For very small temperature difference T^4 can be expanded about T_0 with the help of Taylor series, as follows.

$$T^4=T_0^4+4T_0^3(T-T_0)+6T_0^2(T-T_0)^2+...$$

Ignoring the higher order terms, we have

$$\begin{aligned} T^4 &= T_0^4 + 4T_0^3(T - T_0) \\ &= T_0^4 + 4T_0^3 T - 4T_0^4 \\ &= -3T_0^4 + 4T_0^3 T \\ &= 4T_0^3 T - 3T_0^4. \end{aligned}$$

For the conversion of the mathematical model (3.1) - (3.5) into a system of ODEs, the non-dimensional similarity variables [46] are:

$$\left.\begin{aligned} &\psi(x,y) = \left(\tfrac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}} x f(\kappa), \\ &u = \left(\tfrac{1-\bar{\alpha}t}{bx}\right)^{-1} f'(\kappa), \\ &v = -\left(\tfrac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}} f(\kappa), \kappa = \left(\tfrac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}} y, \\ &\theta(\kappa) = \left(\tfrac{T-T_0}{T_2-T_0}\right), \phi(\kappa) = -1 + \tfrac{C}{C_0}. \end{aligned}\right\} \tag{3.7}$$

Here ψ denotes the stream function, T_0 and C_0 are the reference temperature and reference concentration for nanoparticles and microorganisms respectively. The detailed procedure for the conversion of (3.1)-(3.5) into the dimensionless form, the procedure has been discussed below.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1} f'(\kappa)\right) \\ &= \frac{b}{1-\bar{\alpha}t} f'(\kappa). \end{aligned} \tag{3.8}$$

$$\begin{aligned} \frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\left(-\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}} f(\kappa)\right) \\ &= -\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}} f'(\kappa)\frac{\partial \kappa}{\partial y} \\ &= -\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}} f'(\kappa)\left(\frac{v(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}} \\ &= -\frac{b}{1-\bar{\alpha}t} f'(\kappa). \end{aligned} \tag{3.9}$$

In equation (3.1) is easily satisfied by using (3.8) and (3.9), as shown below.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{bf'(\kappa)}{1-\bar{\alpha}t}+\frac{-bf'(\kappa)}{1-\bar{\alpha}t}=0. \tag{3.10}$$

Now, for the momentum equations (3.2) and (3.3), the following derivatives are required.

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f'(\kappa)\right)$$

$$=\frac{b}{1-\bar{\alpha}t}f'(\kappa). \tag{3.11}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=0 \tag{3.12}$$

$$u\frac{\partial u}{\partial x}=\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f'(\kappa).\frac{b}{1-\bar{\alpha}t}f'(\kappa)$$

$$=\frac{b^2x}{(1-\bar{\alpha}t)^2}\left[f'(\kappa)\right]^2. \tag{3.13}$$

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial t}\left(\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f'(\kappa)\right)$$

$$=-\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-2}\left(\frac{-\bar{\alpha}}{bx}\right)f'(\kappa)+\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f''(\kappa)\frac{\partial\kappa}{\partial t}$$

$$=\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-2}\left(\frac{\bar{\alpha}}{bx}\right)f'(\kappa)+\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f''(\kappa)\left[\frac{1}{2}\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-3}{2}}\frac{\bar{\alpha}\nu}{b}y\right]$$

$$=\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-2}\left(\frac{\bar{\alpha}}{bx}\right)f'(\kappa)+\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f''(\kappa)$$

$$\left[\frac{1}{2}\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}}\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-1}\frac{\bar{\alpha}\nu}{b}y\right]$$

$$=\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-2}\frac{\bar{\alpha}}{bx}f'(\kappa)+\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f''(\kappa)\frac{1}{2}\kappa\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-1}\frac{\bar{\alpha}\nu}{b}$$

$$=\frac{b^2x^2}{(1-\bar{\alpha}t)^2}\frac{\bar{\alpha}}{bx}f'(\kappa)+\frac{bx}{(1-\bar{\alpha}t)}\frac{1}{2}\kappa\frac{b}{\nu(1-\bar{\alpha}t)}\frac{\bar{\alpha}\nu}{b}f''(\kappa)$$

$$=\frac{\bar{\alpha}bx}{(1-\bar{\alpha}t)^2}f'(\kappa)+\frac{\bar{\alpha}bx}{(1-\bar{\alpha}t)^2}\frac{\kappa}{2}f''(\kappa)$$

$$=\frac{\bar{\alpha}bx}{(1-\bar{\alpha}t)^2}.\left[f'(\kappa)+\frac{\kappa}{2}f''(\kappa)\right]. \tag{3.14}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1} f'(\kappa)\right]$$
$$= \left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1} f''(\kappa)\frac{\partial \kappa}{\partial y}$$
$$= \left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1} f''(\kappa)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}$$
$$= \left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}} f''(\kappa). \tag{3.15}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$
$$= \frac{\partial}{\partial y}\left[\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}} f''(\kappa)\right]$$
$$= \left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}} f'''(\kappa)\frac{\partial \kappa}{\partial y}$$
$$= \left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}} f'''(\kappa)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}$$
$$= \left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}.\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{-1} f'''(\kappa)$$
$$= \frac{bx}{1-\bar{\alpha}t}\frac{b}{\nu(1-\bar{\alpha}t)} f'''(\kappa)$$
$$= \frac{b^2 x}{\nu(1-\bar{\alpha}t)^2} f'''(\kappa). \tag{3.16}$$

$$v\frac{\partial u}{\partial y} = -\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}} f(\kappa)\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}} f''(\kappa)$$
$$= -\frac{(1-\bar{\alpha}t)^{\frac{-1}{2}}}{(b\nu)^{\frac{-1}{2}}}\frac{(1-\bar{\alpha}t)^{-1}}{(bx)^{-1}}\frac{(\nu(1-\bar{\alpha}t))^{\frac{-1}{2}}}{b^{\frac{-1}{2}}} f(\kappa) f''(\kappa)$$
$$= -\frac{(1-\bar{\alpha}t)^{-2}}{b^{-2}\nu^{\frac{-1}{2}}x^{-1}}\nu^{\frac{-1}{2}} f(\kappa) f''(\kappa)$$
$$= \frac{-b^2 x}{(1-\bar{\alpha}t)^2} f(\kappa) f''(\kappa). \tag{3.17}$$

$$\kappa = \left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}} y.$$

$$\frac{\partial \kappa}{\partial x} = 0, \tag{3.18}$$

$$\frac{\partial \kappa}{\partial y} = \left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}} \tag{3.19}$$

$$\frac{\partial \kappa}{\partial t} = \frac{-1}{2}\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-3}{2}}\left(\frac{-\bar{\alpha}\nu}{b}\right) y$$
$$= \frac{1}{2}\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-3}{2}}\left(\frac{\bar{\alpha}\nu}{b}\right) y. \tag{3.20}$$

Now,1^{st} momentum equation (3.2) becomes

$$\Rightarrow \rho\left(\frac{\bar{\alpha}bx}{(1-\bar{\alpha}t)^2}\left[f'(\kappa)+\frac{\kappa}{2}f''(\kappa)\right]+\frac{b^2x}{(1-\bar{\alpha}t)^2}\left[f'(\kappa)\right]^2-\frac{b^2x}{(1-\bar{\alpha}t)^2}f(\kappa)f''(\kappa)\right)$$
$$=-P_{xy}-x+u\left(0+\frac{b^2x}{\nu(1-\bar{\alpha}t)^2}f'''(\kappa)\right)-\sigma B_0^2\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f'(\kappa)$$
$$\Rightarrow \rho\left(\frac{\bar{\alpha}bx}{(1-\bar{\alpha}t)^2}\left[f'(\kappa)+\frac{\kappa}{2}f''(\kappa)\right]+\frac{b^2x}{(1-\bar{\alpha}t)^2}\left[f'(\kappa)\right]^2-\frac{b^2x}{(1-\bar{\alpha}t)^2}f(\kappa)f''(\kappa)\right)$$
$$=-P_{xy}-x+u\frac{b^2x}{\nu(1-\bar{\alpha}t)^2}f'''(\kappa)-\sigma B_0^2\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f'(\kappa)$$

Differentiate w.r.t y,

$$\rho\frac{\bar{\alpha}bx}{(1-\bar{\alpha}t)^2}\left\{f''(\kappa)\frac{\partial\kappa}{\partial y}+\frac{1}{2}\left(\frac{\partial\kappa}{\partial y}f''(\kappa)+\rho\kappa f'''(\kappa)\frac{\partial\kappa}{\partial y}\right)\right\}$$
$$+\rho\frac{b^2x}{(1-\bar{\alpha}t)^2}2f'(\kappa)f''(\kappa)\frac{\partial\kappa}{\partial y}-\rho\frac{b^2x}{(1-\bar{\alpha}t)^2}\left\{f'(\kappa)\frac{\partial\kappa}{\partial y}f''(\kappa)+f(\kappa)f'''(\kappa)\frac{\partial\kappa}{\partial y}\right\}$$
$$=-P_{xy}+u\frac{b^2x}{\nu(1-\bar{\alpha}t)^2}f''''(\kappa)\frac{\partial\kappa}{\partial y}-\sigma B_0^2\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f''(\kappa)\frac{\partial\kappa}{\partial y}.$$
$$\Rightarrow \rho\frac{\bar{\alpha}bx}{(1-\bar{\alpha}t)^2}\left\{f''(\kappa)+\frac{1}{2}\left(f''(\kappa)+\kappa f'''(\kappa)\right)\right\}+\rho\frac{2b^2x}{(1-\bar{\alpha}t)^2}f'(\kappa)f''(\kappa)$$
$$-\rho\frac{b^2x}{(1-\bar{\alpha}t)^2}\left\{f'(\kappa)\frac{\partial\kappa}{\partial y}f''(\kappa)+f(\kappa)f'''(\kappa)\right\}\frac{\partial\kappa}{\partial y}$$
$$=-P_{xy}+\left[u\frac{b^2x}{\nu(1-\bar{\alpha}t)^2}f''''(\kappa)-\sigma B_0^2\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f''(\kappa)\right]\frac{\partial\kappa}{\partial y}. \tag{3.21}$$
$$v=-\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}}f(\kappa) \tag{3.22}$$
$$\Rightarrow \frac{\partial v}{\partial x}=0 \tag{3.23}$$
$$\Rightarrow \frac{\partial^2 v}{\partial x^2}=0 \tag{3.24}$$
$$\frac{\partial v}{\partial y}=-\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}}f'(\kappa)\frac{\partial\kappa}{\partial y}$$
$$=-\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}}f'(\kappa)\left(\frac{\nu(1-\alpha t)}{b}\right)^{\frac{-1}{2}}$$
$$=-\frac{(1-\bar{\alpha}t)^{-1}\nu^{\frac{-1}{2}}}{b^{-1}\nu^{\frac{-1}{2}}}f'(\kappa)$$

$$= -\frac{(1-\bar{\alpha}t)^{-1}}{b^{-1}} f'(\kappa). \tag{3.25}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{-b}{1-\bar{\alpha}t} f''(\kappa)\frac{\partial \kappa}{\partial y}$$

$$= \frac{-b}{1-\bar{\alpha}t} f''(\kappa)\frac{\partial \kappa}{\partial y},$$

$$= \frac{-b}{1-\bar{\alpha}t} f''(\kappa)\left[\frac{\nu(1-\bar{\alpha}t))}{b}\right]^{\frac{-1}{2}}.$$

$$= -\frac{b^{\frac{3}{2}}}{\nu^{\frac{1}{2}}(1-\bar{\alpha}t)^{\frac{3}{2}}} f''(\kappa). \tag{3.26}$$

$$u\frac{\partial v}{\partial x} = (\frac{1-\bar{\alpha}t}{bx})^{-1} g'(\kappa)(0) = 0 \tag{3.27}$$

$$v\frac{\partial v}{\partial y} = -(\frac{1-\bar{\alpha}t}{bx})^{\frac{-1}{2}}\frac{-b}{1-\bar{\alpha}t} f(\kappa)f'(\kappa)$$

$$= \frac{b^{\frac{3}{2}}\nu^{\frac{1}{2}}}{(1-\bar{\alpha}t)^{\frac{3}{2}}} f(\kappa)f'(\kappa) \tag{3.28}$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\left[-\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}} f(\kappa)\right]$$

$$= \frac{-1}{2}\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-3}{2}}\left(\frac{-\bar{\alpha}}{b\nu}\right) f(\kappa) + \left\{-\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}}\right\} f'(\kappa)\frac{\partial \kappa}{\partial t}$$

$$= \frac{1}{2}\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-3}{2}}\frac{\bar{\alpha}}{b\nu} f(\kappa) - \left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}} f'(\kappa)\frac{1}{2}\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-3}{2}}\frac{\bar{\alpha}\nu}{b}y$$

$$= \frac{1}{2}\frac{b^{\frac{3}{2}}\nu^{\frac{1}{2}}}{(1-\bar{\alpha}t)^{\frac{3}{2}}}\frac{\bar{\alpha}}{b\nu} f(\kappa) - \left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}} f'(\kappa)\frac{1}{2}\kappa\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-3}{2}}\frac{\bar{\alpha}\nu}{b}$$

$$= \frac{1}{2}\frac{b^{\frac{1}{2}}\nu^{\frac{1}{2}}\bar{\alpha}}{(1-\bar{\alpha}t)^{\frac{3}{2}}} f(\kappa) - \frac{b^{\frac{1}{2}}\nu^{\frac{1}{2}}}{(1-\bar{\alpha}t)^{\frac{1}{2}}}\frac{\kappa}{2}\frac{b}{\nu(1-\bar{\alpha}t)}\frac{\bar{\alpha}\nu}{b} f'(\kappa)$$

$$= \frac{1}{2}\frac{b^{\frac{1}{2}}\nu^{\frac{1}{2}}\bar{\alpha}}{(1-\bar{\alpha}t)^{\frac{3}{2}}} f(\kappa) - \frac{b^{\frac{1}{2}}v^{\frac{1}{2}}\bar{\alpha}}{(1-\bar{\alpha}t)^{\frac{3}{2}}}\frac{\kappa}{2} f'(\kappa)$$

$$= \frac{b^{\frac{1}{2}}\nu^{\frac{1}{2}}\bar{\alpha}}{2(1-\bar{\alpha}t)^{\frac{3}{2}}}\left(f(\kappa) - F'(\kappa)\right) \tag{3.29}$$

Now,1^{st} momentum equation (3.2) becomes:

$$\rho\left(\frac{b^{\frac{1}{2}}\nu^{\frac{1}{2}}\bar{\alpha}}{2\left(1-\bar{\alpha}t\right)^{\frac{3}{2}}}\left\{f(\kappa) - f'(\kappa)\right\} + 0 + \frac{b^{\frac{3}{2}}\nu^{\frac{1}{2}}}{(1-\bar{\alpha}t)^{\frac{3}{2}}} f(\kappa)f'(\kappa)\right)$$

$$
\begin{aligned}
&= -\frac{\partial p}{\partial y} + \mu\left(0 - \frac{b^{\frac{3}{2}}}{\nu^{\frac{1}{2}}\left(1-\bar{\alpha}t\right)^{\frac{3}{2}}} f''\left(\kappa\right)\right)\\
\Rightarrow &\rho\left(\frac{b^{\frac{1}{2}}\nu^{\frac{1}{2}}\bar{\alpha}}{2\left(1-\bar{\alpha}t\right)^{\frac{3}{2}}}\left\{f(\kappa) - f'(\kappa)\right\} + 0 + \frac{b^{\frac{3}{2}}\nu^{\frac{1}{2}}}{\left(1-\bar{\alpha}t\right)^{\frac{3}{2}}} f(\kappa) f'(\kappa)\right)\\
&= -\frac{\partial p}{\partial y} - \mu\frac{b^{\frac{3}{2}}}{\nu^{\frac{1}{2}}\left(1-\bar{\alpha}t\right)^{\frac{3}{2}}} f''\left(\kappa\right)
\end{aligned}
$$

Differentiating above expression w.r.t x,

$$
\begin{aligned}
&\rho\left(\frac{b^{\frac{1}{2}}\nu^{\frac{1}{2}}\bar{\alpha}}{2\left(1-\bar{\alpha}t\right)^{\frac{3}{2}}}\left\{f'(\kappa)\frac{\partial\kappa}{\partial x} - f''(\kappa)\frac{\partial\kappa}{\partial x}\right\}\right)\\
&+\rho\left(\frac{b^{\frac{3}{2}}\nu^{\frac{1}{2}}}{\left(1-\bar{\alpha}t\right)^{\frac{3}{2}}}\left\{f'(\kappa)\frac{\partial\kappa}{\partial x}f'(\kappa) + f\left(\kappa\right)f''\left(\kappa\right)\frac{\partial\kappa}{\partial x}\right\}\right)\\
&= -P_{xy} - \frac{\mu b^{\frac{3}{2}}}{\nu^{\frac{1}{2}}\left(1-\bar{\alpha}t\right)^{\frac{3}{2}}} f''\left(\kappa\right)\frac{\partial\kappa}{\partial x}
\end{aligned}
\tag{3.30}
$$

As,

$$
\begin{aligned}
&\frac{\partial\kappa}{\partial x} = 0\\
\Rightarrow\quad &\rho\left(0\right) = -P_{xy} - 0\\
\Rightarrow &- P_{xy} = 0
\end{aligned}
$$

Using above expression in equation (3.30), we get

$$
\begin{aligned}
&\rho\left[\frac{\bar{\alpha}bx}{\left(1-\bar{\alpha}t\right)^{2}}\left\{\frac{3}{2}f''\left(\kappa\right) + \frac{\kappa}{2}f'''\left(\kappa\right)\right\}\right]\frac{\partial\kappa}{\partial y}\\
&+\rho\left[\frac{b^{2}x}{\left(1-\bar{\alpha}t\right)^{2}}f'\left(\kappa\right)f''\left(\kappa\right) - \frac{b^{2}x}{\left(1-\bar{\alpha}t\right)^{2}}f\left(\kappa\right)f'''\left(\kappa\right)\right]\frac{\partial\kappa}{\partial y}.\\
&=\left[\mu\frac{b^{2}x}{\nu\left(1-\bar{\alpha}t\right)^{2}}f''''\left(\kappa\right) - \frac{\sigma B_{0}^{2}bx}{1-\bar{\alpha}t}f''\left(\kappa\right)\right]\frac{\partial\kappa}{\partial y}
\end{aligned}
$$

$$\Rightarrow \rho \left[\frac{\bar{\alpha} b x}{2\left(1-\bar{\alpha} t\right)^{2}} \left\{ 3 f''(\kappa) + \kappa f'''(\kappa) \right\} + \frac{b^{2} x}{\left(1-\bar{\alpha} t\right)^{2}} f'(\kappa) f''(\kappa) - \frac{b^{2} x}{\left(1-\bar{\alpha} t\right)^{2}} f(\kappa) f'''(\kappa) \right] = \left[\mu \frac{b^{2} x}{\nu \left(1-\bar{\alpha} t\right)^{2}} f''''(\kappa) - \frac{\sigma B_{0}^{2} b x}{1-\bar{\alpha} t} f''(\kappa) \right]$$

$$\Rightarrow \frac{\bar{\alpha} b x}{2\left(1-\bar{\alpha} t\right)^{2}} \left\{ 3 f''(\kappa) + \kappa f'''(\kappa) \right\} + \frac{b^{2} x}{\left(1-\bar{\alpha} t\right)^{2}} f'(\kappa) f''(\kappa) - \frac{b^{2} x}{\left(1-\bar{\alpha} t\right)^{2}} f(\kappa) f'''(\kappa) = \frac{\mu}{\rho \nu} \frac{b^{2} x}{\left(1-\bar{\alpha} t\right)^{2}} f''''(\kappa) - \frac{\sigma B_{0}^{2} b x}{1-\bar{\alpha} t} f''(\kappa).$$

$$\frac{\bar{\alpha}}{2b} \left\{ 3 f''(\kappa) + \kappa f'''(\kappa) \right\} + f'(\kappa) f''(\kappa) - f(\kappa) f'''(\kappa) = f''''(\kappa) - \frac{\sigma B_{0}^{2} \left(1-\bar{\alpha} t\right)}{\rho b} f''(\kappa)$$

As,

$$\frac{\bar{\alpha}}{2b} = \lambda, \; M = \frac{\sigma B_{0}^{2}}{\rho b} \left(1-\bar{\alpha} t\right),$$

$$3\lambda f''(\kappa) + \lambda \kappa f'''(\kappa) + f'(\kappa) f''(\kappa) - f(\kappa) f'''(\kappa) = f''''(\kappa) - M f''(\kappa).$$

$$\Rightarrow f''''(\kappa) - M f''(\kappa) - 3\lambda f''(\kappa) - \lambda \kappa f'''(\kappa) - f'(\kappa) f''(\kappa) + f(\kappa) f'''(\kappa) = 0.$$

$$\Rightarrow f'''' + f f''' - f' f'' - \lambda \kappa f''' - 3\lambda f'' - M f'' = 0. \tag{3.31}$$

Now consider,

$$\theta(\kappa) = \frac{T - T_{0}}{T_{2} - T_{0}}.$$

$$\Rightarrow T - T_{0} = \theta(\kappa) \left(T_{2} - T_{0}\right).$$

$$\Rightarrow T = \theta(\kappa) \left(T_{2} - T_{0}\right) + T_{0}.$$

$$\frac{\partial T}{\partial t} = \left(T_{2} - T_{0}\right) \theta'(\kappa) \frac{\partial \kappa}{\partial t}.$$

$$= \left(T_{2} - T_{0}\right) \theta'(\kappa) \left[\frac{1}{2} \left(\frac{\nu \left(1-\bar{\alpha} t\right)}{b} \right)^{\frac{-3}{2}} \frac{\bar{\alpha} \nu}{b} y \right]$$

$$= \left(T_{2} - T_{0}\right) \theta'(\kappa) . \frac{\bar{\alpha} \nu}{2b} \left(\frac{\nu \left(1-\bar{\alpha} t\right)}{b} \right)^{\frac{-3}{2}} y. \tag{3.32}$$

$$\frac{\partial T}{\partial y} = \left(T_{2} - T_{0}\right) \theta'(\kappa) \frac{\partial \kappa}{\partial y}$$

$$= \left(T_{2} - T_{0}\right) \theta'(\kappa) \left(\frac{\nu \left(1-\bar{\alpha} t\right)}{b} \right)^{\frac{-1}{2}}. \tag{3.33}$$

$$\begin{aligned}
\frac{\partial^2 T}{\partial y^2} &= (T_2 - T_0)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}} \theta''(\kappa)\frac{\partial \kappa}{\partial y} \\
&= (T_2 - T_0)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}} \theta''(\kappa)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}} \\
&= (T_2 - T_0)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{-1} \theta''(\kappa) \\
&= (T_2 - T_0)\left[\frac{b}{\nu(1-\bar{\alpha}t)}\right] \theta''(\kappa). \qquad (3.34)
\end{aligned}$$

$$u\frac{\partial T}{\partial x} = 0. \qquad (3.35)$$

$$\begin{aligned}
v\frac{\partial T}{\partial y} &= -\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}} f(\kappa)(T_2 - T_0)\theta'(\kappa)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}} \\
&= -(T_2 - T_0)\left(\frac{1-\bar{\alpha}t}{b}\right)^{-1} f(\kappa)\theta'(\kappa) \\
&= -(T_2 - T_0)\left(\frac{b}{1-\bar{\alpha}t}\right) f(\kappa)\theta'(\kappa). \qquad (3.36)
\end{aligned}$$

$$\left(\frac{\partial T}{\partial x}\right)^2 = 0 \qquad (3.37)$$

$$\left(\frac{\partial T}{\partial y}\right)^2 = \left[(T_2 - T_0)\theta'(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}}\right]^2. \qquad (3.38)$$

$$\phi(\kappa) = -1 + \frac{C}{C_0}.$$

$$\Rightarrow C = C_0\phi(\kappa) + C_0$$

$$\Rightarrow \frac{\partial C}{\partial x} = 0. \qquad (3.39)$$

$$\begin{aligned}
\frac{\partial C}{\partial y} &= C_0\phi'(\kappa).\frac{\partial \kappa}{\partial y}. \\
&= C_0\phi'(\kappa)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}. \qquad (3.40)
\end{aligned}$$

Now equation (3.4) becomes:

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \hat{\bar{\alpha}}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$+\tau\left[D_B\left\{\frac{\partial C}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}\right\}+\left(\frac{D_T}{T_0}\right)\left\{\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2\right\}\right]$$
$$-\frac{1}{(\rho c_p)_f}\frac{\partial q_{rd}}{\partial y}$$
$$\Rightarrow (T_2-T_0)\,\phi'(\kappa)\,\frac{\bar{\alpha}\nu}{2b}\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-3}{2}}y-(T_2-T_0)\left(\frac{b}{1-\bar{\alpha}t}\right)\theta'(\kappa)\,f(\kappa)$$
$$=\bar{\alpha}\,(T_2-T_0)\,\frac{b}{\nu(1-\bar{\alpha}t)}\theta''(\kappa)$$
$$+\lambda\left[D_B\left\{C_0\phi'(\kappa)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}(T_2-T_0)\,\theta'(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}}\right\}\right]$$
$$+\lambda\left[\left(\frac{D_T}{T_0}\right)\left\{(T_2-T_0)^2\,(\theta'(\kappa))^2\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-1}\right\}\right]$$
$$-\frac{1}{\rho c_p}\frac{16T_0^3\sigma}{3k\star}(T_2-T_0)\left(\frac{b}{\nu(1-\bar{\alpha}t)}\right)\theta''(\kappa)\,.$$
$$\Rightarrow \phi'(\kappa)\,\frac{\bar{\alpha}\nu}{2b}\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-3}{2}}y-\left(\frac{b}{1-\bar{\alpha}t}\right)\theta'(\kappa)\,f(\kappa)$$
$$=\bar{\alpha}\frac{b}{\nu(1-\bar{\alpha}t)}\theta''(\kappa)$$
$$+\lambda\left[D_B\left\{C_0\phi'(\kappa)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}\theta'(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}}\right\}\right]$$
$$+\lambda\left[\left(\frac{D_T}{T_0}\right)\left\{(T_2-T_0)\,(\theta'(\kappa))^2\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-1}\right\}\right]$$
$$-\frac{1}{\rho c_p}\frac{16T_0^3\sigma}{3k^\star}\left(\frac{b}{\nu(1-\bar{\alpha}t)}\right)\theta''(\kappa)\,. \tag{3.41}$$

As,

$$\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-3}{2}}=\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}}\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-1},$$
$$\Rightarrow \phi'(\kappa)\,\frac{\bar{\alpha}\nu}{2b}(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-1}-\left(\frac{b}{1-\bar{\alpha}t}\right)\theta'(\kappa)\,f(\kappa)=\bar{\alpha}\frac{b}{\nu(1-\bar{\alpha}t)}\theta''(\kappa)$$
$$+\lambda\left[D_B\left\{C_0\phi'(\kappa)\,\theta'(\kappa)\,\frac{b}{\nu(1-\bar{\alpha}t)}\right\}+\left(\frac{D_T}{T_0}\right)\left\{(T_2-T_0)\,(\theta'(\kappa))^2\,\frac{b}{\nu(1-\bar{\alpha}t)}\right\}\right]$$
$$-\frac{1}{\rho c_p}\frac{16T_0^3\sigma}{3k^\star}\left(\frac{b}{\nu(1-\bar{\alpha}t)}\right)\theta''(\kappa)\,.$$

Multipling by ν and dividing by $\bar{\alpha}$,

$$P_r\left[\theta'(\kappa)\lambda\kappa-\theta'(\kappa)f(\kappa)\right]=\left(1+\frac{4}{3}Rd\right)\theta''(\kappa)+N_b\phi'\theta'+N_t(\theta')^2$$

$$\Rightarrow\left(1+\frac{4}{3}Rd\right)\theta''+P_r(f-\lambda\kappa)\theta'+N_b\phi'\theta'+N_t(\theta')^2=0. \tag{3.42}$$

Now conssider,

$$\frac{\partial C}{\partial x}=0.$$
$$\frac{\partial^2 C}{\partial x^2}=0. \tag{3.43}$$
$$\frac{\partial C}{\partial y}=C_0.\phi'(\kappa)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}$$
$$\frac{\partial^2 C}{\partial y^2}=C_0\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}\phi''(\kappa)\frac{\partial\kappa}{\partial y}.$$
$$=C_0\phi''(\kappa).\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}$$
$$=C_0\phi''(\kappa)\left[\frac{b}{\nu(1-\bar{\alpha}t)}\right]. \tag{3.44}$$
$$\frac{\partial C}{\partial t}=C_0\phi'(\kappa)\frac{\partial\kappa}{\partial t},$$
$$=C_0\phi'(\kappa)\frac{\bar{\alpha}\nu}{2b}\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-3}{2}}y,$$
$$=C_0\phi'(\kappa)\frac{\bar{\alpha}\nu}{2b}\kappa\left(\frac{b}{\nu(1-\bar{\alpha}t)}\right). \tag{3.45}$$
$$v\frac{\partial C}{\partial y}=\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}}f(\kappa)C_0\phi'(\kappa)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}$$
$$=C_0\left(\frac{b}{(1-\bar{\alpha}t)}\right)\phi'(\kappa)f(\kappa). \tag{3.46}$$

Thus equation (3.5) becomes:

$$C_0\phi'(\kappa)\frac{\bar{\alpha}\nu}{2b}\kappa\left(\frac{b}{\nu(1-\bar{\alpha}t)}\right)+C_0\left(\frac{-b}{(1-\bar{\alpha}t)}\right)\phi'(\kappa)f(\kappa)$$
$$=D_B\left\{C_0\phi''(\kappa)\frac{b}{(1-\bar{\alpha}t)}\right\}+\frac{D_T}{T_0}(T_2-T_0)\frac{b}{\nu(1-\bar{\alpha}t)}\theta''(\kappa).$$
$$\Rightarrow C_0\phi'(\kappa)\lambda\kappa-C_0\phi'(\kappa)f(\kappa)=D_B\left\{C_0\phi''(\kappa)\frac{1}{\nu}\right\}+\frac{D_T}{T_0}\left\{(T_2-T_0)\frac{1}{\nu}\theta''(\kappa)\right\}.$$
$$\Rightarrow\phi'(\kappa)\lambda\kappa-\phi'(\kappa)f(\kappa)=D_B\left\{\phi''(\kappa)\frac{1}{\nu}\right\}+\frac{D_T}{T_0C_0}\left\{(T_2-T_0)\frac{1}{\nu}\theta''(\kappa)\right\}.$$
$$\Rightarrow(\phi'(\kappa)\lambda\kappa-\phi'(\kappa)f(\kappa))\frac{\nu}{D_B}=\phi''(\kappa)+\frac{D_T}{T_0C_0D_B}\{(T_2-T_0)\theta''(\kappa)\}.$$

$$\Rightarrow\phi''+L_e(f-\lambda\kappa)\phi'+\left(\frac{N_t}{N_b}\right)\theta'. \tag{3.47}$$

The corresponding BCs are transformed into the non-dimensional form through the following procedure.

$$v=0 \qquad at \qquad y=0.$$

$$\Rightarrow-\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}}f(\kappa)=0 \qquad at \qquad \kappa=0.$$
$$\Rightarrow f(\kappa)=0 \qquad at \qquad \kappa=0.$$
$$\Rightarrow f(0)=0.$$
$$u=0 \qquad at \qquad y=0.$$
$$\Rightarrow\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f'(\kappa)=0 \qquad at \qquad y=0.$$
$$\Rightarrow f'(\kappa)=0 \qquad at \qquad \kappa=0.$$
$$\Rightarrow f'(0)=0.$$
$$T=T_2 \qquad at \qquad y=0.$$
$$\Rightarrow(T_2-T_0)\theta(\kappa)+T_0=T_2 \qquad at \qquad \kappa=0.$$
$$\Rightarrow\theta(\kappa)=1 \qquad at \qquad \kappa=0.$$
$$\Rightarrow\theta(0)=1$$

As

$$C = C_0(1+\phi(\kappa))$$
$$\frac{\partial C}{\partial y} = C_0(\phi'(\kappa)\frac{\partial \kappa}{\partial y})$$
$$= C_0\phi'(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-\frac{1}{2}}$$
$$\frac{\partial T}{\partial y} = (T_2-T_1)\theta'(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-\frac{1}{2}}$$

Now

$$D_B(\frac{\partial C}{\partial y}) + \frac{D_T}{T_0}(\frac{\partial T}{\partial y}) = 0 \qquad at \qquad y=0.$$
$$\Rightarrow D_B C_0\phi'(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-\frac{1}{2}}$$
$$+\frac{D_T}{T_0}(T_2-T_0)\phi(\kappa)\left(\frac{v(1-\bar{\alpha}t)}{b}\right)^{-\frac{1}{2}} = 0 \qquad at \qquad \kappa=0.$$
$$\Rightarrow D_B(C_0\phi'(\kappa)) + \frac{D_T}{T_0}((T_2-T_0)\phi(\kappa)) = 0 \qquad at \qquad \kappa=0.$$

Multiplying $\frac{(\rho_c)_p}{(\rho_c)_F}.\frac{C_0}{\bar{\alpha}}$ on both sides

$$\Rightarrow \frac{(\rho_c)_p}{(\rho_c)_f}\frac{D_B C_0}{\bar{\alpha}}\phi'(\kappa) + \frac{(\rho_c)_p}{(\rho_c)_f}\frac{D_T(T_2-T_0)}{T_0\bar{\alpha}}\phi'(\kappa) = 0 \qquad at \qquad \kappa=0.$$
$$\Rightarrow N_b\phi'(\kappa) + N_t\phi'(\kappa) = 0 \qquad at \qquad \kappa=0.$$
$$\Rightarrow N_b\phi'(0) + N_t\phi'(0) = 0$$
$$v = \frac{dh}{dt} \qquad at \qquad y=h(t).$$
$$\Rightarrow -(\frac{1-\bar{\alpha}t}{b\nu})^{-\frac{1}{2}}f(\kappa) = \frac{1}{2}(\frac{\nu(1-\bar{\alpha}t)}{b})^{-\frac{1}{2}}(-\frac{\bar{\alpha}\nu}{b}) \qquad at \qquad \kappa=1.$$
$$\Rightarrow f(\kappa) = \frac{\bar{\alpha}\nu}{2b} \qquad at \qquad \kappa=1.$$
$$\Rightarrow f(1) = w$$

$$
\begin{aligned}
& u = 0 && at && y = h(t). \\
\Rightarrow & \left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1} f'(\kappa) = 1 && at && \kappa = 1. \\
\Rightarrow & f'(\kappa) = 0 && at && \kappa = 1. \\
\Rightarrow & f'(1) = 0. && && \\
& T = T_1 && at && y = h(t). \\
\Rightarrow & (T_2 - T_0)\,\theta\,(\kappa) + T_0 = T_1 && at && \kappa = 1. \\
\Rightarrow & \theta(\kappa) = \delta_\theta && at && \kappa = 1. \\
\Rightarrow & \theta(1) = \delta_\theta. && && \\
& C = C_1 && at && y = h(t). \\
\Rightarrow & C_0\phi\,(\kappa) = -C_0 + C_1 && at && \kappa = 1. \\
\Rightarrow & \phi(\kappa) = \frac{-C_0 + C_1}{C_0} && at && \kappa = 1. \\
\Rightarrow & \phi(1) = \delta_\phi && at && \kappa = 1. \\
\Rightarrow & \phi(1) = \delta_\phi && && \qquad (3.48)
\end{aligned}
$$

The final dimensionless form of the governing model is,

$$f'''' + ff''' - f'f'' - \lambda\kappa f''' - 3\lambda f'' - Mf'' = 0, \tag{3.49}$$

$$\left(1 + \frac{4}{3}Rd\right)\theta'' + Pr\,(f - \lambda\kappa\kappa)\,\theta' + Nb\phi'\theta' + Nt\,(\theta')^2 = 0, \tag{3.50}$$

$$\phi'' - Le\,(f - \lambda\kappa)\,\phi' + \left(\frac{Nb}{Nt}\right)\theta'' = 0. \tag{3.51}$$

The associated BCs (3.6) in the dimensionless form are:

$$\left.\begin{aligned} & f\,(0) = 0, f'\,(0) = 0, f'\,(1) = 0, f\,(1) = w, \theta\,(1) = \delta_\theta, \\ & \theta\,(0) = 1, \phi\,(1) = \delta_\phi, \phi'\,(0)\,Nb + \theta'\,(0)\,Nt = 0. \end{aligned}\right\}. \tag{3.52}$$

Different dimensionless parameters used in equations (3.49)-(3.51) are formulated as follow:

$$\lambda = \frac{\alpha}{2b}, \quad M = \frac{\sigma B_0^2}{\rho b}\left(1-\alpha t\right), \quad Rd = \frac{4T^3\sigma}{3\left(\rho c_p\right)_f k\alpha},$$
$$Nt = \frac{\left(\rho c_p\right)_p}{\left(\rho c_p\right)_f}\frac{D_T\left(T_2-T_0\right)}{T_0\alpha}, \quad Nb = \frac{\left(\rho c_p\right)_p}{\left(\rho c_p\right)_f}\frac{D_b C_0}{\alpha}, \quad Pr = \frac{v}{\alpha},$$
$$Le = \frac{v}{D_B}, \quad w = \frac{\alpha H}{2\left(vb\right)^{\frac{1}{2}}}, \quad \delta\phi = \left(\frac{C_1-C_0}{C_0}\right), \quad \delta\theta = \frac{\left(T_1-T_0\right)}{\left(T_2-T_0\right)}.$$

The skin friction coefficient, is given as follows.

$$C_f = 2\frac{\tau_w|_{y=0}}{\rho_f u_w^2(x)}.$$

To achive the dimensionless form of C_f the following steps will be helpful. Since

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0},$$
$$C_{fx} = \frac{1}{\rho u_w(x)^2}\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$
$$= \frac{2}{\rho\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-2}}\mu\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}\left[\frac{\nu\left(1-\bar{\alpha}t\right)}{b}\right]^{\frac{-1}{2}} f''\left(\kappa\right)$$
$$= \frac{2\mu}{\rho\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}}x^{\frac{-1}{2}}\left[\frac{\nu\left(1-\bar{\alpha}t\right)}{bx}\right]^{\frac{-1}{2}} f''\left(\kappa\right)$$

$$= \frac{2\nu}{\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}}x^{\frac{-1}{2}}\nu^{\frac{-1}{2}}\left[\frac{\nu\left(1-\bar{\alpha}t\right)}{b}\right]^{\frac{-1}{2}} f''\left(\kappa\right)$$
$$= \frac{2\nu^{\frac{-1}{2}}}{x^{\frac{-1}{2}}u_w(x)^{\frac{-1}{2}}}f''\left(\kappa\right)$$
$$= \frac{2x^{\frac{-1}{2}}u_w(x)^{\frac{-1}{2}}}{\nu^{\frac{-1}{2}}}f''\left(\kappa\right).$$
$$\Rightarrow \frac{C_{fx}\sqrt{Re_x}}{2} = f''\left(\kappa\right).$$

where Re denotes the Reynolds number defined as $Re = \frac{xu_w(x)}{\nu}$.

To achieve the dimensionless form of Nu_x, the following steps will be helpful. Since Local Nusselt number is defined as follows:

$$
\begin{aligned}
Nu_x &= \frac{xq_w}{K(T_2 - T_0)}. \\
\Rightarrow Nu_x &= \frac{xK\left(T_2 - T_0\right)\theta'\left(\kappa\right)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}}{K\left(T_2 - T_0\right)} \\
&= x^{\frac{1}{2}}\theta'\left(\kappa\right)\nu^{\frac{-1}{2}}\left[\frac{(1-\bar{\alpha}t)}{bx}\right]^{\frac{-1}{2}} \\
&= \frac{\nu^{\frac{-1}{2}}}{x^{\frac{-1}{2}}u_w(x)^{\frac{-1}{2}}}\theta'\left(\kappa\right) \\
&= \frac{1}{\left(Re_x^{\frac{-1}{2}}\right)}\theta'\left(\kappa\right). \\
\Rightarrow \left(Re_x^{\frac{-1}{2}}\right) Nu_x &= \theta'\left(\kappa\right).
\end{aligned}
$$

The local Sherwood number is defined as:

$$
Sh_x = \frac{xq_m}{D_B(C0)}.
$$

To achieve the dimensionless form of Sh_x, the following steps will be helpful. Since

$$
\begin{aligned}
q_m &= -D_B\left(\frac{\partial C}{\partial y}\right)_{y=0}, \\
Sh_x &= -D_B\frac{x\left(C_0\right)\phi'\left(\kappa\right)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}}{D_B\left(C_0\right)} \\
&= -\frac{x\left(C_0\right)\phi'\left(\kappa\right)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}}{D_B\left(C_0\right)} \\
&= -x\phi'\left(\kappa\right)\left[\frac{\nu\left(1-\bar{\alpha}t\right)}{b}\right]^{\frac{-1}{2}} \\
&= -x^{\frac{1}{2}}\phi'\left(\kappa\right)\nu^{\frac{-1}{2}}\left[\frac{(1-\bar{\alpha}t)}{bx}\right]^{\frac{-1}{2}}
\end{aligned}
$$

$$= -\frac{\nu^{\frac{-1}{2}}}{x^{\frac{-1}{2}} u_w(x)^{\frac{-1}{2}}} \theta'(\kappa)$$

$$= -\frac{1}{\left(Re_x^{\frac{-1}{2}}\right)} \phi'(\kappa).$$

$$\Rightarrow \left(Re_x^{\frac{-1}{2}}\right) Sh_x = -\phi'(\kappa).$$

3.3 Solution Methodology

For solving (3.49) with the associated boundary conditions (3.52) the shooting method has been opted.

First of all we convert the fourth order ODE into the system of first order ODEs. In order to solve the IVP, we will use the RK4 method by assuming the missing initial conditions. Then for (3.49), we use the following notations:

$$f = y_1,$$

$$f' = {y_1}' = y_2,$$

$$f'' = {y_1}'' = {y_2}' = y_3,$$

$$f''' = {y_1}''' = {y_2}'' = {y_3}' = y_4.$$

The resulting inital value problem takes the form:

$$\begin{aligned}
{y_1}' &= y_2, & y_1(0) &= 0.\\
{y_2}' &= y_3, & y_2(0) &= 0.\\
{y_3}' &= y_4, & y_3(0) &= r.\\
y_4' &= -y_1 y_4 + y_2 y_3 + \lambda\kappa y_4 + 3\lambda y_3 + M y_3, & y_4(0) &= s.
\end{aligned}$$

Missing conditions r and s are assumed to satisfy the following relations:

$$(y_1(r,s))_{\kappa=\kappa_\infty} = w,$$

$$(y_2(r,s))_{\kappa=\kappa_\infty} = 0,$$

As the numerical computation can not be performed on an unbounded domain, therefore the domain of the above problem has taken as $[0, \kappa_\infty)$ instead of $[0, \infty)$, where κ_∞ is an appropriate initial positive number.To solve the above algebaric equations, we apply the Newton's method which has the following scheme.

$$\begin{bmatrix} r \\ s \end{bmatrix}_{n+1} = \begin{bmatrix} r \\ s \end{bmatrix}_{n} - \begin{bmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial s} \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial s} \end{bmatrix}_{n}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{n} \tag{3.53}$$

where n=0,1,2,...

Now utilize the following notations:

$$\frac{\partial y_1}{\partial r} = y_5, \qquad \frac{\partial y_2}{\partial r} = y_6$$
$$\frac{\partial y_3}{\partial r} = y_7, \qquad \frac{\partial y_4}{\partial r} = y_8$$
$$\frac{\partial y_1}{\partial s} = y_9, \qquad \frac{\partial y_2}{\partial s} = y_{10}$$
$$\frac{\partial y_3}{\partial s} = y_{11}, \qquad \frac{\partial y_4}{\partial s} = y_{12}$$

Using the above notations in (3.53),

$$\begin{bmatrix} r \\ s \end{bmatrix}_{n+1} = \begin{bmatrix} r \\ s \end{bmatrix}_{n} - \begin{bmatrix} y_5 & y_9 \\ y_6 & y_{10} \end{bmatrix}_{n}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{n}$$

Now differentiating the system of four first order ODEs with respect to r and s, we get another system of ODEs, as follows.

$$y_5{}' = y_6, \qquad y_5(0) = 0.$$

$$y_6{}' = y_7, \qquad y_6(0) = 0.$$

$$y_7{}' = y_8, \qquad y_7(0) = 1.$$

$$y_8' = -y_5y_4 - y_1y_8 + y_2y_7 + y_6y_3 + \lambda\kappa y_8 + 3\lambda y_7 + My_7, \qquad y_8(0) = 0.$$

$$y_9{}' = y_{10}, \qquad y_9(0) = 0.$$

$$y_{10}{}' = y_{11}, \qquad y_{10}(0) = 0.$$

$$y_{11}{}' = y_{12}, \qquad y_{11}(0) = 0.$$

$$y_{12}' = -y_1 y_{12} - y_9 y_4 + y_{10} y_3 + y_2 y_{11} + \lambda\kappa y_{12} + 3\lambda y_{11} + M y_{11}, \qquad y_{12}(0) = 1.$$

RK4 method is used to solve the IVP and initial values are choosen arbitrarily. On executing the iterations, these initial values will be updated with the help of Newton's method and the whole process will be continued until the following criteria is achieved.

$$max\{|y_1(\kappa_\infty)|, |y_2(\kappa_\infty)|\} < \epsilon.$$

Throughout this work, ϵ has been taken as 10^{-10} unless otherwise mentioned. Also, for equations (3.50) and (3.51), the following notations have been used

$$\theta = Y_1, \theta' = Y_1' = Y_2, \theta'' = Y_2',$$

$$\phi = Y_3, \phi' = Y3' = Y_4, \phi'' = Y_4',$$

$$A = \left(1 + \frac{4}{3} R_d\right).$$

The system of equations (3.50) and (3.51), can be written in the form of the following first order coupled ODEs,

$$Y_1' = Y_2, \qquad Y_1(0) = 1.$$

$$Y_2'' = -\frac{1}{A}\left[P_r(f - \lambda\kappa)Y_2 + N_b Y_4 Y_2 + N_t (Y_2)^2\right], \qquad Y_2(0) = P.$$

$$Y_3' = Y_4, \qquad Y_3(0) = Q.$$

$$Y_4' = -L_e(f - \lambda\kappa)Y_4$$

$$-\frac{N_t}{N_b}(-\frac{1}{A}\left[P_r(f - \lambda\kappa)Y_2 + N_b Y_4 Y_2 + N_t (Y_2)^2\right], \qquad Y_4(0) = \frac{Nt}{Nb}(P).$$

The RK-4 method has been taken into consideration for solving the above initial value problem. For the above system of equtions, the missing conditions are to be chosen such that.

$$(Y_1(P,Q))_{\kappa=\kappa_\infty} = \delta_\theta,$$

$$(Y_4(P,Q))_{\kappa=\kappa_\infty} = \delta_\phi.$$

To solve the above algebaric equations, we apply the Newton's method which has the following scheme.

$$\begin{bmatrix} P \\ Q \end{bmatrix}_{n+1} = \begin{bmatrix} P \\ Q \end{bmatrix}_n - \begin{bmatrix} \frac{\partial Y_1}{\partial P} & \frac{\partial Y_1}{\partial Q} \\ \frac{\partial Y_4}{\partial P} & \frac{\partial Y_4}{\partial Q} \end{bmatrix}_n^{-1} \begin{bmatrix} Y_1 \\ Y_4 \end{bmatrix}_n$$

Now, introduce the following notations,

$$\frac{\partial Y_1}{\partial P} = Y_5, \qquad \frac{\partial Y_2}{\partial P} = Y_6, \qquad \frac{\partial Y_3}{\partial P} = Y_7, \qquad \frac{\partial Y_4}{\partial P} = Y_8.$$
$$\frac{\partial Y_1}{\partial Q} = Y_9, \qquad \frac{\partial Y_2}{\partial Q} = Y_{10}, \qquad \frac{\partial Y_3}{\partial Q} = Y_{11}, \qquad \frac{\partial Y_4}{\partial Q} = Y_{12}.$$

As the result of these new notations, the Newton's iterative scheme gets the form.

$$\begin{bmatrix} P \\ Q \end{bmatrix}_{n+1} = \begin{bmatrix} P \\ Q \end{bmatrix}_n - \begin{bmatrix} Y_5 & Y_9 \\ Y_8 & Y_{12} \end{bmatrix}_n^{-1} \begin{bmatrix} Y_1 \\ Y_4 \end{bmatrix}_n .$$

Now differentiating the system of four first order ODEs with respect to P and Q we get another system of ODEs, as follows.

$$Y_5' = Y_6 \qquad\qquad Y_5(0) = 0.$$

$$Y_6' = -\frac{1}{A}\left[P_r(f-\lambda\kappa)Y_6 + N_bY_6Y_4 + N_bY_2Y_8 + 2N_tY_2Y_6\right] \qquad\qquad Y_6(0) = 1.$$

$$Y_7' = Y_8 \qquad\qquad Y7(0) = 0.$$

$$Y_8' = -L_e(f-\lambda\kappa)Y_8 - \frac{N_t}{N_b}$$
$$(-\frac{1}{A}[P_r(f-\lambda\kappa)Y_6 + N_bY_6Y_4 + N_bY_2Y_8 + 2N_tY_2Y_6]), \qquad Y_8(0) = -\frac{Nt}{Nb}.$$
$$Y_9' = Y_{10} \qquad Y_9(0) = 0.$$
$$Y_{10}' = -\frac{1}{A}[P_r(f-\lambda\kappa)Y_{10} + N_bY_{10}Y_4 + N_bY_2Y_{12} + 2N_tY_2Y_{10}] \qquad Y_{10}(0) = 0.$$
$$Y_{11}' = Y_{12} \qquad Y_{11}(0) = 1.$$
$$Y_{12}' = -L_e(f-\lambda\kappa)Y_{12} - \frac{N_t}{N_b}(-\frac{1}{A}$$
$$[P_r(f-\lambda\kappa)Y_{10} + N_bY_{10}Y_4 + N_bY_2Y_{12} + 2N_tY_2Y_{12}]) \qquad Y_{12}(0) = 0.$$

The stopping criteria for the Newton's method is set as.

$$max\{|Y_1(\kappa_\infty)|, |Y_4(\kappa_\infty)|\} < \epsilon.$$

3.4 Representation of Graphs and Tables

A thorough discussion on the graphs and tables has been conducted which contains the impact of different non-dimensional parameters on the skin friction coefficient $(Re_x)^{\frac{1}{2}}C_f$ and Nusselt number $(Re_x)^{\frac{-1}{2}}Nu_x$.

Table 3.1 explains the impact of parameter λ, magnetic parameter M, and parameter w on $(Re_x)^{\frac{1}{2}}C_f$. For the rising values of λ, the skin fraction coefficient $(Re_x)^{\frac{1}{2}}C_F$ increases.

In Table 3.2, the impact of significant parameters on Nusselt number $(Re_x)^{\frac{-1}{2}}Nu_x$ as well as Shewrood number $(Re_x)^{\frac{-1}{2}}Sh_x$ has been discussed. The rising pattern is found in $(Re_x)^{\frac{-1}{2}}Nu_x$ and $(Re_x)^{\frac{-1}{2}}Sh_x$ due to increasing values of w.

Figures 3.2-3.4 reflect the behaviour of the velocity profile $f(\kappa)$,concentration profile $\phi(\kappa)$ and temperature profile $\theta(\kappa)$ due to different values of λ with M and w

Figures 3.5 - 3.7 show the impact of w. For the rising values of w, the velocity profile $g(\kappa)$ increases while the temperature profile $\theta(\kappa)$ and the concentration profile $\phi(\kappa)$ decreases.

Figures 3.8 and 3.9 show the impact of thermal radiation Rd on the temperature

profile $\theta(\kappa)$ and concentration profile $\phi(\kappa)$. It can be noticed from Figure 3.8 that the temperature profile increases for increasing values of Rd. This increment in the dimensionless temperature will cause more deformation in liquid. When thermal radiation Rd increases, there is a thickness in momentum boundary layer, whereas with an increase in Rd, an increment in the concentration profile is noticed, leading to an increment in the heat transfer.
Figure 3.10 illustrates the impact of heat generation Pr on $\theta(\kappa)$. It is analysed that for the rising values of Prandtl number Pr, more heat is generated, because of which $\theta(\kappa)$ and the boundary layer thickness increases.
From Figure 3.11, it can be seen that by increasing the values of Prandtl number Pr, the concentration profile also increases. Figure 3.12 represents the impact of δt on $\theta(\kappa)$. In this graph it is analysed that on the rising values of δt, the temperature profile $\theta(\kappa)$ also increase.
Figure 3.13 represents the impact of δt on the concentration distribution. The concentration distribution expands by rising the values of δt.

Table 3.1: Results of $(Re_x)^{\frac{1}{2}}C_f$

λ	M	w	$(Re_x)^{\frac{1}{2}}C_f$
2.0	2.0	0.6	3.79884
4.0			3.80989
6.0			3.87873
8.0			3.98292
	4.0		3.91560
	6.0		4.02894
	8.0		4.13912
		0.7	4.45786
		0.8	5.12430
		0.9	5.79816

TABLE 3.2: Results of $-(Re_x)^{\frac{-1}{2}}Nu_x$ and $-(Re_x)^{\frac{-1}{2}}Sh_x$ with fixed parameter $M = 0.2$

λ	w	Rd	Pr	Nt	Nb	Le	δt	δp	$-(Re_x)^{\frac{-1}{2}}Nu_x$	$-(Re_x)^{\frac{-1}{2}}Sh_x$
1	0.1	0.2	0.5	0.2	0.1	0.2	0.1	0.2	0.84337	-1.68674
2									0.78225	-1.56450
3									0.72256	-1.44512
4									0.66460	-1.32920
	0.5								0.86545	-1.73090
	0.8								0.88230	-1.76461
	1								0.89367	-1.78735
	0.1	0.4							0.85375	-1.70750
		0.6							0.86092	-1.72184
		0.9							0.86829	-1.73654
		0.2	0.1						0.88941	-1.77882
			0.5						0.84337	-1.68674
			0.8						0.80930	-1.61861
			1						0.78686	-1.57372
			0.5	0.4					0.83997	-3.35988
				0.6					0.83614	-5.01688
				0.9					0.82949	-7.46541
				0.2	0.2				0.84337	-0.84337
					0.3				0.84337	-0.56224
					0.4				0.84337	-0.42168
2	0.9		0.6	2.0		2.0	0.5	0.5	0.50103	-0.50103
		5.0				4.0			0.51767	-0.51767
			0.6			6.0			0.54866	-0.54866
			0.6			8.0			0.60405	-0.60405
1	0.1	0.2	0.5	0.2	0.1	0.2	0.1	0.2	0.84337	-1.68674
							0.5		0.46391	-0.93862
							0.8		0.18794	-0.37589
							0.9		0.09400	-0.18801
							0.1		0.84337	-1.68674
								0.4	0.84337	-1.68674
								0.6	0.84337	-1.68674
								0.9	0.84337	-1.68674

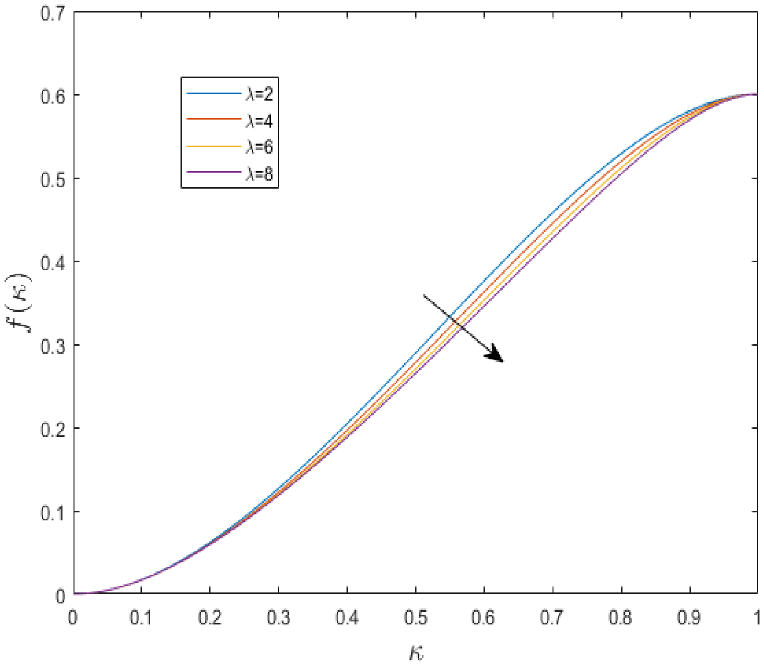

FIGURE 3.2: Effect of λ when $M = 2$, $w = 0.6$,on $f(\kappa)$.

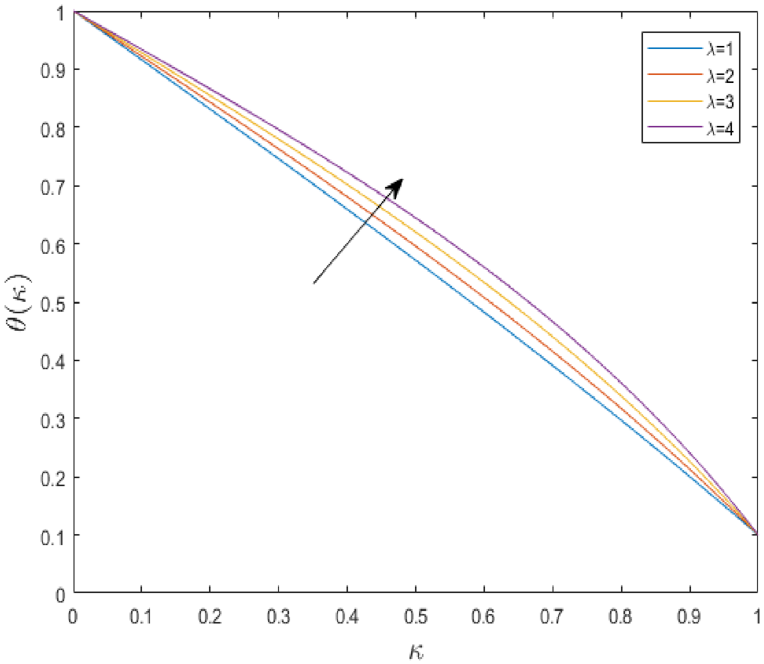

FIGURE 3.3: Effect of λ when $M = 0.2$, $w = 0.1$, $Rd = 0.2$,$Pr = 0.5$,$Nt = 0.2$,$Nb = 0.1$,$Le = 0.2$,$dt = 0.1$, $\delta_\phi = 0.2$, on $\theta(\kappa)$.

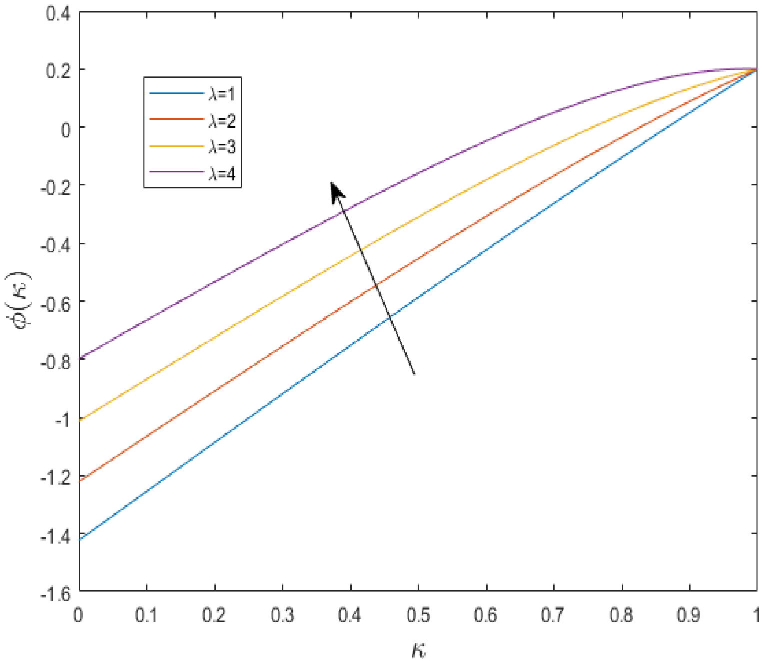

FIGURE 3.4: Effect of λ when $M = 0.2$, $w = 0.1$, $Rd = 0.2$,$Pr = 0.5$,$Nt = 0.2$,$Nb = 0.1$,$Le = 0.2$,$\delta_\theta = 0.1$, $\delta_\phi = 0.2$,on $\phi(\kappa)$.

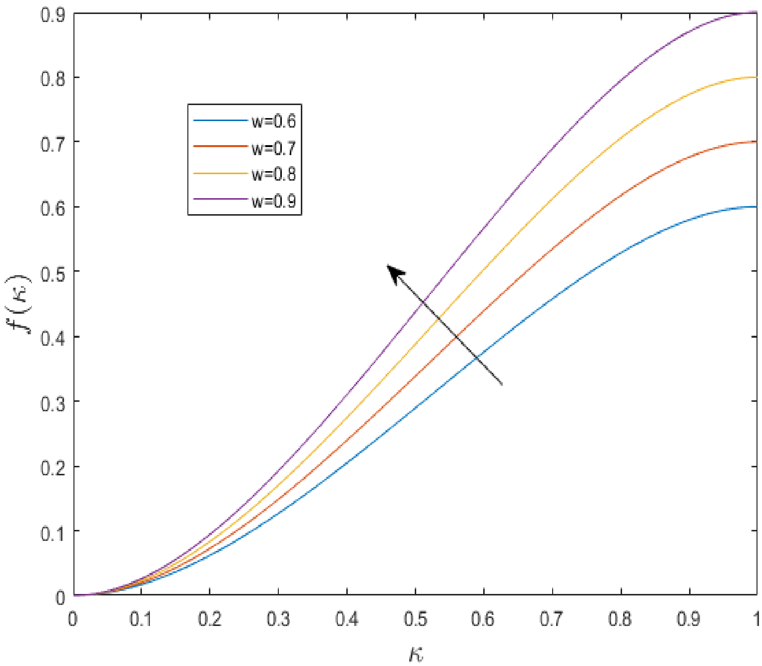

FIGURE 3.5: Effect of w when $M = 2$, $\lambda = 2$,on $f(\kappa)$.

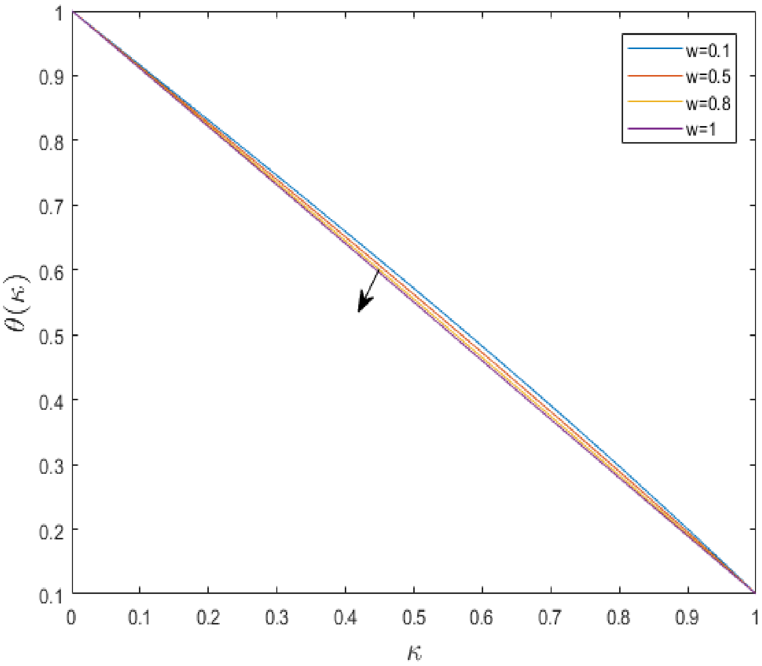

FIGURE 3.6: Effect of w when $M = 0.2$, $\lambda = 1$, $Rd = 0.2$,$Pr = 0.5$,$Nt = 0.2$,$Nb = 0.1$,$Le = 0.2$,$\delta_\theta = 0.1$, $\delta_\phi = 0.2$, on $\theta(\kappa)$.

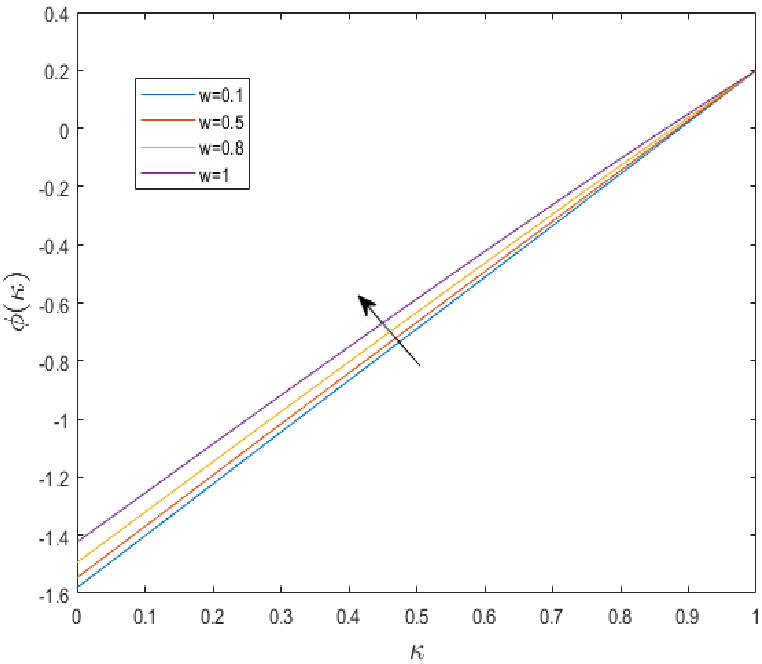

FIGURE 3.7: Effect of w on $\phi(\kappa)$ for $M = 0.2$, $\lambda = 1$, $Rd = 0.2$,$Pr = 0.5$,$Nt = 0.2$,$Nb = 0.1$,$Le = 0.2$,$\delta_\theta = 0.1$, $\delta_\phi = 0.2$.

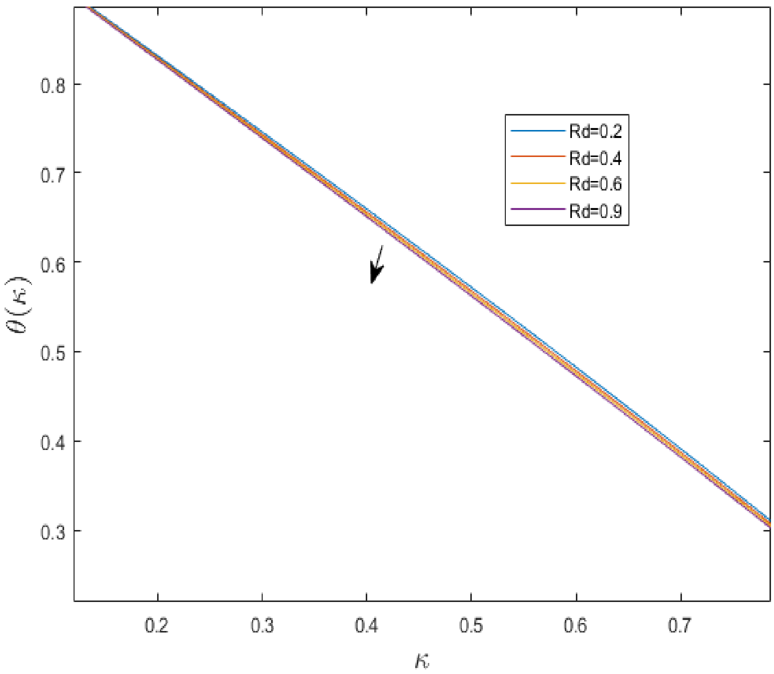

FIGURE 3.8: Effect of Rd when $M = 0.2$, $\lambda = 1$, $w = 0.1$,$Pr = 0.5$,$Nt = 0.2$,$Nb = 0.1$,$Le = 0.2$,$\delta_\theta = 0.1$, $\delta_\phi = 0.2$, on $\theta(\kappa)$.

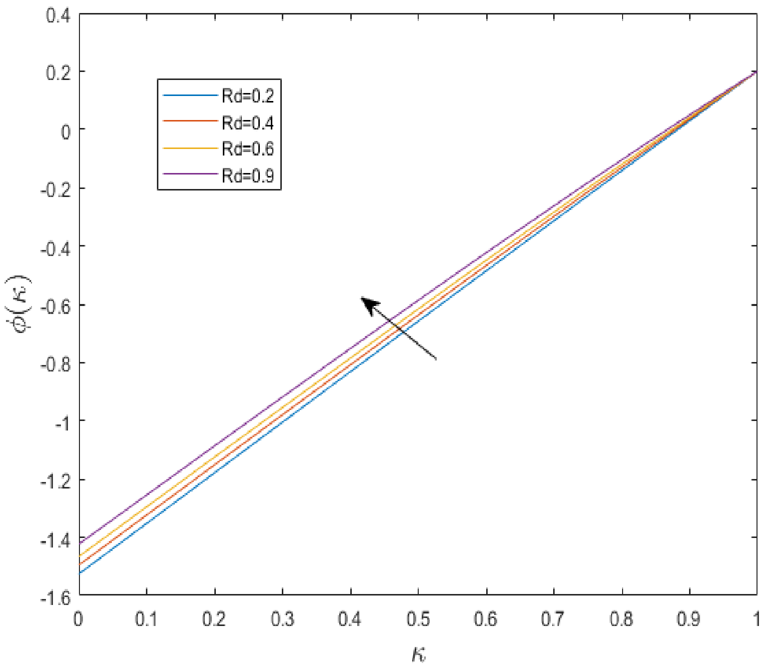

FIGURE 3.9: Effect of Rd on $\phi(\kappa)$ for $M = 0.2$, $\lambda = 1$, $w = 0.1$,$Pr = 0.5$,$Nt = 0.2$,$Nb = 0.1$,$Le = 0.2$,$\delta_\theta = 0.1$, $\delta_\phi = 0.2$.

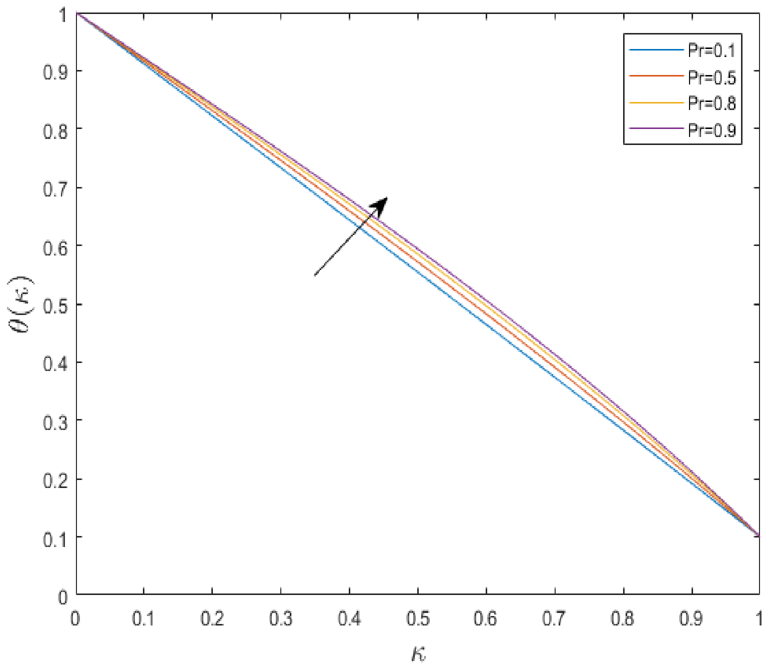

FIGURE 3.10: Effect of Pr on $\theta(\kappa)$ for $M = 0.2$, $\lambda = 1$, $w = 0.1$,$Rd = 0.2$,$Nt = 0.2$,$Nb = 0.1$,$Le = 0.2$,$\delta_\theta = 0.1$, $\delta_\phi = 0.2$.

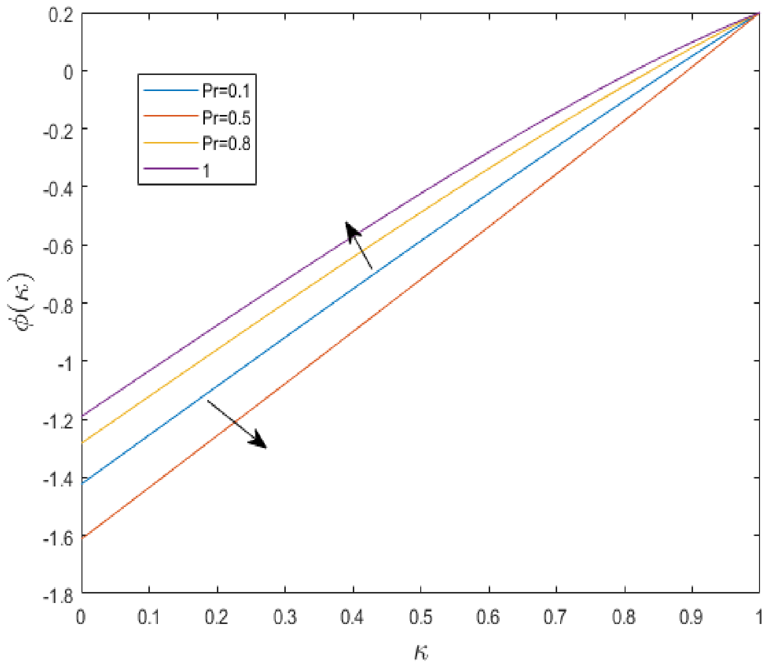

FIGURE 3.11: Effect of Pr when $M = 0.2$, $\lambda = 1$, $w = 0.1$,$Rd = 0.2$,$Nt = 0.2$,$Nb = 0.1$,$Le = 0.2$,$\delta_\theta = 0.1$, $\delta_\phi = 0.2$, on $\phi(\kappa)$.

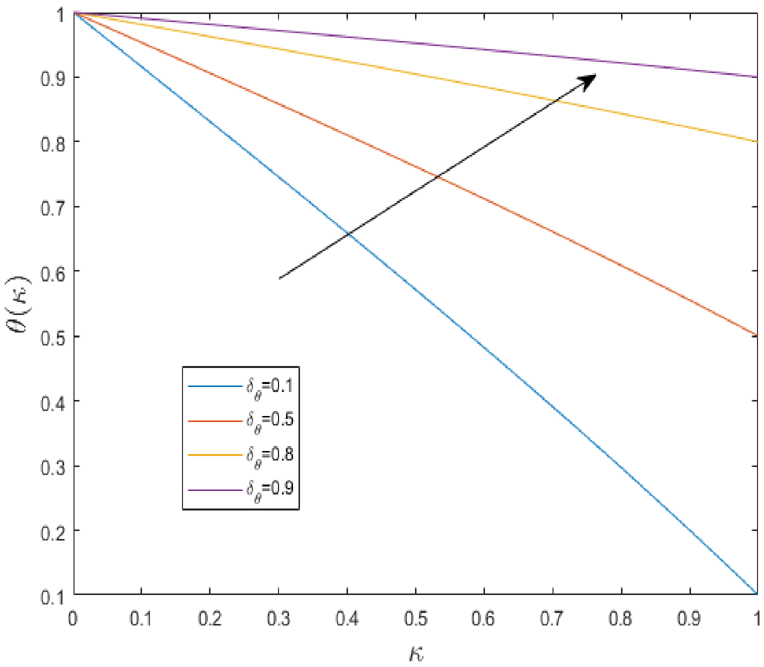

FIGURE 3.12: Effect of δ_θ on $\theta(\kappa)$ for $M = 0.2$, $\lambda = 1$, $w = 0.1$,$Rd = 0.2$,$Nt = 0.2$,$Nb = 0.1$,$Le = 0.2$,$Pr = 0.5$, $\delta_\phi = 0.2$.

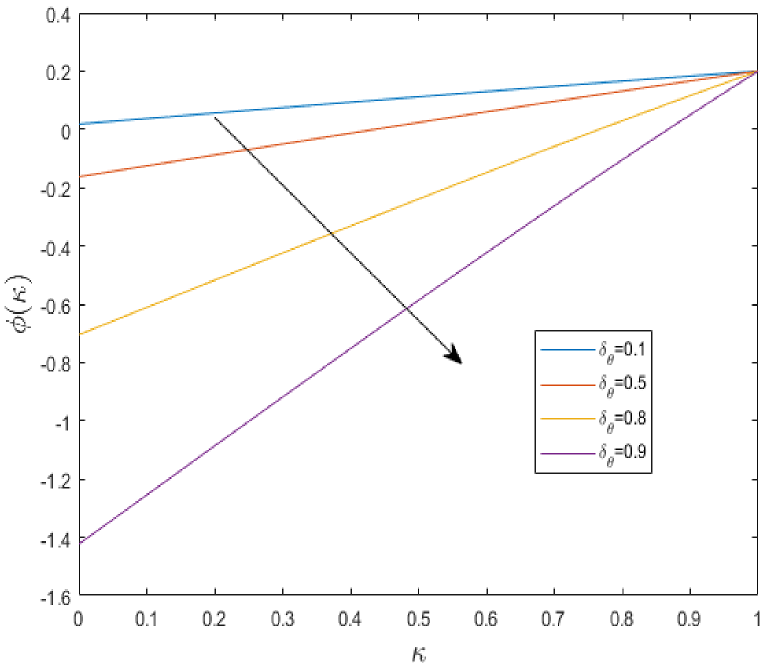

FIGURE 3.13: Effect of δ_θ when $M = 0.2$, $\lambda = 1$, $w = 0.1$,$Rd = 0.2$,$Nt = 0.2$,$Nb = 0.1$,$Le = 0.2$,$Pr = 0.5$, $\delta_\phi = 0.2$, on $\phi(\kappa)$.

Chapter 4

MHD Squeezing Nanofluid Flow between Two Parallel Plates with Cattaneo-Christov Double Diffusion and Thermal Radiation

4.1 Introduction

This chapter contains an extension of the model discussed in [46] by considering aligned magnetic field in the momentum equation. The Cattaneo-Christov Double Diffusion are also included in the temperature equation. The governing nonlinear PDEs are converted into a system of dimensionless ODEs by utilizing the similarity transformations. The numerical solution of ODEs is obtaind by applying numerical method known as shooting method. At the end of this chapter, the final results are discussed for significant parameters affecting $f(\eta)$, $\theta(\eta)$ and $\phi(\eta)$ which are shown in tables and graphs.

4.2 Mathematical Modeling

In this chapter, the effects of MHD and thermal radiations between two parallel plates for unsteady, two-dimensional and symmetric-nature viscous incompressible fluid flow are considered. The lower plate is kept fixed on the horizontal x-axis while upper plate can moved with velocity $v(t) = \frac{dh}{dt}$. The y-axis of the plates is at the perpendicular position to the lower plate. The constant magnetic-field B_0 is acting in the y-direction. The analysis for energy transport is carried out in the presence of thermal radiation, viscous dissipation and Cattaneo-Christov Double Diffusion.

4.3 Model Development

By considering the above assumptions, the governing PDEs are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \sigma B_o^2 u(t), \tag{4.2}$$

$$\rho_{nf}\left(\frac{\partial v}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{4.3}$$

$$\begin{aligned}
&\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) \\
&+ \lambda_T\left[u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} + v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} + u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y} + v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y} + u^2\frac{\partial^2 T}{\partial x^2} + v^2\frac{\partial^2 T}{\partial y^2} + 2uv\frac{\partial^2 T}{\partial x \partial y}\right] \\
&= \hat{\hat{\alpha}}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \\
&+ \tau\left[D_B\left\{\frac{\partial C}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial C}{\partial y}\frac{\partial T}{\partial y}\right\} + \left(\frac{D_T}{T_0}\right)\left\{\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2\right\}\right] \\
&\quad - \frac{1}{(\rho * c_p)_f}\frac{\partial q_{rd}}{\partial y},
\end{aligned} \tag{4.4}$$

$$\left(\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}\right)$$
$$+\lambda_C\left[u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x}+v\frac{\partial u}{\partial y}\frac{\partial C}{\partial x}+u\frac{\partial v}{\partial x}\frac{\partial C}{\partial y}+v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y}+u^2\frac{\partial^2 C}{\partial x^2}+v^2\frac{\partial^2 C}{\partial y^2}+2uv\frac{\partial^2 C}{\partial x\partial y}\right]$$
$$=D_B\left\{\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right\}+\frac{D_T}{T_0}\left\{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right\}. \tag{4.5}$$

The associated boundary conditions are

$$\left.\begin{array}{ll} v=0, u=0, T=T_2, D_B\left(\frac{\partial C}{\partial y}\right)+\frac{D_T}{T_0}\left(\frac{\partial T}{\partial y}\right)=0 & at \quad y=0 \\ v=\frac{dh}{dt}, u=0, C=C_1, T=T_1 & at \quad y=h(t). \end{array}\right\} \tag{4.6}$$

In equations (4.1-4.5), u and v are the velocities in x and y direction respectively, T is taken as temperature at the plates and C represents volumetric fraction of the nano particles. Moreover, ρ represents the density of the nanofluid, μ represents viscosity, D_B is taken as the Brownian diffusion and D_T represents the thermophoretic coefficient.

In Equation (4.4), the radiative heat flux is

$$q_r=-\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}.$$

Here σ^* and k^* are the Stefan-Boltzman constant and the absorption coefficient respectively. For very small temperature difference, T^4 can be expanded about T_0 with the help of Taylor series, as follows.

$$T^4=T_0^4+4T_0^3(T-T_0)+6T_0^2(T-T_0)^2+...$$

Ignoring the higher order terms, we have

$$T^4=T_0^4+4T_0^3(T-T_0)$$

$$=T_0^4+4T_0^3T-4T_0^4$$

$$=-3T_0^4+4T_0^3T$$

$$=4T_0^3T-3T_0^4.$$

Following similarity transformation has been used to convert the PDEs (4.1) to (4.5) into the system of ODEs.

$$\left.\begin{aligned} &\psi\left(x,y\right)=\left(\tfrac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}}xf(\eta),\\ &u=\left(\tfrac{1-\bar{\alpha}t}{bx}\right)^{-1}f'(\kappa),\\ &v=-\left(\tfrac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}}f(\kappa),\\ &\kappa=\left(\tfrac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}}y,\\ &\theta\left(\kappa\right)=\left(\tfrac{T-T_0}{T_2-T_0}\right),\\ &\phi\left(\kappa\right)=-1+\tfrac{C}{C_0}. \end{aligned}\right\}. \tag{4.7}$$

Here ψ stands for stream function, κ denotes the similarity variable, f, θ and ϕ are the dimensionless velocity, temperature as well as concentration respectively. The detailed procedure for the conversion of (4.1) has been discussed in Chapter 3.

The detailed procedure for conversion of (4.2) and (4.3) has been also discussed in Chapter 3.

Included below is the procedure for the conversion of (4.4) into the dimensionless form

$$\frac{\partial T}{\partial t}=\left(T_2-T_0\right)\theta'\left(\kappa\right)\frac{\bar{\alpha}\nu}{2b}\left(\frac{\nu\left(1-\bar{\alpha}t\right)}{b}\right)^{\frac{-3}{2}}y \tag{4.8}$$

$$\frac{\partial T}{\partial y}=\left(T_2-T_0\right)\theta'\left(\kappa\right)\left(\frac{\nu\left(1-\bar{\alpha}t\right)}{b}\right)^{\frac{-1}{2}}. \tag{4.9}$$

$$\frac{\partial^2 T}{\partial y^2}=\left(T_2-T_0\right)\left[\frac{b}{\nu\left(1-\bar{\alpha}t\right)}\right]\theta''\left(\kappa\right). \tag{4.10}$$

$$u\frac{\partial T}{\partial x}=0. \tag{4.11}$$

$$v\frac{\partial T}{\partial y}=-\left(T_2-T_0\right)\left(\frac{b}{1-\bar{\alpha}t}\right)f\left(\kappa\right)\theta'\left(\kappa\right). \tag{4.12}$$

$$\left(\frac{\partial T}{\partial x}\right)^2=0. \tag{4.13}$$

$$\frac{\partial T}{\partial y}^2 = \left[\left(T_2 - T_0\right) \theta'(\kappa) \left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}} \right]^2. \tag{4.14}$$

$$\frac{\partial C}{\partial x} = 0. \tag{4.15}$$

$$\frac{\partial C}{\partial y} = C_0 \phi'(\kappa) \left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}. \tag{4.16}$$

$$\frac{\partial^2 C}{\partial x^2} = 0. \tag{4.17}$$

$$\frac{\partial^2 C}{\partial y^2} = C_0 \phi''(\kappa) \left[\frac{b}{\nu(1-\bar{\alpha}t)}\right]. \tag{4.18}$$

$$\frac{\partial C}{\partial t} = C_0 \phi'(\kappa) \frac{\bar{\alpha}\nu}{2b} \kappa \left(\frac{b}{\nu(1-\bar{\alpha}t)}\right). \tag{4.19}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left(\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1} f'(\kappa) \right). \tag{4.20}$$

$$\frac{\partial u}{\partial y} = \left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1} \left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}} f''(\kappa). \tag{4.21}$$

$$\frac{\partial v}{\partial x} = 0. \tag{4.22}$$

$$\frac{\partial v}{\partial y} = -\frac{(1-\bar{\alpha}t)^{-1}}{b^{-1}} f'(\kappa). \tag{4.23}$$

$$\frac{\partial^2 T}{\partial x \partial y} = 0. \tag{4.24}$$

Now Energy equation (4.4) gets the following form

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \hat{\bar{\alpha}} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \tau \left[D_B \left\{ \frac{\partial C}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial C}{\partial y}\frac{\partial T}{\partial y} \right\} + \left(\frac{D_T}{T_0}\right) \left\{ \left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 \right\} \right] - \frac{1}{(\rho c_p)_f}\frac{\partial q_{rd}}{\partial y}. \tag{4.25}$$

$$\Rightarrow (T_2 - T_0)\phi'(\kappa) \frac{\bar{\alpha}\nu}{2b} \left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-3}{2}} y - (T_2 - T_0)\left(\frac{b}{1-\bar{\alpha}t}\right) \theta'(\kappa) f(\kappa) = \bar{\alpha}(T_2 - T_0) \frac{b}{\nu(1-\bar{\alpha}t)} \theta''(\kappa)$$

$$+\lambda D_B\left\{C_0\phi'(\kappa)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}(T_2-T_0)\,\theta'(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}}\right\}$$
$$+\lambda\left(\frac{D_T}{T_0}\right)\left\{(T_2-T_0)^2(\theta'(\kappa))^2\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-1}\right\}$$
$$-\frac{1}{\rho c_p}\frac{16T_0^3\sigma}{3k\star}(T_2-T_0)\left(\frac{b}{\nu(1-\bar{\alpha}t)}\right)\theta''(\kappa). \tag{4.26}$$

$$\Rightarrow\phi'(\kappa)\frac{\bar{\alpha}\nu}{2b}\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-3}{2}}y-\left(\frac{b}{1-\bar{\alpha}t}\right)\theta'(\kappa)f(\kappa)$$
$$=\bar{\alpha}\frac{b}{\nu(1-\bar{\alpha}t)}\theta''(\kappa)+\lambda D_B\left\{C_0\phi'(\kappa)\left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}\theta'(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}}\right\}$$
$$+\lambda\left(\frac{D_T}{T_0}\right)\left\{(T_2-T_0)(\theta'(\kappa))^2\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-1}\right\}$$
$$-\frac{1}{\rho c_p}\frac{16T_0^3\sigma}{3k^\star}\left(\frac{b}{\nu(1-\bar{\alpha}t)}\right)\theta''(\kappa). \tag{4.27}$$

As,

$$\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-3}{2}}=\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}}\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-1}.$$

$$\Rightarrow\phi'(\kappa)\frac{\bar{\alpha}\nu}{2b}(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-1}-\left(\frac{b}{1-\bar{\alpha}t}\right)\theta'(\kappa)f(\kappa)=\bar{\alpha}\frac{b}{\nu(1-\bar{\alpha}t)}\theta''(\kappa)$$
$$+\lambda\left[D_B\left\{C_0\phi'(\kappa)\,\theta'(\kappa)\frac{b}{\nu(1-\bar{\alpha}t)}\right\}+\left(\frac{D_T}{T_0}\right)\left\{(T_2-T_0)(\theta'(\kappa))^2\frac{b}{\nu(1-\bar{\alpha}t)}\right\}\right]$$
$$-\frac{1}{\rho c_p}\frac{16T_0^3\sigma}{3k^\star}\left(\frac{b}{\nu(1-\bar{\alpha}t)}\right)\theta''(\kappa). \tag{4.28}$$

Multiplying by ν and Dividing by $\bar{\alpha}$,we get

$$P_r\left[\theta'(\kappa)\lambda\kappa-\theta'(\kappa)f(\kappa)\right]=\left(1+\frac{4}{3}Rd\right)\theta''(\kappa)+N_b\phi'\theta'+N_t(\theta')^2. \tag{4.29}$$
$$\Rightarrow\left(1+\frac{4}{3}Rd\right)\theta''+P_r(f-\lambda\kappa)\,\theta'+N_b\phi'\theta'+N_t(\theta')^2=0. \tag{4.30}$$
$$u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x}=0. \tag{4.31}$$

$$v\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} = 0. \tag{4.32}$$

$$u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y} = 0. \tag{4.33}$$

$$v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y} = \left(\frac{(1-\bar{\alpha}t)}{b\nu}\right)^{\frac{-1}{2}} f(\kappa) - \frac{b}{1-\bar{\alpha}t} f'(\kappa)(T_2 - T_0)\theta'(\kappa)\left(\frac{\nu\,(1-\bar{\alpha}t)}{b}\right)^{\frac{-1}{2}}. \tag{4.34}$$

$$= \frac{b^{\frac{1}{2}}\nu^{\frac{1}{2}}}{(1-\bar{\alpha}t)^{\frac{1}{2}}} f(\kappa)\frac{b}{1-\bar{\alpha}t} f'(\kappa)\frac{b^{\frac{1}{2}}}{\nu^{\frac{1}{2}}(1-\bar{\alpha}t)^{\frac{1}{2}}}(T_2 - T_0)\theta'(\kappa). \tag{4.35}$$

$$= \frac{b^2}{(1-\bar{\alpha}t)^2}(T_2 - T_0) f(\kappa) f'(\kappa)\theta'(\kappa). \tag{4.36}$$

$$v^2\frac{\partial^2 T}{\partial y^2} = \left(-\left(\frac{(1-\bar{\alpha}t)}{b\nu}\right)^{\frac{-1}{2}} f(\kappa)\right)^2 (T_2 - T_0)\frac{b}{\nu(1-\bar{\alpha}t)}\theta''(\kappa). \tag{4.37}$$

$$= \frac{b}{\nu(1-\bar{\alpha}t)}\left(f(\kappa)\right)^2 (T_2 - T_0)\frac{b}{\nu(1-\bar{\alpha}t)}\theta''(\kappa). \tag{4.38}$$

$$= \frac{b^2}{(1-\bar{\alpha}t)^2}\left(f(\kappa)\right)^2\theta''(\kappa)(T_2 - T_0). \tag{4.39}$$

$$2uv\frac{\partial^2 T}{\partial x\partial y} = 0. \tag{4.40}$$

Adding (4.31) - (4.40)

$$\lambda_T\left[u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} + v\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} + u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y} + v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y} + v^2\frac{\partial^2 T}{\partial y^2} + 2uv\frac{\partial^2 T}{\partial x\partial y}\right]$$

$$\Rightarrow \lambda_T\Bigg[0 + 0 + 0 + \frac{b^2}{(1-\bar{\alpha}t)^2}(T_2 - T_0) f(\kappa) f'(\kappa)\theta'(\kappa)\frac{b^2}{(1-\bar{\alpha}t)^2} \left(f(\kappa)\right)^2\theta''(\kappa)(T_2 - T_0) + 0\Bigg]$$

$$\Rightarrow \lambda_T\left[\frac{b^2}{(1-\bar{\alpha}t)^2}(T_2 - T_0)\left\{f(\kappa) f'(\kappa)\theta'(\kappa) + \left(f(\kappa)\right)^2\theta''(\kappa)\right\}\right]$$

$$\Rightarrow \wedge_T\left\{f(\kappa) f'(\kappa)\theta'(\kappa) + \left(f(\kappa)\right)^2\theta''(\kappa)\right\}. \qquad \left(\because \wedge_T = \lambda_T\frac{b^2}{(1-\bar{\alpha}t)^2}(T_2 - T_0)\right) \tag{4.41}$$

Adding (4.31) and (4.41)

$$\left(1 + \frac{4}{3}Rd\right)\theta'' + P_r\left(f - \lambda\kappa\right)\theta' + N_b\phi'\theta' + N_t\left(\theta'\right)^2$$
$$+ \wedge_T\left\{f(\kappa) f'(\kappa)\theta'(\kappa) + \left(f(\kappa)\right)^2\theta''(\kappa)\right\} = 0. \tag{4.42}$$

Now, We include the below procedure for the conversion of (4.5) into the dimensionless form:

$$\frac{\partial C}{\partial x} = 0. \tag{4.43}$$

$$\frac{\partial C}{\partial y} = C_0 \phi'(\kappa) \left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}}. \tag{4.44}$$

$$\frac{\partial^2 C}{\partial x^2} = 0. \tag{4.45}$$

$$\frac{\partial^2 C}{\partial y^2} = C_0 \phi''(\kappa) \left[\frac{b}{\nu(1-\bar{\alpha}t)}\right]. \tag{4.46}$$

$$\frac{\partial C}{\partial t} = C_0 \phi'(\kappa) \frac{\bar{\alpha}\nu}{2b} \kappa \left(\frac{b}{\nu(1-\bar{\alpha}t)}\right). \tag{4.47}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1} f'(\kappa)\right). \tag{4.48}$$

$$\frac{\partial u}{\partial y} = \left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1} \left[\frac{\nu(1-\bar{\alpha}t)}{b}\right]^{\frac{-1}{2}} f''(\kappa). \tag{4.49}$$

$$\frac{\partial v}{\partial x} = 0. \tag{4.50}$$

$$\frac{\partial v}{\partial y} = -\frac{(1-\bar{\alpha}t)^{-1}}{b^{-1}} f'(\kappa). \tag{4.51}$$

$$C_0 \phi'(\kappa) \frac{\bar{\alpha}\nu}{2b} \kappa\kappa \left(\frac{b}{\nu(1-\bar{\alpha}t)}\right) + C_0 \left(\frac{-b}{(1-\bar{\alpha}t)}\right) \phi'(\kappa) f(\kappa)$$
$$= D_B \left\{C_0 \phi''(\kappa) \frac{b}{(1-\bar{\alpha}t)}\right\} + \frac{D_T}{T_0}(T_2 - T_0) \frac{b}{\nu(1-\bar{\alpha}t)} \theta''(\kappa) \tag{4.52}$$

$$\Rightarrow C_0 \phi'(\kappa) \lambda\kappa - C_0 \phi'(\kappa) f(\kappa) = D_B \left\{C_0 \phi''(\kappa) \frac{1}{\nu}\right\} + \frac{D_T}{T_0}\left\{(T_2 - T_0)\frac{1}{\nu}\theta''(\kappa)\right\}. \tag{4.53}$$

Dividing by C_0;

$$\phi'(\kappa)\lambda\kappa - \phi'(\kappa) f(\kappa) = \frac{D_B}{\nu}\phi''(\kappa) + \frac{D_T}{T_0 C_0}\left\{(T_2 - T_0)\frac{1}{\nu}\theta''(\kappa)\right\} \tag{4.54}$$

Now multiplying by $\frac{\nu}{D_B}$;

$$\left(\phi'\left(\kappa\right)\lambda\kappa-\phi'\left(\kappa\right)f\left(\kappa\right)\right)\frac{\nu}{D_B}=\phi''\left(\kappa\right)+\frac{D_T}{D_BT_0C_0}\left\{\left(T_2-T_0\right)\frac{1}{\nu}\theta''\left(\kappa\right)\right\} \tag{4.55}$$

$$\Rightarrow\phi''+Le\left(f-\lambda(\kappa)\right)\phi'+\left(\frac{Nt}{Nb}\right)\theta''=0. \tag{4.56}$$

$$u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x}=0. \tag{4.57}$$

$$v\frac{\partial u}{\partial x}\frac{\partial C}{\partial x}=0. \tag{4.58}$$

$$u\frac{\partial v}{\partial x}\frac{\partial C}{\partial y}=0. \tag{4.59}$$

$$v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y}=\left(\frac{(1-\bar{\alpha}t)}{b\nu}\right)^{\frac{-1}{2}}(\kappa)-\frac{b}{1-\bar{\alpha}t}f'(\kappa)C_0\phi'(\kappa)\left(\frac{\nu\left(1-\bar{\alpha}t\right)}{b}\right)^{\frac{-1}{2}}$$

Now consider,

$$\Rightarrow\quad=\frac{b^{\frac{1}{2}}v^{\frac{1}{2}}}{(1-\bar{\alpha}t)^{\frac{1}{2}}}f(\kappa)\frac{b}{1-\bar{\alpha}t}f'(\kappa)\frac{b^{\frac{1}{2}}}{v^{\frac{1}{2}}(1-\bar{\alpha}t)^{\frac{1}{2}}}C_0\phi'(\kappa)$$

$$\Rightarrow\quad=\frac{b^2}{(1-\bar{\alpha}t)^2}C_0f(\kappa)f'(\kappa)\phi'(\kappa). \tag{4.60}$$

$$u^2\frac{\partial^2 C}{\partial x^2}=0. \tag{4.61}$$

$$v^2\frac{\partial^2 C}{\partial y^2}=\left(-\left(\frac{(1-\bar{\alpha}t)}{b\nu}\right)^{\frac{-1}{2}}f(\kappa)\right)^2C_0\frac{b}{\nu(1-\bar{\alpha}t)}\phi''\left(\kappa\right)$$

$$\Rightarrow\quad=\frac{b}{\nu(1-\bar{\alpha}t)}\left(f(\kappa)\right)^2C_0\frac{b}{\nu(1-\bar{\alpha}t)}\phi''\left(\kappa\right)$$

$$\Rightarrow\quad=\frac{b^2}{(1-\bar{\alpha}t)^2}\left(f(\kappa)\right)^2\phi''\left(\kappa\right)C_0. \tag{4.62}$$

$$2uv\frac{\partial^2 C}{\partial x\partial y}=0. \tag{4.63}$$

Adding (4.57) - (4.63)

$$\lambda_C\left[u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x}+v\frac{\partial u}{\partial x}\frac{\partial C}{\partial x}+u\frac{\partial v}{\partial x}\frac{\partial C}{\partial y}+v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y}+v^2\frac{\partial^2 C}{\partial y^2}+2uv\frac{\partial^2 C}{\partial x\partial y}\right]$$
$$\Rightarrow\lambda_C\left[0+0+0+\frac{b^2}{(1-\bar{\alpha}t)^2}C_0 f(\kappa)f'(\kappa)\phi'(\kappa)\frac{b^2}{(1-\bar{\alpha}t)^2}\right.$$
$$\left.(f(\kappa))^2\,\phi''(\kappa)\,C_0+0\right]$$
$$\Rightarrow\lambda_C\left[\frac{b^2}{(1-\bar{\alpha}t)^2}C_0\left\{f(\kappa)f'(\kappa)\phi'(\kappa)+(f(\kappa))^2\,\phi''(\kappa)\right\}\right],$$
$$\Rightarrow\wedge_C\left\{f(\kappa)f'(\kappa)\phi'(\kappa)+(f(\kappa))^2\,\phi''(\kappa)\right\}. \qquad \left(\because\wedge_C=\lambda_C\frac{b^2}{(1-\bar{\alpha}t)^2}C_0\right) \tag{4.64}$$

Adding (4.56) and (4.64)

$$\phi''+Le\,(f-\lambda(\kappa))\,\phi'+\left(\frac{Nt}{Nb}\right)\theta''+\wedge_C\left\{f(\kappa)f'(\kappa)\phi'(\kappa)+(f(\kappa))^2\,\phi''(\kappa)\right\}=0. \tag{4.65}$$

The corresponding BCs are transformed into the non-dimensional form through the following procedure:

$$v=0 \qquad at \qquad y=0.$$
$$\Rightarrow-\left(\frac{1-\bar{\alpha}t}{b\nu}\right)^{\frac{-1}{2}}f(\kappa)=0 \qquad at \qquad \kappa=0.$$
$$\Rightarrow f(\kappa)=0 \qquad at \qquad \kappa=0.$$
$$\Rightarrow f(0)=0.$$
$$u=0 \qquad at \qquad y=0.$$
$$\Rightarrow\left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1}f'(\kappa)=0 \qquad at \qquad y=0.$$
$$\Rightarrow f'(\kappa)=0 \qquad at \qquad \kappa=0.$$
$$\Rightarrow f'(0)=0.$$

$$T = T_2 \qquad at \qquad y = 0.$$

$$\Rightarrow (T_2 - T_0)\,\theta\,(\kappa) + T_0 = T_2 \qquad at \qquad \kappa = 0.$$

$$\Rightarrow \theta(\kappa) = 1 \qquad at \qquad \kappa = 0.$$

$$\Rightarrow \theta(0) = 1$$

As

$$C = C_0(1 + \phi(\kappa))$$

$$\frac{\partial C}{\partial y} = C_0(\phi'(\kappa)\frac{\partial \kappa}{\partial y})$$

$$= C_0\phi'(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-\frac{1}{2}}$$

$$\frac{\partial T}{\partial y} = (T_2 - T_1)\theta'(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-\frac{1}{2}}$$

Now

$$D_B(\frac{\partial C}{\partial y}) + \frac{D_T}{T_0}(\frac{\partial T}{\partial y}) = 0 \qquad at \qquad y = 0.$$

$$\Rightarrow D_B(C_0\phi'(\kappa)\left(\frac{\nu(1-\bar{\alpha}t)}{b}\right)^{-\frac{1}{2}})$$

$$+ \frac{D_T}{T_0}((T_2 - T_0)\phi(\kappa)\left(\frac{v(1-\bar{\alpha}t)}{b}\right)^{-\frac{1}{2}}) = 0 \qquad at \qquad \kappa = 0.$$

$$\Rightarrow D_B(C_0\phi'(\kappa)) + \frac{D_T}{T_0}((T_2 - T_0)\phi(\kappa)) = 0 \qquad at \qquad \kappa = 0.$$

Multiplying $\frac{(\rho_c)_p}{(\rho_c)_f}.\frac{C_0}{\bar{\alpha}}$ on both sides

$$\Rightarrow \frac{(\rho_c)_p}{(\rho_c)_f}\frac{D_B C_0}{\bar{\alpha}}\phi'(\kappa) + \frac{(\rho_c)_p}{(\rho_c)_f}\frac{D_T(T_2 - T_0)}{T_0\bar{\alpha}}\phi'(\kappa) = 0 \qquad at \qquad \kappa = 0.$$

$$\Rightarrow N_b\phi'(\kappa) + N_t\phi'(\kappa) = 0 \qquad at \qquad \kappa = 0.$$

$$\Rightarrow N_b\phi'(0) + N_t\phi'(0) = 0$$

$$v = \frac{dh}{dt} \qquad at \qquad y = h(t).$$

$$\Rightarrow -(\frac{1-\bar{\alpha}t}{b\nu})^{-\frac{1}{2}} f(\kappa) = \frac{1}{2}(\frac{\nu(1-\bar{\alpha}t)}{b})^{-\frac{1}{2}}(-\frac{\bar{\alpha}\nu}{b}) \qquad at \qquad \kappa = 1.$$

$$\Rightarrow f(\kappa) = \frac{\bar{\alpha}\nu}{2b} \qquad at \qquad \kappa = 0.$$

$$\Rightarrow f(1) = w$$

$$u = 0 \qquad at \qquad y = h(t).$$

$$\Rightarrow \left(\frac{1-\bar{\alpha}t}{bx}\right)^{-1} f'(\kappa) = 0 \qquad at \qquad \kappa = 1.$$

$$\Rightarrow f'(\kappa) = 0 \qquad at \qquad \kappa = 1.$$

$$\Rightarrow f'(1) = 0.$$

$$T = T_1 \qquad at \qquad y = h(t).$$

$$\Rightarrow (T_2 - T_0)\,\theta(\kappa) + T_0 = T_1 \qquad at \qquad \kappa = 1.$$

$$\Rightarrow \theta(\kappa) = \delta_\theta \qquad at \qquad \kappa = 1.$$

$$\Rightarrow \theta(1) = \delta_\theta.$$

$$C = C_1 \qquad at \qquad y = h(t).$$

$$\Rightarrow C_0\phi(\kappa) = -C_0 + C_1 \qquad at \qquad \kappa = 1.$$

$$\Rightarrow \phi(\kappa) = \frac{-C_0 + C_1}{C_0} \qquad at \qquad \kappa = 1.$$

$$\Rightarrow \phi(1) = \delta_\phi \qquad at \qquad \kappa = 1.$$

$$\Rightarrow \phi(1) = \delta_\phi$$

The final dimensionless form of the governing model is,

$$f^{iv} + ff''' - f'f'' - \lambda\kappa f''' - 3\lambda f'' - Mf'' = 0. \tag{4.66}$$

$$\left(1 + \frac{4}{3}Rd\right)\theta'' + Pr\,(f - \lambda\kappa)\,\theta' + Nb\phi'\theta' + Nt\,(\theta')^2$$
$$+ \wedge_T \left\{f(\kappa)f'(\kappa)\theta'(\kappa) + (f(\kappa))^2\,\theta''(\kappa)\right\} = 0. \tag{4.67}$$

$$\phi'' - Le\,(f - \lambda\kappa)\,\phi' + \left(\frac{Nb}{Nt}\right)\theta'' + \wedge_C \left\{f(\kappa)f'(\kappa)\phi'(\kappa) + (f(\kappa))^2\,\phi''(\kappa)\right\} = 0. \tag{4.68}$$

The associated BCs in the dimensionless form are,

$$\left\{\begin{array}{l} f(0)=0, f'(0)=0, f'(1)=0, f(1)=w, \theta(1)=\delta_{\theta}, \\ \theta(0)=1, \phi(1)=\delta_{\phi}, \phi'(0)Nb+\theta'(0)Nt=0 \end{array}\right\}. \tag{4.69}$$

Different dimensionless parameters used in equations (4.66)-(4.68) are formulated as follow:

$$\lambda=\frac{\alpha}{2b}, \quad M=\frac{\sigma B_0^2}{\rho b}(1-\alpha t), \quad Rd=\frac{4T^3\sigma}{3(\rho c_p)_f k\alpha},$$
$$Nt=\frac{(\rho c_p)_p}{(\rho c_p)_f}\frac{D_T(T_2-T_0)}{T_0\alpha}, \quad Nb=\frac{(\rho c_p)_p}{(\rho c_p)_f}\frac{D_b C_0}{\alpha}, \quad Pr=\frac{v}{\alpha},$$
$$Le=\frac{v}{D_B}, \quad w=\frac{\alpha H}{2(vb)^{\frac{1}{2}}}, \quad \delta\phi=\left(\frac{C_1-C_0}{C_0}\right), \quad \delta\theta=\frac{(T_1-T_0)}{(T_2-T_0)},$$
$$\wedge_T=\lambda_T\frac{b^2}{(1-\bar{\alpha}t)^2}(T_2-T_0), \wedge_C=\lambda_C\frac{b^2}{(1-\bar{\alpha}t)^2}(C_0).$$

4.4 Solution Methodology

For solving eq. (4.69) with associated boundary conditions (4.72). We used shooting method.

First of all we convert the fourth order ODE into the system of first order ODEs.In order to solve IVP, we used thr RK4 method and assume the missing initial conditions. Now, we used following notations:

$$f=y_1,$$
$$f'={y_1}'=y_2,$$
$$f''={y_1}''={y_2}'=y_3,$$
$$f'''={y_1}'''={y_2}''={y_3}'=y_4.$$

The resulting inital value problem takes the form:

$${y_1}'=y_2, \qquad y_1(0)=0.$$

$$y_2{}' = y_3, \qquad y_2(0) = 0.$$

$$y_3{}' = y_4, \qquad y_3(0) = r.$$

$$y_4' = -y_1y_4 + y_2y_3 + \lambda\kappa y_4 + 3\lambda y_3 + My_3, \qquad y_4(0) = s.$$

Missing conditions $'r'$ and $'s'$ are assumed to satisfy the following relations:

$$(y_1(r,s))_{\kappa=\kappa_\infty} = w$$

$$(y_2(r,s))_{\kappa=\kappa_\infty} = 0$$

To solve the above algebaric equations, we apply the Newton's method which has the following scheme.

$$\begin{bmatrix} r \\ s \end{bmatrix}_{n+1} = \begin{bmatrix} r \\ s \end{bmatrix}_n - \begin{bmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial s} \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial s} \end{bmatrix}_n^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_n \tag{4.70}$$

where n=0,1,2,...

Now utilize the following notations:

$$\frac{\partial y_1}{\partial r} = y_5, \qquad \frac{\partial y_2}{\partial r} = y_6$$
$$\frac{\partial y_3}{\partial r} = y_7, \qquad \frac{\partial y_4}{\partial r} = y_8$$
$$\frac{\partial y_1}{\partial s} = y_9, \qquad \frac{\partial y_2}{\partial s} = y_{10}$$
$$\frac{\partial y_3}{\partial s} = y_{11}, \qquad \frac{\partial y_4}{\partial s} = y_{12}$$

As the result of these new notations, the Newton's iterative scheme gets the form.

$$\begin{bmatrix} r \\ s \end{bmatrix}_{n+1} = \begin{bmatrix} r \\ s \end{bmatrix}_n - \begin{bmatrix} y_5 & y_9 \\ y_6 & y_{10} \end{bmatrix}_n^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_n$$

Now differentiating the system of four first order ODEs with respect to r and s, we get another system of ODEs, as follows.

$$
\begin{aligned}
&{y_5}' = y_6, && y_5(0) = 0.\\
&{y_6}' = y_7, && y_6(0) = 0.\\
&{y_7}' = y_8, && y_7(0) = 1.\\
&y_8' = -y_5y_4 - y_1y_8 + y_2y_7 + y_6y_3 + \lambda\kappa y_8 + 3\lambda y_7 + My_7, && y_8(0) = 0.\\
&{y_9}' = y_{10}, && y_9(0) = 0.\\
&{y_{10}}' = y_{11}, && y_{10}(0) = 0.\\
&{y_{11}}' = y_{12}, && y_{11}(0) = 0.\\
&y_{12}' = -y_1y_{12} - y_9y_4 + y_{10}y_3 + y_2y_{11} + \lambda\kappa y_{12} + 3\lambda y_{11} + My_{11}, && y_{12}(0) = 1.
\end{aligned}
$$

RK4 method is used to solve the IVP and initial values are choosen arbitrarily. On executing the iterations, these initial values will be updated with the help of Newton's method and the whole process will be continued until the following criteria is achieved.

$$max\{|y_1(\kappa_\infty)|, |y_2(\kappa_\infty)|\} < \epsilon.$$

Throughout this work, ϵ has been taken as 10^{-10} unless otherwise mentioned. Also, for equations (4.70) and (4.71) the following notation have been used

$$
\begin{aligned}
&\theta = Y_1, \theta' = Y_1' = Y_2, \theta'' = Y_2'\\
&\phi = Y_3, \phi' = Y_3' = Y_4, \phi'' = Y_4'\\
&A = \left(1 + \frac{4}{3}R_d\right).
\end{aligned}
$$

The system of equations (4.46) and (4.47), can be taken in the form of the following first order coupled ODEs

$$Y_1' = Y_2, \qquad Y_1(0) = 1.$$

$$Y_2' = -\frac{1}{A + \wedge_T f(\kappa)^2}$$

$$\left[P_r(f - \lambda\kappa)Y_2 + N_b Y_4 Y_2 + N_t (Y_2)^2 + \wedge_T (f(\kappa) f'(\kappa) Y_2)\right] \qquad Y_2(0) = P.$$

$$Y_3' = Y_4, \qquad Y_3(0) = Q.$$

$$Y_4' = -\frac{1}{1 + \wedge_c f(\kappa)^2}\{L_e(f - \lambda\kappa)Y_4\}$$

$$-\frac{1}{1 + \wedge_c f(\kappa)^2}\frac{N_t}{N_b}\left[-\frac{1}{A + \wedge_T f(\kappa)^2}\left[P_r(f - \lambda\kappa)Y_2 + N_b Y_4 Y_2 + N_t (Y_2)^2 + \wedge_T f(\kappa) f'(\kappa) Y_2\right]\right]$$

$$-\frac{1}{1 + \wedge_c f(\kappa)^2}\wedge_T f(\kappa) f'(\kappa) Y_4 \qquad Y_4(0) = \frac{Nt}{Nb}(P).$$

The RK-4 method has been taken into consideration for solving the above initial value problem. For the system of equtions, the missing conditions are selected in such a way that

$$(Y_1(P, Q))_{\kappa=\kappa_\infty} = \delta_\theta, \qquad (Y_4(P, Q))_{\kappa=\kappa_\infty} = \delta_\phi.$$

To solve the above algebaric equations, we apply the Newton's method which has the following scheme.

$$\begin{bmatrix} P \\ Q \end{bmatrix}_{n+1} = \begin{bmatrix} P \\ Q \end{bmatrix}_n - \begin{bmatrix} \frac{\partial Y_1}{\partial P} & \frac{\partial Y_1}{\partial Q} \\ \frac{\partial Y_4}{\partial P} & \frac{\partial Y_4}{\partial Q} \end{bmatrix}_n^{-1} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}_n$$

Now, introduce the following notations,

$$\frac{\partial Y_1}{\partial P} = Y_5, \quad \frac{\partial Y_2}{\partial P} = Y_6, \quad \frac{\partial Y_3}{\partial P} = Y_7, \quad \frac{\partial Y_4}{\partial P} = Y_8.$$
$$\frac{\partial Y_1}{\partial Q} = Y_9, \quad \frac{\partial Y_2}{\partial Q} = Y_{10}, \quad \frac{\partial Y_3}{\partial Q} = Y_{11}, \quad \frac{\partial Y_4}{\partial Q} = Y_{12}.$$

As the result of these new notations, the Newton's iterative scheme gets the form.

$$\begin{bmatrix} P \\ Q \end{bmatrix}_{n+1} = \begin{bmatrix} P \\ Q \end{bmatrix}_n - \begin{bmatrix} Y_5 & Y_9 \\ Y_8 & Y_{12} \end{bmatrix}_n^{-1} \begin{bmatrix} Y_1 \\ Y_4 \end{bmatrix}_n$$

Now differentiating the system of four first order ODEs with respect to P and Q we get another system of ODEs, as follows.

$$Y_5' = Y_6 \qquad Y_5(0) = 0.$$

$$Y_6' = -\frac{1}{A + \lambda_T}$$

$$[P_r(f - \lambda\kappa)Y_6 + N_b Y_6 Y_4 + N_b Y_2 Y_8 + 2N_t Y_2 Y_6 + \lambda_T f(\kappa) f'(\kappa) Y_6] \qquad Y_6(0) = 1.$$

$$Y_7' = Y_8 \qquad Y_7(0) = 0.$$

$$Y_8' = -\frac{1}{1 + \wedge_c f(\kappa)^2}\{L_e(f - \lambda\kappa)Y_8\}$$

$$- \frac{1}{1 + \wedge_c f(\kappa)^2}\frac{N_t}{N_b}\left[-\frac{1}{A + \wedge_T f(\kappa)^2}\Big[P_r(f - \lambda\kappa)Y_6 + N_b Y_4 Y_6\right.$$

$$\left. + N_b Y_2 Y_8 + 2N_t Y_2 Y_6\Big] - [\wedge_T f(\kappa) f'(\kappa) Y_6]\right]$$

$$- \frac{1}{1 + \wedge_c f(\kappa)^2} \wedge_T f(\kappa) f'(\kappa) Y_8 \qquad Y_8(0) = -\frac{Nt}{Nb}.$$

$$Y_9' = Y_{10} \qquad Y_9(0) = 0.$$

$$Y_{10}' = -\frac{1}{A + \wedge_T f(\kappa)^2}$$

$$[P_r(f - \lambda\kappa)Y_{10} + N_b Y_{10} Y_4 + N_b Y_2 Y_{12} + 2N_t y_2 Y_{10} + \wedge_T f(\kappa) f'(\kappa) Y_{10}] \qquad Y_{10}(0) = 0.$$

$$Y_{11}' = Y_{12} \qquad Y_{11}(0) = 1.$$

$$Y_{12}' = -\frac{1}{1 + \wedge_c f(\kappa)^2}\{L_e(f - \lambda\kappa)Y_{12}\}$$

$$- \frac{1}{1 + \wedge_c f(\kappa)^2}\frac{N_t}{N_b}\left[-\frac{1}{A + \wedge_T f(\kappa)^2}\Big[P_r(f - \lambda\kappa)Y_{10} + N_b Y_4 Y_{10}\right.$$

$$\left. + N_b Y_2 Y_{12} + 2N_t Y_2 Y_{10}\Big] - [\wedge_T f(\kappa) f'(\kappa) Y_{10}]\right]$$

$$- \frac{1}{1 + \wedge_c f(\kappa)^2} \wedge_T f(\kappa) f'(\kappa) Y_{12} \qquad Y_{12}(0) = 0.$$

The stopping criteria for the Newton's method is set as.

$$max\{|Y_1(\kappa_\infty)|, |Y_4(\kappa_\infty)|\} < \epsilon.$$

4.5 Representation of Graphs and Tables

The principle aim is to study the effect of different parameters against the velocity $f(\kappa)$, temperature $\theta(\kappa)$ and concentration distribution $\phi(\kappa)$. The impact of different factors like magnetic parameter(M) , Prandtl number (Pr), Brownian motion parameter (Nb), thermophoretic parameter (Nt), as well as Levis number (Le) are analysed graphically. The Numerical outcomes for skin friction coefficient, Nusselt number and Sherwood number are shown in tables (4.1)-(4.2) for the distinct values of some fixed parameters.

A thorough discussion on the graphs and tables has been conducted which contains the impact of dimensionless parameters on the C_{fx}, Nu_x and Sh_x.

Table (4.1), explains the impact of λ, M, and w on the C_{fx}. Furthermore, the rising value of the influences of sequeezing fluid parameters λ, the C_{fx} decreases. Due to the rising value of parameter w, while C_{fx} is increased.

Table (4.2), the effect of significant parameters on Nusselt number Nu_x as well as Sherwood number Sh_x have been discussed. The reduction pattern is found in the Nu_x and Sh_x due to extending value of Pr, Le , Rd, Nt and δ_θ while the Nu_x increases and Sh_x increases due to w and $\wedge_T$.

Figures (4.1)- (4.3), represents the effect of λ on velocity profile $f(\kappa)$, temperature profile $\theta(\kappa)$ and concentration profile $\phi(\kappa)$ respectively.By enhancing the values of λ, the velocity profile decreases and increases the boundary layer thickness. Reason behind this behaviour is that, if we increases the value of λ then effective viscosity is increased.Due to increment in viscosity there is more resistance between fluid particles.

Figures (4.4)- (4.6) shows the impact of parameter w. For the rising values of w, the velocity profile $f(\kappa)$ is increased. The concentration profile $\phi(\kappa)$ also increases while there is a decrease in temperature profile $\theta(\kappa)$.

Figure (4.7) displays the impact of Prandtl number Pr on the temperature distribution. By rising the values of Pr, the temperature distribution show the increasing behaviour.

On concentration distribution, the impact of Prandtl number Pr is shown in Figure (4.8) which represent that when increasing the values of Pr,the concentration distribution also increased.

On temperature profile $\theta(\kappa)$, the effect of radiation parameter Rd can be seen in Figure (4.9). It can be seen that when by increasing values of Rd, more heat is generated.Due to this, $\theta(\kappa)$ and thermal boundary layer thickness are increased.

Figure (4.10) shows the effect of radiation parameter Rd on the concentration distribution. The higher values of Rd, shows decreasing behavior of concentration profile.

Figure (4.11) represent the effect of thermophoretic parameter Nt on temperature profile $\theta(\kappa)$. The temperature distribution expands by increasing the values of thermophoretic parameter Nt.

The effect of parameter Nt (thermophoretic) on concentration profile $\phi(\kappa)$ is shown in Figure (4.12). The concentration distribution reduce by increasing the values of thermophoretic parameter Nt.

Figure (4.13) display the influence of Levis number Le on the temperature distributions. The temperature distributions is increases when the Lewis number Le increased.

The relationship among Lewis numbers Le and the dimensional concentration distribution $\phi(\kappa)$ is shown in Figure (4.14). Concentration profile first increasing and then decreasing on rising values of Le and thus we have get a very small molecular diffusion and thermal boundary layer.

Figure (4.15) represent the effect of parameter $\wedge_T$ on the temperature profile $\theta(\kappa)$. An increment is noticed in temperature distribution by rising the values of parameter T.

From Figure (4.16), the effect of parameter $\wedge_T$ on the concentration profile $\phi(\kappa)$ can be seen. An decrement is noticed in concentration distribution by rising the values of parameter T.

Table 4.1: Results of $(Re_x)^{\frac{1}{2}}C_f$

λ	M	w	$(Re_x)^{\frac{1}{2}}C_f$
1.0	2.0	0.5	3.16644
3.0			3.14595
5.0			3.18456
9.0			3.35912
1.0	4.0		3.26187
	6.0		3.35472
	8.0		3.44513
	2.0	0.6	3.82377
		0.7	4.48912
		0.8	5.16248

TABLE 4.2: Results of $-(Re_x)^{\frac{-1}{2}} Nu_x$ and $-(Re_x)^{\frac{-1}{2}} Sh_x$ with fixed parameter $M = 0.1$

λ	w	Rd	Pr	Nt	Nb	Le	$\delta\theta$	$\delta\phi$	$\wedge_T$	$\wedge_C$	Nu_x	$-Sh_x$
0.2	0.2	0.1	5.0	0.1	0.2	2.0	0.2	0.1	0.1	0.2	0.78921	-0.39460
0.3											0.73210	-0.36605
0.6											0.56884	-0.28442
0.9											0.42344	-0.21172
0.2											0.78921	-0.39460
	0.3										0.84187	-0.42090
	0.6										0.99824	-0.49912
	0.9										1.1474	-0.57372
1.0	0.5		1.0	0.5		5.0	0.6	0.3		0.5	0.38066	-0.19033
			3.0					0.3			0.31894	-0.15947
			6.0								0.23632	-0.11816
			9.0								0.16791	-0.08395
						2.0					0.16429	-0.08214
						5.0					0.16791	-0.08955
						7.0					0.17122	-0.08561
						9.0					0.17561	-0.08780
	0.9		1.0						5.0	5.0	0.53092	-2.65463
											0.50249	-2.51229
		2.0									0.45814	-2.29070
		5.0									0.43179	-2.15897
			0.5						0.5	0.2	0.39882	-1.9941
									2.0		0.40233	-2.01168
									5.0		0.40903	-2.04519
									9.0		0.41738	-2.08690
									0.5	0.1	0.38066	-1.9933
										1.0	0.38066	-1.9947
										5.0	0.38066	-1.9946
										9.0	0.38066	-1.9945

λ	w	Rd	Pr	Nt	Nb	Le	$\delta\theta$	$\delta\phi$	$\wedge_T$	$\wedge_C$	Nu_x	$-Sh_x$
0.2			2.0		0.2	5.0		0.1	0.1	0.2	0.84214	-4.21074
				3.0	0.2						0.80236	-12.03547
				5.0							0.76292	-19.07314
				9.0							0.68674	-30.90341
					1.0						0.84214	-4.21074
					3.0						0.84214	-4.21074
					5.0						0.84214	-4.21074
					9.0						0.84214	-4.21074
					0.2				0.1		0.96727	- 0.4836
							0.2		0.1		0.86005	-0.43002
							0.5		0.1		0.53797	-0.26898
							0.9		0.1		0.10772	-0.05386
							0.2		0.1		0.96727	-0.48363
								0.3			0.96727	-0.48363
								0.5			0.96727	-0.48363
								0.9			0.96727	-0.48363

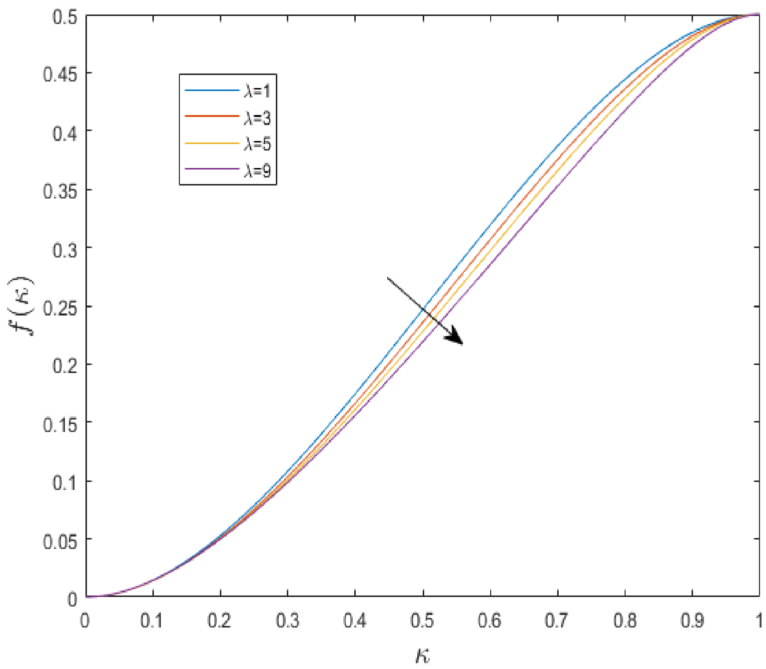

FIGURE 4.1: Effect of λ when $M = 2$, $w = 0.5$,on $f(\kappa)$.

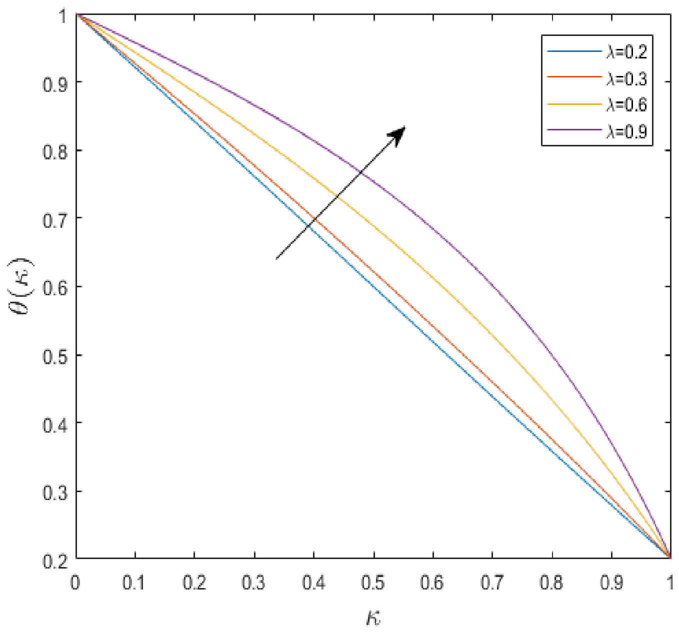

FIGURE 4.2: Effect of $\lambda$when $M = 0.1$, $w = 0.2$, $Rd = 0.1$,$Pr = 5$,$Nt = 0.1$,$Nb = 0.2$,$Le = 2$,$\delta_\theta = 0.2$, $\delta_\phi = 0.1$,$\wedge_T = 0.1$,$\wedge_C = 0.2$, on $\theta(\kappa)$.

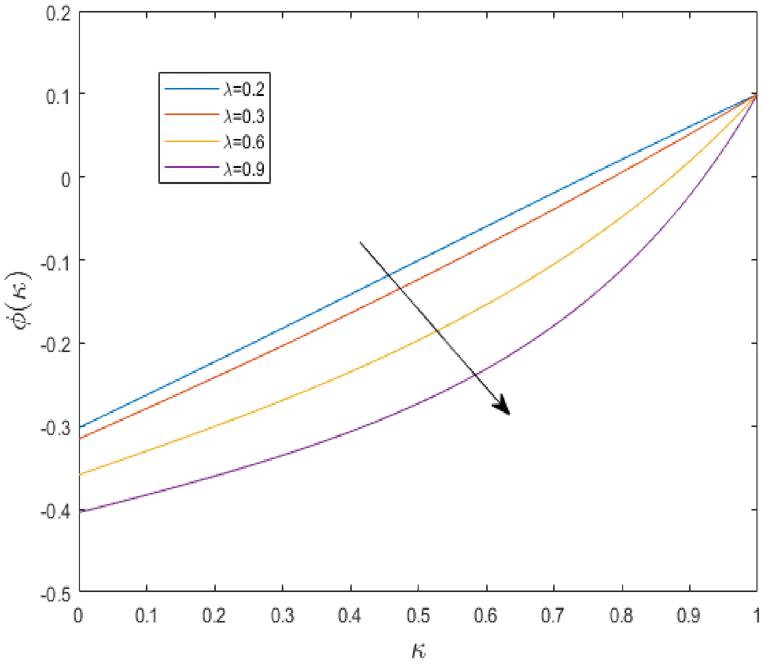

FIGURE 4.3: Effect of λ when $M = 0.1$, $w = 0.2$, $Rd = 0.1$,$Pr = 5$,$Nt = 0.1$,$Nb = 0.2$,$Le = 2$,$\delta_\theta = 0.2$, $\delta_\phi = 0.1$,$\wedge_T = 0.1$,$\wedge_C = 0.2$,on $\phi(\kappa)$.

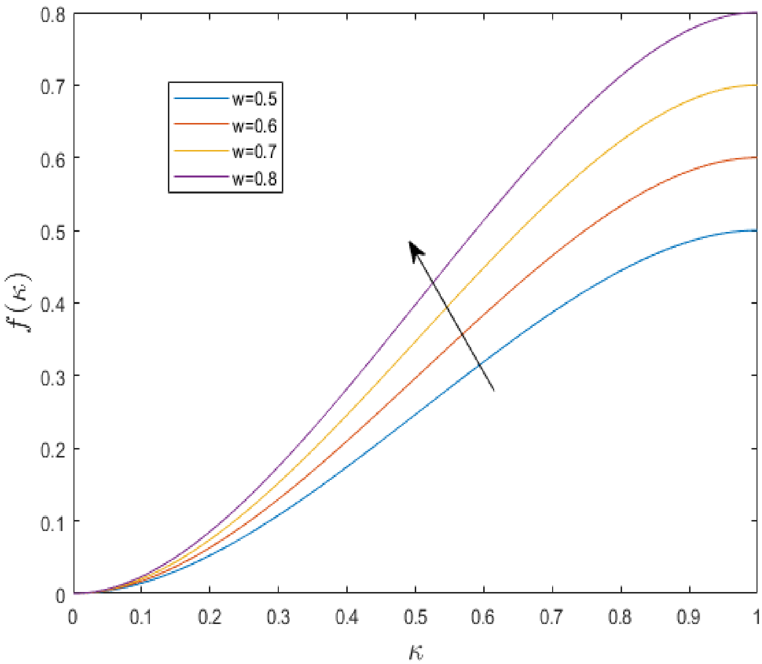

FIGURE 4.4: Effect of w when $M = 2$, $\lambda = 1$, on $f(\kappa)$.

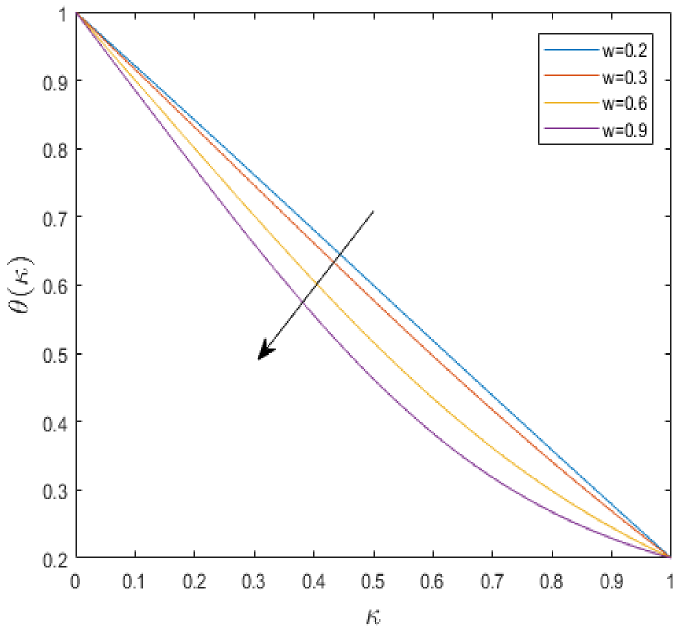

FIGURE 4.5: Effect of w when $M = 0.1$, $\lambda = 0.2$, $Rd = 0.1$,$Pr = 5$,$Nt = 0.1$,$Nb = 0.2$,$Le = 2$,$\delta_\theta = 0.2$, $\delta_\phi = 0.1$,$\wedge_T = 0.1$,$\wedge_C = 0.2$, on $\theta(\kappa)$.

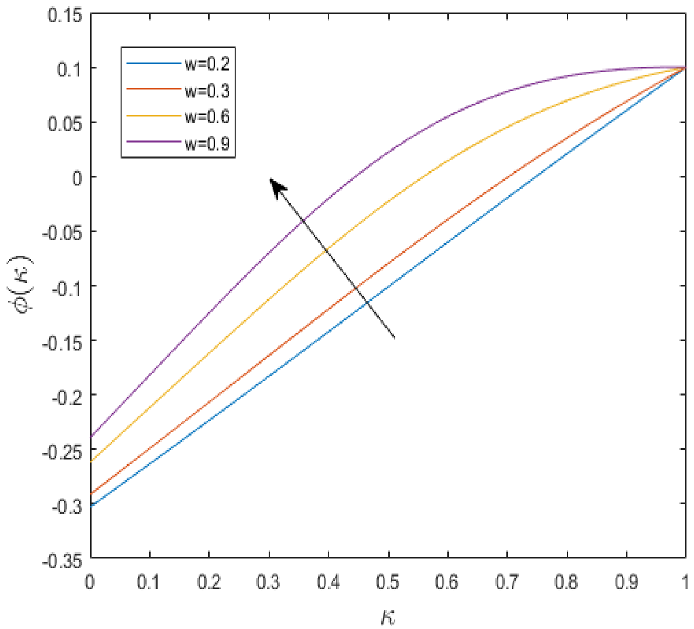

FIGURE 4.6: Effect of w when $M = 0.1$, $\lambda = 0.2$, $Rd = 0.1$,$Pr = 5$,$Nt = 0.1$,$Nb = 0.2$,$Le = 2$,$\delta_\theta = 0.2$, $\delta_\phi = 0.1$,$\wedge_T = 0.1$,$\wedge_C = 0.2$, on $\phi(\kappa)$.

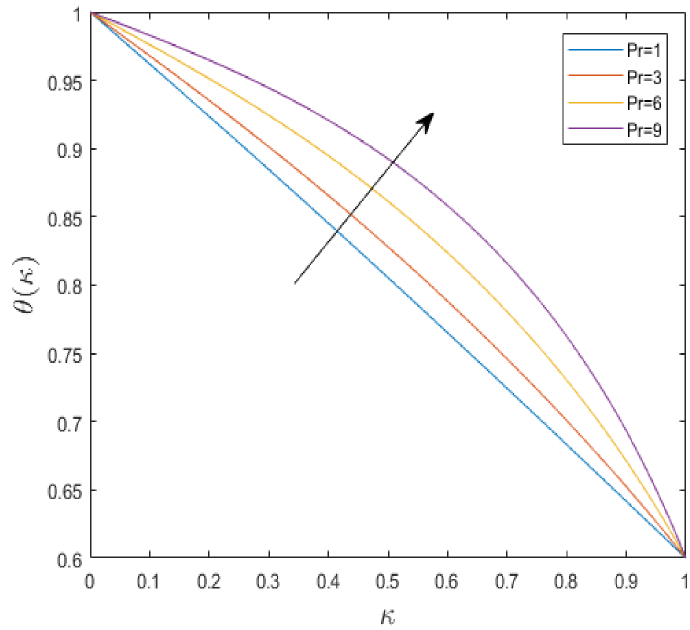

FIGURE 4.7: Effect of Pr on $\theta(\kappa)$ for $M = 0.1$, $\lambda = 1$, $Rd = 0.1$,$w = 0.5$,$Nt = 0.5$,$Nb = 1$,$Le = 5$,$\delta_\theta = 0.6$, $\delta_\phi = 0.3$,$\wedge_T = 0.1$,$\wedge_C = 0.5$.

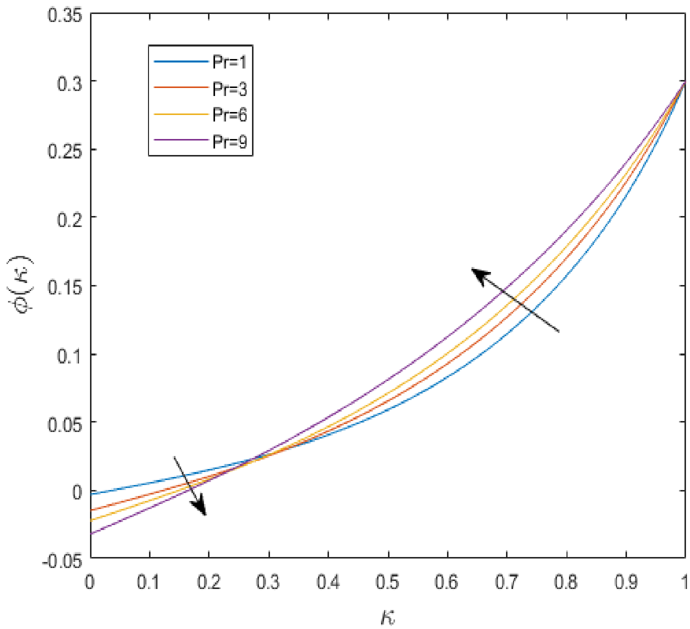

FIGURE 4.8: Effect of Pr on $\phi(\kappa)$ for $M = 0.1$, $\lambda = 1$, $Rd = 0.1$,$w = 0.5$,$Nt = 0.5$,$Nb = 1$,$Le = 5$,$\delta_\theta = 0.6$, $\delta_\phi = 0.3$,$\wedge_T = 0.1$,$\wedge_C = 0.5$..

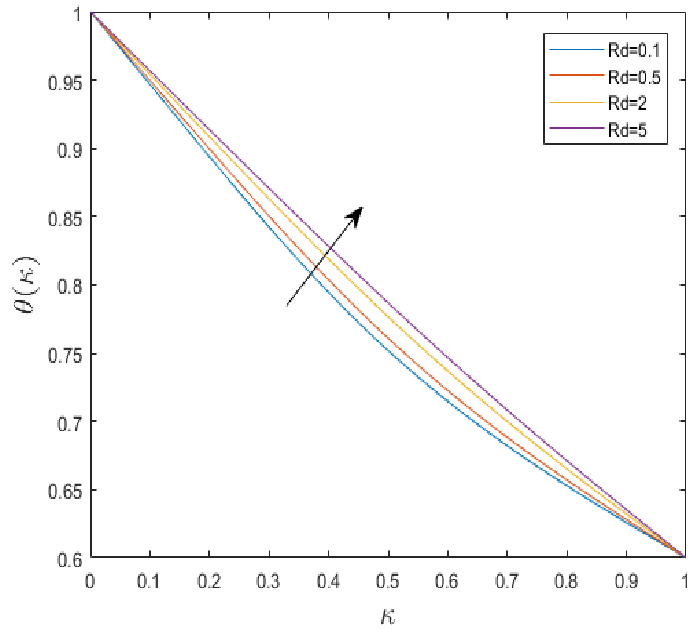

FIGURE 4.9: Effect of Rd on $\theta(\kappa)$ for $M = 0.1$, $\lambda = 1$, $Le = 0.1$,$w = 0.9$,$Nt = 0.5$,$Nb = 0.1$,$Pr = 1$,$\delta_\theta = 0.6$, $\delta_\phi = 0.3$,$\wedge_T = 5$,$\wedge_C = 5$.

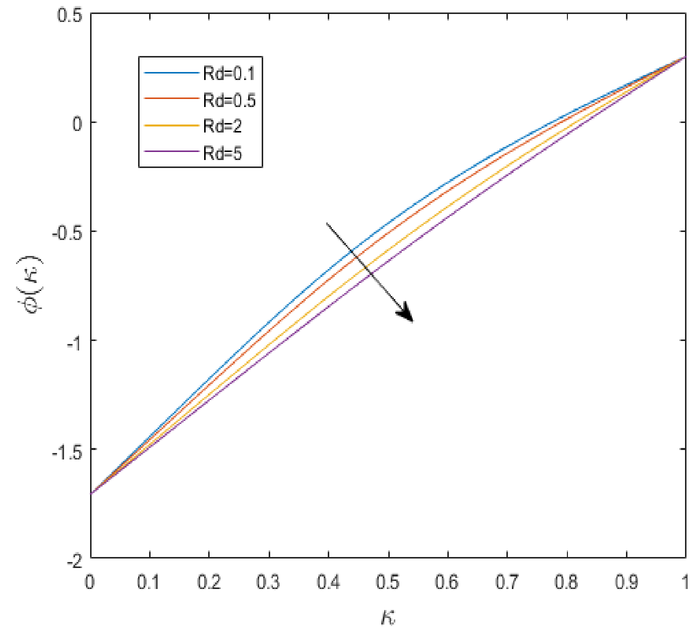

FIGURE 4.10: Effect of Rd on $\phi(\kappa)$ for $M = 0.1$, $\lambda = 1$, $Le = 0.1$,$w = 0.9$,$Nt = 0.5$,$Nb = 0.1$,$Pr = 1$,$\delta_\theta = 0.6$, $\delta_\phi = 0.3$,$\wedge_T = 5$,$\wedge_C = 5$.

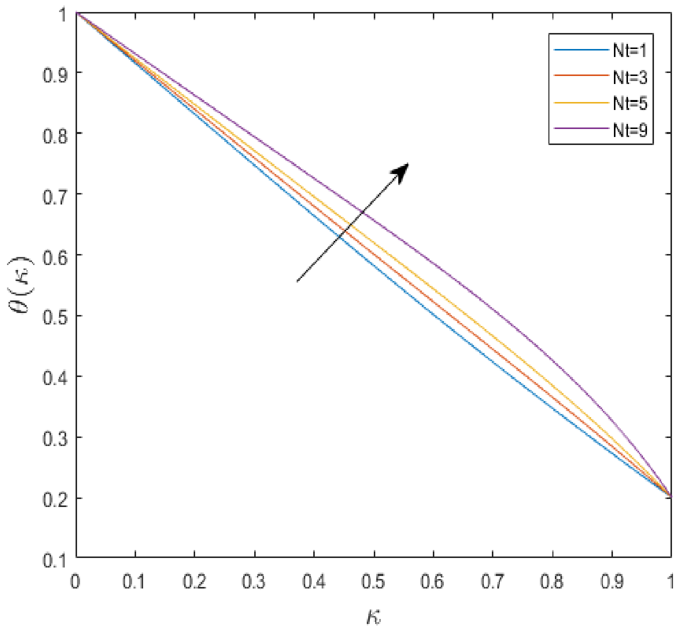

FIGURE 4.11: Nt effect when $M = 0.1$, $\lambda = 0.2$, $Le = 5$,$w = 0.5$,$Rd = 0.1$,$Nb = 0.2$,$Pr = 2$,$\delta_\theta = 0.2$, $\delta_\phi = 0.1$,$\wedge_T = 0.1$,$\wedge_C = 0.2$, on $\theta(\kappa)$.

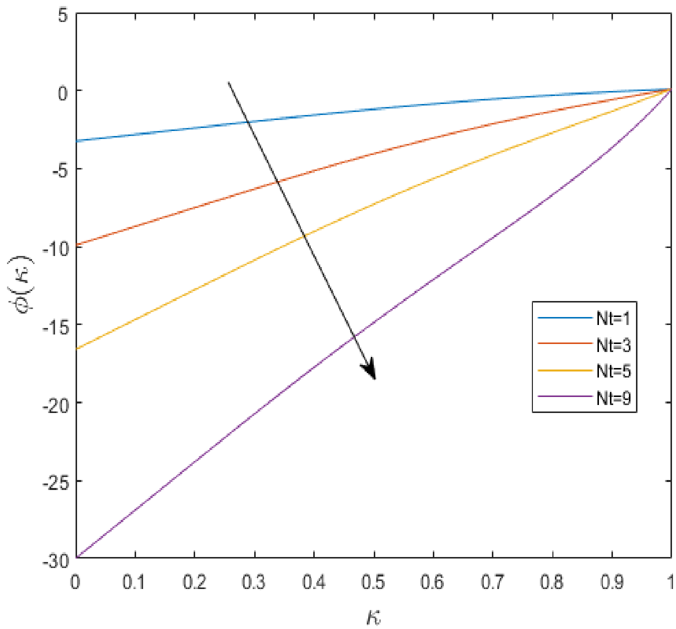

FIGURE 4.12: Nt effect when $M = 0.1$, $\lambda = 0.2$, $Le = 5$,$w = 0.5$,$Rd = 0.1$,$Nb = 0.2$,$Pr = 2$,$\delta_\theta = 0.2$, $\delta_\phi = 0.1$,$\wedge_T = 0.1$,$\wedge_C = 0.2$, on $\phi(\kappa)$.

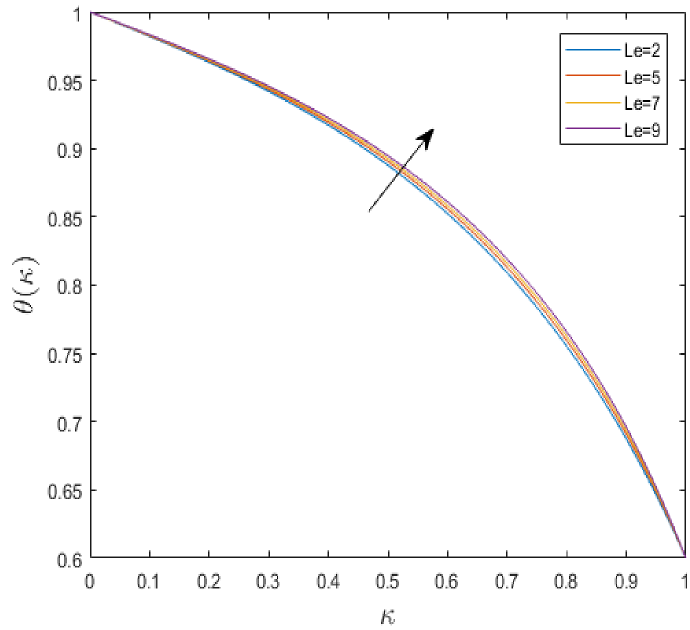

FIGURE 4.13: Le effect on $\theta(\kappa)$ for $M = 0.1$, $\lambda = 1$, $Rd = 0.1$,$w = 0.5$,$Nt = 0.5$,$Nb = 1$,$Pr = 1$,$\delta_\theta = 0.6$, $\delta_\phi = 0.3$,$\wedge_T = 0.1$,$\wedge_C = 0.5$.

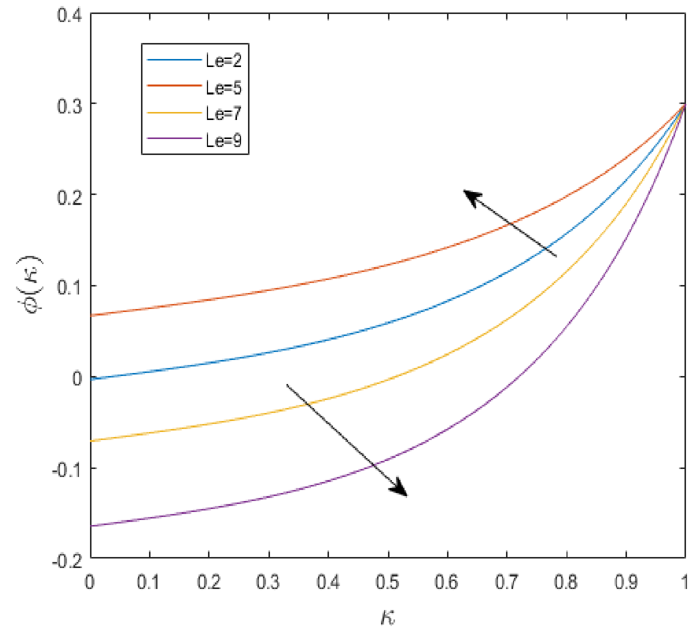

FIGURE 4.14: Le effect on $\phi(\kappa)$ for $M = 0.1$, $\lambda = 1$, $Rd = 0.1$,$w = 0.5$,$Nt = 0.5$,$Nb = 1$,$Pr = 1$,$\delta_\theta = 0.6$, $\delta_\phi = 0.3$,$\wedge_T = 0.1$,$C = 0.5$.

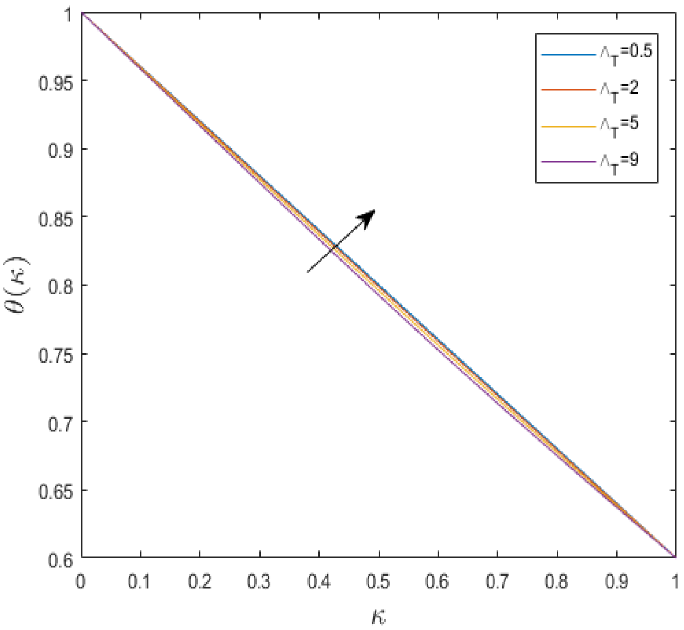

FIGURE 4.15: Effect of $\wedge_T$ on $\theta(\kappa)$ for $M = 0.1$, $\lambda = 1$, $Le = 0.1$,$w = 0.5$,$Nt = 0.5$,$Nb = 0.1$,$Pr = 0.5$,$\delta_\theta = 0.6$, $\delta_\phi = 0.3$,$Rd = 5$,$\wedge_C = 0.2$.

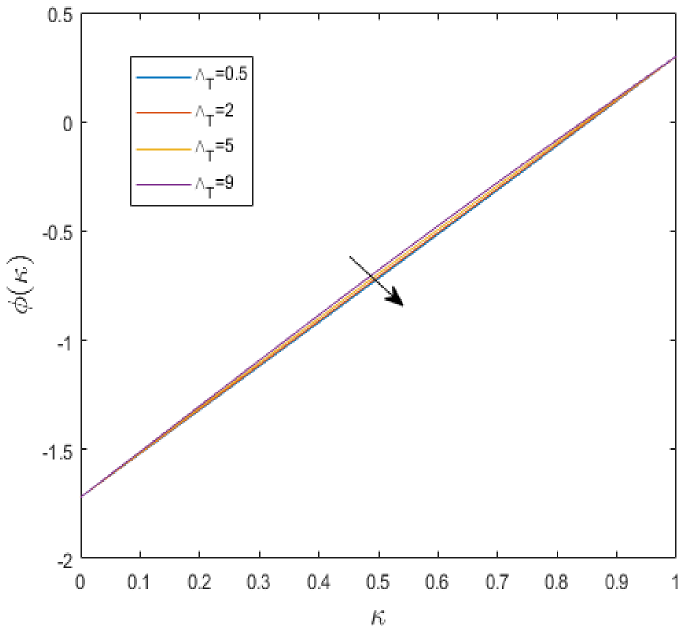

FIGURE 4.16: Effect of $\wedge_T$ on $\theta(\kappa)$ for $M = 0.1$, $\lambda = 1$, $Le = 0.1$,$w = 0.5$,$Nt = 0.5$,$Nb = 0.1$,$Pr = 0.5$,$\delta_\theta = 0.6$, $\delta_\phi = 0.3$,$Rd = 5$,$\wedge_C = 0.2$.

Chapter 5

Conclusion

In this thesis, the work of Sher Muhammad et al. [46] is reviewed and extended with the impact of inclined magnetic field, Cattaneo-Christov heat flux, Brownain motion, thermophoresis diffusion and thermal radiation. First of all, momentum, energy as well as concentration equations are converted into the ODEs with the aid of using the usage of few similarity transformations. By the usage of the shooting technique, numerical solution has been determined for the converted ODEs..The numarical results for velocity,heat distribution as well as concentrarion profiles are presented in the form of tables and graphs by taking different values of relevant physical parameters. The achievements of the present work can be summarized as below:

- Increasing the values of λ, there is a decreases in velocity profile while the increment in temperature profile is observed.
- For the enhancing values of Rd and Le, the temperature distribution is increased.
- On increment of physical parameter w,there is increases in velocity profile.
- The temperature profile is increased on increasing the values of Prandtl number Pr.

- Increasing the magnetic parameter M results in a rise in the skin friction coefficent.
- The Nusselt number Nu is decreased when Prandtl number is increased.
- The temperature distribution is increased when thermal radiation Rd increased.
- By increasing the values of physical parameter w, the concentration profile increased.
- With a rise in Brownian motion Nb, the temperature profile increases.
- Due to the ascending values of Lewis number Le, the numerical values of local sharwood number Sh_x is increased.

Bibliography

[1] R. Pal, "A novel method to determine the thermal conductivity of interfacial layers surrounding the nanoparticles of a nanofluid," *Nanomaterials*, vol. 4, no. 4, pp. 844–855, 2014.

[2] M. H. Bahmani, G. Sheikhzadeh, M. Zarringhalam, O. A. Akbari, A. A. Alrashed, G. A. S. Shabani, and M. Goodarzi, "Investigation of turbulent heat transfer and nanofluid flow in a double pipe heat exchanger," *Advanced Powder Technology*, vol. 29, no. 2, pp. 273–282, 2018.

[3] M. Bahiraei, M. Jamshidmofid, and M. Goodarzi, "Efficacy of a hybrid nanofluid in a new microchannel heat sink equipped with both secondary channels and ribs," *Journal of Molecular Liquids*, vol. 273, pp. 88–98, 2019.

[4] M. Mahmoodi and S. Kandelousi, "Kerosene- alumina nanofluid flow and heat transfer for cooling application," *Journal of Central South University*, vol. 23, no. 4, pp. 983–990, 2016.

[5] S. Choi, D. Singer, H. Wang, *et al.*, "Developments and applications of non-Newtonian flows," *ASME FED*, vol. 66, pp. 99–105, 1995.

[6] Y. Xuan and W. Roetzel, "Conceptions for heat transfer correlation of nanofluids," *International Journal of Heat and Mass Transfer*, vol. 43, no. 19, pp. 3701–3707, 2000.

[7] S. Choi, Z. Zhang, W. Yu, F. Lockwood, and E. Grulke, "Anomalous thermal conductivity enhancement in nanotube suspensions," *Applied Physics Letters*, vol. 79, no. 14, pp. 2252–2254, 2001.

[8] J. Buongiorno, "Convective transport in nanofluids," *Journal of Heat Transfer*, vol. 128, no. 3, pp. 240–250, 2006.

[9] R. K. Tiwari and M. K. Das, "Heat transfer augmentation in a two-sided lid-driven differentially heated square cavity utilizing nanofluids," *International Journal of Heat and Mass Transfer*, vol. 50, no. 9-10, pp. 2002–2018, 2007.

[10] J.-C. Yang, F.-C. Li, W.-W. Zhou, Y.-R. He, and B.-C. Jiang, "Experimental investigation on the thermal conductivity and shear viscosity of viscoelastic-fluid-based nanofluids," *International Journal of Heat and Mass Transfer*, vol. 55, no. 11-12, pp. 3160–3166, 2012.

[11] Khan and Pop, "Boundary-layer flow of a nanofluid past a stretching sheet," *International Journal of Heat and Mass Transfer*, vol. 53, no. 11-12, pp. 2477–2483, 2010.

[12] P. Rana and R. Bhargava, "Finite element simulation of transport phenomena of viscoelastic nanofluid over a stretching sheet with energy dissipation," *Journal of Information and Operations Management*, vol. 3, no. 1, 2012.

[13] M. Hamad and M. Ferdows, "Similarity solutions to viscous flow and heat transfer of nanofluid over nonlinearly stretching sheet," *Applied Mathematics and Mechanics*, vol. 33, no. 7, pp. 923–930, 2012.

[14] F. M. Hady, F. S. Ibrahim, S. M. Abdel-Gaied, and M. R. Eid, "Radiation effect on viscous flow of a nanofluid and heat transfer over a nonlinearly stretching sheet," *Nanoscale Research Letters*, vol. 7, no. 1, p. 229, 2012.

[15] K. Das, "Nanofluid flow over a non-linear permeable stretching sheet with partial slip," *Journal of the Egyptian Mathematical Society*, vol. 23, no. 2, pp. 451–456, 2015.

[16] J. A. Khan, M. Mustafa, T. Hayat, and A. Alsaedi, "Three-dimensional flow of nanofluid over a non-linearly stretching sheet: An application to solar energy," *International Journal of Heat and Mass Transfer*, vol. 86, pp. 158–164, 2015.

[17] A. Malvandi, F. Hedayati, and D. Ganji, "Nanofluid flow on the stagnation point of a permeable non-linearly stretching/shrinking sheet," *Alexandria Engineering Journal*, vol. 57, no. 4, pp. 2199–2208, 2018.

[18] S. U. Khan and S. A. Shehzad, "Brownian movement and thermophoretic aspects in third-grade nanofluid over oscillatory moving sheet," *Physica Scripta*, vol. 94, no. 9, p. 095202, 2019.

[19] M. H. Matin, M. Dehsara, and A. Abbassi, "Mixed convection MHD flow of nanofluid over a nonlinear stretching sheet with effects of viscous dissipation and variable magnetic field," *Mechanics*, vol. 18, no. 4, pp. 415–423, 2012.

[20] A. Z. Ghadi, M. J. Noroozi, and M. H. Esfe, "Nanofluid implementation for heat transfer augmentation of magneto hydrodynamic flows in a lid-driven cavity using experimental-based correlations," *International Journal of Applied Electromagnetics and Mechanics*, vol. 42, no. 4, pp. 589–602, 2013.

[21] N. Sandeep, C. Sulochana, C. Raju, M. J. Babu, and V. Sugunamma, "Unsteady boundary layer flow of thermophoretic MHD nanofluid past a stretching sheet with space and time dependent internal heat source/sink," *Applied Mathematics*, vol. 10, no. 1, pp. 312–327, 2015.

[22] S. Naramgari and C. Sulochana, "Dual solutions of radiative MHD nanofluid flow over an exponentially stretching sheet with heat generation/absorption," *Applied Nanoscience*, vol. 6, no. 1, pp. 131–139, 2016.

[23] M. Khan and M. Azam, "Unsteady heat and mass transfer mechanisms in MHD Carreau nanofluid flow," *Journal of Molecular Liquids*, vol. 225, pp. 554–562, 2017.

[24] A. Jamaludin, R. Nazar, and I. Pop, "Three-dimensional magnetohydrodynamic mixed convection flow of nanofluids over a nonlinearly permeable stretching/shrinking sheet with velocity and thermal slip," *Applied Sciences*, vol. 8, no. 7, p. 1128, 2018.

[25] I. Ahmad, M. Sajid, W. Awan, M. Rafique, W. Aziz, M. Ahmed, A. Abbasi, and M. Taj, "MHD flow of a viscous fluid over an exponentially stretching sheet in a porous medium," *Journal of Applied Mathematics*, vol. 2014, 2014.

[26] Z. Shah, E. Bonyah, S. Islam, W. Khan, and M. Ishaq, "Radiative MHD thin film flow of Williamson fluid over an unsteady permeable stretching sheet," *Heliyon*, vol. 4, no. 10, p. e00825, 2018.

[27] A. Zeeshan, R. Ellahi, and M. Hassan, "Magnetohydrodynamic flow of water/ethylene glycol based nanofluids with natural convection through a porous medium," *The European Physical Journal Plus*, vol. 129, no. 12, p. 261, 2014.

[28] D. Pal and G. Mandal, "Mixed convection–radiation on stagnation-point flow of nanofluids over a stretching/shrinking sheet in a porous medium with heat generation and viscous dissipation," *Journal of Petroleum Science and Engineering*, vol. 126, pp. 16–25, 2015.

[29] R. Bhargava and H. Chandra, "Numerical simulation of MHD boundary layer flow and heat transfer over a nonlinear stretching sheet in the porous medium with viscous dissipation using hybrid approach," *Journal of Applied Mathematics*, 2017.

[30] N. A. Haroun, S. Mondal, and P. Sibanda, "Hydromagnetic nanofluids flow through a porous medium with thermal radiation, chemical reaction and viscous dissipation using the spectral relaxation method," *International Journal of Computational Methods*, vol. 16, no. 06, p. 1840020, 2019.

[31] Y. Geng, A. Hassanvand, M. Monfared, and R. Moradi, "MHD nanofluid heat transfer between a stretching sheet and a porous surface using neural network approach," *International Journal of Modern Physics C*, vol. 30, no. 06, p. 1950048, 2019.

[32] H. R. Patel, "Effects of cross diffusion and heat generation on mixed convective MHD flow of Casson fluid through porous medium with nonlinear thermal radiation," *Heliyon*, vol. 5, no. 4, p. e01555, 2019.

[33] E. Magyari and A. J. Chamkha, "Combined effect of heat generation or absorption and first order chemical reaction on micropolar fluid flows over a uniformly stretched permeable surface," *International Journal of Thermal Sciences*, vol. 49, no. 9, pp. 1821–1828, 2010.

[34] A. J. Chamkha and A. Rashad, "Unsteady heat and mass transfer by MHD mixed convection flow from a rotating vertical cone with chemical reaction and soret and dufour effects," *The Canadian Journal of Chemical Engineering*, vol. 92, no. 4, pp. 758–767, 2014.

[35] K. Das, "Effect of chemical reaction and thermal radiation on heat and mass transfer flow of MHD micropolar fluid in a rotating frame of reference," *International Journal of Heat and Mass Transfer*, vol. 54, no. 15-16, pp. 3505–3513, 2011.

[36] D. F. Young, T. T. H. Okiishi, and W. Huebsch, *Fundamentals of fluid mechanics*. Wiley, 2006.

[37] P. A. Davidson and A. Thess, *Magnetohydrodynamics*, vol. 418. Springer Science & Business Media, 2002.

[38] R. Bansal, *A Textbook of Fluid Mechanics and Dydraulic Machines*. Laxmi publications, 2004.

[39] J. N. Reddy and D. K. Gartling, *The Finite Element Method in Heat Transfer and Fluid Dynamics*. CRC press, 2010.

[40] J. Ahmed and M. S. Rahman, *Handbook of Food Process Design*. John Wiley & Sons, 2012.

[41] J. H. Ferziger, M. Perić, and R. L. Street, *Computational Methods for Fluid Dynamics*, vol. 3. Springer, 2002.

[42] R. W. Fox, A. McDonald, and P. Pitchard, *Introduction to Fluid Mechanics*. 2006.

[43] R. W. Lewis, P. Nithiarasu, and K. N. Seetharamu, *Fundamentals of the Finite Element Method for Heat and Fluid Flow*. John Wiley & Sons, 2004.

[44] J. Kunes, *Dimensionless Physical Quantities in Science and Engineering*. Elsevier, 2012.

[45] M. Gad-el Hak, *Frontiers in Experimental Fluid Mechanics*, vol. 46. Springer Science & Business Media, 2013.

[46] G. A. M. I. Sher Muhammad, Syed Inayat Ali Shah, "Squeezing nanofluid flow between two parallel plates under the influence of mhd and thermal radiation," *Asian Research Journal of Mathematics*, vol. 10, pp. 1–20, 2017.