CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Cattaneo-Christov Double Diffusion, MHD and Thermal Radiation Effects on a Nanofluid Flow

by

Iqra Hanif

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in the

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*I dedicate my dissertation work to my **family** and dignified **teachers**. A special feeling of gratitude to my loving parents who have supported me in my studies.*

CERTIFICATE OF APPROVAL

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Iqra Hanif

(MMT203005)

THESIS EXAMINING COMMITTEE

(a)	External Examiner	Dr. Ehnber Naheed	NSU, Islamabad
(b)	Internal Examiner	Dr. Abdul Rehman Kashif	CUST, Islamabad
(c)	Supervisor	Dr. Muhammad Sagheer	CUST, Islamabad

Dr. Muhammad Sagheer
Thesis Supervisor
October, 2022

Dr. Muhammad Sagheer
Head
Dept. of Mathematics
October, 2022

Dr. M. Abdul Qadir
Dean
Faculty of Computing
October, 2022

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Acknowledgement

Opening my thesis with the mention of the greatest to be praised, the lord of worlds, heavens and prayers. Nothing could have been accomplished without the endless bounty and mercy of **ALLAH** the almighty upon me. I would like to pay my deepest and most sincere invocation to the bonfied human, our beloved **Prophet Muhammad (S.A.W)** whose mercy lead us to a balanced and beautiful existence.

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Abstract

Magnetichydrodynamic and thermal radiation effects of nanofluid over a stretching sheet are theoretically studied by using shooting method. The main aim of the present research is to observe in detail the effect of MHD and thermal radiation by inclucating Cattaneo-Christov double diffusion also taking into consideration, the factors like thermophoresis diffusion, Brownian motion, thermal diffusitivity and chemical reaction on a nanofluid that streams along a stretchy sheet. During the process, similarity transformations have been applied to convert nonlinear partial differential equations into ordinary differential equations. During this study computational techniques are kept in handy to solve the momentum, energy and concenteration equations of moving nanofluid using the shooting method. Tables and graphs clearly depict the effect of parameters such as nonlinear stretching sheet parameter, magnetic field parameter, Heat generation parameter, Prandtl number, thermophoresis parameter, brownian motion parameter, lewis number and chemical reaction parameter on the velocity profile, temperature distribution, concenteration distribution, skin friction coefficient, Nusselt number and Sherwood number. The result shows that increasing the values of the magnetic parameter , the velocity profile decreases while the temperature profile increases. Rising the values of Prandtl number results in a decrease in the temperature profile. Due to the ascending values of the parameter γ_1, the values of the local Nusselt number are increased while the Sherwood number is decreased.

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Abbreviations

IVPs	Initial value problems
MHD	Magnetohydrodynamics
ODEs	Ordineary differential equations
PDEs	Partial differential equations
RK	Runge-Kutta

Symbols

μ	Viscosity
ρ	Density
ν	Kinematic viscosity
τ	Stress tensor
k	Thermal conductivity
α	Thermal diffisuitivity
σ	Electrical conductivity
u	x-component of fluid velocity
v	y-component of fluid velocity
B_0	Magnetic field constant
a	Stretching constant
T_w	Temperature of the wall
T_∞	Ambient temperature of the nanofluid
T	Temperature
C_∞	Ambient concentration
C	Concentration
C_w	Nanoparticles concentration at the stretching surface
q_r	Radiative heat flux
q	Heat generation constant
q_w	Heat flux
q_m	Mass flux
σ^*	Stefan Boltzmann constant
k^*	Absorption coefficient

ψ	Stream function
ξ	Similarity variable
C_f	Skin friction coefficient
Nu	Nusselt number
Nu_x	Local Nusselt number
Sh	Sherwood number
Sh_x	Local Sherwood number
Re	Reynolds number
Re_x	Local Reynolds number
R	Thermal radiation parameter
n	Stretching parameter
M	Magnetic parameter
Pr	Prandtl number
Q	Heat generation parameter
λ_a	Relaxation time parameter
Nb	Brownain motion parameter
Nt	Thermophoresis parameter
L_e	Lewis number
S_r	Soret number
f	Dimensionless velocity
θ	Dimensionless temperature
ϕ	Dimensionless concentration

Chapter 1

Introduction

Colloidal suspension of nanoparticles into base fluid has introduced a new class of fluids called nanofluids. Nanofluid posses remarkable properties that the technology was unlikely to attain through conventional fluids. These conventional fluids when added by nanosized particles exhibit enhanced strength, chemical reactivity. electrical conductivity, supermagnetic characteristics and in particular heat transfer and thermal conductivity. Carbon nanotubes, metal oxides and nano polymers are some examples of nano materials that are dispersed in conventional fluids by methods like dispersion, chemical precipitation/ condensation etc. Applications of nano fluids in sectors like aeronautics, medicene, pharmaceutics and photoelectric has produced marvels for example brake fluids, nuclear reactions, improvements in cooling transformer oil, power plants, diseases generators and even in space technologies. Choi and Eastman [1] is the person who introduced the term nanofluids through their experimental work. This invention opened doors for further researches and provided humanity with platform to extract more out of it. The earliest works on nanofluids were done by Wang and Majumdar [2], Yang et alia [3], Jahani et alia [4] and Das et alia [5] etc. The introductory nanofluid model was put forth by Buongiorno [6] which got preceded by Tiwari and Das [7] model. Buongiorno [6] targeted Brownian motion and thermophoresis to explain fluctuatios in thermal parameters where as Tiwari and Das [7] worked based on volume fractions.

Khan and Pop [8] were able to generate first ever work on laminar flow of nanofluid over a stretching surface emphasizing that the behaviour can also be well observed in nanofluids. Noghrehabadi et al. [9] and M.Hady et al. [10] performed similar experiments depicting behaviours of nanofluids over a stretching sheet. Wang [11] was the first who theoretically and experimentally noted down the flow towards a shrinking sheet. Out of many significant characteristics, the most advanceed to grasp interest are MHD and thermal radiation effects on nanofluids. Nadeem et al. [12] used Homotopy method to observe two dimensional flow of heat transfer considering Williamson nanofluids, these nanofluids are viscous noneleastic fluids. His work was followed by PrasannaKumara et al. [13] analysing chemical activity on a porous medium. Krishnamurty et al [14] provided an extension to this phenomenon. More work on Williamson nanofluids was presented by Kothandapani and Prakash [15] who studied MHD and thermal effects on peristaltic flow. The presentation of heat tranfer on two phase model with effects of the Magnetohydrodynamic and the thermal radiation was made in its earliest form by Sheikholeslami et al. [16]. Kleinstreuer and Feng [17] investigated thermal conductivity improvement experimentally which was taken up by Yu et al. [18] who performed comparative studies keeping in view heat transfer.

Tzeng et al. [19] emphasized on the significance of heat transfer in nanotechnology. They experimented with engine transmission oil in the early 2000. Results depicted the reduced manual transmission temperatures and a better engine efficiency. Mabood et al. [20] who was a pioneer in this area, studied the flow of a magnetohydrodynamic boundary layer via a nonlinear stretching sheet. Zhang et alia [21] preformed a similar sequence of events but using a porous medium whereas Hamad at al. [22] extended Zhang et al. [21] work by taking into account megnatohydrodynamic effects while Nadeem and Haq [23] work targeted a porous shrinking sheet. MHD stagnation point was theoretically and experimentally explored by Ibrahim et al. [24] whereas Srinivasacharya et al. [25] examined MHD boundary layer. Bhatti et al. [26] critically evaluated Reynolds number's relation to the magnetic field. This work was taken on experimental basis by Xuan and Li [27] who took volume percentage into consideration this time. Magnetic field parameter, Brownian motion, heat generation and temperature field were evaluated by Poply et al. [28] under the effect of MHD. Factors that effect velocity and temperature of Cu and Al_20_3 nanoparticles were demonstrated by Thumma et al. [29]. Shateyi et al. [30] and Aly et al. [31] discussed the MHD laminar boundary flow across moving surfaces by implying different approaches and methods. Chamkha [32] and Uddin et al. [33] successfully described MHD boundary layer flow with convective slip flow under the effect of heat. Malik et al. [34] unlike others chose a non-Newtonian fluid for instant Casson nanofluid to discuss velocity changes under MHD effects. Ganga [35], Khan et al. [36] and Ahmed [37] also contributed significantly by considering magnetohydrodynamics in the fluid flow problems. The fisrt people to explain heat and mass transmission, respectively, were Fourier [38] and Fick [39]. They claim that the distributions of temperature and concentration have parabolic equations. Later, Cattaneo [40] modified the Fourier's rule of heat conduction by include the term for thermal relaxation and explored heat transmission with limited speed in thermal waves as a result. In order to reach the material-invariant formulation, Christov [41] developed a new design.

1.1 Thesis Contributions

The analysis of Magnetohydrodynamic radiative nanofluid flow by considering Cattaneo-Christov double diffusion, Brownian motion and Soret effect has not been investigated yet. Keeping in view, present research work is an attempt to fill this gap, and the finding of present study is a noval addition in the literature. During the process, similarity transformations have been applied to convert nonlinear PDEs into system of dimensionless ODEs and the results are produced by using shooting method. The numerical results are deduced graphically by aid of MATLAB. Tables and graphs clearly depict the effect of significant paramters on the velocity profile $f'(\xi)$, temperature profile $\theta(\xi)$, concenteration profile $\phi(\xi)$, Skin friction coefficient C_f, Local Nusselt number Nu_x and Local Sherwood number Sh_x.

1.2 Layout of Thesis

A concise outline of the thesis content is given below.

Chapter 2 contains certain terminologies and fundamental definitions , that will be helpful to understand the concepts discussed later on.

Chapter 3 provides an analytical investigation of MHD nanofluid. The numerical results of the governing flow equations are derived by the shooting method.

Chapter 4 inculdes an extension of the work presented in chapter 3 by considering Cattaneo-Christov double diffusion, Brownian motion and chemical reaction.

Chapter 5 provides the concluding remarks of the thesis.

References used in the thesis are mentioned in **Biblography**.

Chapter 2

Preliminaries

This chapter comprise certain fundamental definitions and governing laws, that will be helpful in the subsequent chapters.

2.1 Some Basic Terminologies

Definition 2.1.1 (Fluid)

"A substance that cannot keep its own shape but instead adopts that of its container is referred to as a fluid." [42]

Definition 2.1.2 (Fluid Mechanics)

"The field of study known as fluid mechanics examines the behaviour of fluids (liquids or gases) both at rest and in motion." [43]

Definition 2.1.3 (Fluid Dynamics)

"The area of mathematics and physics that deals with describing and understanding how liquids and gases move." [43]

Definition 2.1.4 (Fluid Statics)

"The area of fluid mechanics known as fluid statics is responsible for studying incompressible fluids at rest." [43]

Definition 2.1.5 (Viscosity)

"The resistance of a fluid to a change in shape or movement of neighbouring components relative to one another is known as its viscosity.. Mathematically,

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}},$$

where μ is viscosity coefficient, τ is shear stress and $\frac{\partial u}{\partial y}$ represents the velocity gradient." [43]

Definition 2.1.6 (Kinematic Viscosity)

"It is described as the relationship between the fluid's dynamic viscosity and density". It is represented by the symbol ν referred to as **nu**. Mathematically,

$$\nu = \frac{\mu}{\rho}.'' \text{ [43]}$$

Definition 2.1.7 (Thermal Conductivity)

"The Fourier heat conduction law states that the heat flow is proportional to the temperature gradient. The coefficient of proportionality is a material parameter known as the thermal conductivity which may be a function of a number of variables. [44]

Definition 2.1.8 (Thermal Diffusivity)

"The rate at which heat diffuses by conducting through a material depends on the thermal diffusivity and can be defined as,

$$\alpha = \frac{k}{\rho C_p},$$

where α is the thermal diffusivity, k is the thermal conductivity, ρ is the density and C_p is the specifc heat at constant pressure." [45]

2.2 Types of Fluid

Definition 2.2.1 (Ideal Fluid)

"A fluid, which is incompressible and has zero viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity." [43]

Definition 2.2.2 (Real Fluid)

"A fluid is considered to be real if it has viscosity. All of the fluids are actual fluids in real life." [43]

Definition 2.2.3 (Newtonian Fluid)

"A Newtonian fluid is defined as one with constant viscosity, with zero shear rate at zero shear stress." [43]

Definition 2.2.4 (Non-Newtonian Fluid)

"A non-Newtonian fluid is a fluid that does not follow Newton law of viscosity, i.e., constant viscosity independent of stress.

$$\tau_{xy} \propto \left(\frac{du}{dy}\right)^m, \qquad m \neq 1$$
$$\tau_{xy} = \mu\left(\frac{du}{dy}\right)^m." \text{ [43]}$$

Definition 2.2.5 (Magnetohydrodynamics)

"The interaction of magnetic fields and fluid flow is the subject of magnetohydrodynamics (MHD). The fluids under consideration must be both electrically conducting and non-magnetic, which restricts us to liquids, hot ionic gases, and strong electrolyte.” [46]

2.3 Types of Flow

Definition 2.3.1 (Rotational Flow)
“Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis.” [43]

Definition 2.3.2 (Irrotational Flow)
Irrotational flow is that type of flow in which the fluid particles while flowing along stream-lines, do not rotate about their own axis then this type of flow is called irrotational flow.” [43]

Definition 2.3.3 (Compressible Flow)
“Compressiible flow is that type of flow in the the density of the fluid changes form point to point or in other words the density (ρ) is not constant for the fluid, Mathematically,

$$\rho \neq k,$$

where k stands constant.” [43]

Definition 2.3.4 (Incompressible Flow)
“Incompressible flow is that type of flow in which the the density is constant for the fluid. Liquids are generally incompressible while gases are compressibles, Mathemtically,

$$\rho = k,$$

where k is constant.” [43]

Definition 2.3.5 (Steady Flow)

"The flow is referred to as steady flow if the flow properties, such as depth of flow, velocity of flow, and rate of flow at any location in open channel flow, do not fluctuate with regard to time. Mathematically,

$$\frac{\partial Q}{\partial t} = 0,$$

where Q is any fluid property." [43]

Definition 2.3.6 (Unsteady Flow)

"Unsteady flow is defined as flow in an open channel that changes with respect to time at any location in terms of velocity, depth, or rate. Mathematically,

$$\frac{\partial Q}{\partial t} \neq 0,$$

where Q is any fluid property." [43]

Definition 2.3.7 (Internal Flow)

"Flows completely bounded by a solid surfaces are called internal or duct flows." [42]

Definition 2.3.8 (External Flow)

"Flows over bodies immersed in an unbounded fluid are said to be an external flows." [42]

2.4 Modes of Heat Transfer

Definition 2.4.1 (Heat Transfer)

"The subject of physics known as "heat transfer" studies how thermal energy moves from one place in one media to another or from one medium to another when there is a temperature difference.” [44]

Definition 2.4.2 (Conduction)

“The transfer of heat within a medium due to a diffusion process is called conduction.” [44]

Definition 2.4.3 (Convection)

“Convection heat transfer is usually defined as energy transport effected by the motion of a fluid. This convection heat transfer between two dissimilar media is governed by Newtons law of cooling.” [44]

Definition 2.4.4 (Thermal Radiation)

“Thermal Radiation is defined as radiant (electromagnetic) energy emitted by a medium and is solely to the temperature of the medium.” [44]

2.5 Dimensionless Numbers

Definition 2.5.1 (Prandtl Number)

“It is ratio between the momentum diffusivity ν and thermal diffusivity α.
Mathematically, it can be defined as:

$$Pr = \frac{\nu}{\alpha} = \frac{\frac{\mu}{\rho}}{\frac{k}{C_p\rho}} = \frac{\mu C_p}{k}$$

where μ represents the dynamics viscosty, Cp denotes specific heat, k stand for thermal conductivity.
The relative thickness of thermal and momentum boundary layers is controlled by Prandtl number. For small P_r, heat distributed rapidly corresponds to the momentum.” [42]

Definition 2.5.2 (Skin Friction Coefficient)

"The Steady flow of an incompressible gas or liquid in a long pipe of internal D. The mean velocity is denoted by u_w. Skin firiction coeficient can be defined as

$$C_f = \frac{2\tau_0}{\rho u_w^2}$$

where τ_0 denotes the wall shear stress and ρ is the density." [47]

Definition 2.5.3 (Nusselt Number)

"The hot surface is cooled by a cold fluid stream. The heat from the hot surface, which is maintained at a constant temperature, is diffused through a boundary layer and convected away by the cold stream. Mathematically,

$$Nu = \frac{qL}{k}$$

where q stands for the convection heat transfer, L for the characteristic length and k stands for thermal conductivity." [48]

Definition 2.5.4 (Sherwod Number)

"It is the nondimensional quantity which show the ratio of the mass transport by convection to the transfer of mass by diffusion. Mathematically:

$$Sh = \frac{kL}{D}$$

here L is characteristics length, D is the mass diffusivIty and k is the mass transfer coeffcient." [49]

Definition 2.5.5 (Reynolds Number)

"It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. Mathematically,

$$Re = \frac{VL}{\nu},$$

where U denotes the free stream velocity, L is the characteristics length and ν stands for kinematic viscosity." [43]

Definition 2.5.6 (Soret Number)

"The ratio of temperature difference to concentration is known as the Soret number.

Mathematically,

$$S_r = \frac{D_m K_T (T_w - T_\infty)}{T_m \alpha (C_w - C_\infty)},$$

where D_m denotes the mass diffusivity, K_T is the thermal-diffusion and T_m is the fluid mean temperature." [43]

2.6 Governing Laws

Definition 2.6.1 (Continuity Equation)

"The principle of conservation of mass can be stated as the time rate of change of mass in fixed volume is equal to the net rate of flow of mass across the surface. Mathematically, it can be written as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0.\text{"} \ [44]$$

Definition 2.6.2 (Momentum Equation)

"The momentum equation states that the time rate of change of linear momentum of a given set of particles is equal to the vector sum of all the external forces acting on the particles of the set, provided Newtons third law of action and reaction governs the internal forces.

Mathematically, it can be written as:

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla.[(\rho \mathbf{u})\mathbf{u}] = \nabla.\mathbf{T} + \rho g.\text{"} \ [44]$$

Definition 2.6.3 (Energy Equation)

"The law of conservation of energy states that the time rate of change of the total energy is equal to the sum of the rate of work done by the applied forces and change of heat content per unit time.

$$\frac{\partial \rho}{\partial t} + \nabla.\rho\mathbf{u} = -\nabla.\mathbf{q} + Q + \phi,$$

where ϕ is the dissipation function." [44]

2.7 Shooting Method

Consider the following nonlinear boundary value problem to explain the shooting method.

$$\left.\begin{aligned} &g''(t) = f(t, f(t), f'(t)) \\ &g(0) = 0, \qquad g(G) = J. \end{aligned}\right\} \tag{2.1}$$

To reduce the order of the above boundary value problem, introduce the following notations.

$$g = Z_1, \qquad g' = Z_1' = Z_2, \qquad g'' = Z_2'. \tag{2.2}$$

As a result, (2.1) is converted into system of first order ODEs as following:

$$Z_1' = Z_2, \qquad Z_1(0) = 0. \tag{2.3}$$

$$Z_2' = f(t, Z_1, Z_2), \qquad Z_2(0) = w. \tag{2.4}$$

where w is the missing initial condition which will be guessed. The Runge-Kutta 4 method will be used to numerically solve the above initial value problem. The missing condition w should be selected as:

$$Z_1(G, w) = J. \tag{2.5}$$

For convenience, now onward $Y_1(G, w)$ will be denoted by $Y_1(w)$.
Let us further denote $Z_1(w) - J$ by $H(w)$, so that

$$H(w) = 0. \tag{2.6}$$

The above equation can be solved by using Newton's method with the following iterative formula.

$$w_{n+1} = w_n - \frac{H(w_n)}{\frac{\partial H(w_n)}{\partial w}},$$

or

$$w_{n+1} = w_n - \frac{Z_1(w_n) - J}{\frac{\partial Z_1(w_n)}{\partial w}}. \tag{2.7}$$

To find $\frac{\partial Z_1(w_n)}{\partial w}$, introduce the following notations.

$$\frac{\partial Z_1}{\partial w} = Z_3, \quad \frac{\partial Z_2}{\partial w} = Z_4. \tag{2.8}$$

As a result of these new notations the Newton's iterative scheme, will then get the form.

$$w_{n+1} = w_n - \frac{Z_1(w_n) - J}{Z_3(w_n)}. \tag{2.9}$$

Now differentiating the system of two first order ODEs (2.3)-(2.4) with respect to w, we get another system of ODEs, as follows.

$$Z_3' = Z_4, \qquad Z_3(0) = 0. \tag{2.10}$$

$$Z_4' = Z_3\frac{\partial f}{\partial w} + Z_4\frac{\partial f}{\partial w}, \qquad Z_4(0) = 1. \tag{2.11}$$

Writing all the four ODEs (2.3), (2.4), (2.10) and (2.11) together, we have the following initial value problem.

$$Z_1' = Z_2, \qquad Z_1(0) = 0.$$

$$Z_2' = f(t, Z_1, Z_2), \qquad Z_2(0) = w.$$

$$Z_3' = Z_4, \qquad Z_3(0) = 0.$$

$$Z_4' = Z_3\frac{\partial f}{\partial w} + Z_4\frac{\partial f}{\partial w}, \qquad Z_4(0) = 1.$$

The above system together will be solved numerically by Runge-Kutta method of order four.

For the Newton's technique, the stopping criteria is set such that.

$$\mid Y_1(w) - J \mid < \epsilon,$$

where $\epsilon > 0$ is an arbitrarily small positive number.

Chapter 3

MHD Nanofluid Flow in the Thermal Radiation Effects Induced by a Stretching Sheet

3.1 Introduction

Flow examination of Magnetohydrodynamic nanofluid passing through a nonlinear stretching sheet while being subjected to a magnetic field, heat generation and thermal radiation numerical is presented in this chapter. The set of equations for momentum, energy, and concenteration is attained by utilizing the boundary layer approximation. The governing nonlinear PDEs are converted by utilizing the appropriate transformation into a dimensionless ODEs. In MATLAB, the shooting method is employed to resolve ODEs. The numerical solution for significant parameters is discussed for the velocity profile $f'(\xi)$, temperature distribution $\theta(\xi)$ and concentration distribution $\phi(\xi)$. Analysis of the obtained numerical results are given through tables and graphs. The impact of some important physical parameters on Skin friction, Local Nusselt number and Local Sherwood number is also analyzed. This chapter provides a detailed review of the work presented by Krishnarao et al. [50].

3.2 Mathematical Modeling

A $2D$ Magnetohydrodynamic investigations have been made intoo this flow of a nanofluid cover a nonlinear stretching sheet with $y = 0$. Suppose that the variable stretching velocity and variable magnetic field of the nanofluid flow are $U_w(x)=ax^n$ and $B(x)=B_0x^{2n-1}$, respectively. It was believed that the cocentration of expanding surface T_w and C_w was higher than the fixation C_∞ and temperature outside T_∞ beacuse the fluid layer along the stretching surface is kept at a constant temperature.

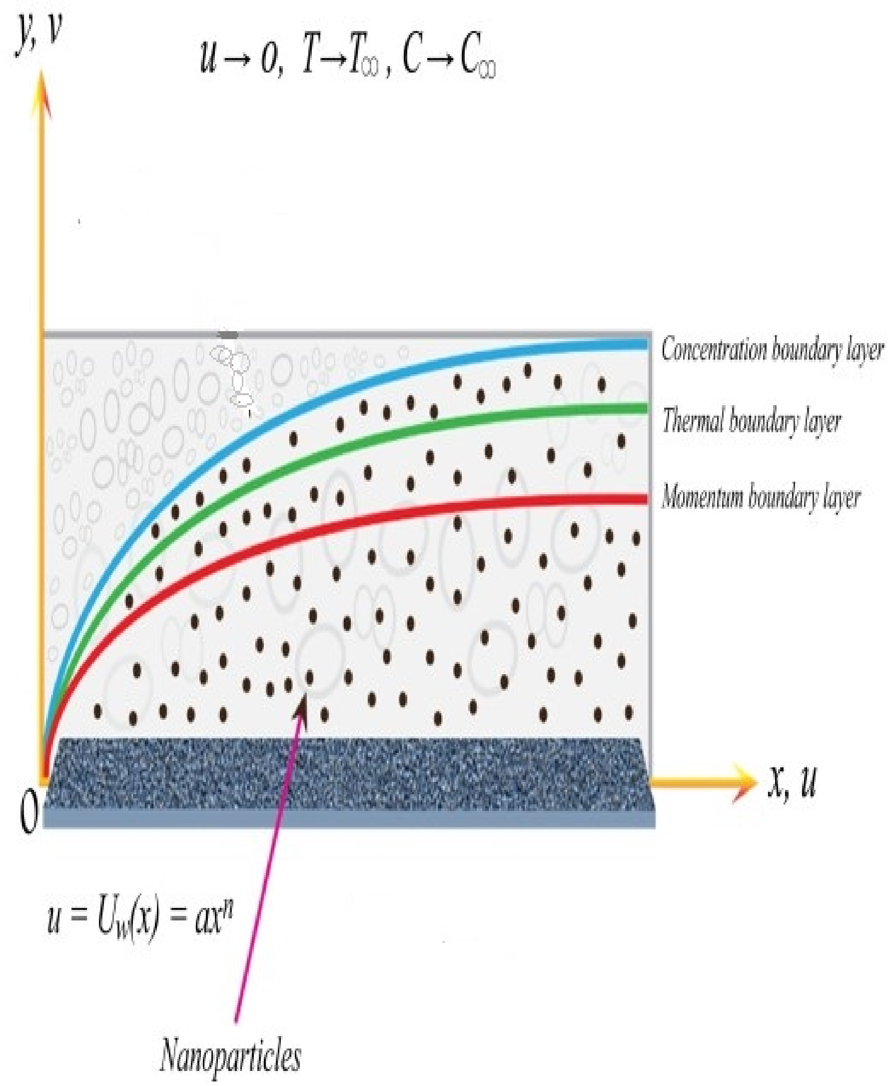

FIGURE 3.1: Systematic representation of physical model.

The set of equations describing the flow are as follows.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\Big(\frac{\partial^2 u}{\partial y^2}\Big) - \frac{\sigma B^2(x)}{\rho}u, \tag{3.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\Big(\frac{\partial^2 T}{\partial y^2}\Big) + \tau\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] - \frac{1}{(\rho C_p)}\Big(\frac{\partial q_r}{\partial y}\Big) + \frac{Q_0}{(\rho C_p)}(T - T_\infty), \tag{3.3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial y^2}\right). \tag{3.4}$$

The associated BCs have been taken as.

$$\left.\begin{array}{l} u = U_w(x) = ax^n, \quad V = V_w(x), \quad T = T_w, \quad C = C_w \quad at \quad y = 0, \\ u \to 0, \quad v \to 0, \quad T \to T_\infty, \quad C \to C_\infty, \quad as \quad y \to \infty. \end{array}\right\} \tag{3.5}$$

In this model, the direction of x is taken to be in a line with the sheet, and y is taken to be perpendicular to it. The components of velocity for the horizontally and vertically axes are u and v, respectively. Here, the radiativie heatflux q_r is constant. expressed as:

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y},$$

where k^* denotes the absorption constant and σ^* the Stefan-Boltzman constant. If the temperature difference is relatively minor, the Taylor series can be used to prolong the temperature T^4 up to T_∞.

$$T^4 = T_\infty^4 + 4T_\infty^3(T - T_\infty) + 6T_\infty^2(T - T_\infty)^2 + ...$$

With the higher order terms ignored, we have

$$\begin{aligned} T^4 &= T_\infty^4 + 4T_\infty^3(T - T_\infty) \\ &= T_\infty^4 + 4T_\infty^3 T - 4T_\infty^4 \\ &= 4T_\infty^3 T - 3T_\infty^4. \end{aligned}$$

The mathematical model (3.1)-(3.4) were transformed into an ODEs system, the following similarity transformation has been used by [50].

$$\left.\begin{aligned}
&\xi = y\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}, \qquad u = ax^n f'(\xi),\\
&v = -\sqrt{\frac{\nu a(n+1)}{2}}x^{\frac{n-1}{2}}\left(f(\xi)+\left(\frac{n-1}{n+1}\right)\xi f'(\xi)\right),\\
&\theta(\xi) = \frac{T-T_\infty}{T_w-T_\infty}, \qquad \phi(\xi) = \frac{C-C_\infty}{C_w-C_\infty}.
\end{aligned}\right\} \tag{3.6}$$

The detailed procedure for the conversion of (3.1)-(3.4) into the dimensionless form has been discussed below.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\left(af'(\xi)x^n\right)\\
&= a\frac{\partial}{\partial x}\left(f'(\xi)x^n\right)\\
&= a\left(nx^{n-1}f'(\xi)+x^n f''(\xi)\frac{\partial \xi}{\partial x}\right)\\
&= a\left(nx^{n-1}f'(\xi)+x^n f''(\xi)y\sqrt{\frac{a(n+1)}{2\nu}}\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}\right)\\
&= a\left(nx^{n-1}f'(\xi)+x^{n-1}f''\xi\left(\frac{n-1}{2}\right)\right)\\
&= ax^{n-1}\left(nf'(\xi)+\xi f''(\xi)\left(\frac{n-1}{2}\right)\right).
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}\left[-x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}}\left(f(\xi)+\left(\frac{n-1}{n+1}\right)\xi f'(\xi)\right)\right]\\
&= -x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}}\left[f'(\xi)\frac{\partial \xi}{\partial y}+\left(\frac{n-1}{n+1}\right)\xi f''(\xi)\frac{\partial \xi}{\partial y}+\left(\frac{n-1}{n+1}\right)f'(\xi)\frac{\partial \xi}{\partial y}\right]\\
&= -x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}}\left[f'(\xi)+\left(\frac{n-1}{n+1}\right)\xi f''(\xi)\right]\sqrt{\frac{(n+1)a}{2\nu}}x^{\frac{n-1}{2}}\\
&\quad - x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}}\left(\left(\frac{n-1}{n+1}\right)f'(\xi)\right)\sqrt{\frac{(n+1)a}{2\nu}}x^{\frac{n-1}{2}}\\
&= -\frac{a}{2}x^{n-1}(n+1)\left(f'(\xi)+\left(\frac{n-1}{n+1}\right)\xi f''(\xi)+\left(\frac{n-1}{n+1}\right)f'(\xi)\right)\\
&= -\frac{a}{2}x^{n-1}\left(f'(\xi)(n+1)+(n-1)\xi f''(\xi)+(n-1)f'(\xi)\right)\\
&= -\frac{a}{2}x^{n-1}f'(\xi)(n+1+n-1)-\frac{a}{2}x^{n-1}(n-1)\xi f''(\xi)\\
&= -\frac{a}{2}x^{n-1}2nf'(\xi)-\frac{a}{2}x^{n-1}(n-1)\xi f''(\xi)
\end{aligned}$$

$$= -ax^{n-1}nf'(\xi) - ax^{n-1}\Big(\frac{n-1}{2}\Big)\xi f''(\xi). \tag{3.8}$$

Equation (3.1) is easily satisfied by using (3.6) and (3.7), as follows

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= ax^{n-1}nf'(\xi) + ax^{n-1}\Big(\frac{n-1}{2}\Big)\xi f''(\xi) - ax^{n-1}nf'(\xi) \\
&\quad - ax^{n-1}\Big(\frac{n-1}{2}\Big)\xi f''(\xi) \\
\Rightarrow \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \qquad (3.9)
\end{aligned}$$

Now, for the momentum equation (3.2) the following derivatives are required.

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}(ax^n f'(\xi)) \\
&= a\frac{\partial}{\partial y}(x^n f'(\xi)) \\
&= ax^n f''(\xi)\frac{\partial \xi}{\partial y} \\
\frac{\partial u}{\partial y} &= ax^n f''(\xi)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}. \qquad (3.10) \\
\frac{\partial^2 u}{\partial y^2} &= ax^n f'''(\xi)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\frac{\partial \xi}{\partial y} \\
&= af'''(\xi)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}x^n\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}} \\
\frac{\partial^2 u}{\partial y^2} &= a^2x^{2n-1}f'''(\xi)\Big(\frac{n+1}{2\nu}\Big). \qquad (3.11) \\
u\frac{\partial u}{\partial x} &= ax^n f'(\xi)\left(ax^{n-1}nf'(\xi) + ax^{n-1}\left(\frac{n-1}{2}\right)\xi f''(\xi)\right) \\
&= a^2x^{2n-1}nf'^2(\xi) + a^2x^{2n-1}\Big(\frac{n-1}{2}\Big)\xi f'(\xi)f''(\xi). \qquad (3.12)
\end{aligned}$$

$$\begin{aligned}
v\frac{\partial u}{\partial y} &= -\sqrt{\frac{a\nu(n+1)}{2}}x^{\frac{n-1}{2}}\left(\xi f'(\xi)\left(\frac{n-1}{n+1}\right) + f(\xi)\right)\left[ax^n f''(\xi)\sqrt{\frac{a(n+1)}{2}}x^{\frac{n-1}{2}}\right] \\
&= -\sqrt{\frac{a\nu(n+1)}{2}}x^{\frac{n-1}{2}}f'(\xi)\xi\left(\frac{n-1}{n+1}\right)ax^n f''(\xi)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}} \\
&\quad - \sqrt{\frac{a\nu(n+1)}{2}}x^{\frac{n-1}{2}}f(\xi)ax^n f''(\xi)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}} \\
&= -\frac{a^2(n+1)}{2}x^{2n-1}f'(\xi)f''(\xi)\xi\left(\frac{n-1}{n+1}\right) - \frac{a^2(n+1)}{2}x^{2n-1}f(\eta)f''(\xi). \qquad (3.13)
\end{aligned}$$

Using (3.12) and (3.13) in the left side of (3.2), it becomes

$$
\begin{aligned}
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= a^2x^{2n-1}nf'^2(\xi)+a^2x^{2n-1}\left(\frac{n-1}{2}\right)\xi f'(\xi)f''(\xi)\\
&-\frac{a^2(n+1)}{2}x^{2n-1}f'(\xi)f''(\xi)\xi\Big(\frac{n-1}{n+1}\Big)-\frac{a^2(n+1)}{2}x^{2n-1}f(\xi)f''(\xi)\\
&= a^2x^{2a-1}nf'^2(\xi)+a^2x^{2n-1}\left(\frac{n-1}{2}\right)\xi f'(\xi)f''(\xi)\\
&-\frac{a^2(n+1)}{2}x^{2n-1}\xi f'(\xi)f''(\xi)\left(\frac{n-1}{n+1}\right)-\frac{a^2(n+1)}{2}x^{2n-1}f(\xi)f''(\xi)\\
&= a^2x^{2n-1}nf'^2(\xi)-\frac{a^2(n+1)}{2}x^{2n-1}f(\xi)f''(\xi)\\
&= a^2x^{2n-1}\left(nf'^2(\xi)-\left(\frac{n+1}{2}\right)f(\xi)f''(\xi)\right).
\end{aligned}
\tag{3.14}
$$

Using (3.11), in the right side of (3.2), we get

$$
\begin{aligned}
\nu\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B^2(x)}{\rho}u &= \nu a^2x^{2n-1}f'''(\xi)\left(\frac{n+1}{2\nu}\right)-\frac{\sigma B_0^2ax^{3n-1}}{\rho}f'(\xi)\\
&= \nu a^2x^{2n-1}f'''(\xi)\frac{n+1}{2\nu}-\frac{\sigma B_0^2ax^{3n-1}}{\rho}f'(\xi).
\end{aligned}
\tag{3.15}
$$

Comparing (3.14) and (3.15), the dimensionless form of (3.2) can be written as:

$$
\begin{aligned}
& a^2x^{2n-1}\Big[nf'^2(\xi)-\frac{n+1}{2}f(\xi)f''(\xi)\Big]=\nu a^2x^{2n-1}f'''(\xi)\Big[\frac{n+1}{2\nu}\Big]-\frac{\sigma B_0^2ax^{3n-1}}{\rho}f'(\xi).\\
\Rightarrow\quad & a^2x^{2n-1}\Big[nf'^2(\xi)-\frac{n+1}{2}f(\xi)f''(\xi)\Big]=a^2x^{2n-1}f'''(\xi)\Big[\frac{n+1}{2}\Big]-\frac{\sigma B^2ax^{3n-1}}{\rho}f'(\xi).\\
\Rightarrow\quad & nf'^2(\xi)-\frac{n+1}{2}f(\xi)f''(\xi)=f'''(\xi)\Big[\frac{n+1}{2}\Big]-\frac{\sigma B^2x^n}{\rho a}f'(\xi).\\
\Rightarrow\quad & \left(\frac{2n}{n+1}\right)f'^2(\xi)-f(\xi)f''(\xi)=f'''(\xi)-\frac{2\sigma B^2x^n}{\rho a(n+1)}f'(\xi).\\
\Rightarrow\quad & f'''(\xi)+f(\xi)f''(\xi)-\left(\frac{2n}{n+1}\right)f'^2(\xi)-Mf'(\xi)=0.
\end{aligned}
\tag{3.16}
$$

Now, for the conversion of energy equation (3.3), the following derivatives are required.

$$
\begin{aligned}
&\theta(\xi)=\frac{T-T_\infty}{T_w-T_\infty}.\\
\Rightarrow\quad & T=\theta(\xi)(T_w-T_\infty)+T_\infty.
\end{aligned}
$$

$$\frac{\partial \xi}{\partial x} = y\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-3}{2}}\left(\frac{n-1}{2}\right)$$

$$\frac{\partial T}{\partial x} = (T_w - T_\infty)\theta'(\xi)\frac{\partial \xi}{\partial x}$$

$$= (T_w - T_\infty)y\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-3}{2}}\left(\frac{n-1}{2}\right)\theta'(\xi). \tag{3.17}$$

$$\frac{\partial \xi}{\partial y} = \sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}$$

$$\frac{\partial T}{\partial y} = (T_w - T_\infty)\theta'(\xi)\frac{\partial \xi}{\partial y}$$

$$= (T_w - T_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\theta'(\xi). \tag{3.18}$$

$$\left(\frac{\partial T}{\partial y}\right)^2 = \left((T_w - T_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\theta'(\xi)\right)^2$$

$$= x^{n-1}\frac{(n+1)a}{2\nu}(T_w - T_\infty)^2\theta'^2(\xi). \tag{3.19}$$

$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\theta''(\xi)\frac{\partial \xi}{\partial y}$$

$$= (T_w - T_\infty)\left(\frac{a(n+1)}{2\nu}\right)x^{n-1}\theta''(\xi). \tag{3.20}$$

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}$$

$$= -\frac{4\sigma^*}{3k^*}\frac{\partial}{\partial y}(4T_\infty^3 T - 3T_\infty^4)$$

$$= -\frac{4\sigma^*}{3k^*}\frac{\partial}{\partial y}(4T_\infty^3 T)$$

$$= -\frac{4\sigma^*}{3k^*}4T_\infty^3\frac{\partial T}{\partial y}$$

$$= -\frac{16\sigma^*}{3k^*}T_\infty^3\frac{\partial T}{\partial y}$$

$$\Rightarrow \quad \frac{\partial q_r}{\partial y} = -\frac{16\sigma^*}{3k^*}T_\infty^3\frac{\partial^2 T}{\partial y^2}$$

$$= -\frac{16\sigma^*}{3k^*}T_\infty^3 x^{n-1}\frac{a(n+1)}{2\nu}(T_w - T_\infty)\theta''(\xi). \tag{3.21}$$

$$(T - T_\infty) = (T_w - T_\infty)\theta(\xi). \tag{3.22}$$

$$\frac{\partial C}{\partial y} = (C_w - C_\infty)\phi'(\xi)\frac{\partial \xi}{\partial y}$$

$$= x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)\phi'(\xi). \tag{3.23}$$

Using (3.17) and (3.18) in the left side of (3.3), we get

$$\begin{aligned}
u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} &= ax^n f'(\xi)\left[(T_w-T_\infty)\left(\frac{n-1}{2x}\right)\xi\theta'(\xi)\right]+\Bigg[-x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}} \\
&\quad \left[\left(\frac{n-1}{n+1}\right)\xi f'(\xi)+f(\xi)\right]\Bigg]\left[(T_w-T_\infty)\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}\theta'(\xi)\right] \\
&= ax^{n-1}(T_w-T_\infty)\left(\frac{n-1}{2}\right)\xi f'(\xi)\theta'(\xi) \\
&\quad -\left(\frac{(n+1)a}{2}\right)x^{n-1}(T_w-T_\infty)\left(\frac{n-1}{n+1}\right)\xi\theta'(\xi)f'(\xi) \\
&\quad -\left(\frac{a(n+1)}{2}\right)x^{n-1}(T_w-T_\infty)f(\xi)\theta'(\xi) \\
&= ax^{n-1}\left(\frac{n-1}{2}\right)(T_w-T_\infty)\xi f'(\xi)\theta'(\xi) \\
&\quad -ax^{n-1}\left(\frac{n-1}{2}\right)(T_w-T_\infty)\xi f'(\xi)\theta'(\xi) \\
&\quad -ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)\theta'(\xi) \\
&= -ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)\theta'(\xi).
\end{aligned} \tag{3.24}$$

Using (3.19)-(3.23) in the right side of (3.3), we get

$$\begin{aligned}
&\alpha\frac{\partial^2 T}{\partial y^2}+\tau\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial x}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right]-\frac{1}{(\rho C)_f}\frac{\partial q_r}{\partial y}+\frac{q}{(\rho C)_f}(T-T_\infty) \\
&=\alpha\left[x^{n-1}\left[\frac{a(n+1)}{2\nu}\right](T_w-T_\infty)\theta''(\xi)\right]+\tau\left[\frac{D_T}{T_\infty}x^{n-1}\frac{(n+1)a}{2\nu}(T_w-T_\infty)^2\theta'^2(\xi)\right] \\
&\quad +\tau\left(D_B\left[x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)\theta'(\xi)\right]\left[x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(C_w-C_\infty)\phi'(\xi)\right]\right) \\
&\quad +\frac{1}{(\rho C)_f}\left[\frac{16\sigma^* T_\infty^3}{3k^*}(T_w-T_\infty)x^{n-1}\left[\frac{a(n+1)}{2\nu}\right]\theta''(\xi)\right]+\frac{Q_0}{(\rho C)_f}(T_w-T_\infty)\theta(\xi).
\end{aligned} \tag{3.25}$$

With the help of (3.24) and (3.25), the dimensionless form of (3.3), is obtained as follows.

$$-ax^{n-1}\left[\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)\theta'(\xi)\right]=\alpha\left[x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)(T_w-T_\infty)\theta''(\xi)\right]$$

$$
\begin{aligned}
&+\tau D_B x^{n-1}\left(\frac{(n+1)a}{2\nu}\right)(T_w-T_\infty)(C_w-C_\infty)\theta'(\xi)\phi'(\xi)\\
&+\tau\frac{D_T}{T_\infty}x^{n-1}\left(\frac{(n+1)a}{2\nu}\right)(T_w-T_\infty)^2\theta'^2(\xi)+\frac{Q_0}{(\rho C)_n}(T_w-T_\infty)\theta(\xi)\\
&+\frac{1}{(\rho C)_f\nu}\left(\frac{16\sigma^* T_\infty^3}{3k^*}(T_w-T_\infty)x^{n-1}\Big[\frac{a(n+1)}{2\nu}\Big]\right)\theta''(\xi).\\
\Rightarrow\quad &-f(\xi)\theta'(\xi)=\frac{\alpha}{\nu}\theta''(\xi)+Nb\theta'(\xi)\phi'(\xi)+Nt\theta'^2(\xi)\\
&+\frac{1}{(\rho C)_f\nu}\left(\frac{16\sigma^* T_\infty^3}{3k^*}\right)\theta''(\xi)+\frac{Q_0}{(\rho C)_n a x^{n-1}}\left(\frac{2}{n+1}\right)\theta(\xi).\\
\Rightarrow\quad &-f(\xi)\theta'(\xi)=\frac{\alpha}{\nu}\theta''(\xi)+Nb\theta'(\xi)\phi'(\xi)+Nt\theta'^2(\xi)\\
&+\frac{k}{(\rho C)_f\nu}\left(\frac{16\sigma^* T_\infty^3}{3kk^*}\right)\theta''(\xi)+\frac{Q_0}{(\rho C)_n a x^{n-1}}\left(\frac{2}{n+1}\right)\theta(\xi).\\
\Rightarrow\quad &\frac{\alpha}{\nu}\left(1+\frac{16\sigma^* T_\infty}{3kk^*}\right)\theta''(\xi)+f(\xi)\theta'(\xi)+Nb\theta'(\xi)\pi'(\xi)+Nt\theta'^2(\xi)+Q\theta(\xi)=0.\\
\Rightarrow\quad &\frac{1}{Pr}\Big(1+\frac{4}{3}R\Big)\theta''(\xi)+f(\xi)\theta'(\xi)+Nb\theta'(\xi)\phi'(\xi)+Nt\theta'^2(\xi)+Q\theta(\xi)=0.
\end{aligned}
\tag{3.26}
$$

Now, we include below the procedure for the conversion of equation (3.4) into the dimensionless form.

$$
\phi(\xi)=\frac{C-C_\infty}{C_w-C_\infty}
$$

$$
\begin{aligned}
\Rightarrow\quad & C=(C_w-C_\infty)\phi(\xi)+C_\infty\\
\Rightarrow\quad & \frac{\partial C}{\partial x}=(C_w-C_\infty)\phi'(\xi)\frac{\partial\xi}{\partial x}\\
&=\Big(\frac{n-1}{2}\Big)x^{\frac{n-3}{2}}y\sqrt{\frac{a(n+1)}{2\nu}}(C_w-C_\infty)\phi'(\xi). \qquad (3.27)\\
&\frac{\partial C}{\partial y}=(C_w-C_\infty)\phi'(\xi)\frac{\partial\xi}{\partial y}\\
&=x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}(C_w-C_\infty)\phi'(\xi). \qquad (3.28)\\
&\frac{\partial^2 C}{\partial y^2}=x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}(C_w-C_\infty)\phi''(\xi)\frac{\partial\xi}{\partial y}\\
&=x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}(C_w-C_\infty)\phi''(\xi)\Big(x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}\Big)\\
&=x^{n-1}\Big(\sqrt{\frac{a(n+1)}{2\nu}}\Big)^2(C_w-C_\infty)\phi''(\xi)
\end{aligned}
$$

$$= x^{n-1}\frac{a(n+1)}{2\nu}(C_w - C_\infty)\phi''(\xi). \tag{3.29}$$

$$\frac{\partial^2 T}{\partial y^2} = x^{n-1}\frac{a(n+1)}{2\nu}(T_w - T_\infty)\theta''(\xi). \tag{3.30}$$

Using (3.27) and (3.28) in left hand side of (3.4),

$$\begin{aligned}
u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} &= ax^n f'(\xi)\left(\left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}y\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)\phi'(\xi)\right) \\
&+ \left(\frac{n-1}{2}x^{n-1}yaf'(\xi) - x^{\frac{n-1}{2}}\frac{n+1}{2}\sqrt{\frac{2\nu a}{n+1}}f(\xi)\right)x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)\phi'(\xi) \\
&= ax^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)f'(\xi)\phi'(\xi) \\
&\quad - x^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)f'(\xi)\phi'(\xi) \\
&\quad - ax^{n-1}\left(\frac{n+1}{2}\right)(C_w - C_\infty)f(\xi)\phi'(\xi) \\
&= -ax^{n-1}\left(\frac{n+1}{2}\right)(C_w - C_\infty)f(\xi)\phi'(\xi).
\end{aligned} \tag{3.31}$$

Using (3.29) and (3.30) in the right hand side of (3.4),

$$\begin{aligned}
D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} &= D_B x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)(C_w - C_\infty)\phi''(\xi) \\
&+ \frac{D_T}{T_\infty}x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)(T_w - T_\infty)\theta''(\xi).
\end{aligned} \tag{3.32}$$

Comparing (3.31) and (3.32)

$$\begin{aligned}
&- ax^{n-1}\left(\frac{n+1}{2}\right)(C_w - C_\infty)f(\xi)\phi'(\xi) = D_B x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)(C_w - C_\infty)\phi''(\xi) \\
&+ \frac{D_T}{T_\infty}x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)(T_w - T_\infty)\theta''(\xi).
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad & -\frac{\nu}{D_B}f(\xi)\phi'(\xi) = \phi''(\xi) + \frac{D_T(T_w - T_\infty)}{T_\infty D_B(C_w - C_\infty)}\theta''(\xi). \\
\Rightarrow \quad & \phi''(\xi) + Lef(\xi)\phi'(\xi) + \frac{D_T\tau(T_w - T_\infty)\nu}{T_\infty \nu D_B\tau(C_w - C_\infty)}\theta''(\xi). \\
\Rightarrow \quad & \phi''(\xi) + Lef(\xi)\phi'(\xi) + \frac{Nt}{Nb}\theta''(\xi) = 0.
\end{aligned} \tag{3.33}$$

The corresponding BCs are transformed into the non-dimensional form through the following procedure.

$$
\begin{aligned}
& u = U_w(x) = ax^n, && at \quad y = 0. \\
\Rightarrow \quad & u = af'(\xi)x^n. && \\
\Rightarrow \quad & ax^n f'(\xi) = ax^n && \\
\Rightarrow \quad & f'(\xi) = 1, && at \quad \xi = 0. \\
\Rightarrow \quad & f'(0) = 1. && \\
& v = v_w(x), && at \quad y = 0. \\
\Rightarrow \quad & -x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}}\left(f(\xi) + \xi f'(\xi)\left(\frac{n-1}{n+1}\right)\right) = v_w(x), && at \quad \xi = 0. \\
\Rightarrow \quad & -x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}} f(\xi) = v_w(x), && at \quad \xi = 0. \\
\Rightarrow \quad & f(\xi) = -\frac{v_w(x)\sqrt{2}}{x^{\frac{n-1}{2}}\sqrt{a\nu(n+1)}}, && \\
\Rightarrow \quad & f(0) = S. &&
\end{aligned}
$$

$$
\begin{aligned}
& T = T_w, && at \quad y = 0. \\
\Rightarrow \quad & \theta(\xi)(T_w - T_\infty) + T_\infty = T_w, && \\
\Rightarrow \quad & \theta(\xi)(T_w - T_\infty) = (T_w - T_\infty), && \\
\Rightarrow \quad & \theta(\xi) = 1, && at \quad \xi = 0. \\
\Rightarrow \quad & \theta(0) = 1. &&
\end{aligned}
$$

$$
\begin{aligned}
& C = C_w, && at \quad y = 0. \\
\Rightarrow \quad & \phi(\xi)(C_w - C_\infty) + C_\infty = C_w, && \\
\Rightarrow \quad & \phi(\xi)(C_w - C_\infty) = (C_w - C_\infty), && \\
\Rightarrow \quad & \phi(\xi) = 1, && at \quad \xi = 0. \\
\Rightarrow \quad & \phi(0) = 1. && \\
& u \to (0), && as \quad y \to \infty. \\
\Rightarrow \quad & af'(\xi)x^n \to (0), && \\
\Rightarrow \quad & ax^n f'(\xi) \to (0), &&
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & f'(\xi) \to (0), && as \quad \xi \to \infty. \\
\Rightarrow \quad & f'(\infty) \to 0. && \\
& T \to T_\infty, && as \quad y \to \infty. \\
\Rightarrow \quad & \theta(\xi)(T_w - T_\infty) + T_\infty \to T_\infty, && \\
\Rightarrow \quad & \theta(\xi)(T_w - T_\infty) \to 0, && as \quad \xi \to \infty. \\
\Rightarrow \quad & \theta(\xi) \to 0, && as \quad \xi \to \infty. \\
\Rightarrow \quad & \theta(\infty) \to 0. &&
\end{aligned}
$$

$$
\begin{aligned}
& C \to C_\infty, && as \quad y \to \infty. \\
\Rightarrow \quad & \phi(\xi)(C_w - C_\infty) + C_\infty \to C_\infty, && \\
\Rightarrow \quad & \phi(\xi)(C_w - C_\infty) \to 0, && \\
\Rightarrow \quad & \phi(\xi) \to 0, && as \quad \xi \to \infty. \\
\Rightarrow \quad & \phi(\infty) \to 0. &&
\end{aligned}
$$

The governing model's ultimate dimensionless form is

$$f'''(\xi) + f(\xi)f''(\xi) - \left(\frac{2n}{n+1}\right) f'^2(\xi) - Mf'(\xi) = 0. \tag{3.34}$$

$$\frac{1}{Pr}\left(1 + \frac{4}{3}R\right) + f(\xi)\theta'(\xi) + Nb\theta'(\xi)\phi'(\xi) + Nt\theta'^2(\xi) + Q\theta(\xi) = 0. \tag{3.35}$$

$$\phi''(\xi) + Lef(\xi)\phi'(\xi) + \frac{Nt}{Nb}\theta''(\xi) = 0. \tag{3.36}$$

The associated BCs (3.5) in the dimensionless form are,

$$
\left.
\begin{aligned}
& f(0) = S, \quad f'(0) = 1, \quad \theta(0) = 1, \quad \phi(0) = 1. \\
& f'(\infty) \to 0, \quad \theta(\infty) \to 0, \quad \phi(\infty) \to 0.
\end{aligned}
\right\} \tag{3.37}
$$

Different dimensionless parameters used in equations (3.34) and (3.36) are formulated as follows.

$$M = \frac{2\sigma B_0^2 x^n}{\rho a(n+1)}, \qquad Le = \frac{\nu}{D_B} \quad R = \frac{4\sigma^* T_\infty^3}{kk^*}, \qquad Pr = \frac{\nu}{\alpha},$$

$$S = -\frac{v_w(x)}{\sqrt{a\nu(n+1)}}\sqrt{2}x^{\frac{n-1}{2}}, \quad Q = \frac{Q_0}{(n+1)\nu(\rho c)_f},$$
$$Nb = \frac{(\rho c)_p D_B(C_w - C_\infty)}{(\rho c)_f \nu}, \quad Nt = \frac{(\rho c)_p D_T(T_w - T_\infty)}{(\rho c)_f T_\infty \nu}.$$

The skin friction coefficient, is given as follows.

$$C_f = \frac{\tau_w}{\rho U_w^2(x)}\bigg|_{y=0}. \tag{3.38}$$

To achive the dimensionless form of C_f, the following steps will be helpful.

where Re denotes the local Reynolds number defined as $Re = \frac{x u_x(x)}{\nu}$

Local Nusselt number is defined as follow.

$$Nu_x = \frac{x q_w}{k(T_w - T_\infty)}. \tag{3.39}$$

To achieve the dimensionless form of Nu_x, the following steps will be helpful.

$$q_w = \left(-\left(k + \frac{16\sigma^* T_\infty^3}{3k^*}\right)\left(\frac{\partial T}{\partial y}\right)\right)_{y=0}. \tag{3.40}$$

$$\begin{aligned}
Nu_x &= -\frac{x\left(k + \frac{16\sigma^* T_\infty^3}{3k^*}\right)\left(\frac{\partial T}{\partial y}\right)_{y=0}}{k(T_w - T_\infty)} \\
&= -\frac{x\left(k + \frac{16\sigma^* T_\infty^3}{3k^*}\right)}{k(T_w - T_\infty)}\left(\frac{\partial T}{\partial y}\right)_{y=0} \\
&= -\frac{x\left(k + \frac{16\sigma^* T_\infty^3}{3k^*}\right)}{k(T_w - T_\infty)}\left(\frac{(n+1)a}{2\nu}\right)^{\frac{1}{2}} x^{\frac{n-1}{2}}(T_w - T_\infty)\theta'(\xi)\Big|_{\xi=0} \\
&= -\left(1 + \frac{4}{3}R\right)\sqrt{\frac{a(n+1)}{2\nu}}\, x^{\frac{n+1}{2}}\theta'(0) \\
&= -\left(1 + \frac{4}{3}R\right)\sqrt{\frac{n+1}{2}}\theta'(0)\sqrt{\frac{ax^{n+1}}{\nu}}, \\
&= -\left(1 + \frac{4}{3}R\right)\sqrt{\frac{n+1}{2}}\theta'(0) Re_x^{\frac{1}{2}} \\
&= -Re_x^{\frac{1}{2}}\left(1 + \frac{4}{3}R\right)\sqrt{\frac{n+1}{2}}\theta'(0)
\end{aligned}$$

$$\Rightarrow \quad Re_x^{\frac{-1}{2}} Nu_x = -\left(1 + \frac{4}{3}R\right)\sqrt{\frac{n+1}{2}}\theta'(0). \tag{3.41}$$

The local Sherwood number is defined as

$$Sh_x = \frac{x q_m}{D_B(C_w - C_\infty)}. \tag{3.42}$$

To achieve the dimensionless form of Sh_x, the following step will be helpful.

$$q_m = -D_B\left(\frac{\partial C}{\partial y}\right)_{y=0}. \tag{3.43}$$

$$\begin{aligned}
Sh_x &= -\frac{x D_B}{D_B(C_w - C_\infty)}\left(\frac{\partial C}{\partial y}\right)_{y=0} \\
&= -\frac{x}{(C_w - C_\infty)} x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)\phi'(\xi)\Big|_{\xi=0}
\end{aligned}$$

$$= -x^{\frac{n+1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}\phi'(0)$$

$$= -\sqrt{\frac{ax^{n+1}}{\nu}}\left(\frac{n+1}{2}\right)^{\frac{1}{2}}\phi'(0)$$

$$= -Re_x^{\frac{1}{2}}\left(\frac{n+1}{2}\right)^{\frac{1}{2}}\phi'(0)$$

$$\Rightarrow \quad \frac{Sh_x}{Re_x^{\frac{1}{2}}} = -\sqrt{\frac{n+1}{2}}\phi'(0)$$

$$\Rightarrow \quad Re_x^{\frac{-1}{2}}Sh_x = -\sqrt{\frac{n+1}{2}}\phi'(0). \tag{3.44}$$

3.3 Solution Methodology

The ordinary differential equation (3.34) has been solved using the shooting method. The domain of original problem is $[0, \infty)$ which is unbounded. As numerical computation cannot be performed on unbounded domain so we will consider the $[0, \xi_\infty]$ as the domain where ξ_∞ is a psoitive real number. The notations below has been taken into consideration.

$$f = Y_1, \qquad f' = Y_1' = Y_2, \qquad f'' = Y_1'' = Y_2' = Y_3, \qquad f''' = Y_3'.$$

Thus, the momentum equation is transformed into a first order ODEs system.

$$Y_1' = Y_2, \qquad Y_1(0) = 0.$$

$$Y_2' = Y_3, \qquad Y_2(0) = 1.$$

$$Y_3' = \left(\frac{2n}{n+1}\right)Y_2^2 - Y_1Y_3 + MY_2, \qquad Y_3(0) = h.$$

The Runge-Kutta 4 method will be used to numerically solve the above initial value problem. The missing condition h should be selected as:

$$Y_2(\xi_\infty, h) = 0.$$

Newton's method will be used to find h. This method has the following iterative scheme.

$$h_{n+1} = h_n - \frac{Y_2(\xi_\infty, h_n)}{\frac{\partial}{\partial h}(Y_2(\xi_\infty, h_n))}.$$

We further introduce the following notations,

$$\frac{\partial Y_1}{\partial h} = Y_4, \qquad \frac{\partial Y_2}{\partial h} = Y_5, \qquad \frac{\partial Y_3}{\partial h} = Y_6.$$

As a result of these new notations, the Newton's iterative scheme gets the form

$$h_{n+1} = h_n - \frac{Y_2(\xi_\infty, h_n)}{Y_5(\xi_\infty, h_n)}.$$

Now differentiating the system of three first order ODEs with respect to h, we get another system of ODEs, as follows.

$$\begin{aligned}
&Y_4' = Y_5, && Y_4(0) = 0.\\
&Y_5' = Y_6, && Y_5(0) = 0.\\
&Y_6' = \left(\frac{4n}{n+1}\right) Y_2Y_5 - Y_4Y_3 - Y_1Y_6 + MY_5, && Y_6(0) = 1.
\end{aligned}$$

The stopping criteria for the Newton's technique is set as:

$$\mid Y_2(\xi_\infty, h) \mid < \epsilon,$$

where $\epsilon > 0$ is an arbitrarily small positive number. From now onward ϵ has been taken as 10^{-10}.

The shooting method will be used to numerically solve equation (3.35) and (3.36) while assuming that f is a known function. For this we use the notations below:

$$\begin{aligned}
&\theta = Z_1, \quad \theta' = Z_1' = Z_2, \quad \theta'' = Z_2'.\\
&\phi = Z_3, \quad \phi' = Z_3' = Z_4, \quad \phi'' = Z_4', \quad A_1 = \left(1 + \frac{4}{3}R\right).
\end{aligned}$$

As a result, the energy equations (3.35) and (3.36) are transformed into the first order ODE system below.

$$
\begin{aligned}
Z_1' &= Z_2, && Z_1(0) = 1.\\
Z_2' &= -\frac{Pr}{A_1}\left[fZ_2 + NbZ_2Z_4 + NtZ_2^2 + QZ_1\right], && Z_2(0) = l.\\
Z_3' &= Z_4, && Z_3(0) = 1.\\
Z_4' &= -LefZ_4 + \frac{Nt}{Nb}\left[\frac{Pr}{A_1}\left[fZ_2 + NbZ_2Z_4 + NtZ_2^2 + QZ_1\right]\right], && Z_4(0) = m.
\end{aligned}
$$

The RK-4 method has been taken into consideration for solving the above initial value problem. The missing conditions are to be chosen for the above system of equations in such a way that.

$$Z_1(\xi_\infty, l, m) = 0, \qquad Z_3(\xi_\infty, l, m) = 0.$$

To solve the above algebaric equations, we apply the Newton's method which has the following scheme.

$$\begin{bmatrix} l \\ m \end{bmatrix}_{(n+1)} = \begin{bmatrix} l \\ m \end{bmatrix}_{(n)} - \begin{bmatrix} \frac{\partial Z_1}{\partial l} & \frac{\partial Z_1}{\partial m} \\ \frac{\partial Z_3}{\partial l} & \frac{\partial Z_3}{\partial m} \end{bmatrix}_{(n)}^{-1} \begin{bmatrix} Z_1 \\ Z_3 \end{bmatrix}_{(n)}$$

Now, introduce the following notations,

$$\frac{\partial Z_1}{\partial l} = Z_5, \qquad \frac{\partial Z_2}{\partial l} = Z_6, \qquad \frac{\partial Z_3}{\partial l} = Z_7, \qquad \frac{\partial Z_4}{\partial l} = Z_8.$$

$$\frac{\partial Z_1}{\partial m} = Z_9, \qquad \frac{\partial Z_2}{\partial m} = Z_{10}, \qquad \frac{\partial Z_3}{\partial m} = Z_{11}, \qquad \frac{\partial Z_4}{\partial m} = Z_{12}.$$

As the result of these new notations, the Newton's iterative scheme gets the form.

$$\begin{bmatrix} l \\ m \end{bmatrix}_{(n+1)} = \begin{bmatrix} l \\ m \end{bmatrix}_{(n)} - \begin{bmatrix} Z_5 & Z_9 \\ Z_7 & Z_{11} \end{bmatrix}_{(n)}^{-1} \begin{bmatrix} Z_1 \\ Z_3 \end{bmatrix}_{(n)}$$

Now differentiating the system of four first order ODEs with respect to l, and m we get another system of ODEs, as follows.

$$Z_5' = Z_6, \qquad Z_5(0) = 0$$

$$Z_6' = -\frac{Pr}{A_1}\left[fZ_6 + Nb(Z_6Z_4 + Z_2Z_8) + 2NtZ_2Z_6 + QZ_5\right], \qquad Z_6(0) = 1.$$

$$Z_7' = Z_8, \qquad Z_7(0) = 0.$$

$$Z_8' = -LefZ_8 + \frac{Nt}{Nb}\left[\frac{Pr}{A_1}\left[fZ_6 + Nb(Z_6Z_4 + Z_2Z_8) + 2NtZ_2Z_6 + QZ_5\right]\right],$$

$$Z_8(0) = 0.$$

$$Z_9' = Z_{10}, \qquad Z_9(0) = 0.$$

$$Z_{10}' = -\frac{Pr}{A_1}\left[fZ_{10} + Nb(Z_{10}Z_4 + Z_2Z_12) + 2NtZ_2Z_{10} + QZ_9\right], \qquad Z_{10}(0) = 0.$$

$$Z_{11}' = Z_{12}, \qquad Z_{11}(0) = 0.$$

$$Z_{12}' = -LefZ_{12} + \frac{Nt}{Nb}\left[\frac{Pr}{A_1}\left[fZ_{10} + Nb(Z_{10}Z_4 + Z_2Z_12) + 2NtZ_2Z_{10} + QZ_9\right]\right],$$

$$Z_{12}(0) = 1.$$

The stopping criteria for the Newton's method is set as.

$$max\{|Z_1(\xi_\infty, l^n, m^n)|, |Z_3(\xi_\infty, l^n, m^n)|\} < \epsilon.$$

3.4 Representation of Graphs and Tables

A thorough discussion on the numerical solution and graphical representation has been conducted which contains the impact of various dimensionless parameters on $(Re_x)^{\frac{1}{2}}C_f$ and $(Re_x)^{\frac{-1}{2}}Nu_x$. Table 3.1 explains the impact of nonlinear stretching parameter n, magnetic parameter M and wall transpiration parameter S on $(Re_x)^{\frac{1}{2}}C_f$. For the rising values of magnetic parameter, $(Re_x)^{\frac{1}{2}}C_f$ decreases. Table 3.1 shows the interval I_f where from the missing condition can be chosen. It is remarkable that the interval mentioned offers a considerable flexibility for the choice of the initial guess. In Table 3.2, the effect of significant parameters on $(Re_x)^{\frac{-1}{2}}Nu_x$ and $(Re_x)^{\frac{-1}{2}}Sh_x$ has been discussed. The rising pattern is found in $(Re_x)^{\frac{-1}{2}}Nu_x$ due to increasing values of R. A decrement is noticed in $(Re_x)^{\frac{-1}{2}}Sh_x$ by increasing the value of Nb and Nt.

The missing initial conditions for $\theta(\xi)$ and $\phi(\xi)$ can be chosen from $[-1.6 - 0.8]$. It is remarkable that the interval mentioned offers a considerable flexibility for the choice of the initial guess.

In Figure 3.2, the velocity profile is shown to drop as the magnetic parameter M is increased. Figure 3.3 shows the impact of M in the temperature profile. Temperature distribution is increased by increasing value of M.

The impact of parameter n on $f'(\xi)$ and $\theta(\xi)$ is depicted in Figures 3.4 and 3.5. By increasing the value of n, the velocity profile decreases but an increment in the temperature distribution due to the effect of S depicted in Figures 3.6 and 3.7 show by increasing the value of S, the velocity profile and temperature distribution decrease.

A significant increase in $\theta(\xi)$ is seen on rising the values of radiation parameter as depicted in Figure 3.8. Figure 3.9 elucidates the effect of Pr on $\theta(\xi)$. By rising the value of Pr, temperature profile is decreased.

The contrasts between the heat generation parameter Q and the temperature profile are shown in Figure 3.10. It can be seen that by rising the value of Q, the temperature profile $\theta(\xi)$ also increases.

Figures 3.11 and 3.12 show the effect of the Brownian parameter Nb, on the temperature and concenteration profiles. Increasing value of Nb is observed to cause an increment in the temperature profile $\theta(\xi)$ but a decrease in the concentration distribution $\phi(\xi)$.

Figures 3.13 shows the effect of Nt on the temperature profiles by increasing the value of Nt it is observed cause an increment in the temperature $\theta(\xi)$.

Figure 3.14 shows impact of Nt on the concentration profile $\phi(\xi)$. Increasing the value of Nt concenteration profile also increases.

Figure 3.15 shows the impact of Lewis numbers Le on the concentration profile

$\phi(\xi)$. The increasing value of Le shows to decrease the concentration profile.

TABLE 3.1: Results of $(Re_x)^{\frac{1}{2}}C_f$ for various parameters

n	M	S	$-(Re_x)^{\frac{1}{2}}C_f$	I_f
1.5	0.5	0.5	1.547729	[-1.6, -0.8]
0.0			1.234501	[-1.6, -0.8]
0.5			1.416913	[-1.3, -1.0]
0.7			1.456598	[-1.7, 1.5]
	0.0		1.336685	[-1.6, 2.6]
	0.25		1.446971	[-1.4, -0.2]
	0.75		1.641091	[-1.5, -0.5]
	1.0		1.728496	[-1.7, -1.0]
		-0.2	1.181757	[-1.2, -0.9]
		0.0	1.276459	[-1.3, -0.9]
		0.2	1.379077	[-1.4, -0.9]
		1.0	1.865900	[-2.0, -0.5]

TABLE 3.2: Results of $-(Re_x)^{\frac{-1}{2}}Nu_x$ and $-(Re_x)^{\frac{-1}{2}}Sh_x$ some fixed parameters $n = 1.5$, $S = 0.5$ and $M = 0.5$

R	Pr	Nt	Nb	Q	Le	$-(Re_x)^{\frac{-1}{2}}Nu_x$	$-(Re_x)^{\frac{-1}{2}}Sh_x$
0.1	2.0	0.2	0.2	0.1	2.0	1.354570	-0.009598
0.0						1.315278	-0.119363
0.5						1.468285	0.273916
0.7						1.551847	0.467323
	1.0					0.772618	0.452331
	1.5					1.080688	-0.208865
	1.75					1.221101	0.097259
		0.0				1.477258	1.004875
		0.1				1.414702	0.473252
		0.15				1.384336	0.225925
			0.5			1.056555	0.697572
			1.0			0.658523	0.916055
			2.0			0.191635	0.997066
				0.0		1.478718	-0.111882
				0.2		1.214521	0.105316
				0.5		0.552410	0.641336
					1.0	1.464162	-0.594562
					3.0	1.287162	0.494303
					5.0	1.211202	1.355628

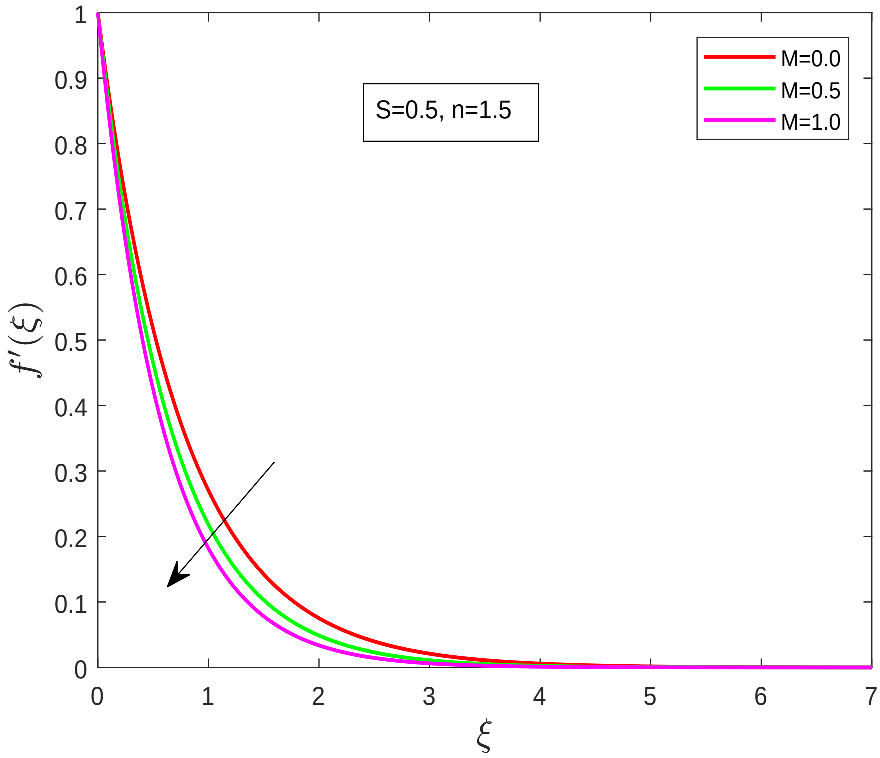

FIGURE 3.2: Change $f'(\xi)$ for M.

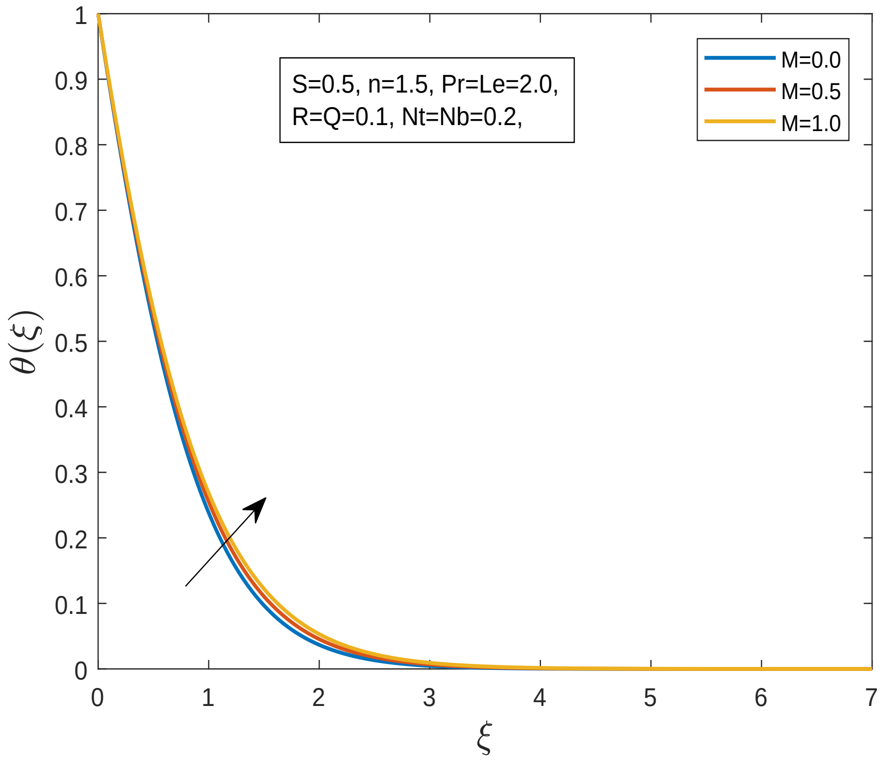

FIGURE 3.3: Change in $\theta(\xi)$ for M.

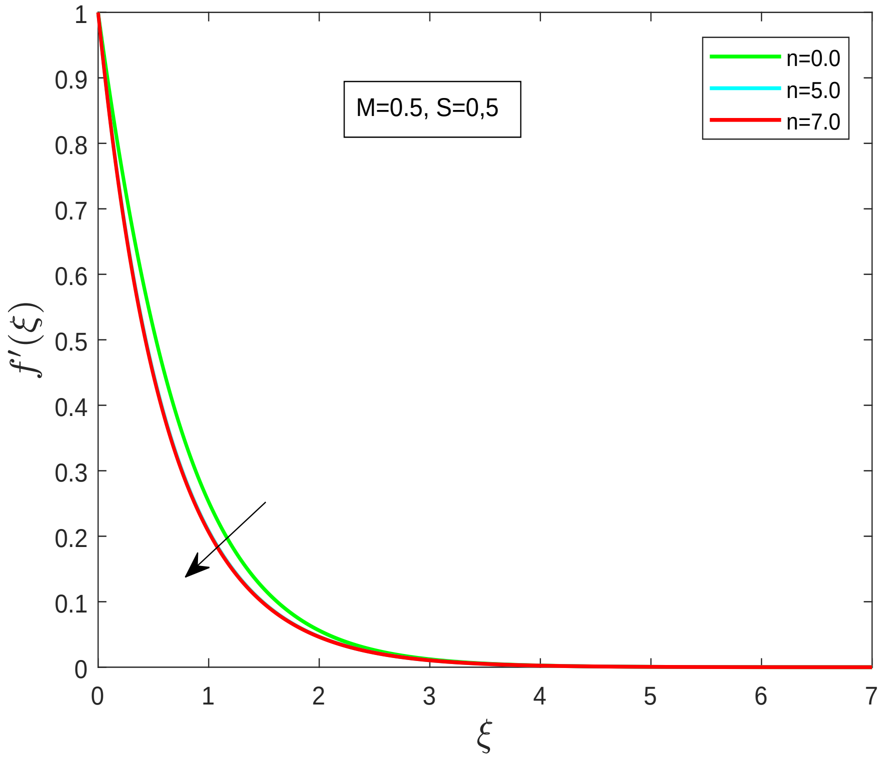

FIGURE 3.4: Change in $f'(\xi)$ for n.

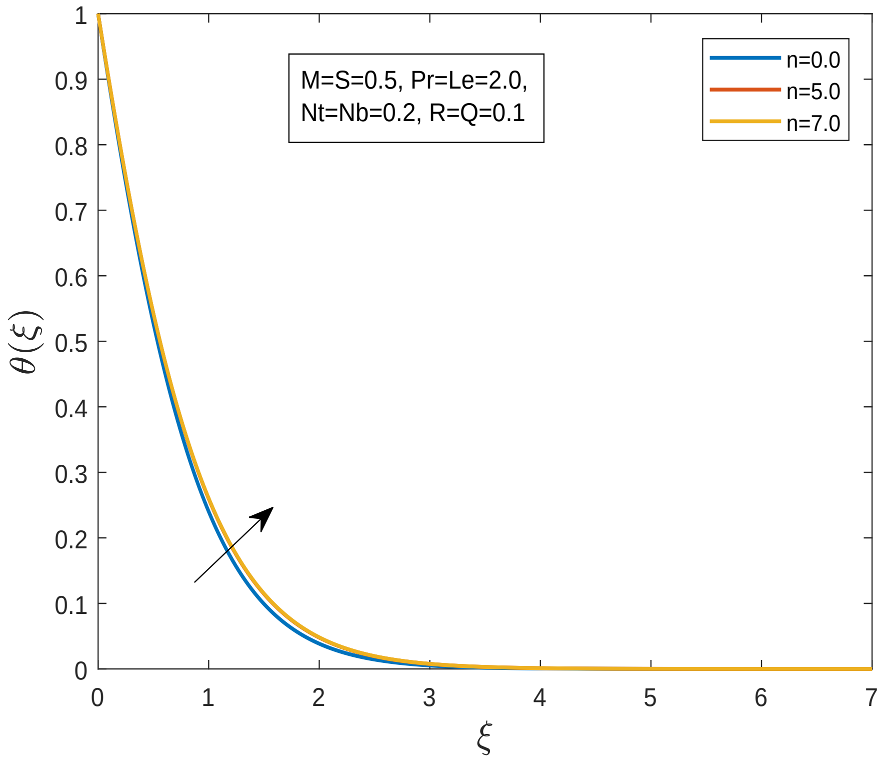

FIGURE 3.5: Change in $\theta(\xi)$ for n.

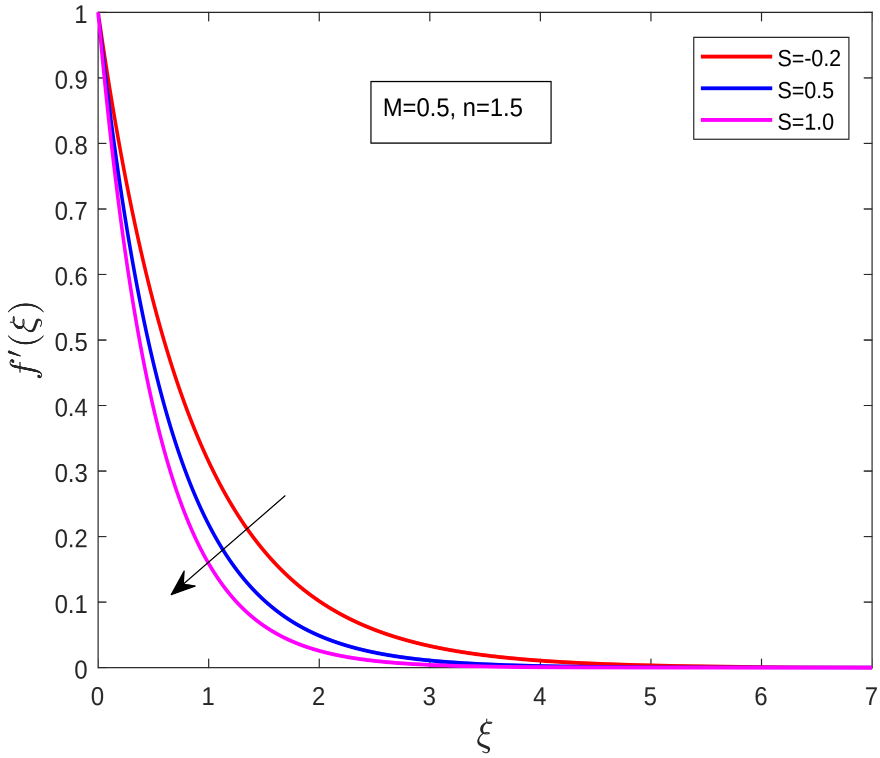

FIGURE 3.6: Change in $f'(\xi)$ for S.

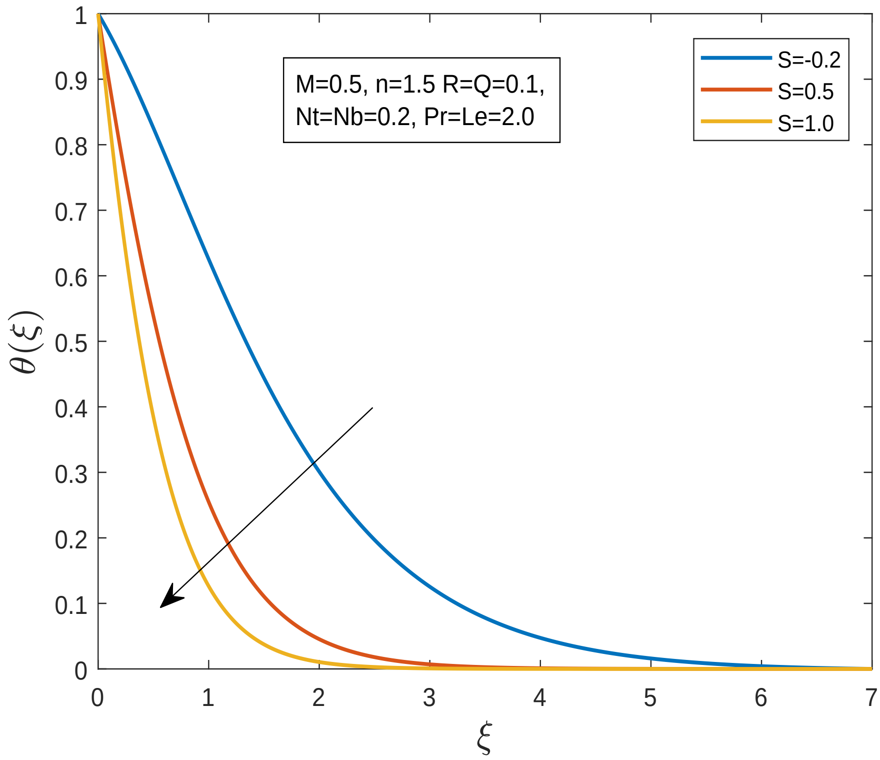

FIGURE 3.7: Change in $\theta(\xi)$ for S.

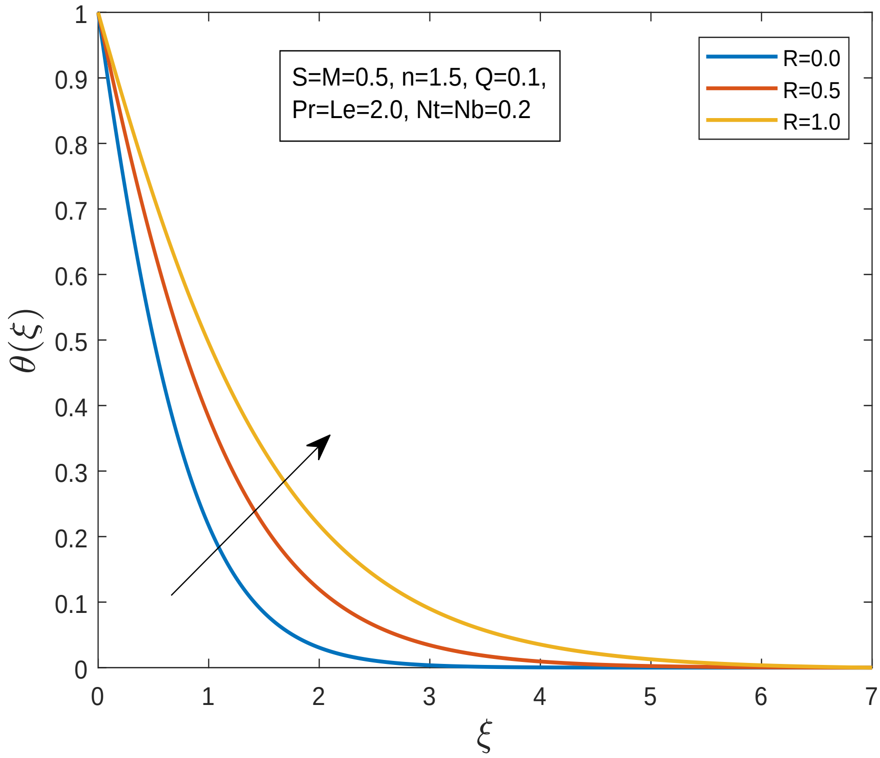

FIGURE 3.8: Change in $\theta(\xi)$ for R.

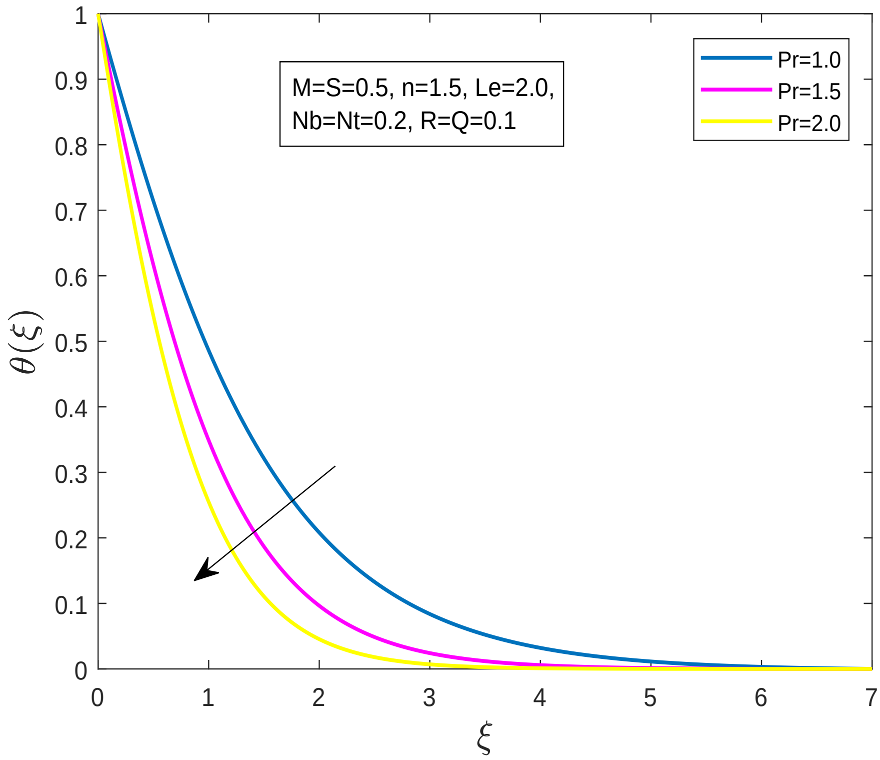

FIGURE 3.9: Change in $\theta(\xi)$ for Pr.

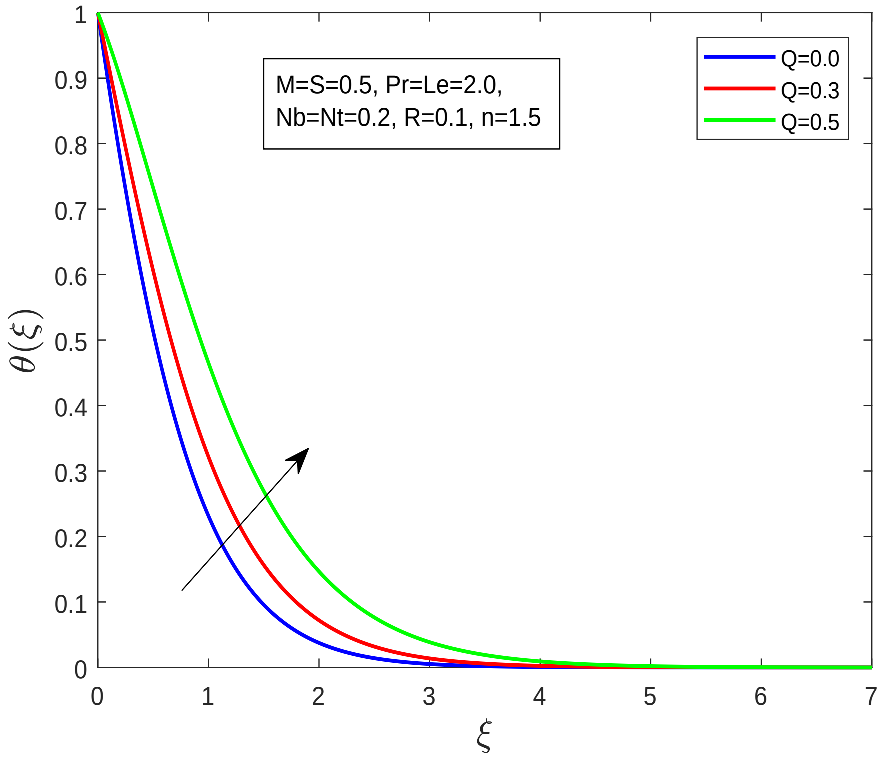

FIGURE 3.10: Change in $\theta(\xi)$ for Q.

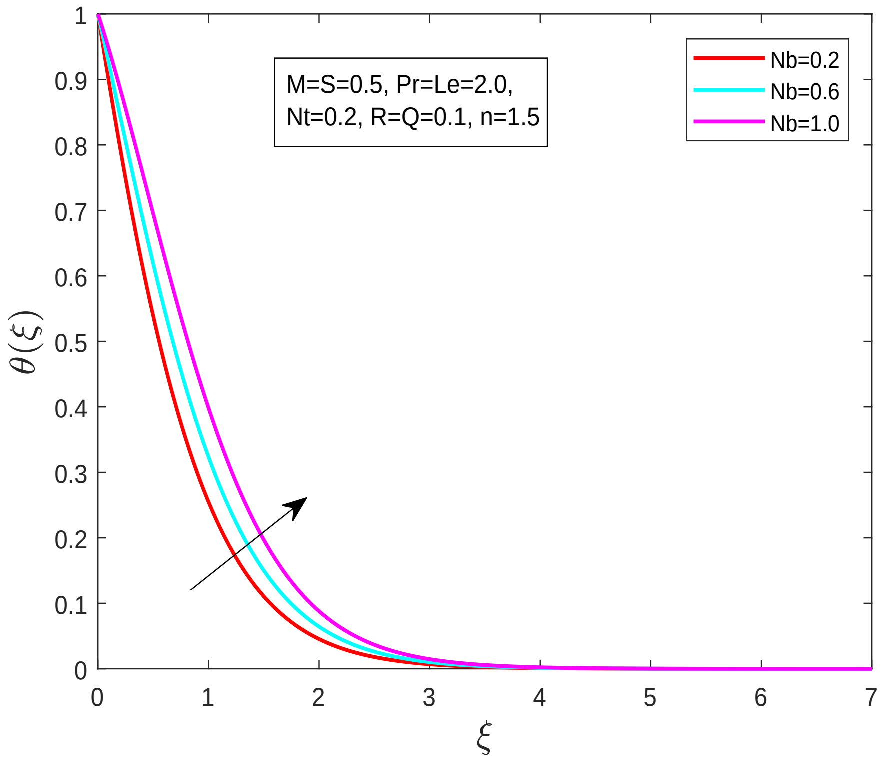

FIGURE 3.11: Change in $\theta(\xi)$ for Nb.

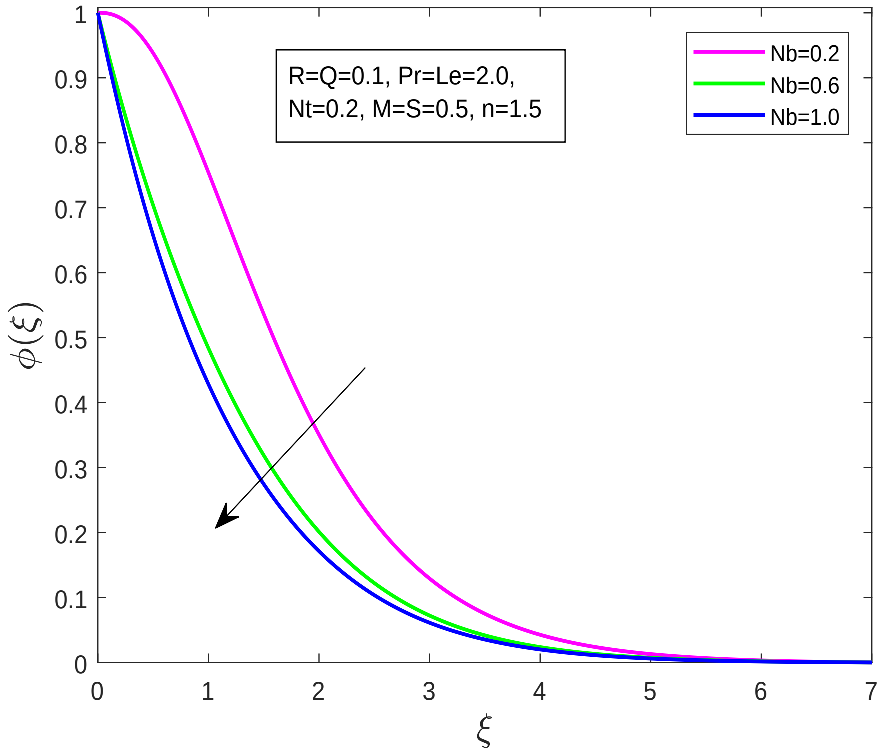

FIGURE 3.12: Change in $\phi(\xi)$ for Nb.

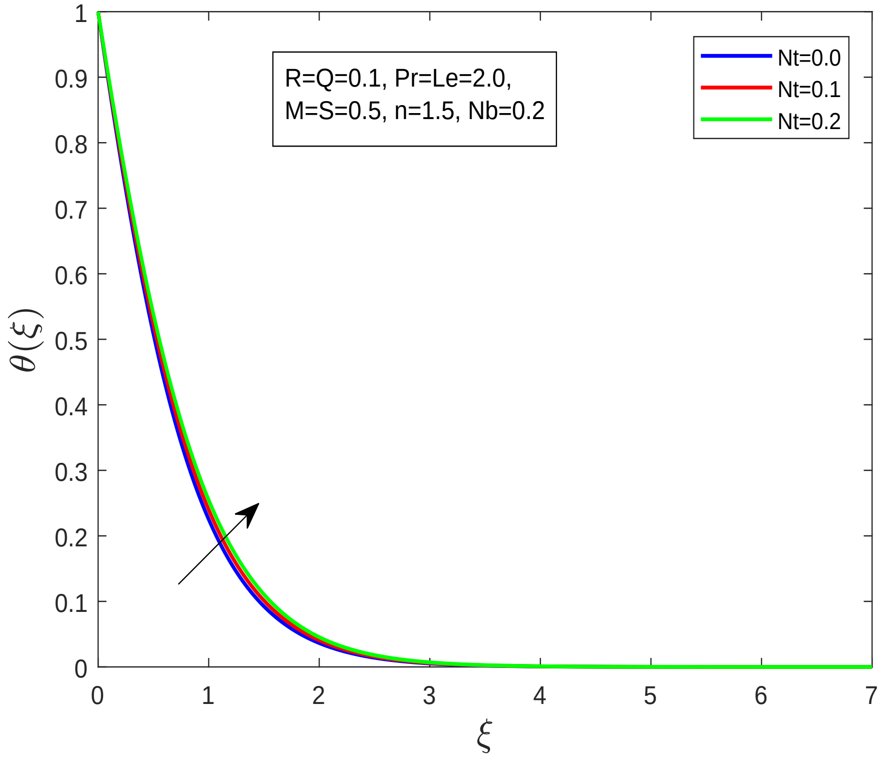

FIGURE 3.13: Change in $\theta(\xi)$ for Nt.

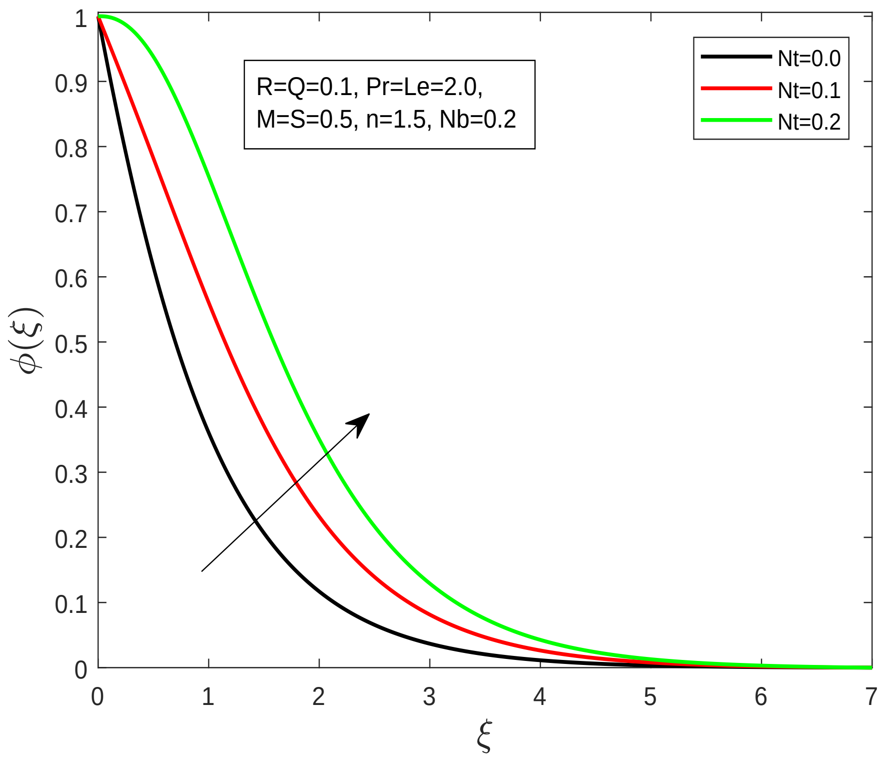

FIGURE 3.14: Change in $\phi(\xi)$ for Nt.

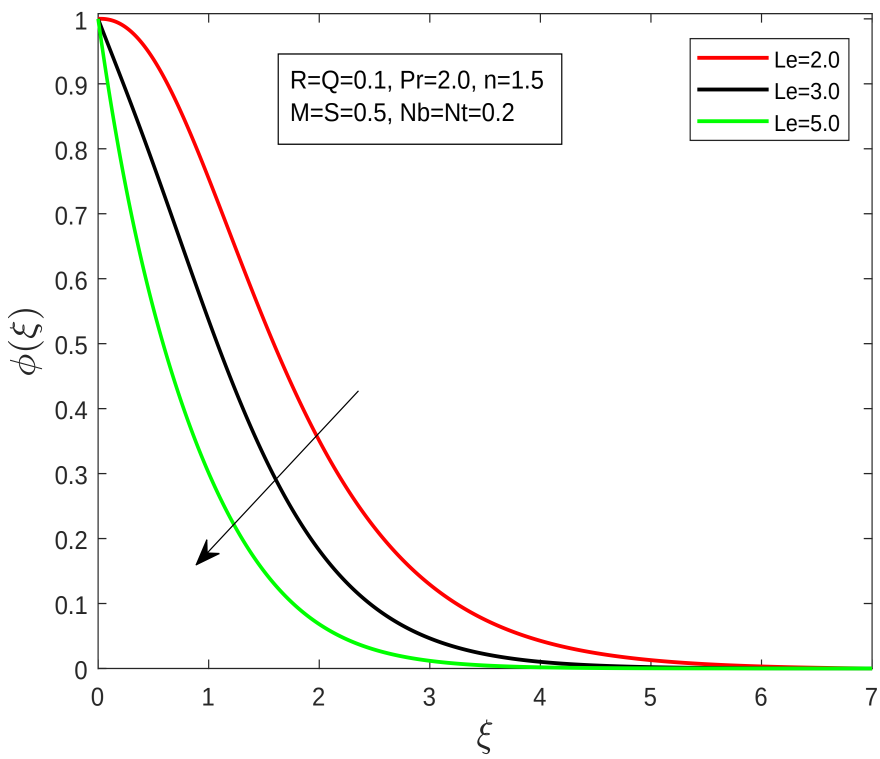

FIGURE 3.15: Change in $\phi(\xi)$ for Le.

Chapter 4

Cattaneo-Christov Double Diffusion, MHD and Thermal Radiation Effects on a Nanofluid Flow

4.1 Introduction

This chapter is an extension of the research article [50] by considering Cattaneo-Christov double diffusion, Brownian motion and Soret number. The set of equations for momentum, energy, and concenteration is attained by utilizing the boundary layer approximation.The governing nonlinear PDEs are converted into a system of dimensionless ODEs by utilizing the similarity transfomations. The numerical solution of ODEs is obtained by applying numerical method known as the shooting method. At the end of this chapter, the final results are discussed for significant parameters effecting velocity profile $f'(\xi)$, temperature distribution $\theta(\xi)$ and concenterartion distribution $\phi(\xi)$ which are shown in tables and graphs. The impact of some important physical parameters on Skin friction, Local Nusselt number and Local Sherwood number is also analyzed.

4.2 Mathematical Modeling

It is aimed to analyse a $2D$, MHD flow of nanofluid past a nonlinear stretching sheet. The flow occupied the space $y > 0$. Magnetic field of strength B is applied through the horizantal axis. Furthermore, the direction of flow is along x-axis and y-axis is normal to the sheet. Energy transport analaysis is also carrieds out in the presence of thermal radiation, heat generation and Cattaneo-Christov double diffusion. Moreover, the concentration of flow is discussed with the help of Cattaneo-Christov.

By considering the above assumptions, the goverining PDEs are.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial y^2}\right) - \frac{\sigma B^2(x)}{\rho}u, \tag{4.2}$$

$$\begin{aligned} &u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \lambda_T\left[u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} + v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y} + u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y} + v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} + 2uv\frac{\partial^2 T}{\partial x \partial y}\right. \\ &\left. + u^2\frac{\partial^2 T}{\partial x^2} + v^2\frac{\partial^2 T}{\partial y^2}\right] = \alpha\left(\frac{\partial^2 T}{\partial y^2}\right) + \tau\left(D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial x} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right) \\ &- \frac{1}{(\rho C)_f}\left(\frac{\partial q_r}{\partial y}\right) + \frac{Q_0}{(\rho C)_f}(T - T_\infty), \end{aligned} \tag{4.3}$$

$$\begin{aligned} &u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + \lambda_C\left[u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x} + v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y} + u\frac{\partial v}{\partial x}\frac{\partial C}{\partial y} + v\frac{\partial u}{\partial y}\frac{\partial C}{\partial x} + 2uv\frac{\partial^2 C}{\partial x \partial y}\right. \\ &\left. + u^2\frac{\partial^2 C}{\partial x^2} + v^2\frac{\partial^2 C}{\partial y^2}\right] = D_B\frac{\partial^2 C}{\partial y^2} + \left(\frac{D_T}{T_\infty}\right)\frac{\partial^2 T}{\partial y^2} + D_m\frac{\partial^2 C}{\partial y^2} + \frac{D_m K_T}{T_m}\frac{\partial^2 T}{\partial y^2}. \end{aligned} \tag{4.4}$$

The associated BCs have been taken as.

$$\left.\begin{aligned} &u = U_w(x) = ax^n, \quad v = v_w(x), \quad T = T_w, \quad C = C_w \quad at \quad y = 0. \\ &u \to 0, \quad T \to T_\infty, \quad C \to C_\infty \quad as \quad y \to \infty. \end{aligned}\right\} \tag{4.5}$$

Following similarity transformation has been used to convert PDEs (4.1)-(4.4) into system of ODEs.

$$\left.\begin{aligned} &\xi = y\sqrt{\frac{a(n+1)}{2\nu}}x^{\frac{n-1}{2}}, \qquad u = ax^n f'(\xi),\\ &v = -x^{n-1}ay\left(\frac{n-1}{2}\right)f'(\xi) - x^{\frac{n-1}{2}}\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu a}{n+1}}f(\xi),\\ &\theta(\xi) = \frac{T-T_\infty}{T_w-T_\infty}, \qquad \phi(\xi) = \frac{C-C_\infty}{C_w-C_\infty}, \end{aligned}\right\} \tag{4.6}$$

The dimensionless velocity, temperature and concentrationwhere are f, θ and ϕ respectively, where ξ stand for the similarity variable.
The detailed procedure for the conversion of (4.1) has been discussed in chapter 3.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4.7}$$

The complete procedure for the conversion of (4.1) discussed in chapter 3.

$$f'''(\xi) + f(\xi)f''(\xi) - \left(\frac{2n}{n+1}\right)f'^2(\xi) - Mf'(\xi) = 0. \tag{4.8}$$

Now, we include below the procedure for the conversion of equation (4.3) into the dimensionless form.

$$\frac{\partial u}{\partial x} = ax^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}f''(\xi) + nax^{n-1}f'(\xi). \tag{4.9}$$

$$\frac{\partial u}{\partial y} = ax^{\frac{3n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}f''(\xi). \tag{4.10}$$

$$\frac{\partial T}{\partial x} = x^{\frac{n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)\theta'(\xi). \tag{4.11}$$

$$\begin{aligned}\frac{\partial^2 T}{\partial x^2} &= x^{n-3}y^2\left(\frac{n-1}{2}\right)^2\frac{a(n+1)}{2\nu}(T_w-T_\infty)\theta''(\xi)\\ &\quad + x^{\frac{n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)\theta'(\xi).\end{aligned} \tag{4.12}$$

$$\frac{\partial T}{\partial y} = x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)\theta'(\xi). \tag{4.13}$$

$$\frac{\partial^2 T}{\partial y^2} = x^{n-1}\frac{a(n+1)}{2\nu}(T_w - T_\infty)\theta''(\xi). \tag{4.14}$$

$$\begin{aligned}\frac{\partial^2 T}{\partial x \partial y} &= x^{\frac{n-3}{2}}\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty)\theta'(\xi) \\ &\quad + x^{n-2}y\left(\frac{n-1}{2}\right)\frac{(n+1)a}{2\nu}(T_w - T_\infty)\theta''(\xi).\end{aligned} \tag{4.15}$$

$$\frac{\partial v}{\partial y} = -nax^{n-1}f'(\xi) - ax^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}f''(\xi). \tag{4.16}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left[-x^{n-1}af'(\xi)y\left(\frac{n-1}{2}\right) - \sqrt{\frac{2\nu a}{n+1}}\left(\frac{n+1}{2}\right)x^{\frac{n-1}{2}}f(\xi)\right],$$

$$\begin{aligned}\frac{\partial v}{\partial x} &= -ax^{\frac{3n-5}{2}}y^2\left(\frac{n-1}{2}\right)^2\sqrt{\frac{(n+1)a}{2\nu}}f''(\xi) - a(n-1)x^{n-2}y\left(\frac{n-1}{2}\right)f'(\xi) \\ &\quad - ax^{n-2}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)f'(\xi) - x^{\frac{n-3}{2}}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu a}{n+1}}f(\xi).\end{aligned} \tag{4.17}$$

$$\begin{aligned}u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} &= ax^n f'(\xi)\left(ax^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}f''(\xi) + nax^{n-1}f'(\xi)\right) \\ &\quad \left(x^{\frac{n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty)\theta'(\xi)\right),\end{aligned}$$

$$\begin{aligned}u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} &= ax^n f'(\xi)\Bigg(ax^{2n-3}y^2\left(\frac{n-1}{2}\right)^2\frac{(n+1)a}{2\nu}f''(\xi)\theta'(\xi)(T_w - T_\infty) \\ &\quad + nax^{\frac{2n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}f'(\xi)\theta'(\xi)(T_w - T_\infty)\Bigg),\end{aligned}$$

$$\begin{aligned}u\frac{\partial u}{\partial x}\frac{\partial T}{\partial x} &= a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(T_w - T_\infty)f'(\xi)f''(\xi)\theta'(\xi) \\ &\quad + na^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty)f'^2(\xi)\theta'(\xi).\end{aligned} \tag{4.18}$$

$$\begin{aligned}v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y} &= \left(-x^{n-1}ay\left(\frac{n-1}{2}\right)f'(\xi) - x^{\frac{n-1}{2}}\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu a}{n+1}}f(\xi)\right) \\ &\quad \left(-nax^{n-1}f'(\xi) - ax^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}f''(\xi)\right) \\ &\quad \left(x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty)\theta'(\xi)\right),\end{aligned}$$

$$\begin{aligned}v\frac{\partial v}{\partial y}\frac{\partial T}{\partial y} &= na^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty)f'^2(\xi)\theta'(\xi) \\ &\quad + a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(T_w - T_\infty)f'(\xi)f''(\xi)\theta'(\xi)\end{aligned}$$

$$+ a^2 x^{\frac{5n-5}{2}} y \left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty) f(\xi) f''(\xi)\theta'(\xi)$$

$$+ n a^2 x^{2n-2}\left(\frac{n+1}{2}\right)(T_w - T_\infty) f(\eta) f'(\xi)\theta'(\xi). \qquad (4.19)$$

$$u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y} = a x^n f'(\xi)\Bigg[- a x^{\frac{3n-5}{2}} y^2 \left(\frac{n-1}{2}\right)^2 \sqrt{\frac{(n+1)a}{2\nu}} f''(\xi)$$

$$- (n-1) x^{n-2} a y \left(\frac{n-1}{2}\right) f'(\xi) - a x^{n-2} y \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) f'(\xi)$$

$$- x^{\frac{n-3}{2}}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu a}{n+1}} f(\xi)\Bigg]\left[x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty)\theta'(\xi)\right],$$

$$u\frac{\partial v}{\partial x}\frac{\partial T}{\partial y} = -a^2 x^{3n-3} y^2 \left(\frac{n-1}{2}\right)^2 \left(\frac{(n+1)a}{2\nu}\right)(T_w - T_\infty) f'(\xi) f''(\xi)\theta'(\xi)$$

$$- (n-1) a^2 x^{\frac{5n-5}{2}} y \left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty) f'^2(\xi)\theta'(\xi)$$

$$- a^2 x^{\frac{5n-5}{2}} y \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty) f'^2(\xi)\theta'(\xi)$$

$$- a^2 x^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(T_w - T_\infty) f(\xi) f'(\xi)\theta'(\xi). \qquad (4.20)$$

$$v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} = \left(-a x^{n-1} y \left(\frac{n-1}{2}\right) f'(\xi) - x^{\frac{n-1}{2}}\sqrt{\frac{2\nu_f a}{n+1}} f(\xi)\right)\left(a x^{\frac{3n-1}{2}}\sqrt{\frac{2\nu a}{n+1}} f''(\xi)\right)$$

$$\left(x^{\frac{n-3}{2}} y \left(\frac{n-1}{2}\right)(T_w - T_\infty)\theta'(\xi)\right),$$

$$v\frac{\partial u}{\partial y}\frac{\partial T}{\partial x} = -a^2 x^{3n-3} y^2 \left(\frac{n-1}{2}\right)^2 \left(\frac{(n+1)a}{2\nu}\right)(T_w - T_\infty) f'(\xi) f''(\xi)\theta'(\xi$$

$$- a^2 x^{\frac{5n-5}{2}} y \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty) f(\xi) f''(\xi)\theta'(\xi).$$

$$(4.21)$$

$$2uv\frac{\partial^2 T}{\partial x \partial y} = 2 a x^n f'(\xi)\left(- x^{n-1} a y \left(\frac{n-1}{2}\right) f'(\xi) - x^{\frac{n-1}{2}}\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu a}{n+1}} f(\xi)\right)$$

$$\left(x^{\frac{n-3}{2}}\frac{n-1}{2}\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty)\theta'(\xi) + x^{n-2} y \frac{n-1}{2}\frac{(n+1)a}{2\nu}(T_w - T_\infty)\theta''(\xi)\right),$$

$$2uv\frac{\partial^2 T}{\partial x \partial y} = \left(- 2a^2 x^{2n-1} y \left(\frac{n-1}{2}\right) f'^2(\xi) - 2 a x^{\frac{3n-1}{2}}\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu a}{n+1}} f(\xi) f'(\xi)\right)$$

$$\left(x^{\frac{n-3}{2}}\sqrt{\frac{(n+1)a}{2\nu}}\theta'(\xi) + x^{n-2} y \frac{(n+1)a}{2\nu}\theta''(\xi)\right)\left(\frac{n-1}{2}\right)(T_w - T_\infty),$$

$$2uv\frac{\partial^2 T}{\partial x \partial y} = -2a^2 x^{\frac{5n-5}{2}} y \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w - T_\infty) f(\xi) f'(\xi)\theta''(\xi)$$

$$- 2a^2y^2x^{3n-3}\left(\frac{n-1}{2}\right)^2\frac{(n+1)a}{2\nu}(T_w-T_\infty)f'^2(\xi)\theta''(\xi)$$
$$- 2a^2x^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)$$
$$- 2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)^2\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi). \tag{4.22}$$

$$u^2\frac{\partial^2 T}{\partial x^2} = a^2x^{2n}f'^2(\xi)\Bigg(x^{n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{a(n+1)}{2\nu}\right)(T_w-T_\infty)\theta''(\xi)$$
$$+ x^{\frac{n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)\theta'(\xi)\Bigg),$$

$$u^2\frac{\partial^2 T}{\partial x^2} = a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(T_w-T_\infty)f'^2(\xi)\theta''(\xi)$$
$$+ a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi). \tag{4.23}$$

$$v^2\frac{\partial^2 T}{\partial y^2} = \left(-x^{n-1}ay\left(\frac{n-1}{2}\right)f'(\xi) - x^{\frac{n-1}{2}}\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu a}{n+1}}f(\xi)\right)^2$$
$$\left(x^{n-1}\frac{a(n+1)}{2\nu}(T_w-T_\infty)\theta''(\xi)\right),$$

$$v^2\frac{\partial^2 T}{\partial y^2} = a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(T_w-T_\infty)f'^2(\xi)\theta''(\xi)$$
$$+ 2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)f(\xi)f'(\xi)\theta''(\xi)$$
$$+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi). \tag{4.24}$$

Adding equations (4.18)-(4.24), we get.

$$na^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi)$$
$$+ a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(T_w-T_\infty)f'(\xi)f''(\xi)\theta'(\xi)$$
$$+ na^2x^{2n-2}\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)$$
$$+ a^2x^{\frac{5n-5}{2}}y\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)f(\xi)f''(\xi)\theta'(\xi)$$
$$- a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(T_w-T_\infty)f'(\xi)f''(\xi)\theta'(\xi)$$

$$
\begin{aligned}
&- (n-1)a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi)\\
&- a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi)\\
&- a^2x^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)\\
&- a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(T_w-T_\infty)f'(\xi)f''(\xi)\theta'(\xi)\\
&- a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)f(\xi)f''(\xi)\theta'(\xi)\\
&- 2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)^2\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi)\\
&- 2a^2y^2x^{3n-3}\left(\frac{n-1}{2}\right)^2\frac{(n+1)a}{2\nu}(T_w-T_\infty)f'^2(\xi)\theta''(\xi)\\
&- 2a^2x^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)\\
&- 2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)f(\xi)f'(\xi)\theta''(\xi)\\
&+ a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(T_w-T_\infty)f'^2(\xi)\theta''(\xi)\\
&+ a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)f'^2(\xi)\theta'(\xi)\\
&+ a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(T_w-T_\infty)f'^2(\xi)\theta''(\xi)\\
&+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)\\
&+ 2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(T_w-T_\infty)f(\xi)f'(\xi)\theta''(\xi)\\
=\ & a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)\\
&\quad + a^2x^{2n-2}\left[\frac{n+1}{2}\right]\left[n-\left(\frac{n-1}{2}\right)-2\left(\frac{n-1}{2}\right)\right](T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)\\
=\ & a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)\\
&\quad + a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(n-\left(\frac{n-1}{2}\right)-(n-1)\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)\\
=\ & a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)
\end{aligned}
$$

$$+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{2n-n+1}{2}-n+1\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)$$
$$= a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)$$
$$+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{2n-n+1-2n+2}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi\theta'(\xi)$$
$$= a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)$$
$$+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{-n+1+2}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)$$
$$= a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)$$
$$+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{-n+3}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)$$
$$= a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)$$
$$- a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{n-3}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi). \tag{4.25}$$

Left hand side of (4.3) is

$$- ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)\theta'(\xi)+\lambda_T a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)$$
$$- \lambda_T a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{n-3}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi). \tag{4.26}$$

Right hand side of (4.3) is

$$ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)\left[\frac{1}{Pr}\left[1+\frac{4}{3}R\right]\theta''(\xi)+Nb\theta'(\xi)\phi'(\xi)+Nt\theta'^2(\xi)+Q\theta(\xi)\right]. \tag{4.27}$$

Comparing (4.26) and (4.27)

$$- ax^{n-1}\left(\frac{n+1}{2}\right)(T_w-T_\infty)f(\xi)\theta'(\xi)+\lambda_T a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(T_w-T_\infty)f^2(\xi)\theta''(\xi)$$
$$- \lambda_T a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{n-3}{2}\right)(T_w-T_\infty)f(\xi)f'(\xi)\theta'(\xi)$$

$$
\begin{aligned}
&= ax^{n-1}\Big(\frac{n+1}{2}\Big)(T_w - T_\infty)\left[\frac{1}{Pr}\Big[1+\frac{4}{3}R\Big]\theta''(\xi) + Nb\theta'\phi' + Nt\theta'^2(\xi) + Q\theta(\xi)\right],\\
&\quad - f(\xi)\theta'(\xi) + \lambda_T a x^{n-1}\Big(\frac{n+1}{2}\Big)f^2(\xi)\theta''(\xi) - \lambda_T a x^{n-1}\left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\theta'(\xi)\\
&= \frac{1}{Pr}\Big(1+\frac{4}{3}R\Big)\theta''(\xi) + Nb\theta'(\xi)\phi'(\xi) + Nt\theta'^2(\xi) + Q\theta(\xi),\\
&\quad - f(\xi)\theta'(\xi) + \lambda_T a x^{n-1}\left[\Big(\frac{n+1}{2}\Big)f^2(\xi)\theta''(\xi) - \left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\theta'(\xi)\right]\\
&= \frac{1}{Pr}\Big(1+\frac{4}{3}R\Big)\theta''(\xi) + Nb\theta'(\xi)\phi'(\xi) + Nt\theta'^2(\xi) + Q\theta(\xi),\\
&\quad - f(\xi)\theta'(\xi) + \gamma_1\left[\Big(\frac{n+1}{2}\Big)f^2(\xi)\theta''(\xi) - \left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\theta'(\xi)\right]\\
&= \frac{1}{Pr}\Big(1+\frac{4}{3}R\Big)\theta''(\xi) + Nb\theta'(\xi)\phi'(\xi) + Nt\theta'^2(\xi) + Q\theta(\xi),\\
&\quad \frac{1}{Pr}\Big(1+\frac{4}{3}R\Big)\theta''(\xi) - \gamma_1\left[\Big(\frac{n+1}{2}\Big)f^2(\xi)\theta''(\xi) - \left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\theta'(\xi)\right]\\
&\quad + f(\xi)\theta'(\xi) + Nb\theta'(\xi)\phi'(\xi) + Nt\theta'^2(\xi) + Q\theta(\xi),\\
&\quad \Big(1+\frac{4}{3}R\Big)\theta''(\xi) - Pr\gamma_1\left[\Big(\frac{n+1}{2}\Big)f^2(\xi)\theta''(\xi) - \left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\theta'(\xi)\right]\\
&\quad + Prf(\xi)\theta'(\xi) + PrNb\theta'(\xi)\phi'(\xi) + PrNt\theta'^2(\xi) + PrQ\theta(\xi).
\end{aligned}
\tag{4.28}
$$

Now, we include below the procedure for the conversion of equation (4.4) into the dimensionless form.

$$
\begin{aligned}
\phi(\xi) &= \frac{C - C_\infty}{C_w - C_\infty},\\
C &= (C_w - C_\infty)\phi(\xi) + C_\infty.\\
\frac{\partial C}{\partial x} &= (C_w - C_\infty)\phi'(\xi)\frac{\partial \xi}{\partial x},\\
\frac{\partial C}{\partial x} &= \left(\frac{n-1}{2}\right)x^{\frac{n-3}{2}}y\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)\phi'(\xi).
\end{aligned}
\tag{4.29}
$$

$$
\frac{\partial C}{\partial y} = x^{\frac{n-1}{2}}\sqrt{\frac{a(n+1)}{2\nu}}(C_w - C_\infty)\phi'(\xi). \tag{4.30}
$$

$$
\frac{\partial^2 C}{\partial y^2} = x^{n-1}\frac{a(n+1)}{2\nu}(C_w - C_\infty)\phi''(\xi). \tag{4.31}
$$

$$
\frac{\partial^2 T}{\partial y^2} = x^{n-1}\frac{a(n+1)}{2\nu}(T_w - T_\infty)\theta''(\xi). \tag{4.32}
$$

$$
\frac{\partial^2 C}{\partial x \partial y} = x^{\frac{n-3}{2}}\sqrt{\frac{(n+1)a}{2\nu}}\left(\frac{n-1}{2}\right)(C_w - C_\infty)\phi'(\xi)
$$

$$+ x^{n-2}y\left(\frac{n-1}{2}\right)\frac{(n+1)a}{2\nu}(C_w - C_\infty)\phi''(\xi). \tag{4.33}$$

$$u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x} = ax^n f'(\xi)\left(ax^{\frac{3n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}f''(\xi) + nax^{n-1}f'(\xi)\right)$$
$$\left(x^{\frac{n-3}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)\phi'(\xi)\right),$$

$$u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x} = ax^n f'(\xi)\Bigg(ax^{2n-3}y^2\left(\frac{n-1}{2}\right)^2\frac{(n+1)a}{2\nu}f''(\xi)\phi'(\xi)(C_w - C_\infty)$$
$$+ nax^{\frac{2n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}f'(\xi)\phi'(\xi)(C_w - C_\infty)\Bigg),$$

$$u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x} = a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(C_w - C_\infty)f'(\xi)f''(\xi)\phi'(\xi)$$
$$+ na^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)f'^2(\xi)\phi'(\xi). \tag{4.34}$$

$$v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y} = \left(-x^{n-1}ay\left(\frac{n-1}{2}\right)f'(\xi) - x^{\frac{n-1}{2}}\sqrt{\frac{2\nu a}{n+1}}\left(\frac{n+1}{2}\right)f(\xi)\right)$$
$$\left(-nax^{n-1}f'(\xi) - ax^{\frac{3n-3}{2}}y\sqrt{\frac{(n+1)a}{2\nu}}\left(\frac{n-1}{2}\right)f''(\xi)\right)$$
$$\left(x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)\phi'(\xi)\right),$$

$$v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y} = na^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)f'^2(\xi)\phi'(\xi)$$
$$+ a^2x^{3n-3}y^2\left(\frac{(n+1)a}{2\nu}\right)\left(\frac{n-1}{2}\right)^2(C_w - C_\infty)f'(\xi)f''(\xi)\phi'(\xi)$$
$$+ a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)f(\xi)f''(\xi)\phi'(\xi)$$
$$+ na^2x^{2n-2}\left(\frac{n+1}{2}\right)(C_w - C_\infty)f(\eta)f'(\xi)\phi'(\xi). \tag{4.35}$$

$$u\frac{\partial v}{\partial x}\frac{\partial C}{\partial y} = ax^n f'(\xi)\Bigg(-ax^{\frac{3n-5}{2}}y^2\left(\frac{n-1}{2}\right)^2\sqrt{\frac{(n+1)a}{2\nu}}f''(\xi)$$
$$-(n-1)x^{n-2}ay\left(\frac{n-1}{2}\right)f'(\xi) - ax^{n-2}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)f'(\xi)$$
$$-x^{\frac{n-3}{2}}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu a}{n+1}}f(\xi)\Bigg)\left(x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)\phi'(\xi)\right),$$

$$u\frac{\partial v}{\partial x}\frac{\partial C}{\partial y} = -a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(C_w - C_\infty)f'(\xi)f''(\xi)\phi'(\xi)$$

$$
\begin{aligned}
&- (n-1)a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)f'^2(\xi)\phi'(\xi)\\
&- a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)f'^2(\xi)\phi'(\xi)\\
&- a^2x^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(C_w - C_\infty)f(\xi)f'(\xi)\phi'(\xi).
\end{aligned}
\tag{4.36}
$$

$$
\begin{aligned}
v\frac{\partial u}{\partial y}\frac{\partial C}{\partial x} &= \left(-ax^{n-1}y\left(\frac{n-1}{2}\right)f'(\xi) - x^{\frac{n-1}{2}}\sqrt{\frac{2\nu_f a}{n+1}}f(\xi)\right)\left(ax^{\frac{3n-1}{2}}\sqrt{\frac{2\nu a}{n+1}}f''(\xi)\right)\\
&\left(x^{\frac{n-3}{2}}y\left(\frac{n-1}{2}\right)(C_w - C_\infty)\phi'(\xi)\right),\\
v\frac{\partial u}{\partial y}\frac{\partial C}{\partial x} &= -a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(C_w - C_\infty)f'(\xi)f''(\xi)\phi'(\xi\\
&- a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)f(\xi)f''(\xi)\phi'(\xi).
\end{aligned}
\tag{4.37}
$$

$$
\begin{aligned}
2uv\frac{\partial^2 C}{\partial x\partial y} &= 2ax^nf'(\xi)\left(-x^{n-1}ay\left(\frac{n-1}{2}\right)f'(\xi) - x^{\frac{n-1}{2}}\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu a}{n+1}}f(\xi)\right)\\
&\left(x^{\frac{n-3}{2}}\frac{n-1}{2}\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)\phi'(\xi) + x^{n-2}y\frac{n-1}{2}\frac{(n+1)a}{2\nu}(C_w - C_\infty)\phi''(\xi)\right),\\
2uv\frac{\partial^2 C}{\partial x\partial y} &= \left[-2a^2x^{2n-1}y\left(\frac{n-1}{2}\right)f'^2(\xi) - 2ax^{\frac{3n-1}{2}}\left(\frac{n+1}{2}\right)\sqrt{\frac{2\nu a}{n+1}}f(\xi)f'(\xi)\right]\\
&\left(x^{\frac{n-3}{2}}\sqrt{\frac{(n+1)a}{2\nu}}\phi'(\xi) + x^{n-2}y\frac{(n+1)a}{2\nu}\phi''(\xi)\right)\left(\frac{n-1}{2}\right)(C_w - C_\infty),\\
2uv\frac{\partial^2 C}{\partial x\partial y} &= -2a^2x^{\frac{5n-5}{2}}y\sqrt{\frac{(n+1)a}{2\nu}}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(C_w - C_\infty)f(\xi)f'(\xi)\phi''(\xi)\\
&- 2a^2y^2x^{3n-3}\left(\frac{n-1}{2}\right)^2\frac{(n+1)a}{2\nu}(C_w - C_\infty)f'^2(\xi)\phi''(\xi)\\
&- 2a^2x^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(C_w - C_\infty)f(\xi)f'(\xi)\phi'(\xi)\\
&- 2a^2x^{\frac{5n-5}{2}}y\sqrt{\frac{(n+1)a}{2\nu}}\left(\frac{n-1}{2}\right)^2(C_w - C_\infty)f'^2(\xi)\phi'(\xi).
\end{aligned}
\tag{4.38}
$$

$$
\begin{aligned}
u^2\frac{\partial^2 C}{\partial x^2} &= a^2x^{2n}f'^2(\xi)\left(x^{n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{a(n+1)}{2\nu}\right)(C_w - C_\infty)\phi''(\xi)\right.\\
&\left.+ x^{\frac{n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty)\phi'(\xi)\right),\\
u^2\frac{\partial^2 C}{\partial x^2} &= a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(C_w - C_\infty)f'^2(\xi)\phi''(\xi)
\end{aligned}
$$

$$+ a^2 x^{\frac{5n-5}{2}} y \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty) f'^2(\xi)\phi'(\xi). \quad (4.39)$$

$$v^2 \frac{\partial^2 C}{\partial y^2} = \left(-x^{n-1} ay \left(\frac{n-1}{2}\right) f'(\xi) - x^{\frac{n-1}{2}} \left(\frac{n+1}{2}\right) \sqrt{\frac{2\nu a}{n+1}} f(\xi)\right)^2$$
$$\left(x^{n-1} \frac{a(n+1)}{2\nu}(C_w - C_\infty)\phi''(\xi)\right),$$

$$\begin{aligned} v^2 \frac{\partial^2 C}{\partial y^2} &= a^2 x^{3n-3} y^2 \left(\frac{n-1}{2}\right)^2 \left(\frac{(n+1)a}{2\nu}\right)(C_w - C_\infty) f'^2(\xi)\phi''(\xi) \\ &+ 2a^2 x^{\frac{5n-5}{2}} y \sqrt{\frac{(n+1)a}{2\nu}} \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(C_w - C_\infty) f(\xi) f'(\xi)\phi''(\xi) \\ &+ a^2 x^{2n-2} \left(\frac{n+1}{2}\right)^2 (C_w - C_\infty) f^2(\xi)\phi''(\xi). \end{aligned} \quad (4.40)$$

Adding equations (4.34)-(4.40), we get.

$$\begin{aligned} & u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x} + v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y} + u\frac{\partial v}{\partial x}\frac{\partial C}{\partial y} + v\frac{\partial u}{\partial y}\frac{\partial C}{\partial x} + 2uv\frac{\partial^2 C}{\partial x \partial y} + u^2\frac{\partial^2 C}{\partial x^2} + v^2\frac{\partial^2 C}{\partial y^2} \\ &= a^2 x^{3n-3} y^2 \left(\frac{n-1}{2}\right)^2 \left(\frac{(n+1)a}{2\nu}\right)(C_w - C_\infty) f'(\xi) f''(\xi)\phi'(\xi) \\ &+ n a^2 x^{\frac{5n-5}{2}} y \left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty) f'^2(\xi)\phi'(\xi) \\ &+ n a^2 x^{\frac{5n-5}{2}} y \left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty) f'^2(\xi)\phi'(\xi) \\ &+ a^2 x^{3n-3} y^2 \left(\frac{n-1}{2}\right)^2 \left(\frac{(n+1)a}{2\nu}\right)(C_w - C_\infty) f'(\xi) f''(\xi)\phi'(\xi) \\ &+ n a^2 x^{2n-2} \left(\frac{n+1}{2}\right)(C_w - C_\infty) f(\xi) f'(\xi)\phi'(\xi) \\ &+ a^2 x^{\frac{5n-5}{2}} y \left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty) f(\xi) f''(\xi)\phi'(\xi) \\ &- a^2 x^{3n-3} y^2 \left(\frac{n-1}{2}\right)^2 \left(\frac{(n+1)a}{2\nu}\right)(C_w - C_\infty) f'(\xi) f''(\xi)\phi'(\xi) \\ &- (n-1) a^2 x^{\frac{5n-5}{2}} y \left(\frac{n-1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty) f'^2(\xi)\phi'(\xi) \\ &- a^2 x^{\frac{5n-5}{2}} y \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w - C_\infty) f'^2(\xi)\phi'(\xi) \\ &- a^2 x^{2n-2} \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(C_w - C_\infty) f(\xi) f'(\xi)\phi'(\xi) \end{aligned}$$

$$
\begin{aligned}
&- a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(C_w-C_\infty)f'(\xi)f''(\xi)\phi'(\xi)\\
&- a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w-C_\infty)f(\xi)f''(\xi)\phi'(\xi)\\
&- 2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)^2\sqrt{\frac{(n+1)a}{2\nu}}(C_w-C_\infty)f'^2(\xi)\phi'(\xi)\\
&- 2a^2y^2x^{3n-3}\left(\frac{n-1}{2}\right)^2\frac{(n+1)a}{2\nu}(C_w-C_\infty)f'^2(\xi)\phi''(\xi)\\
&- 2a^2x^{2n-2}\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)(C_w-C_\infty)f(\xi)f'(\xi)\phi'(\xi)\\
&- 2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w-C_\infty)f(\xi)f'(\xi)\phi''(\xi)\\
&+ a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(C_w-C_\infty)f'^2(\xi)\phi''(\xi)\\
&+ a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w-C_\infty)f'^2(\xi)\phi'(\xi)\\
&+ a^2x^{3n-3}y^2\left(\frac{n-1}{2}\right)^2\left(\frac{(n+1)a}{2\nu}\right)(C_w-C_\infty)f'^2(\xi)\phi''(\xi)\\
&+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(C_w-C_\infty)f^2(\xi)\phi''(\xi)\\
&+ 2a^2x^{\frac{5n-5}{2}}y\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\sqrt{\frac{(n+1)a}{2\nu}}(C_w-C_\infty)f(\xi)f'(\xi)\phi''(\xi),\\
=&\ a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(C_w-C_\infty)f^2(\xi)\phi''(\xi)\\
&+ a^2x^{2n-2}\left[\frac{n+1}{2}\right]\left[n-\left(\frac{n-1}{2}\right)-2\left(\frac{n-1}{2}\right)\right](C_w-C_\infty)f(\xi)f'(\xi)\phi'(\xi),\\
=&\ a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(C_w-C_\infty)f^2(\xi)\phi''(\xi)\\
&+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(n-\left(\frac{n-1}{2}\right)-(n-1)\right)(C_w-C_\infty)f(\xi)f'(\xi)\phi'(\xi),\\
=&\ a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(C_w-C_\infty)f^2(\xi)\phi''(\xi)\\
&+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{2n-n+1}{2}-n+1\right)(C_w-C_\infty)f(\xi)f'(\xi)\phi'(\xi),\\
=&\ a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(C_w-C_\infty)f^2(\xi)\phi''(\xi)\\
&+ a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{2n-n+1-2n+2}{2}\right)(C_w-C_\infty)f(\xi)f'(\xi\phi'(\xi),
\end{aligned}
$$

$$
\begin{aligned}
&= a^2x^{2n-2}\Big(\frac{n+1}{2}\Big)^2(C_w - C_\infty)f^2(\xi)\phi''(\xi)\\
&\quad + a^2x^{2n-2}\Big(\frac{n+1}{2}\Big)\Big(\frac{-n+1+2}{2}\Big)(C_w - C_\infty)f(\xi)f'(\xi)\phi'(\xi),\\
&= a^2x^{2n-2}\Big(\frac{n+1}{2}\Big)^2(C_w - C_\infty)f^2(\xi)\phi''(\xi)\\
&\quad + a^2x^{2n-2}\Big(\frac{n+1}{2}\Big)\Big(\frac{-n+3}{2}\Big)(C_w - C_\infty)f(\xi)f'(\xi)\phi'(\xi),\\
&= a^2x^{2n-2}\Big(\frac{n+1}{2}\Big)^2(C_w - C_\infty)f^2(\xi)\phi''(\xi)\\
&\quad - a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{n-3}{2}\right)(C_w - C_\infty)f(\xi)f'(\xi)\phi'(\xi).
\end{aligned} \tag{4.41}
$$

$$
D_B\frac{\partial^2 C}{\partial y^2} + \left(\frac{D_T}{T_\infty}\right)\frac{\partial^2 T}{\partial y^2} = ax^{n-1}\left(\frac{n+1}{2}\right)\left[\frac{(\rho C)_f}{(\rho C)_p}Nb\phi''(\xi) + \frac{(\rho C)_f}{(\rho C)_p}Nt\theta''(\xi)\right]. \tag{4.42}
$$

$$
D_m\frac{\partial^2 C}{\partial y^2} + \frac{D_m K_T}{T_m}\frac{\partial^2 T}{\partial y^2} = (C_w - C_\infty)x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)\left[D_m\phi''(\xi) + S_r\alpha_T\theta''(\xi)\right]. \tag{4.43}
$$

Using (4.29), (4.30) and (4.41) in the left side of (4.4), we get

$$
\begin{aligned}
&u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + \lambda_c\left[u\frac{\partial u}{\partial x}\frac{\partial C}{\partial x} + v\frac{\partial v}{\partial y}\frac{\partial C}{\partial y} + u\frac{\partial v}{\partial x}\frac{\partial C}{\partial y} + v\frac{\partial u}{\partial y}\frac{\partial C}{\partial x} + 2uv\frac{\partial^2 C}{\partial x\partial y}\right.\\
&\quad \left. + u^2\frac{\partial^2 C}{\partial x^2} + v^2\frac{\partial^2 C}{\partial y^2}\right] = -ax^{n-1}\left(\frac{n+1}{2}\right)(C_w - C_\infty)f(\xi)\phi'(\xi)\\
&\quad + \lambda_c a^2x^{2n-2}\left(\frac{n+1}{2}\right)^2(C_w - C_\infty)f^2(\xi)\phi''(\xi)\\
&\quad - \lambda_c a^2x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{n-3}{2}\right)(C_w - C_\infty)f(\xi)f'(\xi)\phi'(\xi).
\end{aligned} \tag{4.44}
$$

Using (4.42) and (4.43) in the right side of (4.4), we get

$$
\begin{aligned}
&D_B\frac{\partial^2 C}{\partial y^2} + \left[\frac{D_T}{T_\infty}\right]\frac{\partial^2 T}{\partial y^2} + D_m\frac{\partial^2 C}{\partial y^2} + \frac{D_m K_T}{T_m}\frac{\partial^2 T}{\partial y^2} = ax^{n-1}\Big(\frac{n+1}{2}\Big)\Big[\frac{(\rho C)_f}{(\rho C)_p}Nb\phi''(\xi)\\
&\quad + \frac{(\rho C)_f}{(\rho C)_p}Nt\theta''(\xi)\Big] + (C_w - C_\infty)x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)\left[D_m\phi''(\xi) + S_r\alpha_T\theta''(\xi)\right].
\end{aligned} \tag{4.45}
$$

Comparing equation (4.44) and (4.45)

$$
\begin{aligned}
&- ax^{n-1}\left(\frac{n+1}{2}\right)(C_w - C_\infty)f(\xi)\phi'(\xi) + \lambda_c a^2 x^{2n-2}\left(\frac{n+1}{2}\right)^2 (C_w - C_\infty)f^2(\xi)\phi''(\xi)\\
&- \lambda_c a^2 x^{2n-2}\left(\frac{n+1}{2}\right)\left(\frac{n-3}{2}\right)(C_w - C_\infty)f(\xi)f'(\xi)\phi'(\xi) =\\
&+ ax^{n-1}\Big(\frac{n+1}{2}\Big)\Big[\frac{(\rho C)_f}{(\rho C)_p}Nb\phi''(\xi) + \frac{(\rho C)_f}{(\rho C)_p}Nt\theta''(\xi)\Big]\\
&+ (C_w - C_\infty)x^{n-1}\left(\frac{a(n+1)}{2\nu}\right)\left[D_m\phi''(\xi) + S_r\alpha_T\theta''(\xi)\right],\\
&- f(\xi)\phi'(\xi) + \lambda_c a x^{n-1}\left(\frac{n+1}{2}\right)f^2(\xi)\phi''(\xi) - \lambda_c a x^{n-1}\left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\phi'(\xi)\\
&= +\Big(\frac{(\rho C)_f}{(\rho C)_p}Nb\phi''(\xi) + \frac{(\rho C)_f}{(\rho C)_p}Nt\theta''(\xi)\Big) + \frac{1}{\nu}\left[D_m\phi''(\xi) + S_r\alpha_T\theta''(\xi)\right],\\
&\quad - f(\xi)\phi'(\xi) + \lambda_c a x^{n-1}\left(\left(\frac{n+1}{2}\right)f^2(\xi)\phi''(\xi) - \left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\phi'(\xi)\right)\\
&= \frac{(\rho C)_f}{(C_w - C_\infty)(\rho C)_p}\Big(Nb\phi''(\xi) + Nt\theta''(\xi)\Big) + \frac{D_m}{\nu}\phi''(\xi) + \frac{S_r\alpha_T}{\nu}\theta''(\xi),\\
&\quad - f(\xi)\phi'(\xi) + \gamma_2\left(\left(\frac{n+1}{2}\right)f^2(\xi)\phi''(\xi) - \left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\phi'(\xi)\right)\\
&= \frac{(\rho C)_f Nb}{(C_w - C_\infty)(\rho C)_p}\Big(\phi''(\xi) + \frac{Nt}{Nb}\theta''(\xi)\Big) + \frac{D_m}{\nu}\phi''(\xi) + \frac{S_r\alpha_T}{\nu}\theta''(\xi),\\
&\quad - f(\xi)\phi'(\xi) + \gamma_2\left(\left(\frac{n+1}{2}\right)f^2(\xi)\phi''(\xi) - \left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\phi'(\xi)\right)\\
&= \frac{1}{Le}\Big(\phi''(\xi) + \frac{Nt}{Nb}\theta''(\xi)\Big) + \frac{1}{Le}\phi''(\xi) + S_r\frac{1}{Pr}\theta''(\xi),\\
&\quad - Lef(\xi)\phi'(\xi) + Le\gamma_2\left(\left(\frac{n+1}{2}\right)f^2(\xi)\phi''(\xi) - \left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\phi'(\xi)\right)\\
&= \phi''(\xi) + \frac{Nt}{Nb}\theta''(\xi) + \phi''(\xi) + S_r\frac{Le}{Pr}\theta''(\xi),\\
&\quad - PrLef(\xi)\phi'(\xi) + PrLe\gamma_2\left(\left(\frac{n+1}{2}\right)f^2(\xi)\phi''(\xi) - \left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\phi'(\xi)\right)\\
&= 2Pr\phi''(\xi) + \frac{PrNt}{Nb}\theta''(\xi) + S_rLe\theta''(\xi),\\
&\quad 2Pr\phi''(\xi) - PrLe\gamma_2\left[\left(\frac{n+1}{2}\right)f^2(\xi)\phi''(\xi) - \left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\phi'(\xi)\right]\\
&\quad + \left(\frac{PrNt}{Nb} + S_rLe\right)\theta''(\xi) + PrLef(\xi)\phi'(\xi) = 0. \qquad (4.46)
\end{aligned}
$$

Now discussing the procedure for conversion of boundary conditions into dimensionless form.

$$u = U_w(x) = ax^n, \qquad at \quad y = 0.$$
$$u = af'(\xi)x^n,$$
$$\Rightarrow \quad af'(\xi)x^n = ax^n,$$
$$\Rightarrow \quad ax^n f'(\xi) = ax^n,$$
$$\Rightarrow \quad f'(\xi) = 1, \qquad at \quad \xi = 0.$$
$$\Rightarrow \quad f'(0) = 1.$$
$$v = v_w(x), \qquad at \quad y = 0.$$
$$\Rightarrow \quad -x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}}\left(f(\xi) + \xi f'(\xi)\left(\frac{n-1}{n+1}\right)\right) = v_w(x), \qquad at \quad \xi = 0.$$
$$\Rightarrow \quad -x^{\frac{n-1}{2}}\sqrt{\frac{(n+1)\nu a}{2}} f(\xi) = v_w(x), \qquad at \quad \xi = 0.$$
$$\Rightarrow \quad f(\xi) = -\frac{v_w(x)\sqrt{2}}{x^{\frac{n-1}{2}}\sqrt{a}\nu(n+1)},$$
$$\Rightarrow \quad f(0) = S.$$

$$T = T_w, \qquad at \quad y = 0.$$
$$\Rightarrow \quad \theta(\xi)(T_w - T_\infty) + T_\infty = T_w,$$
$$\Rightarrow \quad \theta(\xi)(T_w - T_\infty) = (T_w - T_\infty),$$
$$\Rightarrow \quad \theta(\xi) = 1, \qquad at \quad \xi = 0.$$
$$\Rightarrow \quad \theta(0) = 1.$$
$$C = C_w, \qquad at \quad y = 0.$$
$$\Rightarrow \quad \phi(\xi)(C_w - C_\infty) + C_\infty = C_w,$$
$$\Rightarrow \quad \phi(\xi)(C_w - C_\infty) = (C_w - C_\infty),$$
$$\Rightarrow \quad \phi(\xi) = 1, \qquad at \quad \xi = 0.$$
$$\Rightarrow \quad \phi(0) = 1.$$
$$u \to (0), \qquad as \quad y \to \infty.$$
$$\Rightarrow \quad af'(\xi)x^n \to (0),$$

$$
\begin{aligned}
\Rightarrow \quad & f'(\xi) \to (0), && as \quad \xi \to \infty. \\
\Rightarrow \quad & f'(\xi) \to 0. \\
& T \to T_\infty, && as \quad y \to \infty. \\
\Rightarrow \quad & \theta(\xi)(T_w - T_\infty) + T_\infty \to T_\infty, \\
\Rightarrow \quad & \theta(\xi)(T_w - T_\infty) \to 0, \\
\Rightarrow \quad & \theta(\xi) \to 0, && as \quad \xi \to \infty. \\
\Rightarrow \quad & \theta(\infty) \to 0.
\end{aligned}
$$

$$
\begin{aligned}
& C \to C_\infty, && as \quad y \to \infty. \\
\Rightarrow \quad & \phi(\xi)(C_w - C_\infty) + C_\infty \to C_\infty, \\
\Rightarrow \quad & \phi(\xi)(C_w - C_\infty) \to 0, \\
\Rightarrow \quad & \phi(\xi) \to 0, && as \quad \xi \to \infty. \\
\Rightarrow \quad & \phi(\infty) \to 0.
\end{aligned}
$$

The final dimensionless form of the governing model, is

$$
f'''(\xi) + f(\xi)f''(\xi) - \left(\frac{2n}{n+1}\right)f'^2(\xi) - Mf'(\xi) = 0. \tag{4.47}
$$

$$
\begin{aligned}
&\Big(1 + \frac{4}{3}R\Big)\theta''(\xi) - Pr\gamma_1\left[\Big(\frac{n+1}{2}\Big)f^2(\xi)\theta''(\xi) - \Big(\frac{n-3}{2}\Big)f(\xi)f'(\xi)\theta'(\xi)\right] \\
&+ Prf(\xi)\theta'(\xi) + PrNb\theta'(\xi)\phi'(\xi) + PrNt\theta'^2(\xi) + PrQ\theta(\xi).
\end{aligned} \tag{4.48}
$$

$$
\begin{aligned}
&2Pr\phi''(\xi) - PrLe\gamma_2\left[\left(\frac{n+1}{2}\right)f^2(\xi)\phi''(\xi) - \left(\frac{n-3}{2}\right)f(\xi)f'(\xi)\phi'(\xi)\right] \\
&+ \left[\frac{PrNt}{Nb} + S_rLe\right]\theta''(\xi) + PrLef(\xi)\phi'(\xi) = 0.
\end{aligned} \tag{4.49}
$$

The associated BCs (4.5) in the dimensionless form are,

$$
\left.
\begin{aligned}
& f(0) = S, \quad f'(0) = 1, \quad \theta(0) = 1, \quad \phi(0) = 1 \\
& f'(\infty) \to 0, \quad \theta(\infty) \to 0, \quad \phi(\infty) \to 0.
\end{aligned}
\right\} \tag{4.50}
$$

Different parameters used in equations (4.47)-(4.49) are formulated as follows.

$$M = \frac{2\sigma B_0^2 x^n}{\rho a(n+1)}, \quad Le = \frac{\nu}{D_B} \quad R = \frac{4\sigma^* T_\infty^3}{kk^*}, \quad Pr = \frac{\nu}{\alpha},$$
$$S = -\frac{v_w(x)}{x^{\frac{n-1}{2}}\sqrt{a\nu(n+1)}}\sqrt{2}, \quad Q = \frac{Q_0}{(n+1)\nu(\rho c)_f}, \quad \gamma_1 = \lambda_T a x^{n-1},$$
$$\gamma_2 = \lambda_C a x^{n-1}, \quad Nb = \frac{(\rho c)_p D_B (C_w - C_\infty)}{(\rho c)_f \nu}, \quad Nt = \frac{(\rho c)_p D_T (T_w - T_\infty)}{(\rho c)_f T_\infty \nu}.$$

4.3 Solution Methodology

The ordinary differential equation (4.47) has been resolved using the shooting method.

$$f = Y_1, \qquad f' = Y_1' = Y_2, \qquad f'' = Y_1'' = Y_2' = Y_3, \qquad f''' = Y_3'.$$

As a result the momentum equation is converted into the following system of first order ODEs.

$$Y_1' = Y_2, \qquad Y_1(0) = 0.$$
$$Y_2' = Y_3, \qquad Y_2(0) = 1.$$
$$Y_3' = \left(\frac{2n}{n+1}\right) Y_2^2 - Y_1 Y_3 + M Y_2, \qquad Y_3(0) = w.$$

The above IVP will be numerically solved by RK-4. The missing condition w is to be chosen such that.

$$Y_2(\xi_\infty, w) = 0.$$

Newton's method will be used to find w. This method has the following iterative scheme.

$$w_{n+1} = w_n - \frac{Y_2(\xi_\infty, w_n)}{\frac{\partial}{\partial w_n}(Y_2(\xi_\infty, w_n))}.$$

We further introduce the following notations,

$$\frac{\partial Y_1}{\partial w} = Y_4, \qquad \frac{\partial Y_2}{\partial w} = Y_5, \qquad \frac{\partial Y_3}{\partial w} = Y_6.$$

As a result of these new notations the Newton's iterative scheme gets the form.

$$w_{n+1} = w_n - \frac{Y_2(\xi_\infty, w_n)}{Y_5(\xi_\infty, w_n)}.$$

Now differentiating the system of three first order ODEs with respect to w, we get another system of ODEs, as follows.

$$Y_4' = Y_5, \qquad Y_4(0) = 0.$$
$$Y_5' = Y_6, \qquad Y_5(0) = 0.$$
$$Y_6' = \left(\frac{4n}{n+1}\right) Y_2Y_5 - Y_4Y_3 - Y_1Y_6 + MY_5, \qquad Y_6(0) = 1.$$

The stopping criteria for the Newton's technique is set as.

$$\mid Y_2(\xi_\infty, w) \mid < \epsilon,$$

where $\epsilon > 0$ is an arbitrarily small positive number. From now onward ϵ has been taken as 10^{-10}.

Also, for equations (4.48) and (4.49), the following notation have been used.

$$\theta = Y_1, \quad \theta' = Y_1' = Y_2, \quad \theta'' = Y_2'.$$
$$\phi = Y_3, \quad \phi' = Y_3' = Y_4, \quad \phi'' = Y_4'.$$
$$A_1 = \left(1 + \frac{4}{3}R\right), \quad A_2 = \left(A_1 - Pr\lambda_a \left(\frac{n+1}{2}\right) f^2\right),$$
$$A_3 = \left(2Pr - Pr\lambda_b \left(\frac{n+1}{2}\right) f^2\right), \quad A_4 = \left(\frac{PrNt}{Nb} + S_r Le\right).$$

The system of equations (4.48) and (4.49), can be written in the form of the following first order coupled ODEs.

$$
\begin{aligned}
Y_1' &= Y_2, && Y_1(0) = 1.\\
Y_2' &= -\frac{Pr}{A_2}\left[\lambda_a\Big(\frac{n-3}{2}\Big)ff'Y_2 + fY_2 + NbY_2Y_4 + NtY_2^2 + QY_1\right], && Y_2(0) = p.\\
Y_3' &= Y_4, && Y_3(0) = 1.\\
Y_4' &= -\frac{1}{A_3}\left[Pr\lambda_b\Big(\frac{n-3}{2}\Big)ff'Y_4 + PrLefY_4 - A_4\left[\frac{Pr}{A_2}\left[\lambda_a\Big(\frac{n-3}{2}\Big)ff'Y_2\right.\right.\right.\\
&\qquad \left.\left.\left. + fY_2 + NbY_2Y_4 + NtY_2^2 + QY_1\right]\right]\right], && Y_4(0) = q.
\end{aligned}
$$

The RK-4 method has been taken into consideration for solving the above initial value problem. For the above system of equtions, the missing conditions are to be chosen such that.

$$Z_1(\xi_\infty, p, q) = 0, \qquad Z_3(\xi_\infty, p, q) = 0.$$

To solve the above algebaric equations, we apply the Newton's method which has the following scheme.

$$
\begin{bmatrix} p \\ q \end{bmatrix}_{(n+1)} = \begin{bmatrix} p \\ q \end{bmatrix}_{(n)} - \begin{bmatrix} \frac{\partial Y_1}{\partial p} & \frac{\partial Y_1}{\partial q} \\ \frac{\partial Y_3}{\partial p} & \frac{\partial Y_3}{\partial q} \end{bmatrix}^{-1}_{(n)} \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix}_{(n)}
$$

Now, introduce the following notations,

$$
\begin{aligned}
&\frac{\partial Y_1}{\partial p} = Y_5, && \frac{\partial Y_2}{\partial p} = Y_6, && \frac{\partial Y_3}{\partial p} = Y_7, && \frac{\partial Y_4}{\partial p} = Y_8.\\
&\frac{\partial Y_1}{\partial q} = Y_9, && \frac{\partial Y_2}{\partial q} = Y_{10}, && \frac{\partial Y_3}{\partial q} = Y_{11}, && \frac{\partial Y_4}{\partial q} = Y_{12}.
\end{aligned}
$$

As the result of these new notations, the Newton's iterative scheme gets the form.

$$
\begin{bmatrix} p \\ q \end{bmatrix}_{(n+1)} = \begin{bmatrix} p \\ q \end{bmatrix}_{(n)} - \begin{bmatrix} Y_5 & Y_9 \\ Y_7 & Y_{11} \end{bmatrix}^{-1}_{(n)} \begin{bmatrix} Y_1 \\ Y_3 \end{bmatrix}_{(n)}
$$

Now differentiating the system of four first order ODEs with respect to p, and q we get another system of ODEs, as follows.

$$Y_5' = Y_6, \qquad Y_5(0) = 0.$$

$$Y_6' = -\frac{Pr}{A_2}\left[\lambda_a\Big(\frac{n-3}{2}\Big)ff'Y_6 + fY_6 + Nb(Y_6Y_4 + Y_2Y_8) + 2NtY_2Y_6 + QY_5\right], \qquad Y_6(0) = 1.$$

$$Y_7' = Y_8, \qquad Y_7(0) = 0.$$

$$Y_8' = -\frac{1}{A_3}\left[Pr\lambda_b\Big(\frac{n-3}{2}\Big)ff'Y_8 + PrLefY_8 - A_4\left[\frac{Pr}{A_2}\left[\lambda_a\Big(\frac{n-3}{2}\Big)ff'Y_6 + fY_6 + Nb(Y_6Y_4 + Y_2Y_8) + 2NtY_2Y_6 + QY_5\right]\right]\right], \qquad Y_8(0) = 0.$$

$$Y_9' = Y_{10}, \qquad Y_9(0) = 0.$$

$$Y_{10}' = -\frac{Pr}{A_2}\left[\lambda_a\Big(\frac{n-3}{2}\Big)ff'Y_{10} + fY_{10} + Nb(Y_{10}Y_4 + Y_2Y_{12}) + 2NtY_2Y_{10} + QY_9\right], \qquad Y_{10}(0) = 0.$$

$$Y_{11}' = Y_{12}, \qquad Y_{11}(0) = 0.$$

$$Y_{12}' = -\frac{1}{A_3}\left[Pr\lambda_b\Big(\frac{n-3}{2}\Big)ff'Y_{12} + PrLefY_{12} - A_4\left[\frac{Pr}{A_2}\left[\lambda_a\Big(\frac{n-3}{2}\Big)ff'Y_{10} + fY_{10} + Nb(Y_{10}Y_4 + Y_2Y_{12}) + 2NtY_2Y_{10} + QY_9\right]\right]\right], \qquad Y_{12}(0) = 1.$$

The stopping criteria for the Newton's method is set as.

$$max\{|Z_1(\xi_\infty, p^n, q^n)|, |Z_3(\xi_\infty, p^n, q^n)|\} < \epsilon.$$

4.4 Representation of Graphs and Tables

The principle object is about to examine the effects of different parameters against the velocity $f'(\xi)$, temperature $\theta(\xi)$ and concentration distribution $\phi(\xi)$. The impact of different factors like nonlinear stretching parameter n, magnetic parameter M, thermal radiation R and Lewis number Le is observed graphically. Numerical outcomes of the skin friction coefficient, local Nusselt number and local Sherwood number for the distinct values of some fixed parameters are shown in Table. The missing initial conditions for $\theta(\xi)$ and $\phi(\xi)$ can be chosen from $[-1.6-0.8]$. It is remarkable that the interval mentioned offers a considerable flexibility for the choice of the initial guess.

Figure 4.1 implies the effect of M on velocity distribution.It depict clearly that M and $f'(\xi)$ are reciprocally related such that increasing M will decrease $f'(\xi)$. Figures 4.2 shows the relation between magnetic field and temperature profile.An increase in value of M increase temperature $\theta(\xi)$.

Figures 4.3 and 4.4 emphasize impact of stretching parameter n on $f'(\xi)$ and $\theta(\xi)$. Elevating the value of n results in drop in value of $f'(\xi)$ and $\theta(\xi)$. Figures 4.5 and 4.6 relate wall transpiration parameter S to velocity and temperature profiles boht velocity and temperature profiles shows inverse relation along S.

Figure 4.7 denotes the changing behaviour of temperature on varrying values of R, Rising R will ultimately increase $\theta(\xi)$. Enhancement in Prandtl number Pr value as presented in Figure 4.8 decrease temperature $\theta(\xi)$. Because of direct proportionality to velocity distribution and an inverse proportionality to chemical diffusitivity,rise in Pr causes drop in values of thermal diffusion rate. Temperature falls exponentially .

Effects of Q on $\theta(\xi)$ are elaborated in Figure 4.9. $\theta(\xi)$ increase on increasing Q. Figures 4.10 and 4.11 discuss Brownian motion causing fluctuations in values of $\theta(\xi)$ and $\phi(\xi)$.Faster the motion the lower the values of $\theta(\xi)$ and $\phi(\xi)$.

Figures 4.12 and 4.13 demonstrates changing behaviours of concentration distribution $\phi(\xi)$ and temperature distribution $\theta(\xi)$ on changing thermosphoresis paramter Nt. Both shows an elevation in values of $\theta(\xi)$ and $\phi(\xi)$ with increasing value of Nt. Impact of γ on temperature distribution is depicted in Figure 4.14. increase γ cause a fall in temperature profile $\theta(\xi)$. Figure 4.15 emphasize on the direct proportionality between Soret number Sr and temperature profile $\theta(\xi)$. Increase in one entity increase the other. Figure 4.16 explains the attainment of lower values of moleculer diffusivity and thermal boundary layer. Because of an inverse relationship between cocentration profile $\phi(\xi)$ and lewis number Le.

Table 4.1: Results of $-(Re_x)^{\frac{-1}{2}}Nu_x$ and $-(Re_x)^{\frac{-1}{2}}Sh_x$ some fixed parameters $n = 2.0,\ M = 0.5,\ S = 0.5$

R	Pr	Nt	Nb	Q	Le	S_r	γ	$-(Re_x)^{\frac{-1}{2}}Nu_x$	$-(Re_x)^{\frac{-1}{2}}Sh_x$
0.1	2.0	0.2	0.2	0.1	2.0	0.5	0.1	1.464760	0.211070
0.0								1.426372	0.115312
0.5								1.574809	0.456887
1.0								1.651019	0.621781
	1.0							0.830902	0.444123
	1.5							1.168853	0.319153
	1.75							1.321128	0.263688
		0.0						1.697908	0.741304
		0.1						1.576336	0.453268
		0.15						1.519356	0.326832
			0.5					1.160535	0.677052
			1.0					0.727135	0.896124
			1.5					0.408569	0.998184
				0.0				1.577988	0.137817
				0.2				1.340600	0.291187
				0.5				0.852078	0.604155
					1.0			1.563554	-0.214846
					3.0			1.400355	0.611758
					5.0			1.326035	1.365767
						0.1		1.430799	0.459807
						0.3		1.447675	0.337001
						0.7		1.482057	0.081953
							0.0	1.322942	0.264219
							0.2	1.629795	0.141068
							0.3	1.823468	0.049235

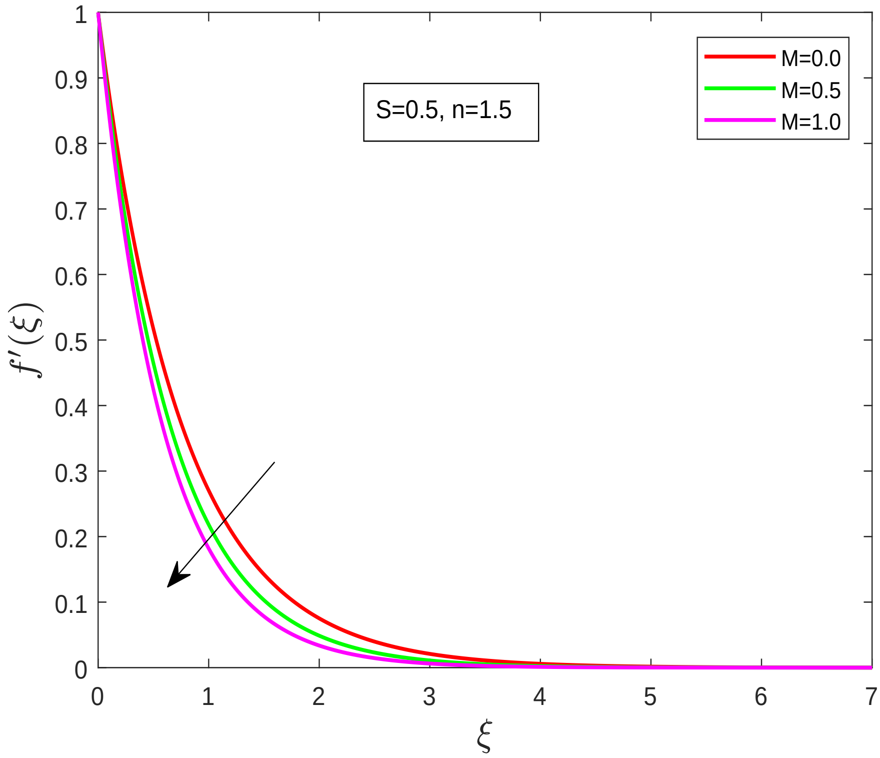

FIGURE 4.1: Change in $f'(\xi)$ for M.

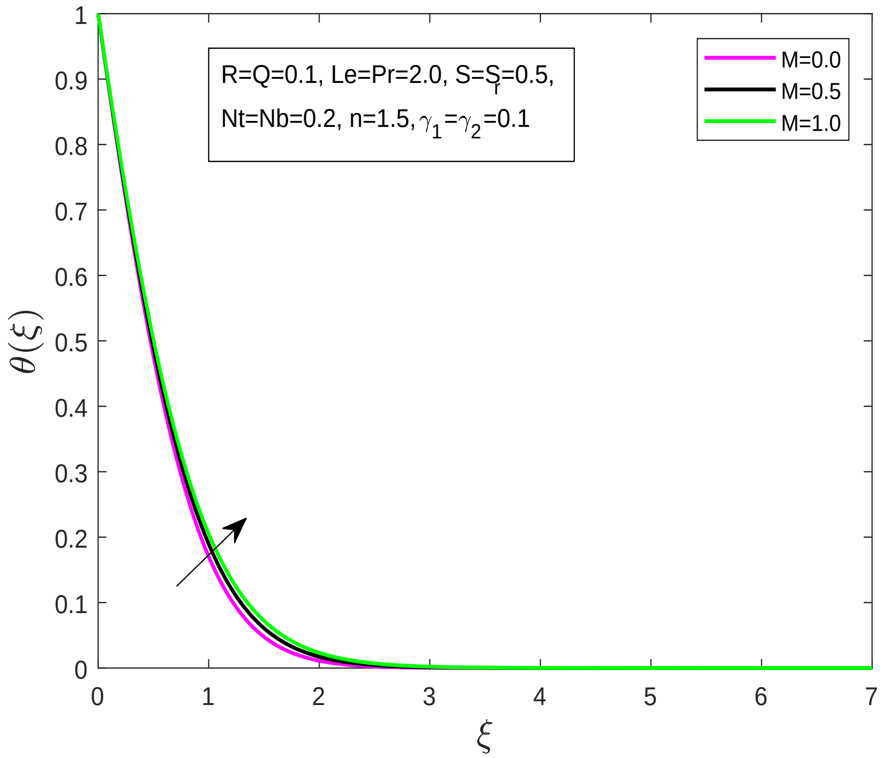

FIGURE 4.2: Change in $\theta(\xi)$ for M.

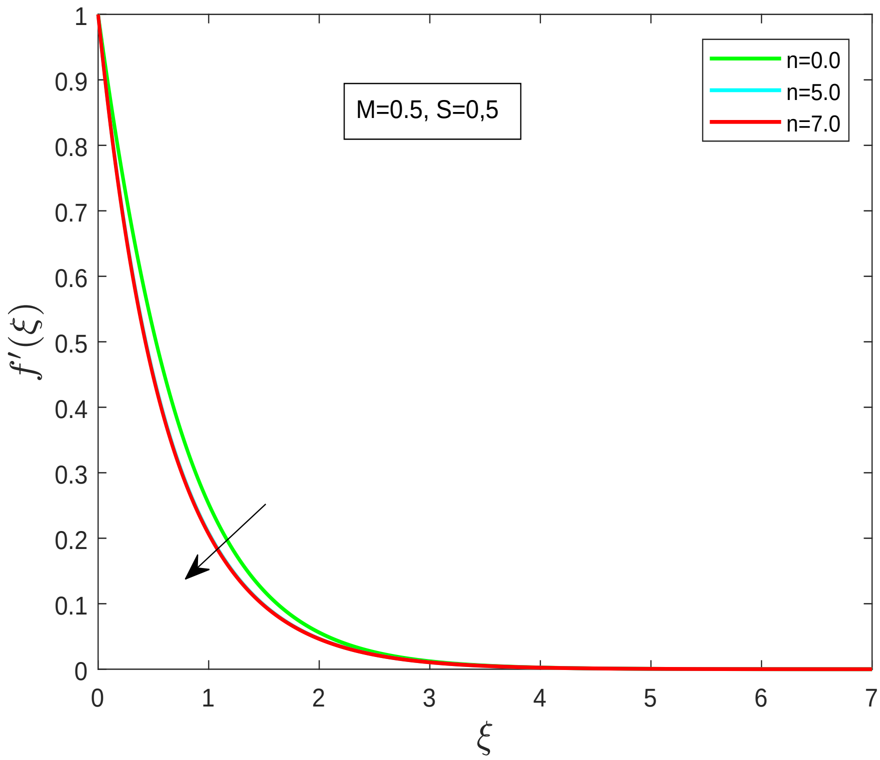

FIGURE 4.3: Change in $f'(\xi)$ for n.

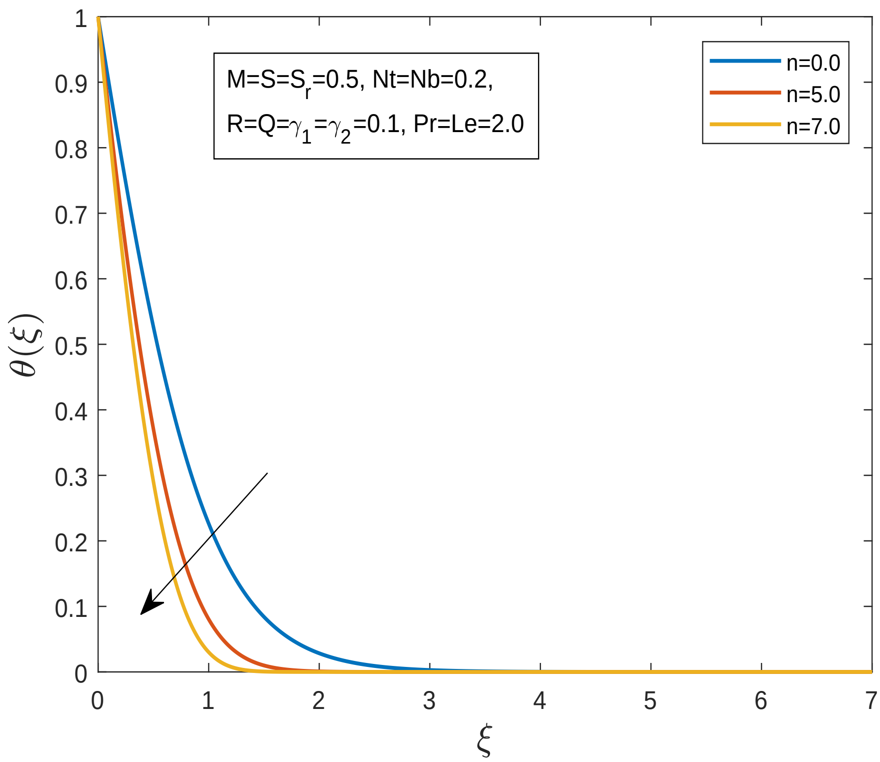

FIGURE 4.4: Change in $\theta(\xi)$ for n.

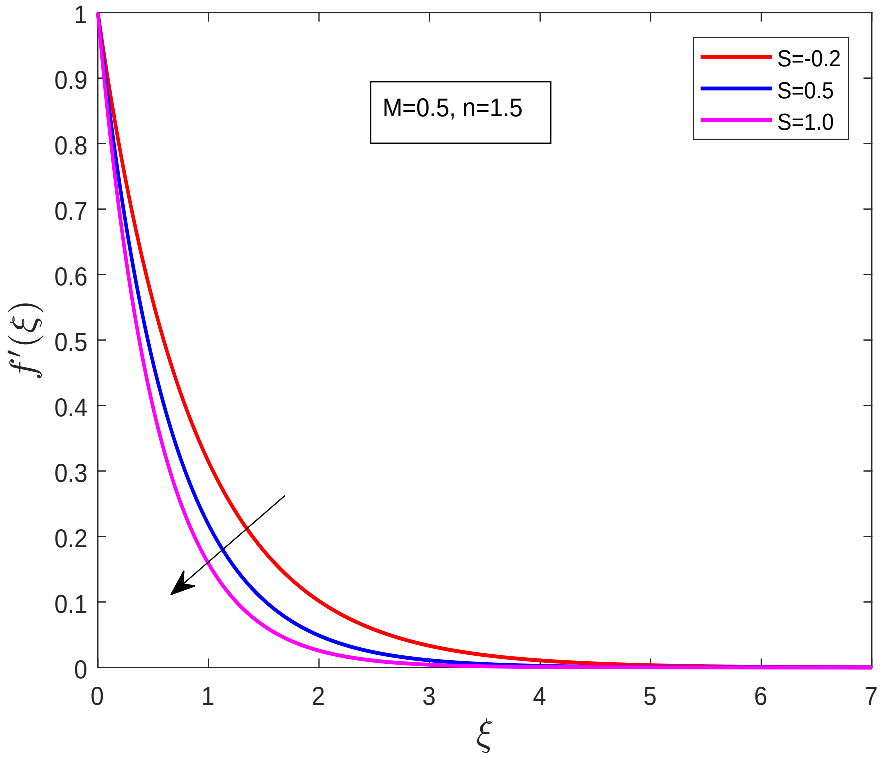

FIGURE 4.5: Change in $f'(\xi)$ for S.

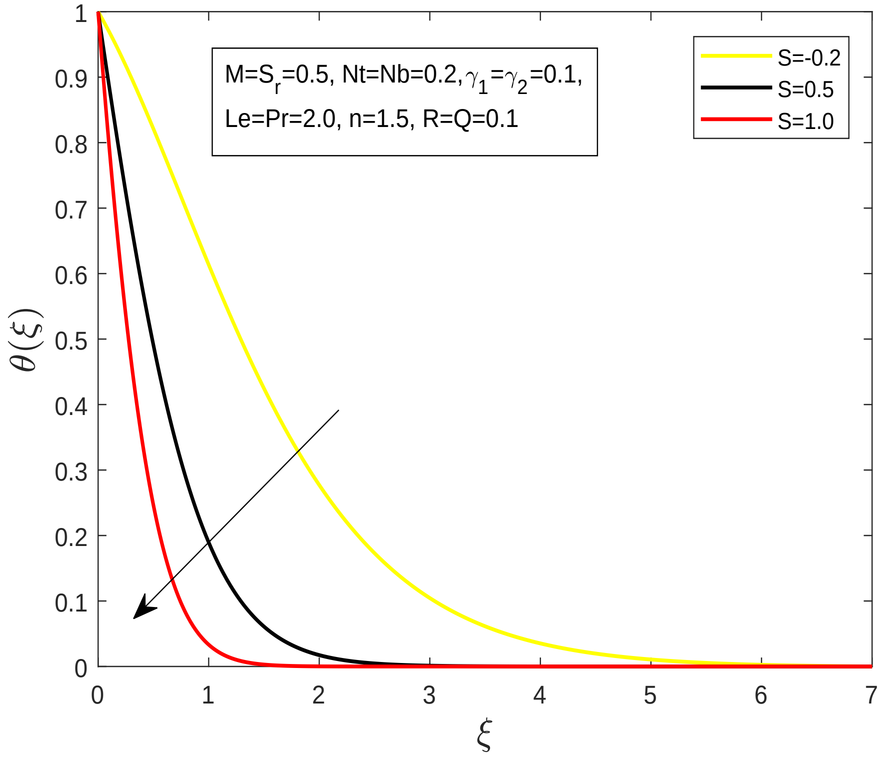

FIGURE 4.6: Change in $\theta(\xi)$ for S.

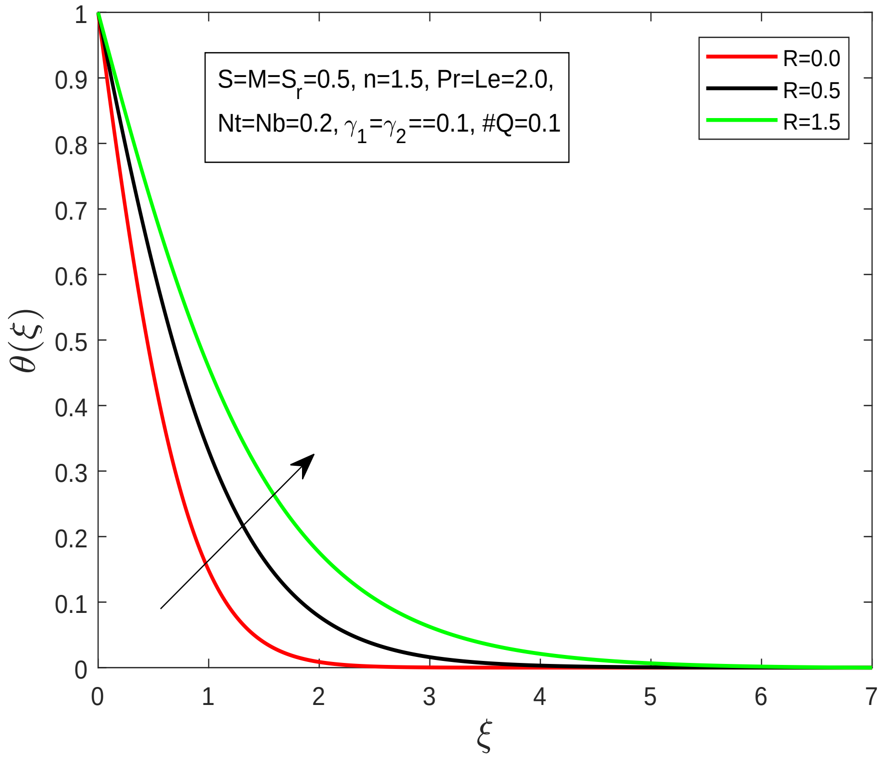

FIGURE 4.7: Change in $\theta(\xi)$ for R.

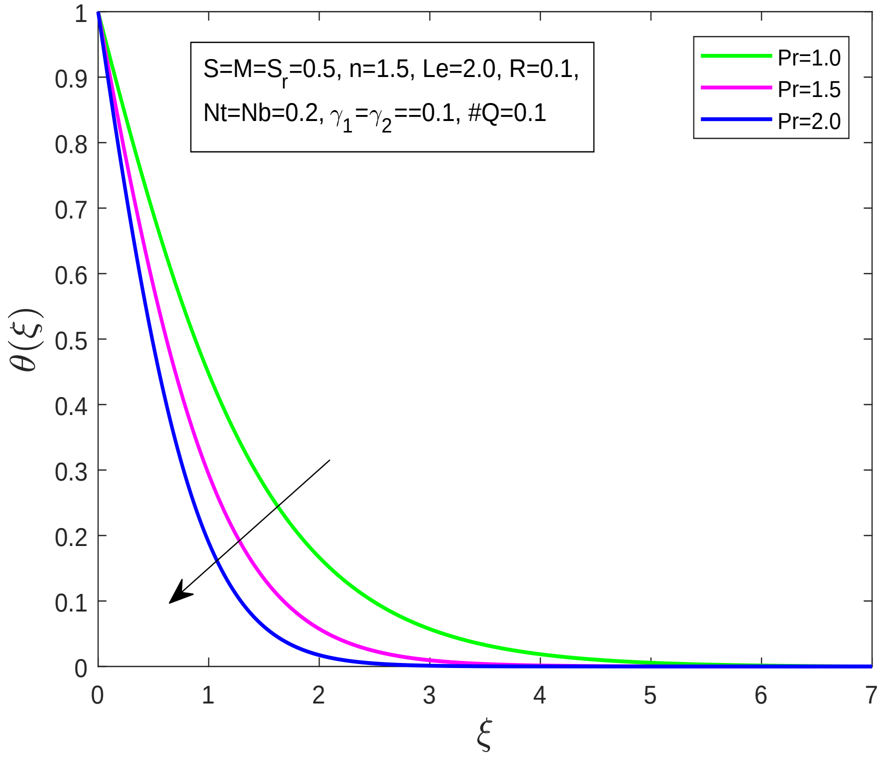

FIGURE 4.8: Change in $\theta(\xi)$ for Pr.

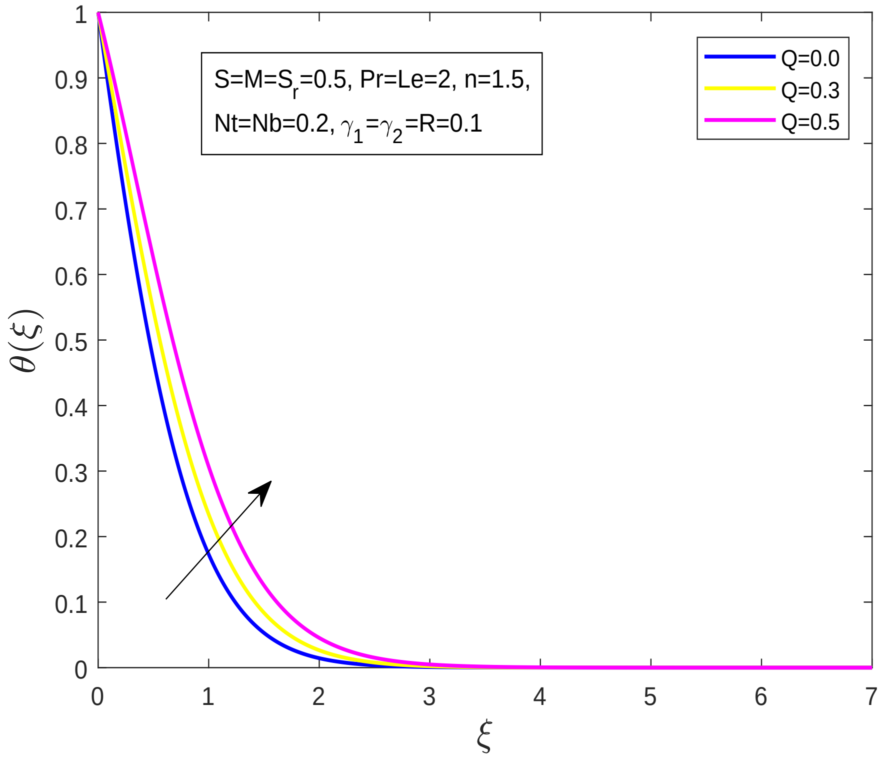

FIGURE 4.9: Change in $\theta(\xi)$ for Q.

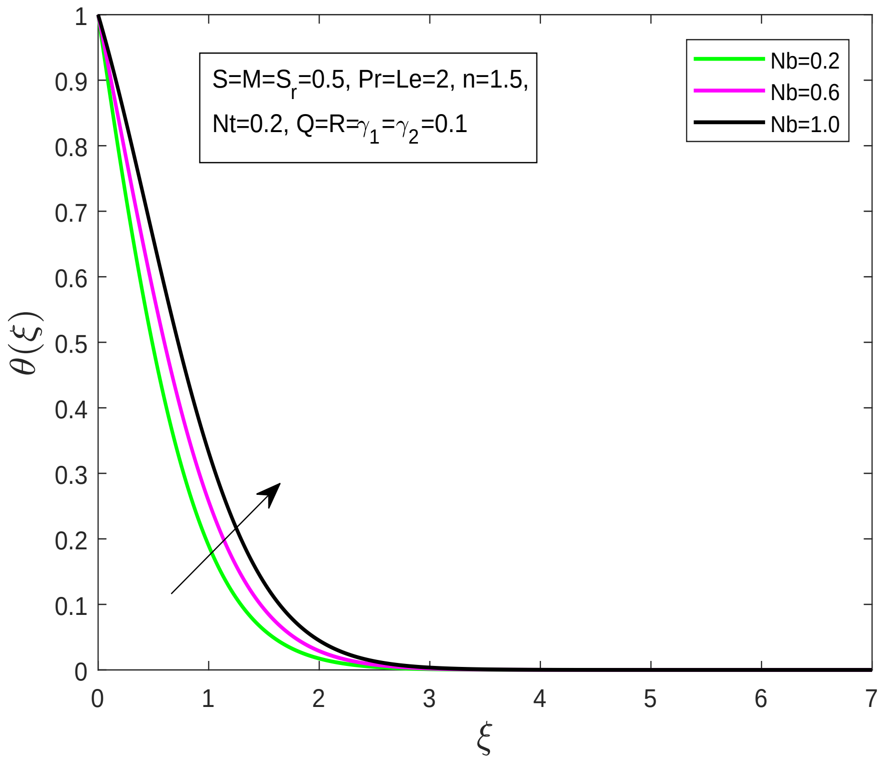

FIGURE 4.10: Change in $\theta(\xi)$ for Nb.

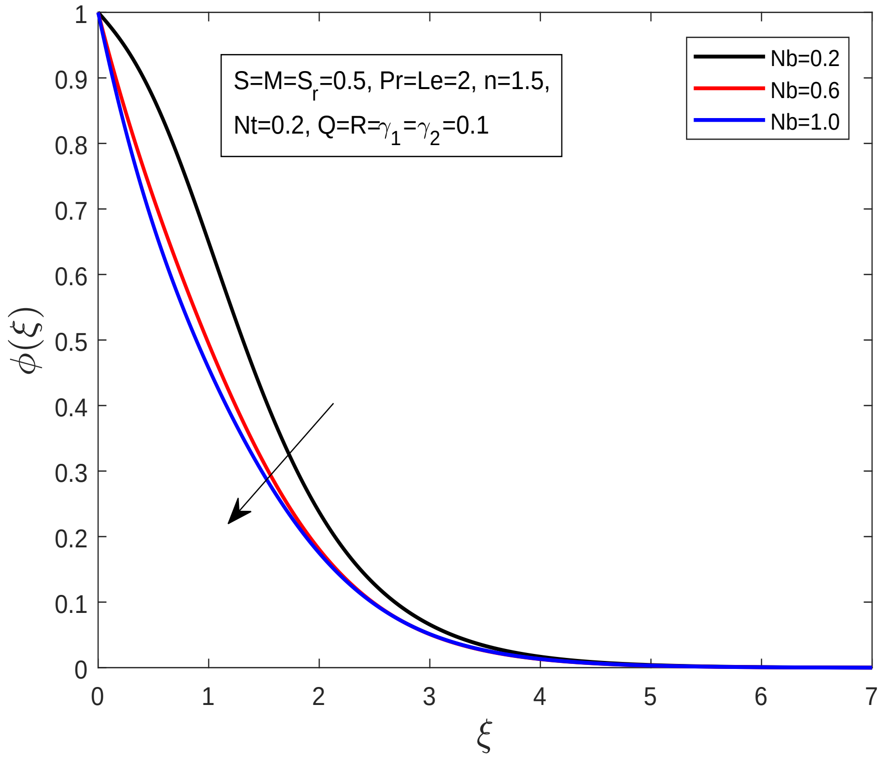

FIGURE 4.11: Change in $\phi(\xi)$ for Nb.

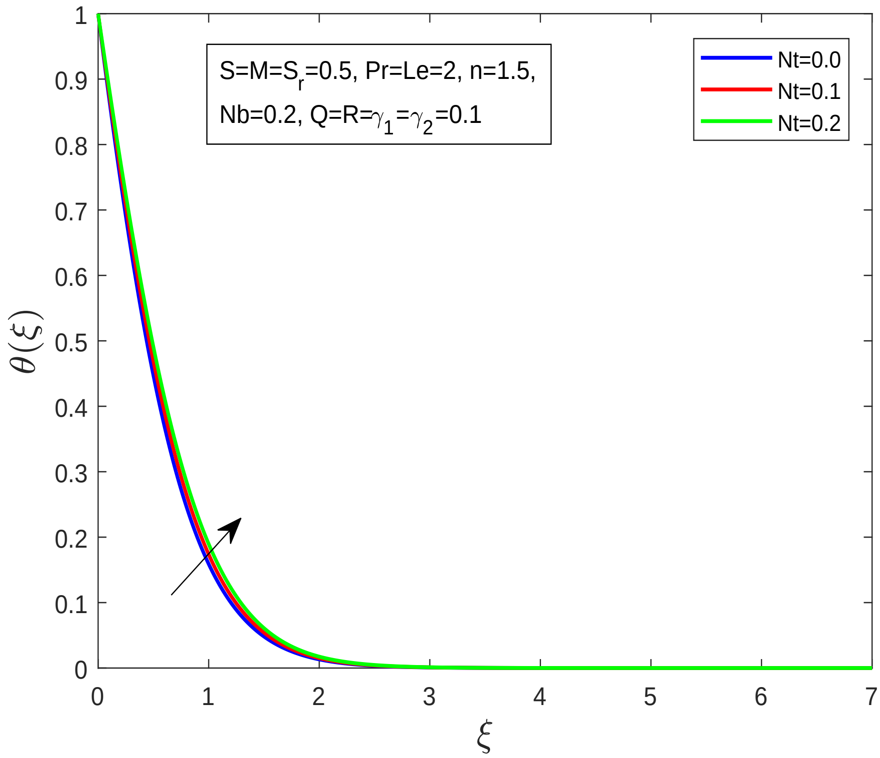

FIGURE 4.12: Change in $\theta(\xi)$ for Nt.

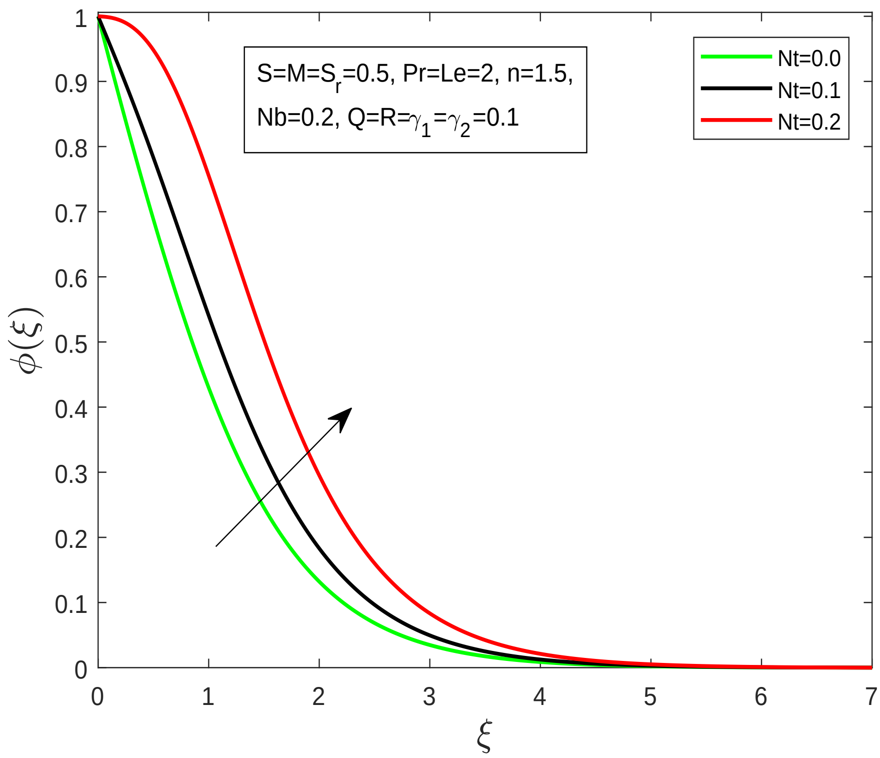

FIGURE 4.13: Change in $\phi(\xi)$ for Nt.

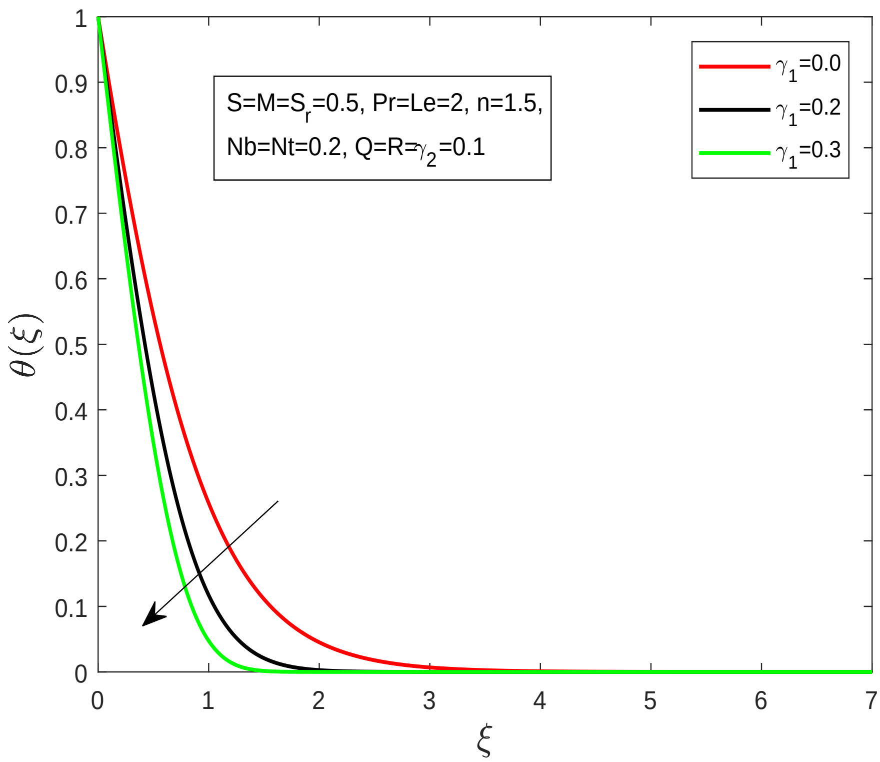

FIGURE 4.14: Change in $\theta(\xi)$ for γ_1.

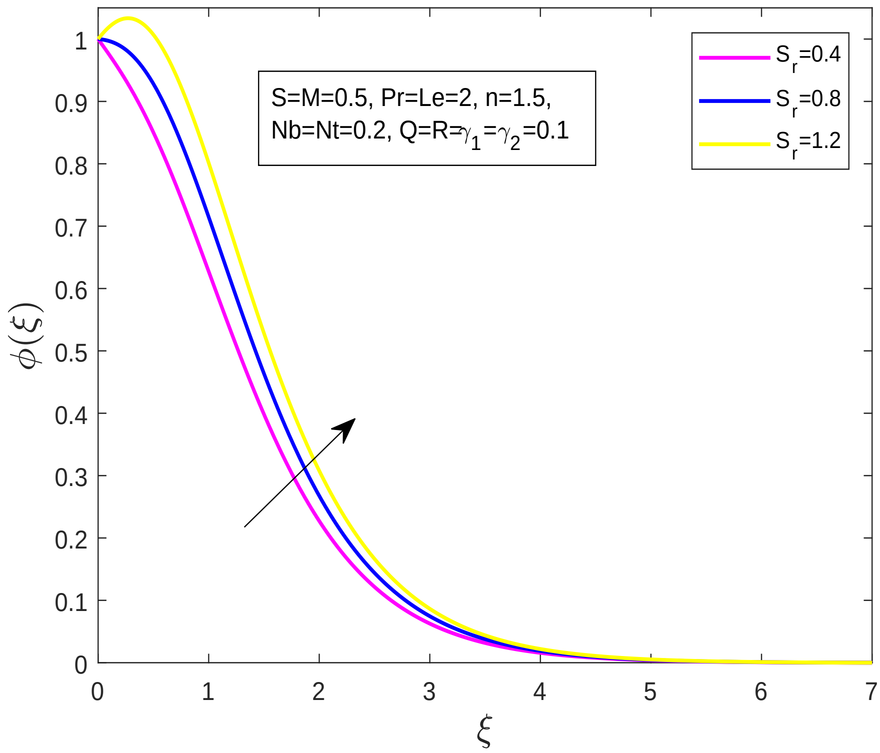

FIGURE 4.15: Change in $\phi(\xi)$ for S_r.

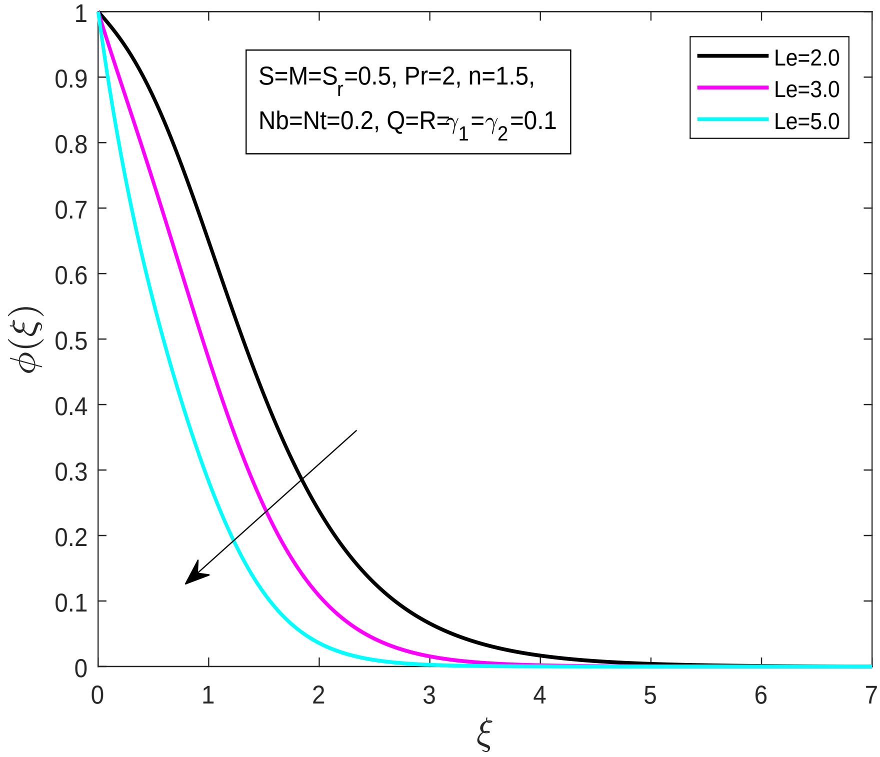

FIGURE 4.16: Change in $\phi(\xi)$ for Le.

Chapter 5

Conclusion

In this thesis, the work of [50] is reviewed and extended by considering Cattaneo-Christov Double Diffusion in the temperature equation and in the concentration equation . First of all, using similarity transformation,the momentum, energy eand concentration equations are transformed into the ODEs. By using the shooting technique, numerical solution has been found for the transformed ODEs. Using different values of the governing parameters, the results are presnted in the form of tables and graphs for velocity, temperature and concentration profiles. The key findings of the current research can be summarized as below:

- Increasing the values of M , the velocity profile decreases while the temperature profile increases.
- $\theta(\xi)$ is increased by rising the values of R and Q.
- The velocity profile and temperature distribution are decreased due to the increasing values of the wall transpiration paramater S.
- The temperature profile decreases while rising the values of Prandtle number.
- An increment is observed in Nu_x by increasing the values of Pr.
- By rising the values of Soret number Sr, an increment is noticed in the concentration profile .

- By increasing the values of Nt, the concentration profile is increased.
- Temperature profile rises as Nb rises.
- Sh_x numerical values increase as by increasing the value of Le.
- As a result of the ascending values of the parameter γ_1, the values of Nu_x are increased while Sh_x is decreased.

Bibliography

[1] S. U. Choi and J. A. Eastman, "Enhancing thermal conductivity of fluids with nanoparticles," tech. rep., Argonne National Lab.(ANL), Argonne, IL (United States), 1995.

[2] X.-Q. Wang and A. S. Mujumdar, "A review on nanofluids-part i: theoretical and numerical investigations," *Brazilian Journal of Chemical Engineering*, vol. 25, pp. 613–630, 2008.

[3] C. Yang, W. Li, Y. Sano, M. Mochizuki, and A. Nakayama, "On the anomalous convective heat transfer enhancement in nanofluids: a theoretical answer to the nanofluids controversy," *Journal of Heat Transfer*, vol. 135, no. 5, 2013.

[4] K. Jahani, M. Mohammadi, M. B. Shafii, and Z. Shiee, "Promising technology for electronic cooling: nanofluidic micro pulsating heat pipes," *Journal of Electronic Packaging*, vol. 135, no. 2, p. 021005, 2013.

[5] W. Yu, D. M. France, S. U. Choi, and J. L. Routbort, "Review and assessment of nanofluid technology for transportation and other applications," tech. rep., Argonne National Lab.(ANL), Argonne, IL (United States), 2007.

[6] J. Buongiorno, "Convective transport in nanofluids," 2006.

[7] R. K. Tiwari and M. K. Das, "Heat transfer augmentation in a two-sided lid-driven differentially heated square cavity utilizing nanofluids," *International Journal of Heat and Mass Transfer*, vol. 50, no. 9-10, pp. 2002–2018, 2007.

[8] W. Khan and I. Pop, "Boundary-layer flow of a nanofluid past a stretching sheet," *International journal of Heat and Mass Transfer*, vol. 53, no. 11-12, pp. 2477–2483, 2010.

[9] A. Noghrehabadi, M. R. Saffarian, R. Pourrajab, and M. Ghalambaz, "Entropy analysis for nanofluid flow over a stretching sheet in the presence of heat generation/absorption and partial slip," *Journal of Mechanical Science and Technology*, vol. 27, no. 3, pp. 927–937, 2013.

[10] F. M. Hady, F. S. Ibrahim, S. M. Abdel-Gaied, and M. R. Eid, "Radiation effect on viscous flow of a nanofluid and heat transfer over a nonlinearly stretching sheet," *Nanoscale Research Letters*, vol. 7, no. 1, pp. 1–13, 2012.

[11] C. Wang, "Free convection on a vertical stretching surface," *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 69, no. 11, pp. 418–420, 1989.

[12] S. Nadeem, R. U. Haq, and Z. Khan, "Numerical solution of non-Newtonian nanofluid flow over a stretching sheet," *Applied Nanoscience*, vol. 4, no. 5, pp. 625–631, 2014.

[13] B. Prasannakumara, B. Gireesha, R. S. Gorla, and M. Krishnamurthy, "Effects of chemical reaction and nonlinear thermal radiation on Williamson nanofluid slip flow over a stretching sheet embedded in a porous medium," *Journal of Aerospace Engineering*, vol. 29, no. 5, p. 04016019, 2016.

[14] M. Krishnamurthy, B. Prasannakumara, B. Gireesha, and R. S. R. Gorla, "Effect of chemical reaction on mhd boundary layer flow and melting heat transfer of Williamson nanofluid in porous medium," *Engineering Science and Technology, an International Journal*, vol. 19, no. 1, pp. 53–61, 2016.

[15] M. Kothandapani and J. Prakash, "Effects of thermal radiation parameter and magnetic field on the peristaltic motion of Williamson nanofluids in a tapered asymmetric channel," *International Journal of Heat and Mass Transfer*, vol. 81, pp. 234–245, 2015.

[16] M. Sheikholeslami, D. D. Ganji, M. Y. Javed, and R. Ellahi, "Effect of thermal radiation on magnetohydrodynamics nanofluid flow and heat transfer by means of two phase model," *Journal of Magnetism and Magnetic Materials*, vol. 374, pp. 36–43, 2015.

[17] C. Kleinstreuer and Y. Feng, "Experimental and theoretical studies of nanofluid thermal conductivity enhancement: a review," *Nanoscale Research Letters*, vol. 6, no. 1, pp. 1–13, 2011.

[18] W. Yu, D. M. France, J. L. Routbort, and S. U. Choi, "Review and comparison of nanofluid thermal conductivity and heat transfer enhancements," *Heat Transfer Engineering*, vol. 29, no. 5, pp. 432–460, 2008.

[19] S.-C. Tzeng, C.-W. Lin, and K. Huang, "Heat transfer enhancement of nanofluids in rotary blade coupling of four-wheel-drive vehicles," *Acta Mechanica*, vol. 179, no. 1, pp. 11–23, 2005.

[20] F. Mabood, W. Khan, and A. M. Ismail, "Mhd boundary layer flow and heat transfer of nanofluids over a nonlinear stretching sheet: a numerical study," *Journal of Magnetism and Magnetic Materials*, vol. 374, pp. 569–576, 2015.

[21] L. Zhang, M. Bhatti, A. Shahid, R. Ellahi, O. A. Beg, and S. M. Sait, "Nonlinear nanofluid fluid flow under the consequences of Lorentz forces and arrhenius kinetics through a permeable surface: A robust spectral approach," *Journal of the Taiwan Institute of Chemical Engineers*, vol. 124, pp. 98–105, 2021.

[22] M. Hamad, "Analytical solution of natural convection flow of a nanofluid over a linearly stretching sheet in the presence of magnetic field," *International Communications in Heat and Mass Transfer*, vol. 38, no. 4, pp. 487–492, 2011.

[23] S. Nadeem and R. Ul Haq, "Mhd boundary layer flow of a nanofluid passed through a porous shrinking sheet with thermal radiation," *Journal of Aerospace Engineering*, vol. 28, no. 2, p. 04014061, 2015.

[24] W. Ibrahim, B. Shankar, and M. M. Nandeppanavar, "Mhd stagnation point flow and heat transfer due to nanofluid towards a stretching sheet," *International Journal of Heat and Mass Transfer*, vol. 56, no. 1-2, pp. 1–9, 2013.

[25] D. Srinivasacharya, U. Mendu, and K. Venumadhav, "Mhd boundary layer flow of a nanofluid past a wedge," *Procedia Engineering*, vol. 127, pp. 1064–1070, 2015.

[26] M. Bhatti, M. Arain, A. Zeeshan, R. Ellahi, and M. Doranehgard, "Swimming of gyrotactic microorganism in mhd Williamson nanofluid flow between rotating circular plates embedded in porous medium: Application of thermal energy storage," *Journal of Energy Storage*, vol. 45, p. 103511, 2022.

[27] Y. Xuan and Q. Li, "Investigation on convective heat transfer and flow features of nanofluids," *J. Heat Transfer*, vol. 125, no. 1, pp. 151–155, 2003.

[28] V. Poply, P. Singh, and A. Yadav, "Stability analysis of mhd outer velocity flow on a stretching cylinder," *Alexandria Engineering Journal*, vol. 57, no. 3, pp. 2077–2083, 2018.

[29] T. Thumma, O. A. Bég, and A. Kadir, "Numerical study of heat source/sink effects on dissipative magnetic nanofluid flow from a non-linear inclined stretching/shrinking sheet," *Journal of Molecular Liquids*, vol. 232, pp. 159–173, 2017.

[30] S. Shateyi and J. Prakash, "A new numerical approach for mhd laminar boundary layer flow and heat transfer of nanofluids over a moving surface in the presence of thermal radiation," *Boundary Value Problems*, vol. 2014, no. 1, pp. 1–12, 2014.

[31] E. H. Aly, "Dual exact solutions of graphene–water nanofluid flow over stretching/shrinking sheet with suction/injection and heat source/sink: Critical values and regions with stability," *Powder Technology*, vol. 342, pp. 528–544, 2019.

[32] A. J. Chamkha and A. Aly, "Mhd free convection flow of a nanofluid past a vertical plate in the presence of heat generation or absorption effects," *Chemical Engineering Communications*, vol. 198, no. 3, pp. 425–441, 2010.

[33] M. J. Uddin, W. A. Khan, and A. I. Ismail, "Mhd free convective boundary layer flow of a nanofluid past a flat vertical plate with Newtonian heating boundary condition," *Plos One*, vol. 7, no. 11, p. e49499, 2012.

[34] M. Malik, M. Khan, T. Salahuddin, and I. Khan, "Variable viscosity and mhd flow in Casson fluid with Cattaneo–Christov heat flux model: using Keller box method," *Engineering Science and Technology, an International Journal*, vol. 19, no. 4, pp. 1985–1992, 2016.

[35] B. Ganga, S. M. Y. Ansari, N. V. Ganesh, and A. A. Hakeem, "Mhd radiative boundary layer flow of nanofluid past a vertical plate with internal heat generation/absorption, viscous and Ohmic dissipation effects," *Journal of the Nigerian Mathematical Society*, vol. 34, no. 2, pp. 181–194, 2015.

[36] M. S. Khan, I. Karim, M. S. Islam, and M. Wahiduzzaman, "Mhd boundary layer radiative, heat generating and chemical reacting flow past a wedge moving in a nanofluid," *Nano Convergence*, vol. 1, no. 1, pp. 1–13, 2014.

[37] S. E. Ahmed, A. K. Hussein, H. A. Mohammed, and S. Sivasankaran, "Boundary layer flow and heat transfer due to permeable stretching tube in the presence of heat source/sink utilizing nanofluids," *Applied Mathematics and Computation*, vol. 238, pp. 149–162, 2014.

[38] J. B. J. Fourier, G. Darboux, *et al.*, *Théorie analytique de la chaleur*, vol. 504. Didot Paris, 1822.

[39] A. Fick, "Poggendorffs flannel," *Physik*, vol. 94, no. 59, p. 297, 1855.

[40] C. Cattaneo, "Sulla conduzione del calore," *Atti Sem. Mat. Fis. Univ. Modena*, vol. 3, pp. 83–101, 1948.

[41] C. Christov, "On frame indifferent formulation of the maxwell–cattaneo model of finite-speed heat conduction," *Mechanics Research Communications*, vol. 36, no. 4, pp. 481–486, 2009.

[42] R. W. Fox, A. McDonald, and P. Pitchard, *Introduction to Fluid Mechanics*. 2006.

[43] R. Bansal, *A Textbook of Fluid Mechanics and Dydraulic Machines*. Laxmi publications, 2004.

[44] J. N. Reddy and D. K. Gartling, *The Finite Element Method in Heat Transfer and Fluid Dynamics*. CRC press, 2010.

[45] J. Ahmed and M. S. Rahman, *Handbook of Food Process Design*. John Wiley & Sons, 2012.

[46] P. A. Davidson and A. Thess, *Magnetohydrodynamics*, vol. 418. Springer Science & Business Media, 2002.

[47] M. Gad-el Hak, *Frontiers in Experimental Fluid Mechanics*, vol. 46. Springer Science & Business Media, 2013.

[48] R. W. Lewis, P. Nithiarasu, and K. N. Seetharamu, *Fundamentals of the Finite Element Method for Heat and Fluid Flow*. John Wiley & Sons, 2004.

[49] J. Kunes, *Dimensionless Physical Quantities in Science and Engineering*. Elsevier, 2012.

[50] B. V. Krishnarao, J. Konda, and G. Charankumar, "Mhd and thermal radiation effects of a nanofluid over a stretching sheet using HAM," *International Journal of Recent Technology and Engineering*, vol. 8, pp. 3489–3496.