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Fixed Point Results for Multivalued $\alpha\mathcal{F}$-Contractions in Partial $\mathfrak{b}$-Metric Spaces

by

Isma Urooj

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Dedicated to my family

CERTIFICATE OF APPROVAL

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In the name of ALLAH, the most beneficent and the most merciful

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Abstract

Altun et al. studied the existence of fixed points for multivalued F-contractions and established some fixed point theorems in complete metric spaces. Kumar et al. expanded these results over the domain of partial metric spaces and demonstrated fixed point theorems for multivalued F-contraction mappings. This research work is an extension of the work of Kumar et al. We established some fixed point results using a combination of alpha admissible mappings under multivalued F contractions in the setting of partial b-metric spaces. Eventually, an application of the main result is elaborated by proving the existence of the solution for an integral equation.

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Abbreviations

𝔣pth	Fixed Point Theorem
BCP	Banach Contraction Principle
MS	Metric Space
PMS	Partial Metric Space
𝔟MS	*b*-metric space
P𝔟MS	Partial *b*-metric space
MVC	Multivalued Contraction
MVCM	Multivalued Contraction Mapping
MVFC	Multivalued F Contraction

Symbols

d	Metric function
$d_{\mathfrak{b}}$	b-Metric function
d_p	Partial Metric function
$p_{\mathfrak{b}}$	Partial b-Metric function
$\mathcal{H}$	Hausdorff metric function
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of non-negative numbers
$C[a, b]$	Collection of continuous functions
$\forall$	For all
$\exists$	There exists
$\in$	Belongs to
$\notin$	Does not belong to
$\times$	Cartesian Product
$\to$	Approaches to
$\iff$	If and only if
$\implies$	Implies
$\sum$	Summation sign
lim	Limit
sup	Supremum
inf	Infimum
max	Maximum

Chapter 1

Introduction

1.1 Historical Background

Functional analysis plays an influential role in the applied sciences as well as in mathematics itself. Functional analysis is an abstract mathematical technique derived from classical analysis. It offers some essential strategies for dealing with difficulties in various mathematical analysis areas. In this study, we will work explicitly on one of the functional analysis units, namely the metric fixed point theory. H. Poincaré [1], a French mathematician, was the first person who examined the domain of fixed point theory in 1886 and explored different results regarding fixed point theorems.

In a broader perspective, by fixed point theorem ($\mathfrak{f}$pth), we mean a formulation that declares that under specific conditions a mapping $\mathfrak{f} : \mathcal{M} \to \mathcal{N}$ acknowledges one or more points m of $\mathcal{M}$ such that $\mathfrak{f}m = m$. For many years, numerous measures have been generalized to the fixed point theorem for different categories of topological spaces and Banach spaces. The solution to various mathematical problems can be traced by adopting the suitable $\mathfrak{f}$pth on the underlying function space with the application of relevant characteristics of a defined mapping. Many researchers have concentrated on this theory due to its practicality in finding unique solution to differential and integral equations and its applications in boundary value problems, approximation theory, and non-linear analysis.

The classical Brouwer fpth [2] confirmed the presence of the fixed point in closed subsets of Euclidean space under continuous mapping and was presented with proof in 1912. Later on, investigations by some authors demonstrated that this result wasn't valid in infinite-dimensional spaces. So researchers began looking at other conditions for the defined mapping and desirable conditions on the ground space. A comprehensive study in the development of fixed point theorems is given in [3]. Stefan Banach [4] is greatly appreciated for setting the concepts of finding the fixed points into an ideal framework, which gave a new direction to the birth of modern fixed point theory. In 1922, he introduced the famous Banach Contraction Principle (BCP), in which contraction mapping was used instead of continuity. Later on, it was considered a powerful tool for finding unique fixed points.

BCP says, in a complete metric space $(\mathcal{M}, d^*)$, a self map $f : \mathcal{M} \to \mathcal{M}$ satisfying the contraction condition on $\mathcal{M}$, i.e.,

$$d^*(f\zeta, f\beta) \leq cd^*(\zeta, \beta),$$

for all $\zeta, \beta \in \mathcal{M}$ provided $c \in [0, 1)$, has a unique fixed point.

Later on, BCP was generalized under different flavors of mappings; the first generalization was given by Edelstein [5] in 1962 by modifying the contraction condition. Another general result dealing with BCP was given by Diaz et al. [6] in 1965. Additional work on BCP was investigated by Kasahara [7] in premetric spaces in 1968, in the same year, Kannan [8] highlighted some improvements regarding the continuity of contraction condition of BCP.

Nadler [9], generalized BCP for set-valued mappings in 1969. Using appropriate fpth for multivalued transformations is extensively more advantageous in optimizations of different problems by reducing the error and allowing researchers to work on the platform of approximation theory. The BCP was extended by Nussbaum [10], under k-set contractions also. One can see [11]- [17] for reading more about generalized BCP in different domains.

On the other hand, researchers began playing with the defined properties of metric spaces presented by M. Fréchet [18] in 1906. In the traditional definition, one may notice the obvious fact that the distance of a point m from itself, i.e., $d(m, m) = 0$.

But in 1992 Matthews [19] introduced the idea of non-zero self distance and presented partial metric space (PMS). The primary pillars of metric space, like open sets, closed sets, convergence criteria and completeness, were also adjusted accordingly, but he demonstrated that BCP could be generalized in complete partial metric space.

In 1999, a relaxed image of the partial metric was presented by Valero et al.[20] in the setting of weak partial metric space. Another direction for some more generalized contractions in partial metric spaces was offered by Altun et al. [21] in 2010, and further revised by Mishra et al. [22] for tracking down unique fixed points under contractive mapping in 2015. Another outstanding work was done by Karapınar et al. [23]. They worked to find the common fixed point in partial metric spaces. For more recent developments in this regard, one can read [24]-[28].

A recommendation of $\mathfrak{b}$ metric space ($\mathfrak{b}$MS) was outlined in 1989 by Bakhtin [29] and modified by Czerwik [30] in 1993, opening new doors of research for others. A new coefficient in triangular inequality of metric space was introduced, which provided a base to $\mathfrak{b}$MS. BCP was beautifully generalized in this direction too. Many researchers started exploring this domain.

An interesting and more generalized result for finding fixed points in this space was given by Akkouchi [31]. The notion of set-valued mapping in the setting of $\mathfrak{b}$MS was introduced by [32] - [34] under different contraction conditions.

A unique idea was given by Shukla [35] by blending both spaces mentioned above together in 2014. He presented the domain of partial $\mathfrak{b}$ metric spaces (P$\mathfrak{b}$MS). In the next year [36] extended his work and used more relations in $\mathfrak{b}$Ms and P$\mathfrak{b}$MS. For working in this new domain, under the setting of multivalued mappings, Aydi et al. [37] gave amazing work in 2019. Exploring P$\mathfrak{b}$MS in more wider sense one can observe the work done by [38] - [40].

An article using multivalued F mapping in partial metric space was presented in April 2021 by Kumar et al. [41]. A good generalization of BCP under this new condition was presented. Encouraged by his work, we offered an idea of extending BCP in the globe of P$\mathfrak{b}$MS by combining the notion of alpha admissible mapping introduced by Samet [42] with the flavor of Multivalued F contractions.

1.2 Thesis Contribution

Our thesis layout is briefly depicted below;

In **Chapter 2**, a short recap of some fundamental ideas used in metric spaces is unveiled. This chapter has three main partitions. Firstly, some essential definitions and examples are referenced, providing a basis for upcoming results. In the 2nd section, a few significant mappings are portrayed with the help of graphical representation. In the 3rd section, a concise history of the development of fixed point theorems is illustrated. Lastly, a quick review of extensions in some important spaces is presented.

Chapter 3 features a historical background of partial metric space, along with remarkable theorems. All required ingredients related to partial metric space are outlined with the help of examples to distinguish it from metric space. The notion of multivalued mappings is clarified with the help of illustrations and diagrams. A detailed review of the work of Kumar [41] is articulated with an application on a Fredholm integral equation.

Chapter 4 gives an overview of all key concepts used in P$\mathfrak{b}$MS. Also, the interconnected ideas concerning Hausdorff distance are presented under the structure of this space. The idea of alpha-admissible mapping is also illustrated with the support of examples and theorems. We proved the result of multivalued alpha F mappings in the setting of P$\mathfrak{b}$MS and worked on locating fixed points in this new direction. An application is also provided to support this idea, which will help in predicting the solution of a Fredholm integral equation.

Chapter 5 recapitulates all expansions of our work and unlocks further recommendations for others.

Chapter 2

Preliminaries

This chapter encloses some elementary definitions coupled with examples and some important results, which are intended to be used in upcoming chapters. The first section of this chapter is framed with some key definitions from metric spaces. The next section is concerned with some preferred mappings, having a vital role in proving subsequent results, and in the next section, a historical review of some theorems is also articulated. Lastly, a quick review of some extensions in metric space is presented.

2.1 Metric space

In 1906, M. Fréchet highlighted the notion of metric spaces, which was a generalization of natural distance. Later on, these spaces acted as a bridge between topological spaces and real analysis and provided an establishment for metric fixed point theory. This construction helped in solving various problems concerning web search tools, graphics arrangements, the convergence of series, approximation problems, etc.

Definition 2.1.1. Metric Space.
"A metric space is a pair $(\mathcal{M}, \check{d})$, where $\mathcal{M}$ is a set and $\check{d}$ is a metric on $\mathcal{M}$ (or distance function on $\mathcal{M}$), that is, a function on $\mathcal{M} \times \mathcal{M}$ such that for all $p, q, r \in \mathcal{M}$, we have:

(M1): $\check{d}$ is real-valued, finite and nonnegative.

(M2): $\check{d}(p,q) = 0$ if and only if $p = q$.

(M3): $\check{d}(p,q) = \check{d}(q,p)$ (Symmetry).

(M4): $\check{d}(p,r) \leq \check{d}(p,q) + \check{d}(q,r)$ (Triangle inequality).

The symbol $\times$ denotes the cartesian product of sets.

Hence $\mathcal{M} \times \mathcal{M}$ is the set of all ordered pairs of elements of $\mathcal{M}$." [18].

Some examples supporting this idea are given below.

Example 2.1.2.

(i) Let $\mathcal{M} = \mathbb{R}$, the set consisting of all real numbers, a metric function $\check{d} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ can be defined as

$$\check{d}(\varrho, \xi) = |\varrho - \xi|,$$

for all $\varrho, \xi \in \mathcal{M}$.

It is easy to verify that this defined metric function satisfies all axioms of metric space and it is known as the usual metric on $\mathcal{M}$.

(ii) Let $\mathcal{M} = \mathbb{R}^2$, a defined metric function $\check{d} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$,

$$\check{d}(l, m) = \sqrt{(\xi_1 - \xi_2)^2 + (\zeta_1 - \zeta_2)^2},$$

where $l = (\xi_1, \zeta_1)$ and $m = (\xi_2, \zeta_2)$, with $\xi_1, \xi_2, \zeta_1, \zeta_2 \in \mathbb{R}$, is a metric on $\mathcal{M} = \mathbb{R}^2$.

(iii) $\mathcal{M}$ is the collection of all real-valued continuous functions $(\zeta, \varrho, ...)$, depending on the real variable θ, on a given closed interval $I = [\alpha, \beta]$. We define a metric function as,

$$\check{d}(\zeta, \varrho) = \max_{\theta \in I} |\zeta(\theta) - \varrho(\theta)|,$$

where max denotes the maximum value of functions at θ in a given interval. Then $(\mathcal{M}, \check{d})$ is a metric space and we would call $\mathcal{C}[\alpha, \beta]$ the function space. One can easily verify that for this particular defined metric function **(M1)**,**(M2)** and **(M3)** are satisfied,

(M4). For $\zeta, \varrho, \upsilon \in \mathcal{M}$. Consider,

$$\begin{aligned}\check{d}(\zeta, \upsilon) &= \max_{\theta \in I} |\zeta(\theta) - \upsilon(\theta)| \\ &= \max_{\theta \in I} |\zeta(\theta) - \varrho(\theta) + \varrho(\theta) - \upsilon(\theta)| \\ &\leq \max_{\theta \in I} (|\zeta(\theta) - \varrho(\theta)| + |\varrho(\theta) - \upsilon(\theta)|) \\ &= \max_{t \in I} |\zeta(\theta) - \varrho(\theta)| + \max_{\theta \in I} |\varrho(\theta) - \upsilon(\theta)| \\ &= \check{d}(\zeta, \varrho) + \check{d}(\varrho, \upsilon),\end{aligned}$$

for all $\zeta, \varrho, \upsilon \in \mathcal{M}$.

Definition 2.1.3. Open and Closed Ball.

"Given a point $m_0 \in \mathcal{M}$ and a real number $a > 0$, an open ball in metric space $(\mathcal{M}, \check{d})$ is defined, as follow

$$B(m_0, a) = \{m \in \mathcal{M} \mid \check{d}(m_0, m) < a\},$$

and a closed ball is defined as

$$B(m_0, a) = \{m \in \mathcal{M} \mid \check{d}(m_0, m) \leq a\}." \ [18]$$

Definition 2.1.4. Open Set and Closed Set.

"A subset U of a metric space $(\mathcal{M}, \check{d})$ is said to be an open set if it contains a ball about each of its points. A subset V of metric space $(\mathcal{M}, \check{d})$ is said to be closed if its complement in $\mathcal{M}$ is open, that is

$$V^c = \mathcal{M} - V$$

is open." [18]

Definition 2.1.5. Diameter of a Set.

"The diameter of a non empty set B in a metric space $(\mathcal{M}, \check{d})$ is defined to be

$$\delta(B) = \sup_{a,b \in B} \check{d}(a, b),$$

B is said to be bounded set if $\delta(B) < \infty$." [18]

Definition 2.1.6. Convergent Sequence.

"A sequence (a_k) in a metric space $\mathcal{M} = (\mathcal{M}, \check{d})$ is said to converge or to be convergent if there is an $a_0 \in \mathcal{M}$, such that $\lim_{k\to\infty} \check{d}(a_k, a_0) = 0$, a_0 is called the limit of (a_k) and we write $\lim_{k\to\infty} a_k = a_0$ or simply $a_k \to a_0$." [18]

Definition 2.1.7. Cauchy Sequence.

"A sequence (a_n) in a metric space $(\mathcal{M}, \check{d})$ is said to be **Cauchy Sequence** (or fundamental) if for every $\epsilon > 0$ there is an $\mathbb{N} = \mathbb{N}(\varepsilon)$, such that $\check{d}(a_m, a_n) < \varepsilon$ for every $m, n \geq \mathbb{N}$." [18]

Example 2.1.8.

Consider a sequence $\{a_\varrho\}$ in $\mathbb{R}$ with usual metric, defined as

$$\{a_\varrho\} = \frac{\varrho^2}{\varrho^2 + 1}.$$

To verify that it is a Cauchy sequence in $\mathbb{R}$,

Consider,

$$\begin{aligned}
|a_\varrho - a_\zeta| &= \left| \frac{\varrho^2}{\varrho^2+1} - \frac{\zeta^2}{\zeta^2+1} \right| \\
&= \left| \frac{\varrho^2\zeta^2 + \varrho^2 - \varrho^2\zeta^2 - \zeta^2}{(\varrho^2+1)(\zeta^2+1)} \right| \\
&= \left| \frac{\varrho^2 - \zeta^2}{(\varrho^2+1)(\zeta^2+1)} \right| \\
&= \left| \frac{\varrho^2}{(\varrho^2+1)(\zeta^2+1)} + \frac{-\zeta^2}{(\varrho^2+1)(\zeta^2+1)} \right| \\
&\leq \left| \frac{\varrho^2}{(\varrho^2+1)(\zeta^2+1)} \right| + \left| \frac{\zeta^2}{(\varrho^2+1)(\zeta^2+1)} \right| \\
&\leq \left| \frac{\varrho^2}{(\varrho^2)(\zeta^2+1)} \right| + \left| \frac{\zeta^2}{(\varrho^2+1)(\zeta^2)} \right| \\
&= \frac{1}{\zeta^2+1} + \frac{1}{\varrho^2+1} \\
&< \frac{1}{\zeta^2} + \frac{1}{\varrho^2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Set $\varrho, \zeta > \sqrt{\frac{2}{\varepsilon}}$ and by Archimedean's property letting $\mathbb{N} > \sqrt{\frac{2}{\varepsilon}}$.

It can be observed that every convergent sequence ultimately becomes a Cauchy sequence, but the converse of this result doesn't hold generally.

Definition 2.1.9. Complete Space.

"The space M is said to be complete if every Cauchy sequence in M converges."[18] The space $\mathbb{C}$ of complex numbers, the space $\mathbb{R}^n$, the space of all real sequences (ξ_k) with metric defined as

$$\breve{d}(\xi_k, \varrho_k) = \sum_{k=1}^{\infty} \frac{|\xi_k - \varrho_k|}{k!(1 + |\xi_k - \varrho_k|)},$$

are some examples of complete spaces.

Not every space is complete; for this, observe the following example.

Example 2.1.10.

Consider $(\mathcal{M}, \breve{d})$, being space of all polynomials depending on variable x on some closed on interval $\mathbf{I} = [0, 1]$, with metric function defined as

$$\breve{d}(\mathcal{P}, \mathcal{S}) = \max_{x \in \mathbf{I}} |\mathcal{P}(x) - \mathcal{S}(x)|.$$

We can easily find a Cauchy sequence $P_k(x) = \frac{x^k}{k!}$,

for large values of k, i.e.,

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

it converges to e^x, which is not a polynomial, so $\mathcal{M}$ is not complete space.

The completeness property of a space not only depends on the space structure but also changes according to the defined metric function. This can be explained with the help of the following example;

Example 2.1.11.

The function space $C\,[a, b]$ with metric function, defined as

$$\breve{d}(x_n, x_p) = \max_{t \in J} |x_n(t) - x_p(t)|,$$

on a finite closed interval $J = [0, 1]$, is a complete space.

But observe that if on the same space in the same closed interval if we define another metric function as

$$\check{d}(x_n, x_p) = \int_0^1 |x_n(t) - x_p(t)| dt,$$

is not a complete metric space.

To verify this, we construct a Cauchy sequence

$$x_p(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{2} \\ p\left(t - \frac{1}{2}\right) & \text{if } \frac{1}{2} \leq t \leq a_p \\ 1 & \text{if } a_p \leq t \leq 1. \end{cases}$$

Where $a_p = \frac{1}{2} + \frac{1}{p}$.

Next to show that it doesn't converge in $C\,[a, b]$.

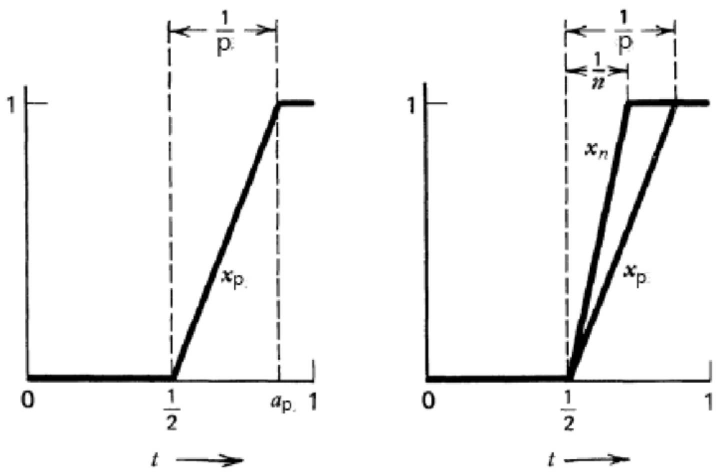

FIGURE 2.1: *Cauchy Sequence*

Consider

$$\begin{aligned} \check{d}(x_p, x) &= \int_0^1 \mid x_p(t) - x(t) \mid dt \\ &= \int_0^{\frac{1}{2}} \mid x_p(t) - x(t) \mid dt + \int_{\frac{1}{2}}^{a_p} \mid x_p(t) - x(t) \mid dt \\ &+ \int_{a_p}^1 \mid x_p(t) - x(t) \mid dt. \end{aligned}$$

Now as

$$\check{d}(x_p, x) \to 0$$

so each integral should approach to 0.

This means (x_p) is not convergent because

$$x_p(t) = 0$$

when t lies in $\left[0, \frac{1}{2}\right]$ and

$$x_p(t) = 1$$

when t is in between $\left[\frac{1}{2}, 1\right]$.

One of the most important ideas in the study of metric space is the distance between sets, which gives us information about the relation in points of a metric space. The following definitions will be helpful in our next discussion.

Definition 2.1.12. Distance of a point and a set.

"The distance $\check{d}(c, A)$ from a point c to a non-empty subset A of $(\mathcal{M}, \check{d})$ is defined to be

$$\check{d}(c, A) = \inf_{a \in A} \check{d}(c, a)." \text{ [18]}$$

Definition 2.1.13. Distance between sets.

"The distance $D(\mathcal{P}, \mathcal{Q})$ between two nonempty subsets $\mathcal{P}$ and $\mathcal{Q}$ of a metric space $(\mathcal{M}, \check{d})$ is defined to be

$$D(\mathcal{P}, \mathcal{Q}) = \inf\{\check{d}(p, q) : p \in \mathcal{P}, q \in \mathcal{Q}\}." \text{ [18]}$$

Definition 2.1.14. Hausdorff distance.

"Let $(\mathcal{M}, \check{d})$ be a metric space and $CB(\mathcal{M})$ denotes the collection of all nonempty closed and bounded subsets of $\mathcal{M}$. For $\mathcal{P}, \mathcal{Q} \in CB(\mathcal{M})$ define

$$\mathcal{H}(\mathcal{P}, \mathcal{Q}) = \max\{\sup_{p \in \mathcal{P}} \check{d}(p, \mathcal{Q}), \sup_{q \in \mathcal{Q}} \check{d}(q, \mathcal{P})\}.$$

Where $\check{d}(p, \mathcal{Q})$ is distance of p to the set $\mathcal{Q}$. It is known that $\mathcal{H}$ is a metric on $CB(\mathcal{M})$, called the Hausdorff metric induced by the metric $\check{d}$." [43]

Example 2.1.15.

Consider $\mathcal{P} = \{0, 1, 2, ..., 10\}$ with usual distance. Select two non-empty closed subsets of $\mathcal{P}$, $\mathcal{D} = \{1, 5\}$ and $\mathcal{M} = \{6, 9\}$. Now,

$$\begin{aligned}\mathcal{H}(\mathcal{D}, \mathcal{M}) &= \max\{\sup_{l \in \mathcal{D}} \check{d}(l, \mathcal{M}), \sup_{m \in \mathcal{M}} \check{d}(\mathcal{D}, m)\} \\ &= \max\{\sup_{l \in \mathcal{D}} \check{d}(l, \{6, 9\}), \sup_{m \in \mathcal{M}} \check{d}(\{1, 5\}, m)\} \\ &= \max\{\sup\{5, 1\}, \sup\{1, 4\}\} \\ &= \max\{5, 4\} \\ &= 5.\end{aligned}$$

2.2 Some Useful Mappings in Metric Space

Some mappings have a major role in developing important results in upcoming chapters; these will highlight the connection between elements of sets and their outputs under defined correspondence.

Definition 2.2.1. Continuous Mapping.

"Let $\mathcal{M} = (\mathcal{M}, d)$ and $\mathcal{N} = (\mathcal{N}, \tilde{d})$ be metric spaces. A mapping $T : \mathcal{M} \to \mathcal{N}$ is said to be continuous at a point $x_0 \in \mathcal{M}$ if for every $\epsilon > 0$ there is a $\delta > 0$, such that $\check{d}(T(x), T(x_0)) < \epsilon$, for all x satisfying $d(x, x_0) < \delta$." [18]

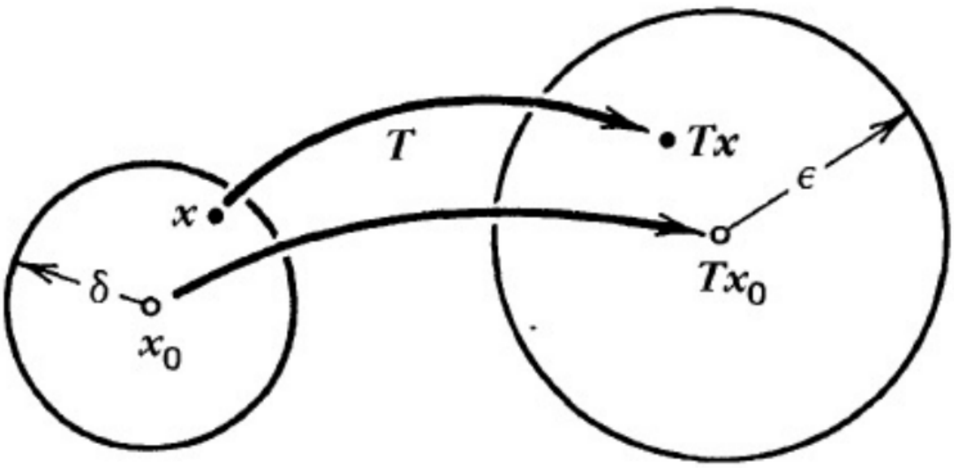

FIGURE 2.2: *Continuous Mapping*

Theorem 2.2.2.

"A mapping $T : \mathcal{M} \rightarrow \mathcal{N}$ of a metric space $(\mathcal{M}, d_1)$ into a metric space $(\mathcal{N}, d_2)$ is continuous at a point $x_0 \in \mathcal{M}$ if and only iff $x_n \longrightarrow x_0 \implies Tx_n \longrightarrow Tx_0$." [18]

Definition 2.2.3. Lipschitzian Mapping.

"Let $(\mathcal{M}, \check{d})$ be a metric space. A mapping $f : \mathcal{M} \rightarrow \mathcal{M}$ is said to be Lipschitzian if there exists a constant $c \geq 0$ with,

$$\check{d}(f(m), f(n)) \leq c\check{d}(m, n) \quad \text{for all} \quad m, n \in \mathcal{M}.$$

The smallest c for which this condition holds is said to be the Lipschitz constant for f." [44]

Example 2.2.4.

Consider $(\mathbb{R}, \check{d})$ with usual metric, we define a self map in $f : \mathbb{R} \rightarrow \mathbb{R}$ as

$$f(\xi) = 3\xi + 5,$$

$$\begin{aligned} \implies \check{d}(f(\xi), f(\eta)) &= |3\xi + 5 - 3\eta - 5| \\ &= |3\xi - 3\eta| \\ &= |3||\xi - \eta| \\ &= 3\check{d}(\xi, \eta), \end{aligned}$$

showing that f is Lipschitzian map with Lipschitz constant 3.

Definition 2.2.5. Contraction Mapping.

"Let $(\mathcal{M}, \check{d})$ be a metric space. A mapping $f : \mathcal{M} \rightarrow \mathcal{M}$ is called contraction on $\mathcal{M}$ if there is a positive real number $c < 1$ such that for all $m, n \in \mathcal{M}$ with

$$\check{d}(f(m), f(n)) \leq c\check{d}(m, n) \tag{2.1}$$

Geometrically this means that any points m and n have images that are closer together than those points m and n." [18]

Example 2.2.6.

Let $\mathcal{M} = [0, 1]$ be endowed with usual metric.

A map $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ defined as

$$\mathcal{T}(m) = \frac{1}{2+m},$$

is a contraction mapping with $c = 0.25 \in [0, 1)$.

Under the same metric structure, another mapping $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ working as

$$\mathcal{T}(m) = \frac{1}{10} + m,$$

is a contraction with $c = 0.1 \in [0, 1)$.

Theorem 2.2.7.

Every contraction mapping is continuous.

Proof. Consider $\mathcal{M}$ be a metric space $(\mathcal{M}, \check{d})$ and let $f : \mathcal{M} \to \mathcal{M}$ be contraction mapping. Let $k, m \in \mathcal{M}$ and $\delta > 0$ be any fixed positive number such that,$\check{d}(m, k) < \delta$. By definition of the contraction mapping

$$\check{d}(f(m), f(k)) \leq c\check{d}(m, k) = c\delta < \epsilon,$$

by choosing $\delta = \epsilon/c$, showing that f is continuous. □

Proposition 2.2.8.

Let $\mathfrak{f} : \mathcal{M} \to \mathcal{M}$ be a continuously differentiable map satisfying $\mid \acute{\mathfrak{f}} \mid < 1$ under the usual metric on $\mathcal{M}$, then $\mathfrak{f}$ is a contraction mapping.

Proof. By mean value theorem, there exists a $z^* \in (0, 1)$ such that

$$\mathfrak{f}(\xi) - \mathfrak{f}(\varrho) = \acute{\mathfrak{f}}(z^*)(\xi - \varrho)$$

$$|\mathfrak{f}(\xi) - \mathfrak{f}(\varrho)| = |\acute{\mathfrak{f}}(z^*)||\xi - \varrho| \leq \gamma|\xi - \varrho|,$$

where

$$\gamma = sup \mid \acute{\mathfrak{f}}(z) \mid < 1$$

for all z in $\mathcal{M}$. □

Example 2.2.9.

Let $\mathcal{M} = (\mathbb{R}, d)$, and define $f : \mathcal{M} \to \mathcal{M}$ as,

$$f(\phi) = \phi - \ln(1 + e^{\phi}),$$

$$\frac{df}{d\phi} = \frac{1}{1 + e^{\phi}} \quad \in (0, 1) \quad \text{for all} \quad \phi \in \mathbb{R}.$$

Hence we can write,

$$| f(\phi_1) - f(\phi_2) | = \left| \frac{df}{d\phi} \right| |\phi_1 - \phi_2 | < | \phi_1 - \phi_2 | \ .$$

Observe that in this case, the Lipschitz constant is exactly equal to 1.

So above mapping is not a contraction mapping. Now we may define another mapping in this regard as follow.

Definition 2.2.10. Contractive Mapping.

"A mapping $\mathbb{F} : \mathcal{M} \to \mathcal{M}$ is said to be contractive if for $m \neq n$, we have,

$$\check{d}(\mathbb{F}(m), \mathbb{F}(n)) < \check{d}(m, n)$$

for all $\quad m, n \in \mathcal{M}$." [45]

Example 2.2.11.

Consider, $\mathcal{M} = [1, \infty)$ with usual metric.

We define $\mathfrak{f} : \mathcal{M} \to \mathcal{M}$ as $\mathfrak{f}(\varrho) = \varrho + \frac{1}{\varrho}$, then we have

$$\begin{aligned}
\check{d}(\mathfrak{f}(\varrho), \mathfrak{f}(\tilde{s})) &= \check{d}(\varrho + \frac{1}{\varrho}, \tilde{s} + \frac{1}{\tilde{s}}) \\
&= | (\varrho + \frac{1}{\varrho}) - (\tilde{s} + \frac{1}{\tilde{s}}) | \\
&= | \varrho - \tilde{s} + \frac{1}{\varrho} - \frac{1}{\tilde{s}} | \\
&= | (\varrho - \tilde{s})(1 - \frac{1}{\varrho\tilde{s}}) | \\
&= | \varrho - \tilde{s} || 1 - \frac{1}{\varrho\tilde{s}} | \\
&< | \varrho - \tilde{s} | \\
&= \check{d}(\varrho, \tilde{s}).
\end{aligned}$$

It can also be observed that it is a contractive mapping but not a contraction. As

$$\frac{d\check{\mathfrak{f}}}{d\varrho} = 1 - \frac{1}{\varrho^2},$$

As,

$$\frac{d\mathfrak{f}}{d\varrho} \in [0, 1).$$

But here Lipschitz constant is approaching 1 for high values of ϱ. It can be shown graphically as

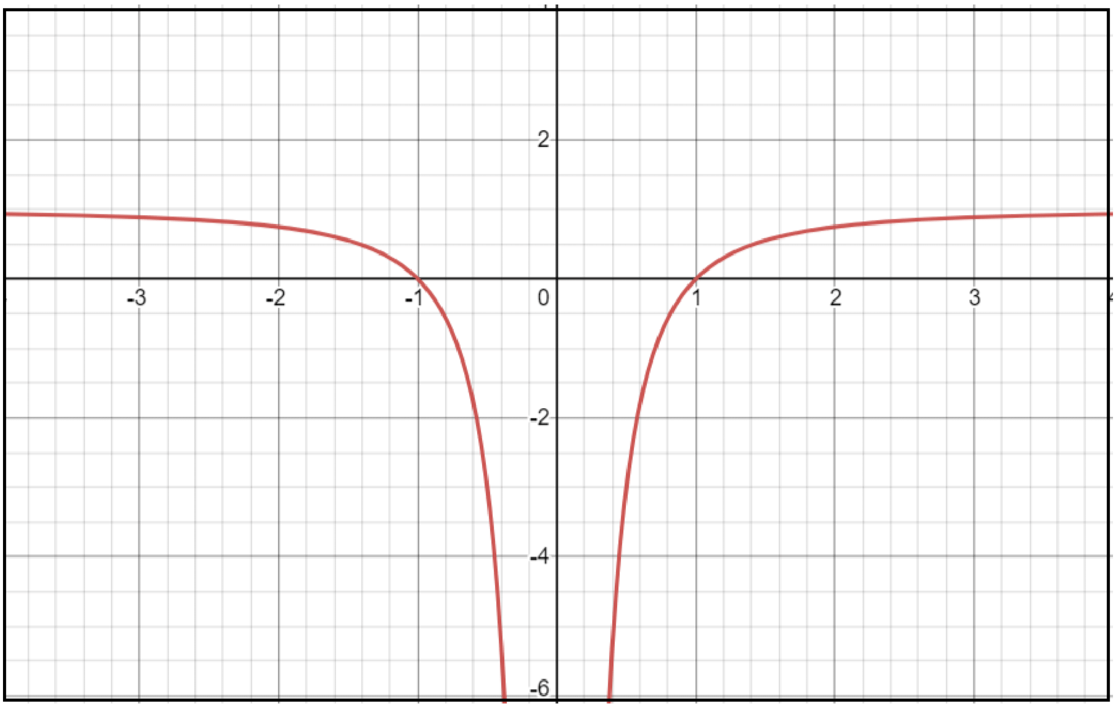

FIGURE 2.3: *Contractive Mapping*

2.3 Fixed Points of a Mapping

In 1880, Henri Poincare revealed that the study of some significant analytical problems could be done by defining a set $\mathcal{M}$ and a function $T : \mathcal{M} \to \mathcal{M}$ so that the solution coincides to the fixed point of the function T.

With this achievement, fixed points become more important for getting the solution to problems occurring in various dimensions of mathematical analysis.

Definition 2.3.1. Fixed Point.

"A fixed point of a mapping $f : \mathcal{M} \to \mathcal{M}$ on a set $\mathcal{M}$ into itself is $m \in \mathcal{M}$ which is mapped onto itself, that is $f(m) = m$, the image $f(m)$ coincides with m." [18]

If we have a real-valued function, then the fixed points are the points of intersection

$$y = m$$

and

$$y = f(m).$$

Let's have a look at the graphical representation of fixed points.

Example 2.3.2.

Consider a real-valued map $f : \mathbb{R} \to \mathbb{R}$, defined as

$$f(m) = m^2 - 5m + 5,$$

has two fixed points.

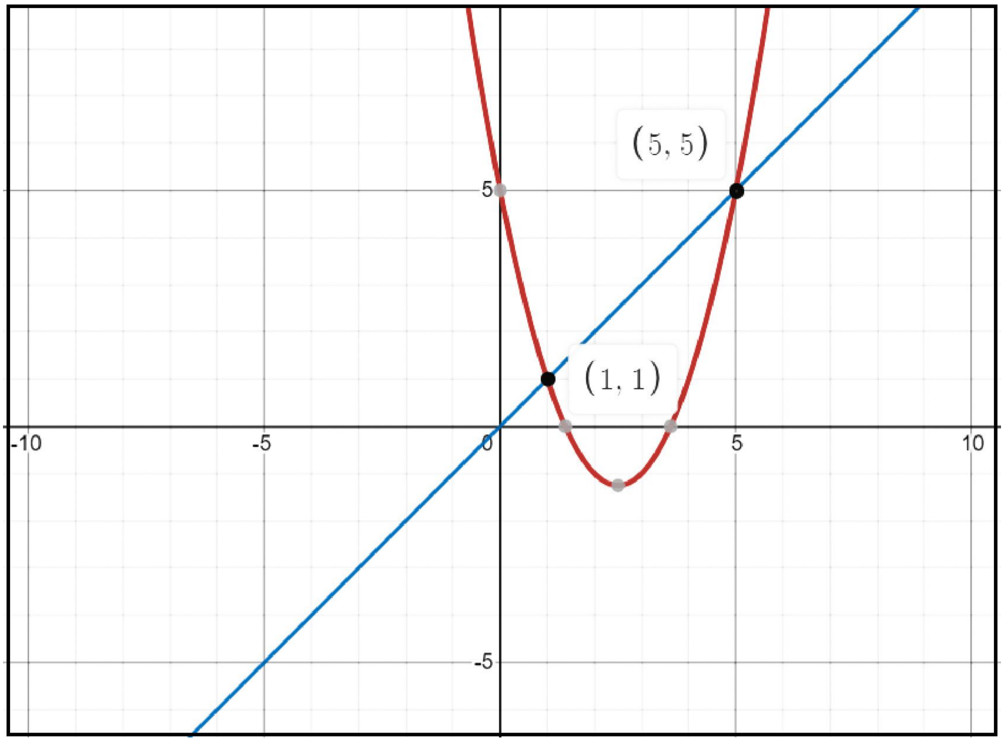

FIGURE 2.4: *2-fixed points*

If we define $f : \mathbb{R} \to \mathbb{R}$, as $f(m) = m - \ln(1 + e^m)$, then there isn't any fixed point.

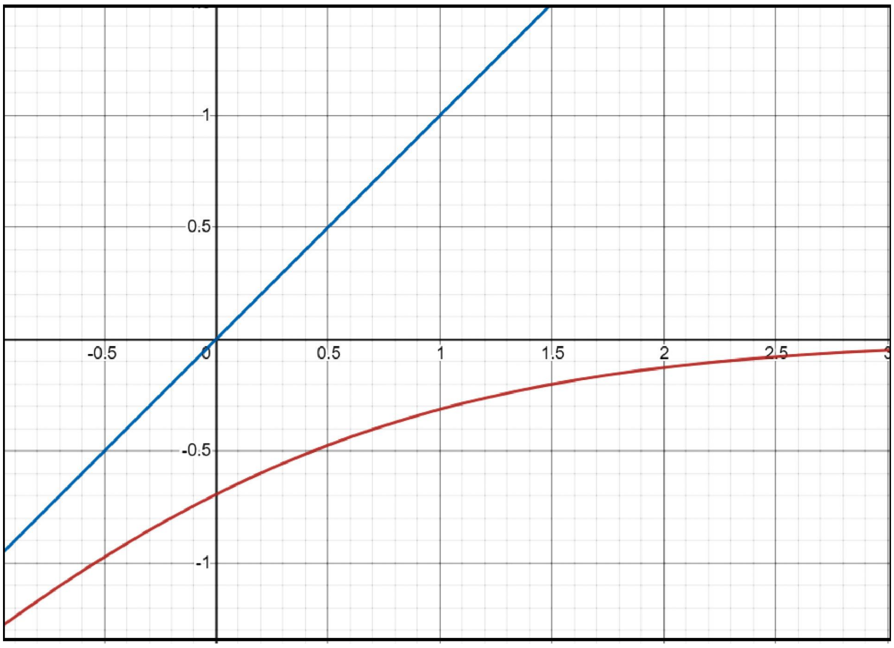

FIGURE 2.5: *No fixed point*

And now for $f : \mathbb{R} \to \mathbb{R}$, defined as $f(m) = 5m(1 - m)$,then fixed point for f and $f(f(m)) = f^2(m)$, are shown below

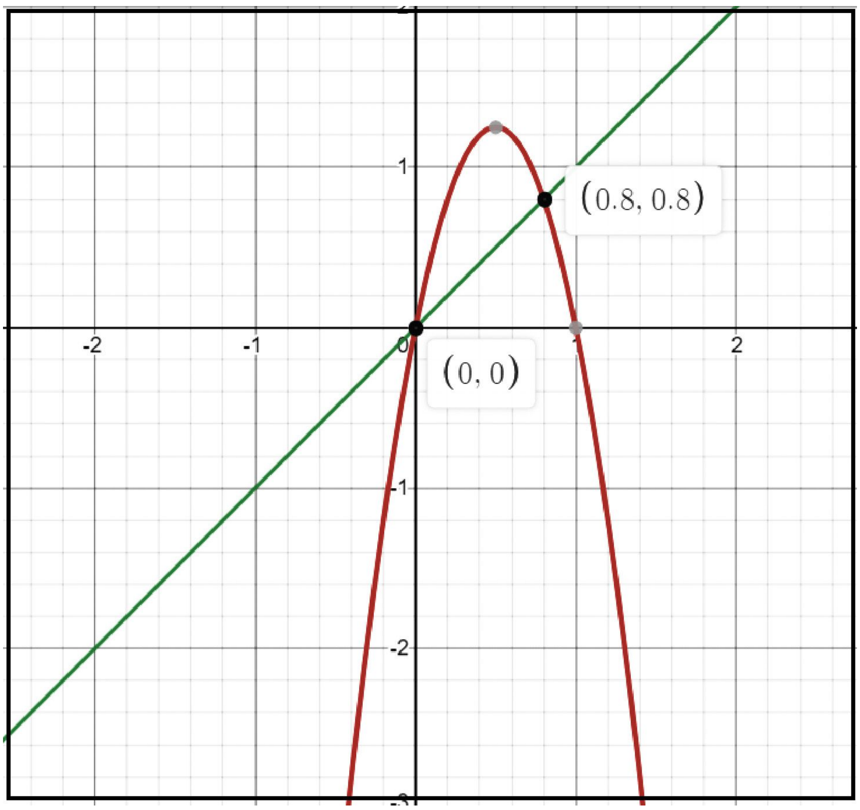

FIGURE 2.6: *Fixed Points of f*

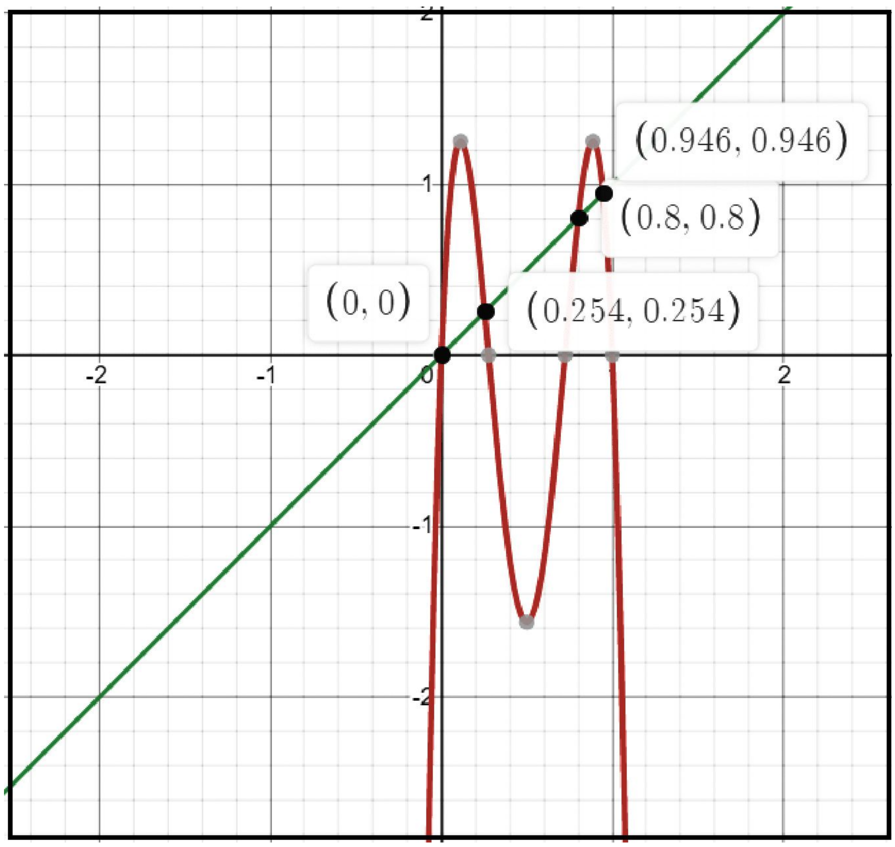

FIGURE 2.7: *Fixed Points of f^2*

Definition 2.3.3. Fixed Point Theorem.

A statement that assures fixed points of a mapping under some certain suitable restrictions in any specified space is called the fixed point theorem.

Some of such important theorems are presented here.

In 1912 famous Brouwer theorem was given. Although this theorem didn't give any information about the location of fixed points, it gave a direction toward the existence of fixed points. He just assumed the continuity condition of mapping on a finite-dimensional space and presented the following theorem.

Theorem 2.3.4.

"Every continuous mapping from a closed ball of Euclidean space into itself has a fixed point." [2]

In 1930, an improved version of the above result was presented by Schauder, with the addition of compactness condition on subsets of the Euclidean space.

Theorem 2.3.5.

"Every continuous function from a convex compact subset of Euclidean space to itself has a fixed point." [46]

In 1922, Stefan Banach presented a theorem known as Banach Contraction Principle (BCP).
He used the contraction condition of a self-map and worked for complete spaces. His approach ensured that fixed points exist and are unique.

Theorem 2.3.6.

"Let $(\mathcal{M}, d)$ be a complete metric space and $\mathbb{F} : \mathcal{M} \to \mathcal{M}$ be a contraction mapping, then $\mathbb{F}$ admits a unique fixed point in $\mathcal{M}$." [4]

Example 2.3.7.

Consider a complete metric space $\mathcal{M} = [-1, 1]$ equipped with the usual distance. We define a self-contraction mapping as

$$f(m) = \frac{1}{2}\cos^2(m).$$

One can check that constant contraction lies between 0 and 0.5. This specified map has a unique fixed point, which can be observed in the following diagram;

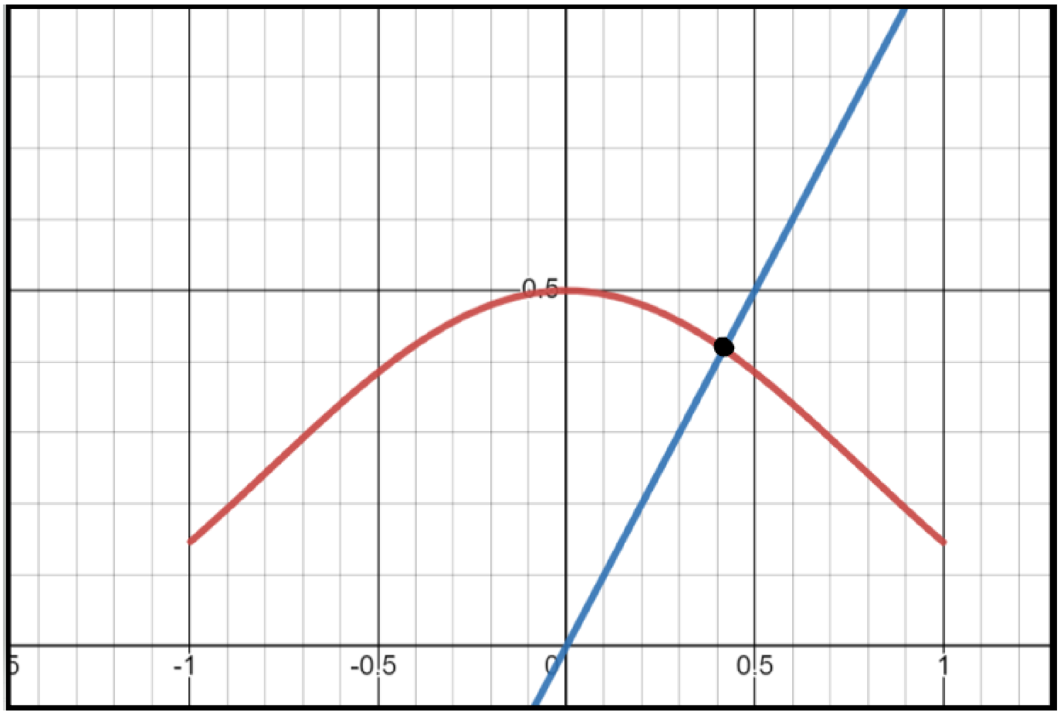

FIGURE 2.8: *A unique fixed point*

The contraction condition used by Banach was modified for the first time by Edelstein in 1962. He presented the following result:

Theorem 2.3.8.
"Let $(\mathcal{M}, \check{d})$ be a metric space and $\mathbb{F}$ be a self-contractive mapping on $\mathcal{M}$ i.e.,

$$\check{d}(\mathbb{F}(m), \mathbb{F}(n)) < \check{d}(m, n) \text{ for all } \quad m, n \in \mathcal{M},$$

and there exists a point m such that the sequence of its iterates contains a subsequence which converges to a point m^* in $\mathcal{M}$, then m^* is a unique fixed point of $\mathbb{F}$".[5]

In 1962, Rakotch [47] proposed a more advanced idea regarding contraction constant. He proved the validity of BCP after replacing the constant $c \in [0, 1)$ with a metric depending monotonically decreasing function $c(m, n) \in [0, 1)$.

He gave the following generalized result.

Theorem 2.3.9.
"Let $(\mathcal{M}, \check{d})$ be a complete metric space and $\mathbb{F}$ be a self contractive mapping on $\mathcal{M}$ such that

$$\check{d}(\mathbb{F}(m), \mathbb{F}(n)) < c(m, n)\check{d}(m, n) \text{ for all } \quad m, n \in \mathcal{M},$$

and there exists a $K \subset \mathcal{M}$ such that for $m_0 \in K$ we have

$$\check{d}(m, m_0) - \check{d}(\mathbb{F}(m), \mathbb{F}(m_0)) \geq 2\check{d}(m_0, \mathbb{F}(m_0)) \text{ for all } \quad m \in K^c,$$

then there exists a unique fixed point of $\mathbb{F}$."

2.4 Some Extensions in Metric spaces

Many researchers tried to modify the conditions of metric spaces and highlighted new possible dimensions. Major metric fixed point theory results were surprisingly valid in these new directions too. Two fine extensions are presented here:

2.4.1 $\mathfrak{b}$-Metric Spaces

The idea of $\mathfrak{b}$MS was suggested by Bakhtin [29] and Czerwik [30], with modification in the triangular property of the metric space. Numerous researchers started working on this ground and produced remarkable work.

In the current section, we will look at some main ideas and results that will be applicable to our upcoming sections, including the basic definition of $\mathfrak{b}$-metric space by attaching a few examples, the convergence criterion, and the completeness property of these spaces.

Definition 2.4.1. $\mathfrak{b}$-metric space.
"Let $\mathcal{M}$ be non empty set and $\mathfrak{b} \geq 1$ be a given real number. A function $d_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ is said to be $\mathfrak{b}$-metric on $\mathcal{M}$, the pair $(\mathcal{M}, d_{\mathfrak{b}})$ is called a $\mathfrak{b}$-metric space if for all $\eta, \xi, \mu \in \mathcal{M}$,
B1: $d_{\mathfrak{b}}(\eta, \xi) = 0$ if and only if $\eta = \xi$.
B2: $d_{\mathfrak{b}}(\eta, \xi) = d_{\mathfrak{b}}(\xi, \eta)$.
B3: $d_{\mathfrak{b}}(\eta, \mu) \leq \mathfrak{b}\{d_{\mathfrak{b}}(\eta, \xi) + d_{\mathfrak{b}}(\xi, \mu)\}$." [48]
One can observe that by setting $\mathfrak{b} = 1$, we will jump back into the domain of metric space.
So the concept of $\mathfrak{b}$-metric is comparatively a stronger idea than the metric space.
Let's have a look at some examples.
We will use the following two well-known inequalities while proving **B3**

$$(m+n)^k \leq 2^{k-1}(m^k + n^k) \qquad k > 1. \tag{2.2}$$

$$\Big(\sum_{k=1}^{\infty} |m_k + n_k|^p\Big)^{1/p} \leq \Big(\sum_{k=1}^{\infty} |m_k|^p\Big)^{1/p} + \Big(\sum_{k=1}^{\infty} |n_k|^p\Big)^{1/p} \tag{2.3}$$

Example 2.4.2.
Let $\mathcal{M} = l_p(\mathbb{R})$, $p \in (0,1)$, the space of all real sequences having the property

$$l_p(\mathbb{R}) = \{m = m_k \subset \mathbb{R}\}$$

such that $\sum_{k=1}^{\infty} |m_k|^p < \infty$.

We define a function $d_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$, as

$$d_{\mathfrak{b}}(m,n) = \big(\sum_{k=1}^{\infty} |m_k - n_k|^p\big)^{1/p}.$$

B1: Consider $d_{\mathfrak{b}}(m,n) = 0$

$$\begin{aligned}
&\implies \big(\sum_{k=1}^{\infty} |m_k - n_k|^p\big)^{1/p} = 0\\
&\iff \sum_{k=1}^{\infty} |m_k - n_k|^p = 0 \qquad : |m_k - n_k|^p \in \mathbb{R}^+\\
&\iff \lim_{n\to\infty} |m_i - n_i| = 0 \qquad \text{for some} \quad i \in \mathbb{N}\\
&\iff |m_i - n_i| = 0\\
&\iff m_i - n_i = 0\\
&\iff m_i = n_i\\
&\iff m = n.
\end{aligned}$$

B2: Symmetry follows from the symmetric relation of absolute value.

B3: Consider

$$\begin{aligned}
d_{\mathfrak{b}}(m,k) &= \big(\sum_{i=1}^{\infty} |m_i - k_i|^p\big)^{1/p}\\
&= \big(\sum_{i=1}^{\infty} |m_i - k_i + n_i - n_i|^p\big)^{1/p}\\
&\leq \big(\sum_{i=1}^{\infty} 2^{p-1}(|m_i - n_i|^p + |n_i - k_i|^p)\big)^{1/p} \quad \text{using (2.2)}\\
&\leq \sum_{i=1}^{\infty} 2^{p-1}\big((|m_i - n_i|^p)^{1/p} + \sum_{i=1}^{\infty}(|n_i - k_i|^p)^{1/p}\big) \quad \text{using (2.3)}\\
&= \mathfrak{b}\Bigg(\sum_{i=1}^{\infty} |m_i - n_i|^p\big)^{1/p} + \big(\sum_{i=1}^{\infty} |n_i - k_i|^p\Bigg)^{1/p}\\
&= \mathfrak{b}\{d_{\mathfrak{b}}(m,n) + d_{\mathfrak{b}}(n,k)\},
\end{aligned}$$

showing that $(\mathcal{M}, d_{\mathfrak{b}})$ is a $\mathfrak{b}$-metric space.

Example 2.4.3.

Let $\mathcal{M} = \mathbb{R}$ and $d_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ be a function defined as

$$d_{\mathfrak{b}}(\eta, \upsilon) = (\eta - \upsilon)^2 \quad \text{for all} \quad \eta, \upsilon \in \mathcal{M},$$

for all $\eta, \upsilon \in \mathcal{M}$. it can be checked easily that $(\mathcal{M}, d_{\mathfrak{b}})$ is a $\mathfrak{b}$-metric space with $\mathfrak{b} = 2$.

Example 2.4.4.

Consider the space of all real functions $\mathcal{M} = l_p(\mathbb{R})$, depending on variable θ, with condition

$$\int_0^1 |h(\theta)|^p dx < \infty,$$

then $(\mathcal{M}, d_{\mathfrak{b}})$ is a $\mathfrak{b}$-metric space with $\mathfrak{b} = 2^{\frac{1}{p}}$.

Where the metric function $d_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is defined, as follow

$$d_{\mathfrak{b}}(h, k) = \left(\int_0^1 |h(\theta) - k(\theta)|^p dx \right)^{1/p}.$$

for all $h, k \in \mathcal{M}$ with $p \in [0, 1]$

Definition 2.4.5. Convergence and Completeness in $\mathfrak{b}$-metric space.

"Let $(\mathcal{M}, d_{\mathfrak{b}})$ be a $\mathfrak{b}$-metric space;

(i) A sequence $\{m_k\}$ in $\mathcal{M}$ is called convergent sequence if and only if there exists a $m_0 \in \mathcal{M}$ such that for all $k \geq k(\epsilon) \in \mathbb{N}$, we have

$$d_{\mathfrak{b}}(m_k, m_0) < \epsilon \quad \text{for} \quad \epsilon > 0.$$

(ii) A sequence $\{m_k\}$ in $\mathcal{M}$ is said to be a Cauchy sequence if and only if there exists $\epsilon > 0$ such that for all $k, l \geq k(\epsilon) \in \mathbb{N}$,

$$d_{\mathfrak{b}}(m_k, m_l) < \epsilon.$$

(iii) If every Cauchy sequence is convergent in $\mathcal{M}$, then $\mathcal{M}$ is said to be a complete $\mathfrak{b}$MS".

In 1993, BCP was generalized by Czerwik [30] in $\mathfrak{b}$-metric space by assuming the completeness property of space.

Theorem 2.4.6.

"Let $(\mathcal{M}, d_{\mathfrak{b}})$ be a complete $\mathfrak{b}$ metric space , and let $\mathbb{T} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ satisfies,

$$d_{\mathfrak{b}}(\mathbb{T}(\upsilon), \mathbb{T}(n)) \leq \psi d_{\mathfrak{b}}(\upsilon, n) \quad \textit{for all} \quad \upsilon, n \in \mathcal{M},$$

where if $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function, such that

$$\lim_{k\to\infty} \psi^k(\upsilon) = 0.$$

Then $\mathbb{T}$ has exactly one fixed point υ^* and

$$\lim_{k\to\infty} d_{\mathfrak{b}}(\mathbb{T}^k(\upsilon), \upsilon^*) = 0." \quad [30]$$

2.4.2 Partial Metric Space

S. Matthews [19] proposed the profound theory of partial metric space in 1992, suggesting the presence of nonzero self-distances in the metric space.
After his work, many authors started exploring more results on the framework of the partial metric space by observing the presence of unique fixed points under different flavors of contraction conditions.

Definition 2.4.7. Partial Metric Space (PMS).
"Let be $\mathcal{M}$ be a non-empty set. A partial metric space is a pair $(\mathcal{M}, d_p)$ where d_p is a function $d_p : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$, called the partial metric, such that for all $s, p^*, k \in \mathcal{M}$. the following axioms hold:
P1: $d_p(s, s) \leq d_p(s, p^*)$.
P2: If $d_p(s, s) = d_p(s, p^*) = d_p(p^*, p^*) \iff s = p^*$.
P3: $d_p(s, p^*) = d_p(p^*, s)$.
P4: $d_p(s, p^*) \leq d_p(s, k) + d_p(k, p^*) - d_p(k, k)$." [19].
Every PMS subsequently becomes metric space with an additional innovation of nonzero self-distance, But converse of this fact doesn't hold generally .

One may notice that the metric space lies in the globe of partial metric space, which is a more generalized concept.

Example 2.4.8.

(i) Let $\mathcal{M} = \mathbb{R}$ and max denotes the maximum function then we define a map as $d_p : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$, such that

$$d_p(\mu, \zeta^*) = \max(\mu, \zeta^*),$$

defines a partial metric on $\mathcal{M}$, which can be shown as follow
P1: Set $\mu > \zeta^*$ then,

$$\begin{aligned} d_p(\mu, \zeta^*) &= \max\{\mu, \zeta^*\} \\ &= \mu \\ &> \zeta^* \\ &= \max\{\zeta^*, \zeta^*\} \\ &= d_p(\zeta^*, \zeta^*) \\ \implies d_p(\mu, \zeta^*) &> d_p(\zeta^*, \zeta^*) \end{aligned}$$

Now let $\mu < \zeta^*$,
then we have,

$$d_p(\mu, \zeta^*) = \max\{\mu, \zeta^*\} = \zeta^* = \max\{\zeta^*, \zeta^*\} = d_p(\zeta^*, \zeta^*),$$

by combining both relations, we conclude
$d_p(\mu, \zeta^*) \geq d_p(\zeta^*, \zeta^*) \quad \text{for all} \quad \mu, \zeta^*$. **P2**: Consider $\mu = \zeta^*$,

$$\begin{aligned} \implies d_p(\mu, \zeta^*) &= d_p(\mu, \mu) \\ &= \max\{\mu, \mu\} \\ &= \mu \\ &= \zeta^* \\ &= \max\{\zeta^*, \zeta^*\} \\ &= d_p(\zeta^*, \zeta^*). \end{aligned}$$

Conversely setting,

$$d_p(\mu, \mu) = d_p(\zeta^*, \zeta^*) = d_p(\mu, \zeta^*)$$

. We can easily get

$$d_p(\mu, \mu) = d_p(\zeta^*, \zeta^*) = d_p(\mu, \zeta^*) \iff \mu = \zeta^*,$$

for all $\mu, \zeta^* \in \mathcal{M}$.

P3: Clearly, symmetry follows because of the symmetric property of the maximum function.

P4: Let μ, ζ^*, θ be any elements of $\mathcal{M}$ and we have a relation

$$\mu \leq \zeta^* \leq \theta$$

. Now consider

$$\begin{aligned} d_p(\mu, \zeta^*) &= \max\{\mu, \zeta^*\} \\ &\leq \zeta^* \\ &= \theta + \zeta^* - \theta \\ &= \max\{\mu, \theta\} + \max\{\zeta^*, \theta\} - \max\{\theta, \theta\} \\ &= d_p(\mu, \theta) + d_p(\zeta^*, \theta) - d_p(\theta, \theta). \\ \implies d_p(\mu, \zeta^*) &\leq d_p(\mu, \theta) + d_p(\zeta^*, \theta) - d_p(\theta, \theta), \end{aligned}$$

for all $\mu, \zeta^*, \theta \in \mathcal{M}$.

(ii) Let $\mathcal{U}$ be the set of closed and bounded intervals in $\mathbb{R}$, i.e.,

$$\mathcal{U} = \{[\zeta, \eta] : \zeta \leq \eta\}$$

with defined metric function $d_p : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ as,

$$d_p([\zeta, \eta], [\xi, \theta]) = \max(\eta, \theta) - \min(\zeta, \xi),$$

for $[\zeta, \eta], [\xi, \theta] \in \mathcal{U}$, is a partial metric over $\mathcal{U}$. In this specific example, the self distance is simply the length of the desired closed interval, the difference of the endpoints of the interval, i.e.,

$$d_p([\nu,\nu],[\xi,\xi]) = \xi - \nu.$$

(iii) Let $\mathcal{M} \neq \phi$ and we have a partial metric $d_p : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$, defined as

$$d_p(\zeta,\xi) = e^{\max(\zeta,\xi)},$$

then $(\mathcal{M}, d_p)$ is a PMS for all $\zeta, \xi \in \mathcal{M}$.

Chapter 3

Some Fixed Point Results under MVF-Contractions in PMS.

In this chapter, a detailed review of the work of Kumar et al. [41] is presented, in which the fixed point theorems for multivalued F-contraction (MVFC) are given. Furthermore, we added useful definitions and important theorems concerning fixed points under MVFCs in the frame of partial metric space.

3.1 Some Tools from Partial Metric Spaces.

In the first section, we will recall some basic ideas from the domain of partial metric space, which includes the convergence criterion and completeness properties of these spaces.
Continuity of a function in these spaces is also portrayed. Some related results are also presented to have a deeper understandings of partial metric space.

Definition 3.1.1. Open p-ball.
Consider $(\mathcal{M}, d_p)$ be a PMS then an open p-ball of radius $\epsilon > 0 \in \mathbb{R}$, with center $\eta \in M$ is defined as

$$B_p(\eta, \epsilon) = \{m \in \mathcal{M} : d_p(\eta, m) < d_p(\eta, \eta) + \epsilon\} \quad \text{for all} \quad m \in \mathcal{M}. \text{ [41]}$$

Remark 3.1.2.

A T_0 topology τ_p on $\mathcal{M} = (\mathcal{M}, d_p)$ can be generated always, where the collection of open p-balls

$$\{B_p(\xi, r^*) : \xi \in \mathcal{M}, r^* > 0\},$$

provides a base for this topology.

Definition 3.1.3. Closed Set.

Let $(\mathcal{M}, d_p)$ is a PMS then a subset $\mathcal{S}$ of $\mathcal{M}$ is said to be a closed in $\mathcal{M}$ if and only if $\bar{\mathcal{S}} = \mathcal{S}$, here $\bar{\mathcal{S}}$ is denoting closure of $\mathcal{S}$. [41]

Theorem 3.1.4.

Let $(\mathcal{M}, d_p)$ be a PMS and $\mathcal{V} \subset \mathcal{M}$ then $w^* \in \bar{\mathcal{V}}$ if and only if $d_p(w^*, w^*) = d_p(w^*, \mathcal{V})$, where $\bar{\mathcal{V}}$ is closure of $\mathcal{V}$. [49]

Proof: Let $\mathcal{V} \subset \mathcal{M}$, $(\mathcal{M}, d_p)$ is a PMS, let $w^* \in \bar{\mathcal{V}}$, then for $\epsilon > 0$. We may write,

$$\begin{aligned}
& B_p(w^*, \epsilon) \cap \mathcal{V} \neq \phi \quad \text{for all} \quad \epsilon > 0 \\
\Longleftrightarrow\ & d_p(w^*, w_0^*) < \epsilon + d_p(w^*, w^*) \quad \text{for some} \quad w_0^* \in \mathcal{V} \\
\Longleftrightarrow\ & d_p(w^*, w_0^*) - d_p(w^*, w^*) < \epsilon \\
\Longleftrightarrow\ & \inf_{w_0^* \in \mathcal{V}} \{d_p(w^*, w_0^*) - d_p(w^*, w^*)\} = 0 \\
\Longleftrightarrow\ & \inf_{w_0^* \in \mathcal{V}} \{d_p(w^*, w_0^*)\} = d_p(w^*, w^*) \\
\Longleftrightarrow\ & d_p(w^*, \mathcal{V}) = d_p(w^*, w^*).
\end{aligned}$$

Definition 3.1.5. p-convergent Sequence.

Let $(\mathcal{M}, d_p)$ be a PMS and $\{a_n\}$ be a sequence in $\mathcal{M}$, it is called to be p-convergent at some m_0 if and only if

$$\lim_{n \to +\infty} d_p(m_0, a_n) = d_p(m_0, m_0). \quad [41]$$

Definition 3.1.6. p-Cauchy Sequence and Completeness in PMS.

Let $(\mathcal{M}, d_p)$ be a PMS and $\{a_k\}$ be any sequence in $\mathcal{M}$.

It becomes a p-Cauchy sequence if

$$\lim_{n,m \to +\infty} d_p(a_k, a_n),$$

exists and is finite.

A PMS is called a complete PMS if every Cauchy sequence $\{a_k\}$ converges in $\mathcal{M}$ w.r.t τ_p, to a point in $m \in \mathcal{M}$ only if

$$d_p(m, m) = \lim_{m,n \to +\infty} d_p(a_k, a_n). \quad [41]$$

Example 3.1.7. Let $\mathcal{M} = \mathbb{R}$ and $d_p : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ is defined as

$$d_p(\zeta, \eta) = \max(\zeta, \eta),$$

is a partial metric on $\mathcal{M}$. A sequence

$$a_n = \frac{1}{4} + \frac{1}{n},$$

is a p-Cauchy sequence in $\mathcal{M}$. As

$$d_p(a_n, a_m) = d_p(\frac{1}{4} + \frac{1}{n}, \frac{1}{4} + \frac{1}{m}) = \max\{\frac{1}{4} + \frac{1}{n}, \frac{1}{4} + \frac{1}{m}\},$$

for higher values of m, n we have

$$\lim_{m,n \to +\infty} d_p(a_n, a_m) = \frac{1}{4}.$$

Showing that a finite limit exists.

Remark *3.1.8.*

Consider d_p, a partial metric function on $\mathcal{M}$. We define a map $d_p^s : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ working as

$$d_p^s(\mu, \zeta^*) = 2d_p(\mu, \zeta^*) - d_p(\zeta^*, \nu) - d_p(\zeta^*, \zeta^*), \tag{3.1}$$

for all $\mu, \nu, \zeta^* \in \mathcal{M}$, which constructs a metric on $\mathcal{M}$.

This shows that we can always associate a metric space with any PMS, and we will call it induced partial space, denoted by $(\mathcal{M}, d_p^s)$. [41]

Definition 3.1.9. Continuous Function in PMS.

Let $(\mathcal{M}, d_p)$ and $(\mathcal{N}, d_p)$ be partial metric spaces and $\mathcal{A} \subset \mathcal{M}$. A function $f : \mathcal{A} \to \mathcal{N}$ is said to be continuous at $c \in \mathcal{A}$ if for any $\epsilon > 0$ there is $\delta > 0$ such that if $a \in \mathcal{A}$:

$$d_p(a,c) - d_p(a,a) < \delta,$$

$$d_p(a,c) - d_p(c,c) < \delta$$

$\Longrightarrow$

$$d_p(f(a), f(c)) - d_p(f(a), f(a)) < \epsilon,$$

$$d_p(f(a), f(c)) - d_p(f(c), f(c)) < \epsilon. \quad [50]$$

Lemma 3.1.10.

(i) A sequence is p-Cauchy in any PMS if and only if it is p-Cauchy in the induced partial space $(\mathcal{M}, d_p^s)$.

(ii) A PMS is complete if and only if the induced partial space $(\mathcal{M}, d_p^s)$ is complete. [41]

Lemma 3.1.11.
Let $(\mathcal{M}, d_p)$ be a PMS with a compact subset $\mathbb{K}$. Consider a subset $B \subset \mathbb{K}$ and define a function $S : B \to \mathbb{K}$ then we have following two equivalent statements;

(i) S is a continuous function.

(ii) For any convergent subsequence $m_{n_k} \to m_0$ we have $Sm_{n_k} \to Sm_0$ for any $m_0 \in B$, using compactness of $\mathbb{K}$. [41]

Proof. By using continuity condition, for any $\epsilon > 0$ there exist a $\delta > 0$ such that,

$$d_p(m, m_0) - d_p(m_0, m_0) < \delta,$$

also

$$d_p(m, m_0) - d_p(m, m) < \delta.$$

$\Longrightarrow$

$$d_p(Sm, Sm_0) - d_p((Sm_0), Sm_0) < \epsilon,$$

and

$$d_p(Sm, Sm_0) - d_p(Sm, Sm) < \epsilon.$$

Now for a convergent subsequence $m_{n_k} \to m_0$, we can write

$$d_p(m_{n_k}, m_0) - d_p(m_0, m_0) < \delta,$$

and

$$d_p(m_{n_k}, m_0) - d_p(m_{n_k}, m_{n_k}) < \delta.$$

Which gives

$$d_p(Sm_{n_k}, Sm_0) - d_p(Sm_0, Sm_0) < \epsilon,$$

and

$$d_p(Sm_{n_k}, Sm_0) - d_p(Sm_{n_k}, Sm_{n_k}) < \epsilon.$$

Showing that,

$$Sm_{n_k} \to Sm_0.$$

So $(i) \implies (ii)$. Now consider $Sm_{n_k} \to Sm_0$ when $m_{n_k} \to m_0$.
To show that S is a continuous function. Suppose on contrary that S is not a continuous function then for $\epsilon > 0$ there must exist a $\delta > 0$ such that, when

$$d_p(m, m_0) - d_p(m_0, m_0) < \delta,$$

and

$$d_p(m, m_0) - d_p(m, m) < \delta.$$

We have,

$$d_p(Sm, Sm_0) - d_p(Sm_0, Sm_0) \geq \epsilon,$$

and

$$d_p(Sm, Sm_0) - d_p(Sm, Sm) \geq \epsilon.$$

Setting $\delta = \frac{1}{m_{n_k}}$ and for any $k \in \mathbb{N}$,
we have

$$d_p(m_{n_k}, m_0) - d_p(m_0, m_0) < \frac{1}{m_{n_k}},$$

also

$$d_p(m_{n_k}, m_0) - d_p(m_{n_k}, m_{n_k}) < \frac{1}{m_{n_k}}.$$

Implying

$$d_p(Sm_{n_k}, Sm_0) - d_p(Sm_0, Sm_0) \geq \epsilon,$$

and

$$d_p(Sm_{n_k}, Sm_0) - d_p(Sm_{n_k}, Sm_{n_k}) \geq \epsilon.$$

Which means, $m_{n_k} \to m_0$ but

$$Sm_{n_k} \nrightarrow Sm_0,$$

which is a contradiction.

Proving that $(ii) \implies (i)$. □

After introducing partial metric space, Matthews gave a fine generalization of BCP, which opened doors for other researchers. One can observe the contraction condition of BCP was used with an additional factor of self distance.

Theorem 3.1.12.
Consider a complete PMS $(\mathcal{M}, d_p)$ and let $\mathbb{S} : \mathcal{M} \to \mathcal{M}$ be a contraction map over $\mathcal{M}$ with $c \in [0, 1)$ i.e.,

$$d_p(\mathbb{S}(\zeta), \mathbb{S}(\rho)) \leq c d_p(\zeta, \rho) \quad \text{for all} \quad \zeta, \rho \in \mathcal{M},$$

then for each $\mathbb{S}$ we must have a unique number $\zeta^* \in \mathcal{M}$ such that $\mathbb{S}(\zeta^*) = \zeta^*$. Also, $d_p(\zeta^*, \zeta^*) = 0$. [19]

3.2 Hausdorff Distance in Partial Metric Space.

In 2012, the idea of the Hausdorff metric was merged in the domain of partial metric space by Aydi et al. [43].

The closedness property of the subsets of a PMS was followed by the induced topology on a given set where a bounded set in partial metric space is defined below.

Definition 3.2.1. Bounded Set in PMS.

Let $(\mathcal{M}, d_p)$ be a PMS and let $\mathcal{N}$ be any non empty subset of $\mathcal{M}$, then $\mathcal{N}$ is called bounded if for any $m_0 \in \mathcal{M}$ there exist an open p-ball centered at m_0 with radius $G \geq 0$ and $n^* \in B_p(m_0, G)$ where n^* be any arbitrary element in $\mathcal{N}$, such that

$$d_p(m_0, n^*) < d_p(n^*, n^*) + G. \quad [43]$$

Definition 3.2.2. Partial Hausdorff Distance.

Let $(\mathcal{M}, d_p)$ be a PMS and $CB_p(\mathcal{M})$ be collection of all non-empty bounded and closed subsets of $\mathcal{M}$. For $\mathcal{P}, \mathcal{P}^* \in CB_p(\mathcal{M})$, partial Hausdorff metric on $CB_p(\mathcal{M})$ induced by d_p is given as follow

$$\mathcal{H}_p(\mathcal{P}, \mathcal{P}^*) = \max\{\sup_{p \in \mathcal{P}} d_p(p, \mathcal{P}^*), \sup_{q \in \mathcal{P}^*} d_p(q, \mathcal{P})\}. \quad [43]$$

Example 3.2.3.

Let $\mathcal{M} = \{0, 1, 4\}$ with partial metric $d_p : \mathcal{M} \times M \to \mathbb{R}^+$, defined as

$$d_p(\zeta, \zeta^*) = \frac{1}{4}|\zeta - \zeta^*| + \frac{1}{2}\max\{\zeta, \zeta^*\},$$

t can be verified easily that $(\mathcal{M}, d_p)$ is a PMS.

Now consider two closed and bounded subset in $\mathcal{M}$,$\mathcal{P} = \{0\}$ and $\mathcal{P}^* = \{0, 1\}$.

Next, to calculate the partial Hausdorff distance between P and Q as,

$$\begin{aligned}
\mathcal{H}(\mathcal{P}, \mathcal{P}^*) &= \max\{\sup_{p \in \mathcal{P}} d_p(p, \mathcal{P}^*), \sup_{q \in \mathcal{P}^*} d_p(q, \mathcal{P})\} \\
&= \max\{\sup_{p \in \mathcal{P}} d_p(p, \{0, 1\}), \sup_{q \in \mathcal{P}^*} d_p(q, \{0\})\} \\
&= \max\{\inf(d_p(\{0\}, \{0, 1\})), \sup(d_p(\{0, 1\}, \{0\}))\} \\
&= \max\{\frac{3}{4}, \frac{3}{4}\} \\
&= \frac{3}{4}.
\end{aligned} \tag{3.2}$$

Example 3.2.4.

Let $M = \{0, 1, 4\}$ with $d_p : M \times M \to \mathbb{R}^+$ defined as,

$$\begin{aligned}
&d_p(0,0) = 0 = d_p(1,1), d_p(4,4) = \frac{1}{4} \\
&d_p(0,1) = d_p(1,0) = \frac{1}{4} \\
&d_p(0,4) = d_p(4,0) = \frac{1}{8} \\
&d_p(4,1) = d_p(1,4) = \frac{1}{16}.
\end{aligned}$$

Consider two closed bounded subsets of $P = \{0\}$ and $Q = \{0, 1\}$, then

$$\begin{aligned}
\mathcal{H}_p(\mathcal{P}, \mathcal{P}^*) &= \max\{\sup_{p \in \mathcal{P}} d_p(p, \mathcal{P}^*), \sup_{q \in \mathcal{P}^*} d_p(q, \mathcal{P})\} \\
&= \max\{\sup_{p \in \mathcal{P}} d_p(p, \{0, 1\}), \sup_{q \in \mathcal{P}^*} d_p(q, \{0\})\} \\
&= \max\{\inf(d_p(\{0\}, \{0, 1\})), \sup(d_p(\{0, 1\}, \{0\}))\} \\
&= \frac{1}{4}.
\end{aligned}$$

Theorem 3.2.5. *Every Hausdorff metric is a partial Hausdorff metric.*

Proof. Let $(\mathcal{M}, \check{d})$ be metric space and $CB(\mathcal{M})$ be collection of all closed and bounded subsets of $\mathcal{M}$. Now for $\mathcal{P}, \mathcal{P}^* \in CB(\mathcal{M})$ we define a Hausdorff metric function $\mathcal{H} : CB_p(\mathcal{M}) \times CB_p(\mathcal{M}) \to \mathbb{R}^+$, as

$$\mathcal{H}(\mathcal{P}, \mathcal{P}^*) = \max\{\sup_{p \in \mathcal{P}} \check{d}(p, \mathcal{P}^*), \sup_{q \in \mathcal{P}^*} \check{d}(q, \mathcal{P})\},$$

then followings conditions are satisfied by $\mathcal{H}$;

P1: $\mathcal{H}(\mathcal{P}, \mathcal{P}^*) \geq \mathcal{H}(\mathcal{P}, \mathcal{P})$.

P2: $\mathcal{H}(\mathcal{P}, \mathcal{P}^*) = \mathcal{H}(\mathcal{P}^*, \mathcal{P})$.

P3: For $\mathcal{P}, \mathcal{P}^*, R \in CB_p(M)$ we have

$$\mathcal{H}(\mathcal{P}, \mathcal{P}^*) \leq \mathcal{H}(\mathcal{P}, \mathcal{R}) + \mathcal{H}(\mathcal{R}, \mathcal{P}^*) - \inf_{r \in \mathcal{R}} \check{d}(r, r),$$

showing that $(\mathcal{H}, CB(\mathcal{M}))$ is a partial metric space. □

The converse of this result generally doesn't hold.

Example 3.2.6.

Consider $\mathcal{M} = [0, 1]$ equipped with partial metric $d_p : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ defined as, $d_p(\zeta, \eta) = \max(\zeta, \eta)$, it is easy to see that $\mathcal{M}$ is a closed and bounded set, so calculating partial Hausdorff distance. We have

$$\begin{aligned}\mathcal{H}_p(\mathcal{M}, \mathcal{M}) &= \sup_{m \in \mathcal{M}} d_p(m, \mathcal{M}) \\ &= 1 \\ &\neq 0.\end{aligned}$$

Showing that $(\mathcal{H}_p, \mathcal{M})$ is not a Hausdorff metric space.

Lemma 3.2.7.

Let $(\mathcal{M}, d_p)$ be a PMS, consider two non empty subsets $\mathcal{P}, \mathcal{S} \in CB_p(\mathcal{M})$ and $k > 1$. For any $p \in \mathcal{P}$, there exists $q \in \mathcal{S}$. Such that

$$d_p(p, q) \leq k\mathcal{H}_p(\mathcal{P}, \mathcal{S}). \quad [43]$$

3.3 Multivalued $\mathcal{F}$-contraction Mapping.

This section aims to illustrate basic definitions related to multivalued $\mathcal{F}$-contraction mapping and to highlight this idea's importance with the help of examples. We will present some related fixed point results also.

Definition 3.3.1. Multivalued Mapping.

Let $\mathcal{S}$ and N be two non empty sets, f is known as multivalued mapping from set $\mathcal{S}$ into $P(N)$ if each element in $\mathcal{S}$ corresponds to any subset $f(s)$ of the set N. [9]

If for each s in $\mathcal{S}$ the set $f(s)$ consists of one element, then f is called single-valued mapping.

Unless otherwise stated, it is always assumed $f(s)$ is nonempty for each $s \in \mathcal{S}$.

Let's have look on some examples;

Example 3.3.2. (i) Consider a map $f : \mathcal{M} \to \mathcal{N}$ which is not one-one. Then the inverse image of f would be a multivalued map.

(ii) Let $\mathcal{M} = \mathbb{R}$ and $N = [1, 10] \subset \mathcal{M}$ we define a multivalued mapping $f : N \to P(N)$ as follow

$$f(\xi) = \{\xi\} \quad \text{for all} \quad \xi \in N.$$

(iii) Let $\mathcal{M} = \mathbb{R}$ and consider any arbitrary c in $\mathcal{M}$, we may define a multivalued map from $\mathcal{M}$ into $N \subset \mathcal{M}$ as follow,

$$fm = \pm\sqrt{c - m^2}$$

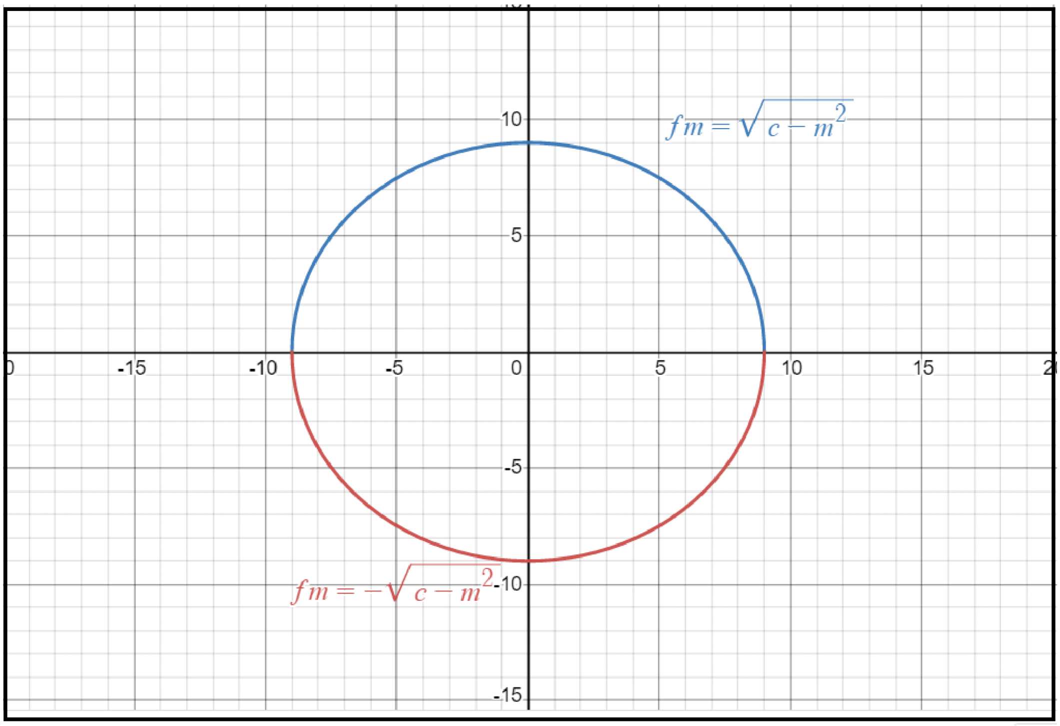

FIGURE 3.1: *Multivalued Map*

(iv) Consider a set $M = [0, 1]$ and let $N \subset M$, we can define a multi-valued map $f : M \to N$ as,

$$fm = \begin{cases} [0, 0.7] & \text{if } m \neq 0.7 \\ (0.7, 1) & \text{if } m = 0.7 \\ 1 & \text{otherwise.} \end{cases}$$

In 1969, Nadler [9] merged the idea of multivalued mapping in the setting of the fixed point results and gave a very strong generalization of BCP under multivalued contraction mappings.

Definition 3.3.3. Fixed Point of Multivalued Mapping

Consider $\mathcal{S}$ and N be two non empty sets and $f : \mathcal{S} \to P(N)$ be a multivalued mapping . Then $s \in \mathcal{S}$ is said to be a fixed point of f if $s \in f(s)$.

Definition 3.3.4. Multivalued Contraction Mapping.

A function $\mathcal{T} : \mathcal{M} \to CB(\mathcal{N})$ is called multivalued contraction mapping of $\mathcal{M}$ into $\mathcal{N}$ if and only if

$$\mathcal{H}(\mathcal{T}p, \mathcal{T}q) \leq \lambda d(p, q), \tag{3.3}$$

for all p, q in $\mathcal{M}$. here $\lambda \in [0, 1)$. Here $CB(\mathcal{N})$ is a collection of all closed and bounded subsets of $\mathcal{N}$ and $\mathcal{H}$ is Hausdorff distance. [9]

Example 3.3.5.

Let $\mathcal{M} = \mathbb{R}$ and $\mathbf{B} = [0, 1] \subseteq \mathbb{R}$ equipped with usual metric and let $\mathfrak{f} : \mathbf{B} \to \mathbf{B}$ is given by

$$\mathfrak{f}(m) = \begin{cases} 0.5m + 0.5, & \text{if } m \in [0, 0.5] \\ -0.5m + 1, & \text{if } m \in [0.5, 1]. \end{cases}$$

$\mathbb{F} : \mathcal{M} \to 2^{\mathcal{M}}$ defined as

$$\mathbb{F}(m) = \{\mathfrak{f}(m)\} \cup \{0\},$$

is a multivalued contraction mapping.

Theorem 3.3.6.

Let $(\mathcal{M}, \tilde{d})$ be a complete metric space and $\mathbb{S} : \mathcal{M} \to CB(\mathcal{M})$ be a multivalued contraction mapping, then $\mathbb{S}$ has a fixed point $c \in \mathcal{M}$ such that $c \in \mathbb{S}(c)$. [9]

A new concept was highlighted by Wardowski [51] in 2012, by introducing Δ_f-family. Which was a stronger idea extending the strictly increasing mappings. He generalized BCP in this new domain too.

Firstly, we define the characteristics of members of this family.

Definition 3.3.7. Δ_f-family.

A Mapping $\mathcal{F}$ from $\mathbb{R}^+$ to $\mathbb{R}$ is member of $\Delta_f - family$ if $\mathcal{F}$ satisfies these properties;

(F_1): $\mathcal{F}$ is strictly increasing, i.e.,

$$\zeta < \eta \implies \mathcal{F}(\zeta) < \mathcal{F}(\eta), \text{ for all } \quad \zeta, \eta \in \mathbb{R}.$$

(F_2): For every positive term sequence $\{a_\eta : \eta \in \mathbb{N}\}$,

$$\lim_{n\to\infty} a_\eta = 0 \quad \iff \quad \lim_{n\to\infty} \mathcal{F}(a_\eta) = -\infty.$$

(F_3): If we have a $c \in (0,1)$ then,

$$\lim_{\eta\to 0^+} \eta^c \mathcal{F}(\eta) = 0. \quad [51].$$

Example 3.3.8.

(i) Let $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ be defined as

$$\mathcal{F}(p) = \ln(p),$$

one can check easily that $\mathcal{F}$ is a member of Δ_f-family.

(F_1): is clearly satisfied as $\ln(p)$ is a strictly increasing function.

(F_2): we can a sequence with positive terms i.e

$$\{a_k = \frac{1}{k} : k \in \mathbb{N}\}$$

one can see easily $\lim\{a_k\} \longrightarrow 0$ for $k \longrightarrow \infty$. Also

$$\lim_{k\to\infty} \mathcal{F}(a_k) = \lim_{k\to\infty} \ln(1/k) = -\lim_{k\to\infty} \ln(k) = -\infty.$$

(F_3): Setting $c = 0.5 \in (0,1)$ and for $p = 1$.

We have,

$$\lim_{p\to 1^+} 1^{0.5} \mathcal{F}(p)$$

$$\implies 1^{0.5} \ln(1) = 0.$$

FIGURE 3.2: *A non-decreasing map ln(p)*

(ii) Consider another map, $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ defined as

$$\mathcal{F}(\xi) = \frac{-1}{\sqrt{\xi}},$$

it is easy to see that this is also a member of $\Delta_f - family$.

Definition 3.3.9. $\mathcal{F}$-contraction.

In a metric space $(\mathcal{M}, d)$ the map $\mathbb{S} : \mathcal{M} \to \mathcal{M}$ is known as $\mathcal{F}$-contraction on $\mathcal{M}$, if for all m, n in $\mathcal{M}$ we have

$$d(\mathbb{S}m, \mathbb{S}n) > 0 \implies \tau + \mathcal{F}(d(\mathbb{S}m, \mathbb{S}n)) \leq \mathcal{F}(d(m, n)), \tag{3.4}$$

where $\mathcal{F} \in \Delta_f$-family and $\tau > 0$. [51]

Theorem 3.3.10.

Consider complete metric space $(\mathcal{M}, d)$, and let $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be an $\mathcal{F}$-contraction. Then $\mathcal{T}$ has a unique fixed point $m^* \in \mathcal{M}$ and for every $m_0 \in \mathcal{M}$ a sequence $\mathcal{T}^n(m_0)$ is convergent to m^*. [51]

Definition 3.3.11. Multivalued $\mathcal{F}$-contraction Mapping.

Consider $(\mathcal{M}, d_p)$ be a PMS and a map $\mathbb{S} : \mathcal{M} \to CB_p(\mathcal{M})$, then $\mathbb{S}$ is called MV$\mathcal{F}$-CM, if there is $\mathcal{F} \in \Delta_f - family$ such that

$$\mathcal{H}_p(\mathbb{S}m, \mathbb{S}n) > 0$$
$$\implies \tau + \mathcal{F}(\mathcal{H}_p(\mathbb{S}m, \mathbb{S}n)) \leq \mathcal{F}(d_p(m, n)). \tag{3.5}$$

where $\tau > 0$. [41]

By integrating the concept of multivalued contraction condition into the $\mathcal{F}$ mappings, Altun et al [52] presented following results in 2015.

Theorem 3.3.12.

Consider a complete metric space $(\mathcal{M}, d)$ and $\mathcal{T} : \mathcal{M} \to \mathcal{K}$ be a multivalued $\mathcal{F}$-contraction mapping, then $\mathcal{T}$ has a fixed point, where $\mathcal{K}$ is any compact subset of $\mathcal{M}$.

Theorem 3.3.13.

Consider a complete metric space $(\mathcal{M}, \tilde{d})$ and let $\mathcal{F} : \mathcal{M} \to CB(\mathcal{M})$ be a set-valued $\mathcal{F}$-contraction map satisfying

$$\mathcal{F}(\inf P) = \inf(\mathcal{F}(P)), \tag{3.6}$$

for all $P \subset \mathbb{R}^+$ and $inf(P) > 0$, then f has a fixed point

3.4 Fixed Point Theorems with $\mathcal{F}$-contraction mapping in PMS.

Two strong theorems and some useful examples taken from [41], are presented here.

An application is also given which emphasizes the importance of these theorems in PMSs.

Theorem 3.4.1.

Let $(\mathcal{M}, d_p)$ be a complete PMS. If $S : \mathcal{M} \to \mathbb{K}(\mathcal{M})$ be a multivalued $\mathcal{F}$-contraction, then S has a fixed point.

Proof. We take an arbitrary $m_0 \in \mathcal{M}$, as Sm being set of all images of $m \in \mathcal{M}$ is non empty for all values in $\mathcal{M}$, we can select $m_1 \in Sm_0$.
If

$$m_1 \in Sm_1,$$

this means that m_1 is the fixed point of S trivially.

Suppose

$$m_1 \notin Sm_1,$$

as Sm_1 is closed so we have

$$d_p(m_1, Sm_1) > 0.$$

Also we know that

$$d_p(m_1, Sm_1) \leq \mathcal{H}_p(Sm_0, Sm_1). \tag{3.7}$$

Now let $\mathcal{F} \in \Delta_f - family$, and as $\mathcal{F}$ so by its property of being non-decreasing, we have

$$\mathcal{F}(d_p(m_1, Sm_1)) \leq \mathcal{F}(\mathcal{H}_p(Sm_0, Sm_1)), \tag{3.8}$$

from the contractive condition

$$\mathcal{F}(d_p(m_1, Sm_1)) \leq \mathcal{F}(\mathcal{H}_p(Sm_0, Sm_1)) \leq \mathcal{F}(d_p(m_0, m_1)) - \tau. \tag{3.9}$$

As Sm_1 is compact so there exists $m_2 \in Sm_1$, such that

$$d_p(m_1, m_2) = d_p(m_1, Sm_1),$$

$$\mathcal{F}(d_p(m_1, m_2)) \leq \mathcal{F}(\mathcal{H}_p(Sm_0, Sm_1)) \leq \mathcal{F}(d_p(m_1, m_0)) - \tau. \tag{3.10}$$

Following the same procedure repeatedly, we have $\{m_\eta\} \in \mathcal{M}$ such that $m_{\eta+1} \in Sm_\eta$, with

$$\mathcal{F}(d_p(m_\eta, m_{\eta+1}) \leq \mathcal{F}(d_p(m_\eta, m_{\eta-1})) - \tau \qquad for \quad all \quad \eta \in \mathbb{N}. \tag{3.11}$$

If there exists $\eta_0 \in \mathbb{N}$ for which we have $m_{\eta_0} \in Sm_{\eta_0}$, then m_{η_0} will be the fixed point S trivially. So set $m_{\eta_0} \notin Sm_{\eta_0}$ for every $\eta \in \mathbb{N}$.

For convenience we are setting $v_\eta = d_p(m_\eta, m_{\eta+1})$ where $\eta = 0, 1, 2.....$ Clearly $v_\eta > 0$ for all $\eta \in \mathbb{N}$. Now, substituting these in the above equation, we have

$$\begin{aligned} \mathcal{F}(v_\eta) &\leq \mathcal{F}(v_{\eta-1}) - \tau \\ &\leq \mathcal{F}(v_{\eta-2}) - 2\tau \leq \\ &. \\ &. \\ &.. \\ &\leq \mathcal{F}(v_0) - \eta\tau, \end{aligned} \tag{3.12}$$

$$\implies \lim_{\eta\to\infty} \mathcal{F}(v_\eta) = -\infty.$$

Then by 2nd property of $\mathcal{F} \in \Delta_f - family$, we have

$$\lim_{\eta\to\infty} v_\eta = 0,$$

then by using 3rd property of $\mathcal{F} \in \Delta_f - family$ we have $\alpha \in (0, 1)$ such that

$$\lim_{\eta\to\infty} {v_\eta}^\alpha \mathcal{F}(v_\eta) = 0,$$

for all $\eta \in \mathbb{N}$, we can write

$${v_\eta}^\alpha(\mathcal{F}(v_\eta) - \mathcal{F}(v_0)) \leq -{v_\eta}^\alpha \eta\tau \leq 0, \tag{3.13}$$

for higher values of η we have

$${v_\eta}^\alpha \eta = 0.$$

So there exists a $\eta_1 \in \mathbb{N}$ such that,

$$v_\eta{}^\alpha \eta \leq 1$$

for all $\eta \geq \eta_1$,

$$v_\eta \leq \frac{1}{\eta^{\frac{1}{\alpha}}}.$$

Now we will prove that $\{m_\eta\}$ is a Cauchy sequence in $\mathcal{M}$. For this let $\eta, l \in \mathbb{N}$ provided that $\eta > l \geq \eta_1$,

consider generalized triangular inequality of PMS

$$\begin{aligned}
d_p(m_\eta, m_l) &\leq d_p(m_\eta, m_{\eta+1}) + d_p(m_{\eta+1}, m_{\eta+2}) + \ldots.. + d_p(m_{l-1}, m_l) - \sum_{j=\eta+1}^{l-1} d_p(m_j, m_j) \\
&\leq d_p(m_\eta, m_{\eta+1}) + d_p(m_{\eta+1}, m_{\eta+2}) + \ldots.. + d_p(m_{l-1}, m_l) \\
&= v_\eta + v_{\eta+1} + v_{\eta+2} + \ldots\ldots v_{l-1} \\
&= \sum_{\beta=\eta}^{l-1} v_\beta \\
&\leq \sum_{\beta=\eta}^{\infty} v_\beta \\
&\leq \sum_{\beta=\eta}^{\infty} \frac{1}{\beta^{\frac{1}{\alpha}}}.
\end{aligned}$$

Which is a convergent series showing that

$$\lim_{\eta \to \infty} d_p(m_\eta, m_l) = 0.$$

Now using (3.1) for any $\eta, l \in \mathbb{N}$, we have

$$p^s(m_\eta, m_l) = 2d_p(m_\eta, m_l) - d_p(m_\eta, m_\eta) - d_p(m_l, m_l) \leq 2d_p(m_\eta, m_l) \longrightarrow 0,$$

for $n \longrightarrow \infty$.

Which shows $\{m_\eta\}$ is Cauchy sequence w.r.t $(\mathcal{M}, d_p^s)$ and hence convergent in $(\mathcal{M}, d_p)$, so there exists $m^* \in \mathcal{M}$ such that

$$\lim_{\eta \to \infty} p^s(m_\eta, m^*) = 0.$$

Now for all $m, n \in \mathcal{M}$ with $\mathcal{H}_p(Sm, Sn) > 0$, We have

$$\mathcal{H}_p(Sm, Sn) < d_p(m, n).$$

Thus

$$d_p(m_{\eta+1}, Sm^*) \leq \mathcal{H}_p(Sm_\eta, Sm^*) \leq d_p(m_\eta, m^*), \tag{3.14}$$

it can be observed for high values of η, we will get

$$d_p(m^*, Sm^*) = 0.$$

This gives $m^* \in \bar{S}m^* = Sm^*$.
Proving that m^* if the fixed point of S.

□

Theorem 3.4.2.

Let $(\mathcal{M}, d_p)$ be a complete PMS and let $S : \mathcal{M} \to CB(\mathcal{M})$ be a multivalued $\mathcal{F}$-contraction mapping where $B \subset (0, \infty)$ with $\inf B > 0$ if $\mathcal{F}$ satisfies

$$\mathcal{F}(\inf B) = \inf \mathcal{F}(B), \tag{3.15}$$

then S has a fixed point.

Proof. We take an arbitrary $m_0 \in \mathcal{M}$, as Sm being set of all images of $m \in \mathcal{M}$, is non empty for all elements in $\mathcal{M}$, we can choose $m_1 \in Sm_0$. If that $m_1 \in Sm_1$ this means that m_1 is the fixed point of S trivially.
Suppose $m_1 \notin Sm_1$, as m_1 is closed so we have

$$d_p(m_1, Sm_1) > 0.$$

As we know that

$$d_p(m_1, Sm_1) \leq \mathcal{H}_p(Sm_0, Sm_1).$$

Now let $\mathcal{F} \in \Delta_f - family$, so by its property of being non-decreasing, we have

$$\mathcal{F}(d_p(m_1, Sm_1)) \leq \mathcal{F}(\mathcal{H}_p(Sm_0, Sm_1)),$$

from the contractive condition,

$$\mathcal{F}(d_p(m_1, Sm_1)) \leq \mathcal{F}(\mathcal{H}_p(Sm_0, Sm_1)) \leq \mathcal{F}(d_p(m_0, m_1)) - \tau.$$

Using (3.15)

$$\mathcal{F}\inf(d_p(m_1, Sm_1)) = \inf_{g \in Sm_1} \mathcal{F}(d_p(m_1, g)).$$

We may write as,

$$\inf_{g \in Sm_1} \mathcal{F}(d_p(m_1, g)) \leq \mathcal{F}(d_p(m_1, m_0)) - \tau \leq \mathcal{F}(d_p(m_1, m_0)) - \frac{\tau}{2}.$$

Now for some $m_2 \in Sm_1$, such that

$$\mathcal{F}(d_p(m_1, m_2)) \leq \mathcal{F}(d_p(m_1, m_0)) - \frac{\tau}{2}.$$

If $m_2 \in Sm_2$, then we have nothing to prove, so set $m_3 \in Sm_2$, we will get

$$\mathcal{F}(d_p(m_2, m_3)) \leq \mathcal{F}(d_p(m_2, m_1)) - \frac{\tau}{2}.$$

Continuing in same manner we have a sequence $\{m_\eta\}$ in $\mathcal{M}$ such that

$$m_{\eta+1} \in Sm_\eta,$$

and for all $\eta \in \mathbb{N}$, we have

$$\mathcal{F}(d_p(m_\eta, m_{\eta+1})) \leq \mathcal{F}(d_p(m_\eta, m_{\eta-1})) - \frac{\tau}{2}.$$

If there exists $\eta_0 \in \mathbb{N}$ for which we have $m_{\eta_0} \in Sm_{\eta_0}$, then m_{η_0} will be the fixed point S trivially, so let

$$m_{\eta_0} \notin Sm_{\eta_0}$$

for every $\eta \in \mathbb{N}$.

For the convenience we are setting

$$w_\eta = d_p(m_\eta, m_{\eta+1})$$

where $\eta = 0, 1, 2.....$ Clearly $w_\eta > 0$ for all $\eta \in \mathbb{N}$.

Now substituting these in above equation, we have

$$\begin{aligned} \mathcal{F}(w_\eta) &\leq \mathcal{F}(w_{\eta-1}) - \frac{\tau}{2} \\ &\leq \mathcal{F}(w_{\eta-2}) - \frac{2\tau}{2} \\ &. \\ &. \\ &\leq \mathcal{F}(w_0) - \frac{\eta\tau}{2}. \end{aligned}$$

$$\implies \lim_{n\to\infty} \mathcal{F}(w_\eta) = -\infty.$$

Then by 2nd property of $\mathcal{F} \in \Delta_f - family$, we have

$$\lim_{\eta\to\infty} w_\eta = 0,$$

then by using 3rd property of $\mathcal{F} \in \Delta_f - family$ we have $\beta \in (0, 1)$ such that

$$\lim_{\eta\to\infty} {w_\eta}^\beta \mathcal{F}(w_\eta) = 0,$$

for all $\eta \in \mathbb{N}$.

$$\begin{aligned} {w_\eta}^\beta(\mathcal{F}(w_\eta) - \mathcal{F}(w_0)) &\leq -{w_\eta}^\beta \frac{\eta\tau}{2} \\ &\leq 0 \end{aligned}$$

for higher values of η we have

$${w_\eta}^\beta \eta = 0$$

So there exists a $\eta_1 \in \mathbb{N}$ such that,

$${w_\eta}^\beta \eta \leq 1$$

for all $\eta \geq \eta_1$, So

$$w_\eta \leq \frac{1}{\eta^{\frac{1}{\beta}}}.$$

Now we will prove that $\{m_\eta\}$ is a Cauchy sequence in $\mathcal{M}$.

For this let $\eta, l \in \mathbb{N}$ provided that

$$\eta > l \geq \eta_1.$$

Consider the generalized triangular inequality of PMS,

$$\begin{aligned}
d_p(m_\eta, m_l) &\leq d_p(m_\eta, m_{\eta+1}) + d_p(m_{\eta+1}, m_{\eta+2}) + ... + d_p(m_{l-1}, m_l) - \sum_{j=\eta+1}^{l-1} d_p(m_j, m_j) \\
&\leq d_p(m_\eta, m_{\eta+1}) + d_p(m_{\eta+1}, m_{\eta+2}) + ... + d_p(m_{l-1}, m_l) \\
&= w_\eta + w_{\eta+1} + w_{\eta+2} + ...w_{l-1} \\
&= \sum_{\beta=\eta}^{l-1} w_\gamma \\
&\leq \sum_{\beta=i}^{\infty} w_\gamma \\
&\leq \sum_{\beta=\eta}^{\infty} \frac{1}{\gamma^{\frac{1}{\beta}}}.
\end{aligned}$$

Which is a convergent series, showing that

$$\lim_{\eta\to\infty} d_p(m_\eta, m_l) = 0.$$

Now using (3.1) for any $\eta, l \in \mathbb{N}$, we have

$$p^s(m_\eta, m_l) = 2d_p(m_\eta, m_l) - d_p(m_\eta, m_\eta) - d_p(m_l, m_l) \leq 2d_p(m_\eta, m_l) \longrightarrow 0,$$

$$p^s(m_\eta, m_l) \longrightarrow 0,$$

for $n \longrightarrow \infty$.

Which shows $\{m_\eta\}$ is Cauchy sequence w.r.t $(\mathcal{M}, d_p^s)$ and hence convergent in $(\mathcal{M}, d_p)$.

So there exists $u^* \in \mathcal{M}$, such that

$$d_p(u^*, u^*) = \lim_{\eta\to\infty} d_p(m_\eta, u^*) = \lim_{\eta,l\to\infty} d_p(m_\eta, m_l).$$

Now we will prove that u^* is the fixed point of S. As, we have

$$\tau + \mathcal{F}(\mathcal{H}_p(Sm_\eta, Su^*)) \leq \mathcal{F}(d_p(m_\eta, u^*)).$$

We get $\lim_{n\to\infty} \mathcal{H}_p(Sm_\eta, Su^*) = 0$, from $m_{\eta+1} \in Sm_\eta$,

$$d_p(m_{\eta+1}, Su^*) \leq d_p(Sm_\eta, Su^*) \leq \mathcal{H}_p(Sm_\eta, Su^*).$$

Showing that

$$\lim_{\eta\to\infty}(Sm_{\eta+1}, Su^*)$$

.

By using the triangular inequality of PMS, we have

$$d_p(u^*, Su^*) \leq d_p(u^*, m_{\eta+1}) + d_p(m_{\eta+1}, Su^*) - d_p(m_{\eta+1}, m_{\eta+1}).$$

For $n \longrightarrow \infty$, we observe

$$d_p(u^*, Su^*) \longrightarrow 0$$

. This gives

$$u^* \in \bar{S}u^* = Su^*.$$

Proving that u^* is the fixed point of S. □

We will apply the above result to the following examples to illustrate its importance.

Through these examples one can easily understand the existence of fixed points with help of above defined results.

Example 3.4.3. Let $M = \{0, 1, 2, 3\}$ be our ground set, we define a metric $d_p : M \times M \to \mathbb{R}^+$ as

$$d_p(\zeta, \varrho^*) = \frac{1}{4}|\zeta - \varrho^*| + \frac{1}{2}\max\{\zeta, \varrho^*\},$$

firstly we will show that above defined metric is a partial metric on M for all $\zeta, \varrho^* \in M$.

P1: Let $\zeta \geq \varrho^*$ then

$$\begin{aligned} d_p(\zeta, \varrho^*) &= \frac{1}{4}|\zeta - \varrho^*| + \frac{1}{2}\max\{\zeta, \varrho^*\} \\ &= \frac{1}{4}|\zeta - \varrho^*| + \frac{1}{2}\zeta \\ &\geq \frac{1}{4}|\zeta - \varrho^*| + \frac{1}{2}\varrho^* \\ &= d_p(\varrho^*, \varrho^*). \end{aligned}$$

$$\implies d_p(\zeta, \varrho^*) \geq d_p(\varrho^*, \varrho^*),$$

i.e., self distance is less than the distance of two distinct elements in M.

P2: Consider $\zeta = \varrho^*$, then

$$\begin{aligned} d_p(\zeta, \varrho^*) &= \frac{1}{4}|\zeta - \varrho^*| + \frac{1}{2}\max\{\zeta, \varrho^*\} \\ &= \frac{1}{4}|\zeta - \zeta| + \frac{1}{2}\zeta \\ &= \frac{1}{2}\zeta \\ &= \frac{1}{2}\varrho^* \\ &= \frac{1}{4}|\varrho^* - \varrho^*| + \frac{1}{2}\varrho^* \\ &= d_p(\varrho^*, \varrho^*) \\ &= d_p(\zeta, \zeta). \end{aligned}$$

Conversely, setting

$$d_p(\zeta, \zeta) = d_p(\varrho^*, \varrho^*) = d_p(\zeta, \varrho^*),$$

by comparing the first two, we will easily get

$$\zeta = \varrho^*$$

so comparing the other two, we have

$$\frac{1}{4}|\zeta - \varrho^*| + \frac{1}{2}\max\{\zeta, \varrho^*\} = \frac{1}{4}|\varrho^* - \varrho^*| + \frac{1}{2}\max\{\varrho^*, \varrho^*\}.$$

On the contrary, suppose that

$$\zeta \neq \varrho^*$$

this means either $\zeta < \varrho^*$ or $\zeta > \varrho^*$, considering the first case,

$$\frac{1}{4}|\zeta - \varrho^*| + \frac{1}{2}\max\{\zeta, \varrho^*\} = \frac{1}{2}\varrho^*,$$

or

$$\frac{1}{4}|\zeta - \varrho^*| + \frac{1}{2}\varrho^* = \frac{1}{2}\varrho^*.$$

$$\implies |\zeta - \varrho^*| = 0.$$

Which shows

$$\zeta = \varrho^*,$$

which is a contradiction.

We will get the same for the second possible case. So we have

$$d_p(\zeta, \zeta) = d_p(\varrho^*, \varrho^*) = d_p(\zeta, \varrho^*) \iff \zeta = \varrho^*$$

for all values on M.

P3: Clearly, symmetry follows because of the symmetric property of maximum function and the absolute value of elements.

P4: Let $\zeta, \varrho^*, \gamma \in M$ having a relationship $\zeta \geq \gamma \geq \varrho^*$.

Consider

$$\begin{aligned}
d_p(\zeta, \varrho^*) &= \frac{1}{4}|\zeta - \varrho^*| + \frac{1}{2}\max\{\zeta, \varrho^*\} \\
&= \frac{1}{4}|\zeta - \varrho^*| + \frac{1}{2}\zeta \\
&= \frac{1}{4}|\zeta - \gamma + \gamma - \varrho^*| + \frac{1}{2}\zeta \\
&\leq \frac{1}{4}|\zeta - \gamma| + \frac{1}{4}|\gamma - \varrho^*| + \frac{1}{2}\zeta \\
&= \frac{1}{4}|\zeta - \gamma| + \frac{1}{4}|\gamma - \varrho^*| + \frac{1}{2}\zeta + \frac{1}{2}\gamma - \frac{1}{2}\gamma \\
&= d_p(\zeta, \gamma) + d_p(\gamma, \varrho^*) - d_p(\gamma, \gamma),
\end{aligned}$$

so for all $\zeta, \varrho^*, \gamma \in M$.

We have

$$d_p(\zeta, \varrho^*) \leq d_p(\zeta, \gamma) + d_p(\gamma, \varrho^*) - d_p(\gamma, \gamma).$$

Showing that (M, d_p) is a PMS.

Now, we define a multivalued mapping $S : M \to CB^p(M)$

$$Sm = \begin{cases} \{2,3\} & \text{if } m = 3 \\ \{3\} & \text{if } m \neq 3. \end{cases}$$

Also we need to define a $\mathcal{F}$ map.

So consider a function $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ as

$$\mathcal{F}(\alpha) = \ln(\alpha),$$

for $\alpha \in \mathbb{R}^+$.

Next, to show that $\mathcal{F}$ satisfies the condition of multivalued $\mathcal{F}$-contraction. For this consider $m \neq 3$ and $\tau = \ln\left(\frac{4}{35}\right)$, we have

$$\begin{aligned}
\tau + \mathcal{F}(\mathcal{H}_p(Sm, S3)) &= \tau + \ln(\mathcal{H}_p(Sm, S3) \\
&= \tau + \ln\big(\max\{\inf d_p(\{3\}, \{2,3\}), \sup d_p(\{2,3\}, \{3\})\}\big) \\
&= \tau + \ln\big(\max\{\inf\{d_p\{(3,2), d_p(3,3)\}, \sup\{d_p\{(3,2), d_p(3,3)\}\}\big) \\
&= \tau + \ln\big(\max\{\inf\{\frac{7}{4}, \frac{3}{2}\}, \sup\{\frac{7}{4}, \frac{3}{2}\}\}\big) \\
&= \tau + \ln\big(\max\{\frac{3}{2}, \frac{7}{4}\}\big) \\
&= \tau + \ln\big(\frac{7}{4}\big) \\
&= \ln\big(\frac{4}{35}\big) + \ln\big(\frac{7}{4}\big) \\
&= \ln\big(\frac{1}{5}\big) \\
&\leq \ln\big(\frac{1}{4}\big) \\
&\leq \ln\big(\frac{1}{4}|m-3| + \frac{3}{2}\big) \\
&= \ln d_p(m, 3) \\
&= \mathcal{F} d_p(m, 3),
\end{aligned}$$

showing that S satisfies the multivalued $\mathcal{F}$-contraction condition.

Hence $m = 3$ is a fixed point of S.

Example 3.4.4.

Consider a complete PMS (M, d_p), where

$$M = \left\{ m_k = 1 - \left(\frac{1}{2}\right)^k \quad : k \in \mathbb{N} \right\},$$

and metric function is defined as

$$d_p(\zeta, \eta) = |\zeta - \eta|.$$

We define a multivalued mapping $S : M \to CB_p(M)$ as

$$Sm = \begin{cases} \{m_1\} & \text{if } m = m_1 \\ \{m_\eta, m_{\eta-1}\} & \text{if } m = m_\eta. \end{cases}$$

Now define a $\mathcal{F}$ map, as $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$

$$\mathcal{F}(\alpha) = \ln(\alpha),$$

for $\alpha \in \mathbb{R}^+$.

Next to show that $\mathcal{F}$ satisfies condition of multivalued $\mathcal{F}$-contraction, i.e. if

$$\mathcal{H}_p(Sm, Sn) > 0$$

$$\begin{aligned} \implies \tau + \mathcal{F}(\mathcal{H}_p(Sm, Sn)) &\leq \mathcal{F}(d_p(m, n)) \\ \tau + \ln(\mathcal{H}_p(Sm, Sn)) &\leq \ln(d_p(m, n)) \\ \mathcal{H}_p(Sm, Sn) &\leq d_p(m, n)e^{-\tau}. \end{aligned}$$

We have the following two cases;

Case:-I For $\eta, l \in \mathbb{N}$ with $\eta = 1, l > \eta + 1$ and $\mathcal{H}_p(Sm_\eta, Sm_l) > 0$,

$$\mathcal{H}_p(Sm_\eta, Sm_1) \leq d_p(m_\eta, m_1)e^{-\tau},$$

then by calculating the Hausdorff distance, we have

$$\begin{aligned}\mathcal{H}_p(Sm_\eta, Sm_1) &= \max\{\sup d_p(\{m_\eta, m_{\eta-1}\}, \{m_1\}), \inf d_p(\{m_\eta, m_{\eta-1}\}, \{m_1\})\} \\ &= \max\{\sup\{|m_\eta - m_1|, |m_{\eta-1}, m_1|\}, \inf\{|m_\eta - m_1|, |m_{\eta-1}, m_1|\} \\ &= \max\{|m_\eta - m_1|, |m_{\eta-1}, m_1|\} \\ &= |m_{\eta-1} - m_1|.\end{aligned}$$

Setting back,

$$|m_{\eta-1} - m_1| \le d_p(m_\eta, m_1)e^{-\tau},$$

or

$$|m_{\eta-1} - m_1| \le |m_\eta - m_1|e^{-\tau}.$$

It follows that for $\eta \in \mathbb{N}$

$$1 - (\frac{1}{2})^{\eta-1} \le \left(1 - (\frac{1}{2})^\eta\right)e^{-\tau}.$$

Case:-II For $\eta, l \in \mathbb{N}$, provided that

$$\eta > l > 1$$

and

$$\mathcal{H}_p(Sm_\eta, Sm_l) > 0$$

,

$$\mathcal{H}_p(Sm_{\eta-1}, Sm_{l-1}) \le d_p(m_\eta, m_1)e^{-\tau}.$$

Again calculating Hausdorff distance, we easily get

$$\mathcal{H}_p(Sm_{\eta-1}, Sm_{l-1}) = |m_{\eta-1} - m_{l-1}|,$$

finally setting these values back in required condition, we get

$$(\frac{1}{2})^{\eta-1} - (\frac{1}{2})^{l-1} \le \left((\frac{1}{2})^\eta - (\frac{1}{2})^l\right)e^{-\tau}.$$

One can observe that in both cases, the required condition is satisfied.

3.5 Application

This section is designed to highlight the importance of the above-proved theorem with the help of the following application.

Suppose $I=[0,1]$, and $\mathcal{M}=\mathcal{C}(I,\mathbb{R}^2)$ be the collection of all continuous functions defined from I to $\mathbb{R}^2$, endowed with usual sup-norm. We define a partial metric function on $\mathcal{M}$, as

$$p_{\mathfrak{b}}(\phi,\psi)=\parallel \phi-\psi \parallel_{\infty}$$

$$\Longrightarrow \qquad p_{\mathfrak{b}}(\phi,\psi)=\sup_{m\in I}\{e^{-mp}|\phi(m)-\psi(m)|\} \quad p>1,$$

for all $\phi,\psi\in\mathcal{M}$.

It is easy to verify that $(\mathcal{M},p_{\mathfrak{b}})$ is a complete PMS.

Theorem 3.5.1.

Suppose that the following conditions hold:

(i) *Let $\mathcal{K}_{\phi}: I\times I\times\mathbb{R}^2\to\mathbb{R}^2$ and $f: I\to\mathbb{R}^2$ be continuous;*

(ii) *there exists $\phi_0\in\mathcal{M}$ such that $\phi_k\in S\phi_{k-1}$;*

(iii) *there exists a continuous function $\mathfrak{f}: I\times I\to I$ such that*

$$|k_{\phi}(\xi,x^*,\phi(x^*))-k_{\psi}(\xi,x^*,\psi(x^*))|\leq\sup_{x^*\in I}\mathfrak{f}(\phi(x^*),\psi(x^*))|\phi(x^*)-\psi(x^*)|,$$

for each $\xi,x^*\in I$ and $\mathfrak{f}(\phi(x^*),\psi(x^*))\leq\gamma$.

Then the integral equation

$$\phi(\xi)=\omega(\xi)+\nu\int_0^1\mathcal{K}_{\phi}(\xi,x^*,\phi(x^*))dx^*, \tag{3.16}$$

has a solution.

Proof. Let $(\mathcal{M},d_p)$ be a complete PMS. Firstly, we define $S:\mathcal{M}\to\mathcal{B}(\mathcal{M})$, as

$$S(\phi(\xi))=\{\phi^*(\xi):\phi^*(\xi)\in\omega(\xi)+\int_0^1\mathcal{K}_{\phi}(\xi,x^*,\phi(x^*))dx^*\}, \tag{3.17}$$

for each $\phi(\xi) \in \mathcal{M}$. Also, for every $\mathcal{K}_\phi : I \times I \times \mathbb{R}^2 \to \mathcal{B}(\mathcal{M})$ there exists

$$k_\phi(\xi, x^*, \phi^*) \in \mathcal{K}_\phi(\xi, x^*, \phi^*).$$

We need to define $\mathcal{F}$ map also, so let $\mathcal{F}(\xi) = \ln(\xi)$, for all $\xi \in \mathcal{M}$.

So after going through natural logarithm, our condition will be,

$$\mathcal{H}_{d_p}(S((\phi), S(\psi)) \leq e^{-\tau} d_p(\psi, \phi).$$

Now for $\phi^* \in S(\phi)$,we have

$$\begin{aligned}
d_p((\phi^*(\xi), S(\psi(\xi)))) &\leq d_p(\phi^*(\xi), (\psi^*(\xi))) \\
&= \sup_{\xi \in I} e^{-\xi\gamma} |\phi^*(\xi) - \psi^*(\xi)| \\
&= \sup_{\xi \in I} e^{-\xi\gamma} |\int_0^1 k_\phi(\xi, x^*, \phi(x^*)) - k_\psi(\xi, x^*, \psi(x^*)) dx^*| \\
&= \sup_{\xi \in I} e^{-\xi\gamma} \int_0^1 |e^{-x^*\gamma + x^*\gamma} k_\phi(\xi, x^*, \phi(x^*)) - k_\psi(\xi, x^*, \psi(x^*))| dx^* \\
&\leq \sup_{\xi \in I} e^{-\xi\gamma} \int_0^1 e^{x^*\gamma} \mathfrak{f}(\phi(x^*), \psi(x^*)) \sup_{x^* \in I} e^{-x^*\gamma} |\phi(x^*) - \psi(x^*)| dx^* \\
&\leq \gamma \parallel \phi(x^*) - \psi(x^*) \parallel_\infty \sup_{\xi \in I} e^{-\xi\gamma} \int_0^1 e^{x^*\gamma} dx^* \\
&= d_p(\phi(x^*), \psi(x^*))(1)(e^\gamma - 1) \\
&\leq d_p(\phi(x^*), \psi(x^*)) e^\gamma
\end{aligned}$$

Also, as ϕ^* is arbitrary, so we have

$$\delta_{d_p}(S(\phi), S(\psi)) \leq e^\gamma d_p(\phi, \psi)$$

Similarly, one can calculate

$$\delta_{d_p}(S(\psi), S(\phi)) \leq e^\gamma d_p(\phi, \psi),$$

finally, we have,

$$\mathcal{H}_{d_p}(S(\phi), S(\psi)) \leq e^\gamma d_p(\phi, \psi).$$

Or equally $\mathcal{H}_{d_p}(S(\phi), S(\psi)) \leq e^{-\tau} d_p(\phi, \psi)$.

This shows our desired contraction condition is satisfied by choice of $-\tau = \gamma$

All conditions of 3.4.1 are satisfied, showing that integral equation (3.16) has a solution. $\square$

Chapter 4

Fixed Point Results for Multivalued $\alpha\mathcal{F}$-Contraction in P𝔟MS

A direction for the extension of the metric space was given by Satish Shukla [35] in 2014. He coordinated the directions of both partial metric space and $\mathfrak{b}$-metric space together and presented a new version of BCP in partial $\mathfrak{b}$-metric space. A detailed review of his work is unveiled in the forthcoming section.

After that, we headlined the idea of the Hausdorff distance under the umbrella of partial $\mathfrak{b}$-metric space. In this chapter, the idea of α-admissible mappings is also portrayed under multivalued $\mathcal{F}$ contractive maps along with a new extension of BCP. Lastly, we mentioned an application of this theorem which features the existence of the solution to a Fredholm integral equation.

4.1 Partial $\mathfrak{b}$-metric space

We begin this section with the concept of P𝔟MS, in which we consider the importance of the non-zero self distance likewise the coefficient $\mathfrak{b} \geq 1$ appearing in triangular inequality of $\mathfrak{b}$-metric space.

Definition 4.1.1. Partial $\mathfrak{b}$-metric space

Let $\mathcal{M} \neq \phi$ and $\mathfrak{b} \geq 1$ be any real number, the metric function $p_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ satisfying following properties on $\mathcal{M}$ is called a partial $\mathfrak{b}$ metric on $\mathcal{M}$.

$\acute{p}_b(1)$: $p_{\mathfrak{b}}(\mu, \upsilon) = p_{\mathfrak{b}}(\mu, \mu) = p_{\mathfrak{b}}(\upsilon, \upsilon)$ if and only if $\mu = \upsilon$;

$\acute{p}_b(2)$: $p_{\mathfrak{b}}(\mu, \upsilon) \geq p_{\mathfrak{b}}(\mu, \mu)$;

$\acute{p}_b(3)$: $p_{\mathfrak{b}}(\mu, \upsilon) = p_{\mathfrak{b}}(\upsilon, \mu)$;

$\acute{p}_b(4)$: $p_{\mathfrak{b}}(\mu, \upsilon) \leq \mathfrak{b}\{p_{\mathfrak{b}}(\mu, k) + p_{\mathfrak{b}}(k, \upsilon)\} - p_{\mathfrak{b}}(k, k)$,

for all $\mu, \upsilon, k \in \mathcal{M}$.

The pair $(\mathcal{M}, p_{\mathfrak{b}})$ is called a partial $\mathfrak{b}$-metric space with coefficient $\mathfrak{b} \geq 1$.

Remark *4.1.2.*

If we ignore the self distance by assuming it zero P$\mathfrak{b}$MS will become $\mathfrak{b}$MS, and if we set the coefficient of P$\mathfrak{b}$MS equals to 1, then it will be simply a PMS. But the converse of this fact isn't true in general. For this, consider the following example;

Example 4.1.3.

Let $\mathcal{M} = \mathbb{R}^+$ we define a the metric function $p_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ working as

$$p_{\mathfrak{b}}(\xi, \mu) = \mid \xi - \mu \mid^q + [\max\{\xi, \mu\}]^q \quad \text{for all} \quad \xi, \mu \in \mathcal{M},$$

$q > 1$ be any constant.

We will show that this defines a partial $\mathfrak{b}$-metric function on $\mathcal{M}$.

$\acute{p}_b(1)$: Setting $\xi = \mu$

$$\begin{aligned} p_{\mathfrak{b}}(\xi, \mu) &= |\xi - \mu|^q + [\max\{\xi, \mu\}]^q \\ &= |\xi - \xi|^q + [\max\{\xi, \xi\}]^q \\ &= [\max\{\xi, \xi\}]^q \\ &= \xi^q \\ &= \mu^q \\ &= [\max\{\mu, \mu\}]^q \\ &= |\mu - \mu|^q + [\max\{\mu, \mu\}]^q \\ &= p_{\mathfrak{b}}(\mu, \mu) \\ &= p_{\mathfrak{b}}(\xi, \xi). \end{aligned}$$

Conversely, setting,

$p_{\mathfrak{b}}(\xi,\xi) = p_{\mathfrak{b}}(\mu,\mu) = p_{\mathfrak{b}}(\xi,\mu)$,

by comparing the first two, we will easily get $\xi = \mu$, so consider the other two we have

$$p_{\mathfrak{b}}(\mu,\mu) = p_{\mathfrak{b}}(\xi,\mu),$$

this will give

$$|\mu - \mu|^q + [\max\{\mu,\mu\}]^q = |\xi - \mu|^q + [\max\{\xi,\mu\}]^q.$$

On the contrary, suppose that

$$\xi \neq \mu$$

.

This means either $\xi < \mu$ or $\xi > \mu$, for the first case

$$\mu^q = |\xi - \mu|^q + \mu^q,$$

$$\implies |\xi - \mu|^q = 0 \quad \text{or} \quad |\xi - \mu| = 0$$

which shows $\xi = \mu$ which is a contradiction. We will get the same for the second possible case.

So we have

$$p_{\mathfrak{b}}(\xi,\xi) = p_{\mathfrak{b}}(\mu,\mu) = p_{\mathfrak{b}}(\xi,\mu) \iff \xi = \mu.$$

$\acute{p}_b(1)$: Let $\xi \geq \mu$, then

$$\begin{aligned} p_{\mathfrak{b}}(\xi,\mu) &= |\xi - \mu|^q + [\max\{\xi,\mu\}]^q \\ &= |\xi - \mu|^q + \xi^q \\ &\geq |\xi - \mu|^q + \mu^q \\ &\geq |\xi - \mu|^q + [\max\{\mu,\mu\}]^q \\ &= p_{\mathfrak{b}}(\mu,\mu), \end{aligned}$$

$$\implies p_{\mathfrak{b}}(\xi,\mu) \geq p_{\mathfrak{b}}(\mu,\mu),$$

i.e., self distance is less than the distance of two distinct elements in $\mathcal{M}$.

$\acute{p}_b(1)$: It is easy to see that symmetry follows because of the symmetric property of maximum function and the absolute value of elements.

$\acute{p}_b(1)$: Let for arbitrary $\xi, \mu, \gamma \in \mathcal{M}$ have a relationship $\xi \geq \gamma \geq \mu$.
Consider,

$$\begin{aligned} p_{\mathfrak{b}}(\xi, \mu) &= |\xi - \mu|^q + [\max\{\xi, \mu\}]^q \\ &= |\xi - \mu - \gamma + \gamma|^q + [\max\{\xi, \mu\}]^q \\ &\leq \{2^{q-1}|\xi - \gamma|^q + |\gamma - \mu|^q\} + \xi^q \\ &= \mathfrak{b}\{|\xi - \gamma|^q + |\gamma - \mu|^q\} + \xi^q \\ &\leq \mathfrak{b}\{|\xi - \gamma|^q + |\gamma - \mu|^q + \xi^q\} \\ &= \mathfrak{b}\{|\xi - \gamma|^q + \xi^q\} + \mathfrak{b}|\gamma - \mu|^q \\ &\leq \mathfrak{b}\{|\xi - \gamma|^q + \xi^q\} + \mathfrak{b}\{|\gamma - \mu|^q + \gamma^q|\} - \gamma^q \\ &= \mathfrak{b}\{p_{\mathfrak{b}}(\xi, \gamma) + p_{\mathfrak{b}}(\gamma, \mu)\} - p_{\mathfrak{b}}(\gamma, \gamma), \end{aligned}$$

so for all $\xi, \mu, \gamma \in \mathcal{M}$.
We have,

$$p_{\mathfrak{b}}(\xi, \mu) \leq \mathfrak{b}\{p_{\mathfrak{b}}(\xi, \gamma) + p_{\mathfrak{b}}(\gamma, \mu)\} - p_{\mathfrak{b}}(\gamma, \gamma),$$

showing that $(\mathcal{M}, p_{\mathfrak{b}})$ is a P$\mathfrak{b}$MS with

$$\mathfrak{b} = 2^{q-1}.$$

It is also easy to check that this defined metric neither satisfies the condition of PMS nor of $\mathfrak{b}$MS. One can easily verify the triangular inequalitie of both spaces may fail here.

Proposition 4.1.4.
Let $\mathcal{M}$ be a non empty set with some defined metric function on it such that $(\mathcal{M}, d_p)$ and $(\mathcal{M}, d_{\mathfrak{b}})$ are PMS and $\mathfrak{b}$MS respectively.
Then we can always construct a P$\mathfrak{b}$MS on $\mathcal{M}$ as follow,

$$p_{\mathfrak{b}}(m, n) = d_p(m, n) + d_{\mathfrak{b}}(m, n), \tag{4.1}$$

for all $m, n \in \mathcal{M}$ and $\mathfrak{b} > 1$. [35]

Proof. Let $(\mathcal{M}, d_p)$ be a PMS and $(\mathcal{M}, d_{\mathfrak{b}})$ be a $\mathfrak{b}$MS with coefficient $\mathfrak{b} > 1$. One can easily verify that $p_b(1)$- $p_b(3)$ are obviously true for (4.1).
For $p_b(4)$ let $\xi, \eta, k \in M$, then using triangular inequality,

$$\begin{aligned} p_{\mathfrak{b}}(\xi, \eta) &= d_p(\xi, \eta) + d_{\mathfrak{b}}(\xi, \eta) \\ &\leq d_p(\xi, k) + d_p(k, \eta) - d_p(k, k) + \mathfrak{b}\{d_{\mathfrak{b}}(\xi, k) + d_{\mathfrak{b}}(k, \eta)\} \\ &\leq \mathfrak{b}\{d_p(\xi, k) + d_p(k, \eta) - d_p(k, k) + d_{\mathfrak{b}}(\xi, k) + d_{\mathfrak{b}}(k, \eta)\} \\ &= \mathfrak{b}\{p_{\mathfrak{b}}(\xi, k) + p_{\mathfrak{b}}(k, \eta) - p_{\mathfrak{b}}(k, k)\} \\ &\leq \mathfrak{b}\{p_{\mathfrak{b}}(\xi, k) + p_{\mathfrak{b}}(k, \eta)\} - p_{\mathfrak{b}}(k, k), \end{aligned}$$

which completes the proof. □

Proposition 4.1.5.

Let $(\mathcal{M}, d_p)$ be a PMS and consider any positive number $k \geq 1$. Then we can always construct a P𝔟MS on $\mathcal{M}$ from this partial metric by the following condition

$$p_{\mathfrak{b}}(m, n) = \{d_p(m, n)\}^k,$$

with $\mathfrak{b} = 2^{k-1}$.

Definition 4.1.6. Open partial 𝔟-ball.

Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a P𝔟MS then an open partial 𝔟-ball of radius $\epsilon > 0 \in \mathbb{R}$ with center $\mu \in \mathcal{M}$, is defined as,

$$B_{p\mathfrak{b}}(\mu, \epsilon) = \{n \in \mathcal{M} : p_{\mathfrak{b}}(\mu, n) < p_{\mathfrak{b}}(\mu, \mu) + \epsilon\},$$

for all $\mu \in \mathcal{M}$.

Remark 4.1.7.

We can always induce a T_0 topology $\tau_{\mathfrak{b}}$ on every P𝔟MS provided that family of open partial 𝔟-ball

$$\{B_{p\mathfrak{b}}(\mu, \epsilon) : \mu \in \mathcal{M}, \epsilon > 0\},$$

is subbase for this topology.

Definition 4.1.8. Convergence and Completeness P𝔟MS.

Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a P𝔟MS with $\mathfrak{b} \geq 1$. Consider $\{m_k\}$ be any sequence in $\mathcal{M}$ also some $m_0 \in \mathcal{M}$ be any arbitrary element.

Then we have the following results for convergence:

(i) The sequence $\{m_k\}$ in is called convergent sequence with respect to $\tau_{\mathfrak{b}}$ with limit m_0 if

$$\lim_{k\to\infty} p_{\mathfrak{b}}(m_k, m_0) = p_{\mathfrak{b}}(m_0, m_0).$$

For example, consider $\mathcal{M} = [0, 1]$ and let

$$m_\eta = \{\frac{1}{\eta} : \eta \in \mathbb{N}\},$$

we define a map $p_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ as

$$p_{\mathfrak{b}}(\xi, \zeta) = |\xi - \zeta|^5 + v,$$

it is easy to see that $(\mathcal{M}, p_{\mathfrak{b}})$ is a P𝔟MS with $\mathfrak{b}$=2^4.

As

$$\lim_{\eta\to\infty} p_{\mathfrak{b}}(m_\eta, 0) = \lim_{\eta\to\infty} p_{\mathfrak{b}}\Big(\frac{1}{\eta}, 0\Big) = \lim_{\eta\to\infty} \left[\left|\frac{1}{\eta} - 0\right| + v\right] = p_{\mathfrak{b}}(0, 0).$$

Showing that $\{m_\eta\}$ is a convergent sequence in $(\mathcal{M}, p_{\mathfrak{b}})$.

(ii) A sequence $\{m_k\}$ in $\mathcal{M}$ becomes a Cauchy sequence if

$$\lim_{k,l\to\infty} p_{\mathfrak{b}}(m_k, m_l),$$

exists and is finite.

(iii) $(\mathcal{M}, p_{\mathfrak{b}})$ is called a complete P𝔟MS if every Cauchy sequence converges in $\mathcal{M}$.

The famous BCP was generalized by Shukla [35] in his paper after introducing the P𝔟MS, which is given below.

Theorem 4.1.9.

Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a P𝔟MS with $\mathfrak{b} \geq 1$. Consider a self map $\mathbb{F} : \mathcal{M} \to \mathcal{M}$ with following condition

$$p_{\mathfrak{b}}(\mathbb{F}(\zeta), \mathbb{F}(\xi)) \leq \alpha(p_{\mathfrak{b}}(\zeta, \xi)),$$

for all $\zeta, \xi \in \mathcal{M}$ and $\alpha \in [0, 1)$, then $\mathbb{F}$ has a unique fixed point with $p_{\mathfrak{b}}(\zeta, \zeta) = 0$.

4.2 Hausdorff metric in P𝔟MS.

The idea of Hausdorff distance emerged in the setting of partial 𝔟-metric space by Felhi [53] in 2016 by extending the work of Aydi [43]. He presented some properties of the Hausdorff metric function in the context of P𝔟MS.
His work was expanded with some more advanced results by Saeed et al. in February 2022 [54]. Some useful results from this article are mentioned below which are going to be used in demonstrating our main theorem.

Definition 4.2.1. Closed set in P𝔟MS.
Let $(\mathcal{M}, p_{\mathfrak{b}})$ be P𝔟MS. A set $\mathcal{K} \subset \mathcal{M}$ is called closed in $(\mathcal{M}, p_{\mathfrak{b}})$ if and only if $\mathcal{K} = \bar{\mathcal{K}}$, here $\bar{\mathcal{K}}$ represents closure of $\mathcal{K}$.

Definition 4.2.2. Bounded set in P𝔟MS.
Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a P𝔟MS and let $\phi \neq \mathcal{N} \subset \mathcal{M}$, then $\mathcal{N}$ is called bounded if for any $m_0 \in \mathcal{M}$ there exist an open partial 𝔟-ball centered at m_0 with radius $R^* \geq 0$ for $n \in B_p(m_0, R^*)$ where n be any arbitrary element in $\mathcal{N}$, such that

$$d_p(m_0, n) < d_p(n, n) + R^*.$$

Definition 4.2.3. Hausdorff distance in P𝔟MS.
Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a P𝔟MS with $\mathfrak{b} \geq 1$, and $CB_{p_{\mathfrak{b}}}(\mathcal{M})$ be the collection of all non-empty bounded and closed subsets of $\mathcal{M}$, For $\mathcal{P}, \mathcal{Q} \in CB_{p_{\mathfrak{b}}}(\mathcal{M})$, partial 𝔟 Hausdorff 𝔟 metric on $CB_{p_{\mathfrak{b}}}(\mathcal{M})$ induced by $p_{\mathfrak{b}}$ is given as follow,

$$\mathcal{H}_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{Q}) = \max\{\delta_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{Q}), \delta_{p_{\mathfrak{b}}}(\mathcal{Q}, \mathcal{P})\},$$

where

$$\delta_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{Q}) = \sup\{p_{\mathfrak{b}}(p, \mathcal{Q}) : p \in \mathcal{P}\},$$

and

$$\delta_{p_{\mathfrak{b}}}(\mathcal{Q}, \mathcal{P}) = \sup\{p_{\mathfrak{b}}(q, \mathcal{P}) : q \in \mathcal{Q}\}.$$

Example 4.2.4.

Let $\mathcal{M} = \{0, 1, 4\}$ with partial metric $p_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$, defined as

$$p_{\mathfrak{b}}(\zeta, \mu) = |\zeta - \mu|^2 + \max\{\zeta, \mu\}^2,$$

one can check that $(M, p_{\mathfrak{b}})$ is a P$\mathfrak{b}$MS with $\mathfrak{b} = 2$.

Now consider two closed and bounded subset in $\mathcal{M}$ as, $\mathcal{P} = \{0\}$ $\mathcal{P}^* = \{0, 1\}$. Next, to calculate the partial Hausdorff distance between $\mathcal{P}$ and $\mathcal{P}^*$, that is

$$\begin{aligned} \mathcal{H}_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{P}^*) &= \max\{\delta_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{P}^*), \delta_{p_{\mathfrak{b}}}(\mathcal{P}^*, \mathcal{P})\} \\ &= \max\{\sup_{p \in \mathcal{P}} d_p(p, \{0, 1\}), \sup_{q \in \mathcal{P}^*} d_p(q, \{0\})\} \\ &= \max\{\inf(d_p(\{0\}, \{0, 1\})), \sup(d_p(\{0, 1\}, \{0\}))\} \\ &= \max\{0, 2\} \\ &= 2. \end{aligned} \tag{4.2}$$

Lemma 4.2.5.

Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$, Consider two non empty subsets $\mathcal{P}, \mathcal{P}^* \in CB_{p_{\mathfrak{b}}}(\mathcal{M})$ and $k^* > 1$. For any $p \in \mathcal{P}$ there exists $q \in \mathcal{P}^*$ such that

$$p_{\mathfrak{b}}(p, q) \leq k^* \mathcal{H}_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{P}^*).$$

Proof. We have the following two cases,

Case I: If $\mathcal{P} = \mathcal{P}^*$, then

$$\mathcal{H}_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{P}^*) = \delta_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{P}^*) = \sup_{p \in \mathcal{P}} d_p(p, p) \leq k^* \sup_{p \in \mathcal{P}} d_p(p, p),$$

for $\quad k^* > 1.$

Showing that,

$$\mathcal{H}_{p_\mathfrak{b}}(\mathcal{P},\mathcal{P}^*) \leq k^*\mathcal{H}_{p_\mathfrak{b}}(\mathcal{P},\mathcal{P}^*).$$

Case II: If $\mathcal{P} \neq \mathcal{P}^*$,

On the contrary suppose that there exists any $p \in \mathcal{P}$, such that

$$d_p(p,q) > k^*\mathcal{H}_{p_\mathfrak{b}}(\mathcal{P},\mathcal{P}^*),$$

then we may write as

$$\inf_{q\in\mathcal{P}^*} d_p(p,\mathcal{P}^*) > k^*\mathcal{H}_{p_\mathfrak{b}}(\mathcal{P},\mathcal{P}^*),$$

for all $\quad q \in \mathcal{P}^*.$

$$\implies d_p(p,\mathcal{P}^*) \geq k^*\mathcal{H}_{p_\mathfrak{b}}(\mathcal{P},\mathcal{P}^*).$$

Consider,

$$\mathcal{H}_{p_\mathfrak{b}}(\mathcal{P},\mathcal{P}^*) \geq \sup_{p\in P} d_p(p,\mathcal{P}^*) \geq d_p(p,\mathcal{P}^*) \geq k^*\mathcal{H}_{p_\mathfrak{b}}(\mathcal{P},\mathcal{P}^*),$$

as $\mathcal{P} \neq \mathcal{P}^*$ this means $\mathcal{H}_{p_\mathfrak{b}}(\mathcal{P},\mathcal{P}^*) \neq 0$ then we must have $k^* \leq 1$.

Which is a contradiction. So from both cases, we conclude

$$d_p(p,q) \leq k^*\mathcal{H}_{p_\mathfrak{b}}(\mathcal{P},\mathcal{P}^*),$$

for $k^* > 1$. □

Lemma 4.2.6.

Consider $(\mathcal{M}, p_\mathfrak{b})$ *be a partial* $\mathfrak{b}$*-metric space with* $\mathfrak{b} \geq 1$*, then for two non empty subsets* $\mathcal{P},\mathcal{P}^* \in CB_{p_\mathfrak{b}}(\mathcal{M})$ *we have,*

$$\mathcal{H}_{p_\mathfrak{b}}(\mathcal{P},\mathcal{P}^*) = 0 \iff \mathcal{P} = \mathcal{P}^*.$$

Lemma 4.2.7.

Let $(\mathcal{M}, p_\mathfrak{b})$ *be a partial* $\mathfrak{b}$*-metric space with* $\mathfrak{b} \geq 1$*, and we have two non empty subsets* $\mathcal{P},\mathcal{P}^* \in CB_{p_\mathfrak{b}}(\mathcal{M})$.

Then for each $p \in \mathcal{P}$ we have,

$$p_{\mathfrak{b}}(p, \mathcal{P}^*) \leq \mathcal{H}_{p_{\mathfrak{b}}}(\mathcal{P}, \mathcal{P}^*).$$

4.3 Some Necessary Ingredients.

Before stating our main result, we are going to mention some necessary definitions; without these ideas, our theorem would be fragmented.
Firstly, we will look at the concept of α-admissible mappings given in 2012 by Samet et al. [42].

Definition 4.3.1. α-admissible mapping.
Let $\mathcal{M}$ be a non empty set we define a self map $\mathcal{K}_\alpha : \mathcal{M} \to \mathcal{M}$. Also define $\alpha : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ then $\mathcal{K}_\alpha$ is called an α-admissible mapping if following condition holds for $\xi, \zeta \in \mathcal{M}$

$$\alpha(\xi, \zeta) \geq 1$$

$$\implies \alpha(\mathcal{K}_\alpha(\xi), \mathcal{K}_\alpha(\zeta)) \geq 1.$$

Example 4.3.2.
Let $\mathcal{M} = \{(0, b) : b > 0\} \subset \mathbb{R}^+$, we define a self map $\mathcal{K}_\alpha : \mathcal{M} \to \mathcal{M}$ as

$$\mathcal{K}_\alpha(m) = \ln(m)$$

for all $m \in \mathcal{M}$, and $\alpha : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ is defined as follow

$$\alpha(m, n) = \begin{cases} 2 & \text{if } m \geq n \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to observe that $\ln(m)$ being an increasing function will satisfy the condition as

$$m \geq n \quad \implies \quad \mathcal{K}_\alpha(m) \geq \mathcal{K}_\alpha(n)$$

so whenever we have

$$\alpha(m, n) \geq 1$$

we must have

$$\alpha(\mathcal{K}_\alpha(m), \mathcal{K}_\alpha(n)) \geq 1.$$

Thus $\mathcal{K}_\alpha$ is an α-admissible mapping.

Example 4.3.3.

Let $\{\mathcal{M} = [0, b) : b > 0\} \subset \mathbb{R}^+$ and a self map $\mathcal{K}_\alpha : \mathcal{M} \to \mathcal{M}$, is defined as

$$\mathcal{K}_\alpha(m) = \sqrt{m} \quad \text{for all} \quad m \in \mathcal{M},$$

and $\alpha : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ is defined as follow

$$\alpha(m, n) = \begin{cases} e^{m-n} & \text{if } m \geq n \\ 0 & \text{otherwise.} \end{cases}$$

This defined $\mathcal{K}_\alpha$ is an α-admissible mapping.

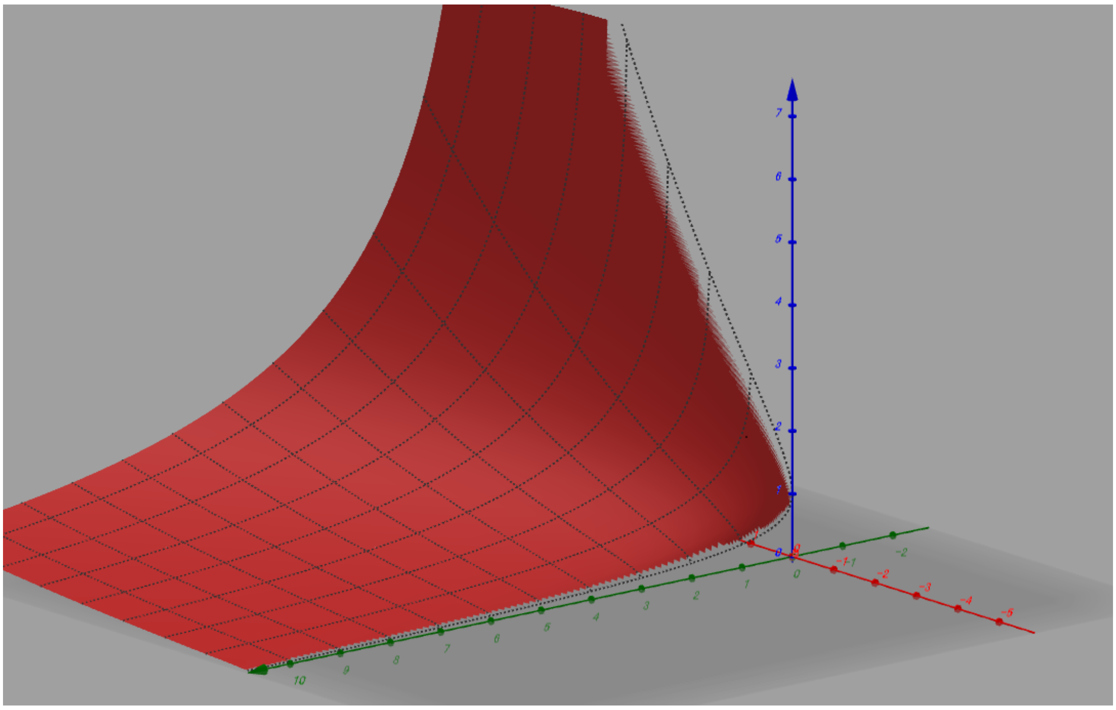

FIGURE 4.1: *A graph showing α-admissibility*

Remark 4.3.4.

Every non-decreasing self map is an α-admissible map.

Definition 4.3.5. Multivalued α-admissible mapping.

Consider a set $\mathcal{M} \neq \phi$ and let $\mathcal{T}_\alpha : \mathcal{M} \to 2^{\mathcal{M}}$ be a multivalued mapping. Also, define $\alpha : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ then $\mathcal{T}_\alpha$ is called multivalued α-admissible mapping if for $m, n \in \mathcal{M}$ we have

$$\alpha(m,n) \geq 1 \implies \alpha(m_0, n_0) \geq 1,$$

where $m_0 \in \mathcal{T}_\alpha(m)$ and $n_0 \in \mathcal{T}_\alpha(n)$.

Definition 4.3.6. $\mathcal{F}$-Contraction in P$\mathfrak{b}$MS.

Let $(\mathcal{M}, p_\mathfrak{b})$ be a P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$, a self map $S : \mathcal{M} \to \mathcal{M}$ is said to be an $\mathcal{F}$-contraction on $\mathcal{M}$, if there exists a $\tau > 0$ such that for all $m, n \in \mathcal{M}$

$$p_\mathfrak{b}(m,n) > 0 \implies \tau + \mathcal{F}(p_\mathfrak{b}(Sm, Sn)) \leq \mathcal{F}(p_\mathfrak{b}(m,n)), \tag{4.3}$$

where $\mathcal{F} \in \Delta_f$-family.

Definition 4.3.7. Multivalued $\mathcal{F}$-Contraction Mapping.

Let $(\mathcal{M}, p_\mathfrak{b})$ be a P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$ and define a map $S : \mathcal{M} \to K(\mathcal{M})$, then S is said to be a MV$\mathcal{F}$-contraction mapping, if $\mathcal{F} \in \Delta_f - family$ and for any $\tau > 0$,

$$\mathcal{H}_{p\mathfrak{b}}(Sm_1, Sm_2) > 0 \implies \tau + \mathcal{F}(\mathfrak{b}\mathcal{H}_{p\mathfrak{b}}(Sm_1, Sm_2)) \leq \mathcal{F}(\mathbb{M}(m_1, m_2)), \tag{4.4}$$

where

$$\mathbb{M}(m_1, m_2) = \max\big\{p_\mathfrak{b}(m_1, m_2), p_\mathfrak{b}(m_1, Sm_1), p_\mathfrak{b}(m_2, Sm_2), \frac{p_\mathfrak{b}(m_1, Sm_2) + p_\mathfrak{b}(m_2, Sm_1)}{2\mathfrak{b}}\big\}.$$

Definition 4.3.8. Multivalued $\alpha\mathcal{F}$-Contraction Mapping.

Let $(\mathcal{M}, p_\mathfrak{b})$ be a P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$, then $S : \mathcal{M} \to K(\mathcal{M})$ is said to be a MV$\alpha\mathcal{F}$-contraction mapping, if $\mathcal{F} \in \Delta_f - family$ and for any $\tau > 0$ such that,

$$\mathcal{H}_{p\mathfrak{b}}(Sm_1, Sm_2) > 0 \implies \tau + \mathcal{F}(\alpha(m_1, m_2)(\mathfrak{b}\mathcal{H}_{p\mathfrak{b}}(Sm_1, Sm_2))) \leq \mathcal{F}(\mathbb{M}(m_1, m_2)), \tag{4.5}$$

where

$$\mathbb{M}(m_1, m_2) = \max\big\{p_\mathfrak{b}(m_1, m_2), p_\mathfrak{b}(m_1, Sm_1), p_\mathfrak{b}(m_2, Sm_2), \frac{p_\mathfrak{b}(m_1, Sm_2) + p_\mathfrak{b}(m_2, Sm_1)}{2\mathfrak{b}}\big\}.$$

4.4 Fixed Point Results for Multivalued $\alpha\mathcal{F}$ Contraction Mapping in P$\mathfrak{b}$MS

Lemma 4.4.1. *Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a complete P$\mathfrak{b}$MS with $\mathfrak{b} \geq 1$ and $S : \mathcal{M} \to K(\mathcal{M})$ be a MV$\mathcal{F}$-contraction mapping, then*

$$\lim_{\xi\to\infty} \mathfrak{b}^{\xi} v_{\xi} = 0,$$

where $v_{\xi} = p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})$ and $\xi = 0, 1, 2.....$

Proof. We take an arbitrary $m_0 \in \mathcal{M}$, as Sm being set of all images of $m \in \mathcal{M}$ is non empty for all values in $\mathcal{M}$, we can choose $m_1 \in Sm_0$. If $m_1 \in Sm_1$ this means that m_1 is the fixed point of S trivially. Suppose $m_1 \notin Sm_1$, as Sm_1 is closed, so we have $p_{\mathfrak{b}}(m_1, Sm_1) > 0$. Also, we know that

$$p_{\mathfrak{b}}(m_1, Sm_1) \leq \mathcal{H}_{p_{\mathfrak{b}}}(Sm_0, Sm_1). \tag{4.6}$$

As Sm_1 is compact, so there exists $m_2 \in Sm_1$ such that

$$p_{\mathfrak{b}}(m_1, m_2) = p_{\mathfrak{b}}(m_1, Sm_1),$$

$$\implies p_{\mathfrak{b}}(m_1, m_2) \leq \mathcal{H}_{p_{\mathfrak{b}}}(Sm_0, Sm_1).$$

Similarly for $m_3 \in Sm_2$, we get

$$p_{\mathfrak{b}}(m_2, m_3) \leq \mathcal{H}_{p_{\mathfrak{b}}}(Sm_1, Sm_2),$$

which ultimately gives

$$p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}) \leq \mathcal{H}_{p_{\mathfrak{b}}}(Sm_{\xi}, Sm_{\xi+1}).$$

$$\implies \mathfrak{b}(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})) \leq \mathfrak{b}(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_{\xi}, Sm_{\xi+1})).$$

(F_1) $\implies$

$$\mathcal{F}(\mathfrak{b}(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}))) \leq \mathcal{F}(\mathfrak{b}(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_{\xi}, Sm_{\xi+1}))). \tag{4.7}$$

by (4.4),

$$\mathcal{F}(\mathfrak{b}(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}))) \leq \mathcal{F}(\mathbb{M}(m_{\xi+1}, m_{\xi})) - \tau. \tag{4.8}$$

Where

$$\begin{aligned}
\mathbb{M}(m_{\xi}, m_{\xi+1}) &= \max\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi}, Sm_{\xi}), p_{\mathfrak{b}}(m_{\xi+1}, Sm_{\xi+1}), \\
&\quad \frac{p_{\mathfrak{b}}(m_{\xi}, Sm_{\xi+1}) + p_{\mathfrak{b}}(m_{\xi+1}, Sm_{\xi})}{2\mathfrak{b}}\} \\
&= \max\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}), \\
&\quad \frac{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}) + p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})}{2\mathfrak{b}}\} \\
&\leq \max\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}), \\
&\quad \mathfrak{b}[\frac{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}) + p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})}{2\mathfrak{b}}]\}. \\
&= \max\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\}.
\end{aligned}$$

Assume,

$$\max\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\} = p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}),$$

(4.8) $\implies$

$$\tau + \mathcal{F}(\mathfrak{b}(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}))) \leq \mathcal{F}(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})),$$

which is a contradiction. Therefore,

$$\max\{p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}), p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})\} = p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1}),$$

$$\implies \mathcal{F}(\mathfrak{b}(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}))) \leq \mathcal{F}(p_{\mathfrak{b}}(m_{\xi}, m_{\xi+1})).$$

For the convenience, we are setting

$$v_{\xi} = p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}),$$

where $\xi = 0, 1, 2.....$ Clearly, $v_{\xi} > 0$ for all $\xi \in \mathbb{N}$, now substituting this in above equation we have

$$\tau + \mathcal{F}(\mathfrak{b}(v_{\xi})) \leq \mathcal{F}(v_{\xi-1}).$$

Iteratively,

$$\tau + \mathcal{F}(\mathfrak{b}^{\xi}(v_{\xi})) \leq \mathcal{F}(\mathfrak{b}^{\xi-1}(v_{\xi-1})).$$

We will get

$$\mathcal{F}(\mathfrak{b}^{\xi}(v_{\xi}) \leq \mathcal{F}(\mathfrak{b}^{\xi-1}(v_{\xi-1})) - \tau \leq \mathcal{F}(\mathfrak{b}^{\xi-2}(v_{\xi-2})) - 2\tau \leq .... \leq \mathcal{F}(v_0) - \xi\tau \tag{4.9}$$

$$\implies \lim_{\xi\to\infty} \mathcal{F}\mathfrak{b}^{\xi}(v_{\xi}) = -\infty,$$

we have

$$\lim_{\xi\to\infty} \mathfrak{b}^{\xi} v_{\xi} = 0, \text{ by } F_2.$$

□

Theorem 4.4.2. *Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a complete P𝔟MS with $\mathfrak{b} \geq 1$, such that $p_{\mathfrak{b}}$ be a continuous mapping and $S : \mathcal{M} \to K(\mathcal{M})$ be a multivalued $\alpha\mathcal{F}$-contraction mapping, suppose that*

(i) S is continuous;

(ii) S is an α-admissible mapping;

(iii) there exists a $m_0 \in \mathcal{M}$ and $m_1 \in Sm_0$ such that $\alpha(m_0, m_1) \geq 1$,

then S has a fixed point.

Proof. For $m_0 \in \mathcal{M}$ by assumption, $\alpha(m_0, m_1) \geq 1$ for some $m_1 \in Sm_0$, similarly for $m_2 \in Sm_1$ we have $\alpha(m_1, m_2) \geq 1$ and for any sequence $m_{\xi+1} \in Sm_{\xi}$, we get

$$\alpha(m_{\xi}, m_{\xi+1}) \geq 1 \quad \text{for all} \quad \xi \in \mathbb{N} \cup \{0\}. \tag{4.10}$$

Now by contraction condition (4.5), we have

$$\tau + \mathcal{F}(\alpha(m_{\xi}, m_{\xi+1})\mathfrak{b}(\mathcal{H}_{p\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}))) \leq \mathcal{F}(\mathbb{M}(m_{\xi+1}, m_{\xi})),$$

(4.10) $\implies$

$$\tau + \mathcal{F}(\mathfrak{b}(\mathcal{H}_{p\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}))) \leq \mathcal{F}(\mathbb{M}(m_{\xi+1}, m_{\xi})),$$

where $\mathfrak{b} \geq 1$.

$$\implies \mathcal{F}(\mathfrak{b}(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}))) \leq \mathcal{F}(\mathbb{M}(m_{\xi+1}, m_{\xi})) - \tau. \tag{4.11}$$

by lemma (4.4.1) $\lim_{\xi\to\infty}\mathfrak{b}^{\xi}\upsilon_{\xi}=0$. By F_3, for any $\gamma\in(0,1)$

$$\lim_{\xi\to\infty}(\mathfrak{b}^{\xi}\upsilon_{\xi})^{\gamma}\mathcal{F}\mathfrak{b}^{\xi}(\upsilon_{\xi})=0\ \ \forall\ \xi\in\mathbb{N},$$

by (4.9)

$$(\mathfrak{b}^{\xi}\upsilon_{\xi})^{\gamma}(\mathcal{F}\mathfrak{b}^{\xi}(\upsilon_{\xi})-\mathcal{F}(\upsilon_{0}))\leq-(\mathfrak{b}^{\xi}\upsilon_{\xi})^{\gamma}\xi\tau\leq 0. \tag{4.12}$$

Now as $\tau>0$, we have $\lim_{\xi\to\infty}(\mathfrak{b}^{\xi}\upsilon_{\xi})^{\gamma}\xi=0$. So there exists a $\xi_1\in\mathbb{N}$, such that

$$(\mathfrak{b}^{\xi}\upsilon_{\xi})^{\gamma}\xi\leq 1 \qquad \forall\ \xi\geq\xi_1.$$

$$\implies\ \mathfrak{b}^{\xi}\upsilon_{\xi}\leq\frac{1}{\xi^{\frac{1}{\gamma}}}. \tag{4.13}$$

Now we will prove that $\{m_{\xi}\}$ is a Cauchy sequence in $\mathcal{M}$. For this let $\xi,l\in\mathbb{N}$ provided that $\xi>l\geq\xi_1$.

Consider triangular inequality of P$\mathfrak{b}$MS,

$$\begin{aligned}
p_{\mathfrak{b}}(m_{\xi},m_{\eta}) &\leq \mathfrak{b}\{p_{\mathfrak{b}}(m_{\xi},m_{\xi+1})+p_{\mathfrak{b}}(m_{\xi+1},m_{\eta})\}-p_{\mathfrak{b}}(m_{\xi+1},m_{\xi+1})\\
&\leq \mathfrak{b}\{p_{\mathfrak{b}}(m_{\xi},m_{\xi+1})+p_{\mathfrak{b}}(m_{\xi+1},m_{\eta})\}\\
&\leq \mathfrak{b}p_{\mathfrak{b}}(m_{\xi},m_{\xi+1})+\mathfrak{b}^2\{p_{\mathfrak{b}}(m_{\xi+1},m_{\xi+2})+p_{\mathfrak{b}}(m_{\xi+2},m_{\eta})\}\\
&-p_{\mathfrak{b}}(m_{\xi+2},m_{\xi+2})\\
&\leq \mathfrak{b}p_{\mathfrak{b}}(m_{\xi},m_{\xi+1})+\mathfrak{b}^2\{p_{\mathfrak{b}}(m_{\xi+1},m_{\xi+2})+p_{\mathfrak{b}}(m_{\xi+2},m_{\eta})\}\\
&.\\
&.\\
&.\\
&= \mathfrak{b}p_{\mathfrak{b}}(m_{\xi},m_{\xi+1})+\mathfrak{b}^2\{p_{\mathfrak{b}}(m_{\xi+1},m_{\xi+2})+.....\mathfrak{b}^{l-\xi}p_{\mathfrak{b}}(m_{\eta-1},m_{\eta})\\
&= \sum_{\beta=\xi}^{\eta-1}\mathfrak{b}^{\beta-\xi+1}p_{\mathfrak{b}}(m_{\beta},m_{\beta+1})\\
&\leq \sum_{\beta=\xi}^{\infty}\mathfrak{b}^{\beta}p_{\mathfrak{b}}(m_{\beta+1},m_{\beta+2})\\
&= \sum_{\beta=\xi}^{\infty}\mathfrak{b}^{\beta}\upsilon_{\beta}\\
&\leq \sum_{\beta=\xi}^{\infty}\frac{1}{\beta^{\frac{1}{\gamma}}},
\end{aligned}$$

The convergence of the series $\sum_{\beta=1}^{\infty} \frac{1}{\beta^{\frac{1}{\gamma}}}$ implies that $\lim_{\xi\to\infty} p_{\mathfrak{b}}(m_\xi, m_\eta) = 0$, which shows $\{m_\xi\}$ is Cauchy sequence in $\mathcal{M}$. Since $\mathcal{M}$ is complete , so there exists $m^* \in \mathcal{M}$ such that

$$\lim_{\xi\to\infty} p_{\mathfrak{b}}(m_\xi, m^*) = 0.$$

So by definition, we must have,

$$\lim_{\xi\to\infty} p_{\mathfrak{b}}(m_\xi, m^*) = p_{\mathfrak{b}}(m^*, m^*) = 0. \tag{4.14}$$

We claim that m^* is a fixed point of S that is

$$p_{\mathfrak{b}}(m^*, Sm^*) = p_{\mathfrak{b}}(m^*, m^*).$$

Suppose $p_{\mathfrak{b}}(m^*, Sm^*) > 0$ this means there exists $k_0 \in \mathbb{N}$ such that we have $p_{\mathfrak{b}}(m_\xi, Sm^*) > 0$ for all $\xi > k_0$.

$$p_{\mathfrak{b}}(m_\xi, Sm^*) \leq \mathcal{H}_{p\mathfrak{b}}(Sm_{\xi+1}, Sm^*).$$

By using our contraction condition and taking limit $\xi \to \infty$, we have,

$$\begin{aligned}
\tau + \mathcal{F}(p_{\mathfrak{b}}(m^*, Sm^*)) &\leq \tau + \mathcal{F}(\alpha(m^*, m^*)\mathcal{H}_{p\mathfrak{b}}(Sm^*, Sm^*)) \\
&\leq \mathcal{F}(\mathbb{M}(m^*, m^*)) \\
&\leq \mathcal{F}(p_{\mathfrak{b}}(m^*, Sm^*)),
\end{aligned}$$

where,

$$\begin{aligned}
\mathbb{M}(m^*, m^*) = \max\{&p_{\mathfrak{b}}(m^*, m^*), p_{\mathfrak{b}}(m^*, Sm^*), p_{\mathfrak{b}}(m^*, Sm^*), \\
&\frac{p_{\mathfrak{b}}(m^*, Sm^*) + p_{\mathfrak{b}}(Sm^*, m^*)}{2\mathfrak{b}}\} \\
\leq\ & p_{\mathfrak{b}}(m^*, Sm^*).
\end{aligned}$$

$$\implies \tau + \mathcal{F}(p_{\mathfrak{b}}(m^*, Sm^*)) \leq \mathcal{F}(p_{\mathfrak{b}}(m^*, Sm^*)).$$

Since $\tau > 0$. The above relation yields a contradiction, therefore $p_{\mathfrak{b}}(m^*, Sm^*) = 0$, also

$$p_{\mathfrak{b}}(m^*, m^*) = 0.$$

This gives $m^* \in \bar{S}m^* = Sm^*$. Proving that m^* is a fixed point of S. □

Example 4.4.3. Let $\mathcal{M} = \{0, 1, 2, 3....\}$ and $p_{\mathfrak{b}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ is defined as

$$p_{\mathfrak{b}}(\zeta, \nu) = \mid \zeta - \nu \mid^q + [\max\{\zeta, \nu\}]^q \quad \text{for all} \quad \zeta, \nu \in \mathcal{M},$$

it is easy to check that $(\mathcal{M}, p_{\mathfrak{b}})$ be a complete P$\mathfrak{b}$MS with $\mathfrak{b} = 2^{q-1}$, where $q > 1$.

We also define a multivalued map $S : \mathcal{M} \to 2^{\mathcal{M}}$, as follow

$$S\zeta = \begin{cases} \{0, 1\}, & \text{if } \zeta = 0, 1 \\ \{\zeta - 1, \zeta\} & \text{otherwise.} \end{cases}$$

Consider, $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$, working as

$$\alpha(\zeta, \nu) = \begin{cases} 2, & \text{if } \zeta, \nu \in \{0, 1\} \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Let $\zeta_0 = 0,\ \zeta_1 = 1$, then $S\zeta_0 = \{0, 1\}$ also $\zeta_1 = \{0, 1\}$, giving $\alpha(\zeta_0, \zeta_1) = \alpha(0, 1) = 2 > 1$, also for some $\zeta_2 = 0 \in S\zeta_1$, we get $\alpha(\zeta_1, \zeta_2) = \alpha(1, 0) = 2 > 1$, showing that S is an α-admissible map.
Define $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ as $\mathcal{F}(\zeta) = \ln(\zeta) + \zeta$, it can be observed easily that $\mathcal{F}$ is a member of Δ_f-family. Now applying $\mathcal{F}$ on our contraction condition,

$$\tau + \mathcal{F}(\alpha(\zeta, \nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta, S\nu)) \leq \mathcal{F}(\mathbb{M}(\zeta, \nu))$$

$$\begin{aligned} &\implies \tau + \ln\{\alpha(\zeta, \nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta, S\nu)\} + \alpha(\zeta, \nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta, S\nu) \\ &\qquad \leq \ln(\mathbb{M}(\zeta, \nu)) + \mathbb{M}(\zeta, \nu) \\ &\implies \tau + \alpha(\zeta, \nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta, S\nu) - \mathbb{M}(\zeta, \nu) \\ &\qquad \leq \ln(\mathbb{M}(\zeta, \nu)) - \ln\{\alpha(\zeta, \nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta, S\nu)\} \\ &\implies e^{\tau + \alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta, S\nu) - \mathbb{M}(\zeta,\nu)} \\ &\qquad \leq \frac{\mathbb{M}(\zeta, \nu)}{\alpha(\zeta, \nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta, S\nu)} \end{aligned}$$

$$\implies \frac{\alpha(\zeta, \nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta, S\nu)}{\mathbb{M}(\zeta, \nu)} e^{\alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu) - \mathbb{M}(\zeta,\nu)} \leq e^{-\tau}. \tag{4.15}$$

Now

$$\begin{aligned}\delta_{p_{\mathfrak{b}}}(\mathcal{P},\mathcal{P}^*) &= \delta_{p_{\mathfrak{b}}}(S\zeta,S\nu)\\ &= \max\{p_{\mathfrak{b}}(\zeta,S\nu),p_{\mathfrak{b}}(\zeta-1,S\nu)\}\\ &= \max\{\inf\{p_{\mathfrak{b}}(\zeta,\nu),p_{\mathfrak{b}}(\zeta,\nu-1)\},\inf\{p_{\mathfrak{b}}(\zeta-1,\nu),p_{\mathfrak{b}}(\zeta-1,\nu-1)\}\}\\ &= \max\{|\zeta-\nu|^q+\zeta^q,|\zeta-\nu-2|^q+\zeta^q\}\\ &= |\zeta-\nu|^q+\zeta^q.\end{aligned}$$

Similarly, we can calculate

$$\delta_{p_{\mathfrak{b}}}(\mathcal{P}^*,\mathcal{P}) = |\zeta-\nu|^q+\zeta^q.$$

$$\begin{aligned}\implies \mathcal{H}_{p_{\mathfrak{b}}}(\mathcal{P},\mathcal{P}^*) &= \max\{|\zeta-\nu|^q+\zeta^q,|\zeta-\nu|^q+\zeta^q\}\\ &= |\zeta-\nu|^q+\zeta^q.\end{aligned} \tag{4.16}$$

Also,

$$\mathbb{M}(\zeta,\nu) \geq p_{\mathfrak{b}}(\zeta,\nu) = |\zeta-\nu|^q+\zeta^q. \tag{4.17}$$

Setting these both in contraction condition,

we get,

$$\begin{aligned}&\frac{\alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu)}{\mathbb{M}(\zeta,\nu)}e^{(\alpha(\zeta,\nu)\mathcal{H}_{p_{\mathfrak{b}}}(S\zeta,S\nu))-\mathbb{M}(\zeta,\nu)}\\ &= \frac{|\zeta-\nu|^q+\zeta^q}{2\mathbb{M}(\zeta,\nu)}e^{\frac{1}{2}(|\zeta-\nu|^q+\zeta^q)-\mathbb{M}(\zeta,\nu)} \quad \text{using} \quad (4.16)\\ &\leq \frac{|\zeta-\nu|^q+\zeta^q}{2|\zeta-\nu|^q+\zeta^q}e^{\frac{1}{2}(|\zeta-\nu|^q+\zeta^q)-|\zeta-\nu|^q+\zeta^q} \quad \text{using} \quad (4.17)\\ &= \frac{1}{2}e^{\frac{-1}{2}(|\zeta-\nu|^q+\zeta^q)}\\ &= \frac{1}{2}e^{-\tau}\\ &< e^{-\tau}.\end{aligned}$$

This implies (4.15) is satisfied with $\tau = \frac{1}{2}(|\zeta-\nu|^q+\zeta^q)$, which is a positive number for $\zeta \neq \nu$.

All conditions of (4.4.2) are true, and 0 and 1 are two fixed points of S.

Theorem 4.4.4.

Let $(\mathcal{M},p_{\mathfrak{b}})$ be a complete $P\mathfrak{b}MS$ with $\mathfrak{b} \geq 1$, such that $p_{\mathfrak{b}}$ be a continuous mapping.

Let $S : \mathcal{M} \to CB_{p_\mathfrak{b}}(\mathcal{M})$ be a $MV\alpha\mathcal{F}$-contraction mapping and $B \subset (0, \infty)$ with $\inf B > 0$*, suppose that*

(i) *S is continuous;*

(ii) *S is an α-admissible mapping;*

(iii) *there exists a $m_0 \in \mathcal{M}$ and $m_1 \in Sm_0$ such that $\alpha(m_0, m_1) \geq 1$;*

(iv) $\mathcal{F}(\inf B) = \inf \mathcal{F}(B)$*, where $\mathcal{F} \in \Delta_f - family$,*

then S has a fixed point.

Proof. We take an arbitrary $m_0 \in \mathcal{M}$, as Sm being set of all images of $m \in \mathcal{M}$ is non empty for all values in $\mathcal{M}$, we can choose $m_1 \in Sm_0$. If $m_1 \in Sm_1$ this means that m_1 is the fixed point of S, So suppose $m_1 \notin Sm_1$, as Sm_1 is closed, so we have

$$p_\mathfrak{b}(m_1, Sm_1) > 0.$$

Also, we know that

$$p_\mathfrak{b}(m_1, Sm_1) \leq \mathcal{H}_{p_\mathfrak{b}}(Sm_0, Sm_1).$$

$$\mathcal{F}(p_\mathfrak{b}(m_1, Sm_1)) \leq \mathcal{F}(\mathcal{H}_{p_\mathfrak{b}}(Sm_0, Sm_1)), \quad \text{by} F_2. \tag{4.18}$$

using (*4*)

$$\mathcal{F}(p_\mathfrak{b}(m_1, Sm_1)) = \inf_{g \in Sm_1} \mathcal{F}(p_\mathfrak{b}(m_1, g)).$$

$\Longrightarrow$

$$\inf_{g \in Sm_1} \mathcal{F}(p_\mathfrak{b}(m_1, g)) \leq \mathcal{F}(\mathcal{H}_{p_\mathfrak{b}}(Sm_0, Sm_1)). \tag{4.19}$$

As Sm_1 is compact so we can find a $m_2 \in Sm_1$ such that

$$\inf_{g \in Sm_1} \mathcal{F}(p_\mathfrak{b}(m_1, g)) = \mathcal{F}(p_\mathfrak{b}(m_1, m_2)).$$

(4.18) gives,

$$\mathcal{F}(p_\mathfrak{b}(m_1, m_2)) \leq \mathcal{F}(\mathcal{H}_{p_\mathfrak{b}}(Sm_0, Sm_1)). \tag{4.20}$$

Similarly for $m_3 \in Sm_2$, we get

$$\mathcal{F}(p_{\mathfrak{b}}(m_2, m_3)) \leq \mathcal{F}(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_1, Sm_2)),$$

which ultimately gives

$$\mathcal{F}(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2})) \leq \mathcal{F}(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_{\xi}, Sm_{\xi+1})).$$

As $\mathfrak{b} \geq 1$, so we can write

$$\mathcal{F}(\mathfrak{b}(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}))) \leq \mathcal{F}(\mathfrak{b}(\mathcal{H}_{p_{\mathfrak{b}}}(Sm_{\xi}, Sm_{\xi+1}))). \tag{4.21}$$

For $m_0 \in \mathcal{M}$ by assumption, $\alpha(m_0, m_1) \geq 1$ for some $m_1 \in Sm_0$, similarly for some $m_2 \in Sm_1$ we have $\alpha(m_1, m_2) \geq 1$ and for any sequence $m_{\xi+1} \in Sm_{\xi}$, we may write we get

$$\alpha(m_{\xi}, m_{\xi+1}) \geq 1 \quad \text{for all} \quad \xi \in \mathbb{N} \cup \{0\}. \tag{4.22}$$

Using (4.5),

$$\tau + \mathcal{F}(\alpha(m_{\xi}, m_{\xi+1})(\mathcal{H}_{p\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}))) \leq \mathcal{F}(\mathbb{M}(m_{\xi+1}, m_{\xi})),$$

(4.22) $\implies$

$$\tau + \mathcal{F}(\mathfrak{b}(\mathcal{H}_{p\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}))) \leq \mathcal{F}(\mathbb{M}(m_{\xi+1}, m_{\xi})),$$

using (4.21), we have

$$\mathcal{F}(\mathfrak{b}(p_{\mathfrak{b}}(m_{\xi+1}, m_{\xi+2}))) \leq \mathcal{F}(\mathbb{M}(m_{\xi+1}, m_{\xi})) - \tau. \tag{4.23}$$

Now using lemma (4.4.1)

$$\lim_{\xi \to \infty} \mathfrak{b}^{\xi} v_{\xi} = 0,$$

Now by F_3, for any $\gamma \in (0, 1)$ and for all $\xi \in \mathbb{N}$,

$$\lim_{\xi \to \infty} (\mathfrak{b}^{\xi} v_{\xi})^{\gamma} \mathcal{F} \mathfrak{b}^{\xi}(v_{\xi}) = 0,$$

$$\implies (\mathfrak{b}^{\xi} v_{\xi})^{\gamma}(\mathcal{F}\mathfrak{b}^{\xi}(v_{\xi}) - \mathcal{F}(v_0)) \leq -(\mathfrak{b}^{\xi} v_{\xi})^{\gamma}\xi\tau \leq 0. \tag{4.24}$$

As $\tau > 0$, we have

$$(\mathfrak{b}^{\xi} v_{\xi})^{\gamma}\xi = 0.$$

So there exists a $\xi_1 \in \mathbb{N}$, such that $(\mathfrak{b}^{\xi} v_{\xi})^{\gamma}\xi \leq 1$ for all $\xi \geq \xi_1$. So

$$\mathfrak{b}^{\xi} v_{\xi} \leq \frac{1}{\xi^{\frac{1}{\gamma}}}. \tag{4.25}$$

Next, to prove that $\{m_{\xi}\}$ is a Cauchy sequence in $\mathcal{M}$. For this, follow the same steps as done in theorem (4.4.2). One can easily have

$$\lim_{\xi \to \infty} p_{\mathfrak{b}}(m_{\xi}, m^*) = p_{\mathfrak{b}}(m^*, m^*) = 0. \tag{4.26}$$

We claim that m^* is a fixed point of S. Suppose that $p_{\mathfrak{b}}(m^*, Sm^*) > 0$ this means there exists $k_0 \in \mathbb{N}$ such that we have $p_{\mathfrak{b}}(m_{\xi}, Sm^*) > 0$ for all $\xi > k_0$.

$$p_{\mathfrak{b}}(m_{\xi}, Sm^*) \leq \mathcal{H}_{p\mathfrak{b}}(Sm_{\xi+1}, Sm^*).$$

Using (4.5) and taking limit $\xi \to \infty$, we have,

$$\begin{aligned} \tau + \mathcal{F}(p_{\mathfrak{b}}(m^*, Sm^*)) &\leq \tau + \mathcal{F}(\alpha(m^*, m^*)\mathcal{H}_{p\mathfrak{b}}(Sm^*, Sm^*)) \\ &\leq \mathcal{F}(\mathbb{M}(m^*, m^*)) \\ &\leq \mathcal{F}(p_{\mathfrak{b}}(m^*, Sm^*)). \end{aligned}$$

Where,

$$\begin{aligned} \mathbb{M}(m^*, m^*) = \max\{&p_{\mathfrak{b}}(m^*, m^*), p_{\mathfrak{b}}(m^*, Sm^*), p_{\mathfrak{b}}(m^*, Sm^*) \\ &, \frac{p_{\mathfrak{b}}(m^*, Sm^*) + p_{\mathfrak{b}}(Sm^*, m^*)}{2\mathfrak{b}}\} \\ \leq\ & p_{\mathfrak{b}}(m^*, Sm^*). \end{aligned}$$

$$\implies \tau + \mathcal{F}(p_{\mathfrak{b}}(m^*, Sm^*)) \leq \mathcal{F}(p_{\mathfrak{b}}(m^*, Sm^*)).$$

Since $\tau > 0$. The above relation yields a contradiction.

$$\implies p_{\mathfrak{b}}(m^*, Sm^*) = 0.$$

Also $p_{\mathfrak{b}}(m^*, m^*) = 0$. This gives $m^* \in \bar{S}m^* = Sm^*$. Proving that m^* is a fixed point of S. □

4.5 Application

Now we apply our main result to find a solution to an integral equation of Fredholm type.

Suppose $I = [0, 1]$, and $\mathcal{M} = \mathcal{C}(I, \mathbb{R}^2)$ be the space of all continuous functions defined from I to $\mathbb{R}^2$, endowed with usual sup-norm.

We define a partial $\mathfrak{b}$ metric on $\mathcal{M}$, as

$$p_{\mathfrak{b}}(\phi, \psi) = \parallel \phi - \psi \parallel_{\infty} = \sup_{m \in I}\{e^{-mp}|\phi(m) - \psi(m)|^q\} \quad p, q > 1,$$

for all $\phi, \psi \in \mathcal{M}$. It is easy to verify that $(\mathcal{M}, p_{\mathfrak{b}})$ is a complete P$\mathfrak{b}$MS.

Consider a Fredholm Integral inclusion

$$\phi(\zeta) \in f(\zeta) + \int_0^1 k_{\phi}(\zeta, x^*, \phi(x^*))dx^*, \tag{4.27}$$

such that for every $\mathcal{K}_{\phi} : I \times I \times \mathbb{R}^2 \to K$ there exists $k_{\phi}(\zeta, x^*, \phi^*) \in \mathcal{K}_{\phi}(\zeta, x^*, \phi^*)$.

Define a multivalued mapping $S : \mathcal{M} \to K(\mathcal{M})$, as

$$S(\phi(\zeta)) = \left\{\phi^*(\zeta) : \phi^*(\zeta) \in \omega(\zeta) + \int_0^1 \mathcal{K}_{\phi}(\zeta, x^*, \phi(x^*))dx^*\right\}. \tag{4.28}$$

Theorem 4.5.1.

Suppose that the following conditions hold:

(i) Let $\mathcal{K}_{\phi} : I \times I \times \mathbb{R}^2 \to \mathbb{R}^2$ and $f : I \to \mathbb{R}^2$ be continuous;

(ii) there exists $\phi_0 \in \mathcal{M}$ such that $\phi_k \in S\phi_{k-1}$;

(iii) there exists a continuous function $\mathfrak{f} : I \times I \to I$ such that

$$|k_{\phi}(\zeta, x^*, \phi(x^*)) - k_{\psi}(\zeta, x^*, \psi(x^*))|^q \leq \sup_{x^* \in I} \mathfrak{f}(\phi(x^*), \psi(x^*))|\phi(x^*) - \psi(x^*)|^q,$$

for each $\zeta, x^* \in I$ *and* $\mathfrak{f}(\phi(x^*), \psi(x^*)) \leq \gamma$.

Then the integral inclusion (4.27) *has a solution.*

Proof. Let $(\mathcal{M}, p_{\mathfrak{b}})$ be a complete P𝔟MS. We define $\mathcal{F}$ map as, $\mathcal{F}(\zeta) = \ln(\zeta)$, for all $\zeta \in \mathcal{M}$. So after going through a natural logarithm, our condition will be,

$$\mathcal{H}_{p_{\mathfrak{b}}}(S(\phi(\zeta), S\psi(\zeta))) \leq e^{-\tau} M(\phi, \psi),$$

with $\alpha(\phi, \psi) = 1$.

Next to show that S satisfies this condition, Let $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then for $\phi^* \in S(\phi)$,we have

$$\begin{aligned}
p_{\mathfrak{b}}((\phi^*(\zeta), S(\psi(\zeta)))) &\leq p_{\mathfrak{b}}(\phi^*(\zeta), (\psi^*(\zeta))) \\
&= \sup_{\zeta \in I} e^{-\zeta\gamma} |\phi^*(\zeta) - \psi^*(\zeta)|^q \\
&= \sup_{\zeta \in I} e^{-\zeta\gamma} \left| \int_0^1 k_\phi(\zeta, x^*, \phi(x^*)) - k_\psi(\zeta, x^*, \psi(x^*)) \right|^q dx^* \\
&\leq \sup_{\zeta \in I} e^{-\zeta\gamma} \Bigg[(\int_0^1 |1|^p dx^*)^{\frac{1}{p}} \\
&\quad \int_0^1 \left(|k_\phi(\zeta, x^*, \phi(x^*)) - k_\psi(\zeta, x^*, \psi(x^*))|^q \right)^{\frac{1}{q}} \Bigg]^q dx^* \\
&= \sup_{\zeta \in I} e^{-\zeta\gamma} \int_0^1 |k_\phi(\zeta, x^*, \phi(x^*)) - k_\psi(\zeta, x^*, \psi(x^*))|^q dx^* \\
&= \sup_{\zeta \in I} e^{-\zeta\gamma} \int_0^1 |e^{-x^*\gamma + x^*\gamma} k_\phi(\zeta, x^*, \phi(x^*)) - k_\psi(\zeta, x^*, \psi(x^*))|^q dx^* \\
&\leq \sup_{\zeta \in I} e^{-\zeta\gamma} \int_0^1 e^{x^*\gamma} \mathfrak{f}(\phi(x^*), \psi(x^*)) \\
&\quad \sup_{x^* \in I} e^{-x^*\gamma} \big|\phi(x^*) - \psi(x^*)\big|^q dx^* \\
&= \gamma \parallel \phi(x^*) - \psi(x^*) \parallel_\infty \sup_{\zeta \in I} e^{-\zeta\gamma} \int_0^1 e^{x^*\gamma} dx^* \\
&= p_{\mathfrak{b}}(\phi(x^*), \psi(x^*))(1)(e^\gamma - 1) \\
&\leq p_{\mathfrak{b}}(\phi(x^*), \psi(x^*)) e^\gamma \\
&\leq e^\gamma \mathbb{M}(\phi(x^*), \psi(x^*))
\end{aligned}$$

Where,

$$\mathbb{M}(\phi(x^*),\psi(x^*)) = \max\Big\{p_{\mathfrak{b}}(\phi(x^*),\psi(x^*)), p_{\mathfrak{b}}(\phi(x^*),S(\phi(x^*))), p_{\mathfrak{b}}(\psi(x^*),S(\psi(x^*))), \frac{p_{\mathfrak{b}}(\phi(x^*),S(\psi(x^*))) + p_{\mathfrak{b}}(\psi(x^*),S(\phi(x^*)))}{2\mathfrak{b}}\Big\}.$$

Also, as ϕ^* is arbitrary, so we have

$$\delta_{p_{\mathfrak{b}}}(S(\phi),S(\psi)) \leq e^{\gamma}\mathbb{M}(\phi,\psi).$$

Similarly, one can calculate

$$\delta_{p_{\mathfrak{b}}}(S(\psi),S(\phi)) \leq e^{\gamma}\mathbb{M}(\psi,\phi),$$

$$\implies \mathcal{H}_{p_{\mathfrak{b}}}(S(\phi),S(\psi)) \leq e^{\gamma}\mathbb{M}(\phi,\psi),$$

or equally $\mathcal{H}_{p_{\mathfrak{b}}}(S(\phi),S(\psi)) \leq e^{-\tau}M(\phi,\psi)$.

This shows our desired contraction condition is satisfied by choice of $-\tau = \gamma$.

All conditions of the theorem (4.4.2) are satisfied, showing that integral inclusion (4.27) has a solution. □

Chapter 5

Conclusions

This research work arrives at its end in the following fashion:

- Our work started with a concise presentation of basic ideas, bringing up the related historical results and some important work done by numerous researchers.
- A quick history is referenced for a brief discussion on fixed point theory. This facilitates comprehending the idea of the existence and uniqueness of the fixed point under various conditions in a space.
- Some suitable results are presented for a better understanding of continuous, Lipschitzian, contraction and contractive mappings, which are utilized in our main theorems.
- A thorough examination of metric space, partial metric space, $\mathfrak{b}$-metric space, and partial $\mathfrak{b}$ metric space is presented. Primary tools for these spaces are demonstrated with examples to differentiate their format.
- A quick review of F mappings by explaining their properties is provided. Some important theorems and results are also linked for a finer understanding of this notion.

- All the properties of the Hausdorff distance are illustrated separately in the domain of the metric space, partial metric space, and partial $\mathfrak{b}$ metric space, with support of examples.

- A segment dealing with alpha admissible mappings is also articulated, which is later integrated into our main theorem.

- Getting motivation from the work of S. Kumar et al., an extension is constructed on the platform of partial $\mathfrak{b}$ metric space under multivalued alpha admissible F contraction maps. Our theorem is equipped with an example to have a more satisfactory understanding. An application is also attached to validate our result.

Future Work

- In the future, one can attempt to find common fixed points in partial $\mathfrak{b}$ metric space under multivalued alpha F contraction mapping.

- Expanding these results in the setting of "extended partial $\mathfrak{b}$ metric space" would be a wonderful idea.

- By introducing more properties in F mapping, one can play with the validity of our results.

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