CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Generalization of BCP in the Setup of Complete b-Metric Space

by

Umaira Latif

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

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Dedicated to my parents

CAPITAL UNIVERSITY OF SCIENCE & TECHNOLOGY
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Abstract

In this present dissertation, we introduce JS-contraction in b-metric spaces. JS-contraction played an important role in the extension and generalization of Banach contraction principle. We have extended the notion of JS-contraction in generalized b-metric spaces and establish and prove fixed point results for such contraction in the setting of generalized b-metric spaces. We introduce a new family for modified JS-contraction and prove fixed point results. Furthermore, we propose generalized modified JS-contraction in b-metric spaces and establish and prove fixed point result for such contraction in the framework of complete b-metric spaces. All our results are extensions and generalization of various results in the literature of fixed point theorems.

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Chapter 1

Introduction

Mathematics has great significance in scientific knowledge which has several applications for humanity and in every field of life. Mathematics is further divided into various branches which have their own significance according to their implementation. Functional analysis is one of foremost branch of mathematics which has substantial uses in different fields. It is widely used in finding solutions of linear and non-linear partial differential equations. It is widely applicable in numerical analysis such as finding solutions of linear and non-linear partial differential equation, error estimation of polynomial, interpolation and finite difference method. Functional analysis accomplishes the beauty of combination of geometry and analysis. The valuable concept of fixed point theory in functional analysis has great importance because of its use in various fields of sciences which enhances the significance of functional analysis such as mathematical economics, game theory, optimization theory, approximation theory and in variational inequalities etc.

Firstly, fixed point theory was considered as entirely pure analytical theory but later on it was divided into different branches which are metric, discrete and topological fixed point theory. One of the most valuable theorem in fixed point theory is fixed point theorem "The Banach Contraction principle" which has significant consequences in metric fixed point theory. This principle states that

"On a complete metric space a contraction mapping has a unique fixed point."

This is a widely known principle which is an essential tool in the development of nonlinear analysis in general and metric fixed point theory. It was first appeared in 1922, in an explicit form in Banach's [7] thesis where solution for an integral equation was obtained by using this theorem. Therefore according to its significance and convenience extensions of the Banach contraction principle have been established either by generalizing the domain of the mapping or by extending the contractive condition on the mapping.

Bakhtin [6] first introduced the concept of b-metric space, then implemented by Czerwick [16] in 1974, Ekeland proposed the variational principle in b-metric space and fixed point theory is one of the application of Elceland's variational principle. It used as the main tool in the proof of the fixed point theorem in complete metric space. The use of different aspects of b-metric space in literature is obvious. Many author's research are found on b-metric space in the field of fixed point theory.

In 2000, Branciari [9] proposed the new concept of metric space this refined metric is known as generalized metric space as well as rectangular metric space, in generalized metric spaces the triangular inequality is substituted by the the inequality $d(x, z) \leq d(x, r) + d(r, s) + d(s, z)$ for all pairwise unique points $x, z, r, s \in X$. Many fixed theorems are proved by many author's in generalized metric space by taking different contractions mapping. [[5], [15], [18], [17]].

In last few year the "Banach contraction principle" has been generalized in many ways by changing the nature of contraction mapping, but we will discuss only those which we used in our thesis work.

In 2013, Jeli and Samet [22] proposed a new type of contraction named as JS-contraction and prove Banach contraction principle for such contraction in the setting of generalized b-metric space. In 2015 Hussain et al. [20] modified JS-contraction and prove fixed point result for such contraction. In 2016, Ahmad et al. [3] prove common fixed point results for a pair of self mapping in the setup of complete metric space by using generalized modified JS-contraction.

In this dissertation, we review the paper of Jleli and Samet [22], Hussain et al. [20] and Ahmad et al [3]. We have extended the result of Jleli and Samet [22] by changing generalized metric space into generalized b-metric space. Further more we have extended the results of Hussain et al.[20] and Ahmad et al.[3] by replacing metric space into b-metric space.

The thesis is organized as follows.

- In Chapter 2, we focused on definition with examples and review of papers.
- In Chapter 3, we have extended and explained briefly the results of Jleli and Samat [22].
- In Chapter 4, deal with an extension of results proved by Hussain et al. [20].
- In Chapter 5, a brief conclusion of an extension work of Ahmad et al. [3] is given and ends with the conclusion.

Chapter 2

Preliminaries

This chapter is divided into four section. The first section includes, metric space and rectangular metric space with some examples. The second section is devoted to the notions of b-metric space, rectangular b-metric space and some related stuff. In the third and fourth section our aim is to review JS-contraction and modified JS-contraction which were defined by Jleli and Samet and modified by Hussain et al. respectively. We have also reviewed the results of fixed point problem for JS-contraction and modified Js-contraction.

2.1 Metric Space and Generalized Metric Space

In this section, we recall the notion of a metric which is nonempty set X equipped with a distance function d satisfying some properties. Throughout $\mathbb{R}$ mean to the set of real number.

Definition 2.1.1. [26] **(Metric Space)**

"A metric space (X, d) consists of a non-empty set X and a function $d\colon X \times X \to \mathbb{R}$ such that:

(i) $d(x, y) \geq 0, d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$ (Positivity)

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$ (Symmetry)

(iii) $d(x,y) \leq d(x,z) + d(z,x)$ for all $x, y \in X$ (Triangle inequality)

A function d satisfying conditions $(i) - (iii)$, is called a metric on X."

Example 2.1.2. Let $X = \mathbb{R}$ and define $d_* \colon X \times X \to \mathbb{R}$ as

$$d_*(t,u) = |t - u|$$

then $(\mathbb{R}, d_*)$ be a metric space and d is called Usual metric on $\mathbb{R}$.

Example 2.1.3. Let $X = \mathbb{R}^2$, define $d_* \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$$d_*(t,u) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}.$$

where $t = (\xi_1, \xi_2), u = (\eta_1, \eta_2) \in \mathbb{R}^2$
Then d_* be a metric on $\mathbb{R}$ and $(\mathbb{R}^2, d_*)$ is a Euclidean metric space.

Example 2.1.4. Let X consists of all bounded sequences of complex numbers i.e,

$$t = \{\xi_i\}_{i \in \mathbb{N}} \quad \text{or} \quad t = (\xi_1, \xi_2, \ldots) \quad \text{and} \quad |\xi_i| \leq c_t \quad \forall \quad i \in \mathbb{N}$$

Define $d_* \colon X \times X \to \mathbb{R}$ by

$$d_*(t,u) = \sup_{i \in \mathbb{N}} |\xi_i - \eta_i|$$

Where $t, u \in X$, $t = \{\xi_i\}$, $u = \{\eta_i\}$ and the sup denote the supremum (least upper bound).

Example 2.1.5. "Let $X = C[a,b]$ be the set of all real-valued continuous function defined on a close interval $[a,b]$. The function $d \colon X \times X \to \mathbb{R}$ given by

$$d(x,y) = \max_{t \in [a,b]} |x(t) - y(t)| \quad x, y \in C[a,b]$$

is a metric on X and (X,d) is a metric space denoted by $C[a,b]$."

Example 2.1.6. [26] "Let $X = B(A)$ be the set of all bounded functions defined on the set A then $d\colon B(A) \times B(A) \to \mathbb{R}$ given by

$$d(x,y) = \sup_{t \in A} |x(t) - y(t)|$$

is a metric on $B(A)$. For a set $A = [a,b] \subseteq \mathbb{R}$; $B(A)$ is denoted as $B[a,b]$."

Example 2.1.7. [10] "The space of real or complex number sequences $x = \{\xi_n\}_{n=1}^{\infty}$ such that for some $p \geq 1$ the infinite series $\sum_{n=1}^{\infty} |\xi_n|^p$ converges. The space is denoted by ℓ^p.

The metric $d\colon \ell^p \times \ell^p \to \mathbb{R}$ is given by

$$d(x,y) = \left(\sum_{n=1}^{\infty} |\xi_n - \eta_n|^p \right)^{1/p} \qquad x, y \in \ell^p$$

Where $y = \{\eta_n\}$ and $\sum |\eta_n|^p < \infty$.

For $p = 2$, we get the Hilbert sequence space ℓ^2 with metric given by

$$d(x,y) = \sqrt{\sum_{n=1}^{\infty} |\xi_n - \eta_n|^2}."$$

In 2000," Branciari [9] introduced the idea of rectangular metric space by changing the sum of right hand side of the triangular inequality in metric space by the three terms expression."

Definition 2.1.8. [26]**(Rectangular Metric Space)**

"Let X be a nonempty set and the mapping $d\colon X \times X \to [0, \infty)$ satisfies:

$(M1)$ $d(x,y) \geq 0$, $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;

$(M2)$ $d(x,y) = d(y,x)$ for all $x, y \in X$;

$(M3)$ $d(x,y) \leq d(x,r) + d(u,v) + d(s,y)$ for all $x, y \in X$ and all distinct point $u, v \in X \setminus \{x, y\}$

Then d is called rectangular metric on X and (X,d) is called a rectangular metric space (in short RMS)."

Example 2.1.9. [29] “Let (X, ρ) be a bounded metric space and let M be a real number satisfying

$$\sup\{\rho(x,y) : x, y \in X\}$$

Let A and B be subset of X with $X = A \cup B$ and $A \cap B = \phi$

Define a function d from $X \times X$ into $[0, \infty)$ by

$$\begin{cases} d(x,y) & = 0 \\ d(x,y) & = d(y,x) = \rho(x,y) \quad if \quad x \in A, y \in B \\ d(x,y) & = M \quad otherwise \end{cases}$$

Then (X, d) is a generalized metric space.”

2.2 b-Metric Space and Rectangular b-Metric Space

Definition 2.2.1. **(b-Metric Space)**

“Let X be a non-empty set and a mapping $d \colon X \times X \to [0, \infty)$ satisfies:

$(bM1)$ $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;

$(bM2)$ $d(x,y) = d(y,x)$ for all $x, y \in X$;

$(bM3)$ there exist a real number $b \geq 1$ such that

$$d(x,y) \leq b[d(x,z) + d(z,y)] \text{ for all } x, y, z \in X$$

Then d is called a b-metric on X and (X, d) is called a b-metric space (in short bMS) with co-efficient s.”

Note that every metric space is b-metric space (with coefficient $s = 1$).

Example 2.2.2. "Let $X = \{0, 1, 2\}$, and let

$$d(x, y) = \begin{cases} 2, & \text{if } x = y = 0 \\ \frac{1}{2}, & \text{if } otherwise. \end{cases}$$

Then (X, d) is a b-metric with coefficient $b = 2$."

Example 2.2.3. [10]" Let ℓ_p, $(0 < p < 1)$

$$\ell_p = \{(\xi_n) \subset R \colon \sum_{n=1}^{\infty} |\xi_n|^p < \infty\},$$

together with the function $d \colon \ell_p \times \ell_p \to \mathbb{R}$
where

$$d(x, y) = \left\{ \sum_{n=1}^{\infty} |\xi_n - \eta_n|^p \right\}^{1/p}$$

where $x = \xi_n$, $y = \eta_n \in \ell_p$ is b-metric space. By an elementary calculation we obtain that

$$d(x, z) = 2^{1/p}[d(x, y) + d(y, z)]$$

Example 2.2.4. The space ℓ_p , $(0 < p < 1)$ of all real functions $x(t)$, $t \in [0, 1]$ such that

$$\int_0^1 |\xi(t)|^p \, dx < \infty$$

is a b-metric space if we take

$$d(x, y) = \left(\int_0^1 |\xi(t) - \eta(t)|^p \, dt \right)^{1/p}$$

for each $x, y \in \ell_p$."

Definition 2.2.5. Let (X, d) be a metric space or b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

1. **Convergent Sequence**

   "The sequence $\{x_n\}$ is said to be convergent in (X, d) and convergent to x, if for every $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$."

2. **Cauchy Sequence**

   " The sequence $\{x_n\}$ is said to be Cauchy sequence if for every $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that for each $n, m \geq n_0$ we have $d(x_n, x_p) < \epsilon$."

3. **Completeness**

   "(X, d) is said to be complete b-metric space if every Cauchy sequence in X converges to some $x \in X$."

Definition 2.2.6. [19] **(Generalized b-Metric Space)**

"Let X be a non-empty set and a mapping $d\colon X \times X \to [0, \infty)$ satisfies:

$(bM1)$ $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

$(bM2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;

$(bM3)$ there exist a real number $s \geq 1$ such that

$$d(x, y) \leq b[d(x, u) + d(u, v) + d(v, y)]$$

for all $u, v \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$.

Then d is called a rectangular b-metric on X and (X, d) is called a b-metric space (in short $GbMS$) with co-efficient b.

Note that every metric space is rectangular metric space and every generalized metric space is a rectangular b-metric space (with coefficient $b = 1$). However the converse of the above implication is not necessarily true."

Example 2.2.7. [19] "Let $X = \mathbb{N}$, define $d\colon X \times X \to X$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 10\alpha, & \text{if } x = 1, y = 2 \\ \alpha, & \text{if } x \in \{1, 2\} \textit{ and } y \in \{3\} \\ 2\alpha, & \text{if } x \in \{1, 2, 3\} \textit{ and } y \in \{4\} \\ 3\alpha, & \text{if } x \textit{ or } y \notin \{1, 2, 3, 4\} \textit{ and } x \neq y \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a generalized b-metric space with coefficient $b = 2 > 1$."

Example 2.2.8. [19] "Let $X = \mathbb{N}$, define $d\colon X \times X \to X$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 4\alpha, & \text{if } x, y \in \{1,2\} \textit{ and } x \neq y \\ \alpha, & \text{if } x \textit{ or } y \notin \{1,2\} \textit{ and } x \neq y \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a rectangular b-metric space with coefficient $b = \frac{4}{3} > 1$, but (X, d) is not a rectangular metric space, as $d(1,2) = 4\alpha > 3\alpha = d(1,3) + d(3,4) + d(4,2)$."

The limit in the b-metric space is not unique, so every convergent sequence in b-metric space is not Cauchy. It is clear from the Example 2.2.2.
In Example 2.2.2, let $x_n = 2$ for each $n = 1, 2, ...$, then is clear that $\lim\limits_{n\to+\infty} d(x_n, 2) = 1/2$ and $\lim\limits_{n\to+\infty} d(x_n, 0) = 2$, hence in b-metric limit is not necessarily unique.

Definition 2.2.9. [19] Let (X, d) be a generalized metric space or generalized b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

1. **Convergent Sequence**
   "The sequence $\{x_n\}$ is said to be convergent in (X, d) and converges to x, if for every $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n > n_0$ or this fact is represented by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$."

2. **Cauchy Sequence**
   "The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, d) if for every $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \epsilon$ for all $n > n_0, p > 0$ or equivalently, $\lim\limits_{n\to\infty} d(x_n, x_{n+m}) = 0$ for all $p > 0$."

3. **Completeness**
   "(X, d) is said to be complete generalized b-metric space if every Cauchy sequence in X converges to some $x \in X$."

Note that, limit of a sequence in generalized b-metric space is not necessarily unique. It is clear from the following example.

Example 2.2.10. [19]“ Let $X = A \cup B$, where $A = \left\{ \dfrac{1}{n} \colon n \in \mathbb{N} \right\}$ and B is the set of all positive integers. Define $d \colon X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 2\alpha, & \text{if } x, y \in A; \\ \dfrac{\alpha}{2n}, & \text{if } x \in A \text{ and } y \in \{2,3\} \\ \alpha, & \text{otherwise} \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is generalized b-metric space with coefficient $b = 2 > 1$.” The sequence $\left\{\frac{1}{n}\right\}$ converges to 2 and 3 in generalized b-metric and so limit is not unique.

2.3 A new Generalization of BCP in Generalized Metric Space

We will present here the review of JS-contraction and fixed point results which were established and proved by Jleli and Samet [22], for such contraction in the setup of complete metric space. We have reviewed the results of Jleli and Samet.

2.3.1 JS-contraction

In 2013, Jleli and Samet [22] gave the idea of JS-contraction and prove fixed point results by using such contraction in the setup of complete metric space.
“We denote by Θ the set of functions $\phi \colon (0, \infty) \to (1, \infty)$ satisfying the following conditions.[22]

(θ) θ is non-decreasing.

(θ) for each sequence $\{t_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to+\infty} \phi(t_n) = 1$if and only if $\lim_{n\to\infty}(t_n) = 0$.

(θ) there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{\alpha\to 0^+} \dfrac{\theta(t)-1}{t^r} = \ell$."

Definition 2.3.1. " Let (X,d) be a rectangular metric space and a given self mapping $V\colon X \to X$ is said to a JS-contraction if there exist a function $\theta \in \Theta$ and for any constant $k \in (0,1)$ such that

$$d(Tx,Ty) \neq 0 \quad \Rightarrow \quad \theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^\alpha$$

for every $x, y \in X$."

Theorem 2.3.2. *[22] "Let (X,d) be a complete g.m.s metric space and $T\colon X \to X$ be a given map. Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$\textit{for all } x,y \in X, \quad d(Tx,Ty) \neq 0 \quad \Rightarrow \quad \theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k \tag{2.1}$$

Then T has only one fixed point."

Proof. See Theorem 3.1.8 □

Since a metric space is rectangular metric space, from Theorem 2.3.2. the following result has been concluded.

Corollary 2.3.3. *[22]" Let (X,d) be a complete metric space and $T\colon X \to X$ be a given self map. Assume that there exist $\Theta \in \theta$ and $k \in (0,1)$ such that*

$$\textit{for all } \; x,y \in X, \quad d(Tx,Ty) \neq 0 \quad \Rightarrow \quad \theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^k \tag{2.2}$$

Then T has only one fixed point."

Let $f\colon X \to X$ be a self mapping and (X, d_*) be metric space. Notice that from Corollary 2.3.3, the Banach Contraction contraction principle follows directly. Certainly if T is a Banach Contraction then for any $\mu \in (0,1)$ such that

$$d_*(fx,fy) \leq \mu d_*(x,y), \quad \forall x,y \in X$$

This implies that

$$e^{d_*(fx,fy)} \le \left[e^{d_*(x,y)}\right]^{\mu}, \quad \forall x, y \in X$$

It is clear that the function $\phi\colon (0,\infty) \to (1,\infty)$ defined by $\phi(u) = e^{\sqrt{u}}$ belongs to the family Φ. The Corollary 2.3.3 shows the existence and uniqueness. it is also shown from example that the Corollary 2.3.3, be a real "generalization of the Banach contraction principle".

Example 2.3.4. Let us define the set Y

$$Y = \{\kappa \in \mathbb{N}\}$$

where

$$\kappa_m = \frac{m(m+1)}{2}, \quad \text{for every} \ \ m \in \mathbb{N}$$

The metric $d\colon Y \times Y \to Y$ is defined by $d(u,t) = |u - t|$ for every $u, t \in Y$. We can show easily that (Y, d) is a complete metric space. Let $V : Y \to Y$ be the self mapping defined as follows

$$V\kappa_1 = \kappa_1, \quad V\kappa_m = \kappa_{m-1}, \quad \text{for all} \quad m \ge 2$$

We can check easily that Banach contraction does not hold.

$$\lim_{m\to\infty} \frac{d(V\kappa_m, V\kappa_1)}{d(\kappa_m, \kappa_1)} = 1$$

Now, take a function $\psi\colon (0,\infty) \to (1,\infty)$ defined by $\psi(u) = e^{\sqrt{ue^u}}$. Then it can be shown easily that $\phi \in \Phi$. Now, our aim is show V fulfill the condition of the result 2.3.3, i.e

$$d(V\kappa_m, V\kappa_n) \ne 0 \quad \Rightarrow \quad e^{\sqrt{d(V\kappa_m,V\kappa_n)e^{d(V\kappa_m,V\kappa_n)}}} \le e^{\alpha\sqrt{d(\kappa_m,\tau_n)e^{d(\kappa_n,\kappa_n)}}}$$

for any $\alpha \in (0,1)$.

From the above inequality, implies that

$$d(V\kappa_m, V\kappa_n)e^{d(V\kappa_m, V\kappa_n)} \leq \alpha^2 d(\kappa_m, \kappa_n)e^{d(\kappa_m, \kappa_n)}$$

So, we have to check that

$$d(V\kappa_m, V\kappa_n) \neq 0 \quad \Rightarrow \quad \frac{d(V\kappa_m, V\kappa_n)e^{d(V\kappa_m, V\kappa_n)-d(\kappa_m,\kappa_n)}}{d(\kappa_m, \kappa_n)} \leq \alpha^2 \tag{2.3}$$

for any $\alpha \in (0,1)$. Let us considering two cases.

Case i. $m = 1$ and $n > 2$. Check for this case, we have

$$\frac{d(V\kappa_m, V\kappa_n)e^{d(V\kappa_m, V\kappa_n)-d(\kappa_m,\kappa_n)}}{d(\kappa_m, \kappa_n)} = \frac{n^2 - n - 2}{n^2 + n - 2}e^{(n^2-2)(-n)} \leq e^{-1}$$

Case ii. $n > m > 1$. Now, check for this case, we have

$$\frac{d(V\kappa_m, V\kappa_n)e^{d(V\kappa_m, V\kappa_n)-d(\kappa_m,\kappa_n)}}{d(\kappa_m, \kappa_n)} = \frac{n + m - 1}{n + n + 1}e^{(m^2-n^2)(n-m)} \leq e^{-1}$$

Hence, the inequality (2.3) is fulfilled for $\alpha = e^{-1/2}$. Corollary 2.3.3 implies that V has at most one fixed point. Observe that for this example κ_1 is the fixed point of V.

2.4 Fixed point Results and Modified JS-contraction

In this section we will review the generalization of Ciric, Chatterjea and Reich contraction. Accordent with [22], Hussain et al. [20] introduced and proved fixed point theorem for self mapping in the setup of complete metric spaces. We present here some results of Hussain.

2.4.1 ϕ-Contractive Condition

The family Φ of the functions ϕ which are defined under some conditions. Hussain et al. [20] modified and extended the conditions of the functions $\phi\colon [0,\infty) \to [1,\infty)$ which are defined as follows.

(ψ_1') "ψ is non-decreasing and $\psi(t) = 1 \quad \Leftrightarrow \quad t = 0$;

(ψ_4') $\psi(a+b) \leq \psi(a)\psi(b)$ for all $a, b > 0$."

The above two conditions are known as ϕ-contractive conditions.The condition $(\psi_1' - \psi_4)$ satisfying by all functions $\phi\colon [0,\infty) \to [1,\infty)$ is denoted by is denoted Ψ. The following fixed point theorem were established and proved by Hussain et al. [20] for ϕ-contraction in the setup of complete metric space.

Theorem 2.4.1. *Let (X, d) be a complete metric space, a $f\colon X \to X$ be a given self mapping. Assume that there exist "a function $\psi \in \Psi$ and positive real number k_1, k_2, k_3 and k_4 with $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$ such that*

$$\begin{aligned} &\psi(d_b(fx, fy)) \\ \leq &[\psi(d(x,y))]^{k_1}[\psi(d(x,fx))]^{k_2}[\psi(d(y,fy))]^{k_3}[\psi(d(x,fy)+d(y,fx))]^{k_4} \end{aligned} \tag{2.4}$$

for each $x, y \in X$", then f has only one fixed point.

Proof. Taking $b = 1$ in Theorem 4.1.2, the proof follows immediately. $\square$

Definition 2.4.2. " Let (X, d) be a metric space. A mapping $f\colon X \to X$ is said to be:

(i) A C-contraction(see[13]) if there exist $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:

$$d(fx, fy) \leq \alpha[d(x, fy) + d(y, fx)];$$

(ii) A K-contraction([24]) if there exist $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:

$$d(fx, fy) \leq \alpha[d(x, fx) + d(y, fy)];$$

(iii) A Reich contraction([27]) iff for all $x, y \in X$ there exist nonnegative numbers q, r, s such that $q + r + s + 2t < 1$ and

$$d(fx, fy) \leq qd(x, y) + rd(x, fx) + sd(y, fy);$$

(iv) A Ciric(see[11]) contraction if and only if for all $x, y \in X$ there exist nonnegative numbers $q + r + s$ and t such that $q + r + t_3 + s + 2t < 1$ and

$$d(fx, fy) \leq qd(x, y) + rd(x, fx) + sd(x, fx) + td(x, fy) + d(y, fx)].\text{''}$$

Theorem 2.4.3. *[20] "Let (X, d) be a complete metric space and $f : Y \to Y$ be a continuous mapping. Suppose that there exist a positive real number k_1, k_2, k_3, k_4 with $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$, such that*

$$\begin{aligned} &\sqrt{d(fx, fy)} \\ &\leq k_1\sqrt{d(x, y)} + k_2\sqrt{d(x, fx)} + k_3\sqrt{d(y, fy)} + k_4\sqrt{(d(x, fy) + d(y, fx))} \end{aligned} \tag{2.5}$$

for all $x, y \in X$, then f has unique fixed point."

Proof. Taking $\psi_b(t) = e^{\sqrt{t}}$ in Theorem 2.4.1 we get the Ciric [11] result. $\square$

Remark 2.4.4. [20] Observe that the following result follows from the condition (2.5).

$$\text{“}d(fx,fy) \leq {k_1}^2 d(x,y) + {k_2}^2 d(x,fx) + {k_3}^2 d(y,fy) + {k_4}^2[d(x,fy)+d(y,fx)]$$
$$+ 2k_1k_2\sqrt{d(x,y)d(x,fx)} + 2k_1k_3\sqrt{d(x,y)d(y,Vy)}$$
$$+ 2k_1k_4\sqrt{d(x,y)[d(x,fy)+d(y,fx)]} + 2k_2k_3\sqrt{d(y,fx)d(y,fy)}$$
$$+ 2k_2k_4\sqrt{d(x,fx)[d(x,fy)+d(y,fx)]}$$
$$+ 2k_3k_4\sqrt{d(y,fy)[d(x,fy)+d(y,fx)]}.\text{”}$$

Further, observe on the Remark 2.4.4, taking $k_1 = k_4 = 0$ in Theorem 2.4.3 follows the Kannan [24] result.

Theorem 2.4.5. *[20] “Let (X,d) be a complete metric space and $f\colon X \to X$ be a given self mapping. Suppose that that there exist positive real numbers k_2, k_3, with $0 < k_2 + k_3 < 1$, such that*

$$d(fx,fy) \leq {k_2}^2 d(x,fx) + {k_3}^2 d(y,fy) + 2k_2k_3\sqrt{d(x,fx)d(y,fy)} \tag{2.6}$$

for all $x, y \in X$. Then f has only one fixed point.”

On another way, by taking $k_1 = k_2 = t_3 = 0$ in Theorem 2.4.3 follows the following Chetterjea[13] result.

Theorem 2.4.6. *[20]“ Let (X,d) be a complete metric space and $f\colon X \to X$ be a continuous mapping. Suppose taht there exist $k_4 \in [0, \frac{1}{2})$ such that*

$$d_b(fx,fy) \leq {k_4}^2[d(x,fy)+d(y,fx)]$$

for all $x, y \in X$. Then f has only one fixed point.”

The following extension of Reich result follows from Theorem 2.4.3 By taking $k_4 = 0$.

Theorem 2.4.7. *[20]“ Let (X,d) be a complete metric space and $f\colon X \to X$ be a continuous mapping. Suppose that there exist positive real number k_1, k_2, k_3,* *with* $0 < k_1 + k_2 + k_3 < 1$*, such that*

$$\begin{aligned} d(fx, fy) \le {} & {k_1}^2 d(x, y) + {k_2}^2 d(x, fx) + {k_3}^2 d_b(y, fy) \\ & + 2k_1k_2\sqrt{d(x, y)d(x, fx)} + 2k_1k_3\sqrt{d(x, y)d(y, fy)} \\ & + 2k_2k_3\sqrt{d(x, fx)d(y, fy)} \end{aligned}$$

for all $x, y \in X$*. Then* f *has only unique fixed point."*

Theorem 2.4.8. *[20]" Let* (X, d) *be a complete metric space and* $f\colon X \to X$ *be a continuous mapping. Suppose that there exist positive real number* k_1, k_2, k_3, k_4 *with* $0 < k_1 + k_2 + k_3 + 2k_4 < 1$*, such that*

$$\sqrt[n]{d(fx, fy)} \le k_1 \sqrt[n]{d(x, y)} + k_2 \sqrt[n]{d(x, fx)} + k_3 \sqrt[n]{d(y, fy)} + k_4 \sqrt[n]{(d(x, fy) + d(y, fx))} \tag{2.7}$$

for all $x, y \in X$*, then* f *has only one fixed point."*

Proof. Taking $\psi(u) = e^{\sqrt[n]{u}}$ in the Theorem 2.4.3, the proof follows immediately. □

Chapter 3

A New Generalization of BCP in $GbMS$

In this chapter we establish and prove Banach contraction principle using JS-contraction in the setup of complete rectangular b-metric spaces. Our aim is to extend the results of Jleli and Samet [22] by changing rectangular metric spaces into rectangular b-metric spaces. An example is also given which illustrates our result.

3.1 JS-Contractions

We will define JS-contraction in rectangular b-metric spaces and then establish and prove fixed point theorem for such contraction in the setup of complete rectangular b-metric spaces.

Ler Φ be the family of all functions $\phi\colon (0,\infty) \to (1,\infty)$ satisfying the following assertions:

(ϕ_1) ϕ is non-decreasing.

(ϕ_2) For each sequence $\{\beta_n\} \subseteq \mathbb{R}^+$, $\lim\limits_{n\to\infty} \phi(\beta_n) = 1 \;\Leftrightarrow\; \lim\limits_{n\to\infty} (\beta_n) = 0.$

(ϕ_3) There exist $0 < h < 1$ and $\ell \in (0, \infty]$ such that $\lim_{\beta \to 0} \dfrac{\phi(\beta) - 1}{\beta^h} = \ell$.

Example 3.1.1. The following are some functions from the family Φ.

(i) $\phi\colon (0, \infty) \to (1, \infty)$ defined by $\phi(u) = e^{\sqrt{u}}$.

(ii) $\phi\colon (0, \infty) \to (1, \infty)$ defined by $\phi(u) = e^{\sqrt{ue^u}}$.

(iii) $\phi\colon (0, \infty) \to (1, \infty)$ defined by $\phi(u) = e^u$.

(iv) $\phi\colon (0, \infty) \to (1, \infty)$ defined by $\phi(u) = \cosh u$.

(v) $\phi\colon (0, \infty) \to (1, \infty)$ defined by $\phi(u) = 1 + \ln(1 + u)$.

(vi) $\phi\colon (0, \infty) \to (1, \infty)$ defined by $\phi(u) = e^{ue^u}$.

Definition 3.1.2. Let $V\colon Y \to Y$ be a given self mapping and (Y, d_b) be a rectangular b-metric with $b \geq 1$, whenever there exist any constant $\alpha \in (0, 1)$ and function $\phi \in \Phi$ satisfying:

$$d_b(Vx, Vy) \neq 0 \quad \Rightarrow \quad \phi(d_b(Vx, Vy)) \leq [\phi(d_b(x, y))]^\alpha.$$

$\forall\, x, y \in Y$, then V is called JS-contraction.

Example 3.1.3. Let $d_b^*\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by $d_b^*(x, z) = (x - z)^2$. Then (Y, d_b^*) be a rectangular b-metric with coefficient $b = 4$ and $V\colon Y \to Y$ be a self mapping defined by $Vy = \dfrac{y}{2}$.

Assume that the function $\phi\colon (0, \infty) \to (1, \infty)$ is defined by $\phi(u) = e^{\sqrt{u}}$. ϕ satisfying the conditions of 3.1. So, $\phi \in \Phi$. Our aim is to prove V is JS-contraction. From Definition 3.1.3, we have

$$\forall\, x, z \in Y, \quad d_b^*(Vx, Vz) \neq 0 \quad \Rightarrow \quad e^{\sqrt{d_b^*(Vx, Vz)}} \leq e^{\alpha\sqrt{d_b^*(x,z)}},$$

for any $\alpha \in (0, 1)$. From the above inequality, we have

$$\begin{aligned} \sqrt{d_b^*(Vx, Vz)} &\leq \alpha\sqrt{d_b^*(x, z)} \\ \frac{\sqrt{d_b^*(Vx, Vz)}}{\sqrt{d_b^*(x, z)}} &\leq \alpha. \end{aligned} \tag{3.1}$$

Consider

$$\frac{\sqrt{d_b^*(Vx,Vz)}}{\sqrt{d_b^*(x,z)}} = \frac{\sqrt{\left(\frac{x}{2}-\frac{z}{2}\right)^2}}{\sqrt{(x-z)^2}} = \frac{\left(\frac{x}{2}-\frac{z}{2}\right)}{(x-z)} = \frac{1}{2}.$$

This implies that the inequality 3.1 hold for $\alpha = \dfrac{1}{2} \in (0,1)$. Hence V is JS-contraction.

Theorem 3.1.4. *Let $V : Y \to Y$ be a given self mapping and (Y, d_b) be a complete rectangular b-metric space with $b \geq 1$, whenever there exist $\phi \in \Phi$ and for any $\alpha \in (0,1)$ satisfying:*

$$d_b(Vx,Vy) \neq 0 \quad \Rightarrow \quad \phi(d_b(Vx,Vy)) \leq [\phi(d_b(x,y))]^\alpha \tag{3.2}$$

$\forall\ x, y \in Y$, then V has only one fixed point.

Proof. Assume that $y_0 \in Y$ be arbitrary. Let us consider a sequence $\{y_m\}$ by $y_{m+1} = Vy_m$ for all $m \geq 0$. We want to prove $\{y_m\}$ is Cauchy sequence. If $y_m = y_{m+1}$ then y_m is fixed point of V, so there is nothing to prove. So, suppose that $y_m \neq y_{m+1}$ for all $m \geq 0$. Setting $d_b(y_m, y_{m+1}) = d_{bm}$ and using 3.2

$$\begin{aligned} 1 < \phi(d_b(y_m, y_{m+1})) &= \phi(d_b(Vy_{m-1}, Vy_m)) \\ &\leq [\phi(d_b(y_{m-1}, y_m))]^\alpha \\ d_{bm} &\leq d_{b(m-1)}^\alpha \end{aligned}$$

Repeating this process

$$d_{bm} \leq d_{b0}^{\alpha^n}$$

$$1 < \phi(d_b(y_m, y_{m+1}) \leq [\phi(d_b(y_1, y_0))]^{\alpha^m}. \tag{3.3}$$

Taking $m \to \infty$ in the above inequality and using Sandwich Theorem, we get

$$\Rightarrow \quad \lim_{m\to\infty} [\phi(d_b(y_0, y_1))]^{\alpha^m} \to 1$$

since $0 < \alpha < 1$, $\alpha^m \to 0$ as $m \to \infty$

$$\Rightarrow \quad \lim_{m\to\infty} [\phi(d_b(y_m, y_{m+1}))] \to 1$$

From the condition (ϕ_2)

$$\lim_{m\to\infty} d_b(y_m, y_{m+1}) = 0.$$

There exist $0 < h < 1$ and $\ell \in (0, \infty]$ from the condition (ϕ_3) such that

$$\lim_{m\to\infty} \frac{\phi(d_b(y_m, y_{m+1})) - 1}{d_b(y_m, y_{m+1})^h} = \ell.$$

Let $\ell < \infty$. Then by definition of limit, choosing $r = \dfrac{\ell}{2}$ there exist a non negative integer $m_0 \in \mathbb{N}$ such that $m > m_0$

$$\left| \frac{\phi(d_b(y_m, y_{m+1})) - 1}{d_b(y_m, y_{m+1})^h} - \ell \right| \leq r.$$

$$-r \leq \frac{\phi(d_b(y_m, y_{m+1})) - 1}{d_b(y_m, y_{m+1})^h} - \ell \leq r$$

Consider

$$\frac{\phi(d_b(y_m, y_{m+1})) - 1}{d_b(y_m, y_{m+1})^h} - \ell \geq -r.$$

$$\frac{\phi(d_b(y_m, y_{m+1})) - 1}{d_b(y_m, x_{m+1})^h} \geq \ell - r.$$

$$\frac{\phi(d_b(y_m, y_{m+1})) - 1}{d_b(y_m, y_{m+1})^h} \geq \ell - \frac{\ell}{2} = r.$$

Then

$$d_b(y_m, y_{m+1})^h \leq s[\phi(d_b(y_m, y_{m+1})) - 1], \quad \text{for all} \quad m > m_0$$

Where $s = \dfrac{1}{r}$.

Let $\ell = \infty$. Then by definition of limit, choosing $r > 0$ there exist a non negative integer $m_0 \in \mathbb{N}$ such that $m > m_0$

$$\frac{\phi(d_b(y_m, y_{m+1})) - 1}{d_b(y_m, y_{m+1})^h} \geq r.$$

This implies that, for all $m \geq m_0$

$$(d_b(y_m, y_{m+1}))^h \leq s[\phi(d_b(y_m, y_{m+1})) - 1].$$

Observe that for each case, $s > 0$ and $m_0 \in \mathbb{N}$ such that

$$(d_b(y_m, y_{m+1}))^h \leq s[\phi(d_b(y_m, y_{m+1})) - 1], \quad \text{for all} \quad m \geq m_0.$$

Using (3.3) in the above inequality, we get

$$(d_b(y_m, y_{m+1}))^h \leq s\left[[\phi(d_b(y_0, y_1))]^{\alpha^m} - 1\right], \quad \text{for all} \quad m \geq m_0. \tag{3.4}$$

Since $b \geq 1$ and $0 < h < 1$, then $b^h > 0$. Then for all $m > m_0$

$$b^h m(d_b(y_m, y_{m+1}))^h \leq sb^h m\left[[\phi(d_b(y_0, y_1))]^{\alpha^m} - 1\right]. \tag{3.5}$$

Taking $m \to \infty$ in the above inequality

$$\begin{aligned}
\lim_{m\to\infty} km[\phi(d_b(y_0, y_1))^{\alpha^m} - 1] &= k \lim_{m\to\infty} \frac{[\phi(d_b(y_1, y_0))^{\alpha^m} - 1]}{\frac{1}{m}} \\
&= k \lim_{m\to\infty} \frac{\alpha^m \ln(\alpha)\ln(\phi(d_b(y_0, y_1))[\phi(d_b(y_0, y_1))]^{\alpha^m}}{\frac{-1}{m^2}} \\
&= k \lim_{m\to\infty} -m^2\alpha^m \ln(\alpha)\ln(\phi(d_b(y_0, y_1))[\phi(d_b(y_0, y_1))]^{\alpha^m} \\
&= k \lim_{m\to\infty} \frac{-m^2 \ln(\alpha)\ln(\phi(d_b(y_0, y_1))[\phi(d(y_0, y_1))]^{\alpha^m}}{\alpha_1^m} \\
&= k \lim_{n\to\infty} \frac{-m^2}{\alpha_1^m} . \lim_{m\to\infty} \ln(\alpha)\ln(\phi(d_b(y_0, y_1))[\phi(d(y_0, y_1))]^{\alpha^m} \\
&= k.0. \ln(\alpha)\ln(\phi(d_b(y_0, y_1)) \\
&= 0 \quad (\text{where} \quad \alpha_1 = \frac{1}{\alpha} \quad \textit{and} \quad k = sb^h).
\end{aligned}$$

$$\Rightarrow \quad \lim_{m\to\infty} m[\phi(d_b(y_0,y_1))^{\alpha^m}-1]=0. \tag{3.6}$$

From (3.5), we have

$$\lim_{m\to\infty} b^h m(d_b(y_m,y_{m+1}))^h=0.$$

Then by definition of limit there exist $\epsilon>0$, choosing $\epsilon\in(0,1)$ and there is an $m_1\in\mathbb{N}$ such that for each $m\geq m_1$

$$\begin{aligned}
\left|b^h m(d_b(y_m,y_{m+1}))^h-0\right| &< \epsilon\\
b^h m(d_b(y_m,y_{m+1}))^h &< \epsilon\\
bm^{\frac{1}{h}}(d_b(y_m,y_{m+1})) &< \epsilon' \quad (\epsilon^{1/h}=\epsilon')\\
d_b(y_m,y_{m+1}) &< \frac{\epsilon'}{bm^{\frac{1}{h}}}.
\end{aligned}$$

$$d_b(y_m,y_{m+1})<\frac{1}{bm^{\frac{1}{h}}}, \quad \text{for all} \quad m\geq m_1. \tag{3.7}$$

Replacing m with $m+1$ in (3.7), we get

$$d_b(y_{m+1},x_{m+2})<\frac{1}{b(m+1)^{\frac{1}{h}}}, \quad \text{for all} \quad m\geq m_1. \tag{3.8}$$

From (3.4) $\qquad (d_b(y_m,y_{m+1}))^h\leq s\left[[\phi(d_b(y_0,y_1))]^{\alpha^m}-1\right], \quad \text{for all} \quad m\geq m_0$

Since $b\geq 1$, $0<h<1$ then $b^{2h}>0$. Then for all $m>m_0$

$$b^{2h}m(d_b(y_m,y_{m+1}))^h\leq sb^{2h}m\left[[\phi(d_b(y_0,y_1))]^{\alpha^n}-1\right].$$

Again, taking $m\to\infty$ in the above inequality and using (3.6).

$$\lim_{m\to\infty} b^{2h}m(d_b(y_m,y_{m+1}))^h=0$$

Then by definition of limit there exist $m_1\in\mathbb{N}$ such that

$$d_b(y_m,y_{m+1})<\frac{1}{b^2m^{\frac{1}{h}}}, \quad \text{for all} \quad m\geq m_1.$$

Replacing m with $m+2$ and $m+3$, we get

$$d_b(y_{m+2}, y_{m+3}) < \frac{1}{b^2(m+2)^{\frac{1}{h}}}, \quad \text{for all} \quad m \geq m_1. \tag{3.9}$$

$$d_b(y_{m+3}, y_{m+4}) < \frac{1}{b^2(m+3)^{\frac{1}{h}}}, \quad \text{for all} \quad m \geq m_1. \tag{3.10}$$

Continuing in this way, we get

$$d_b(y_{n+2I}, y_{n+2I+1}) < \frac{1}{b^I(n+2I)^{\frac{1}{h}}}, \quad \text{for all} \quad m \geq m_1. \tag{3.11}$$

Also, let us assume that y_0 is not a periodic point of V. Indeed, if $y_0 = y_m$ then using (3.2), for all $m \geq 2$, we have

$$\begin{aligned} \phi(d_b(y_0, Vy_0)) &= \phi(d_b(y_m, Vy_m)) \\ \phi(d_b(y_0, y_1)) &= \phi(d_b(y_m, y_{m+1})) \\ d_{b0} &= d_{bm} \\ d_{b0} &\leq d_{b0}^{\alpha^m} \\ \ln d_{b0} &\leq \alpha^m \ln d_{b0} \leq \ln d_{b0} \end{aligned}$$

which contradict to our supposition.

This implies that $d_0 = 0$, i.e $y_0 = y_1$, and y_0 is a fixed point of V. Assume that $y_m \neq y_n$ for all distinct $n, m \in \mathbb{N}$ such that $m \neq n$.

Again setting $\phi(d_b(y_m, y_{m+2})) = d'_{bm}$.

$$\begin{aligned} \phi(d_b(y_m, y_{m+2})) &\leq [\phi(Vy_{m-1}, Vy_{m+1}) \leq [\phi(d_b(y_{m-1}, y_{m+1}))]^\alpha \\ d'_m &= [\phi(y_{m-1}, y_{m+1}))]^\alpha \\ d'_{bm} &\leq d'^{\alpha}_{b(m-1)}. \end{aligned}$$

Continuing in this way, we get

$$d'_m \leq d'^{\alpha^m}_0$$

$$\Rightarrow \quad 1 < \psi(d'(y_m, y_{m+2})) \leq [\psi(d'(y_0, y_2))]^{\alpha^m}.$$

Taking $m \to \infty$ on both sides of the above inequality and then using Sandwich Theorem, we obtain

$$\lim_{m\to\infty} \phi(d'(y_m, y_{m+2})) = 1.$$

From the condition (ϕ_2), we get

$$\lim_{m\to\infty} d'(y_m, y_{m+2}) = 0.$$

Similarly, from condition (ϕ_3), there exist $m_2 \in \mathbb{N}$ such that

$$d_b'(y_m, y_{m+2}) \leq \frac{1}{bm^{\frac{1}{h}}}, \quad \text{for all} \quad m \geq m_2.$$

Replacing m with $m + 2I - 2$, we have

$$d_b'(y_{m+2I}, y_{m+2I-2}) \leq \frac{1}{b^{I-1}m^{\frac{1}{h}}}, \quad \text{for all} \quad m \geq m_2. \tag{3.12}$$

Let $N = \max\{m_0, m_1\}$.

For the sequence $\{y_m\}$, as d_b is rectangular metric space so consider $d_b(y_m, y_{m+q})$ into two case.

Case-i:

If $q > 2$ is odd say $q = 2I + 1, I \geq 1$, using (3.7), (3.8),..., (3.11) for all $m > N$, we obtain

$$\begin{aligned}
&d_b(y_m, y_{m+q})\\
&\leq b[d_b(y_m, y_{m+1}) + d_b(y_{m+1}, y_{m+2}) + d_b(y_{m+2}, y_{m+2I+1})]\\
&= b[d_{bm} + d_{b(m+1)}] + bd(y_{m+2}, y_{m+2I+1})]\\
&\leq b[d_m + d_{b(m+1)}] + b^2[d_b(y_{m+2}, y_{m+3}) + d_b(y_{m+3}, y_{m+4}) + d_b(y_{m+4}, y_{m+2L+1})]\\
&= b[d_{bm} + d_{b(m+1)}] + b^2[d_{b(m+2)} + d_{b(m+3)}] + b^2[d_b(y_{m+4}, y_{m+2I+1})]
\end{aligned}$$

Continuing in this way, we obtain

$$\begin{aligned}
&d_b(y_m, y_{m+2I+1})\\
&\leq b[d_{bm} + d_{b(m+1)}] + b^2[d_{b(m+2)} + d_{b(m+3)}] + b^3[d_{b(m+4)} + d_{b(m+5)}] + \cdots + b^I d_{b(m+2I)}
\end{aligned}$$

$$
\begin{aligned}
&= [bd_{bm} + b^2 d_{b(m+2)} + b^3 d_{b(m+4)} + \cdots\cdot] \\
&\quad + [bd_{b(m+1)} + b^2 d_{b(m+3)} + b^3 d_{b(m+5)} + \cdots\cdot + b^I d_{b(m+2I)}] \\
&= \left(\frac{b}{bm^{\frac{1}{h}}} + \frac{b^2}{b^2(m+2)^{\frac{1}{h}}} + \frac{b^3}{b^3(m+4)^{\frac{1}{h}}} + \cdots \right) \\
&\quad + \left(\frac{b}{b(m+1)^{\frac{1}{h}}} + \frac{b^2}{b^2(m+3)^{\frac{1}{h}}} + \frac{b^3}{b^3(m+5)^{\frac{1}{h}}} + \cdots + \frac{b^I}{b^I(m+2I)^{\frac{1}{h}}} \right) \\
&= \left(\frac{1}{m^{\frac{1}{h}}} + \frac{1}{(m+1)^{\frac{1}{h}}} + \frac{1}{(m+2)^{\frac{1}{h}}} + \cdots + \frac{1}{(m+2I)^{\frac{1}{h}}} \right) \\
&= \sum_{j=m}^{m+2I} \frac{1}{j^{\frac{1}{h}}} \\
&\leq \sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{h}}}.
\end{aligned}
$$

Case-ii:

If $q > 2$ is even say $2I, I \geq 2$, using (3.7), (3.8), ... and (3.12) for all $m > N$

$$
\begin{aligned}
&d_b(y_m, y_{(m+2I)}) \\
&\leq b[d_{bm} + d_{b(m+1)}] + b^2[d_{b(m+2)} + d_{b(m+3)}] + b^3[d_{b(m+4)} + d_{b(m+5)}] + \cdots + b^{I-1} d'_{b(m+2I-2)} \\
&= [bd_{bm} + b^2 d_{b(m+2)} + b^3 d_{b(m+3)} + \cdots\cdot] \\
&\quad + [bd_{b(m+1)} + b^2 d_{b(m+2)} + b^3 d_{b(m+3)} + \cdots\cdot + b^{I-1} d'_{b(m+2I-2)}] \\
&= \left(\frac{b}{bm^{\frac{1}{h}}} + \frac{b^2}{b^2(m+2)^{\frac{1}{h}}} + \frac{b^3}{b^3(m+4)^{\frac{1}{k}}} + \cdots \right) \\
&\quad + \left(\frac{b}{b(m+1)^{\frac{1}{h}}} + \frac{b^2}{b^2(m+3)^{\frac{1}{h}}} + \frac{b^3}{b^3(m+5)^{\frac{1}{h}}} + \cdots + \frac{b^{I-1}}{b^{I-1}(m+2I-2)^{\frac{1}{h}}} \right) \\
&= \left(\frac{1}{m^{\frac{1}{h}}} + \frac{1}{(m+1)^{\frac{1}{h}}} + \frac{1}{(m+2)^{\frac{1}{h}}} + \cdots + \frac{1}{(m+2I-2)^{\frac{1}{h}}} \right) \\
&= \sum_{j=n}^{m+2I-2} \frac{1}{j^{\frac{1}{h}}} \\
&\leq \sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{h}}}.
\end{aligned}
$$

Thus, combining all these cases, we have

$$d_b(y_m, y_{m+q}) \leq \sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{h}}}, \quad \text{for all} \quad m \geq N, q \in \mathbb{N}$$

Since $0 < h < 1$, then $\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{h}}}$ converges.
This implies that $\lim\limits_{m\to\infty} d_b(x_m, x_{m+q}) \to 0$ for all $q > 0$. Thus we proved that $\{y_m\}$ is a Cauchy sequence in Y. The completeness of Y make insure there exist $y_0 \in Y$ such that $y_m \to y_0$ as $m \to \infty$. First we show that y_0 is a fixed point of V. Contrary suppose $y_0 \neq V y_0$.
Then

$$\begin{aligned} 1 < \phi(d_b(y_m, V y_0)) &= \phi(d_b(V y_{m-1}, V y_0)) \\ &\leq [\phi(d_b(y_{m-1}, y_0))]^{\alpha} \end{aligned}$$

Taking $m \to \infty$ in the above inequality, we get

$$1 < \phi(d_b(y_0, V y_0)) \leq 1$$

which contradict to our supposition. Thus x_0 is the fixed point of V.
For uniqueness. Suppose there exist another fixed point x_0 of V different from y_0 that is $x_0 = V x_0$.
Then

$$\begin{aligned} 1 < \phi(d_b(y_0, x_0)) &= \phi(d_b(V y_0, V x_0)) \\ &\leq [\phi(d_b(y_0, x_0))]^{\alpha} \\ &< \phi(d_b(y_0, x_0)) \end{aligned}$$

$$\Rightarrow \quad 1 < \phi(d_b(y_0, x_0)) < \phi(d_b(y_0, x_0))$$

which contradict to our supposition that $y_0 \neq x_0$. Thus $y_0 = x_0$. Thus V has only one fixed point, which ends the proof. □

Definition 3.1.5. Let $V : Y \to Y$ given self mapping and (Y, d_b) be a b-metric with $b \geq 1$, whenever there exist a function $\phi \in \Phi$ and for any constant $\alpha \in (0,1)$ satisfying:

$$d_b(Vx, Vy) \neq 0 \quad \Rightarrow \quad \phi(d_b(Vx, Vy)) \leq [\phi(d_b(x,y))]^\alpha$$

$\forall \;\; x, y \in Y$, then V is called JS-contraction.

Since a b-metric space with coefficient b is a rectangular b-metric space with coefficient b^2. The following result has been concluded from Theorem 3.1.4.

Corollary 3.1.6. *Let $V\colon Y \to Y$ be a given self mapping and (Y, d_b) be a b-metric space with co-efficient $b \geq 1$, whenever their exist $\phi \in \Phi$ and any constant $\alpha \in (0,1)$ satisfying:*

$$d_b(Vx, Vy) \neq 0 \quad \Rightarrow \quad \phi(d_b(Vx, Vy)) \leq [\phi(d_b(x,y))]^\alpha$$

for all $x, y \in Y$. Then V has only one fixed point.

Definition 3.1.7. Let $V\colon Y \to Y$ be a given self mapping and (Y, d) be a rectangular metric space, whenever there exist a function $\phi \in \Phi$ and any constant $\alpha \in (0,1)$ satisfying:

$$d(Vx, Vy) \neq 0 \quad \Rightarrow \quad \phi(d_b(Vx, Vy)) \leq [\phi(d(x,y))]^\alpha$$

for all $x, y \in Y$., then V is called JS-contraction.

The main result by Jleli and Samat [22] can now be established as the following Corollary of our result.

Corollary 3.1.8. *[22] "Let (X, d) be a complete g.m.s and $T\colon X \to X$ be a given map. Suppose that there exist $\theta \in \Theta$ and any constant $k \in (0,1)$ such that*

$$x, y \in X, \quad d_b(Tx, Ty) \neq 0 \quad \Rightarrow \quad \theta(d(Vx, Vy)) \leq [\theta(d(x,y))]^k. \tag{3.13}$$

Then T has only one fixed point."

Proof. Taking $b = 1$ in Theorem 3.1.4, the proof follows immediately. $\square$

Definition 3.1.9. Let $V : Y \to Y$ be a given self mapping and (Y, d) be metric space, whenever there exist a function $\phi \in \Phi$ and any constant $\alpha \in (0, 1)$ satisfying:

$$d(Vx, Vy) \neq 0 \quad \Rightarrow \quad \phi(d(Vx, Vy)) \leq [\phi(d(x, y))]^{\alpha}$$

for all $x, y \in Y.$, then V is called JS-contraction.

Theorem 3.1.10. *[22] "Let (Y, d) be a complete metric space and $T : X \to X$ be a given map. Suppose that there exist $\theta \in \Theta$ and any constant $k \in (0, 1)$ such that*

$$x, y \in X, \qquad d(Vx, Vy) \neq 0 \quad \Rightarrow \quad \phi(d(Vx, Vy)) \leq [\phi(d(x, y))]^{\alpha}. \tag{3.14}$$

Then T has unique fixed point."

Proof. The result follows from Corollary 3.1.6 by taking $b = 1$. $\square$

Example 3.1.11. Let Y be the set defined by

$$Y = \{\kappa \in \mathbb{N}\}$$

where

$$\kappa_m = \frac{m(m+1)}{2}, \quad \text{for all} \quad m \in \mathbb{N}$$

Let $d : Y \times Y \to Y$ defined by $d(x, y) = (x - y)^2$. It is b-metric with coefficient $b = 2$. Let $V : Y \to Y$ be the mapping defined by

$$V\kappa_1 = \kappa_1, \quad V\kappa_m = \kappa_{m-1}, \quad \text{for all} \quad m \geq 2$$

We can check easily that Banach contraction does not hold.

$$\lim_{m \to \infty} \frac{d_b(V\kappa_m, V\kappa_1)}{d_b(\kappa_m, \tau_1)} = 1.$$

Consider a function $\psi : (0, \infty) \to (1, \infty)$ defined by $\psi(u) = e^{\sqrt{ue^u}}$. Then it is easy to show that $\phi \in \Phi$. Our aim is to prove V fulfill the condition of the result 3.1.4, that is

$$d_b(V\kappa_m, V\kappa_n) \neq 0 \quad \Rightarrow \quad e^{\sqrt{d_b(V\kappa_m, V\tau_n)e^{d_b(V\kappa_m, V\kappa_n)}}} \leq e^{\alpha\sqrt{d_b(\kappa_m,\kappa_n)e_b^d(\kappa_m,\kappa_n)}},$$

for any $\alpha \in (0,1)$. Then the above condition is equivalent to

$$d_b(V\kappa_m, V\kappa_n)e^{d_b(V\kappa_m, V\kappa_n)} \leq \alpha^2 d(\kappa_m, \kappa_n)e^{d_b(\kappa_m,\kappa_n)}.$$

So, we have to check that

$$d_b(V\kappa_m, V\kappa_n) \neq 0 \quad \Rightarrow \quad \frac{d_b(V\kappa_m, V\kappa_n)e^{d_b(V\kappa_m, V\kappa_n)-d_b(\kappa_m,\kappa_n)}}{d_b(\kappa_m, \kappa_n)} \leq \alpha^2, \tag{3.15}$$

for any $\alpha \in (0,1)$. We discuss two cases.

Case i. $m = 1$ and $n > 2$. For this case, we have

$$\frac{d_b(V\kappa_m, V\kappa_n)e^{d_b(V\kappa_m, V\kappa_n)-d_b(\kappa_m,\kappa_n)}}{d_b(\kappa_m, \kappa_n)} = \left(\frac{n^2-n-2}{n^2+n-2}\right)^2 e^{(n^2-2)(-n)} \leq e^{-1}$$

Case ii. $m > n > 1$. For this case, we have

$$\frac{d_b(V\kappa_m, V\kappa_n)e^{d_b(V\kappa_m, V\kappa_n)-d_b(\kappa_m,\kappa_n)}}{d_b(\kappa_m, \kappa_m)} = \left(\frac{n+m-1}{n+n+1}\right)^2 e^{(m^2-n^2)(n-m)} \leq e^{-1}$$

Hence, the inequality (3.15) holds for $\alpha = e^{-1/2}$. Corollary (3.1.6) implies that V has only one fixed point. It is clear that κ_1 is the fixed point of V.

Chapter 4

Fixed Point Results and Modified JS-Contraction

In this chapter we introduce a family Ψ_b of the functions ψ_b which is defined under some conditions and then modify and extend those conditions which are known as ψ_b-contractive conditions or mappings. Further we will define modified form of JS-contraction and will prove a new fixed point result for self mapping that satisfies modified JS-contraction in the setup of complete b-metric spaces. Our result is an extension of the results proved in [20].

4.1 Modified JS-Contraction

We will define modified form of JS-contraction and establish and prove fixed point theorems for such contraction in the setting of complete b-metric space.
Let Ψ_b be the family of all functions $\psi_b\colon (0,\infty) \to (b^{\frac{\alpha}{1-\alpha}}, \infty)$, where $0 \leq \alpha < 1$ and $b \geq 1$ satisfying the following assertions:

(ψ_{b_1}) ψ_b is non-decreasing.

(ψ_{b_2}) For each sequence $\{\beta_n\} \subseteq \mathbb{R}^+$, $\lim\limits_{n\to\infty} \psi_b(\beta_n) = b^{\frac{\alpha}{1-\alpha}}$ if and only if $\lim\limits_{n\to\infty} (\beta_n) = 0$.

(ψ_{b_3}) There exist $0 < h < 1$ and $\ell \in (0, \infty]$ such that $\lim_{\beta \to 0^+} \frac{\psi_b(\beta) - 1}{\beta^h} = \ell$.

Note. For $b = 1$ the family ψ_b becomes the family ψ which is introduced by Jleli and Samat [22].

4.1.1 ψ_b-contractive conditions

We modify and extend the family Ψ_b of function $\psi_b \colon [0, \infty) \to [b^{\frac{\alpha}{1-\alpha}}, \infty)$ and proved the following fixed point theorem for self mapping that holds ψ_b-contractive condition in the context of complete b-metric spaces.

(ψ'_{b_1}) ψ_b is non-decreasing and $\psi_b(u) = b^{\frac{\alpha}{1-\alpha}}$ if and only if $u = 0$.

(ψ_{b_4}) $\psi_b(bx + by) \leq b\psi_b(x)\psi_b(y)$ for all $x, y > 0$ and $b \geq 1$.

The set of all functions $\psi \colon [0, \infty) \to [b^{\frac{\alpha}{1-\alpha}}, \infty)$ satisfying the conditions $(\psi'_{b_1} - \psi_{b_4})$ is denoted by Ψ'_b.

Note. For $b = 1$ the contractive conditions coincides with the conditions introduced by Hussain et al. [20].

Definition 4.1.1. Let $V \colon Y \to Y$ be a given self mapping and (Y, d_b) be a b-metric space with co-efficient $b \geq 1$, whenever there exist positive real numbers t_1, t_2, t_3 and t_4 with $0 \leq t_1 + t_2 + t_3 + 2t_4 < 1$ and a function $\psi_b \in \Psi'_b$ satisfying:

$$\begin{aligned} &\psi_b(d_b(Vx, Vy)) \\ \leq &[\psi_b(d_b(x, y))]^{t_1} [\psi_b(d_b(x, Vx))]^{t_2} [\psi_b(d_b(y, Vy))]^{t_3} [\psi_b(d_b(x, Vy) + d_b(y, Vx))]^{t_4} \end{aligned}$$

for all $x, y \in Y$, then V is called JS-contraction.

Theorem 4.1.2. *Let $V : Y \to Y$ be a self mapping and (Y, d_b) be a complete b-metric space with $b \geq 1$, whenever there are any positive real numbers t_1, t_2, t_3 and t_4 with $0 \leq t_1 + t_2 + t_3 + 2t_4 < 1$ and a function $\psi_b \in \Psi_b'$ satisfying:*

$$\begin{aligned} &\psi_b(d_b(Vx, Vy)) \\ \leq &[\psi_b(d_b(x, y))]^{t_1}[\psi_b(d_b(x, Vx))]^{t_2}[\psi_b(d_b(y, Vy))]^{t_3}[\psi_b(d_b(x, Vy) + d_b(y, Vx))]^{t_4} \end{aligned} \tag{4.1}$$

for all $x, y \in Y$, then V has only one fixed point.

Proof. Let $y_0 \in Y$ be arbitrary. Let us consider a sequence $\{y_m\}$ by $y_{m+1} = Vy_m$ for all $m \geq 0$.

We want to prove $\{y_m\}$ is a Cauchy sequence. If $y_m = y_{m+1}$ then y_m is the fixed point of V, so there is nothing to prove. So suppose that $y_m \neq y_{m+1}$, for all $m \geq 0$. Setting $d_b(y_m, y_{m+1}) = d_{bm}$.

It follows form (4.1)

$$\begin{aligned} \psi_b(d_b(y_m, y_{m+1})) = \psi_b(d_b(Vy_{m-1}, Vy_m)) \\ &\leq [\psi_b(d_b(y_{m-1}, y_m))]^{t_1}.[\psi_b(d_b(y_{m-1}, Vy_{m-1}))]^{t_2}.[\psi_b(d_b(y_m, Vy_m))]^{t_3} \\ &\quad .[\psi_b(d_b(y_{m-1}, Vy_m) + d_b(y_m, Vy_{m-1}))]^{t_4}. \end{aligned}$$

By using triangular inequality of b-metric space, we get

$$\begin{aligned} &\psi_b(d_b(y_m, y_{m+1})) \\ =&[\psi_b(d_b(y_{m-1}, y_m))]^{t_1}.[\psi_b(d_b(y_{m-1}, y_m))]^{t_2}.[\psi_b(d_b(y_m, y_{m+1}))]^{t_3} \\ &\quad .[\psi_b(d_b(y_{m-1}, y_{m+1}) + d_b(y_m, y_m))]^{t_4} \\ \leq&[\psi_b(d_b(y_{m-1}, y_m))]^{t_1}.[\psi_b(d_b(y_{m-1}, y_m))]^{t_2}.[\psi_b(d_b(y_m, y_{m+1}))]^{t_3} \\ &\quad .[\psi_b(b(d_b(y_{m-1}, y_m) + d_b(y_m, y_{m+1})))]^{t_4} \\ =&[\psi_b(d_b(y_{m-1}, y_m))]^{t_1}.[\psi_b(d_b(y_{m-1}, y_m))]^{t_2}.[\psi_b(d_b(y_m, y_{m+1}))]^{t_3} \\ &\quad .[\psi_b(bd_b(y_{m-1}, y_m) + bd_b(y_m, y_{m+1}))]^{t_4} \\ \leq&b^{t_4}[\psi_b(d_b(y_{m-1}, y_m))]^{t_1}.[\psi_b(d_b(y_{m-1}, y_m))]^{t_2}.[\psi_b(d_b(y_m, y_{m+1}))]^{t_3} \\ &\quad .[\psi_b(d_b(y_{m-1}, y_m)]^{t_4}[\psi_b(d_b(y_m, y_{m+1}))]^{t_4} \end{aligned}$$

$$
\begin{aligned}
&= b^{t_4}[\psi_b(d_b(y_{m-1}, y_m)]^{t_1+t_2+t_4}.[\psi_b(d_b(y_m, y_{m+1}))]^{t_3+t_4} \\
&< b^{t_1+t_2+t_4}[\psi_b(d(y_{m-1}, y_m)]^{t_1+t_2+t_4}.[\psi_b(d(y_m, y_{m+1}))]^{t_3+t_4} \\
&= [b\ \psi_b(d(y_{m-1}, y_m)]^{t_1+t_2+t_4}.[\psi_b(d_b(y_m, y_{m+1})]^{t_3+t_4}.
\end{aligned}
$$

Taking natural log on both sides of above inequality, we have

$$
\begin{aligned}
&\ln \psi_b(d_b(y_m, y_{m+1})) \\
&\leq (t_1+t_2+t_4)\ln[b\ \psi(d_b(y_{m-1}, y_m))] + (t_3+t_4)\ln[\psi_b(d_b(y_m, y_{m+1})].
\end{aligned}
$$

$$
\begin{aligned}
&\ln[\psi_b(d_b(y_m, y_{m+1}))] - (t_3+t_4)\ln[\psi_b(d_b(y_m, y_{m+1})] \\
&\leq (t_1+t_2+t_4).\ln[b\ \psi_b(d_b(y_{m-1}, y_m))]
\end{aligned}
$$

$$(1-t_3-t_4)\ln[\psi_b(d_b(y_m, y_{m+1}))] \leq (t_1+t_2+t_4)\ln[b\ \psi_b(d_b(y_{m-1}, y_m))]$$

$$\ln[\psi_b(d_b(y_m, y_{m+1}))] \leq \frac{(t_1+t_2+t_4)}{(1-t_3-t_4)}\ln[b\psi_b(d_b(y_{m-1}, y_m))]$$

$$\psi_b(d_b(y_m, y_{m+1})) \leq [b\psi_b(d_b(y_{m-1}, y_m))]^{\frac{(t_1+t_2+t_4)}{(1-t_3-t_4)}}.$$

Let $\alpha = \dfrac{(t_1+t_2+t_4)}{(1-t_3-t_4)} < 1.$

$$\psi_b(d_b(y_m, y_{m+1})) \leq [b\psi_b(d_b(y_{m-1}, y_m))]^{\alpha}.$$

Repeating this process, we get

$$
\begin{aligned}
b^{\frac{\alpha}{1-\alpha}} < \psi_b(d_b(y_m, y_{m+1})) &\leq b^{\alpha}[\psi_b(d_b(y_{m-1}, y_m))]^{\alpha} \\
&\leq b^{\alpha+\alpha^2}[\psi_b(d_b(y_{m-2}, y_{m-1}))]^{\alpha^2} \\
&\ \vdots \\
&\ \vdots \\
&\leq b^{\alpha+\alpha^2+\cdots+\alpha^n}[\psi_b(d_b(y_0, y_1))]^{\alpha^n}. \qquad (4.2)
\end{aligned}
$$

Taking $m \to \infty$ in the above inequality, we get

$$\lim_{m\to\infty} b^{\alpha+\alpha^2+\cdots+\alpha^m}[\psi_b(d_b(y_0,y_1))]^{\alpha^n} = \lim_{m\to\infty} b^{\frac{\alpha(1-\alpha^m)}{1-\alpha}}[\psi_b(d_b(y_0,y_1))]^{\alpha^m}$$

$$= \lim_{m\to\infty} b^{\frac{\alpha(1-\alpha^m)}{1-\alpha}}. \lim_{m\to\infty}[\psi_b(d_b(y_0,y_1))]^{\alpha^m}$$

$$= b^{\frac{\alpha}{1-\alpha}}.1 \quad (\text{since } \alpha^m \to 0 \text{ as } m \to \infty)$$

$$= b^{\frac{\alpha}{1-\alpha}}.$$

By using Sandwich Theorem, we get

$$\Rightarrow \lim_{m\to\infty} \psi_b(d_b(y_m,y_{m+1})) = b^{\frac{\alpha}{1-\alpha}}.$$

$$\Rightarrow \lim_{m\to\infty} d_b(y_m,y_{m+1}) = 0.$$

From the condition (ψ_{b_3}), there exist $0 < h < 1$ and $\ell \in (0,\infty]$ such that

$$\lim_{m\to\infty} \frac{\psi_b(d_b(y_m,y_{m+1}))-1}{d_b(y_m,y_{m+1})^h} = \ell.$$

Let $\ell < \infty$. Then by definition of limit, choosing $r = \dfrac{\ell}{2}$ there exist a positive integer $m_0 \in \mathbb{N}$ such that $m > m_0$

$$\left|\frac{\psi_b(d_b(y_m,y_{m+1}))-1}{d_b(y_m,y_{m+1})^h} - \ell\right| \leq r.$$

$$-r \leq \frac{\psi_b(d_b(y_m,y_{m+1}))-1}{d_b(y_m,y_{m+1})^h} - \ell \leq r$$

$$\frac{\psi_b(d_b(y_m,y_{m+1}))-1}{d_b(y_m,y_{m+1})^h} - \ell \geq -r.$$

$$\frac{\psi_b(d_b(y_m,y_{m+1}))-1}{d_b(y_m,y_{m+1})^h} - \ell \leq r.$$

Consider

$$\frac{\psi_b(d_b(y_m,y_{m+1}))-1}{d_b(y_m,y_{m+1})^h}-\ell\geq -r.$$

$$\frac{\psi_b(d_b(y_m,y_{m+1}))-1}{d_b(y_m,y_{m+1})^h}\geq \ell-r.$$

$$\frac{\psi_b(d_b(y_m,y_{m+1}))-1}{d_b(y_m,y_{m+1})^h}\geq \ell-\frac{\ell}{2}=r.$$

for all $m>m_0$. Then

$$d_b(y_m,y_{m+1})^h\leq s[\psi_b(d_b(y_m,y_{m+1}))-1],$$

Where $s=\dfrac{1}{r}$.

Using (4.2) in the the above inequality, we have

$$(d_b(y_m,y_{m+1}))^h\leq sb^{\alpha+\alpha^2+\cdots+\alpha^n}\left[[\psi_b(d_b(y_0,y_1))]^{\alpha^n}-1\right],\quad \text{for all}\quad m\geq m_0. \tag{4.3}$$

Since $b\geq 1$ and $0<h<1$, then $b^h>0$. Then for all $m>m_0$.

Multiplying an inequality (4.3) by $b^h m$, we have

$$b^h m(d_b(y_m,y_{m+1}))^h\leq sb^h b^{\alpha+\alpha^2+\cdots+\alpha^m} m\left[[\psi_b(d_b(y_0,y_1))]^{\alpha^n}-1\right].$$

Taking $m\to\infty$ in the above inequality, we have

$$\lim_{m\to\infty} b^h m(d_b(y_m,y_{m+1}))^h\leq sb^h\lim_{n\to\infty} b^{\alpha+\alpha^2+\cdots+\alpha^m} m\left[[\psi_b(d_b(y_0,y_1))]^{\alpha^m}-1\right]$$

$$=sb^h\lim_{n\to\infty} b^{\alpha+\alpha^2+\cdots+\alpha^m}.\lim_{m\to\infty} m\left[[\psi_b(d_b(y_0,y_1))]^{\alpha^m}-1\right]$$

$$=sb^h.b^{\frac{\alpha}{1-\alpha}}.\lim_{m\to\infty} m\left[[\psi_b(d_b(y_0,y_1))]^{\alpha^m}-1\right].$$

Consider

$$\begin{aligned}
\lim_{m\to\infty} m[\psi_b(d_b(y_0,y_1))^{\alpha^m}-1] &= \lim_{m\to\infty}\frac{[\psi_b(d_b(y_0,y_1))^{\alpha^m}-1]}{\frac{1}{m}}\\
&= \lim_{m\to\infty}\frac{\alpha^m\ln(\alpha)\ln(\psi_b(d_b(y_0,y_1))[\psi_b(d_b(y_0,y_1))]^{\alpha^m}}{\frac{-1}{m^2}}\\
&= \lim_{m\to\infty} -m^2\alpha^m\ln(\alpha)\ln(\psi_b(d_b(y_0,y_1))[\psi_b(d_b(y_0,y_1))]^{\alpha^m}\\
&= \lim_{m\to\infty}\frac{-m^2\alpha^m\ln(\psi_b(d_b(y_0,y_1))[\psi_b(d_b(y_0,y_1))]^{\alpha^m}}{\alpha_1^m}\\
&= \lim_{n\to\infty}\frac{-m^2}{\alpha_1^m}.\lim_{m\to\infty}\ln(\alpha)\ln(\psi_b(d_b(y_0,y_1))[\psi_b(d_b(y_0,y_1))]^{\alpha^m}\\
&= 0.\ln(\alpha)\ln(\psi_b(d_b(y_0,y_1))\\
&= 0 \quad (\text{where} \quad \alpha_1=\frac{1}{\alpha}).
\end{aligned}$$

This implies that

$$\lim_{m\to\infty} m[\psi_b(d_b(y_0,y_1))^{\alpha^m}-1]=0. \tag{4.4}$$

$$\Rightarrow \lim_{m\to\infty} b^h m(d_b(y_m,y_{m+1}))^h = 0.$$

Then by definition of limit there exist $\epsilon>0$, choosing $\epsilon\in(0,1)$ and there is an $m_1\in\mathbb{N}$ such that for all $m\geq m_1$

$$\begin{aligned}
\left|b^h m(d_b(y_m,y_{m+1}))^h-0\right| &< \epsilon\\
b^h m(d_b(y_m,y_{m+1}))^h &< \epsilon\\
bm^{\frac{1}{h}}(d_b(y_m,y_{m+1})) &< \epsilon^{'} \quad (\text{where} \quad \epsilon^{1/h}=\epsilon^{'})\\
d_b(y_m,y_{m+1}) &< \frac{\epsilon^{'}}{bm^{\frac{1}{h}}}
\end{aligned}$$

$$d_b(y_m,y_{m+1}) < \frac{1}{bm^{\frac{1}{h}}}, \quad \text{for all} \quad m\geq m_1. \tag{4.5}$$

From (4.3)

$(d_b(y_m,y_{m+1}))^h \leq sb^{\alpha+\alpha^2+\cdots+\alpha^m}\left[[\psi_b(d_b(y_0,y_1))]^{\alpha^m}-1\right], \quad \text{for all} \quad m\geq m_0.$

Since $b \geq 0$, $0 < h < 1$ then $b^{2h} > 0$. Then for all $m > m_0$.

$$b^{2h} m(d_b(y_m, y_{m+1}))^h \leq s b^{2h} b^{\alpha+\alpha^2+\cdots+\alpha^m} m \left[[\psi_b(d_b(y_0, y_1))]^{\alpha^m} - 1 \right].$$

Taking $m \to \infty$ in the above inequality and using (4.4), we get

$$\lim_{m\to\infty} b^{2h} m(d_b(x_m, x_{m+1}))^h = 0.$$

Then there exist $m_1 \in \mathbb{N}$ such that

$$d_b(y_m, y_{m+1}) < \frac{1}{b^2 m^{\frac{1}{h}}}, \quad \text{for all} \quad m \geq m_1.$$

Replacing m with $m+1$, we get

$$d_b(y_{m+1}, y_{m+2}) < \frac{1}{b^2 (m+1)^{\frac{1}{h}}}, \quad \text{for all} \quad m \geq m_1. \tag{4.6}$$

Continuing in this way, we obtain

$$d_b(y_{n-1}, y_n) < \frac{1}{b^{n-m} (n-1)^{\frac{1}{h}}}, \quad \text{for all} \quad m > m_1. \tag{4.7}$$

Let $N = \max\{m_0, m_1\}$.

Let me to prove $\{y_m\}$ is a Cauchy sequence . For $n > m > N$ and using (4.5), (4.6) and (4.7), we have

$$\begin{aligned} d_b(x_m, x_n) &\leq b\ d_b(x_m, x_{m+1}) + b^2 d_b(x_{m+1}, x_{m+2}) + \cdots + b^{n-m} d_b(x_{n-1}, x_n) \\ &\leq \left(\frac{b}{b(m)^{1/h}} + \frac{b^2}{b^2(m+1)^{1/h}} + \cdots + \frac{b^{n-m}}{b^{n-m}(n-1)^{1/h}} \right) \\ &= \sum_{j=m}^{n-1} \frac{1}{j^{1/h}} \\ &\leq \sum_{j=1}^{\infty} \frac{1}{j^{1/h}}. \end{aligned}$$

Since $0 < h < 1$, then $\sum_{j=1}^{\infty} \dfrac{1}{j^{1/h}}$ converges.

Therefore $d_b(y_m, y_n) \to \infty$ as $m, n \to 0$.

Hence we have proved $\{y_m\}$ is a Cauchy sequence in Y. The completeness of Y admits that there exist $y_0 \in Y$ such that $y_m \to \infty$.

First we prove that y_0 is a fixed point of V. Contrary suppose that $y_0 \neq Vy_0$, then

$$\begin{aligned}1 < \psi_b(d_b(Vy_0, y_m)) &= \psi_b(d_b(Vy_0, Vy_{m+1}))\\ &\leq [\psi_b(d_b(y_0, y_{m+1})]^{t_1}.[\psi_b(d_b(y_0, Vy_0))]^{t_2}.[\psi_b(d_b(y_{m+1}, Vy_{m+1}))]^{t_3}\\ &\quad .[\psi_b(d_b(y_0, Vy_{m+1}) + d_b(y_{m+1}, Vy_0)]^{t_4}.\end{aligned}$$

Taking $m \to \infty$ in the above inequality, we get

$$1 < \psi_b(d_b(Vy_0, y_0)) \leq [\psi_b(d_b(y_0, Vy_0))]^{t_2+t_4} \leq \psi_b(d_b(Vy_0, y_0))$$

which contradict to our supposition. Hence we have $y_0 = Vy_0$. Therefore, y_0 is fixed point of V. For uniqueness, let x_0 be another fixed point of V.

Then

$$\begin{aligned}1 < \psi_b(d_b(x_0, y_0)) &= \psi_b(d_b(Vx_0, Vy_0))\\ &\leq [\psi_b(d_b(x_0, y_0))]^{t_1}.[\psi_b(d_b(x_0, Vx_0))]^{t_2}.[\psi_b(d_b(y_0, Vy_0))]^{t_3}\\ &\quad .[\psi_b(d_b(x_0, Vy_0) + d_b(y_0, Vx_0))]^{t_4}\\ &= [\psi_b(d_b(x_0, y_0))]^{t_1}[\psi_b(d_b(x_0, Vx_0))]^{t_2}[\psi_b(d_b(y_0, Vy_0))]^{t_3}\\ &\quad .[\psi_b(d_b(x_0, y_0)) + \psi_b(d_b(y_0, x_0))]^{t_4}\\ &\leq [\psi_b(d_b(x_0, y_0))]^{t_1}[\psi_b(d_b(x_0, Vx_0))]^{t_2}[\psi_b(d_b(y_0, Vy_0))]^{t_3}\\ &\leq 2^{t_4}[\psi_b(d_b(x_0, y_0))]^{t_1}.\end{aligned}$$

$$1 < \psi_b(d_b(x_0, y_0)) \leq 2^{t_4}[\psi_b(d_b(x_0, y_0))]^{t_1} < \psi(d_b(x_0, y_0))$$

which contradict to our supposition. Hence $x_0 = y_0$. Thus V has unique fixed point.

□

Definition 4.1.3. Let $V \colon Y \to Y$ be self mapping and (Y, d_b) be a b-metric space. Then V is called:

(i) A C-contraction if for any $\alpha \in (0, \frac{1}{2})$ satisfying the following inequality:

$$d_b(Va, Vb) \leq \alpha[d_b(a, Vb) + d_b(b, Va)]$$

for all $a, b \in Y$.

(ii) A K-contraction if for any $\alpha \in (0, \frac{1}{2})$ satisfying the following inequality:

$$d_b(Va, Vb) \leq \alpha[d_b(a, Va) + d_b(b, Vb)]$$

for all $a, b \in Y$.

(iii) A Reich contraction if and only if there exist a non-negative real numbers t_1, t_2, t_3 with $t_1 + t_2 + t_3 < 1$ satisfying the following inequality

$$d_b(Va, Vb) \leq t_1 d_b(a, b) + t_2 d_b(a, Va) + t_3 d_b(b, Vb)$$

for all $a, b \in Y$.

(iv) A Ciric contraction if and only if there exist non negative real numbers t_1, t_2, t_3 and t_4 such that $t_1 + t_2 + t_3 + t_4 < 1$ satisfying:

$$d_b(Va, Vb) \leq t_1 d_b(a, b) + t_2 d_b(a, Va) + t_3 d_b(a, Va) + t_4 d_b(a, Vb) + d_b(b, Va)]$$

for all $a, b \in Y$.

Theorem 4.1.4. *Let $V\colon Y \to Y$ be a given self mapping and (Y, d_b) be a complete b-metric space with $b \geq 1$, whenever there are positive real numbers t_1, t_2, t_3 and t_4 with $0 \leq t_1 + t_2 + t_3 + 2t_4 < 1$ satisfying:*

$$\begin{aligned} &\sqrt{d_b(Vx, Vy)} \\ &\leq t_1\sqrt{d_b(x, y)} + t_2\sqrt{d_b(x, Vx)} + t_3\sqrt{d_b(y, Vy)} + t_4\sqrt{(d_b(x, Vy) + d_b(y, Vx))} \end{aligned} \tag{4.8}$$

for all $x, y \in Y$, then V has only one fixed point.

Proof. The result follows from Theorem 4.1.2 by taking $\psi_b(u) = b^{\frac{\alpha}{1-\alpha}} e^{\sqrt{u}}$. □

Remark 4.1.5. The following result follows from the condition (4.8).

$$\begin{aligned} d_b(Vx, Vy) \leq {t_1}^2 d_b(x, y) &+ {t_2}^2 d_b(x, Vx) + {t_3}^2 d_b(y, Vy) + {t_4}^2 [d_b(x, Vy) + d_b(y, Vx)] \\ &+ 2t_1 t_2 \sqrt{d_b(x, y) d_b(x, Vx)} + 2t_1 t_3 \sqrt{d_b(x, y) d_b(y, Vy)} \\ &+ 2t_1 t_4 \sqrt{d_b(x, y)[d_b(x, Vy) + d_b(y, Vx)]} + 2t_2 t_3 \sqrt{d_b(y, Vx) d_b(y, Vy)} \\ &+ 2t_2 t_4 \sqrt{d_b(x, Vx)[d_b(x, Vy) + d_b(y, Vx)]} \\ &+ 2t_3 t_4 \sqrt{d_b(y, Vy)[d_b(x, Vy) + d_b(y, Vx)]}. \end{aligned}$$

Next, in view Remark of 4.1.5, by taking $t_1 = t_4 = 0$ in Theorem 4.1.4, we get the following extension.

Theorem 4.1.6. *Let $V : Y \to Y$ be a given self mapping and (Y, d_b) be a complete b-metric space with $b \geq 1$, whenever there are any positive real numbers t_2, t_3 with $0 < t_2 + t_3 < 1$, satisfying:*

$$d_b(Vx, Vy) \leq {t_2}^2 d_b(x, Vx) + {t_3}^2 d_b(y, Vy) + 2t_2 t_3 \sqrt{d_b(x, Vx) d_b(y, Vy)} \tag{4.9}$$

for all $x, y \in Y$. Then V has only one fixed point.

Any other way, the result follows from Theorem 4.1.4 by taking $t_1 = t_2 = t_3 = 0$.

Theorem 4.1.7. *Let $V : Y \to Y$ be a given self mapping and (Y, d_b) be a complete b-metric space with $b \geq 1$, whenever there are positive real number t_2, t_3 with $0 < t_2 + t_3 < 1$, satisfying:*

$$d_b(Vx, Vy) \leq {t_2}^2 d_b(x, Vx) + {t_3}^2 d_b(y, Vy) + 2t_2 t_3 \sqrt{d_b(x, Vx) d_b(y, Vy)}$$

for all $x, y \in Y$. Then V has only one fixed point.

The following result follows from Theorem 4.1.4 by taking $t_1 = t_2 = t_3 = 0$.

Theorem 4.1.8. *Let $V: Y \to Y$ be a given self mapping and (Y, d_b) be a complete b-metric space with $b \geq 1$ and there is any $t_4 \in [0, \frac{1}{2})$ satisfying*

$$d_b(Vx, Vy) \leq {t_4}^2[d_b(x, Vy) + d_b(y, Vx)]$$

for every $x, y \in Y$. Then V has only one fixed point.

The following result follows from Theorem 4.1.4, by taking $k_4 = 0$.

Theorem 4.1.9. *Let $V: Y \to Y$ be a given self mapping and (Y, d_b) be a complete b-metric space with $b \geq 1$, whenever there are positive real numbers t_1, t_2, k_3 with $0 < t_1 + t_2 + t_3 < 1$, satisfying:*

$$\begin{aligned} d_b(Vx, Vy) \leq\ & {t_1}^2 d_b(x, y) + {t_2}^2 d_b(x, Vx) + {t_3}^2 d_b(y, Vy) \\ & + 2t_1t_2\sqrt{d_b(x, y)d(x, Vx)} + 2t_1t_3\sqrt{d_b(x, y)d(y, Vy)} \\ & + 2t_2t_3\sqrt{d_b(x, Vx)d_b(y, Vy)} \end{aligned}$$

for all $x, y \in Y$, then V has only one fixed point.

The following Corollary follows from Theorem 4.1.2 by taking $\psi(u) = e^{\sqrt[n]{u}}$.

Corollary 4.1.10. *Let $V: Y \to Y$ be a self mapping and (Y, d_b) be a complete b-metric space with $b \geq 1$, whenever there are positive real numbers t_1, t_2, t_3 and t_4 with $0 < t_1 + t_2 + t_3 + 2t_4 < 1$ satisfying:*

$$\sqrt[n]{d_b(Vx, Vy)} \leq t_1\sqrt[n]{d_b(x, y)} + t_2\sqrt[n]{d_b(x, Vx)} + t_3\sqrt[n]{d_b(y, Vy)} + t_4\sqrt[n]{(d_b(x, Vy) + d(y, Vx))} \tag{4.10}$$

for all $x, y \in Y$, then V has only one fixed point.

Chapter 5

Common Fixed Point Theorem and Generalized JS-Contraction

In this chapter, we define the generalized modified JS-contraction in b-metric spaces for a pair of self mapping satisfying ψ_b-contractive condition and prove common fixed point results for such contraction in the framework of complete b-metric spaces. The obtained result extend the result of Ahmad et al.[3]. The presented result are generalization of recent fixed point result due to Hussain et al. [20]. We have also concluded the given results of Ahmad et al. [3].

5.1 Main result

Very recently, Ahmad et al. [3] defined two families $G(U,V)$ and $H(U,V)$ which are defined as follows.
The family $G(U,V)$ defined by all functions $t\colon Y\times Y\to[0,1)$ such that

$$t(x,VUy)\leq t(x,y)\quad\text{and}\quad t(UVx,y)\leq t(x,y)\ \text{ for all }\ x,y\in Y$$

and the family $H(U,V)$ defined by all functions $\gamma\colon Y\to[0,1)$ such that

$$\gamma(VUy)\leq\gamma(y)$$

For two given self mapping $U, V : Y \to Y$ and a b-metric space (Y, d_b).

Proposition 5.1. *Let (Y, d) be a b-metric space and $U, V : Y \to Y$ be given self mappings. Let $y_0 \in Y$, take a sequence $\{y_m\}$ defined by*

$$y_{2m+1} = Uy_{2m}, \quad y_{2m+2} = Vy_{2m+1}, \quad \textit{for each non-negative integer } m.$$

If $t \in G(U, V)$, then $t(x, y_{2m}) \le t(x, y_0)$ and $t(y_{2m+1}, y) \le t(y_1, y)$ for each $x, y \in Y$ and non-negative integer m.

Definition 5.1.1. Let (Y, d_b) be a b-metric space with co-efficient $b \ge 1$ and a given self mappings $U, V : Y \to Y$ is called generalized modified JS-contraction whenever there are mappings $t_1, t_2, t_3, t_4 \in G(U, V)$ with
$0 \le t_1(x, y) + t_2(x, y) + t_3(x, y) + 2t_4(x, y) < 1$ and there exist a function $\psi_b \in \Psi_b'$ satisfying

$$\begin{aligned}\psi_b(d_b(Ux, Vy)) \le [\psi_b(d_b(x, y))]^{t_1(x,y)}.[\psi_b(d_b(x, Ux))]^{t_2(x,y)}.[\psi_b(d_b(y, Vy))]^{t_3(x,y)} \\ .[\psi_b(d_b(x, Vy) + d_b(y, Ux))]^{t_4(x,y)}\end{aligned}$$

for all $x, y \in Y$.

Note. See 4.1 and 4.1.1 in Chapter 4 there is defined a family Ψ_b and Ψ_b'.
Now we state our main theorem.

Theorem 5.1.2. *Let (Y, d_b) be complete b-metric space with co-efficient $b \ge 1$ and let $U, V : Y \to Y$ be a given self mappings, whenever there are mappings $t_1 + t_2 + t_3 + t_4 \in G(U, V)$ and a function $\psi_b \in \Psi_b'$ satisfying:*

(a) $t_1(x, y) + t_2(x, y) + t_3(x, y) + 2t_4(x, y) < 1$

(c) $\psi_b(d_b(Ux, Vy))$
$\le [\psi_b(d_b(x, y))]^{t_1(x,y)}.[\psi_b(d_b(x, Ux)]^{t_2(x,y)}.[\psi_b(d_b(y, Vy)]^{t_3(x,y)}$
$.[\psi_b(d_b(x, Vy) + d_b(y, Ux)]^{t_4(x,y)}$

for all $x, y \in Y$, then U and V have only one fixed point.

Proof. Let $y_0 \in Y$, we define the sequence $\{y_n\}$ by

$$y_{2m+1} = Uy_{2m} \quad and \quad y_{2m+2} = Vy_{2m+1}$$

for every non-negative integer m. From Proposition 5.1 for every non-negative integer m, we have

$$\begin{aligned}
1 &< \psi_b(d_b(y_{2m}, y_{2m+1})) = \psi_b(d_b(Vy_{2m-1}, Uy_{2m})) = \psi_b(d_b(Uy_{2m}, Vy_{2m-1})) \\
&\leq [\psi_b(d_b(y_{2m}, y_{2m-1}))]^{t_1(y_{2m},y_{2m-1})}.[\psi_b(d_b(y_{2m}, Uy_{2m}))]^{t_2(y_{2m},y_{2m-1})} \\
&\quad .[\psi_b(d_b(y_{2m-1}, Vy_{2m-1}))]^{t_3(y_{2m},y_{2m-1})} \\
&\quad .[\psi_b(d_b(y_{2m}, Vy_{2m-1}) + d_b(y_{2m-1}, Ux_{2m}))]^{t_4(y_{2m},y_{2m-1})} \\
&= [\psi_b(d_b(y_{2m}, y_{2m-1}))]^{t_1(y_{2m},y_{2m-1})}.[\psi_b(d_b(y_{2m}, x_{2m+1}))]^{t_2(y_{2m},y_{2m-1})} \\
&\quad .[\psi_b(d_b(y_{2m-1}, y_{2m}))]^{t_3(y_{2m},y_{2m-1})}.[d_b(y_{2m-1}, y_{2m+1}))]^{t_4(y_{2m},y_{2m-1})} \\
&\leq [\psi_b(d_b(y_{2m}, y_{2m-1}))]^{t_1(y_{2m},y_{2m-1})}.[\psi_b(d_b(y_{2m}, y_{2m+1}))]^{t_2(y_{2m},y_{2m-1})} \\
&\quad .[\psi_b(d_b(y_{2m-1}, y_{2m}))]^{t_3(y_{2m},y_{2m-1})} \\
&\quad .[\psi_b(b((d_b(y_{2m-1}, y_{2m}) + d_b(y_{2m}, y_{2m+1})))]^{t_4(y_{2m},y_{2m-1})} \\
&\leq [\psi_b(d_b(y_{2m}, y_{2m-1}))]^{t_1(y_0,y_{2m-1})}.[\psi_b(d_b(y_{2m}, y_{2m+1}))]^{t_2(y_0,y_{2m-1})} \\
&\quad .[\psi_b(d_b(y_{2m-1}, y_{2m}))]^{t_3(y_0,y_{2m-1})} \\
&\quad .b^{t_4(y_0,y_{2m-1})}.[\psi_b(d_b(y_{2m-1}, y_{2m})]^{t_4(y_0,y_{2m-1})} \\
&\quad .[d_b(y_{2m}, y_{2m+1}))]^{t_4(y_0,y_{2m-1})}. \\
&\leq [\psi_b(d_b(y_{2m}, y_{2m-1}))]^{t_1(y_0,y_1)}.[\psi_b(d_b(y_{2m}, y_{2m+1}))]^{t_2(y_0,y_1)} \\
&\quad .[\psi_b(d_b(y_{2m-1}, y_{2m}))]^{t_3(y_0,y_1)} \\
&\quad .[b(y_0, y_1)]^{t_4(y_0,y_1)}.[\psi_b(d_b(y_{2m-1}, y_{2m})]^{t_4(y_0,y_1)}.[d_b(y_{2m}, y_{2m+1}))]^{t_4(y_0,y_1)}. \\
&= b^{t_4(y_0,y_1)}.[\psi_b(d_b(y_{2m-1}, y_{2m})]^{t_1(y_0,y_1)+t_3(y_0,y_1)+t_4(y_0,y_1)} \\
&\quad .[\psi(d_b(y_{2m}, y_{2m+1}))]^{t_2(y_0,y_1)+t_4(y_0,y_1)} \\
&\leq b^{t_1(y_0,y_1)+t_3(y_0,y_1)+t_4(y_0,y_1)}[\psi_b(d_b(y_{2m-1}, y_{2m}))]^{t_1(y_0,y_1)+t_3(y_0,y_1)+t_4(y_0,y_1)} \\
&\quad .[\psi_b(d_b(y_{2m}, y_{2m+1}))]^{t_2(y_0,y_1)+t_4(y_0,y_1)}.
\end{aligned}$$

Taking ln on both sides of the above inequality, we have

$\ln[\psi_b(y_{2m}, y_{2m+1}))] - (t_3(y_0, y_1) + t_4(y_0, y_1)) \ln[\psi_b(d_b(y_{2m}, y_{2m+1})]$

$\leq (t_1(y_0, y_1) + t_2(y_0, y_1) + t_4(y_0, y_1)) \ln[b\psi_b(d_b(y_{2m-1}, y_{2m}))].$

$(1-t_3(y_0, y_1)-t_4(y_0, y_1)) \ln[\psi_b(d_b(y_{2m}, y_{2m+1}))] \leq (t_1(y_0, y_1) + t_2(y_0, y_1) + t_4(y_0, y_1))$
$. \ln[b\psi_b(d_b(y_{2m-1}, y_{2m}))].$

$\ln[\psi_b(d_b(y_m, y_{m+1}))] \leq \dfrac{(t_1(y_0, y_1) + t_2(y_0, y_1) + t_4(y_0, y_1))}{(1 - t_3(y_o, y_1) - t_4(y_0, y_1))} \ln[b\psi_b(d_b(y_{2m-1}, y_{2m}))].$

$\psi_b(d_b(y_{2m}, y_{2m+1})) \leq [b\psi_b(d_b(y_{2m-1}, y_{2m}))]^{\frac{(t_1(y_0,y_1)+t_2(y_0,y_1)+t_4(y_0,y_1))}{(1-t_3(y_0,y_1)-t_4(y_0,y_1))}}.$

Let

$$\alpha = \frac{t_1(y_0, y_1) + t_3(y_0, y_1) + t_4(y_0, y_1)}{1 - t_3(y_0, y_1) - t_4(y_0, y_1)} < 1.$$

$$\text{Thus} \quad [\psi_b(d_b(y_{2m}, y_{2m+1}))] \leq [b\psi_b(d_b(y_{2m-1}, y_{2m}))]^{\alpha}. \tag{5.1}$$

Similarly, we have

$$\begin{aligned}
1 &< \psi_b(d_b(y_{2m+1}, y_{2m+2})) = \psi_b(d_b(Uy_{2m}, Vy_{2m+1})) \\
&\leq [\psi_b(d_b(y_{2m}, y_{2m+1}))]^{t_1(y_{2m},y_{2m+1})}.[\psi_b(d_b(y_{2m}, Uy_{2m}))]^{t_2(y_{2m},y_{2m-1})} \\
&\quad .[\psi_b(d_b(y_{2m+1}, Vy_{2m+1}))]^{t_3(y_{2m},y_{2m-1})} \\
&\quad .[\psi_b(d_b(y_{2m}, Vy_{2m+1}) + d(y_{2m+1}, Uy_{2m}))]^{t_4(y_{2m+1},y_{2m})} \\
&= [\psi_b(d_b(y_{2m}, y_{2m+1}))]^{t_1(y_{2m},y_{2m+1})}.[\psi_b(d_b(y_{2m}, y_{2m+1}))]^{t_2(y_{2m},y_{2m+1})} \\
&\quad .[\psi_b(d_b(y_{2m+1}, y_{2m+2}))]^{t_3(y_{2m},y_{2m+1})}.[\psi_b(d_b(y_{2m}, y_{2m+2}))]^{t_4(y_{2m+1},y_{2m})} \\
&\leq [\psi_b(d_b(y_{2m}, y_{2m+1}))]^{t_1(y_0,y_{2m+1})}.[\psi_b(d_b(y_{2m}, y_{2m+1}))]^{t_2(y_0,y_{2m+1})} \\
&\quad .[\psi_b(d_b(y_{2m+1}, y_{2m+2}))]^{t_3(y_0,y_{2m+1})} \\
&\quad .[\psi_b(b(d_b(y_{2m}, y_{2m+1}) + d_b(y_{2m+1}, y_{2m+2}))]^{t_4(y_0,y_{2m+1})} \\
&\leq [\psi_b(d_b(y_{2m}, y_{2m+1}))]^{a_1(y_0,y_1)}.[\psi_b(d_b(y_{2m}, y_{2m+1}))]^{t_2(y_0,y_1)} \\
&\quad .[\psi_b(d_b(y_{2m+1}, y_{2m+2}))]^{t_3(y_0,y_1)} \\
&\quad .b^{t_4(y_0,y_1)}.[\psi_b(d_b(y_{2m}, y_{2m+1})]^{t_4(y_0,y_1)}.[d_b(y_{2m+1}, y_{2m+2})]^{t_4(y_0,y_1)}
\end{aligned}$$

$$
\begin{aligned}
&\leq [\psi_b(d(y_{2m}, y_{2m+1}))]^{t_1(y_0,y_1)}.[\psi_b(d(y_{2m}, y_{2m+1}))]^{t_2(y_0,y_1)} \\
&\quad .[\psi_b(d(y_{2m+1}, y_{2m+2}))]^{t_3(y_0,y_1)} \\
&\quad .b^{t_4(y_0,y_1)}[\psi_b(d_b(y_{2m}, y_{2m+1})]^{t_4(y_0,y_1)}[d_b(y_{2m+1}, y_{2m+2})]^{t_4(y_0,y_1)} \\
&\leq b^{t_1(y_0,y_1)+t_2(y_0,y_1)+t_4(y_0,y_1)}.[\psi_b(d_b(y_{2m}, y_{2m+1}))]^{t_1(y_0,y_1)+t_2(y_0,y_1)+t_4(y_0,y_1)} \\
&\quad .[\psi_b(d_b(y_{2m+1}, y_{2m+2}))]^{t_3(y_0,y_1)+t_4(y_0,y_1)}.
\end{aligned}
$$

Taking ln on both sides of the above inequality, we have

$$
\begin{aligned}
&\ln[\psi_b(d_b(y_{2m+1}, y_{2m+2}))] - (t_3(y_0,y_1) + t_4(y_0,y_1)) \ln[\psi_b(d_b(y_{2m+1}, y_{2m+2})] \\
&\leq (t_1(y_0,y_1) + t_2(y_0,y_1) + t_4(y_0,y_1)) \ln[b\psi_b(d_b(y_{2m}, y_{2m+1}))].
\end{aligned}
$$

$$
\begin{aligned}
&(1-t_3(y_0,y_1)-t_4(y_0,y_1)) \ln[\psi_b(d_b(y_{2m+1}, y_{2m+2}))] \leq (t_1(y_0,y_1) + t_2(y_0,y_1) + t_4(y_0,y_1)) \\
&. \ln[b\psi_b(d_b(y_{2m}, y_{2m+1}))].
\end{aligned}
$$

$$
\ln[\psi_b(d_b(y_m, y_{m+1}))] \leq \frac{(t_1 + t_2 + t_4)}{(1 - t_3 - t_4)} \ln[b\psi_b(d_b(y_{2m-1}, y_{2m}))].
$$

$$
\psi_b(d_b(y_{2m+1}, y_{2m+2})) \leq [b\psi_b(d_b(y_{2m}, y_{2m+1}))]^{\frac{(t_1(y_0,y_1)+t_2(y_0,y_1)+t_4(y_0,y_1))}{(1-t_3(y_0,y_1)-t_4(y_0,y_1))}}.
$$

Thus

$$
\psi_b(d_b(y_{2m+1}, y_{2m+2}) \leq [b\psi_b(d(y_{2m}, y_{2m+1}))]^{\frac{t_1(y_0,y_1)+t_3(y_0,y_1)+t_4(y_0,y_1)}{1-t_3(y_0,y_1)-t_4(y_0,y_1)}}. \tag{5.2}
$$

Let

$$
\alpha = \frac{t_1(y_0,y_1) + t_3(y_0,y_1) + t_4(y_0,y_1)}{1 - t_3(y_0,y_1) - t_4(y_0,y_1)} < 1.
$$

Then from 5.1 and 5.2, we get

$$
\begin{aligned}
1 < \psi_b(d_b(y_m, y_{m+1})) &\leq b^{\alpha}[\psi_b(d(y_{m-1}, y_m))]^{\alpha} \\
&\leq b^{\alpha+\alpha^2}[\psi_b(d_b(y_{m-2}, y_{m-1}))]^{\alpha^2} \\
&\ \ \vdots \\
&\leq b^{\alpha+\alpha^2\cdots+\alpha^m}[\psi_b(d_b(y_0, y_1))]^{\alpha^m}. \qquad (5.3)
\end{aligned}
$$

Taking $m \to \infty$ in the above inequality, we get

$$\begin{aligned}
\lim_{m\to\infty} b^{\alpha+\alpha^2+\cdots+\alpha^m}[\psi_b(d_b(y_0,y_1))]^{\alpha^m} &= \lim_{m\to\infty} b^{\frac{\alpha(1-\alpha^m)}{1-\alpha}}[\psi_b(d_b(y_0,y_1))]^{\alpha^m} \\
&= \lim_{m\to\infty} b^{\frac{\alpha(1-\alpha^m)}{1-\alpha}}. \lim_{m\to\infty}[\psi_b(d_b(y_0,y_1))]^{\alpha^m} \\
&= b^{\frac{\alpha}{1-\alpha}}.1 \\
&= b^{\frac{\alpha}{1-\alpha}}.
\end{aligned}$$

Using Sandwitch Theorem, we get

$$\lim_{m\to\infty} \psi_b(d_b(y_m,y_{m+1})) = b^{\frac{\alpha}{1-\alpha}}.$$

By using condition 4.1. From condition of (ψ_{b_1}), we have

$$\lim_{m\to\infty} d_b(y_m,y_{m+1}) = 0.$$

From the condition (ψ_{b_3}), there exist $0 < h < 1$ and $\ell \in (0,\infty]$ such that

$$\lim_{m\to\infty} \frac{\psi_b(d_b(y_m,y_{m+1}))-1}{d(y_m,y_{m+1})^h} = \ell.$$

$$\left|\frac{\psi_b(d_b(y_m,y_{m+1}))-1}{d_b(y_m,y_{m+1})^h} - \ell\right| \le r.$$

$$-r \le \frac{\psi_b(d_b(y_m,y_{m+1}))-1}{d_b(y_m,y_{m+1})^h} - \ell \le r.$$

Consider

$$\frac{\psi_b(d_b(y_m,y_{m+1}))-1}{d_b(y_m,y_{m+1})^h} - \ell \ge -r.$$

$$\frac{\psi(d(y_m,y_{m+1}))-1}{d(y_m,y_{m+1})^h} \ge \ell - r.$$

$$\frac{\psi_b(d(y_m, y_{m+1})) - 1}{d(y_m, y_{m+1})^h} \geq \ell - \frac{\ell}{2} = r.$$

$$d(y_m, y_{m+1})^h \leq \frac{1}{r}\psi_b(d(y_m, y_{m+1})) - 1$$

for all $m > m_0$. Then

$$d_b(y_m, y_{m+1})^h \leq s[\psi_b(d_b(y_m, y_{m+1})) - 1],$$

Where $s = \frac{1}{r}$.

Suppose that $\ell = \infty$. Let $r > 0$ be an arbitrary positive number. From the definition of limit, there exist $m_0 \in \mathbb{N}$ such that

$$\frac{\psi_b(d_b(y_m, y_{m+1})) - 1}{d(y_m, y_{m+1})^h} \geq r.$$

This implies that, for all $n \geq n_0$

$$(d_b(y_m, y_{m+1}))^h \leq s[\psi_b(d_b(y_m, y_{m+1})) - 1].$$

Hence for all cases there exist, $s > 0$ and $m_0 \in \mathbb{N}$ such that

$$(d_b(y_m, y_{m+1}))^h \leq s[\psi_b(d_b(y_m, y_{m+1})) - 1].$$

Using (5.2) in the above inequality, we get

$$(d_b(y_m, y_{m+1}))^h \leq sb^{\alpha+\alpha^2+\cdots+\alpha^m}\left[[\psi_b(d(y_0, y_1))]^{\alpha^m} - 1\right]. \tag{5.4}$$

Since $b \geq 1$ and $0 < h < 1$, then $b^h > 0$. Then for all $m > m_0$

$$b^h m(d_b(y_m, y_{m+1}))^h \leq sb^h b^{\alpha+\alpha^2+\cdots+\alpha^m} m\left[[\psi_b(d_b(y_0, y_1))]^{\alpha^n} - 1\right].$$

Taking $m \to \infty$ in the above inequality.

$$\lim_{m\to\infty} b^h m(d_b(y_m, y_{m+1}))^h \leq sb^h \lim_{n\to\infty} b^{\alpha+\alpha^2+\cdots+\alpha^m} m\left[[\psi_b(d_b(y_0, y_1))]^{\alpha^m} - 1\right]$$

$$\begin{aligned}\lim_{m\to\infty} b^h m(d_b(y_m, y_{m+1}))^h &= sb^h \lim_{n\to\infty} b^{\alpha+\alpha^2+\cdots+\alpha^m}. \lim_{m\to\infty} m\left[[\psi_b(d_b(y_0, y_1))]^{\alpha^m} - 1\right] \\ &= sb^h.b^{\frac{\alpha}{1-\alpha}}. \lim_{m\to\infty} m\left[[\psi_b(d_b(y_0, y_1))]^{\alpha^m} - 1\right].\end{aligned}$$

Consider

$$\begin{aligned}\lim_{m\to\infty} m[\psi_b(d_b(y_0, y_1))^{\alpha^m} - 1] &= \lim_{m\to\infty} \frac{[\psi_b(d_b(y_0, y_1))^{\alpha^m} - 1]}{\frac{1}{m}} \\ &= \lim_{m\to\infty} \frac{\alpha^m \ln(\alpha) \ln(\psi_b(d_b(y_0, y_1))[\psi_b(d_b(y_0, y_1))]^{\alpha^m}}{\frac{-1}{m^2}} \\ &= \lim_{m\to\infty} -m^2\alpha^m \ln(\alpha) \ln(\psi_b(d_b(y_0, y_1))[\psi_b(d_b(y_0, y_1))]^{\alpha^m} \\ &= \lim_{m\to\infty} \frac{-m^2\alpha^m \ln(\psi_b(d_b(y_0, y_1))[\psi_b(d_b(y_0, y_1))]^{\alpha^m}}{\alpha_1^m} \\ &= \lim_{m\to\infty} \frac{-m^2}{\alpha_1^m}. \lim_{m\to\infty} \ln(\alpha) \ln(\psi_b(d(y_0, y_1))[\psi_b(d(y_0, y_1))]^{\alpha^m} \\ &= 0. \ln(\alpha) \ln(\psi_b(d_b(y_0, y_1)) \\ &= 0 \quad (\text{where} \quad \alpha_1 = \frac{1}{\alpha}).\end{aligned}$$

$$\lim_{m\to\infty} m[\psi_b(d_b(y_0, y_1))^{\alpha^m} - 1] = 0. \tag{5.5}$$

This implies that

$$\lim_{m\to\infty} b^h m(d_b(y_m, y_{m+1}))^h = 0.$$

Then by definition of limit there exist $\epsilon > 0$, choosing $\epsilon \in (0, 1)$ and there is an $m_1 \in \mathbb{N}$ such that for every $m \geq m_1$

$$\begin{aligned}\left|b^h m(d(y_m, y_{m+1}))^h - 0\right| &< \epsilon \\ b^h m(d_b(y_m, y_{m+1}))^h &< \epsilon \\ bm^{\frac{1}{h}}(d_b(y_m, y_{m+1})) &< \epsilon' \quad (\text{where} \quad \epsilon^{1/h} = \epsilon') \\ d_b(y_m, y_{m+1}) &< \frac{\epsilon'}{bm^{\frac{1}{h}}} \\ \Rightarrow \quad d_b(y_m, y_{m+1}) &< \frac{1}{bm^{\frac{1}{h}}}, \quad \text{for all} \quad m \geq m_1.\end{aligned} \tag{5.6}$$

From (5.4) $\quad (d_b(y_m, y_{m+1}))^h \leq sb^{\alpha+\alpha^2+\cdots+\alpha^m}\left[[\psi_b(d_b(y_0, y_1))]^{\alpha^m} - 1\right].$

Since $b \geq 0$, $0 < h < 1$ then $b^{2h} > 0$. Then for all $m > m_0$

$$b^{2h} m(d_b(y_m, y_{m+1}))^h \leq sb^{2h} b^{\alpha+\alpha^2+\cdots+\alpha^m} m\left[[\psi_b(d_b(y_0, y_1))]^{\alpha^m} - 1\right].$$

Taking lim $m \to \infty$ and using (5.5)

$$\lim_{m\to\infty} b^{2h} m(d(y_m, y_{m+1}))^h = 0.$$

Then there exist $m_1 \in \mathbb{N}$ such that

$$d_b(y_m, y_{m+1}) < \frac{1}{b^2 m^{\frac{1}{h}}}, \quad \text{for all} \quad m \geq m_1.$$

Replacing m with $m+1$, we get

$$d_b(y_{m+1}, y_{m+2}) < \frac{1}{b^2(m+1)^{\frac{1}{h}}}, \quad \text{for all} \quad m \geq m_1. \tag{5.7}$$

Continuing in this way, we obtain

$$d_b(y_{n-1}, y_n) < \frac{1}{b^{n-m}(n-1)^{\frac{1}{h}}}, \quad \text{for all} \quad m \geq m_1. \tag{5.8}$$

Let $N = \max\{m_0, m_1\}$.

Now we prove that $\{y_m\}$ is a Cauchy sequence . For $n > m > N$ and using (5.6), (5.7) and (5.8), we have

$$\begin{aligned}
d_b(y_m, y_n) &\leq b\ d_b(y_m, y_{m+1}) + b^2 d_b(y_{m+1}, y_{m+2}) + \cdots + b^{n-m} d(y_{n-1}, y_n) \\
&\leq \left(\frac{b}{b(m)^{1/h}} + \frac{b^2}{b^2(m+1)^{1/h}} + \cdots + \frac{b^{n-m}}{b^{n-m}(n-1)^{1/h}}\right) \\
&= \left(\frac{1}{(m)^{1/h}} + \frac{1}{(m+1)^{1/h}} + \cdots + \frac{1}{(n-1)^{1/h}}\right) \\
&= \sum_{j=m}^{n-1} \frac{1}{j^{1/h}} \\
&\leq \sum_{j=1}^{\infty} \frac{1}{j^{1/h}}.
\end{aligned}$$

Since $0 < h < 1$, then $\sum_{j=1}^{\infty} \dfrac{1}{j^{1/h}}$ converges.
Therefore $d_b(y_m, y_n) \to \infty$ as $m, n \to 0$.
Thus it is proved that $\{y_m\}$ is a Cauchy sequence in Y. The completeness of Y insure that there exist $y_0 \in Y$ such that $y_m \to \infty$. First we show that y_0 is a fixed point of U. Contrary suppose that $y_0 \neq Uy_0$

$$\begin{aligned}
1 &< \psi_b(d_b(Uy_0, y_{2m+2})) = \psi_b(d_b(Uy_0, Vy_{2m+1})) \\
&\leq [\psi_b(d_b(y_0, y_{2m+1})]^{t_1(y_0,y_{2m+1})}.[\psi_b(d(y_0, Uy_0))]^{t_2(y_0,y_{2m+1})} \\
&\quad .[\psi_b(d(y_{2m+1}, Vy_{2m+1}))]^{t_3(y_0,y_{2m+1})}.[\psi_b(d_b(y_0, Vy_{2m+1}) + d_b(y_{2m+1}, Uy_0)]^{t_4(y_0,y_{2m+1})} \\
&= [\psi_b(d_b(y_0, y_{2m+1})]^{t_1(y_0,y_{2m+1})}.[\psi_b(d(y_0, Uy_0))]^{t_2(y_0,y_{2m+1})} \\
&\quad .[\psi_b(d_b(y_{2m+1}, y_{2m+2}))]^{t_3(y_0,y_{2m+1})}.[\psi_b(d(y_0, y_{2m+2}) + d_b(x_{2m+1}, Uy_0)]^{t_4(y_0,y_{2m+1})} \\
&\leq [\psi_b(d_b(y_0, y_{2n+1})]^{t_1(y_0,y_1)}.[\psi_b(d_b(y_0, Vy_0))]^{t_2(y_0,y_1)} \\
&\quad .[\psi_b(d_b(y_{2m+1}, y_{2m+2}))]^{t_3(y_0,y_1)}.[\psi_b(d_b(y_0, y_{2m+2}) + d_b(y_{2m+1}, Uy_0)]^{t_4(y_0,y_1)}.
\end{aligned}$$

Taking lim $n \to +\infty$, in the above inequality, we get

$$1 < \psi_b(d_b(Uy_0, y_0)) \leq [\psi_b(d_b(y_0, Uy_0))]^{t_2(y_0,y_1)+t_4(y_0,y_1)} < \psi_b(d_b(Uy_0, y_0))$$

which contradict to our supposition $y_0 \neq Uy_0$.
Hence $y_0 = Uy_0$. We also show that x_0 is the fixed point of V, suppose $y_0 \neq Vy_0$, then by the Propsition 5.1, we have

$$\begin{aligned}
1 &< \psi_b(d_b(y_{2m+1}, Vy_0)) = \psi_b(d_b(Uy_{2m}, Vy_0)) \\
&\leq [\psi_b(d(y_{2m}, y_0)]^{a_1(y_{2m},y_0)}.[\psi_b(d(y_{2m}, Uy_{2n}))]^{a_2(y_{2m},y_0)}.[\psi_b(d_b(y_0, Vy_0))]^{a_3(y_{2m},y_0)} \\
&\quad .[\psi_b(d(y_{2m}, Vy_0) + d_b(y_0, Uy_{2m})]^{a_4(y_{2m},y_0)} \\
&= [\psi_b(d_b(y_{2m}, y_0)]^{a_1(y_{2m},y_0)}.[\psi_b(d_b(y_{2m}, y_{2m+1}))]^{a_2(y_{2m},y_0)}.[\psi_b(d_b(y_0, Vy_0))]^{a_3(y_{2m},y_0)} \\
&\quad .[\psi_b(d_b(y_{2m}, Vy_0) + d_b(y_0, y_{2m+1})]^{a_4(y_{2m},y_0)} \\
&\leq [\psi_b(d_b(y_{2m}, y_0)]^{a_1(y_0,y_0)}.[\psi_b(d_b(y_0, Vy_0))]^{a_2(y_0,y_0)}.[\psi_b(d_b(y_0, Vy_0))]^{a_3(y_0,y_0)} \\
&\quad .[\psi_b(d_b(y_1, Vy_0)]^{a_4(y_0,y_0)}[d_b(y_0, y_{2n+1})]^{a_4(y_0,y_0)}.
\end{aligned}$$

Taking $m \to +\infty$, in the above inequality, we get

$$1 < \psi_b(d_b(y_0, Vy_0)) \leq [\psi_b(d_b(y_0, Vy_0))]^{a_3(y_0,y_0)+a_4(y_0,y_1)} < [\psi_b(d_b(y_0, Vy_0))] \quad (5.9)$$

which contradict to our supposition $y_0 \neq Vy_0$.
Hence $y_0 = Vy_0$. Therefore, y_0 is the fixed point of U and V.
Now we shall show y_0 is unique fixed point of U and V, for this let x_0 be another fixed point of U and V. This implies $x_0 = Ux_0 = Vx_0$

$$\begin{aligned}
1 < \psi_b(d_b(x_0, y_0)) &= \psi_b(d_b(Ux_0, Vy_0)) \\
&\leq [\psi_b(d_b(x_0, y_0))]^{t_1(x_0,y_0)}.[\psi_b(d(x_0, Ux_0))]^{t_2(x_0,y_0)}.[\psi_b(d(y_0, Vy_0))]^{t_3(x_0,y_0)} \\
&\quad .[\psi_b(d_b(x_0, Vy_0) + d(y_0, Ux_0))]^{t_4(x_0,y_0)} \\
&\leq [\psi_b(d_b(x_0, y_0))]^{t_1(x_0,y_0)}.[\psi_b(d(x_0, y_0))]^{t_4(x_0,y_0)}.[\psi_b(d(x_0, y_0))]^{t_4(x_0,y_0)} \\
&= [\psi_b(d_b(x_0, y_0))]^{t_1(x_0,y_0)+2t_4(x_0,y_0)} < [\psi_b(d_b(x_0, y_0))].
\end{aligned}$$

Which contradict to our supposition. Hence U and V have only one fixed point. □

The following results has been concluded from above result.

Corollary 5.1.3. *Let (Y, d_b) be a complete b-metric space with coefficient $b \geq 1$ and $U\colon Y \to Y$ be the given self mappings, whenever there are mappings $t_1, t_2, t_3, t_4 \in M(U, V)$ and a function $\psi_b \in \Psi_b'$ satisfying:*

(a) $t_1(x, y) + t_2(x, y) + t_3(x, y) + t_3(x, y) + 2t_4(x, y) < 1$

(b) $\psi_b(d_b(Ux, Uy))$
$\leq [\psi_b(d_b(x, y))]^{t_1(x,y)}.[\psi_b(d_b(x, Ux)]^{t_2(x,y)}.[\psi_b(y, Uy)]^{t_3(x,y)}$
$.[\psi_b(d_b(x, Uy) + d_b(y, Ux)]^{t_4(x,y)}$

for all $x, y \in Y$.

Proof. The result follows from Theorem 5.1.2 by taking $U = V$. □

Theorem 5.1.4. *Let (Y, d_b) be a complete b-metric space with coefficient $b \geq 1$ and $U, V\colon Y \to Y$ be the given self mapping, whenever there are mapping $t_1, t_2, t_3, t_4 \in M(U, V)$ satisfying:*

(a) $t_1(x,y) + t_2(x,y) + t_3(x,y) + t_3(x,y) + 2t_4(x,y) < 1$

(b) $$\sqrt{\psi_b(d_b(Ux,Vy))}$$
$$\leq t_1(x,y)\sqrt{(d_b(x,y))} + t_2(x,y)\sqrt{d(x,Ux)} + t_3(x,y)\sqrt{(y,Vy)}$$
$$+ t_4(x,y)\sqrt{d_b(x,Vy) + d(y,Ux)}$$

for all $x, y \in Y$, then U and V have a unique fixed point.

Proof. Taking $\psi(t) = b^{\frac{\alpha}{1-\alpha}} e^{\sqrt{t}}$ in Theorem 5.1.2, the proof follows immediately.

□

Corollary 5.1.5. *Let (Y, d_b) be a complete b-metric space with coefficient $b \geq 1$ and $U\colon Y \to Y$ be the given self mapping, whenever there are mappings $t_1, t_2, t_3, t_4 \in M(U, V)$ satisfying:*

(a) $t_1(x,y) + t_2(x,y) + t_3(x,y) + 2t_4(x,y) < 1$

(b) $$\sqrt{(d_b(Ux,Uy))}$$
$$\leq t_1(x,y)\sqrt{d_b(x,y))} + t_2(x,y)\sqrt{d_b(x,Ux)} + t_3(x,y)\sqrt{d_b(y,Uy)}$$
$$+ t_4(x,y)\sqrt{d(x,Uy) + d(y,Ux)}$$

for all $x, y \in Y$, then U has only one fixed point.

Proof. The proof follows from corollary 5.1.3 by taking $\psi_b(t) = b^{\frac{\alpha}{1-\alpha}} e^{\sqrt{t}}$. □

Remark 5.1.6. From the above Corollary, we deduce the following result

$$\begin{aligned} d_b(Ux,Vy) \leq & a_1^2(x,y)d(x,y) \\ & + a_2^2(x,y)^2 d_b(x,Ux) + a_3^3(x,y)^2 d_b(y,Vy) \\ & + a_4^4(x,y)^2 [d_b(x,Vy) + d_b(y,Ux)] \end{aligned}$$

$$
\begin{aligned}
d_b(Ux, Vy) \leq &+ 2a_1(x,y)a_2(x,y)\sqrt{d_b(x,y)d(x,Ux)} \\
&+ 2a_1(x,y)a_3(x,y)\sqrt{d_b(x,y)d_b(y,Vy)} \\
&+ 2a_1(x,y)a_4(x,y)\sqrt{d_b(x,y)[d_b(x,Vy)+d_b(y,Ux)} \\
&+ 2a_2(x,y)a_3(x,y)\sqrt{d_b(x,Ux)d_b(y,Vy)} \\
&+ 2a_2(x,y)a_4(x,y)\sqrt{d_b(x,Ux)[d_b(x,Vy)+d_b(y,Ux)} \\
&+ 2a_3(x,y)a_4(x,y)\sqrt{d_b(y,Vy)[d_b(x,Vy)+d_b(y,Ux)}.
\end{aligned}
$$

Theorem 5.1.7. *Let (Y, d_b) be a complete b-metric space with coefficient $b \geq 1$ and $U, V : Y \to Y$ be a given self mappings, whenever there are mappings $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in N(U, V)$ satisfying:*

(a) $\gamma_1(y) + \gamma_2(y) + \gamma_3(y) + \gamma_3(y) + 2\gamma_4(y) < 1$

(b) $\psi_b(d_b(Ux, Vy))$
$\leq [\psi_b(d_b(x,y))]^{\gamma_1(x)}.[\psi_b(d_b(x,Ux)]^{\gamma_2(x)}.[\psi_b(y,Vy)]^{\gamma_3(y)}$
$.[\psi_b(d_b(x,Vy)+d_b(y,Ux)]^{\gamma_4(y)}$

for all $x, y \in Y$. and $\psi_b \in \Psi_b'$, then U and V have only one fixed point.

Proof. Define $t_1, t_2, t_3, t_4 \colon Y \times Y \to [0,1)$ by $t_1(x,y) = \gamma_1(y), t_2(x,y) = \gamma_2(y)$, $t_3(x,y) = \gamma_3(y)$ and $t_4(x,y) = \gamma_4(y)$ for all $x, y \in Y$. Then for every $x, y \in Y$.

$t_1(x, VUy) = \gamma_1(VUx) \leq \gamma_1(y) = t_1(x,y)$ *and* $t_1(UVx, y) = \gamma_1(y) = t_1(x,y)$
$t_2(x, VUy) = \gamma_1(VUx) \leq \gamma_2(y) = t_2(x,y)$ *and* $t_2(UVx, y) = \gamma(y)_2 = t_2(x,y)$
$t_3(x, VUy) = \gamma_3(VUx) \leq \gamma_3(y) = t_3(x,y)$ *and* $t_3(UVx, y) = \gamma_3(y) = t_3(x,y)$
$t_4(x, VUy) = \gamma_4(VUx) \leq \gamma_4(y) = t_4(x,y)$ *and* $t_4(UVx, y) = \gamma_4(y) = t_4(x,y)$
$t_1(x,y) + t_2(x,y) + t_3(x,y) + t_4(x,y) = \gamma_1(y) + \gamma_2(y) + \gamma_3(y) + \gamma_4(y) < 1$
Thus
$\psi_b(d_b(Ux, Vy))$
$\leq [\psi_b(d_b(x,y))]^{\gamma_1}.[\psi_b(d_b(x,Ux)]^{\gamma_2}.[\psi_b(d_b(y,Vy)]^{\gamma_3}.[\psi_b(d_b(x,Vy)+d_b(y,Ux)]^{\gamma_4}$
$\psi_b(d_b(Ux, Vy))$
$\leq [\psi_b(d_b(x,y))]^{t_1(x,y)}.[\psi_b(d_b(x,Ux)]^{t_2(x,y)}.[\psi_b(d_b(y,Vy)]^{t_3(x,y)}$

$.[\psi_b(d_b(x, Vy) + d_b(y, Ux)]^{t_4(x,y)}$

Then by Theorem 5.1.2 U and V have only one fixed point. □

Replacing $\gamma_1(y), \gamma_2(y), \gamma_3(y)$ and $\gamma_4(y)$ by $\gamma_1, \gamma_2, \gamma_3$ and γ_4 respectively.

Corollary 5.1.8. *Let (Y, d_b) be a complete b-metric space with coefficient $b \geq 1$ and $U\colon Y \to Y$ be a given self mappings. Whenever there are a mappings $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in N(U, V)$ and there exist a function $\psi_b \in \Psi_b'$ satisfying:*

(a) $\gamma_1(y) + \gamma_2(y) + \gamma_3(y) + \gamma_3(y) + 2\gamma_4(y) < 1$

(b) $\psi_b(d_b(Ux, Uy))$
$\leq [\psi_b(d_b(x, y))]^{\gamma_1(x)}.[\psi_b(d_b(x, Ux)]^{\gamma_2(x)}.[\psi_b(y, Uy)]^{\gamma_3(x)}.[\psi_b(d_b(x, Uy)+d_b(y, Ux)]^{\gamma_4(x)}$

for all $x, y \in Y$, then U and V have only one fixed point.

Corollary 5.1.9. *Let (Y, d_b) be a complete b-metric space with co-efficient with $b \geq 1$ and $V\colon Y \to Y$ be a given mapping . Whenever there is a constant $\gamma \in [0, 1)$ and there exist a function $\psi_b \in \Psi_b'$ satisfying:*

$$d_b(Vx, Vy) \neq 0 \quad \Rightarrow \quad \psi_b(d_b(Vx, Vy)) \leq [\psi_b(d_b(x, y))]^{\gamma}$$

for all $x, y \in Y$, then V has only one fixed point.

Taking $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma$ in Corollary 5.1.8.

The main result by Ahmad et al. [3] can now be established as the following Corollary of our result. Since for $b = 1$ the family ψ_b becomes the family ψ given in [3].

Corollary 5.1.10. *[3] "Let (Y, d) be complete metric space and let $S, T\colon X \to X$ be a given self mappings. If there exist mappings $a_1, a_2, a_3, a_4 \in G(S, T)$ and a function $\psi \in \Psi$ such that for all $x, y \in X$:*

(a) $a_1(x, y) + a_2(x, y) + a_3(x, y) + 2a_4(x, y) < 1$

(c) $\psi(d(Sx, Ty))$
$\leq [\psi(d(x,y))]^{a_1(x,y)}.[\psi(d(x,Sx)]^{a_2(x,y)}.[\psi(d(y,Ty)]^{a_3(x,y)}$
$.[\psi(d(x,Ty) + d(y,Sx)]^{a_4(x,y)}$.

Then S and T have unique fixed point."

Proof. The result follows from Theorem 5.1.2 by taking $b = 1$. □

The results proved by Ahmad et al. [3] follows from above results by taking $b = 1$. Now, we introduce an example which illustrate our result.

Example 5.1.11. Take a sequence

$$\begin{aligned}
S_1^* &= 1 \times 2 \\
S_2^* &= 1 \times 2 + 2 \times 3 \\
S_3^* &= 1 \times 2 + 2 \times 3 + 3 \times 4 \\
S_m^* &= 1 \times 2 + 2 \times 3 + \cdots + m \times (m+1) = \frac{m(m+1)(m+2)}{3}
\end{aligned}$$

Let $Y = \{S_m^* \colon m \in \mathbb{N}\}$ and $d_b(x,y) = (x-y)^2$. Then (Y, d_b) is a complete b-metric space with coefficient $b = 2$. Define the mapping $V \colon Y \to Y$ by,

$$V(S_1^*) = S_1^*, V(S_m^*) = S_{m-1}^*, \quad \forall \;\; m \geq 2$$

It is clear that the Banach contraction is not fulfilled. indeed, it is not difficult to check.
$$\lim_{n\to\infty} \frac{d_b(V(S_m^*), V(S_1^*))}{d_b(S_m^*, S_1^*)} = \lim_{n\to\infty} \frac{d_b(S_{m-1}^*, S_1^*)}{d_b(S_m^*, S_1^*)} = \lim_{n\to\infty} \frac{((m-1)m(m+1)-6)^2}{(m(m+1)(m+2)-6)^2} = 1$$
Let us take a function $\psi \colon (0,\infty) \to (1,\infty)$ defined by $\psi(u) = e^{\sqrt{ue^u}}$. We can show $\psi \in \Psi'$. We shall prove that V fulfill the condition of the result 5.1.9 ,i.e

$$d_b(V(S_m^*), (S_n^*)) \neq 0 \quad \Rightarrow \quad e^{\sqrt{d_b(V(S_m^*),V(S_n^*))e^{d_b(V(S_n^*),V(S_m^*))}}} \leq e^{\alpha\sqrt{d_b(S_m^*,S_n^*)e^{d_b(S_m^*,S_n^*)}}}$$

for some $\alpha \in (0,1)$. From above inequality, we have

$$d_b(V(S_m^*), V(S_n^*)) \neq 0 \quad \Rightarrow \quad \frac{d_b(V(S_n^*), V(S_m^*))e^{d_b(V(S_m^*),V(S_n^*))}}{d_b(S_m^*, S_n^*)e^{d_b(S_m^*,S_n^*)}} \leq \alpha^2$$

We discuss two cases.

Case i: For $1 = m < n$, we have

$$d_b(V(S_m^*) - V(S_n^*)) = (S_{n-1}^* - (S_1^*))^2 = (2 \times 3 + 3 \times 4 + \cdots + (n-1)n)^2$$

and

$$d(S_n^*, S_1^*) = (S_n^* - S_1^*)^2 = (2 \times 3 + 3 \times 4 + \cdots + (n)(n+1))^2$$

Thus

$$\frac{d_b(V(S_m^*), V(S_n^*))e^{d_b(V(S_m^*),V(S_n^*))}}{d_b(S_m^*, S_n^*)e^{d_b(S_m^*,S_n^*)}} = \frac{e^{(4\times 3+6\times 4+\cdots+2n(n-1)+n(n+1))(-n(n+1))}}{(2n)^2(2n-1)^2} \leq e^{-1}$$

Case ii: For $n > m > 1$, we have

$$d_b(V(S_m^*) - V(S_n^*)) = ((2m-1)2m + (2m+1)(2m+1) + \cdots + (2n-3)(2n-2))^2$$

and

$$d_b(S_n^*, S_1) = ((2m+1)(2m+2) + (2m+3)(2m+4) + \cdots + (2n-1)(2n))^2$$

Since $m > n > 1$, we have

$$\begin{aligned}
&\frac{d_b(V(S_m^*), V(S_n^*))e^{d_b(V(S_m^*),V(S_n^*))}}{d_b(S_m^*, S_n^*)e^{d_b(S_m^*,S_n^*)}} \\
=&\frac{(2m-1)^2(2m)^2 e^{((2m-1)2m+2(2m+1)(2m+2)+\cdots+2(2n-2)(2n-1)+2n(2n-1))((2m-1)2m-(2n-1)2n)}}{(2n)^2(2n-1)^2} \\
\leq& e^{-1}
\end{aligned}$$

It fulfill all conditions of the Theorem 5.1.9, this implies that S_1^* is only the fixed point of V.

5.2 Conclusion

We have introduced JS-contraction, modified JS-contraction and generalized modified JS-contraction in b-metric spaces and established and proved fixed point and commom fixed point results for all these contraction in the setting of complete b-metric space. We have provided examples which support our result. We have extended the results of Jleli and Samet[22], Hussain et al.[20] and Ahmad et al.[3] in the setup of complete b-metric space. The results proved in this thesis may helpful for solving fixed point problem in b-metric space.

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