CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Non Symmetric Collinear Central Configurations for Five Bodies

by

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A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

2021

Dedicated to my beloved parents

CERTIFICATE OF APPROVAL

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Acknowledgement

First of all, I would like to pay my cordial gratitude to the almighty **Allah** who created us as a human being with the great boon of intellect. I would like to pay my humble gratitude to the Allah almighty, for blessing us with the holy prophet **Hazrat Mohammad (Sallallahu Alaihy Wa'alihi wassalam)** for whom the whole universe is being created. He (Sallallahu Alaihy Wa'alaihi wassalam) removed the ignorance from the society and brought us out of darkness. Thanks again to that Monorealistic power for granting me with a strength and courage whereby I dedicatly completed my MPhil thesis with positive and significant result.

I owe honour, reverence and indebtedness to my acomplished supervisor and mentor **Dr. Abdul Rehman Kashif** whose affectionate guidence, authentic supervision, keen interest and ingenuity was a source of inspiration for commencement, advancement and completion of the present study. I would like to acknowledge CUST to provide me such a favourable environment to conduct this research.

Thanks to my seniors especially Irtiza ul Hassan, Muhammad Sohail Abid and Ateeq ur Rehman for helping me. Thanks to my beloved parents, brothers, sister, wife and kind persons. Thanks to my father Muhammad Yousaf, my great mother, brothers Abdul Wahab and Ameer Hamza and my dear sister for their ever encouragement and overall support during my entire university life. May almighty Allah show his blessing and prosperity on all those who assisted me during the complition of this thesis.

(Waqas Muawiya)

Abstract

We review the inverse problem of the central configuration of the general colinear 4 and 5 body problems. A central configuration for n-body problems is established if each particle's position vector is a standard scalar multiple of its acceleration with respect to the center of mass. We consider a collinear 5-body problem and define regions in phase space where positive masses can be chosen to make the configuration central. We derive, in the symmetric case, a critical value for the central mass above which there are no central configurations. We also show that there is no such limit on the value of the central mass in general.

Contents

List of Figures

Abbreviations

2BPs	Two-Body Ptoblems
3BPs	Three-Body Problems
4BPs	Four-Body Problems
5BPs	Five-Body Problems
C5BP	Collinear Five Body Problem
C.O.M	Centre of Mass
CCs	Central Configurations
D	Denominator
Ms	Mass of the Sun
N	Numerator
R	Region
SI	System International

Symbols

Symbol	Name	Unit
G	Universal gravitational constant	$m^3kg^{-1}s^{-2}$
F	Gravitational force	Newton
r	Distance	Meter
P	Linear momentum	$kgms^{-1}$
L	Angular momentum	kgm^2s^{-1}
m_i	Point masses	kg
Δ_{ijk}	Area of the triangle	unit square
μ_i	Ratio of the masses	
$\mathbb{R}$	Real number	
$\ni$	Such that	
$\forall$	For all	
$\in$	Belongs to	

Chapter 1

Introduction

In classical mechanics, the 2-body problem (2BP) predicts the movement of two large objects considered particles. The 2BP is most common in the gravitational field that occurs in astronomy to find orbits around objects such as satellites, planets and stars. Newton has solved 2BP with its basic law of gravity. Newtonian mechanics are a mathematical model that aims to answer the movements of various objects in the universe. The basic idea of this model was first expressed by Sir Isaac Newton in a book entitled "Philosophiae Naturalis Principia Mathematica"[1]. This work was published in 1687. The problem does not have a significant solution if $n \geq 3$. Although we have a restricted 3-body problems (R3BP) [2], it may provide us with a particular solution. The 3BP is a problem of taking first positions and the speeds of three-point piles and resolving their next proposal according to Newton's laws of motion Newton's law of gravity. 3BP is a special case of NPB. The 3BP is one of the oldest problems in classical dynamics that continues to throw surprises. It has challenged scientists from Newton's time to date [3]. It emerged in an effort to understand the effect of the Suns on the motion of the Moon around the Earth. N-body problem is also known as multiple body problem [4]. Many physical problems were first handled by Newton. In its own way where an object entails a series of points: "it may be stated as given at any time the position and velocities of three or more massive particles moving under their mutual gravitational forces, the mass also being known, calculated their positions and velocities at any other time". The NBP predicts every movement of the celestial system that attracts its own energy. The problem statement says, "What could be the path, if we were not given the celestial objects that meet each other under gravity". Mathematicians and astronomers continued to work on the NBP for the past four hundred years. First, Kepler in his planetary laws described the elliptical paths of the planets around the sun. The most important works in the history of science where Newton discovered and developed Kepler's law [5]. Newton turned his attention to more complex plans, following Kepler's rules. However, he was unable to achieve any of the 3BP outbreaks throughout his life after a major struggle. After a few refinements, his work turned to a lunar eclipse, which was Newton's goal. By the 19th century, many famous astronomers and mathematicians were working on NBP [6].

1.1 Central Configuration

One of the most common and fundamental themes in the analysis of few-body problem is centre configuration (CCs). Therefore, over the years [7–9], less physical problems in general and more information in particular CC have become available. Studies in N-body (NBP) CC are limited (up to $n \geq 4$), because a large number of problems involve complex bodies. The literature available for $n \geq 4$ focus primarily on a limited number of issues; See example [10–12]. This provides an opportunity to examine the central suspension of the five-body problem. For this reason, in this thesis, we will use the method described [13] to investigate the suspensions between collinear and 5 body problems.

Multiple methods and strategies for restricting the body issue were used to examine. In a particular case of the (5BPs) Roberts addressed for example the relative equilibria in [14], which contains 4 bodies, i.e. $(m_1, m_2, m_3, m_4) = (1, 1, 1, 1)$ at the rhombus vertices and a central body i.e. m_5 at $-1/4$, with the same mass opposite vertically. Roberts demonstrated the presence of a degenerate family of relative equilibriums with one parameter, with 4 equal masses put at a rhombus vertex and the rest of the body at the center. In [15] the main configurations of the (1 + 4)-body problem were addressed by Albouy and Llibre. They held 4 equal masses on a sphere, whose center is the big mass. They discovered four symmetric central configurations and determined that each of them has at least one plane of symmetry [16].

More recently Jensen and Hampton in [17], have shown that the number of central spatial configurations is finite in the 5-body problem, with the exception of some special cases. In [13], Ouyang and Xie [18] considered the inverse problem of CCs of collinear 4-bodies and defined possible conditions for choosing positive masses while retaining a CCs. Dependant on position x and centre of mass u, the authors define a 4-mass expression, which gives central configurations for the 4-body collinear problem. We use a similar approach to model our problem and suggest a method for determining central configurations for a collinear 5 body problem. The model being proposed has the fifth mass set at the center of mass (C.O.M).

1.2 Thesis Contribution

We have reviewed [16] a collinear five body problem (C5BP) involving two pairs of masses in two symmetrical configurations. The masses are respectively m_0, m_1, m_2, m_3 and m_4. The mass m_0 at the centre of mass of the system is stationary. For the rest of the 4 bodies we choose the coordinates. We investigate a C5BP and can discover places within the phase space in which positive masses might be chosen to make the central configuration. In this symmetrical scenario, a critical value is derived for the central mass over which there is no central configuration. We further demonstrate that the value of the central mass is not generally restricted.

1.3 Dissertation Outlines

We have classified thesis into 5 chapters.

In **Chapter-1** the objectives are discussed in the introduction. First of all we have discussed the basics concept and history of the (2BP), (3BP) and (NBP).

In **Chapter-2** contains several fundamental concepts related to celestial mechanics, the laws of Newton of motion, and the Kepler laws of planetary motion . The two-body is briefly discussed in the last portion of this chapter.

In **Chapter-3** we reviewed a research paper [16].

In **Chapter-4** we investigate a (C5BP) and can discover places within the phase space in which positive masses might be chosen to make the central configuration. In this symmetrical scenario, a critical value is derived for the central mass over which there is no central configuration. We further demonstrate that the value of the central mass is not generally restricted.

In **Chapter-5** provides the concluding remarks of the thesis.

Chapter 2

Preliminaries

This chapter contains some important definitions, concepts, governing laws which are essential to understand the work presented in next chapters.

2.1 Basics Definitions

Motion

"Motion is the action used to change the location or position of an object with respect to the surroundings over time".[19]

Mechanics

"Mechanics is a branch of physics concerned with motion or change in position of physical objects."[19]

Scalar

"Various quantities of physics, such as length, mass and time, requires for their specification a single real number (apart from units of measurement which are decided upon in advance). Such quantities are called **Scalars** and the real number is called the magnitude of the quantity".[19]

Vector

"Other quantities of physics, such as displacement, velocity, momentum, force etc require for their specification a direction as well as magnitude. Such quantities are called **Vectors**".[19]

Field

"A field is a physical quantity associated with every point of spacetime. The physical quantity may be either in vector form, scalar form or tensor form".[19]

Scalar Field

"If at every point in a region, a scalar function has a defined value, the region is called a scalar field. i.e.,

$$f : \mathbb{R}^3 \to \mathbb{R}.\text{"}\ [20]$$

Vector Field

"If at every point in a region, a vector function has a defined value, the region is called a vector field,

$$f : \mathbb{R}^3 \to \mathbb{R}^3.\text{"}\ [20]$$

Conservative Vector Field

"A vector field $\mathbf{V}$ is conservative if and only if there exists a contentiously differentiable scalar field f such that $\mathbf{V}$ = -∇f or equivalently,

$$\nabla \times \mathbf{V} = curl\mathbf{V} = \mathbf{0}.\text{"}[20]$$

Uniform Force Field

"A force field which has constant magnitude and direction is called a uniform or constant force field. If the direction of the field is taken as negative z direction and magnitude is constant $F_0 > 0$, then the force field is given by:

$$\mathbf{F} = -F_0\hat{\mathbf{k}}.\text{"}[20]$$

Central Force

"Suppose that a force acting on a particle of mass m such that

(a) it is always directed from m towards or away from a fixed point O,

(b) its magnitude depends only on the distance r from O.

Then we call the force a central force or central force field with the O as the center of the force field. Mathematically, $\mathbf{F}$ is central force if and only if

$$\mathbf{F} = f(r)\mathbf{r}_1 = f(r)\frac{\mathbf{r}}{r},$$

where $\mathbf{r}_1 = \frac{\mathbf{r}}{r}$ is a unit vector in the direction of $\mathbf{r}$. The central force is one of attraction towards O or repulsion from O according as $f(r) < 0$ or $f(r) > 0$ respectively." [20]

Degree of Freedom

"The number of coordinates required to specify the position of a system of one or more particles is called number of degree of freedom of the system." [20]

Center of Mass

"Let $r_1, r_2, ..., r_n$ be the position vector of a system of n particles of masses $m_1, m_2, ..., m_n$ respectively. The center of mass or centroid of the system of particles is defined as that point having position vector,

$$\hat{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + ... + m_n\mathbf{r}_n}{m_1 + m_2 + ... + m_n} = \frac{1}{\mathbf{M}}\sum_{\nu=1}^{n} m_\nu \mathbf{r}_\nu,$$

where

$$\mathbf{M} = \sum_{\nu=1}^{n} m_\nu,$$

is the total mass of the system.” [20]

Center of Gravity

“If a system of particles is in a uniform gravitational field, the center of mass is sometimes called the center of gravity.” [19]

Torque

“If a particle with a position vector r moves in a force field $\mathbf{F}$, we define τ as torque or moment of the force as

$$\tau = \mathbf{r} \times \mathbf{F}.$$

The magnitude of τ is

$$\tau = rF\sin\theta.$$

The magnitude of torque is a measure of the turning effect produced on the particle by the force.” [19]

Momentum

“The linear momentum $\mathbf{P}$ of an object with mass m and velocity $\mathbf{v}$ is defined as:

$$\mathbf{P} = m\mathbf{v}.$$

Under certain circumstances the linear momentum of a system is conserved. The linear momentum of a particle is related to the net force acting on that object:

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}.$$

The rate of change of linear momentum of a particle is equal to the net force acting on the object, and is pointed in the direction of the force. If the net force acting on an object is zero, its linear momentum is constant (conservation of linear momentum). The total linear momentum **P** of a system of particles is defined as the vector sum of the individual linear momentum:

$$\mathbf{P} = \sum_{i=1}^{n} \mathbf{P}_i.\text{"}[20]$$

Point-like Particle

"A point-like particle is an idealization of particles mostly used in different fields of physics. Its defining features is the lacks of spatial extension:being zero-dimensional, it does not take up space. A point-like particle is an appropriate representation of an object whose structure, size and shape is irrelevant in a given context. e.g., from far away, a finite-size mass (object) will look like a point-like particle".[19]

Angular Momentum

"Angular momentum **L** of a particle of mass m and linear momentum **p** is a vector quantity defined as:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

Where **r** is a position vector of a particle relative to an origin O, that is in an inertial frame. Where magnitude of **L** is given by

$$\mathbf{L}\text{=m}\mathbf{vr}\ \sin\phi.\text{" [21]}$$

Angular Velocity

Angular velocity $\boldsymbol{\omega}$ is a vector quantity and is described as the rate of change of angular displacement which specifies the angular speed or rotational speed of an object and the axis about which the object is rotating. The amount of change of angular displacement of the particle at a given period of time is called angular velocity.

Principle of Conservation of Momentum

"If the net external force acting on a particle is zero, the angular momentum will remain unchanged. This is called the principle of conservation of momentum." [20]

Galilean Transformation

"In physics, a Galilean transformation is used to transform between the coordinates of two reference frames which differ only by constant relative motion within the constructs of Newtonian physics. Without the translations in space and time the group is the homogeneous Galilean group. Galilean transformations, also called Newtonian transformations, set of equations in classical physics that relate the space and time coordinates of two systems moving at a constant velocity relative to each other."[19]

Celestial Mechanics

"Celestial mechanics is the branch of astronomy that deals with the motions of objects in outer space. Historically, celestial mechanics applies principles of physics (classical mechanics) to astronomical objects, such as stars and planets. Actually celestial mechanics is the science devoted to the study of the motion of the celestial bodies on the basis of the laws of gravitation. It was founded by Newton and it is the oldest concept of Physical Astronomy".[19]

Lagrange Points

"Let us search for possible equilibrium points of the mass m_3 in the rotating reference frame. Such points are termed Lagrange points. Hence, in the rotating frame, the mass m_3 would remain at rest if placed at one of the Lagrange points. It is, thus, clear that these points are fixed in the rotating frame. The Lagrange points satisfy $\dot{\mathbf{r}} = \ddot{\mathbf{r}} = \mathbf{0}$ in the rotating frame." [22]

Equilibrium Solution

The **Equilibrium solution** can lead us through the behavior of the equation that describes the problem without really solving it. These solutions are only possible if we satisfy the necessary condition of all rates being equal to zero. If we have two variables then

$$x' = y' = x'' = y'' = ... = x^{(n)} = y^{(n)} = 0.$$

These solutions may be stable or unstable. The stable solutions in celestial mechanics assist us in locating parking spaces where a satellite or other object may be put and remain there indefinitely. These type of places are also found along the Jupiter's orbital path where bodies called trojan are present. These equilibrium points with respect to Celestial Mechanics are also called Lagrange points named after a French mathematician and astronomer Joseph-Louis Lagrange. He was first to find these equilibrium points for the Sun-Earth system. He found that three of these five points were collinear.

Procedure for Stability Analysis and Equilibrium Points:

We need to follow the following steps to check the stability of equilibrium points.

1) Determine the equilibrium points, $\mathbf{u}^*$, solving $\phi(\mathbf{u}^*) = \mathbf{0}$.

2) Construct the Jacobian matrix, $J(\mathbf{u}^*) = \frac{\partial \phi}{\partial \mathbf{u}^*}$.

3) Compute eigenvalues of $\phi(\mathbf{u}^*)$: $det|\phi(\mathbf{u}^*) - \beta\ I| = 0$.

4) Stability or instability of $\mathbf{u}^*$ based on the real parts of eigenvalues.

5) Point is stable, if all eigenvalues have real parts negative.

6) Unstable, If at least one eigenvalue has a positive real part.

7) Otherwise, there is no conclusion, (i.e, require an investigation of higher order terms).

Lorentz Transformation

"Lorentz transformation is the relationship between two different coordinate frames that move at a constant velocity and are relative to each other. The name of the transformation comes from a Dutch physicist Hendrik Lorentz. There are two frames of reference, which are"

Inertial Frame of Reference

"A frame of reference that remains at rest or moves with constant velocity with respect to other frames of reference is called inertial frame of reference. Actually, an unaccelerated frame of reference is an inertial frame of reference. In this frame of reference a body does not acted upon by external forces. Newton's laws of motion are valid in all inertial frames of reference. All inertial frames of reference are equivalent. A frame which is not inertial is called non inertial frame".[19]

Holonomic and Non Holonomic Constraints "In classical mechanics, a constraint on a system is a parameter that the system must obey. The limitation on the motion are often called constraints. If the constraints condition can be expressed as an equation,

$$\phi(r_1, r_2, ..., r_n, t) = 0,$$

connecting the position vector of the particles and the time, then the constraints are called holonomic, otherwise non-holonomic".[19]

2.2 Kepler's Laws of Planetary Motion

"Kepler's three laws of planetary motion can be described as follows:[23]

1. The orbit of planet around the Sun is an ellipse with the Sun at one of its foci.

2. Each planet revolves so that the line joining it to the Sun sweeps out equal areas in equal intervals of time.

3. The squares of the sidereal periods (of revolution) of the planets are directly proportional to the cubes of their mean distances from the Sun. Mathematically, Kepler's third law can be written as:

$$T = \left(\frac{4\pi^2}{GM_s}\right) r^3,$$

where T is the time period, r is the semi major axis, M_s is the mass of sun and G is the universal gravitational constant."

2.3 Newton's Laws of Motion

"The following three laws of motion given by Newton are considered the axioms of mechanics:[24]

1. First law of motion

Every body continues in its state of rest, or of uniform motion in a straight line, unless compelled by an applied force to change that state.

2. Second Law of Motion

The rate of change of momentum is proportional to the applied force, and takes place in the direction in which force acts.

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}.$$

If m is independent of time the above expression becomes becomes

$$\mathbf{F} = m\frac{d}{dt}(\mathbf{v}) = m\mathbf{a},$$

where a is the acceleration of the particle

2. Third Law of Motion

To every action corresponds an equal and opposite reaction."

2.3.1 Newton's Universal Law of Gravitation

"The Newtonian law of gravitational attraction is: Every two particles in the universe attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of distance between them. Mathematically can be written as:

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}\mathbf{r},$$

where G is universal gravitational constant. Its numerical value in SI units is $6.67408\times10^{-11}m^3kg^{-1}s^{-2}$." [24]

2.4 Two Body Problem

"An isolated dynamical system consisting of two freely moving point objects exerting forces on one another is conventionally termed a two-body problem. Suppose that the first object is of mass m_1 and is located at position vector $\mathbf{r}_1$. Likewise, the second object is of mass m_2 and is located at position vector $\mathbf{r}_2$. Let the first object exert a force $\mathbf{f}_{21}$ on the second. The equations of motion of our two objects are thus

$$m_1\frac{d^2\mathbf{r}_1}{dt^2} = -\mathbf{f}$$

$$m_2\frac{d^2\mathbf{r}_2}{dt^2} = \mathbf{f}.$$" [22]

2.4.1 The Solution to the Two Body Problem [25]

The governing law for the two-body is Newtons universal gravitational law:

$$\mathbf{F} = G\frac{\mathcal{M}_1\mathcal{M}_2}{q^3}\mathbf{q}, \tag{2.1}$$

for two masses $\mathcal{M}_1$ and $\mathcal{M}_1$ separated by a distance of q, and the universal gravitational constant is G.

If the initial positions and velocities are known, the aim is to find the path of the particles at any time t. The force of attraction $\mathbf{F}_{12}$ in Figure 2.1, is directed towards $\mathcal{M}_1$ along $\mathbf{q}$, while the force $\mathbf{F}_{21}$ on $\mathcal{M}_2$ is directed in the opposite direction. According to Newton's third law of motion,

$$\mathbf{F}_1 = -\mathbf{F}_2. \tag{2.2}$$

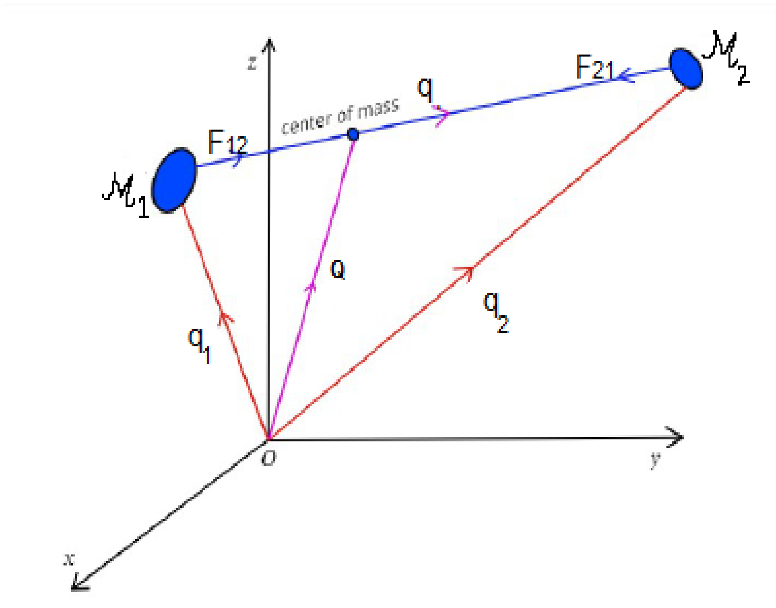

FIGURE 2.1: Center of mass of two body system;

From Figure 2.1,

$$\mathbf{F}_{12} = G\frac{\mathcal{M}_1\mathcal{M}_2}{q^3}\mathbf{q}, \tag{2.3}$$

Now let vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ be directed from some fixed reference point O to the particles of mass m_1 and mass m_2 respectively. Using the Newton's 2nd law of motion and equations (2.2), (2.3), the equations of motion of the particles under their mutual gravitational attractions are then given by the two equations

$$\mathcal{M}_1\mathbf{q}_1'' = \mathcal{M}_1\frac{d^2\mathbf{q}_1}{dt^2} = \mathbf{F} = G\frac{\mathcal{M}_1\mathcal{M}_2}{q^3}\mathbf{q}, \tag{2.4}$$

$$\mathcal{M}_2\mathbf{q}_2'' = \mathcal{M}_2\frac{d^2\mathbf{q}_2}{dt^2} = \mathbf{F} = -G\frac{\mathcal{M}_1\mathcal{M}_2}{q^3}\mathbf{q}, \tag{2.5}$$

Adding equations (2.4) and (2.5) gives

$$\mathcal{M}_1\mathbf{q}_1'' + \mathcal{M}_2\mathbf{q}_2'' = \mathbf{0}. \tag{2.6}$$

The above equations are integrated to give

$$\mathcal{M}_1\mathbf{q}_1' + \mathcal{M}_2\mathbf{q}_2' = \mathbf{c}_1, \tag{2.7}$$

$$\mathcal{M}_1\mathbf{q}_1 + \mathcal{M}_2\mathbf{q}_2 = \mathbf{c}_1 t + \mathbf{c}_2, \tag{2.8}$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ are constant vectors. But if $\mathbf{D}$ is the position vector of G (the centre of mass of the two masses m_1 and m_2), $\mathbf{D}$ is defined as

$$\begin{aligned}(\mathcal{M}_1 + \mathcal{M}_2)\mathbf{Q} &= \mathcal{M}_1\mathbf{q}_1 + \mathcal{M}_2\mathbf{q}_2,\\ \mathcal{M}_t\mathbf{Q} &= \mathcal{M}_1\mathbf{q}_1 + \mathcal{M}_2\mathbf{q}_2,\end{aligned} \tag{2.9}$$

where $\mathcal{M}_t = \mathcal{M}_1 + \mathcal{M}_2$. Differentiate the equation (2.9) and compare it with equation (2.7)

$$\mathcal{M}_t\mathbf{Q}' = \mathbf{c}_1 \qquad \Rightarrow \qquad \mathbf{Q}' = \frac{\mathbf{c}_1}{\mathcal{M}_t} = constant.$$

These relations show that the centre of mass of the system moves with constant velocity. Equations (2.4) and (2.5) may be written as

$$\mathbf{q}_1'' = G\frac{\mathcal{M}_2}{q^3}\mathbf{q}, \tag{2.10}$$

,

$$\mathbf{q}_2'' = -G\frac{\mathcal{M}_1}{q^3}\mathbf{q}, \tag{2.11}$$

,

Subtracting equation (2.10) from equation (2.11) gives

$$\mathbf{q}_1'' - \mathbf{q}_2'' = G\frac{\mathcal{M}_2}{q^3}\mathbf{q} + G\frac{\mathcal{M}_1}{q^3}\mathbf{q}, \tag{2.12}$$

$$\mathbf{q}_1'' - \mathbf{q}_2'' = G(\mathcal{M}_1 + \mathcal{M}_2)\frac{\mathbf{q}}{q^3},$$

$$\Rightarrow -\mathbf{q}'' = \alpha\frac{\mathbf{q}}{q^3},$$

$$\Rightarrow \mathbf{q}'' + \alpha\frac{\mathbf{q}}{q^3} = 0, \tag{2.13}$$

where $\alpha = G(m_1 + m_2)$ is described as a reduction in mass and $\mathbf{q}_1 - \mathbf{q}_2 = -\mathbf{q}$, as seen in Figure 2.1.

Taking the vector product of $\mathbf{q}$ with equation (2.13) we obtain

$$\mathbf{q} \times \alpha\mathbf{q}'' + \frac{\alpha^2}{q^3}\mathbf{q} \times \mathbf{q} = \mathbf{0},$$

$$\Rightarrow \mathbf{q} \times \mathbf{q}'' = \mathbf{0}, \tag{2.14}$$

Integrating, we have

$$\Rightarrow \mathbf{q} \times \mathbf{q}' = \mathbf{T}, \tag{2.15}$$

where $\mathbf{T}$ is a constant vector. The equation (2.14) should be written as

$$\mathbf{q} \times \alpha \mathbf{q}'' = \mathbf{0}, \tag{2.16}$$

$$\mathbf{q} \times \mathbf{F} = \mathbf{0}, \tag{2.17}$$

where $\mathbf{F} = \alpha \mathbf{q}'' = \alpha \mathbf{a}$.

The description of angular momentum and torque is taken from Chapter 2

$$\tau = \frac{d\mathbf{T}}{dt} = \mathbf{q} \times \mathbf{F}, \tag{2.18}$$

when equations (2.16) and (2.17) are compared, we get

$$\tau = \frac{d\mathbf{T}}{dt} = \mathbf{q} \times \mathbf{F} = \mathbf{0}, \tag{2.19}$$

$$\frac{d\mathbf{T}}{dt} = \mathbf{0}, \tag{2.20}$$

$$\mathbf{T} = constant.$$

This means that angular momentum is constant.

Radial and Transverse Components of Velocity and Acceleration: If polar coordinates q and θ are taken in this plane as in Figure 2.2, the velocity components along and perpendicular to the radius vector joining $\mathcal{M}_1$ to $\mathcal{M}_2$ are q' and $q\theta'$, then

$$\mathbf{q}' = \frac{d\mathbf{q}}{dt} = \mathbf{q}'\mathbf{I} + q\theta'\mathbf{J}, \tag{2.21}$$

where $\mathbf{I}$ and $\mathbf{J}$ are unit vectors along and perpendicular to the radius vector. Thus, by means of equations (2.15) and (2.18)

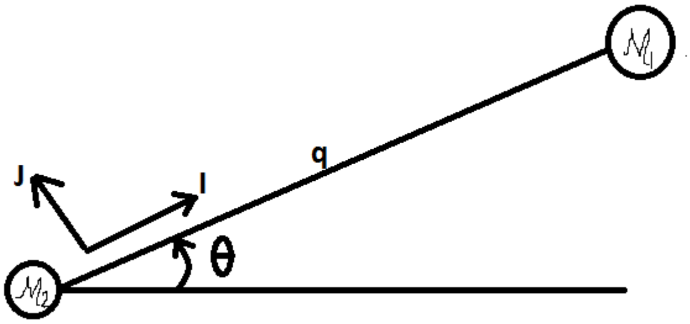

FIGURE 2.2: radial and transverse components of velocity and acceleration

$$\mathbf{q} \times (\mathbf{q}'\mathbf{I} + s\theta'\mathbf{J}) = q^2\theta'\mathbf{K} = T\mathbf{K}, \tag{2.22}$$

where $\mathbf{K}$ is the unit vector perpendicular to the orbit's plane, which can be written as

$$q^2\theta' = T. \tag{2.23}$$

where the constant H is seen to be twice the rate of description of area by the radius vector. This is the mathematical form of Keplers second law. If the scalar product of $\mathbf{d}'$ with equation (2.13) is now taken, we obtain

$$\mathbf{q}'.\frac{d^2\mathbf{q}}{dt^2} + \alpha\frac{\mathbf{q}'.\mathbf{q}}{q^3} = \mathbf{0}.$$

After integrating, we have get

$$\frac{1}{2}\mathbf{q}'.\mathbf{q}' - \frac{\alpha}{q} = C, \tag{2.24}$$

or

$$\frac{1}{2}v^2 - \frac{\alpha}{q} = C, \tag{2.25}$$

where C is a constant of integration. This is a sort of energy system preservation. The C quantity does not include absolute energy, $\frac{1}{2}\alpha^2$ is associated with KE, and $\frac{-\alpha}{q}$ is associated with PE of the system's, i.e., the system's total energy is constant. Recall that from celestial mechanics, components of acceleration vector along and perpendicular to the radius vector (see Figure 2.2):

$$\mathbf{a} = (q'' - q\theta^{'})\mathbf{I} + \frac{1}{q}\frac{d}{dt}(q^2\theta'')\mathbf{J}.$$

Using above equation in (2.13), we get

$$(q'' - q\theta') = -\frac{\alpha}{q^2}, \tag{2.26}$$

$$\frac{1}{q}\frac{d}{dt}(q^2\theta') = 0. \tag{2.27}$$

Integrating equation (2.24) gives the angular momentum integral

$$q^2\theta' = T, \tag{2.28}$$

making the usual substitution of

$$w = \frac{1}{q}, \tag{2.29}$$

the absence of time between equations (2.23) and (2.24), therefore, means that:

$$\frac{q^2 w}{q\theta^2} + w = \frac{\alpha}{T^2}. \tag{2.30}$$

The general solution of above equation is:

$$w = \frac{\alpha}{T^2} + B\cos(\theta - \theta_0), \tag{2.31}$$

where B and θ_0 are two constants of integration. Substitute $w = \frac{1}{q}$ in above equation:

$$\frac{1}{q} = \frac{\alpha}{T^2} + B\cos(\theta - \theta_0), \tag{2.32}$$

$$\Rightarrow q = \frac{\frac{T^2}{\alpha}}{1 + \frac{T^2 B}{\alpha}\cos(\theta - \theta_0)},$$

is the conic equation's polar form, can be expressed as

$$q = \frac{p}{1 + \epsilon\cos(\theta - \theta_0)},$$

where

$$p = \frac{T^2}{\alpha},$$

$$\epsilon = \frac{h^2 B}{\alpha}.$$

Here ϵ is the eccentricity of the orbit that classifies the trajectory of one celestial body around to another celestial body. Thus:

(i) The motion of an orbit is elliptical, if $0 < \epsilon < 1$

(ii) The motion of an orbit is parabolic, if $\epsilon = 1$

(iii) The motion of an orbit is hyperbolic, if $\epsilon > 1$.

Therefore, that is the conic section to the solution of the 2BP, including the Kepler's first law of motion as a special case.

Chapter 3

Non Symmetric Collinear Central Configurations for Five Bodies

In this chapter, we will discuss the problem of the central configuration for the collinear 5 body problem There are five masses m_0, m_1, m_2, m_3, and m_4 at the position of q_0, q_1, q_2, q_3 and q_4 respectively. Then we will discuss the solution of the general equations for 5 body collinear central configurations and positivity of their masses. We will also discuss the symmetric case, the position of the masses on the both sides of the axis will be same and in the special case the value of the mass m_0 will be zero.

3.1 General Equations for 5 Body Collinear Central Configurations

The classical equation of motion for the n-body problem is as follows.

$$m_i \frac{d^2\mathbf{q}_i}{dt^2} = \sum_{j\neq i} \frac{m_i m_j(\mathbf{q}_i - \mathbf{q}_j)}{\mid \mathbf{q}_i - \mathbf{q}_j \mid^3} \qquad i = 1, 2, 3...., n, \tag{3.1}$$

where the units are chosen so that the gravitational constant is equal to one and $q_i \in R^d$ $(1 \leq d \leq 3)$, $i = 1, 2, 3, ..., n$ represents R^d of n masses of m_i in the Euclidean space. An arrangement at the centre $\mathbf{q}$=$(q_1, q_2, q_3, ..., q_n) \in R^{nd}$ is a configuration of n bodies in which each body's acceleration vector is proportional to its position vector, and the proportionality constant is the same for all n bodies. Therefore, a central configuration satisfy the equation,

$$\sum_{j=1, j\neq i}^{n} \frac{m_j(\mathbf{q}_j - \mathbf{q}_i)}{\mid \mathbf{q}_j - \mathbf{q}_i \mid^3} = -\lambda(\mathbf{q}_i - \mathbf{c}) \qquad i = 1, 2, 3, ..., n, \tag{3.2}$$

where λ is the same for all particles and

$$\mathbf{c} = \frac{\sum_{i=1}^{n} m_i \mathbf{q}_i}{\sum_{i=1}^{n} m_i} \qquad i = 1, 2, 3, ..., n. \tag{3.3}$$

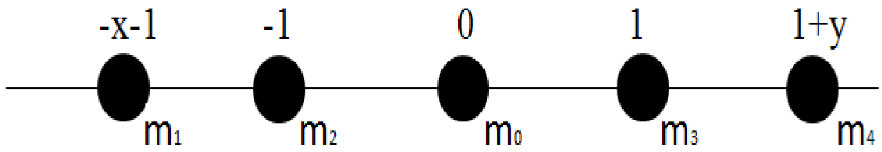

FIGURE 3.1: geometry of the problem

Consider 5 masses of collinear bodies, m_0, m_1, m_2, m_3, and m_4. The mass m_0 at the centre of the mass of the system is stationary. For the rest of the 4 bodies we choose the coordinates as follows (see figure 3.1):

$$q_0 = 0, q_1 = -x - 1, q_2 = -1, q_3 = 1, \text{ and } q_4 = 1 + y \text{ where } x, y > 0. \tag{3.4}$$

For $i = 1$ and $n = 4$, equation (3.2) becomes,

$$\sum_{j=0}^{4} \frac{m_j(\mathbf{q}_j - \mathbf{q}_1)}{\mid \mathbf{q}_j - \mathbf{q}_1 \mid^3} = -\lambda(\mathbf{q}_1 - \mathbf{c}),$$

$$\frac{m_0(q_0 - q_1)}{\mid q_0 - q_1 \mid^3} + \frac{m_2(q_2 - q_1)}{\mid q_2 - q_1 \mid^3} + \frac{m_3(q_3 - q_1)}{\mid q_3 - q_1 \mid^3} + \frac{m_4(q_4 - q_1)}{\mid q_4 - q_1 \mid^3} = -\lambda(q_1 - c),$$

using equation (3.4),

$$\frac{m_0(0-(-x-1))}{\mid 0-(-x-1)\mid^3}+\frac{m_2(-1-(-x-1))}{\mid -1-(-x-1)\mid^3}+\frac{m_3(1-(-x-1))}{\mid 1-(-x-1)\mid^3} +\frac{m_4(1+y-(-x-1))}{\mid 1+y-(-x-1)\mid^3}=-\lambda(-x-c-1),$$

$$\frac{m_0(x+1)}{\mid x+1\mid^3}+\frac{m_2(x)}{\mid x\mid^3}+\frac{m_3(x+2)}{\mid x+2\mid^3}+\frac{m_4(x+y+2)}{\mid x+y+2\mid^3}=\lambda(1+x+c),$$

$$\frac{m_0}{(x+1)^2}+\frac{m_2}{x^2}+\frac{m_3}{(x+2)^2}+\frac{m_4}{(x+y+2)^2}=\lambda(1+x+c). \tag{3.5}$$

For $i=2$ and $n=4$, equation (3.2) becomes,

$$\sum_{j=0}^{4}\frac{m_j(\mathbf{q}_j-\mathbf{q}_2)}{\mid \mathbf{q}_j-\mathbf{q}_2\mid^3}=-\lambda(\mathbf{q}_2-\mathbf{c}),$$

$$\frac{m_0(q_0-q_2)}{\mid q_0-q_2\mid^3}+\frac{m_1(q_1-q_2)}{\mid q_1-q_2\mid^3}+\frac{m_3(q_3-q_2)}{\mid q_3-q_2\mid^3}+\frac{m_4(q_4-q_2)}{\mid q_4-q_2\mid^3}=-\lambda(q_2-c),$$

using equation (3.4),

$$\frac{m_0(0-(-1))}{\mid 0-(-1)\mid^3}+\frac{m_1(-1-x-(-1))}{\mid -1-x-(-1)\mid^3}+\frac{m_3(1-(-1))}{\mid 1-(-1)\mid^3} +\frac{m_4(1+y-(-1))}{\mid 1+y-(-1)\mid^3}=-\lambda(-c-1),$$

$$m_0+\frac{m_1(-x)}{\mid -x\mid^3}+\frac{m_3(2)}{\mid 2\mid^3}+\frac{m_4(2+y)}{\mid 2+y\mid^3}=\lambda(c+1),$$

$$m_0-\frac{m_1}{x^2}+\frac{m_3}{4}+\frac{m_4}{(2+y)^2}=\lambda(c+1). \tag{3.6}$$

For $i=3$ and $n=4$, equation (3.2) becomes,

$$\sum_{j=0}^{4}\frac{m_j(\mathbf{q}_j-\mathbf{q}_3)}{\mid \mathbf{q}_j-\mathbf{q}_3\mid^3}=-\lambda(\mathbf{q}_3-\mathbf{c}),$$

$$\frac{m_0(q_0-q_3)}{\mid q_0-q_3\mid^3}+\frac{m_1(q_1-q_3)}{\mid q_1-q_3\mid^3}+\frac{m_2(q_2-q_3)}{\mid q_2-q_3\mid^3}+\frac{m_4(q_4-q_3)}{\mid q_4-q_3\mid^3}=-\lambda(q_3-c),$$

using equation (3.4),

$$\frac{m_0(0-(1))}{\mid 0-(1)\mid^3}+\frac{m_1(-x-1-(1))}{\mid -x-1-(1)\mid^3}+\frac{m_2(-1-(1))}{\mid -1-(1)\mid^3}$$
$$+\frac{m_4(1+y-(1))}{\mid 1+y-(1)\mid^3}=-\lambda(1-c),$$

$$-\frac{m_0}{\mid -1\mid^3}-\frac{m_1(2+x)}{\mid 2+x\mid^3}-\frac{m_2(2)}{\mid 2\mid^3}+\frac{m_4(y)}{\mid y\mid^3}=-\lambda(1-c),$$

$$m_0+\frac{m_1}{(2+y)^2}+\frac{m_2}{4}-\frac{m_4}{y^2}=\lambda(1-c). \tag{3.7}$$

For $i=4$ and $n=4$, equation (3.2) becomes,

$$\sum_{j=0}^{4}\frac{m_j(\mathbf{q}_j-\mathbf{q}_4)}{\mid \mathbf{q}_j-\mathbf{q}_4\mid^3}=-\lambda(\mathbf{q}_4-\mathbf{c}),$$

$$\frac{m_0(q_0-q_4)}{\mid q_0-q_4\mid^3}+\frac{m_1(q_1-q_4)}{\mid q_1-q_4\mid^3}+\frac{m_2(q_2-q_4)}{\mid q_2-q_4\mid^3}+\frac{m_3(q_3-q_4)}{\mid q_3-q_4\mid^3}=-\lambda(q_4-c),$$

using equation (3.4),

$$\frac{m_0(0-(1+y))}{\mid 0-(1+y)\mid^3}+\frac{m_1(-x-1-(1+y))}{\mid -x-1-(1+y)\mid^3}+\frac{m_2(-1-(1+y))}{\mid -1-(1+y)\mid^3}$$
$$+\frac{m_3(1-(1+y))}{\mid 1-(1+y)\mid^3}=-\lambda(1+y-c),$$

$$\frac{m_0(-1-y)}{\mid -1-y\mid^3}+\frac{m_1(-2-y-x)}{\mid -2-y-x\mid^3}+\frac{m_2(-2-y)}{\mid -2-y\mid^3}+\frac{m_3(-y)}{\mid -y\mid^3}=-\lambda(1+y-c),$$

$$\frac{m_0}{(1+y)^2}+\frac{m_1}{(2+y+x)^2}+\frac{m_2}{(2+y)^2}+\frac{m_3}{y^2}=\lambda(1+y-c). \tag{3.8}$$

Put λ=1 in equations (3.5) - (3.8), we get

$$\frac{m_0}{(x+1)^2} + \frac{m_2}{x^2} + \frac{m_3}{(x+2)^2} + \frac{m_4}{(x+y+2)^2} = (1+x+c), \tag{3.9}$$

$$m_0 + \frac{m_1}{x^2} + \frac{m_3}{4} + \frac{m_4}{(2+y)^2} = (c+1), \tag{3.10}$$

$$m_0 + \frac{m_1}{(2+x)^2} + \frac{m_2}{4} - \frac{m_4}{y^2} = (1-c), \tag{3.11}$$

$$\frac{m_0}{(1+y)^2} + \frac{m_1}{(2+y+x)^2} + \frac{m_2}{(2+y)^2} + \frac{m_3}{y^2} = (1+y-c). \tag{3.12}$$

To solve equation (3.9) - equation (3.12) for m_1, m_2, m_3 and m_4, we have

$$\begin{aligned} m_1 = \; & \frac{-1-x+m_0}{y^2} + \frac{-1+c+m_0}{(2+y)^2} + \frac{1}{4}(1-c+y-\frac{m_0}{(1+y)^2})/ \\ & \frac{1}{x^2y^2} - \frac{4}{(2+x)^2(2+y)^24y^2} + \frac{1}{(2+x+y)^2}, \end{aligned}$$

$$\begin{aligned} = \; & \frac{-1-c}{y^2} + \frac{-1+x}{(2+y)^2} + \frac{1-c+y}{4} + m_0(\frac{1}{y^2} + \frac{1}{(2+y)^2} - \frac{1}{4(1+y)^2})/ \\ & \frac{4(2+y)^2(2+x+y)^2 - 4x^2(2+x+y)^2 + 4x^2y^2(2+y)^2(2+x)^2}{4x^2y^2(2+x)^2(2+y)^2(2+x+y)^2}, \end{aligned}$$

$$\begin{aligned} = \; & \frac{x^2(2+x)^2(2+x+y)^2}{4(2+y)^2(2+x+y)^2 - 4x^2(2+x+y)^2 + 4x^2y^2(2+y)^2(2+x)^2} \\ & \Big[4(-1-c)(y^2+4y+4) + 4y^2(-1+c) + y^2(1-c+y)(y^2+4y+4) \\ & + m_0\left(4y^2 + \frac{4(2+y)^2(1+y)^2 - y^2(2+y)^2}{(1+y)^2}\right)\Big], \end{aligned}$$

$$= \left[\frac{x^2(2+x)^2(2+x+y)^2}{4(2+y)^2(2+x+y)^2-4x^2(2+x+y)^2+4x^2y^2(2+y)^2(2+x)^2}\right]$$
$$.\Big[-16(1+c)-4y^2(1+c)-16y(1+c)+y^4+4y^3-y^4c-4cy^3$$
$$+4y^3+y^5+4y^4+m_0\left(4y^2+\frac{(2+y)^2((2+2y)^2-y^2)}{(1+y)^2}\right)\Big],$$

$$= \left[\frac{x^2(2+x)^2(2+x+y)^2}{4(2+y)^2(2+x+y)^2-4x^2(2+x+y)^2+4x^2y^2(2+y)^2(2+x)^2}\right]$$
$$.\Big[y^5-y^4(-5+c)-y^3(-8+4c)-y^2(4+4c)-16y(1+c)$$
$$-16(1+c)+m_0\left(4y^2+\frac{(2+y)^2(2+2y+y)(2+2y-y)}{(1+y)^2}\right)\Big],$$

we can rewrite m_1 as

$$m_1(m_0,x,y,c)=\frac{N_{m_1}(m_0,x,y,c)}{D_m(x,y)}, \tag{3.13}$$

where

$$N_{m_1}(m_0,x,y,c)= A_1A_2x^2(y^5-y^4(-5+c)-4y^3(-2+c)-4y^2(1+c)$$
$$-16y(1+c)-16(1+c)+m_0\left(4y^2+\frac{(2+y)^3(2+3y)}{(1+y)^2})\right),$$

$$D_m(x,y)= 256+512x+384x^2+128x^3+16x^4+(512+896x+576x^2$$
$$+160x^3+16x^4)y+(384+576x+304x^2+64x^3+4x^4)y^2$$
$$+(128+160x+64x^2+16x^3+4x^4)y^3$$
$$+(16+16x+4x^2+4x^3+x^4)y^4,$$

and

$$A_1 = \ (2+x)^2, A_2 = (2+x+y)^2, A_3 = (2+y)^2.$$

$$\begin{aligned}
m_2 =& \frac{4x^2(2+y)^2}{D_m(x,y)}\Bigg[\bigg((2+x)^4(1+c+x)+2(2+x)^3(1+c+x)y \\
&+(2+x)^2(1+c+x)y^2-(2+x)(4-2c+x)y^3 \\
&+(-5+x-2x)y^4-y^5\bigg)+m_0\bigg(-(2+x)^2(1+x)^2(1+y)^2y^2 \\
&-(2+x)^2(2+x+y)^2(1+y)^2+y^2(2+x+y)^2(1+x)^2\bigg) \\
&/(1+x)^2(1+y)^2\Bigg],
\end{aligned}$$

$$\begin{aligned}
=&\frac{4x^2A_3}{D_m(x,y)}\Bigg[(2+x)^2(1+x+c)((2+x)^2+2(2+x)y+y^2) \\
&+y^3(-y^2-(2+y)(4-2c+x)+(-5+c-2x)y) \\
&+m_0\bigg(-(2+x)^2y^2-\frac{(2+x)^2(2+x+y)^2}{(1+x)^2}+\frac{(2+x+y)^2y^2}{(1+y)^2}\bigg)\Bigg],
\end{aligned}$$

$$\begin{aligned}
=&\frac{4x^2A_3}{D_m(x,y)}\Bigg[(2+x)^2(1+x+c)(2+x+y)^2+y^3(-8+4c-2x-4x+2xc \\
&-x^2-5y+cy-2xy-y^2)+m_0\bigg(-\frac{A_1A_2}{(1+x)^2}-A_1y^2+\frac{A_2y^2}{(1+y)^2}\bigg)\Bigg],
\end{aligned}$$

$$\begin{aligned}
=&\frac{4x^2A_3}{D_m(x,y)}\Bigg[(2+x)^2(1+x+c)(2+x+y)^2+y^3(-8-6x-5y-x^2-y^2 \\
&-2xy-4c+2xc+yc)+m_0\bigg(-\frac{A_1A_2}{(1+x)^2}-A_1y^2+\frac{A_2y^2}{(1+y)^2}\bigg)\Bigg],
\end{aligned}$$

$$\begin{aligned}
=&\frac{4x^2A_3}{D_m(x,y)}\Bigg[A_1A_2(1+x+c)-y^3(4+4x+x^2+y^2+2xy+4y)+y^3(-2x \\
&-4-y+4c+2xc+yc)+m_0\bigg(-\frac{A_1A_2}{(1+x)^2}-A_1y^2+\frac{A_2y^2}{(1+y)^2}\bigg)\Bigg],
\end{aligned}$$

$$=\frac{4x^2A_3}{D_m(x,y)}\bigg[A_1A_2(1+x+c)-y^3(2+x+y)^2+y^3(-2x(1-c)-4(1-c)$$
$$-y(1-c))+m_0\bigg(-\frac{A_1A_2}{(1+x)^2}-A_1y^2+\frac{A_2y^2}{(1+y)^2}\bigg)\bigg],$$

$$=\frac{4x^2A_3}{D_m(x,y)}\bigg[A_1A_2(1+x+c)-y^3A_3-y^2(1-c)(2xy+4y+y^2)$$
$$+m_0\bigg(-\frac{A_1A_2}{(1+x)^2}-A_1y^2+\frac{A_2y^2}{(1+y)^2}\bigg)\bigg],$$

$$=\frac{4x^2A_3}{D_m(x,y)}\bigg[A_1A_2(1+x+c)-y^3A_3-y^2(1-c)(y^2+4ty2xy+x^2-x^2$$
$$+4x-4x+4-4)+m_0\bigg(-\frac{A_1A_2}{(1+x)^2}-A_1y^2+\frac{A_2y^2}{(1+y)^2}\bigg)\bigg],$$

$$=\frac{4x^2A_3}{D_m(x,y)}\bigg[A_1A_2(1+x+c)-y^3A_3-((2+x+y)^2-(2+x)^2)(1-c)y^2$$
$$+m_0\bigg(-\frac{A_1A_2}{(1+x)^2}-A_1y^2+\frac{A_2y^2}{(1+y)^2}\bigg)\bigg],$$

$$=\frac{4x^2A_3}{D_m(x,y)}\bigg[A_1A_2(1+x+c)-y^3A_3-(A_2-A_1)(1-c)y^2$$
$$+m_0\bigg(-\frac{A_1A_2}{(1+x)^2}-A_1y^2+\frac{A_2y^2}{(1+y)^2}\bigg)\bigg],$$

we can rewrite m_2 as

$$m_2(m_0,x,y,c)=\frac{N_{m_2}(m_0,x,y,c)}{D_m(x,y)}, \tag{3.14}$$

where

$$N_{m_2}(m_0,x,y,c)=4x^2A_3(A_1A_2(1+x+c)-y^3A_3-(A_2-A_1)(1-c)y^2$$
$$+m_0(-\frac{A_1A_2}{(1+x)^2}-A_1y^2+\frac{A_2y^2}{(1+y)^2})).$$

$$m_3 = \frac{4A_1y^2}{D_m(x,y)}\Big(-x^5 + y^2(1+y)(2+y)^2 + 2x(1+y)(2+y)^3 + (1+y)(2+y)^4$$
$$- x^3(2+y)(4+y) - x^4(5+2y) - c(x^4 + 2x^3(2+y) + x^2(2+y)^2 + 2x(2+y)^3$$
$$+ (2+y)^4) + m_0(-(2+y)^2(1+x)^2(1+y)^2x^2 + (2+x+y)^2(1+y)^2x^2$$
$$- (2+x+y)^2(2+x)^2(1+x)^2)/(1+x)^2(1+y)^2\Big),$$

$$= \frac{4A_1y^2}{D_m(x,y)}\Big(x^2(1+y)(2+y)^2 + 2x(1+y)(2+y)^3 + (1+y)(2+y)^4 - c(x^2(2+y)^2$$
$$+ 2x(2+y)^3 + (2+y)^4) - x^3(2+y)(4+y+2c) - x^4(5+2y+c) - x^5$$
$$+ m_0(-x^2(2+y)^2 + \frac{x^2(2+x+y)^2}{(1+x)^2} - \frac{(2+x+y)^2(2+x)^2}{(1+y)^2})\Big),$$

$$= \frac{4A_1y^2}{D_m(x,y)}\Big((1+y)(2+y)^2(x^2 + 2x(2+y) + (2+y)) - c(2+y)^2(x^2 + 2x(2+y)$$
$$+ (2+y)) - x^3(8 + 2y + 4c + 4y + y^2 + 2cy + cx + x^2 + 5x + 2xy)$$
$$+ m_0(-x^2A_3 + \frac{x^2A_2}{(1+x)^2} - \frac{A_1A_2}{(1+y)^2})\Big),$$

$$= \frac{4A_1y^2}{D_m(x,y)}\Big((2+y)^2(x^2 + 2x(2+y) + (2+y))(1+y-c) - x^3(4 + 4x + y^2 + x^2$$
$$+ 4y + 2xy) - x^3(4 + 2y + 4c + 2cy + x + cx) + m_0(-x^2A_3 + \frac{x^2A_2}{(1+x)^2} - \frac{A_1A_2}{(1+y)^2})\Big),$$

$$= \frac{4A_1y^2}{D_m(x,y)}\Big(A_3(2+x+y)^2(1+y-c) - x^3(2+x+y)^2 - x^3(4(1+c) + 2y(1+c)$$
$$+ x(1+c)) + m_0(-x^2A_3 + \frac{s^2A_2}{(1+x)^2} - \frac{A_1A_2}{(1+y)^2})\Big),$$

$$= \frac{4A_1y^2}{D_m(x,y)}\Big(A_2A_3(1+y-c) - A_2x^3 - x^3(1+c)(4+2y+x)$$
$$+ m_0(-x^2A_3 + \frac{x^2A_2}{(1+x)^2} - \frac{A_1A_2}{(1+y)^2})\Big),$$

$$=\frac{4A_1y^2}{D_m(x,y)}\Big(A_2A_3(1+y-c)-A_2x^3-x^2(1+c)(x^2+4x+2xy+y^2-y^2$$
$$+4y-4y+4-4)+m_0(-x^2A_3+\frac{x^2A_2}{(1+x)^2}-\frac{A_1A_2}{(1+y)^2})\Big),$$

$$=\frac{4A_1y^2}{D_m(x,y)}\Big(A_2A_3(1+y-c)-A_2x^3-x^2(1+c)((2+x+y)^2-(2+y)^2)$$
$$+m_0(-x^2A_3+\frac{x^2A_2}{(1+x)^2}-\frac{A_1A_2}{(1+y)^2})\Big),$$

$$=\frac{4A_1y^2}{D_m(x,y)}\Big(A_2A_3(1+y-c)-(A_2-A_3)(1+c)x^2-A_2x^3$$
$$+m_0(-x^2A_3+\frac{x^2A_2}{(1+x)^2}-\frac{A_1A_2}{(1+y)^2})\Big),$$

we can rewrite m_3 as

$$m_3(m_0,x,y,c)=\frac{N_{m_3}(m_0,x,y,c)}{D_m(x,y)}, \tag{3.15}$$

where

$$N_{m_3}(m_0,x,y,c)=4A_1y^2(A_2A_3(1+y-c)-(A_2-A_3)(1+c)x^2-A_2x^3$$
$$+m_0(-x^2A_3+\frac{x^2A_2}{(1+x)^2}-\frac{A_1A_2}{(1+y)^2})).$$

$$m_4=\frac{y^2(2+y)^2(2+x+y)^2}{D_m(x,y)}\Big(4(-1+c)(2+x)^2+4x^2(-1-c)+x^2(1+x+c)(2+x)^2$$
$$+m_0(4x^2+4(2+x)^2-\frac{x^2(2+x)^2}{(1+x)^2})\Big),$$

$$=\frac{y^2A_2A_3}{D_m(x,y)}\Big(4(-1+c)(4+x^2+4x)+4x^2(-1-c)+x^2(1+x+c)(4+x^2+4x)$$
$$+m_0(4x^2+\frac{4(1+x)^2(2+x)^2-x^2(2+x)^2}{(1+x)^2})\Big),$$

$$=\frac{x^2A_2A_3}{D_m(x,y)}\Big(16(-1+c)+16x(-1+c)+4x^2(-1+c)-4x^2-4cx^2+4x^2+4cx^2$$
$$+8x^3+5x^4+cx^4+x^5+4cx^3+m_0(4x^2+\frac{4(1+x)^2(2+x)^2-x^2(2+x)^2}{(1+x)^2})\Big),$$

$$=\frac{y^2A_2A_3}{D_m(x,y)}\Big(16(-1+c)+16x(-1+c)+4x^2(-1+c)+8x^3+4cx^3+5x^4+cx^4$$
$$+x^5+m_0(4x^2+\frac{(2+x)^2((2+2x)^2-x^2)}{(1+x)^2})\Big),$$

$$=\frac{y^2A_2A_3}{D_m(x,y)}\Big(16(-1+c)+16x(-1+c)+4x^2(-1+c)+4x^3(2+c)+x^4(5+c)$$
$$+x^5+m_0(4x^2+\frac{(2+x)^2(2+2x+x)(2+2x-x)}{(1+x)^2})\Big),$$

$$=\frac{y^2A_2A_3}{D_m(x,y)}\Big(16(-1+c)+16x(-1+c)+4x^2(-1+c)+4x^3(2+c)+x^4(5+c)$$
$$+x^5+m_0(4x^2+\frac{(2+x)^3(2+3x)}{(1+x)^2})\Big),$$

we can rewrites m_4 as

$$m_4(m_0,x,y,c)=\frac{N_{m_4}(m_0,x,y,c)}{D_m(x,y)}, \tag{3.16}$$

where

$$N_{m_4}(m_0,x,y,c)= y^2A_2A_3(16(-1+c)+16x(-1+c)+4x^2(-1+c)+4x^3(2+c)$$
$$+x^4(5+c)+x^5+m_0(4x^2+\frac{(2+x)^3(2+3x)}{(1+x)^2})).$$

Equations $(3.13)-(3.16)$ gives the general solutions of masses m_1, m_2, m_3 and m_4 in terms of m_0, x and y.These equations give the xym_0-space regions of central configurations for fixed values of c. In other words, provided the values x, y, and m_0 values, the values of m_1, m_2, m_3, and m_4 can be found from equations $(3.13)-(3.16)$, which will make the configurations central. The values obtained from m_i $(i=1,2,3,4)$ can also be negative and not realistic. Therefore, we would like to identify regions that would keep the masses m_1, m_2, m_3 and m_4 are positive. In the next segment, we will discuss the special case where m_1, m_2, m_3 and m_4 are symmetric about the center of mass.

Fully Symmetric Collinear 4 and 5 Body Problems

For symmetric case we choose $x = y$ in which m_0 is constant at the center of mass. It is taken to be the center of mass at the origin. As a consequence, it can shown that for $x = y$, $m_1 = m_4$ and $m_2 = m_3$. Furthermore, the masses (m_1, m_4) and (m_2, m_3) are the symmetric about the center of the system. Therefore, only m_1 and m_2 should be evaluated as a function of $m_0 \geq 0$ and $y > 0$. The results derived from equations (3.13) – (3.16) of masses m_1 and m_2 are given below.

$$m_1 = \frac{4y^2(1+y)^2(2+y)^2(y^5+5y^4+8y^3-4y^2-16y-16+m_0(4x^2+\frac{(2+y)^3(2+3y)}{(1+x)^2}))}{256+1024y+1664y^2+1408y^3+656y^4+160y^5+24y^6+8y^7+y^8},$$

$$\begin{aligned} m_1 =& \frac{4y^2(2+y)^2}{D_m^*(y)}\Big[m_0(4y^2(1+y)^2+(8+y^3+6y^2+12y)(2+3y)) \\ &+(1+y)^2(y^5+5y^4+8y^3-4y^2-16y-16)\Big], \end{aligned}$$

where

$$D_m^*(y) = 256+1024y+1664y^2+1408y^3+656y^4+160y^5+24y^6+8y^7+y^8. \tag{3.17}$$

$$\begin{aligned} m_1 =& \frac{4y^2(2+y)^2}{D_m^*(y)}\Big(m_0(4y^2+4y^4+8y^3+16+24y+3y^4+2y^3+24y \\ &+36y^2+12y^2+18y^3)+(1+y)^2(y^5+5y^4+8y^3-4y^2-16y-16)\Big), \end{aligned}$$

$$=\frac{4y^2(2+y)^2}{D_m^*(y)}\Big[m_0(16+48y++52y^2+28y^3+7y^4)+(1+y)^2(y^5+5y^4$$
$$+8y^3-4y^2-16y-16)\Big],$$

we can rewrite m_1 as

$$m_1=\frac{N_{m_1}^*(m_0,y)}{D_m^*(y)}, \tag{3.18}$$

where

$$N_{m_1}^*(m_0,y)=4y^2(2+y)^2\Big(m_0(16+48y++52y^2+28y^3+7y^4)$$
$$+(1+y)^2(y^5+5y^4+8y^3-4y^2-16y-16)\Big), \tag{3.19}$$

$$P_1(y)=-16-16y-4y^2+8y^3+5y^4+y^5.$$

$$m_2=\frac{4y^2(2+y)^2}{D_m^*(y)}\Big(-4y^3(1+y^2+2y)+4(4+y^2+4y)(1+y)^3-y^3(3y^2+4y)$$
$$-m_0(y^2(4+y^2+4y)-4y^2+4(4+y^2+4y)\Big),$$
$$=\frac{4y^2(2+y)^2}{D_m^*(y)}\Big(-4y^3-4y^5-8y^4+(16+16y+4y^2)(1+3y+3y^2+y^3)$$
$$-3y^4+4y^3)-m_0(16+16y+4y^2+4y^3+y^4)\Big),$$

$$=\frac{4y^2(2+y)^2}{D_m^*(y)}\Big((16+64y+100y^2+68y^3+17y^4)$$
$$-m_0(16+16y+4y^2+4y^3+y^4)\Big),$$

we can rewrite m_2 as

$$m_2=\frac{N_{m_2}^*(m_0,y)}{D_m^*(y)}, \tag{3.20}$$

where

$$N_{m_2}^*(m_0,y)=4y^2(2+y)^2(16+64y+100y^2+68y^3+17y^4)$$
$$-4m_0y^2(2+y)^2(16+16y+4y^2+4y^3+y^4). \tag{3.21}$$

Lemma 1 suppose that $P_1(y) = -16 - 16y - 4y^2 + 8y^3 + 5y^4 + y^5$. $P_1(y)$ is always positive, for any $y > 1.39681$.

Proof $P_1(y)$ is a 5 degree polynomial in y and the sign of its coefficient only varies once; thus it can have only 1 real root, according to Descartes' sign rule, $y = 1.39681$. It can be easily shown that $P_1(y)$ is always positive for $y > 1.39681$. For instance, $y = 1$, $P_1(y) < 0$, and $y = 2$, $P1(y) > 0$.

D^*_m is positive for all values of $y > 0$, according to equation (3.17). So, only we are $N^*_{m_1}$and $N^*_{m_2}$ need to be analysed for $m_0 \geq 0$ and $y > 0$.

The term $4y^2(2 + y)^2$ is always positive in equation (3.19); therefore, it has no effect on the sign of $N^*_{m_1}$. Similarly, the term $m_0(16+48y+52y^2+28y^3+7y^4)$ is always positive. The only term that may become negative in $N^*_{m_1}$ is $(1+y)^2P_1(y)$. Therefore, by Lemma 1, $N^*_{m_1}$ will be positive for all $m_0 \geq 0$ and $y > 1.39681$. Hence, for $m_0 \geq 0$ and $y > 1.39681$, m_1 will also be positive. For $0 < y \leq 1.39681$, the positivity of $N^*_{m_1}$ and thus m_1 is shown in figure 3.2, where m_1 on the right side of the curve is positive. From figure 3.2, it can be deduced that m_1 is positive for $m_0 \geq 1$ and $y > 0$.

Following the above procedure, we get the following relation between m_0 and y for m_2 to be positive:

$$m_2 > 0,$$

from (3.20), we have

$$\frac{N^*_{m_2}(m_0, y)}{D^*_m(y)} > 0,$$

$\Longrightarrow$

$$N^*_{m_2}(m_0, y) > 0.$$

From (3.21),

$$4y^2(2+y)^2(16+64y+100y^2+68y^3+17y^4)-4m_0y^2(2+y)^2(16+16y+4y^2+4y^3+y^4) > 0$$

$$(16+64y+100y^2+68y^3+17y^4) > m_0(16+16y+4y^2+4y^3+y^4),$$

$$m_0 < \frac{(16+64y+100y^2+68y^3+17y^4)}{(16+16y+4y^2+4y^3+y^4)}. \tag{3.22}$$

Careful analyses of equation (3.21) and equation (3.22) show that m_0 must be less than or equal to 17 for m_2 to be positive. This can be seen in figure 3.3. Positive masses, which will make the configuration central, can not be found in the white region of figure 3.3. Figure 3.4 gives the common region in which m_1 and m_2 are both positive.

In the special case if $m_0 = 0$, which is the symmetric case of 4 bodies, the expressions for m_1 and m_2 are reduced to

$$m_1 = \frac{4y^2(1+y)^2(2+y)^2 P_1(y)}{D_m(y)}. \tag{3.23}$$

$$m_2 = \frac{4y^2(2+y)^2(16+64y+100y^2+68y^3+17y^4)}{D_m(y)}. \tag{3.24}$$

In this case the m_1 and m_2 solutions can be analysed very easily. The only term that can turn negative in m_1 is $P_1(y)$. By Lemma 1, $m_1 > 0$ for $y > 1.39681$. This is shown in figure 3.5, numerically. Since m_2 is positive for all values of y (see figure 3.6), both m_1 and m_2 for $y > 1.39681$ will be positive.

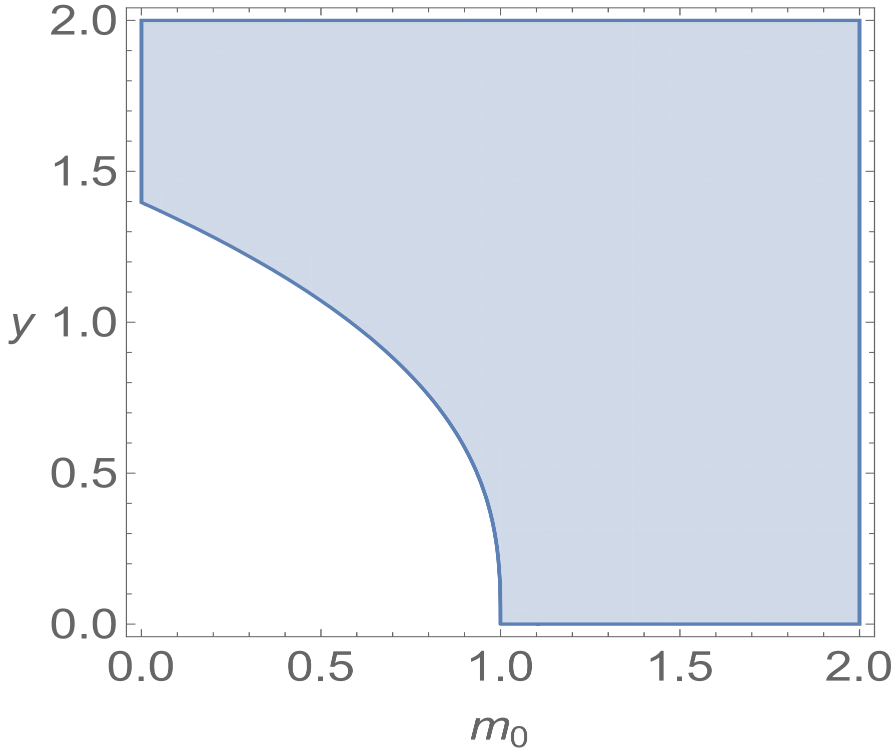

FIGURE 3.2: Solution space where m_1 is positive

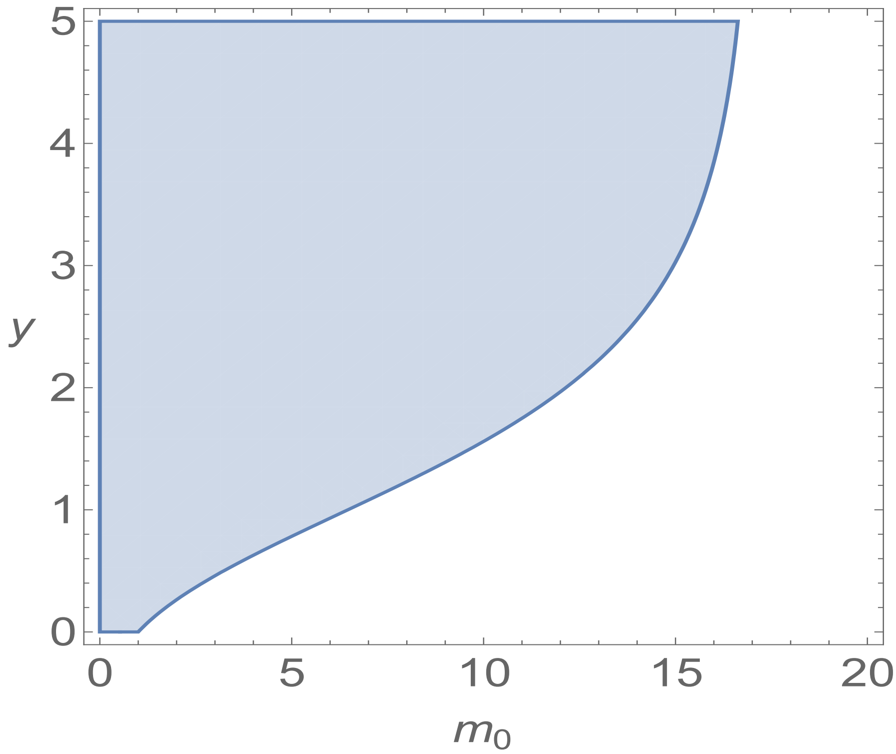

FIGURE 3.3: Solution space where m_2 is positive

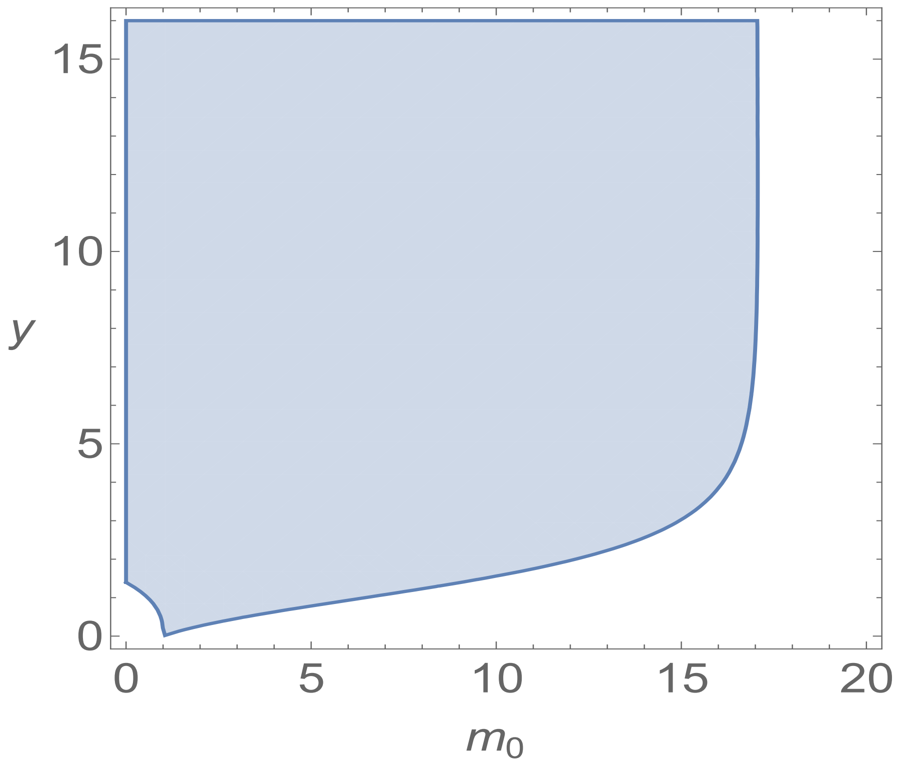

FIGURE 3.4: Solution space where m_1 and m_2 are both positive

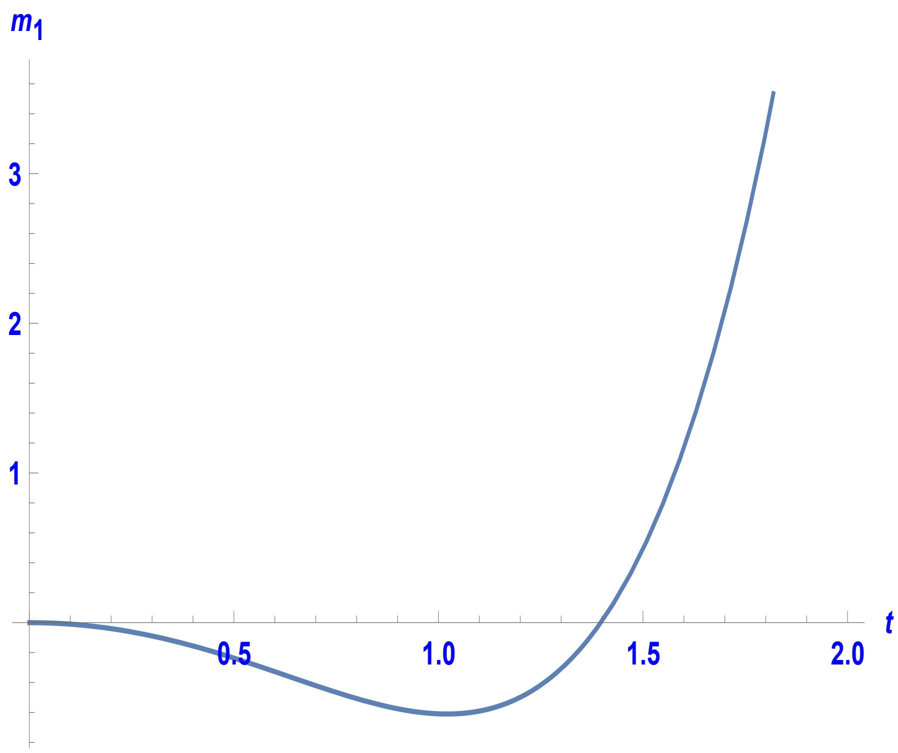

FIGURE 3.5: Solution space for m_1 when $m_0 = 0$ and $x = y$.

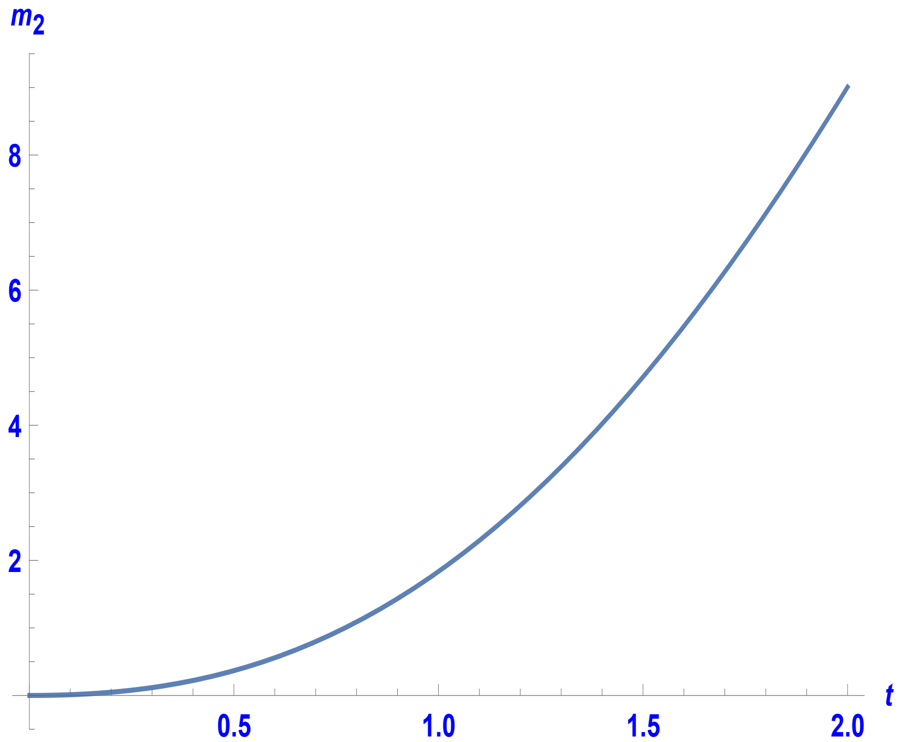

FIGURE 3.6: Solution space for m_2 when $m_0 = 0$ and $x = y$.

Chapter 4

Universal Collinear 4 and 5 Body Problem

In this chapter, we will find that region in the xym_0-space where all masses m_1, m_2, m_3 and m_4 are positive. We will examine the 4 masses separately, analytically as well as numerically. Finally, there will be an intersection of all 4 regions, which will show the regions where positive masses can have CCs. Positive masses can not have a CCs in complementing these regions. We do not consider when we have $m_0 = 0$, which is the 4-body case of this 5BP, since Ouyang and Xie discussed it in detail in [13].

4.1 General Collinear 4 and 5 Body Problems

In equations (3.13) – (3.16), the general solutions for masses m_1, m_2, m_3, and m_4 with arbitrary mass m_0 are given. There is only one symmetry of these equations with $x \neq y$. The common denominator $D_m(x, y)$ of m_i (where $i = 1, 2, 3, 4$) is a polynomial of positive coefficients in x and y. So, $D_m(x, y) > 0$ for all x,$y > 0$. Therefore we just need to analyze the numerators $N_{m_i}(m_0, x, y, c)$. We will examine them one by one.

The only part that can become negative of $N_{m_1}(m_0, x, y, c)$, and $N_{m_4}(m_0, x, y, c)$ is:

$$Neg_{m_1}(y, c) = y^5 - y^4(-5 + c) - 4y^3(-2 + c) - 4y^2(1 + c) - 16y(1 + c) - 16(1 + c),$$

$$Neg_{m_4}(x, c) = x^5 + x^4(5 + c) + 4x^3(2 + c) + 4x^2(-1 + c) + 16x(-1 + c) + 16(-1 + c),$$

$Neg_{m_1}(y, c)$ is a degree 5 polynomial in y; its coefficients change the sign only once at $-1 < c < 1$ and all of them are positive at $c < -1$. Thus, it has only 1 real positive root according to Descarte's sign law, which is $y = 1.39681$ at $c = 0$. For $y > 1.39681$, $Neg_{m_1}(t, 0)$ is positive and therefore N_{m_1} is also positive. $Neg_{m_1}(y, c) > 0$ can be seen easily by y_0 when the $c(y)$ function for the fixed c value is monotonically increased.

$$Neg_{m_1}(y, c) = 0,$$

$$y^5 + 5y^4 - cy^4 + 8y^3 - 4cy^3 - 4y^2 - 4cy^2 - 16y - 16cy - 16 - 16c = 0,$$

$$c(y^4 + 4y^3 + 4y^2 + 16y + 16) = -16 - 16y - 4y^2 + 8y^3 + 5y^4 + y^5,$$

$$c(y) = \frac{-16 - 16y - 4y^2 + 8y^3 + 5y^4 + y^5}{y^4 + 4y^3 + 4y^2 + 16y + 16}.$$

It is easy to show that $c(y)$ increases the function of $\frac{dc(y)}{dy} > 0$ for all y monotonically. This implies that m_1 for all $m_0 \geq 0$ and $y > y_0$ is positive. When

$Neg_{m_1}(t,c) < 0$, does not mean automatically that $N_{m_1}(t,c) < 0$. We must have $t < t_0$

$$N_{m_1}(y,c) > 0,$$

$$y^5 + 5y^4 - cy^4 + 8y^3 - 4cy^3 - 4y^2 - 4cy^2 - 16y - 16cy - 16 - 16c \\ +m_0\left(4y^2 + \frac{(2+y)^3(2+3y)}{(1+y)^2}\right) > 0,$$

$$y^5 + 5y^4 + 8y^3 - 4y^2 - 16y - 16 - c(y^4 + 4y^3 + 4y^2 + 16y + 16) \\ +m_0\left(4y^2 + \frac{(2+y)^3(2+3y)}{(1+y)^2}\right) > 0,$$

$$(1+y)^2(y^5 + 5y^4 + 8y^3 - 4y^2 - 16y - 16 - c(y^4 + 4y^3 + 4y^2 + 16y + 16)) \\ +m_0(4y^2(1+y)^2 + (2+y)^3(2+3y)) > 0,$$

$$(1+y)^2(y^5 + 5y^4 + 8y^3 - 4y^2 - 16y - 16 - c(y^4 + 4y^3 + 4y^2 + 16y + 16)) \\ +m_0(4y^2(1+y^2+2y) + (8+y^3+6y^2+12y)(2+3y)) > 0,$$

$$(1+y)^2(y^5 + 5y^4 + 8y^3 - 4y^2 - 16y - 16 - c(y^4 + 4y^3 + 4y^2 + 16y + 16)) \\ +m_0(7y^4 + 28y^3 + 52y^2 + 48y + 16) > 0,$$

$$m_0 > \frac{(1+y)^2(16 + 16y + 4y^2 - 8y^3 - 5y^4 - y^5 + (y^4 + 4y^3 + 4y^2 + 16y + 16)c)}{(7y^4 + 28y^3 + 52y^2 + 48y + 16)}.$$

See figure 4.1 for the behavior of m_1 if $m_0 > 0$ and $c = 0$. At $c = 0$ the behaviour in $Neg_{m_4}(x,c)$ is similar to $Neg_{m_1}(y,c)$. The region in which

$Neg_{m_4}(x,c) > 0$ is bounded by $c(x)$, a monotonically increasing function of x, i.e. $\forall\ m_0 \geq 0$ and $x > x_0$, $m_4 > 0$.

$$c(x) = \frac{16 + 16x + 4x^2 - 8x^3 - 5x^4 - x^5}{x^4 + 4x^3 + 4x^2 + 16x + 16}.$$

The value of x_0 is obtained similarly as y_0. If $Neg_{m_4}(s,c) < 0$, it does not mean that $N_{m_4}(s,c) < 0$ automatically. For $x < x_0$, we must have

$$m_0 > \frac{(1+x)^2(16 + 16x + 4x^2 - 8x^3 - 5x^4 - x^5 - (x^4 + 4x^3 + 4x^2 + 16x + 16)c)}{(7x^4 + 28x^3 + 52x^2 + 48x + 16)},$$

See figure 4.4, for the behavior of m_4 when $m_0 > 0$ and $c = 0$.

The expression $m_2(m_0, x, y, c)$ that gives the value of m_2 as a complex function of m_0, x, y and c. We initially take $c = 0$ to understand the actions of m_2. We can see that m_2 can be written as follows after certain simplifications:

$$m_2(m_0, x, y) = \frac{4A_2x^2}{D_m(x,y)}\left(Neg_{m_2}(x,y) - \frac{m_0 C_{m_0}(x,y)}{(1+x)^2(1+y)^2}\right), \tag{4.1}$$

where

$$\begin{aligned} Neg_{m_2}(x,y) = \;& (1+x)(2+x)^4 + 2(1+x)(2+x)^3y + (1+x)(2+x)^2y^2 \\ & - (4+x)(2+x)y^3 - (2x+5)y^4 - y^5. \end{aligned}$$

$$\begin{aligned} C_{m_0}(x,y) = \;& (2+x)^4 + 2(3+x)(2+x)^3y + (13 + 8x + x^2)(2+x)^2y^2 \\ & + (2+x)(7 + 8x + 4x^2 + x^3)y^3 \\ & + (7 + 14x + 13x^2 + 6x^3 + x^4)y^4. \end{aligned}$$

The coefficient of m_0 is always negative in the m_2 above. Other than the coefficient of m_0, which is always negative, $Neg_{m_2}(x,y)$ is the term that can become negative. Consider $Neg_{m_2}(x,y)$ as a polynomial with variable coefficients in y. Given $x > 0$, the coefficients of y^0, y, y^2 are positive and the coefficients of y^3, y^4, y^5 are negative. Therefore, according to Descartes' sign rule $Neg_{m_2}(x,y)$,

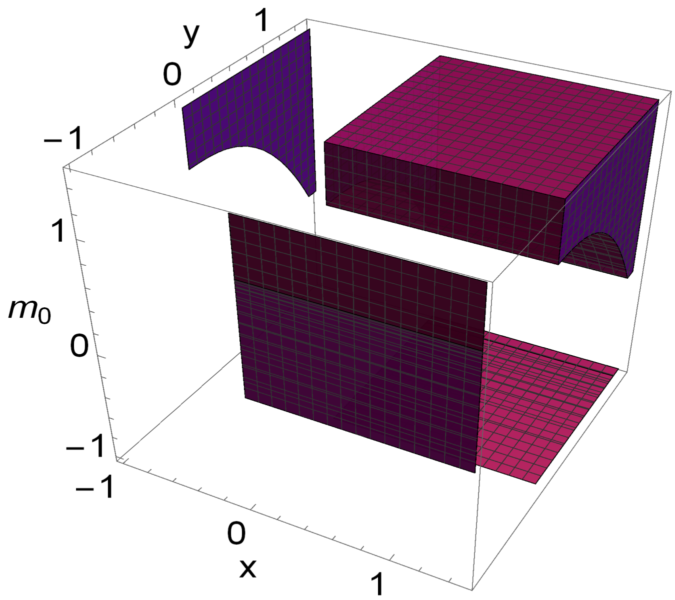

FIGURE 4.1: Solution space for $m_1 > 0$ when $m_0 > 0$ is arbitrary and $x \neq y$.

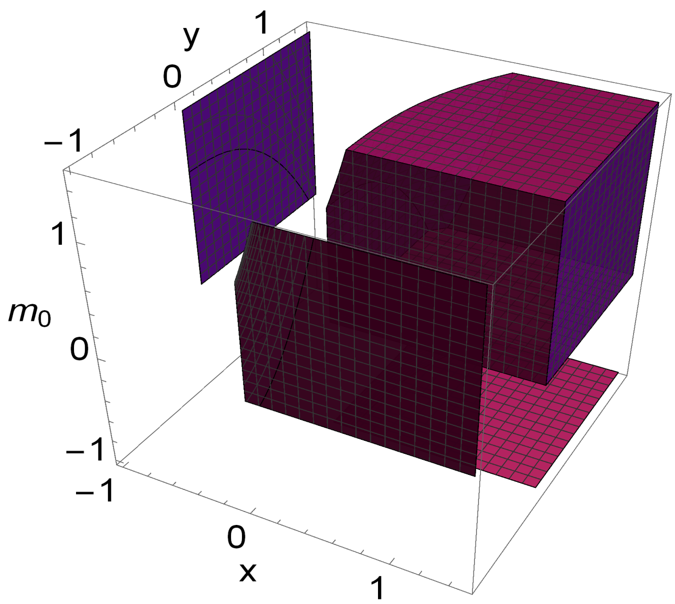

FIGURE 4.2: Solution space for $m_2 > 0$ when $m_0 > 0$ is arbitrary and $x \neq y$.

for every value of x, there will be only 1 positive root, which defines a monotone increasing function $y = f(x)$. The function $f(x) \approx x + 1.4$ defines the limit from which $Neg_{m_2}(x, y)$ can have negative and positive values. If $y > f(x)$, $Neg_{m_2}(x, y)$

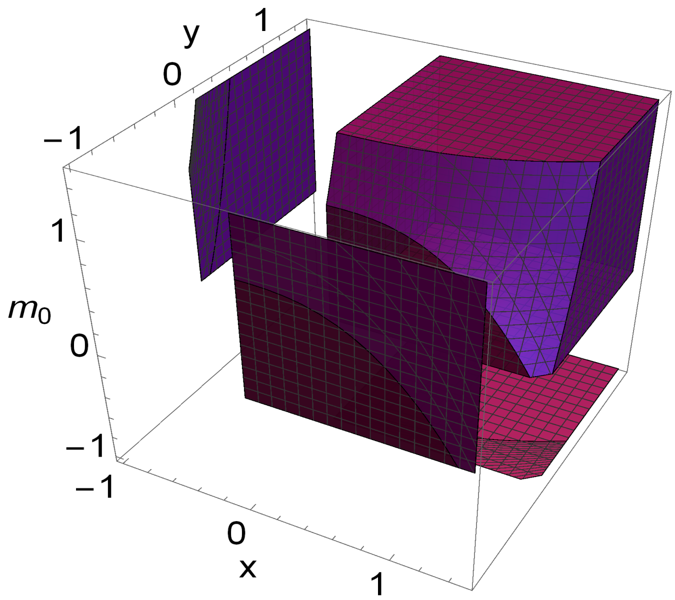

FIGURE 4.3: Solution space for $m_3 > 0$ when $m_0 > 0$ is arbitrary and $x \neq y$.

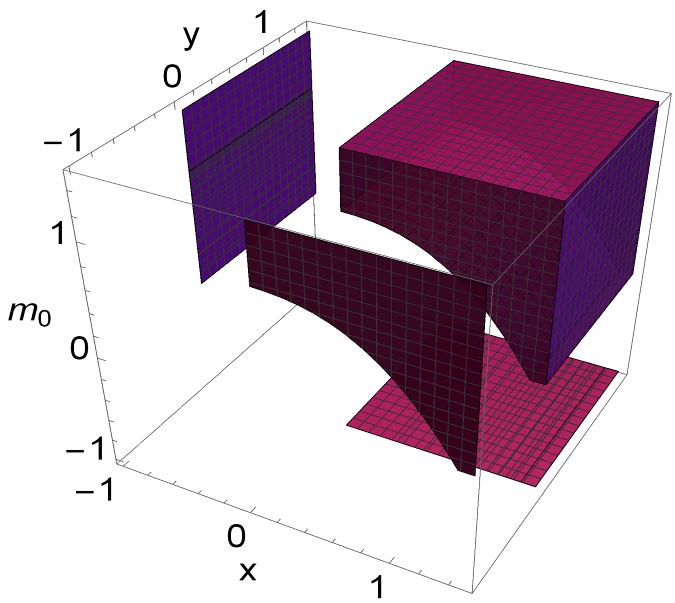

FIGURE 4.4: Solution space for $m_4 > 0$ when $m_0 > 0$ is arbitrary and $x \neq y$.

is negative and then m_2 is negative, since the second part of m_2 involving m_0 is always negative. For $y < f(x)$, $Neg_{m_2}(x, y)$ is always positive, but it does not

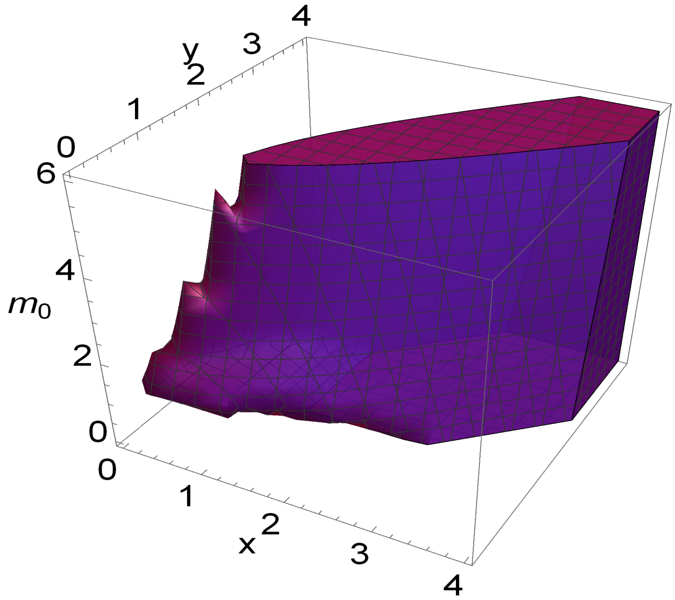

FIGURE 4.5: Solution space for $m_1 > 0$ when ,$m_1 > 0$,$m_2 > 0$,$m_3 > 0$ and $m_4 > 0$ is arbitrary and $x \neq y$.

guarantee that $Neg_{m_2}(x, y)$ and therefore m_2 will be positive as well. For m_2 to be positive, we must also have

$$Neg_{m_2}(x, y) - \frac{m_0 C_{m_0}(x, y)}{(1+x)^2(1+y)^2} > 0,$$

$$Neg_{m_2}(x, y) > \frac{m_0 C_{m_0}(x, y)}{(1+x)^2(1+y)^2},$$

$$m_0(x, y) < \frac{(1+x)^2(1+y)^2 Neg_{m_2}(x, y)}{C_{m_0}(x, y)}. \tag{4.2}$$

As shown in chapter 3, the above inequality gives an upper bound of 17 on m_0 in the special case of $x = y$, but no such bound on m_0 exists in the general case. The above inequality for each value of x and y will give an upper bound of m_0. Therefore, it can be concluded that we will find a suitable $m_0 > 0$ for all $y < f(x)$ which will make m_2 positive. Conversely, we will find $x, y > 0$ for all $m_0 > 0$, which will make m_2 positive. For regions with in xym_0-space where m_2 is positive, please refer to figure 4.2. The coefficient of m_0 is always negative in the general case when $c \neq 0$, so only $Neg_{m_2}(s,t,c)$ that is mentioned below must be analysed.

$$\begin{aligned} Neg_{m_2}(x,y,c) = & -y^5 - y^4(5+2x-c) - y^3(2+x)(4+x-2c) \\ & + y^2(2+x)^2(1+x+c) + 2y(2+x)^3(1+x+c) \\ & + (2+x)^4(1+x+c). \end{aligned}$$

$Neg_{m_2}(x,y,c)$ is similar to the polynomial $Neg_{m_2}(x,y)$ in y it has variable coefficients that are functions of the two variables in x and c. It can be observed by the careful analysis of $Neg_{m_2}(x,y,c)$, that the coefficient of y changes sign only once for each value of the other two variables x and c. According to Descartes' rule of signs, $Neg_{m_2}(x,y,c)$ will have only one positive root for each x and c, which determines a smooth monotonous increasing function $y = f(x,c)$. The function $f(x,c)$ defines a limit from the positive to negative values of m_2 that m_0 satisfies the following inequality:

$$m_0(x,y) < \frac{(1+x)^2(1+y)^2 Neg_{m_2}(x,y,c)}{C_{m_0}(x,y)}. \tag{4.3}$$

As $m_2(x,y,c) = m_3(x,y,-c)$, m_3 analysis are similar to m_2 analysis.

For example, the upper bound on m_0 is given by

$$m_0(x,y) < \frac{(1+x)^2(1+y)^2 Neg_{m_3}(x,y,c)}{C_{m_0}(x,y)}, \tag{4.4}$$

where $Neg_{m_2}(x,y,c)$=$Neg_{m_3}(x,y,-c)$.The above inequalities for fixed values of x, y, and c will give an upper limit of m_0.

For those regions where m_3 and m_4 are positive see figure 4.3 and figure 4.4. Numerically, regions of central configuration for the general 5-body collinear problem are given in figure 4.5. In figures 3.2 - 4.21, keeping c is equal to zero and in figures 4.6 - 4.21 m_0 varies from 0 to 16.5 that gives the cross-sections of the region as shown in figure 4.5.

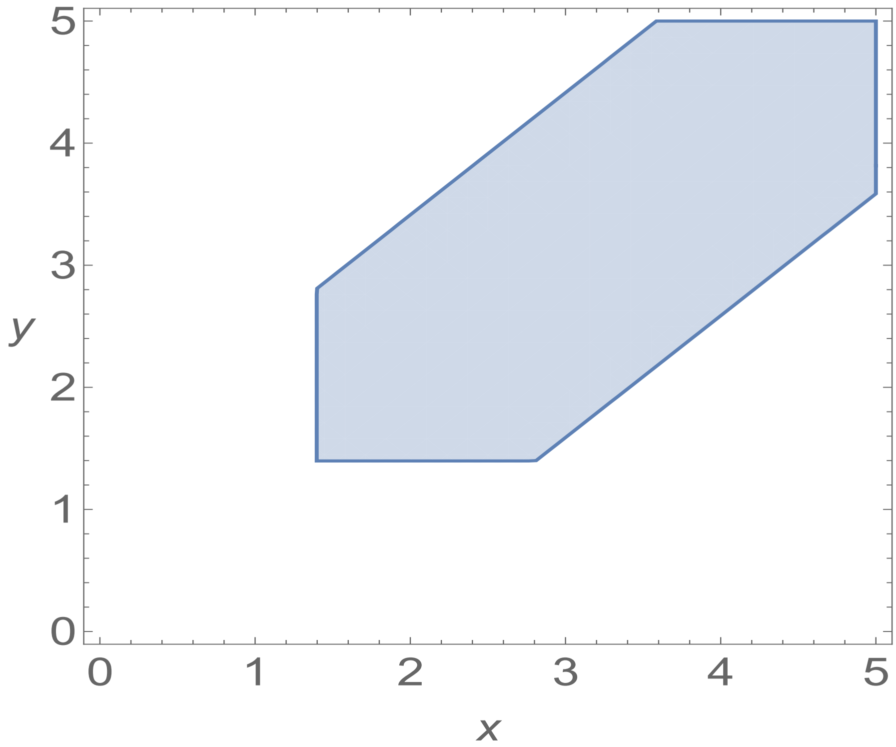

FIGURE 4.6: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 0$

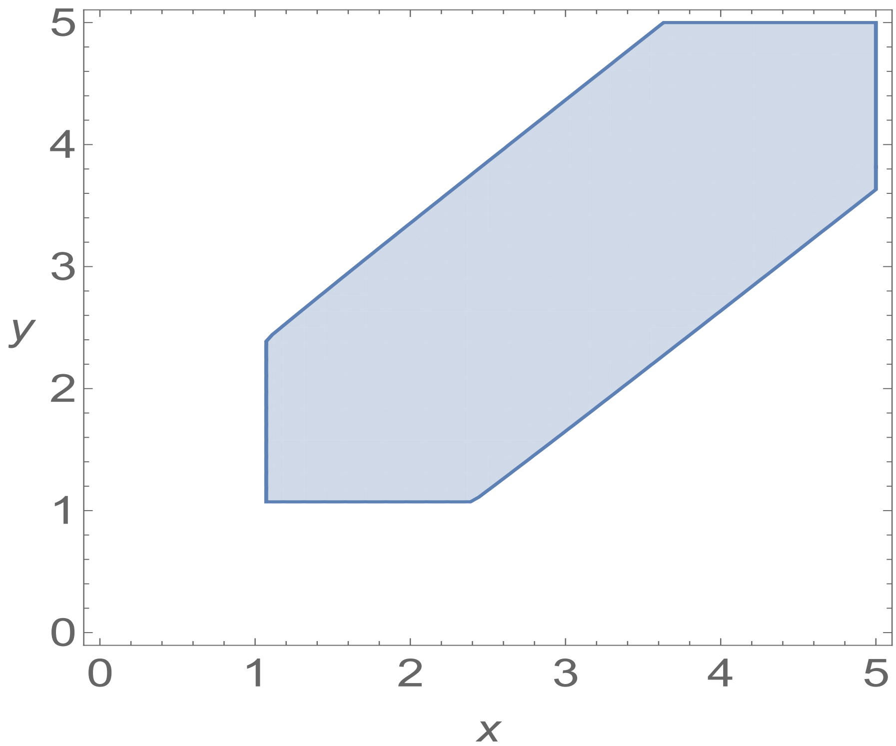

FIGURE 4.7: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 0.5$

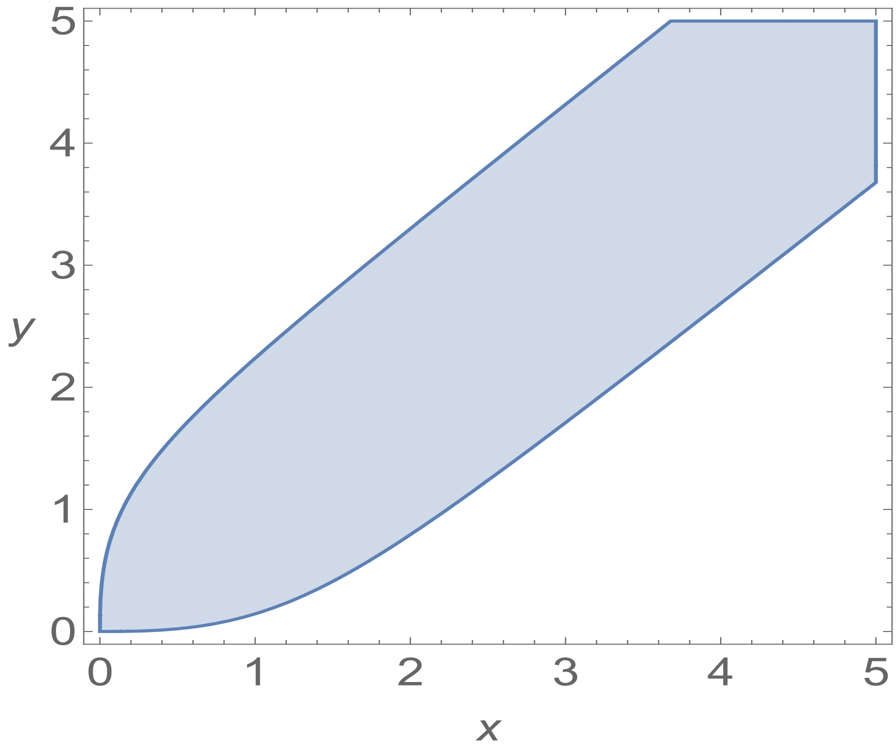

FIGURE 4.8: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 1$

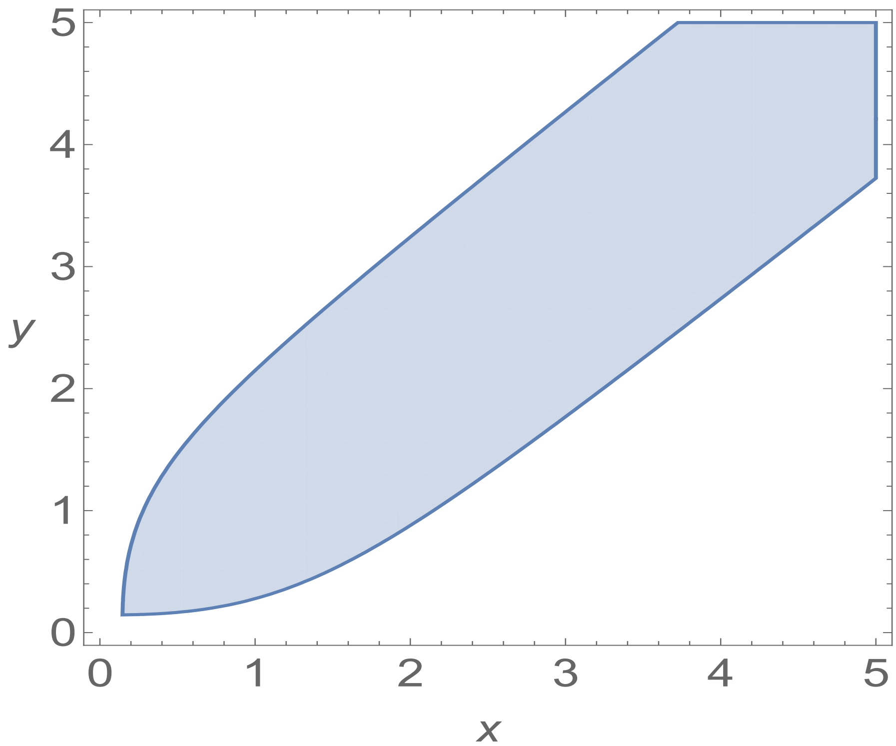

FIGURE 4.9: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 1.5$

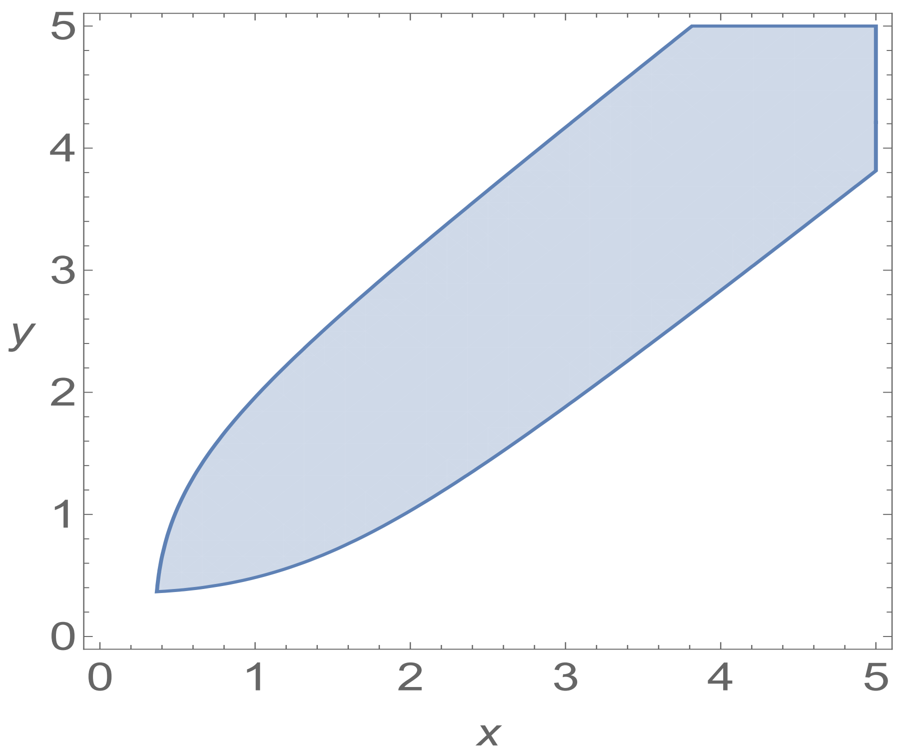

FIGURE 4.10: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 2.5$

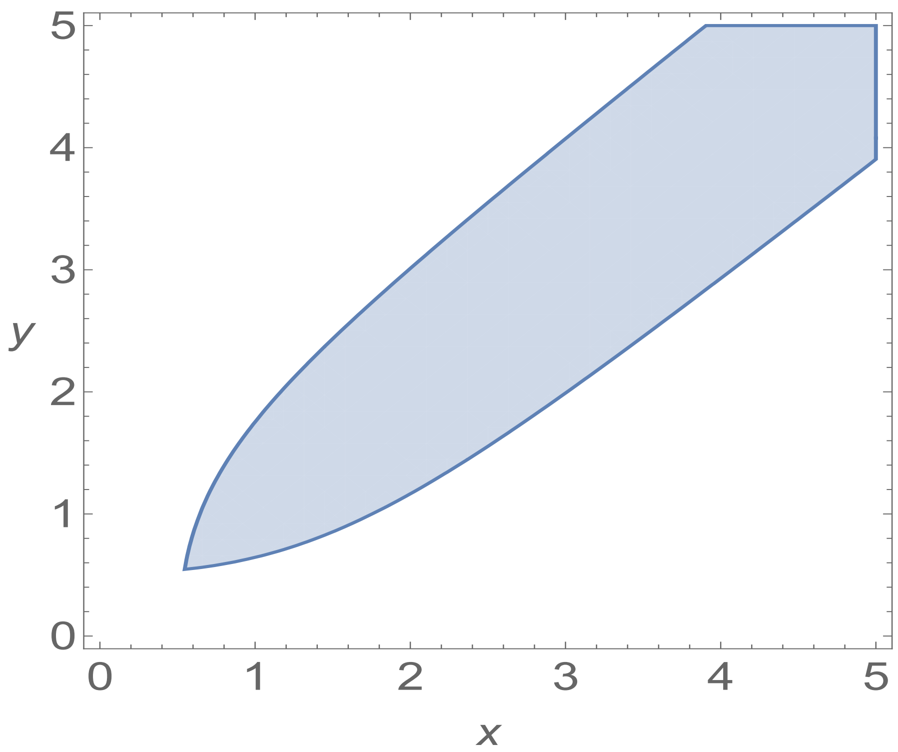

FIGURE 4.11: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 3.5$

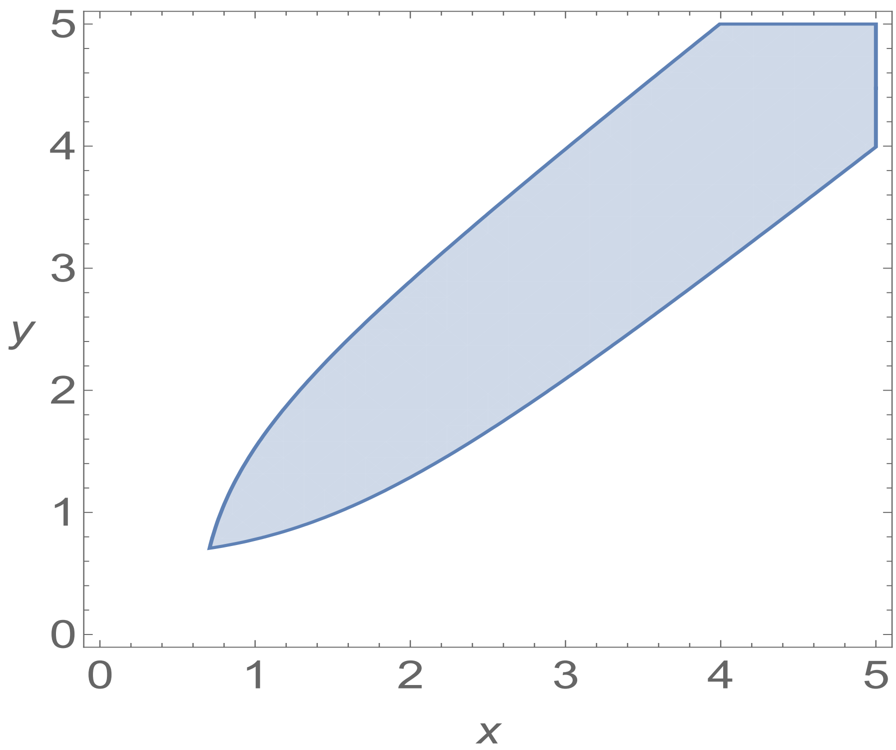

FIGURE 4.12: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 4.5$

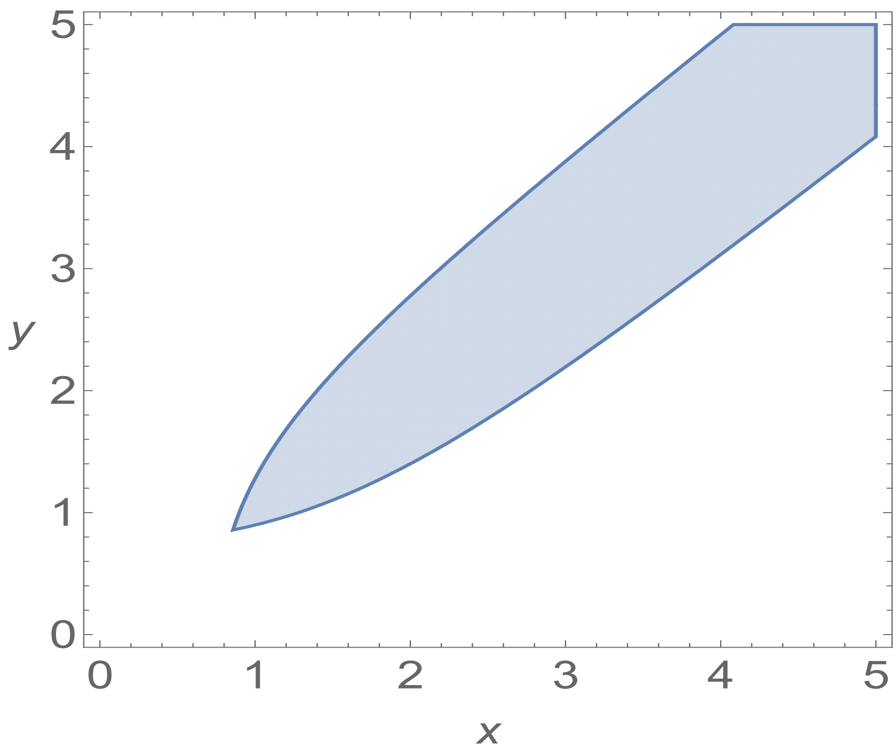

FIGURE 4.13: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 5.5$

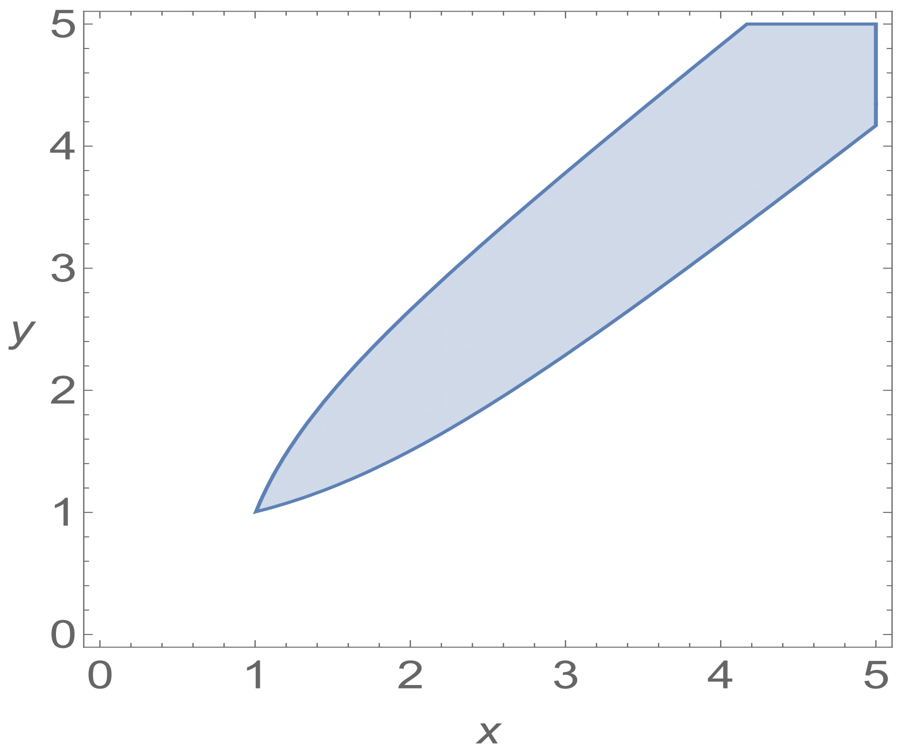

FIGURE 4.14: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 6.5$

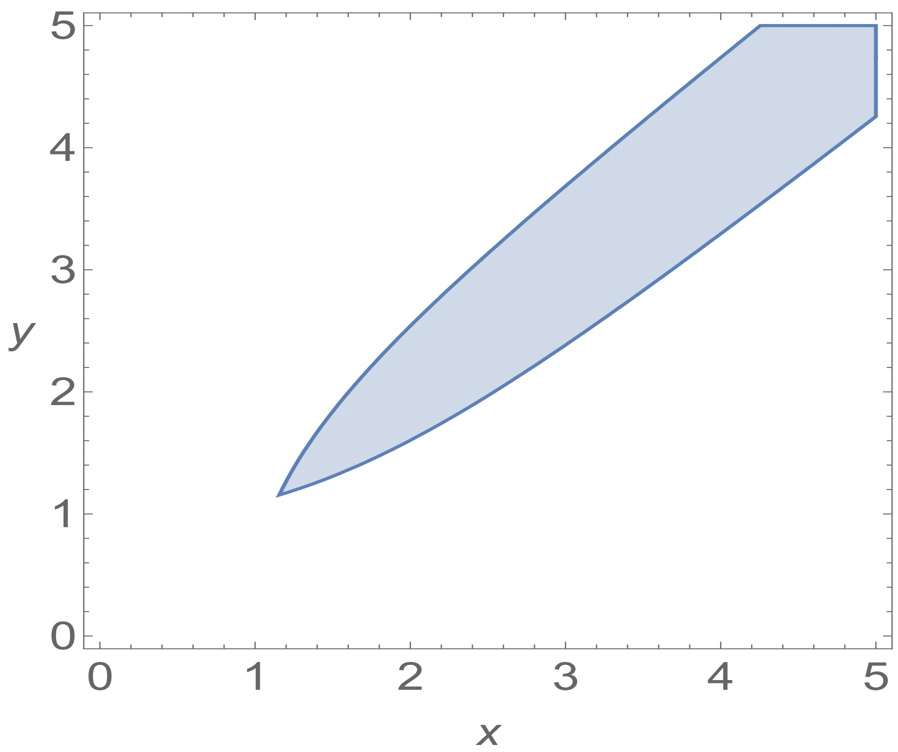

FIGURE 4.15: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 7.5$

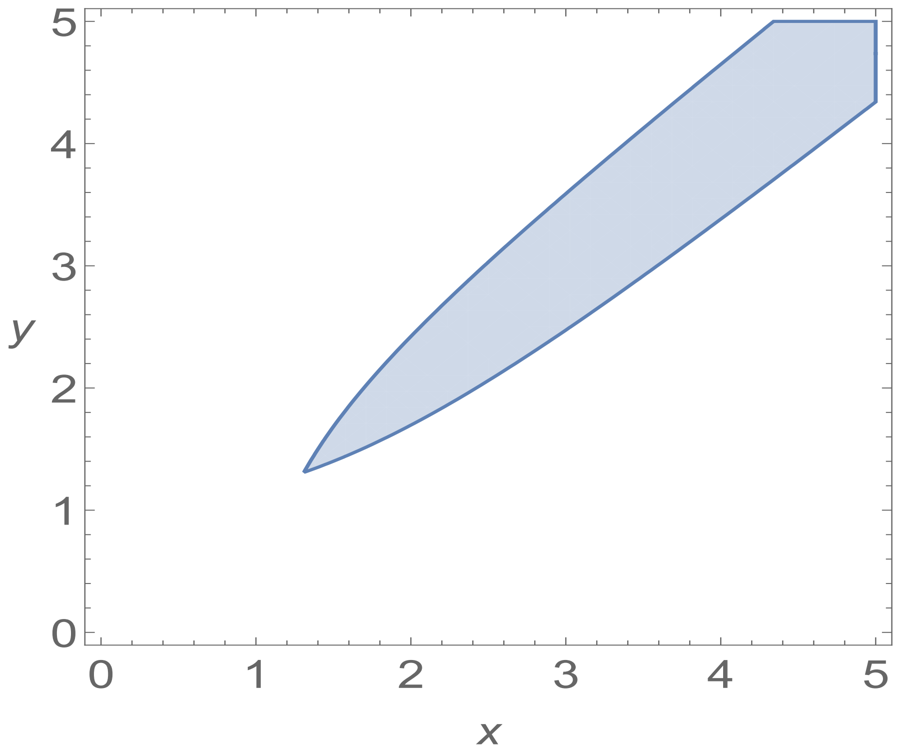

FIGURE 4.16: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 8.5$

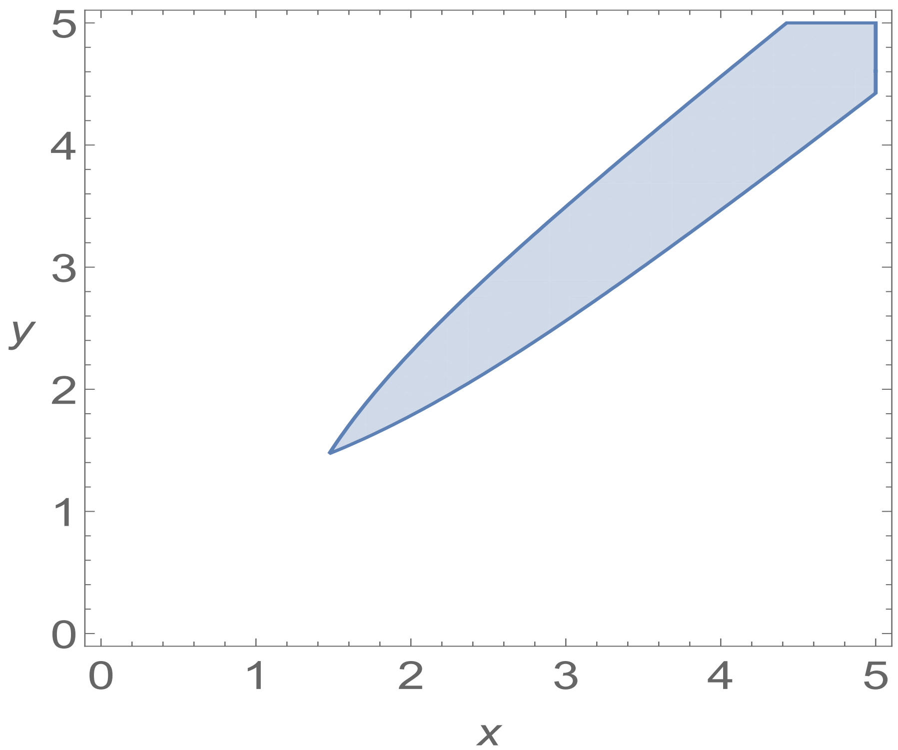

FIGURE 4.17: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 10$

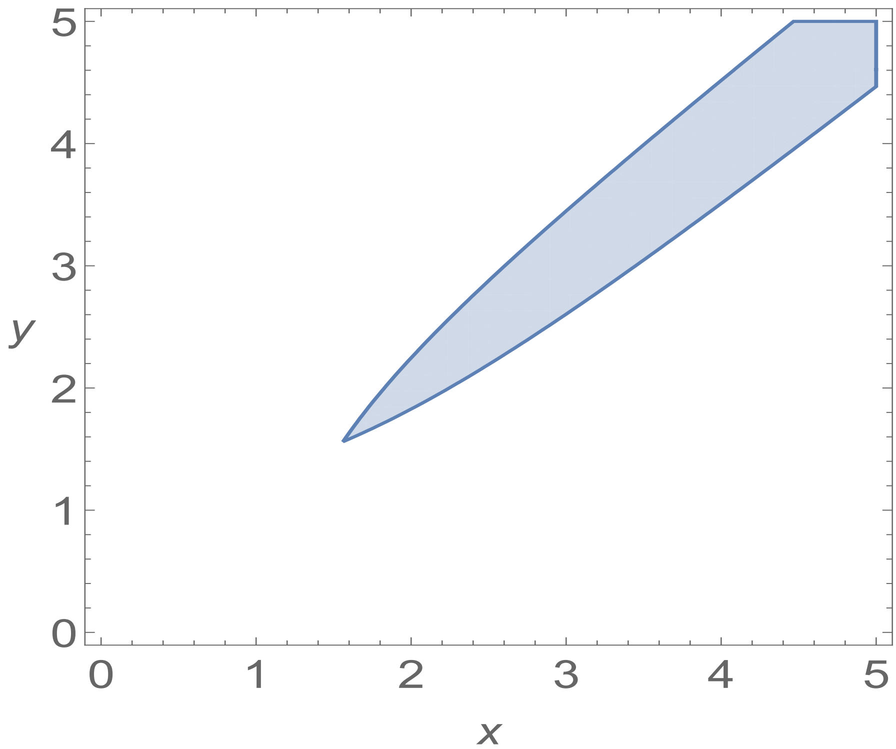

FIGURE 4.18: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 11.5$

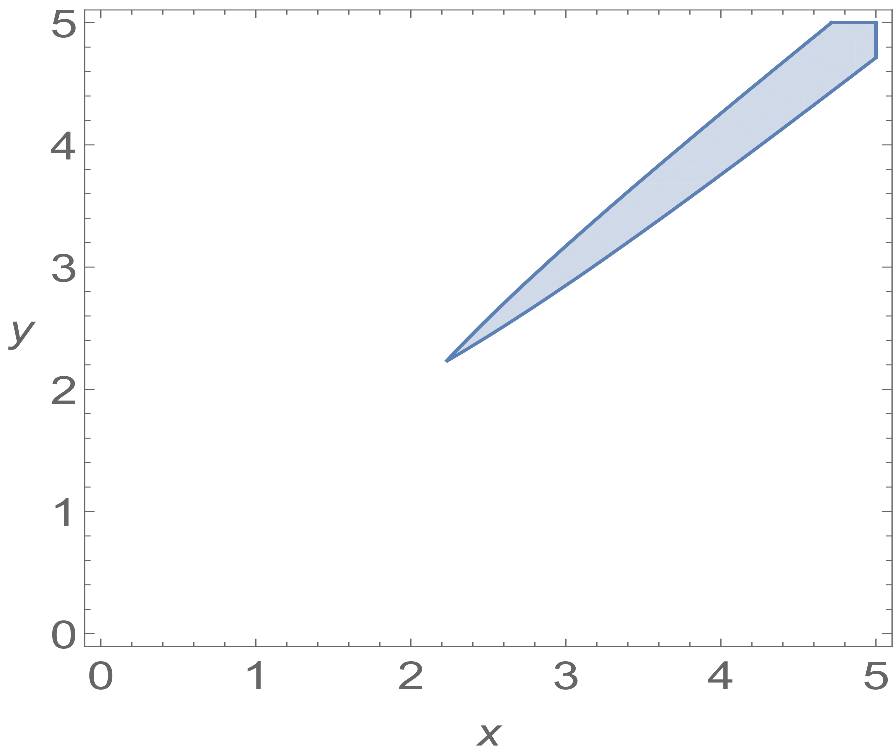

FIGURE 4.19: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 13$

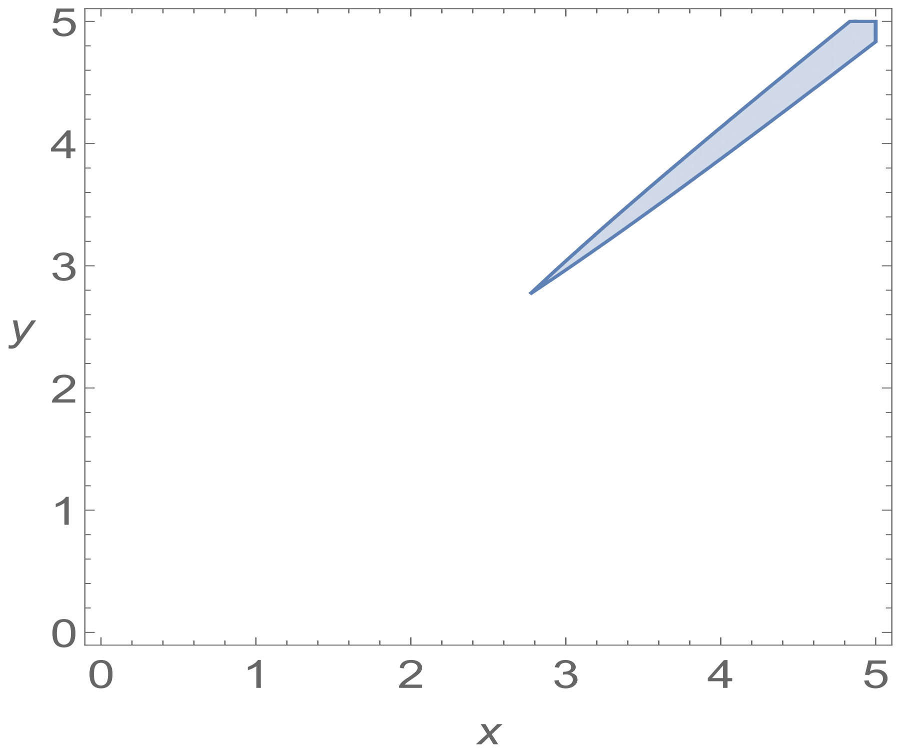

FIGURE 4.20: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 14.5$

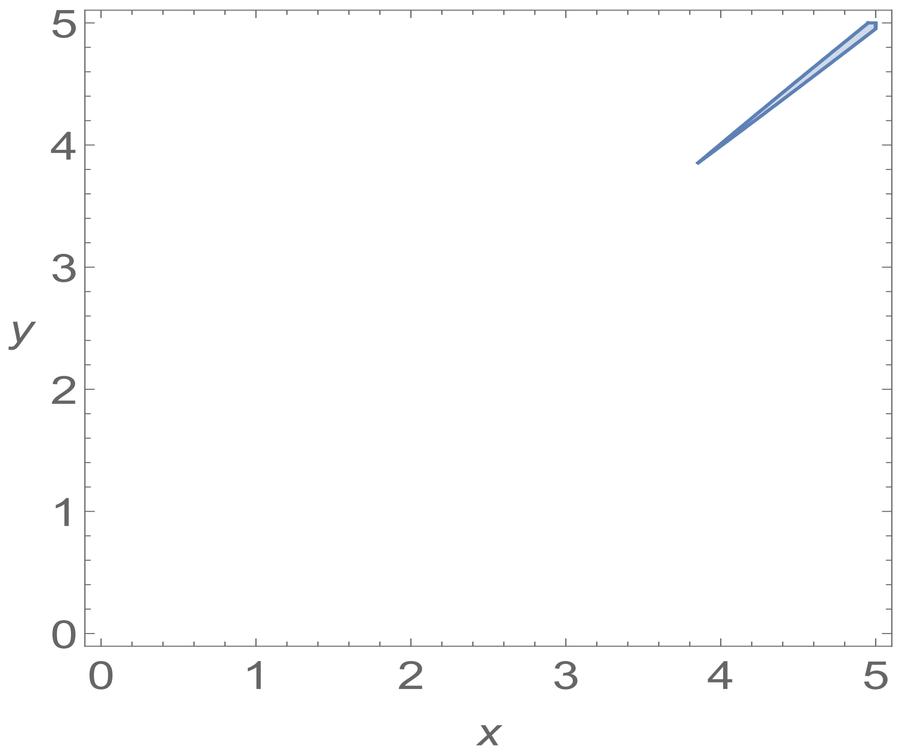

FIGURE 4.21: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 16$

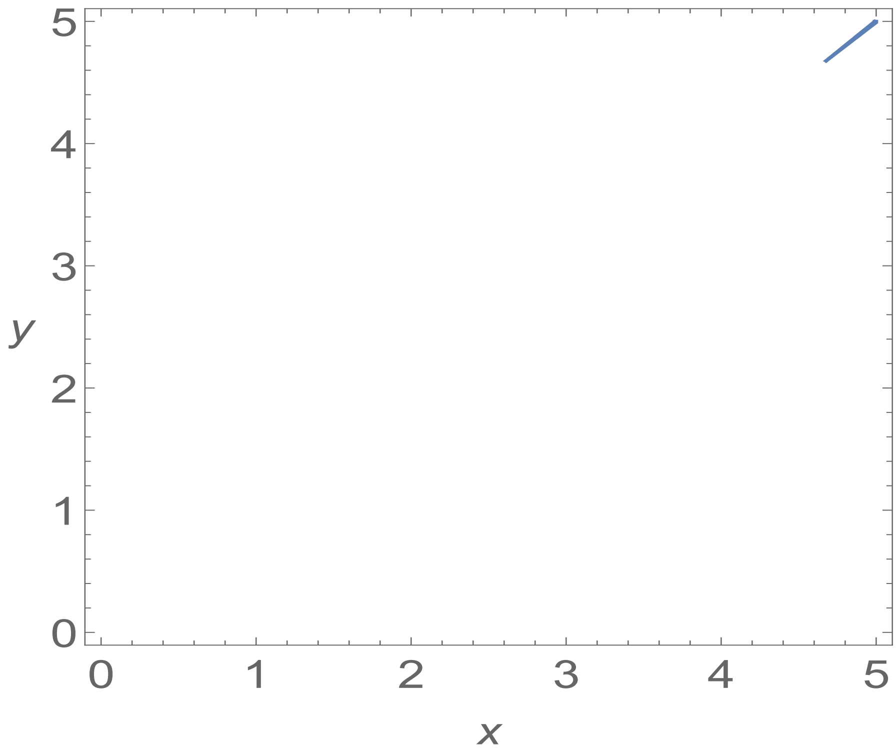

FIGURE 4.22: Solution space for $m_i > 0$, $i = 1, 2, 3, 4$ when $m_0 = 16.5$

Chapter 5

Conclusions

We describe a central configuration of general collinear 5-body problem (C5BP) in which four masses are placed in a straight line and the fifth mass is stationary at the C.O.M. As functions of x, y, and m_0, we derive formulas for $m_i,\ (i = 1, 2, 3, 4)$ which give CCs in the C5BP. If we take all 5 masses are positive in the fully symmetric form of this C5BP, there exist regions in the ym_0-plane where no central configurations can be found. Conversely, it is always possible to choose positive masses in the complement to the above mentioned region. It is also shown that there are no central configurations for $m_0 > 17$ except that we allow some of the masses to become negative. Similarly, in the general 5-body collinear problem, we analyze $m_i, i = 1, 2, 3, 4$. If we restrict all the masses are to be positive, we may identify the regions in the xym_0-space where no central configurations are possible. Positive masses can always be chosen as a complement to these regions.

Bibliography

[1] I. Newton, "Philosophiae naturalis principia mathematica.," *University of California digital Library*, vol. 2000, pp. 203–224.

[2] J. Simmons, A. McDonald, and J. Brown, "The restricted 3-body problem with radiation pressure," *Celestial mechanics*, vol. 35, no. 2, pp. 145–187, 1985.

[3] G. S. Krishnaswami and H. Senapati, "An introduction to the classical three-body problem," *Resonance*, vol. 24, no. 1, pp. 87–114, 2019.

[4] J. Wisdom and M. Holman, "Symplectic maps for the n-body problem," *The Astronomical Journal*, vol. 102, pp. 1528–1538, 1991.

[5] J. L. Russell, "Kepler's laws of planetary motion: 1609–1666," *The British journal for the history of science*, vol. 2, no. 1, pp. 1–24, 1964.

[6] R. G. Littlejohn and M. Reinsch, "Gauge fields in the separation of rotations andinternal motions in the n-body problem," *Reviews of Modern Physics*, vol. 69, no. 1, p. 213, 1997.

[7] L. Mohn and J. Kevorkin, "Some limiting cases of the restricted four-body problem," *AJ*, vol. 72, p. 959, 1967.

[8] F. R. Moulton, "The straight line solutions of the problem of n bodies," *The Annals of Mathematics*, vol. 12, no. 1, pp. 1–17, 1910.

[9] M. Shoaib, B. Steves, and A. Széll, "Stability analysis of quintuple stellar and planetary systems using a symmetric five-body model," *New Astronomy*, vol. 13, no. 8, pp. 639–645, 2008.

[10] K. Glass, "Equilibrium configurations for a system of n particles in the plane," *Physics Letters A*, vol. 235, no. 6, pp. 591–596, 1997.

[11] J. I. Palmore, "Collinear relative equilibria of the planarn-body problem," *Celestial mechanics*, vol. 28, no. 1-2, pp. 17–24, 1982.

[12] M. Shoaib and I. Faye, "Collinear equilibrium solutions of four-body problem," *Journal of Astrophysics and Astronomy*, vol. 32, no. 3, pp. 411–423, 2011.

[13] T. Ouyang and Z. Xie, "Collinear central configuration in four-body problem," *Celestial Mechanics and Dynamical Astronomy*, vol. 93, no. 1-4, pp. 147–166, 2005.

[14] G. E. Roberts, "A continuum of relative equilibria in the five-body problem," *Physica D: Nonlinear Phenomena*, vol. 127, no. 3-4, pp. 141–145, 1999.

[15] A. Albouy and J. Llibre, "Spatial central configurations for the 1+ 4 body problem," *cmds*, vol. 292, p. 1, 2002.

[16] M. Shoaib, A. Sivasankaran, and A. Kashif, "Central configurations in the collinear 5-body problem," *Turkish Journal of Mathematics*, vol. 38, no. 3, pp. 576–585, 2014.

[17] M. Hampton and A. Jensen, "Finiteness of spatial central configurations in the five-body problem," *Celestial Mechanics and Dynamical Astronomy*, vol. 109, no. 4, pp. 321–332, 2011.

[18] W. Z. Khan, M. K. Khan, F. T. B. Muhaya, M. Y. Aalsalem, and H.-C. Chao, "A comprehensive study of email spam botnet detection," *IEEE Communications Surveys & Tutorials*, vol. 17, no. 4, pp. 2271–2295, 2015.

[19] M. R. Spiegel, *Schaum's outline of theory and problems of theoretical mechanics: with an introduction to Lagrange's equations and Hamiltonian theory.* McGraw-Hill Companies, 1967.

[20] M. R Spiegel, "Theory and problems of theoretical mechanics (schaums outline)," 2021.

[21] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of physics.* John Wiley & Sons, 2013.

[22] R. Fitzpatrick, *An introduction to celestial mechanics.* Cambridge University Press, 2012.

[23] G. Beutler, *Methods of celestial mechanics: volume I: physical, mathematical, and numerical principles.* Springer Science & Business Media, 2004.

[24] D. Brouwer and G. M. Clemence, *Methods of celestial mechanics.* Elsevier, 2013.

[25] A. E. Roy, *Orbital motion.* CRC Press, 2020.