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Microeconomics

Static and Dynamic Analysis

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Contents

Mathematical Symbols

$\forall$	Universal quantifier: "for all"
$\exists$	Existential quantifier: "there exists"
$\exists_1$	Existential quantifier: "there exists exactly one"
$\exists_{>1}$	Existential quantifier: "there exists more than one"
$\neg$	Negation; $\neg P$ "not P"
$\wedge$	Conjuction: "and"
$\vee$	Disjunction: "or"
$\Rightarrow$	Implication; $p \Rightarrow q$ "if p then q"
$\Leftrightarrow$	Equivalence: "if and only if"
$a, x \in \mathbb{R}$	Numbers, scalars
$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n_+$	Vector of parameters
$\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_+$	Vector of variables
A, X	Sets
$\mathbf{A}, \mathbf{X}$	Matrices (with dimensions m by) with m rows and n columns
$\sim$	Indifference relation
$\mathbf{x} \sim \mathbf{y}$	Vector $\mathbf{x}$ is indifferent with respect to vector $\mathbf{y}$
$\succ$	Strong preference relation
$\mathbf{x} \succ \mathbf{y}$	Vector $\mathbf{x}$ is strongly preferred over vector $\mathbf{y}$
$\succsim$	(weak) Preference relation
$\mathbf{x} \succsim \mathbf{y}$	Vector $\mathbf{x}$ is (weakly) preferred over vector $\mathbf{y}$
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of nonnegative real numbers
$\text{int } \mathbb{R}_+$	Set of positive real numbers
$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$	n-Dimensional space of real numbers, Cartesian product of set $\mathbb{R}$
$\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \geq 0\} \subset \mathbb{R}^n$	Nonnegative orthant (subspace) of set $\mathbb{R}^n$
$\text{int } \mathbb{R}^n_+ \subset \mathbb{R}^n_+$	Interior of set $\mathbb{R}^n$

$f : X \to Y$	Function, X—Domain of a function (set of arguments of a function) Y—Codomain of a function (set of values of a function)
$y = f(x)$	Scalar function of one variable x—Independent variable (argument/input of a function) y—Independent variable (value of function)
$y = f(x_1, x_2)$	Scalar function of two variables x_1, x_2—Independent variables (arguments/inputs of a function) y—Independent variable (value of function)
$\frac{\mathrm{d}y}{\mathrm{d}x},\ y',\ f'(x)$	Derivative of first order of function $y = f(x)$
$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2},\ y'',\ f''(x)$	Derivative of second order of function $y = f(x)$
$\frac{\partial y}{\partial x_i},\ \frac{\partial f(x_1,x_2)}{\partial x_i}$	Partial derivatives of first order for function $y = f(x_1, x_2)$ with respect to variable $x_i, i = 1, 2$
$\frac{\partial^2 f(x_1,x_2)}{\partial x_i \partial x_j},\ \frac{\partial^2 (x_1,x_2)}{\partial x_i^2},\quad i = 1, 2$	Partial derivatives of second order for function $y = f(x_1, x_2)$,
$H(x_1, x_2) = \begin{bmatrix} \frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} & \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x_1,x_2)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} \end{bmatrix}$	Hessian, symmetric matrix of partial derivatives of 2nd order for function $y = f(x_1, x_2)$
$\left.\frac{df(x)}{dx}\right\|_{x=\bar{x}}$	Value of a derivative of one variable function at point $\bar{x}$
$\left.\frac{\partial f(x_1,x_2)}{\partial x_i}\right\|_{\mathbf{x}=\bar{\mathbf{x}}}$	Value of a derivative of one variable function at point $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$
$\langle \mathbf{p}, \mathbf{x} \rangle = \sum_{i=1}^{2} p_i x_i$	Scalar product of two vectors $\mathbf{p}, \mathbf{x} \in \mathbb{R}_+^2$
$\det(\mathbf{A})$ or $\|\mathbf{A}\|$	Determinant of matrix $\mathbf{A}$
$d_E(\mathbf{x}^1, \mathbf{x}^2) = \left(\sum_{i=1}^{2} \left(x_i^1 - x_i^2 \right)^2 \right)^{\frac{1}{2}}$	Euclidean metric (distance)
$d(\mathbf{x}^1, \mathbf{x}^2) = \max_{i=1,2} \{ \|x_i^1 - x_i^2\| \}$	Non-Euclidean metric (distance)
$\\|\mathbf{x}_E\\| = \left(\sum_{i=1}^{2} (x_i)^2 \right)^{\frac{1}{2}}$	Euclidean norm

$\|\mathbf{x}\| = \max\limits_{i=1,2}\{\|x_i\|\}$	Non-Euclidean norm
$t = 0, 1, 2, \ldots$	Time as discrete variable
$t \in [0; +\infty)$	Time as continuous variable

1 Introduction

In this chapter you will learn:

– what the criteria according to which one distinguishes three branches in economic theory are: microeconomics, mesoeconomics and macroeconomics
– what the basic thematic fields of microeconomics in neoclassical economic theory are
– what the scope of topics covered in this textbook is and its learning objectives.

"*Common sense is the most fairly distributed thing in the world: because everyone thinks he is so well endowed, that even those who are hardest to satisfy in everything else, have no habit of desiring more than they have*".

Discourse on the Method

René Descartes

Economics is a discipline concerning rational methods of use of resources by an individual or society, with the aim to produce goods and services and to distribute them among those individuals or groups constituting a society in a given time horizon.

The term "economics" was first used by Xenophon. According to some, it comes from the Greek words: *οίκος* (*oikos*)—house, household and νομος (nomos)—law, rule, and means the rules of running a household. According to others, the word "economics" is a combination of the words oikos—home, household and *nomeus*—the person who manages it. Thus, the concepts of *oikonomeo* and *oikonomikos* should be associated with management of a household.

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The contemporary understanding of economics has significantly expanded over the ages. This results from the fact that, since the times of ancient Greece, experience and knowledge about the methods and effects of running a business by various actors of economic processes have improved remarkably. Particular individuals, households, social groups, societies and also civilizations have been undergoing various processes of evolution in such fields as technology, organization, management, information, communication, institutions or economic policy.

There are many different criteria for classifying areas of economic knowledge. One of them is the division of economics according to the type of entity, from the point of view of which an analysis of management processes is carried out as well as the management processes themselves.

From this point of view, economics distinguishes between:

(1) microeconomics that concerns management processes in which the main entities are specifically defined units taking different roles: consumers, producers, rentiers, employees, employers, traders, farmers, households, etc. These are called economic agents, business entities or management units,
(2) mesoeconomics (or mezzoeconomics)[1] that deals with management processes by entities operating within separate sectors or branches of the economy,
(3) macroeconomics that concerns the management processes taking place in the economy regarded most broadly as a whole.

Those specializing in microeconomics focus on the activities of individual consumers (households), producers (enterprises, undertakings) and on analysis of given markets.

In microeconomic research—as opposed to macroeconomics—the economy of a given country or region is treated as a set of separate entities, not as a single organism. In the currently leading trend of neoclassical economics, we often encounter mathematical models of the behaviour of sets of economic entities or agents: consumers, producers, public and private institutions, etc.

The main research areas of microeconomics include:

- consumer demand theory (choices made by consumers),
- production theory (choices made by producers),
- theory of exchange (choices made by consumers and producers),
- market structures (monopoly, duopoly, oligopoly, perfect competition),
- forms of economic activities (perfect competition, imperfect competition),
- partial or general equilibrium in the markets of: consumer goods, production factors or intermediate and end products,

[1] Many Economists, however, do not distinguish such area and in their work refer to either “microeconomics” or “macroeconomics” only.

- the role of public (government, public administration) and non-public institutions (associations, non-public administration) in rational decision-making processes by individual economic agents,
- risk and uncertainty in business activities.

The main goals of microeconomics include:

- analysis and prediction of the behaviour of economic agents operating in a specific economic, technological and social environment,
- analysis and prediction of social interactions between economic agents, resulting from their behaviour,
- analysis of the effects of these interactions from the viewpoint of (1) institutions responsible for management of these interactions or (2) outcomes of these interactions that have less formal character compared to the exchange itself.

Microeconomics is a discipline of economics that abounds in textbooks written by eminent economists with a recognized scientific and professional position in the world.[2] At the same time, this is the starting point for all economists to begin their economic education at the academic level.

The list of microeconomics textbooks released worldwide in English after 2010 is impressively high and proves the constant search for alternative methods of economic education at undergraduate, graduate and doctoral levels. On the Internet, we can easily find an extensive editorial offer taking the shape of various rankings either prepared by experts or based on insights regarding current purchases.

The most useful rankings of this type include: https:/www.wallstreet.orgmojo.com/microeconomics-books Top 10 Microeconomics Books according to WallStreetMojo, https:/bookautority.org/books/new-microeconomics-books 10 Best New Microeconomics Books to Read in 2020 as well as https:/bookautority.org/books/best-microeconomics-books 100 Best Microeconomics Books of All Time, according to CNN, Forbes and Inc-BookAuthority.

Among the many valuable items, we would like to highlight the textbook (Acemoglu et al., 2017) along with additional didactic materials from MyLab Economics that allow you to independently study the problems as well as methods for their description and solution, as outlined in the textbook. As a model business study, we recommend the textbook (Porter, 2008), in which economic education is focused on competences and practical skills based on microeconomic foundations.

The present textbook is based on a series of lectures and workshops in microeconomics conducted by the authors for students of Computer Science and Econometrics at the Poznań University of Economics and Business. It is closely

[2] A classic textbook on microeconomics is an excellent work by one of the greatest French economists, Edmund Malinvaud, Leçons de théorie microéconomique, Dunod, Paris, 1969, which was published in numerous, revised and expanded editions. An example of a highly valued textbook, unfortunately less known in Poland, is D.M. Kreps, A Course in Microeconomic Theory, Princeton University Press, 1990.

related to the textbooks published in Polish (Malaga, 2010, 2012; Malaga & Sobczak, 2020) and a script published in English (Malaga & Sobczak, 2021).

Due to the fact that this lecture opens the cycle of education in the field of economic theory its content is firmly rooted in the traditional, neoclassical trend of microeconomics. Its distinguishing feature, compared to most of the available academic textbooks, is that it relates, to a slightly greater extent, to the knowledge of linear algebra, mathematical analysis, as well as the basics of mathematical economics.

The language of mathematics, like any other type of language, requires knowledge of precisely defined concepts (vocabulary), rules of using them (grammar), the ability to communicate in this language (communication) and, finally, creativity. The ability to use the language of mathematics efficiently should be one of the basic features of a well-educated economist. The more so that, even despite certain limitations resulting from the use of mathematics in economics, today no one will question its usefulness from the perspective of describing economic phenomena and processes, as well as formulating and solving economic problems. However, in order to acquire this ability, it is necessary to start formulating intuitions and economic knowledge in a mathematical language as early as possible. By acting in this way, one can properly recognize the limits of knowledge, which are determined by the quality and type of language that we use to teach and describe the economic reality, including the language of mathematics.

An important supplement to the lecture on microeconomics should be the work of René Descartes "The Discourse on the Method of Proper Managing the Reason and Seeking Truth in Sciences".

Every student of economics and management should have the ability to observe and analyze the surrounding socio-economic reality. In the face of universal access to information, facilitated by the Internet, attention should be paid to the subjectivity of every person in the process of developing an individual way of properly following their head and searching for the truth about the surrounding reality, also in science. One of the elements of this process should include equipping the student of economics and management with the ability to think abstractly and use formal constructions, which, however, should be confronted with the student's knowledge and intuition.

The textbook attempts to harmoniously combine elementary knowledge of traditional microeconomics, linear algebra and mathematical analysis.

For the sake of simplicity and clarity in the lecture, the textbook is limited to case studies in which commodity spaces and factor spaces constitute the $\mathbb{R}^2_+$, metric spaces, whereas the classes of consumers, producers, or markets are two-element or less. This type of simplification allows for a relatively simple generalization of the considerations into metric spaces $\mathbb{R}^n_+$, $n > 2$ and to the collection of consumers, producers and markets with any, but limited, number of entities.

The main goals of this textbook include:

- equipping the reader with the knowledge of fundamental microeconomic categories and the principles of inference using the basic concepts of linear algebra and mathematical analysis,
- preparing and encouraging the reader to study monographs and articles in the field of economic theory, using the conceptual apparatus of more advanced mathematics than linear algebra or mathematical analysis,
- familiarizing the reader with the principles of deductive analysis based on the adopted system of assumptions,
- drawing the reader's attention to the way in which conclusions derived from specific scenarios of behaviour of individual consumers, producers, groups of consumers and producers are conditional and limited by the system of adopted assumptions,
- presentation of the fundamental theoretical achievements of microeconomics in terms of elements of the demand theory, production theory, partial equilibrium theory and general competitive equilibrium theory,
- outlining the key features of contemporary microeconomics and indicating the likely directions of its further development.

The textbook consists of seven chapters, Bibliography, Mathematical appendix, Glossary and Index of terms. This chapter presents an introduction. Chapter 7 presents a conclusion. Each of Chaps. 2–6 is concluded with a set of questions and exercises that the readers may solve on their own.

Chapter 2 deals with the problem of describing the rational behaviour of an individual consumer. In particular, it concerns: the relation of consumer preferences, the utility function as a numerical characteristic of the relation of consumer preferences, the Marshallian demand function understood as the optimal solution for maximizing the utility of consumption, the Hicksian demand function as the optimal solution for minimizing consumer spending, relations between the Marshallian and Hicksian demand functions, as well as the substitution and income effects of changes in commodity prices, discussed on the basis of the Slutsky equation. These considerations are carried out with the assumption that the limited supply of goods does not restrict the consumer when deciding to choose the optimal basket of goods. The said assumption is then rebutted in the exercises at the end of the chapter.

Chapter 3 examines the problem of making rational decisions by a group of consumers. Using the example of the simple exchange model and the static Arrow-Hurwicz model, it presents the Walras general equilibrium law. Considerations presented in the chapter are complemented with presentation of selected issues related to the Arrow-Hurwicz dynamic model in a discrete and continuous version, in particular the problem of global asymptotic stability of the Walrasian general equilibrium. In the considered models of competitive equilibrium, a limited supply of goods does restrict consumers (traders) when making choices of optimal baskets of goods, equated with the Marshallian demand functions.

Chapter 4 is devoted to the description of the rational behaviour of a single producer. The more detailed considerations include: production space, the production function as a set of technologically effective processes, as well as a description of rational decisions made by enterprises based on the criterion of profit maximization or minimization of the costs of production of a given number of units under the conditions of either perfect competition or monopoly, with or without limitations on the resources of production factors at the disposal of each enterprise. In the case of enterprises that make rational choices under perfect competition, the characteristics of optimal decisions have been extended to include the analysis of their sensitivity to changes in the relevant parameters. In the analysed models, the demand for product is limited, but it does not restrict enterprises when making decisions concerning the optimal supply of products.

Chapter 5 deals with rational decisions made by individual producers in the conditions of perfect competition, monopoly or duopoly with exogenously defined demand functions. It considers the following problems: determining the production volume by an enterprise operating in perfect competition with an exogenously defined function of product demand, setting the product price and production volume by a monopolistic enterprise with an exogenous demand function for the product produced by the monopolist, as well as price discrimination by the monopolist which is selling a single product in two different markets. It also addresses the issues of quantitative competition in the Cournot and Stackelberg duopoly models or price competition in the Bertrand duopoly model. The focus is on the problems of partial equilibrium of an enterprise operating in the conditions of perfect competition, monopoly or duopoly, sensitivity of optimal variables to changes in model parameters and comparative analysis of the analysed market structures. Market models discussed in this chapter are characterized by the fact that producers make decisions about the optimal supply of products when the demand for them is limited and described by exogenously defined demand functions specific to each market.

Chapter 6 is devoted to the description and analysis of the rational behaviour of the community of consumers and producers. Considerations conducted in this chapter are based on two different types of models. The first type includes static and dynamic two-commodity market models with exogenously defined product demand and supply functions. The second group of models, on the other hand, includes static and dynamic models of the market of two goods with endogenously defined functions of demand and supply of products, known as Arrow-Debreu-McKenzie general equilibrium models. Due to the important role of time in the theory of economics, the distinguished types of dynamic models are presented in two alternative versions: a discrete one (time is a discrete variable) or in a continuous one (time is a continuous variable). The notion of Walrasian general equilibrium is at the centre of attention. In particular, we consider the problem of existence, uniqueness and asymptotically global stability of Walrasian equilibrium. This chapter is, in a way, the crowning achievement of the textbook. It discusses general equilibrium models in which the demand and supply of goods are limited, and therefore binding for the community of consumers and producers. At the same time, the demand and supply are defined by the exogenous or endogenous functions of product demand and supply.

Chapter 7 presents a conclusion and includes an assessment of the traditional approach to microeconomics, as well as the description of characteristics of contemporary microeconomics and the likely directions of its further development, reaching well beyond the framework of traditional microeconomics.

The thematic scope of the textbook defined in this way fits into the traditional approach to microeconomics, which is a synthesis made within the framework of neoclassical mathematical economics of the nineteenth century marginalism movement and the theory of general equilibrium by L. Walras and V. Pareto. The Nobel laureates in economics, Paul Samuelson (1970) and Hicks (1972), are widely believed to be the precursors of the traditional microeconomics.

While handing over the textbook into the hands of the readers, we would like to thank the Springer Publishing House for its publication, in particular to the following representatives of Springer: Selveraj Vijay Kumar, Dr. Johannes Glaeser, Judith Kripp, Katrin Petermann and Sindhu Sundar. Thank you for your inspiring support and professionalism. We would also like to thank Professors Tomasz Tokarski and Adam Krawiec from the Jagiellonian University in Kraków and Professor Łukasz Lach from the AGH University of Science and Technology in Kraków for reliable and insightful reviews, as well as to the students of Computer Science and Econometrics at the Poznań University of Economics and Business, who had a significant impact on the final form of the textbook.

2 Rationality of Choices Made by Individual Consumer

In this chapter you will learn:

- how to describe preferences of a consumer towards consumption bundles, a utility he/she derives from having a consumption bundle and her/his demand for consumer goods
- what substitute goods, perfect substitutes, complementary goods, perfect complements are
- on the basis of what criteria a consumer chooses an optimal consumption bundle and how he/she makes this choice
- how to formulate a consumption utility maximization problem and a consumer's expenditure minimization problem
- what is a Marshallian demand function and a Hicksian demand function
- how to classify consumer goods according to reactions of the consumer demand for goods to changes in goods' prices and in a consumer's income
- what is stated in Gossen's first and second laws
- what the paths of price expansion of demand and the path of income expansion of demand are
- what relationships are described by the Slutsky equation
- what price, substitution and income effects are.

The subject of our considerations in the whole textbook is an individual and groups of individuals who can play various social roles of workers, producers, employers, owners of material and non-material resources, rentiers or consumers.

The focus on an individual follows the logic of methodological individualism which is based on a belief that in order to understand social reality one has to concentrate on an individual, not on society as a whole. It is because society is an outcome of activities of individuals who undergo various transformations being a result of these activities. Methodological individualism is typical for traditional

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microeconomics and in general for the neoclassical school. It is opposed to holism which focuses on a belief that society is not a simple sum of individuals who form it and that characteristic features of a given society have much influence on the behaviour and activities of the individuals.

The elementary character of the analysis we conduct in this book allows us not to enter into a discussion on anthropological questions when and why one can describe an individual as *homo oeconomicus* or *homo socialis*.[1] We only assume that an individual belongs to the *Homo sapiens* group.

Our study is conducted in terms of analysis of activities of a representative individual since we do not try to identify various types of behaviour and activities of particular individuals. The focus on an individual, as presented in our textbook, is not an expression of a view that economic individualism is a more appropriate approach than holism. We believe that they are complementary to each other unless treated in a doctrinal manner when one of them is given an exaggerated importance.

We begin our analysis by describing rational choices made by individual consumers.[2] The consumer choice is identified with a decision. We call it rational when it is made on the basis of a distinct criterion (usually a single one) and when a consumer is aware of conditions which constrain making a rational and hence an optimal choice.[3]

2.1 Preliminary Terms

For the sake of simplicity we consider rational choices[4] made by an individual consumer on a market of two consumer goods[5] denoted by subscript $i = 1, 2$.

Let us introduce basic terms which set the frame of an analysis we conduct in this chapter and Chap. 3.

[1] A brief explanation of these and other terms is given in the glossary annexed at the end of the textbook.

[2] In fact we consider a representative consumer, her/his behaviour and activities. Conclusions about them have general sense and can be transferred to conclusions about behaviour of a group of consumers.

[3] A rational choice is a decision made on the basis of subjective choice criterion, in conditions constraining this choice.

[4] By that we assume that the most rational choice is the same as the optimal decision. But any choice that satisfies the constraints, called a feasible decision, does not have to be the most rational choice. From further analyses it follows that a set of feasible decisions does not have to be a single element set, in some cases it happens to be a set of infinitely many elements.

[5] The analysis conducted for two goods can be easily generalized to a case of a market with any large but finite number of consumer goods.

Definition 2.1 A **bundle of consumer goods** (a **consumption bundle**) is a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$, in which i -th component $x_i \geq 0$, $i = 1, 2$ indicates a non-negative expressed in physical units amount of i-th good in the consumption bundle $\mathbf{x}$.

Definition 2.2 A **consumer goods space** is a set $X = \mathbb{R}^2_+$ of all bundles of goods available on the market along with a metric specified on it[6]:

$$d_E(\mathbf{x}^1, \mathbf{x}^2) = \sqrt{\sum_{i=1}^{2} (x_i^1 - x_i^2)^2} = \sqrt{(x_1^1 - x_1^2)^2 + (x_2^1 - x_2^2)^2}, \tag{2.1}$$

or

$$d(\mathbf{x}^1, \mathbf{x}^2) = \max_{i=1,2}\{|x_i^1 - x_i^2|\} = \max_{i=1,2}\{|x_1^1 - x_1^2|, |x_2^1 - x_2^2|\}, \tag{2.2}$$

being a measure of distance between two consumption bundles.[7]

Definition 2.3 A **Cartesian product** determined on the goods space $X = \mathbb{R}^2_+$ is such a set:

$$X \times X = \{(\mathbf{x}^1, \mathbf{x}^2) | \mathbf{x}^1 \in X, \mathbf{x}^2 \in X\}, \tag{2.3}$$

of all ordered pairs of consumption bundles in which both bundles (the first one and the second one in the pair) belong to the goods space.

Definition 2.4 A **relation of consumer (weak) preference** is a set:

$$P = \{(\mathbf{x}^1, \mathbf{x}^2) \in X \times X | \mathbf{x}^1 \succsim \mathbf{x}^2\} \subset X \times X, \tag{2.4}$$

of all ordered pairs of consumption bundles in which the first bundle is not worse (weakly preferred) than the second bundle.

[6] A metric is a term defined in Mathematical appendix in Definition A.15.

[7] The first metric is the Euclidean metric. It does not apply to measure distance (differentiation) of those consumption bundles which presents amounts of goods expressed in different physical units. Everyone knows that it is not possible to "add dogs and cats together" and in general to add together any quantities expressed in different units of measure. The other metric, which we call for simplicity non-Euclidean one, does not create any problems regarding volumes of goods.

Definition 2.5 A **relation of consumer strong preference** is a set:

(2.5) $$P_s = \{(\mathbf{x}^1, \mathbf{x}^2) \in X \times X | \mathbf{x}^1 \succ \mathbf{x}^2\} \subset X \times X,$$

of all ordered pairs of consumption bundles in which the first bundle is better (strongly preferred) than the second bundle.

Definition 2.6 A **relation of consumer indifference** is a set:

(2.6) $$I = \{(\mathbf{x}^1, \mathbf{x}^2) \in X \times X | \mathbf{x}^1 \sim \mathbf{x}^2\} \subset X \times X,$$

of all ordered pairs of consumption bundles in which the first bundle is as good (indifferent) as the second bundle.

Note 2.1 $P, P_s, I \subset X \times X$, which means that relations of consumer: weak preference, strong preference and indifference are subsets of the Cartesian product $X \times X$.

Note 2.2 The weak preference relation is a union of the strong preference relation and the indifference relation:

(2.7) $$P = P_s \cup I.$$

Definition 2.7 The relation P of consumer (weak) preference is a relation of a **total preorder**,[8] which means that it is complete and transitive

(2.8) $$\forall \mathbf{x}^1, \mathbf{x}^2 \in X = \mathbb{R}^2_+ \quad \mathbf{x}^1 \succsim \mathbf{x}^2 \vee \mathbf{x}^2 \succsim \mathbf{x}^1 \quad \text{(completeness)},$$

(2.9) $$\forall \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in X = \mathbb{R}^2_+ \quad \mathbf{x}^1 \succsim \mathbf{x}^2 \wedge \mathbf{x}^2 \succsim \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \succsim \mathbf{x}^3 \quad \text{(transitivity)}.$$

The completeness means that a consumer, when asked about her/his preferences with regard to two different consumption bundles, is always able to determine that the first bundle is not worse than the second one or that the second bundle is not worse than the first one. In other words, he/she can also point to the strongly preferred bundle or state that both bundles are equally good (indifferent).

The transitivity means that a consumer is able to order the bundles in terms of her/his preferences.

[8] In Mathematical appendix we give brief descriptions of basic types of preference relations.

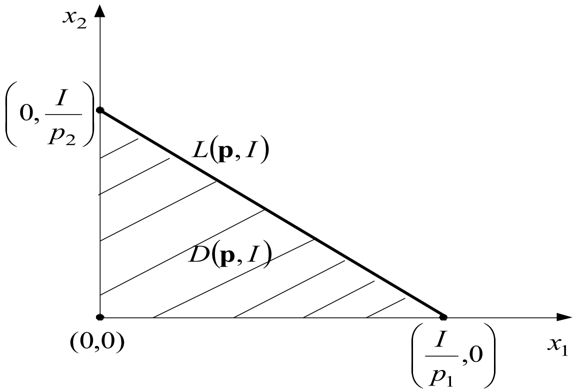


Fig. 2.1 Budget set $D(\mathbf{p}, I)$and budget line $L(\mathbf{p}, I)$ in goods space $X = \mathbb{R}_+^2$

Assumption 2.1 Let us assume that by given prices $\mathbf{p} = (p_1, p_2) \in \operatorname{int} \mathbb{R}_+^2$ of goods and a given consumer's income[9] $I \in \operatorname{int} \mathbb{R}_+$ supply of any good is bounded but always bigger than demand reported by a consumer for this good.[10]

Definition 2.8 A **budget set** is a set[11]:

(2.10) $$D(p_1, p_2, I) = \left\{(x_1, x_2) \in \mathbb{R}_+^2 | p_1 x_1 + p_2 x_2 \leq I\right\} \subset X = \mathbb{R}_+^2$$

of all consumption bundles whose money value, by given prices of consumer goods, is not greater than the consumer's income.

Note 2.3 The budget set is assumed to be not empty,[12] compact (closed and bounded) and convex.

Definition 2.9 A **budget line (budget constraint)** is a set:

(2.11) $$L(p_1, p_2, I) = \left\{(x_1, x_2) \in \mathbb{R}_+^2 | p_1 x_1 + p_2 x_2 = I\right\} \subset X = \mathbb{R}_+^2$$

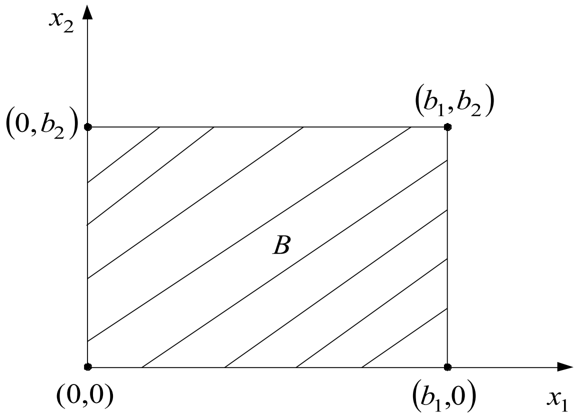
of all consumption bundles whose money value, by given prices of consumer goods, is equal to the consumer's income (Fig. 2.1).

A consumer's goal is to choose such a consumption bundle $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2) \in D(p_1, p_2, I)$ which is the best in the set of all consumption bundles and at the same time whose value is not greater than the consumer's income.

---

[9] Here we do not specify sources of income.
[10] Strictly speaking the supply of all goods is sufficiently big (see Note 2.5).
[11] It is a set of all consumption bundles that can be purchased by a consumer with a given income.
[12] If the budget set $D(\mathbf{p}, I)$ was empty, then an individual consumer would not be able to make a choice of any consumption bundle. As a consequence the only rational decision would be to abandon the choice because of not satisfying conditions which make the choice possible.

Fig. 2.2a Supply set



Definition 2.10 An optimal consumption bundle in the budget set $D(\mathbf{p}, I) \subset X = \mathbb{R}^2_+$ is such a consumption bundle $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2) \in D(\mathbf{p}, I)$ that

(2.12) $$\forall \mathbf{x} = (x_1, x_2) \in D(\mathbf{p}, I) \quad \bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2) \succsim (x_1, x_2) = \mathbf{x}.$$

Note 2.4 The consumption bundle which is optimal,[13] regarding the consumer's preferences, in the budget set $D(\mathbf{p}, I) \subset X = \mathbb{R}^2_+$, is therefore not worse than any other consumption bundle belonging to this set.

Assumption 2.2 Let us assume that by given prices $\mathbf{p} = (p_1, p_2) \in \operatorname{int} \mathbb{R}^2_+$ of goods and a given consumer's income[14] $I \in \operatorname{int} \mathbb{R}_+$ supply of any good is constrained. Hence, it can be lower than, equal to or higher than demand reported by the consumer for a given good.

Definition 2.11 A **supply set** is a set:

(2.13) $$B = \{(x_1, x_2) \in \mathbb{R}^2_+ | x_1 \leq b_1, x_2 \leq b_2\}$$

of all consumption bundles in which quantity x_i of i-th good is not greater than non-negative supply of this good $b_i \geq 0, i = 1, 2$ (Figs. 2.2a, 2.2b and 2.2c).

From the point of view of a consumer, who in fact needs to take into account both supply and budget constraints when choosing the optimal consumption bundle, it is essential what the relationships between the budget and the supply sets are.

[13] Optimal means the best regarding a given optimality criterion. Thus one should not use a name "the most optimal".
[14] Here we do not specify sources of income.

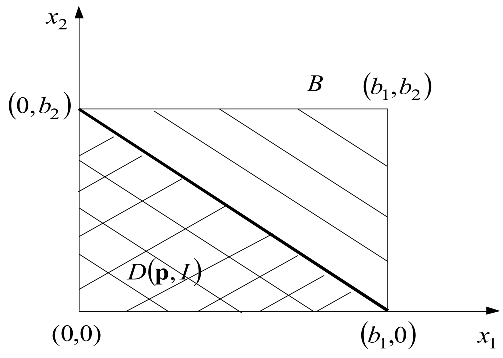


Fig. 2.2b Budget set as subset of supply set when $\frac{I}{p_i} = b_i,\ i = 1, 2$

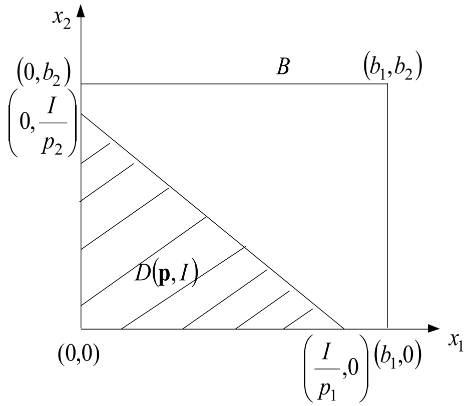


Fig. 2.2c Budget set as subset of supply set when $\frac{I}{p_i} < b_i,\ i = 1, 2$

Let us consider the following cases:

Case 1

The budget set is a subset of the supply set: $D(\mathbf{p}, I) \subseteq B$, which means that the supply of each good is sufficiently big in comparison to the consumer's income. It is the case when:

$$0 < \frac{I}{p_1} \leq b_1 \wedge 0 < \frac{I}{P_2} \leq b_2.$$

Then, the budget constraint is binding and a consumer chooses the optimal consumption bundle in the budget set $D(\mathbf{p}, I)$.

Case 2

The supply set is a proper subset of the budget set: $B \subset D(\mathbf{p}, I)$, which means that the supply of each good is sufficiently small in comparison to the consumer's income. It is the case when:

$$\forall \alpha, \beta \geq 0, \alpha + \beta = 1 \quad 0 < b_1 < \frac{\alpha I}{p_1} \quad \wedge \quad 0 < b_2 < \frac{\beta I}{P_2}.$$

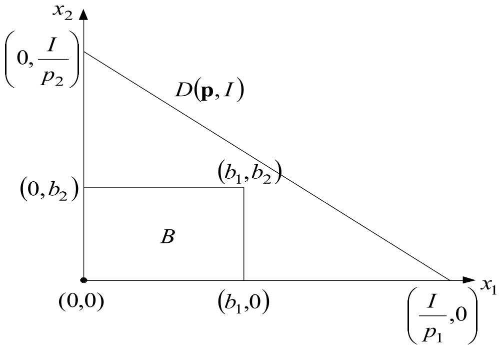


Fig. 2.3 Supply set as subset of budget set when $b_1 \in \left(0; \frac{\alpha I}{p_1}\right) \wedge b_2 \in \left(0; \frac{\beta I}{p_2}\right)$

Then, the supply constraint is binding and a consumer chooses the optimal consumption bundle in the supply set B (Fig. 2.3).

Case 3
The budget and the supply set are not disjoint but at the same time none of them is the proper subset of the other. It is the case when:

$$0 < b_1 < \frac{I}{p_1} \quad \wedge \quad 0 < \frac{I}{P_2} < b_2 \quad \text{or} \quad 0 < \frac{I}{p_1} < b_1 \quad \wedge \quad 0 < b_2 < \frac{I}{P_2}.$$

Then both constraints are binding and a consumer chooses the optimal consumption bundle in a set $B \cap D(\mathbf{p}, I)$ which is not equal to B nor to $D(\mathbf{p}, I)$ (Figs. 2.4a and 2.4b).

Case 4
The budget and the supply set are not disjoint but at the same time none of them is the proper subset of the other. It is the case when:

$$0 < \frac{I - p_2 b_2}{p_1} < b_1 < \frac{I}{p_1} \text{ and } 0 < \frac{I - p_1 b_1}{P_2} < b_2 < \frac{I}{p_2}.$$

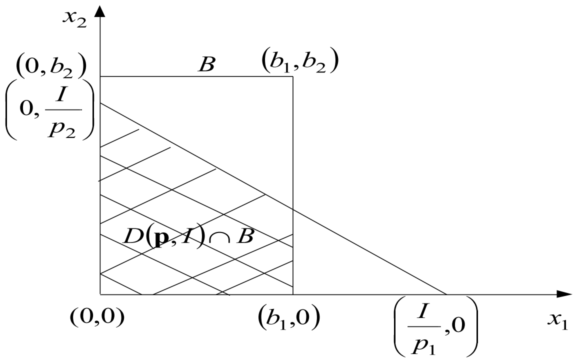


Fig. 2.4a Supply set not being subset of budget set (and conversely) when $0 < b_1 < \frac{I}{p_1} \wedge 0 < \frac{I}{P_2} < b_2$

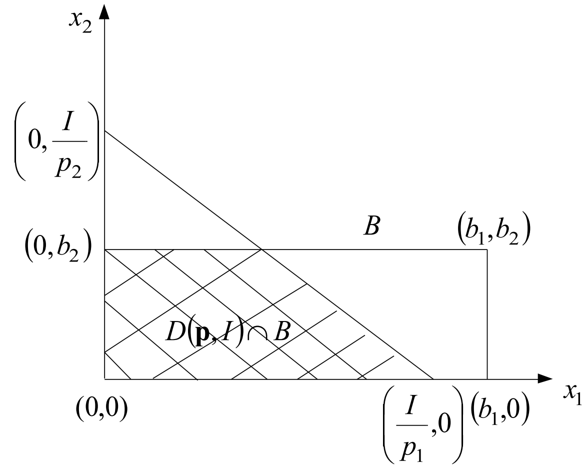


Fig. 2.4b Supply set not being subset of budget set (and conversely) when $0 < \frac{I}{p_1} < b_1 \wedge 0 < b_2 < \frac{I}{P_2}$

Then both constraints are binding and a consumer chooses the optimal consumption bundle in a set $B \cap D(\mathbf{p}, I)$ which is not equal to B nor to $D(\mathbf{p}, I)$ (Fig. 2.5).

Note 2.5 From this moment we further assume in Chap. 1 that a consumer choosing the optimal consumption bundle is not bounded by the supply of goods. It is the case

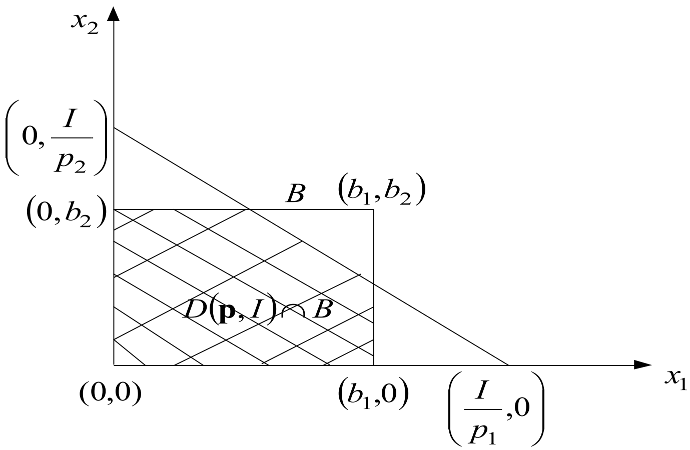


Fig. 2.5 Supply set not being subset of budget set (and conversely) when $0 < \frac{I - p_2 b_2}{p_1} < b_1 < \frac{I}{p_1}$ and $0 < \frac{I - p_1 b_1}{p_2} < b_2 < \frac{I}{p_2}$

when the supply of each good is not smaller than the demand reported for this good. However, it does not mean that the supply of goods is unbounded. The matter of how the binding supply constraint influences the choice of the optimal consumption bundle is considered in the exercises given at the end of this chapter.

2.2 Utility Function

Definition 2.12 A **consumer's utility function** (defined on the goods space $X = \mathbb{R}^2_+$) is a mapping $u:\mathbb{R}^2_+ \to \mathbb{R}$ such that

(2.14) $$\forall \mathbf{x}^1, \mathbf{x}^2 \in X = \mathbb{R}^2_+ \quad \mathbf{x}^1 \succsim \mathbf{x}^2 \Leftrightarrow u\left(\mathbf{x}^1\right) \geq u\left(\mathbf{x}^2\right),$$

(2.15) $$\forall \mathbf{x}^1, \mathbf{x}^2 \in X = \mathbb{R}^2_+ \quad \mathbf{x}^1 \succ \mathbf{x}^2 \Leftrightarrow u\left(\mathbf{x}^1\right) > u\left(\mathbf{x}^2\right),$$

(2.16) $$\forall \mathbf{x}^1, \mathbf{x}^2 \in X = \mathbb{R}^2_+ \quad \mathbf{x}^1 \sim \mathbf{x}^2 \Leftrightarrow u\left(\mathbf{x}^1\right) = u\left(\mathbf{x}^2\right).$$

Some properties of the utility function

Definition 2.13 A utility function $u:\mathbb{R}^2_+ \to \mathbb{R}$ is called **continuous at point x** $\in \mathbb{R}^2_+$ if for any sequence $\left\{\mathbf{x}^i\right\}_{i=1}^{+\infty}$, where $\mathbf{x}^i \in X = \mathbb{R}^2_+$, it is satisfied:

(2.17) $$\lim_{i \to +\infty} \mathbf{x}^i = \mathbf{x} \Rightarrow \lim_{i \to +\infty} u\left(\mathbf{x}^i\right) = u(\mathbf{x}).$$

Definition 2.14 A utility function $u:\mathbb{R}^2_+ \to \mathbb{R}$ is called **continuous on the goods space** $X = \mathbb{R}^2_+$ if it is continuous at every point of this space.

Definition 2.15 A utility function $u:\mathbb{R}^2_+ \to \mathbb{R}$ is called **differentiable on the goods space** $X = \mathbb{R}^2_+$ if its partial first-order derivatives:

(2.18) $$\frac{\partial u(x_1, x_2)}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1},$$

(2.19) $$\frac{\partial u(x_1, x_2)}{\partial x_2} = \lim_{\Delta x_2 \to 0} \frac{u(x_1, x_2 + \Delta x_2) - u(x_1, x_2)}{\Delta x_2}$$

are continuous on this space.

Definition 2.16 A **marginal utility of *i*-th good** in a consumption bundle $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ is a partial first-order derivative of the utility function:

(2.20) $$\frac{\partial u(x_1, x_2)}{\partial x_i} \quad i = 1, 2,$$

which describes by approximately how many units the utility of a consumption bundle $\mathbf{x} \in \mathbb{R}_+^2$ changes when quanity of i-th good increases by one (notional) unit and quantity of the other good in the bundle does not change.

Definition 2.17 A utility function $u{:}\mathbb{R}_+^2 \to \mathbb{R}$ is called **twice differentiable on the goods space** $X = \mathbb{R}_+^2$ if its partial second-order derivatives:

$$\text{(2.21)} \qquad \frac{\partial^2 u(x_1, x_2)}{\partial x_i^2}, \frac{\partial^2 u(x_1, x_2)}{\partial x_i \partial x_j} = \frac{\partial^2 u(x_1, x_2)}{\partial x_j \partial x_i} \quad i, j = 1, 2, \quad i \neq j,$$

are continuous on this space.

Note 2.6 If a utility function $u{:}\mathbb{R}_+^2 \to \mathbb{R}$ is twice differentiable and:

$$\text{(2.22)} \qquad \forall \mathbf{x} \in X = \mathbb{R}_+^2 \quad \frac{\partial^2 u(x_1, x_2)}{\partial x_i^2} < 0 \quad i = 1, 2,$$

then it means that the marginal utility of the i-th good decreases with an increase of quantity of this good in a consumption bundle $\mathbf{x} \in \mathbb{R}_+^2$. This property is called **first Gossen's law (law of decreasing marginal utility).**

Example 2.1 Justify by geometric and analytical means that for a logarithmic utility function:

(a) $u(x) = a \ln x, \quad a > 0, \quad x \in \mathbb{R}_+,$
(b) $u(x_1, x_2) = a_1 \ln x_1 + a_2 \ln x_2, \quad a_i > 0, \quad x_i \in \mathbb{R}_+, \quad i = 1, 2,$

the first Gossen's law is satisfied.

Ad (a) The logarithmic one-variable utility function is increasing since $\frac{\mathrm{d}u(x)}{\mathrm{d}x} = \frac{a}{x} > 0$ (Figs. 2.6a and 2.6b).

The first-order derivative of this utility function is decreasing since $\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = -\frac{a}{x^2} < 0$ (Figs. 2.6b and 2.6c). This means that the marginal utility of a good decreases with an increase of its quantity.

Graphs and forms of the first- and second-order derivatives show that the first Gossen's law is satisfied for a logarithmic function $u\colon \mathrm{int}\,\mathbb{R}_+ \to \mathbb{R}$.

Ad (b) The logarithmic two-variable utility function is increasing in each variable since $\frac{\partial u(\mathbf{x})}{\partial x_i} = \frac{a}{x_i} > 0, \quad i = 1, 2$ (Figs. 2.7a and 2.7b).

The first-order partial derivative of this utility function is decreasing in the quantity of i-th good since $\frac{\partial^2 u(\mathbf{x})}{\partial x_i^2} = -\frac{a}{x_i^2} < 0, \quad i = 1, 2$ (Figs. 2.7b and 2.7c). This means that the marginal utility of i-th good decreases with an increase of its quantity in a consumption bundle $\mathbf{x} \in \mathbb{R}_+^2$.

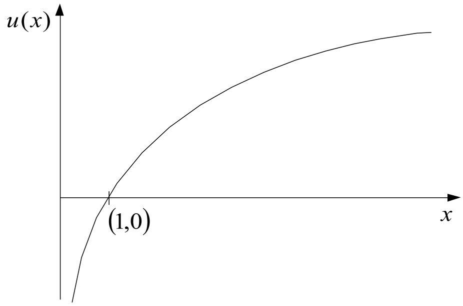


Fig. 2.6a Graph of utility function $u(x) = a \ln x, \quad a > 0, \quad x \in \mathbb{R}_+$

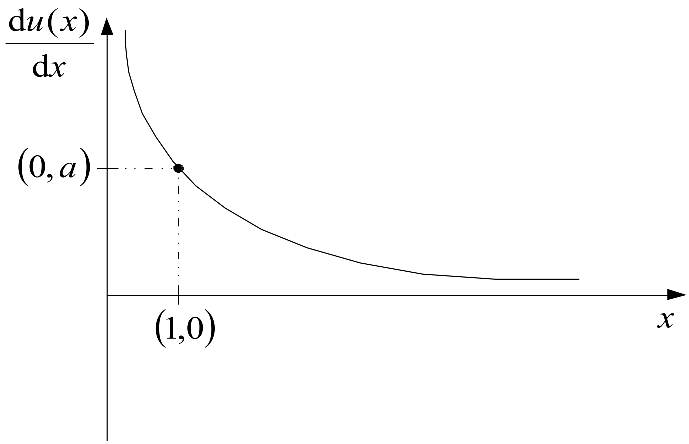
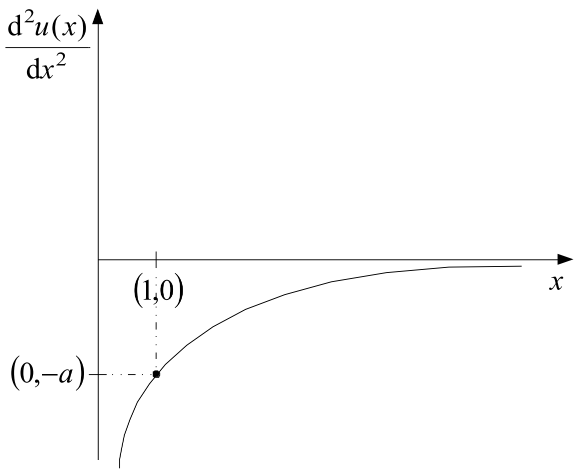


Fig. 2.6b Graph of marginal utility function with respect to the quantity of goods

Fig. 2.6c Graph of second-order derivative of utility function



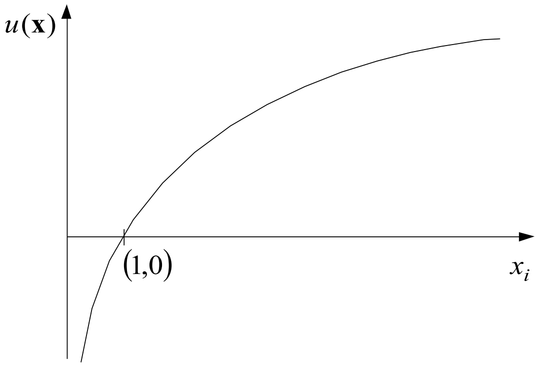


Fig. 2.7a Graph of a projection of logarithmic utility function on plane

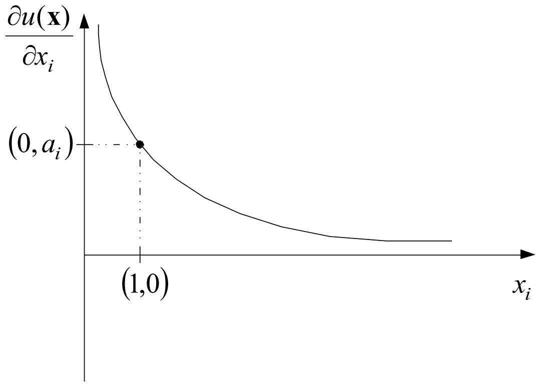


Fig. 2.7b Graph of a projection of marginal utility function on plane

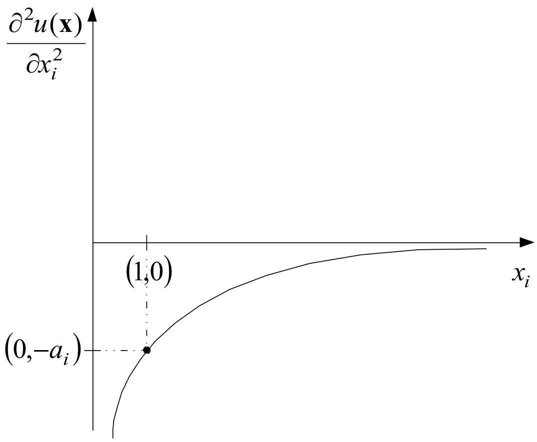


Fig. 2.7c Graph of second-order partial derivative of utility function with respect to the quantity of i-th good

Graphs and forms of the first- and second-order partial derivatives show that the first Gossen's law is satisfied for a two-variable logarithmic function $u\colon \text{int}\,\mathbb{R}^2_+ \to \mathbb{R}$.

Definition 2.18 A utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ is called:

(a) **concave** in $\mathbb{R}^2_+$ if:

$$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2_+ \quad \forall \alpha, \beta \geq 0, \alpha + \beta = 1 \quad u(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2) \geq \alpha u(\mathbf{x}^1) + \beta u(\mathbf{x}^2),$$

(b) **convex** in $\mathbb{R}^2_+$ if:

$$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2_+ \quad \forall \alpha, \beta \geq 0, \alpha + \beta = 1 \qquad u(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2) \leq \alpha u(\mathbf{x}^1) + \beta u(\mathbf{x}^2),$$

(c) **strictly concave** in $\mathbb{R}^2_+$ if:

$$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2_+, \mathbf{x}^1 \neq \mathbf{x}^2 \quad \forall \alpha, \beta > 0, \; \alpha + \beta = 1$$
$$u(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2) > \alpha u(\mathbf{x}^1) + \beta u(\mathbf{x}^2),$$

(d) **strictly convex** in $\mathbb{R}^2_+$ if:

$$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2_+, \mathbf{x}^1 \neq \mathbf{x}^2 \quad \forall \alpha, \beta > \; 0, \alpha + \beta = 1$$
$$u(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2) < \; \alpha u(\mathbf{x}^1) + \beta u(\mathbf{x}^2).$$

Definition 2.19 A utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ is called:

(a) **increasing** in $\mathbb{R}^2_+$ if[15] $\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2_+ \quad \mathbf{x}^1 \gneqq \mathbf{x}^2 \Rightarrow u(\mathbf{x}^1) > u(\mathbf{x}^2)$,
(b) **decreasing** in $\mathbb{R}^2_+$ if $\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2_+ \quad \mathbf{x}^1 \gneqq \mathbf{x}^2 \Rightarrow u(\mathbf{x}^1) < u(\mathbf{x}^2)$,
(c) **weakly increasing** in $\mathbb{R}^2_+$ if $\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2_+ \quad \mathbf{x}^1 \gneqq \mathbf{x}^2 \Rightarrow u(\mathbf{x}^1) \geq u(\mathbf{x}^2)$,
(d) **weakly decreasing** in $\mathbb{R}^2_+$ if $\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2_+ \quad \mathbf{x}^1 \gneqq \mathbf{x}^2 \Rightarrow u(\mathbf{x}^1) \leq u(\mathbf{x}^2)$.

Note 2.7 If a utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ is differentiable on its domain, then it is (Table 2.1):

(a) increasing, when $\frac{\partial u(\mathbf{x})}{\partial x_i} > 0, i = 1, 2,$
(b) decreasing, when $\frac{\partial u(\mathbf{x})}{\partial x_i} < 0, i = 1, 2,$
(c) weakly increasing, when $\frac{\partial u(\mathbf{x})}{\partial x_i} \geq 0, i = 1, 2,$
(d) weakly decreasing, when $\frac{\partial u(\mathbf{x})}{\partial x_i} \leq 0, i = 1, 2.$

[15] An inequality $\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n_+ \quad \mathbf{x}^1 \gneqq \mathbf{x}^2$ means that at least one component x_i^1 of a vector $\mathbf{x}^1$ is bigger than the corresponding component x_i^2 of a vector $\mathbf{x}^2$ while the other components x_j^1 ($j = 1, 2, \ldots, n, j \neq i$, here $n = 2$) are bigger or equal to corresponding components x_j^2.

Table 2.1 Examples of utility functions*

Type of a utility function	$u: \mathbb{R}_+ \to \mathbb{R}$	$u: \mathbb{R}_+^2 \to \mathbb{R}$
Linear	$u(x) = ax + b$ $a, b > 0$	$u(x_1, x_2) = a_1x_1 + a_2x_2$ $a_i > 0,\ i = 1, 2$
Power function	$u(x) = ax^\alpha$ $a, \alpha > 0$	$u(x_1, x_2) = ax_1^{\alpha_1}x_2^{\alpha_2}$ $a, \alpha_i > 0,\ i = 1, 2$
Logarithmic	$u(x) = ax + b$ $a > 0, x > 0$	$u(x_1, x_2) = a_1 \ln x_1 + a_2 \ln x_2$ $a_i > 0, x_i > 0,\ i = 1, 2$
Subadditive	–	$u(x_1, x_2) = a_1x_1^\alpha + a_2x_2^\alpha$ $a_i, \alpha > 0,\ i = 1, 2$
Koopmans-Leontief function	$u(x) = \min ax = ax$ $a > 0$	$u(x_1, x_2) = \min\{a_1x_1, a_2x_2\}$ $a_i > 0,\ i = 1, 2$
CES (constant elasticity of substitution) function	$u(x) = ax^\theta$ $a, \theta > 0$	$u(x_1, x_2) = \left(a_1x_1^\gamma + a_2x_2^\gamma\right)^{\frac{\theta}{\gamma}}$ $\theta, a_i > 0, i = 1, 2$ $\gamma \in (-\infty; 0) \cup (0; 1)$

* In case of two-variable functions it is assumed that the free term $b = 0$

Note 2.8 The utility function is just a numerical characteristic of consumer's preferences. It means that for every consumption bundle there is some real (not necessarily positive) number assigned. A utility of a consumption bundle determined in this way is useful if it allows to determine whether utilities of two different consumption bundles are the same or different. Absolute values of utilities of two bundles are not important, just a comparison of these values is, because a consumer compares bundles according to her/his preferences, not determining numerical relationships amongst bundles.

One can notice then, as a conclusion, that if there exists some utility function describing a relation of consumer preference then any function, derived as a composition of this utility function and any increasing function, is also a utility function describing the same relation of consumer preference. In other words, there exist infinitely many utility functions describing the same relation of consumer preference and each of these functions assigns a different number to a given consumption bundle.

Let us assume that a power function $u: \mathbb{R}_+^2 \to \mathbb{R}$, of a form $u(\mathbf{x}) = ax_1^{\alpha_1}x_2^{\alpha_2}$, describes a preference relation of a consumer. From the definition of a utility function it follows that:

$$\forall \mathbf{x}^1, \mathbf{x}^2 \in X = \mathbb{R}_+^2 \qquad \mathbf{x}^1 \succsim \mathbf{x}^2 \Leftrightarrow u\left(\mathbf{x}^1\right) \geq u\left(\mathbf{x}^2\right). \tag{2.23}$$

Let us consider a logarithmic function $g: \operatorname{int} \mathbb{R}_+ \to \mathbb{R}$, of a form $g(x) = \ln x$, which, as we know, is increasing. Then, a function $v: \mathbb{R}_+^2 \to \mathbb{R}$, derived as a composition $g \circ u$ of the power utility function u and the increasing logarithmic function g, is also a utility function and describes the same relation of consumer preference as the function u. It has a form:

$$\begin{aligned}\nu(\mathbf{x}) &= (g \circ u)(\mathbf{x}) = g(u(\mathbf{x})) = \ln(u(\mathbf{x})) = \ln\left(a x_1^{\alpha_1} x_2^{\alpha_2}\right) \\ &= \ln a + \alpha_1 \ln x_1 + \alpha_2 \ln x_2\end{aligned}$$

and is a logarithmic utility function describing the same relation of consumer preference as the function u, because:

(2.24) $$\forall \mathbf{x}^1, \mathbf{x}^2 \in X = \mathbb{R}_+^2 \qquad \mathbf{x}^1 \succsim \mathbf{x}^2 \Leftrightarrow u\left(\mathbf{x}^1\right) \geq u\left(\mathbf{x}^2\right) \Leftrightarrow \nu\left(\mathbf{x}^1\right) \geq \nu\left(\mathbf{x}^2\right).$$

Let us notice that there is a constant $\ln a > 0$ in the formula of a function ν and that it matters for value of the utility function but it does not matter for determining which of two consumption bundles $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}_+^2$ is not worse than the other. Hence, it is more convenient to handle a logarithmic utility function of a form[16]:

(2.25) $$\omega(\mathbf{x}) = \nu(\mathbf{x}) - \ln a = \ln \frac{u(\mathbf{x})}{a} = \ln\left(x_1^{\alpha_1} x_2^{\alpha_2}\right) = \alpha_1 \ln x_1 + \alpha_2 \ln x_2.$$

Definition 2.20 An indifference **curve for the reference utility** u_0 by a utility function $u\colon \mathbb{R}_+^2 \to \mathbb{R}$ is a set:

(2.26) $$G(u_0) = \left\{(x_1, x_2) \in X = \mathbb{R}_+^2 | u(x_1, x_2) = u_0 = \text{const.}\right\},$$

of all these consumption bundles whose utility is the same and equal to $u_0 = \text{const.}$

Example 2.2 Sketch graphs of a two-variable utility function which is: linear, power, logarithmic, subadditive, Koopmans-Leontief, in space $\mathbb{R}_+^3$ and graphs of indifference curves $G(u) = \left\{(x_1, x_2) \in X = \mathbb{R}_+^2 | u(x_1, x_2) = u = \text{const.}\right\}$ in the goods space $X = \mathbb{R}_+^2$ (Figs. 2.8a, 2.8b, 2.8c, 2.8d, 2.8e, 2.8f, 2.8g and 2.8h).

Let us present these selected characteristics of the utility function which are important in the consumer theory. Definitions of absolute and relative increments of values of one-variable or two-variable utility functions are given in Table 2.2.

We are given a utility function $u\colon \mathbb{R}_+^2 \to \mathbb{R}$ about which we know that it is differentiable.

Definition 2.21 A **marginal utility of i-th good** in a consumption bundle $\mathbf{x} \in \mathbb{R}_+^2$ is a partial first-order derivative of the utility function:

(2.27) $$T_i(\mathbf{x}) = \lim_{\Delta x_i \to 0} \frac{u\left(x_i + \Delta x_i, x_j\right) - u(x_i, j)}{\Delta x_i} = \frac{\partial u(\mathbf{x})}{\partial x_i} \quad i, j = 1, 2, \ i \neq j$$

[16] We have shown that in the formula of the logarithmic utility function, derived as a composition of the power utility function and the increasing logarithmic function, there are parameters equal to exponents of the power utility function. However, when using a general form of a logarithmic utility function, we denote its parameters as a_i instead of α_i ($a_i, \alpha_i > 0, i = 1, 2$).

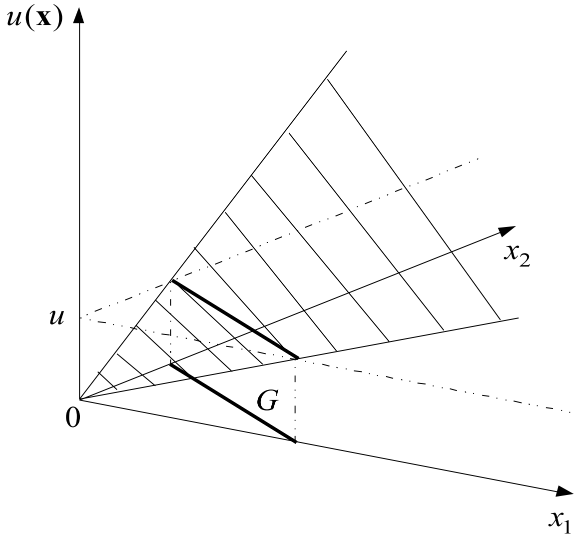


Fig. 2.8a Graph of linear utility function and its indifference curve for $u(\mathbf{x}) = u > 0$

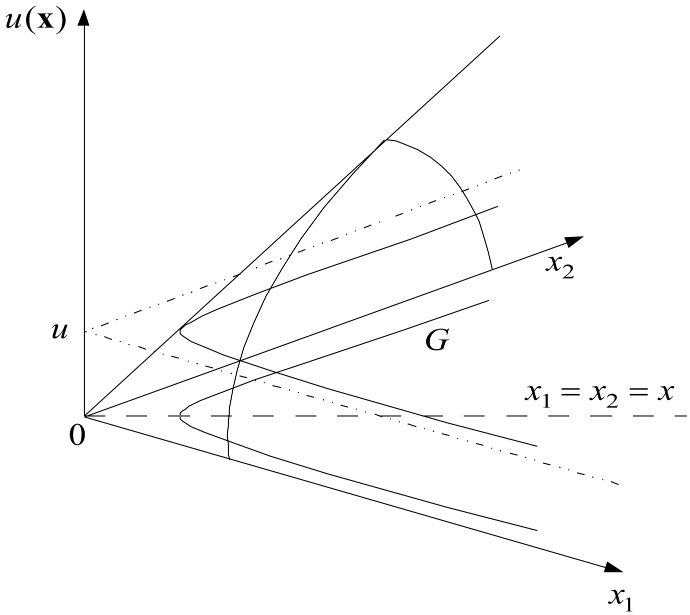


Fig. 2.8b Graph of a power utility function and its indifference curve for $u(\mathbf{x}) = u > 0, \alpha_1 + \alpha_2 = 1$

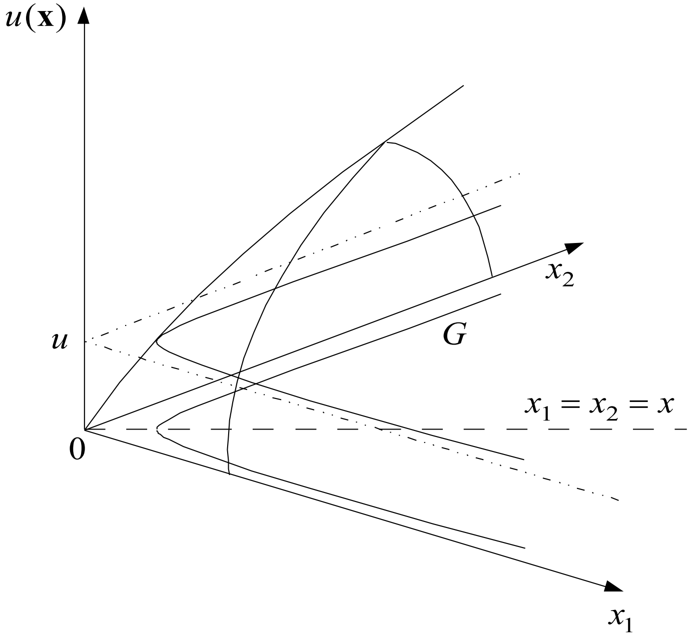


Fig. 2.8c Graph of power utility function and its indifference curve for $u(\mathbf{x}) = u > 0, \alpha_1 + \alpha_2 < 1$

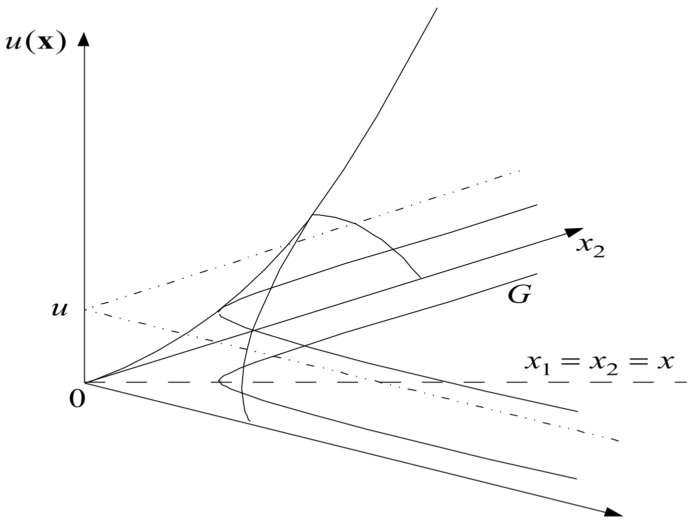


Fig. 2.8d Graph of power utility function and its indifference curve for $u(\mathbf{x}) = u > 0, \alpha_1 + \alpha_2 > 1$

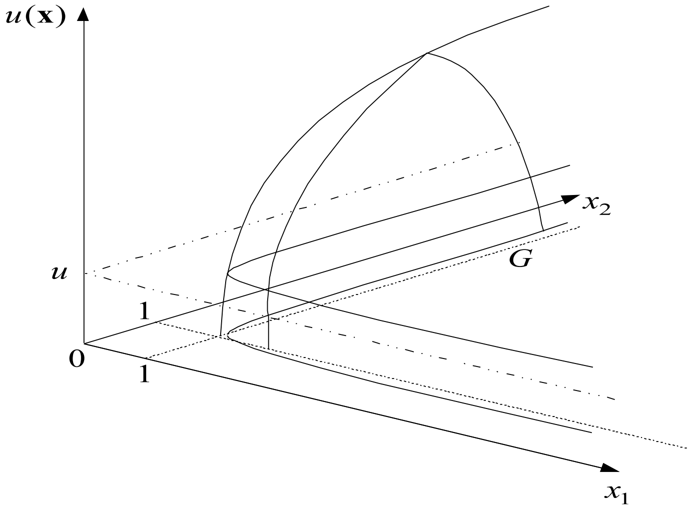


Fig. 2.8e Graph of logarithmic utility function and its indifference curve for $u(\mathbf{x}) = u$

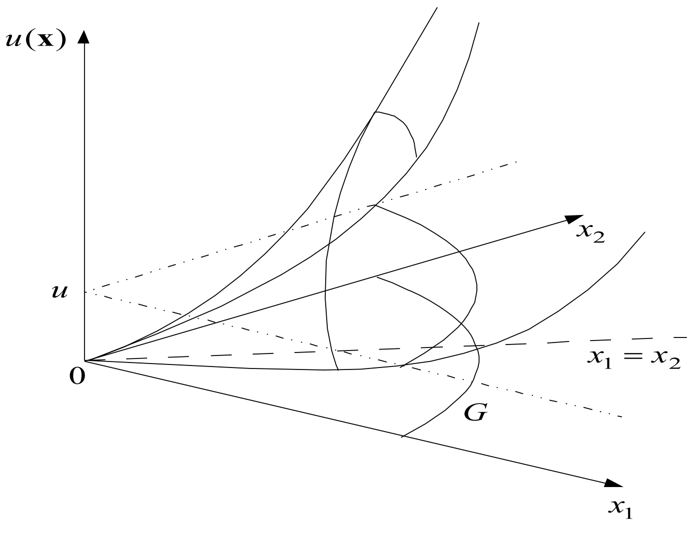


Fig. 2.8f Graph of subadditive utility function[17] and its indifference curve for $u(\mathbf{x}) = u$, $\alpha > 1$

which describes by approximately how many units the utility of a consumption bundle $\mathbf{x} \in \mathbb{R}^2_+$ changes (increases, decreases, or remains unchanged) when quantity

[17] When $\alpha = 1$ then the subadditive utility function is linear.

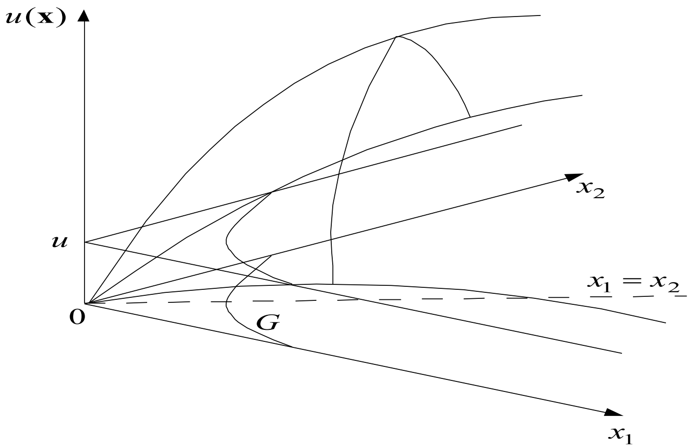


Fig. 2.8g Graph of subadditive utility function and its indifference curve for $u(\mathbf{x}) = u$, $\alpha < 1$

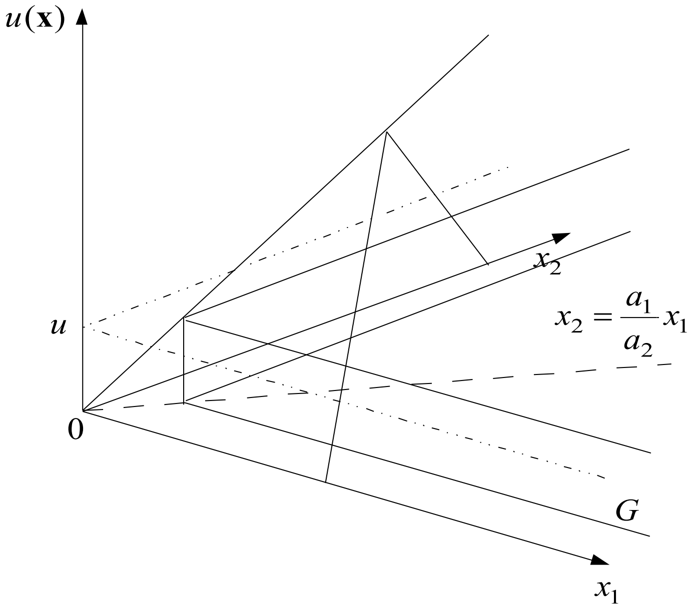


Fig. 2.8h Graph of Koopmans-Leontief utility function and its indifference curve for $u(\mathbf{x}) = u > 0$

Table 2.2 Absolute and relative increments of values of utility functions

	$u\colon \mathbb{R}_+ \to \mathbb{R}$	$u\colon \mathbb{R}^2_+ \to \mathbb{R}$
Absolute increment	$\Delta u = u(x + \Delta x) - u(x)$ $\Delta x \in \mathbb{R}$	$\Delta_1 u = u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)$ $\Delta_2 u = u(x_1, x_2 + \Delta x_2) - u(x_1, x_2)$
Relative increment	$\frac{\Delta u}{\Delta x} = \frac{u(x+\Delta x)-u(x)}{\Delta x}$	$\frac{\Delta_1 u}{\Delta x_1} = \frac{u(x_1+\Delta x_1,x_2)-u(x_1,x_2)}{\Delta x_1}$ $\frac{\Delta_2 u}{\Delta x_2} = \frac{u(x_1,x_2+\Delta x_2)-u(x_1,x_2)}{\Delta x_2}$

of i-th good increases by one (notional) unit and quantity of the other good in the bundle does not change.

Definition 2.22 A **growth rate**[18] **of consumption bundle utility** with respect to quantity of i-th good in a bundle $\mathbf{x} \in \mathbb{R}^2_+$ is an expression:

$$\begin{aligned} S_i(\mathbf{x}) &= \lim_{\Delta x_i \to 0} \frac{u\big(x_i + \Delta x_i, x_j\big) - u\big(x_i, x_j\big)}{\Delta x_i} \cdot \frac{1}{u(\mathbf{x})} \\ &= \frac{\partial u(\mathbf{x})}{\partial x_i} \cdot \frac{1}{u(\mathbf{x})} = \frac{T_i(\mathbf{x})}{u(\mathbf{x})} \quad i, j = 1, 2, \quad i \neq j \end{aligned} \tag{2.28}$$

which describes by approximately what % the utility of a consumption bundle $\mathbf{x} \in \mathbb{R}^2_+$ changes (increases, decreases, or remains unchanged) when quantity of i-th good increases by one (notional) unit and quantity of the other good in the bundle does not change.

Definition 2.23 An **elasticity of consumption bundle utility** with respect to quantity of i-th good in a bundle $\mathbf{x} \in \mathbb{R}^2_+$ is an expression:

$$\begin{aligned} E_i(\mathbf{x}) &= \lim_{\Delta x_i \to 0} \frac{u\big(x_i + \Delta x_i, x_j\big) - u\big(x_i, x_j\big)}{\Delta x_i} \cdot \frac{x_i}{u(\mathbf{x})} \\ &= \frac{\partial u(\mathbf{x})}{\partial x_i} \cdot \frac{x_i}{u(\mathbf{x})} = S_i(\mathbf{x}) \cdot x_i \quad i, j = 1, 2, \; i \neq j \end{aligned} \tag{2.29}$$

which describes by approximately what % the utility of a consumption bundle $\mathbf{x} \in \mathbb{R}^2_+$ changes (increases, decreases, or remains unchanged) when quantity of i-th good increases by 1% and quantity of the other good in the bundle does not change.

[18] The growth rate of a utility function is a term used very rarely in the consumer theory.

Note 2.9 The growth rate and the elasticity of consumption bundle utility with respect to quantity of i-th good in a bundle $\mathbf{x} \in \mathbb{R}^2_+$ are measures of a relative increment of utility function value. The increment is caused, respectively, by one-unit or by 1% increase in quantity of i-th good in the consumption bundle $\mathbf{x}$ (Table 2.3).

Note 2.10 The Koopmans-Leontief utility function is not differentiable, thus is not possible to determine the marginal utility, the growth rate, and the elasticity of utility using Definitions 2.21–2.23

2.3 Substitute, Independent and Complementary Goods

The substitutability concerns only these consumption bundles $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ whose utility is the same.

Any two goods are called **substitute goods (substitutes)** if in order to keep some given level of utility of a consumption bundle $\mathbf{x} \in \mathbb{R}^2_+$ when quantity of one of the goods is reduced (raised) one needs to compensate this change by appropriate increase (decrease) in quantity of the other good in a consumption bundle $\mathbf{x}$.

If in order to keep some given level of utility of a consumption bundle $\mathbf{x} \in \mathbb{R}^2_+$ when quantity of one of the goods is reduced (raised) one does need to compensate for this change by any increase (decrease) in quantity of the other good in a consumption bundle $\mathbf{x}$, then such two consumer goods are called **independent goods**.

Any two goods are called **complementary goods (complements)** if in order to change the utility of a given consumption bundle $\mathbf{x} \in \mathbb{R}^2_+$ one needs to simultaneously raise or reduce quantities of both goods in the bundle.

Note 2.11 When classifying consumption bundles in substitutes or independent goods one takes into account only these goods which are considered in bundles with the same utility level (bundles indifferent to each other).

Note 2.12 To state if any two goods in a consumption bundle $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ are complements to each other one needs to determine whether an increase (decrease) in the utility of this bundle requires a simultaneous raise (reduction) in quantities of both goods. If there is no such need then the goods are called not complementary.

Let us define measures of the substitutability of consumer goods. For this purpose let us assume that we are given:

(1) a differentiable utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$,

Table 2.3 Marginal utilities, growth rates, and elasticities[19] of consumption bundle utility with respect to quantity of i-th good in bundle $\mathbf{x} \in \mathbb{R}_+^2$ for selected utility functions

Type of utility function $u:\mathbb{R}_+^2 \to \mathbb{R}$	Marginal utility	Growth rate of utility	Elasticity of utility
Linear $u(\mathbf{x}) = a_1 x_1 + a_2 x_2 \;\; a_i > 0,\; i = 1,2$	$T_i(\mathbf{x}) = a_i$ $i = 1,2$	$S_i(\mathbf{x}) = \frac{a_i}{a_1+a_2}$ $i = 1,2$	$E_i(\mathbf{x}) = \frac{a_i x_i}{a_1+a_2}$ $i = 1,2$
Power function $u(\mathbf{x}) = a x_1^{\alpha_1} x_2^{\alpha_2} \;\; a, \alpha_i > 0,\; i = 1,2$	$T_1(\mathbf{x}) = \alpha_1 a x_1^{\alpha_1 - 1} x_2^{\alpha_2}$ $T_2(\mathbf{x}) = \alpha_2 a x_1^{\alpha_1} x_2^{\alpha_2 - 2}$	$S_1(\mathbf{x}) = \frac{\alpha_1}{x_1}$ $S_2(\mathbf{x}) = \frac{\alpha_2}{x_2}$	$E_1(\mathbf{x}) = \alpha_1$ $E_2(\mathbf{x}) = \alpha_2$
Logarithmic $u(\mathbf{x}) = \sum_{i=1}^2 a_i \ln x_i$ $a_i > 0, \quad x_i \in \operatorname{int} \mathbb{R}_+, \;\; i = 1,2$	$T_i(\mathbf{x}) = \frac{a_i}{x_i}$ $i = 1,2$	$S_i(\mathbf{x}) = \frac{a_i}{x_i \sum_{i=1}^2 a_i \ln x_i}$ $i = 1,2$	$E_i(\mathbf{x}) = \frac{a_i}{\sum_{i=1}^2 a_i \ln x_i}$ $i = 1,2$
Subadditive $u(\mathbf{x}) = \sum_{i=1}^2 a_i x_i^{\alpha} \;\; a_i, \alpha > 0,\; i = 1,2$	$T_i(\mathbf{x}) = \alpha a_i x_i^{\alpha - 1}$ $i = 1,2$	$S_i(\mathbf{x}) = \frac{\alpha a_i x_i^{\alpha-1}}{\sum_{i=1}^2 a_i x_i^{\alpha}}$ $i = 1,2$	$E_i(\mathbf{x}) = \frac{\alpha a_i x_i^{\alpha}}{\sum_{i=1}^2 a_i x_i^{\alpha}}$ $i = 1,2$
CES $u(\mathbf{x}) = \left(\sum_{i=1}^2 a_i x_i^{\gamma}\right)^{\frac{\theta}{\gamma}} \;\; a_i > 0, \quad i = 1,2$ $\theta > 0,\; \gamma \in (-\infty; 0) \cup (0; 1)$	$T_i(\mathbf{x}) = \theta\left(\sum_{i=1}^2 a_i x_i^{\gamma}\right)^{\frac{\theta}{\gamma}-1} a_i x_i^{\gamma-1}$ $i = 1,2$	$S_i(\mathbf{x}) = \frac{\theta\left(\sum_{i=1}^2 a_i x_i^{\gamma}\right)^{\frac{\theta}{\gamma}-1} a_i x_i^{\gamma-1}}{(a_1 x_1^{\gamma} + a_2 x_2^{\gamma})^{\frac{\theta}{\gamma}}}$ $i = 1,2$	$E_i(\mathbf{x}) = \frac{\theta\left(\sum_{i=1}^2 a_i x_i^{\gamma}\right)^{\frac{\theta}{\gamma}-1} a_i x_i^{\gamma}}{(a_1 x_1^{\gamma} + a_2 x_2^{\gamma})^{\frac{\theta}{\gamma}}}$ $i = 1,2$

[19] The characteristics presented in Table 2.3 are scalar and two-variable functions of quantities of goods in a consumption bundle $\mathbf{x} \in \mathbb{R}_+^2$. However, one should remember that economic interpretation concerns values of these functions, but not the functions themselves.

(2) an indifference curve - a set

$$G(u) = \{\mathbf{x} \in \mathbb{R}^2_+ | u(\mathbf{x}) = u = \text{const.} > 0\}. \tag{2.30}$$

of all bundles with the same reference utility $u = \text{const.} > 0$.

Theorem 2.1 An indifference curve $G(u) = \{\mathbf{x} \in \mathbb{R}^2_+ | u(\mathbf{x}) = u = \text{const.} > 0\}$ is given. Then there exists a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ of a form $x_2 = g(x_1)$ which describes the relationship between quantities of the second and of the first good in any consumption bundle with the same reference utility $u = \text{const.} > 0$.

Example 2.3 Sketch graphs of a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ when a utility function is (Figs. 2.9a, 2.9b, 2.9c, 2.9d, 2.9e, 2.9f and 2.9g):

1. linear: $u(\mathbf{x}) = a_1x_1 + a_2x_2 = u \Leftrightarrow x_2 = g(x_1) = \frac{u - a_1x_1}{a_2}$,
2. power function: $u(\mathbf{x}) = ax_1^{\alpha_1}x_2^{\alpha_2} = u \Leftrightarrow x_2 = g(x_1) = \left(\frac{u}{a}\right)^{\frac{1}{\alpha_2}} x_1^{-\frac{\alpha_1}{\alpha_2}}$,
3. logarithmic: $u(\mathbf{x}) = a_1 \ln x_1 + a_2 \ln x_2 = u \Leftrightarrow x_2 = g(x_1) = e^{\frac{u}{a_2}} x_1^{-\frac{a_1}{a_2}}$,
4. subadditive: $u(\mathbf{x}) = a_1x_1^{\alpha} + a_2x_2^{\alpha} = u \Leftrightarrow x_2 = g(x_1) = \left(\frac{u - a_1x_1^{\alpha}}{a_2}\right)^{\frac{1}{\alpha}}$,
5. Koopmans-Leontief function: $u(\mathbf{x}) = \min\{a_1x_1, a_2x_2\} = u \Leftrightarrow x_2 = g(x_1) = \frac{u}{a_2}$,
6. CES function: $u(\mathbf{x}) = \left(a_1x_1^{\gamma} + a_2x_2^{\gamma}\right)^{\frac{\theta}{\gamma}} = u \Leftrightarrow x_2 = g(x_1) = \left(\frac{u^{\frac{\gamma}{\theta}} - a_1x_1^{\gamma}}{a_2}\right)^{\frac{1}{\gamma}}$.

Definition 2.24 A **marginal rate of substitution of the first good by the second good** in a consumption bundle $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ with a utility level $u(\mathbf{x}) = \text{const.} > 0$ is an expression:

$$s_{12}(x_1, x_2) = -\lim_{\substack{\Delta x_1 \to 0 \\ \Delta x_1 < 0}} \frac{\Delta x_2}{\Delta x_1} \cong -\frac{dx_2}{dx_1}, \tag{2.31}$$

which describes by approximately how many units one should raise the quantity of the second good in a consumption bundle $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ when the quantity of the first good has been reduced by one (notional) unit, in order to keep the consumption bundle utility unchanged (Table 2.4).

Note 2.13 The sign "–" in Definition 2.24 results from the fact that the first good quantity has been reduced.

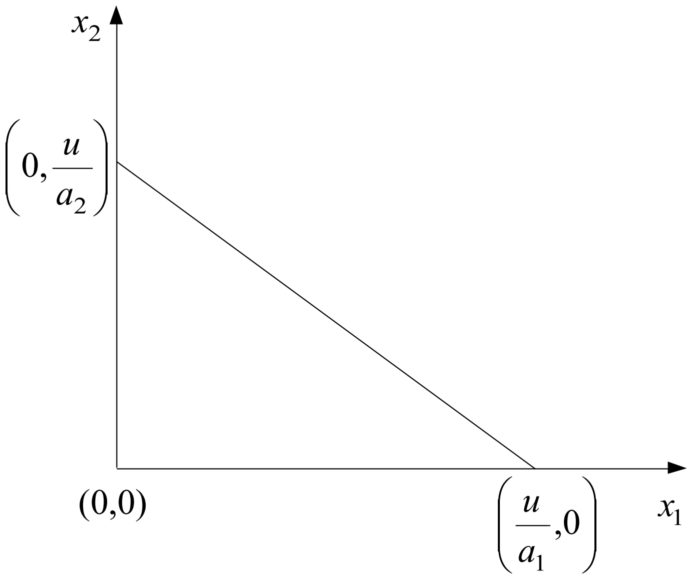


Fig. 2.9a Graph of function $g: \mathbb{R}_+ \to \mathbb{R}_+$ related to linear utility function

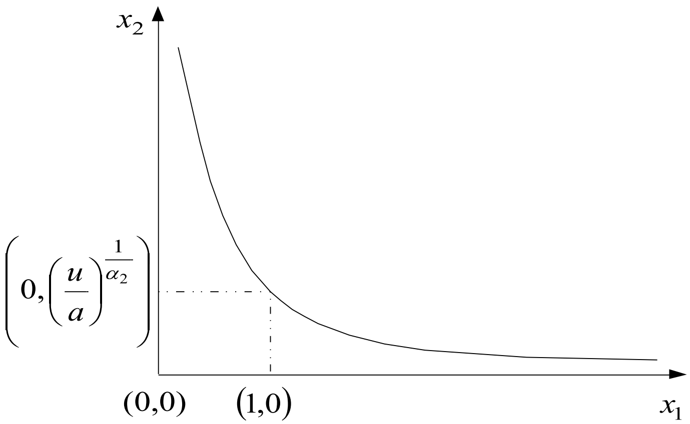


Fig. 2.9b Graph of function $g: \mathbb{R}_+ \to \mathbb{R}_+$ related to power utility function

Note 2.14 If a utility function $u(x_1, x_2)$ and a reference utility $u = \text{const.} > 0$ are given, then a total differential of a given value of the utility function is of a form:

(2.32)
$$\mathrm{d}u = \frac{\partial u(x_1, x_2)}{\partial x_1}\mathrm{d}x_1 + \frac{\partial u(x_1, x_2)}{\partial x_2}\mathrm{d}x_2.$$

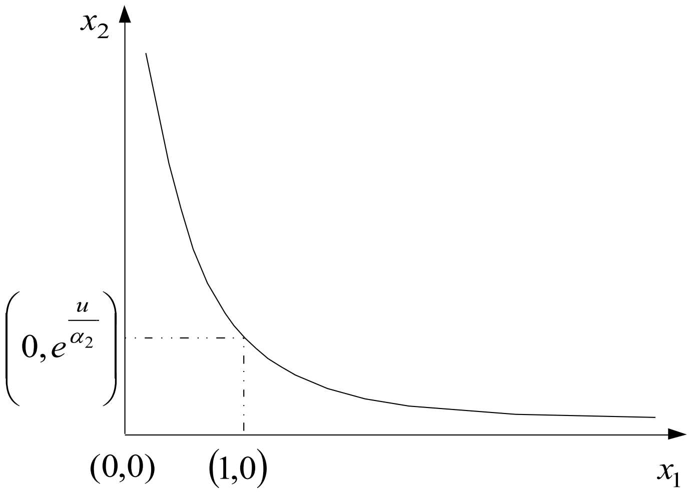


Fig. 2.9c Graph of function $g: \mathbb{R}_+ \to \mathbb{R}_+$ related to logarithmic utility function

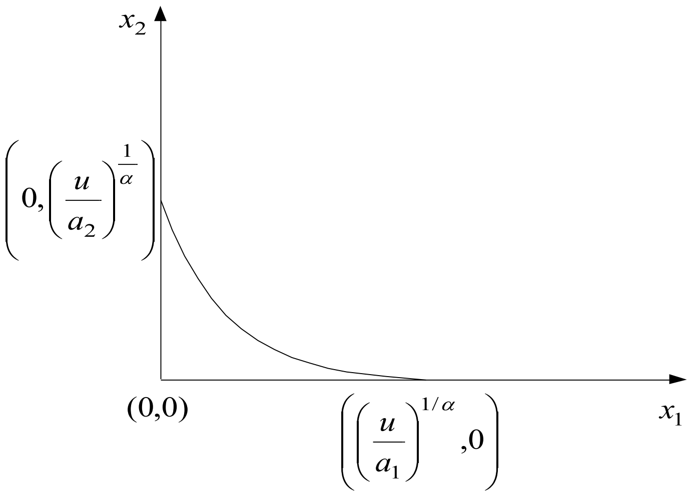


Fig. 2.9d Graph of function $g: \mathbb{R}_+ \to \mathbb{R}_+$ related to subadditive utility function

Since we are interested in consumption bundles with unchanged utility level, then:

(2.33) $$\mathrm{d}u = 0 \Leftrightarrow \frac{\partial u(x_1, x_2)}{\partial x_1}\mathrm{d}x_1 + \frac{\partial u(x_1, x_2)}{\partial x_2}\mathrm{d}x_2 = 0$$

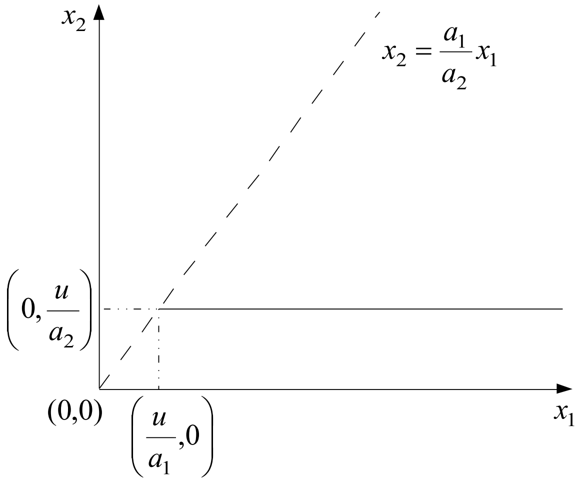


Fig. 2.9e Graph of function $g: \mathbb{R}_+ \to \mathbb{R}_+$ related to Koopmans-Leontief utility function

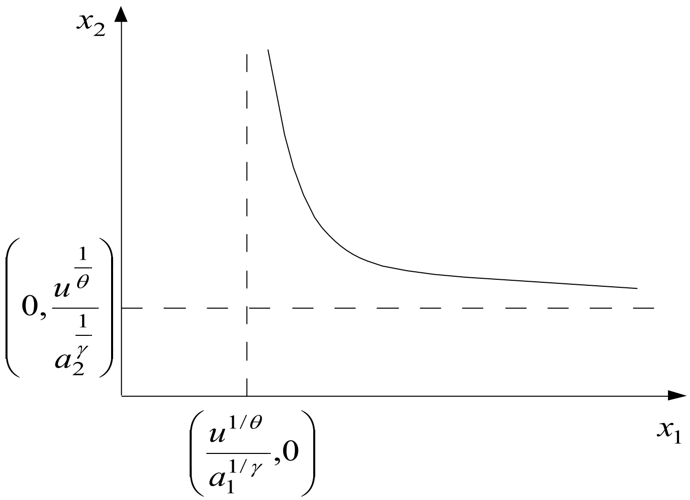


Fig. 2.9f Graph of function $g: \mathbb{R}_+ \to \mathbb{R}_+$ related to CES utility function when $\gamma \in (0; +\infty)$

and hence we get

$$s_{12}(x_1, x_2) = -\frac{dx_2}{dx_1} = \frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}}$$

(2.34) or

$$s_{21}(x_1, x_2) = -\frac{dx_1}{dx_2} = \frac{\frac{\partial u(x_1,x_2)}{\partial x_2}}{\frac{\partial u(x_1,x_2)}{\partial x_1}}.$$

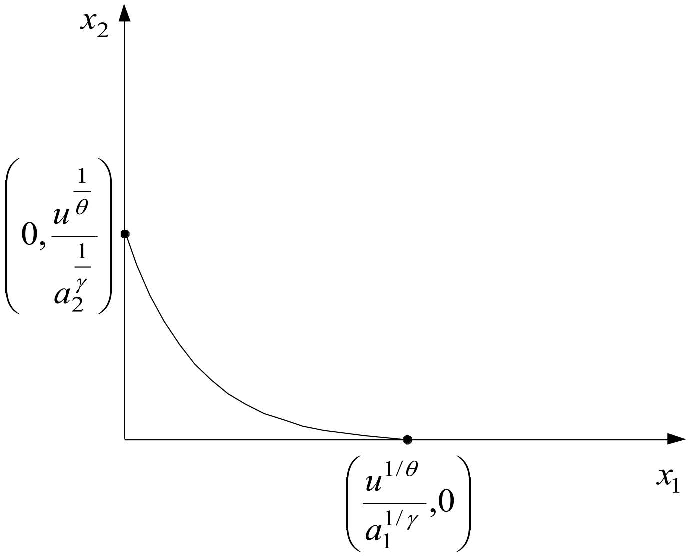


Fig. 2.9g Graph of function $g: \mathbb{R}_+ \to \mathbb{R}_+$ related to CES utility function when $\gamma \in (-1; 0)$

Example 2.4 In view of Example 2.2 sketch a graph of the marginal rate of substitution of the first good by the second good in a consumption bundle $\mathbf{x}^0 \in \mathbb{R}^2_+$, for a utility function which is: linear, power, subadditive, or logarithmic (Figs. 2.10a, 2.10b, 2.10c and 2.10d).

Let us notice that for all utility functions considered above, except the linear one, we have $dx_1 = \Delta x_1$ and $dx_2 \cong \Delta x_2$. Hence, $s_{12}(\mathbf{x}) \cong -\frac{dx_2}{dx_1} = \text{tg}\alpha$. It means that the measure of substitution of the first good by the second good, in a consumption bundle $\mathbf{x}^0 \in \mathbb{R}^2_+$ with a given utility level $u(\mathbf{x}^0) = u > 0$, is approximately equal to a tangent of an angle α between the tangent line at point $\mathbf{x}^0$ and the horizontal axis.[20]

Theorem 2.2 If a utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ is differentiable in $\text{int}\,\mathbb{R}^2_+$, then:

$$s_{12}(\mathbf{x}) = \frac{\frac{\partial u(\mathbf{x})}{\partial x_1}}{\frac{\partial u(\mathbf{x})}{\partial x_2}} = \left(\frac{\frac{\partial u(\mathbf{x})}{\partial x_2}}{\frac{\partial u(\mathbf{x})}{\partial x_1}}\right)^{-1} = \frac{1}{s_{21}(\mathbf{x})}. \tag{2.35}$$

Note 2.15 The marginal rate of substitution of the first good by the second good is equal to the ratio of the marginal utility of the first good and the marginal utility of the second good in a consumption bundle $\mathbf{x} \in G \subset \mathbb{R}^2_+$.

[20] In the case of the linear utility function the marginal rate of substitution $s_{12}(\mathbf{x})$ of the 1st good by the 2nd good is exactly equal to $-\frac{dx_2}{dx_1} = \text{tg}\alpha$. In the case of other utility functions, for which the substitutability can also be regarded, this equality is just an approximation.

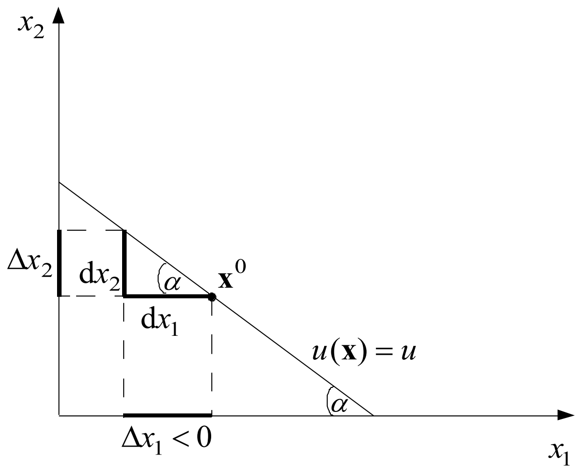


Fig. 2.10a Marginal rate of substitution of first good by second good in consumption bundle $\mathbf{x}^0 \in \mathbb{R}^2_+$ for linear utility function

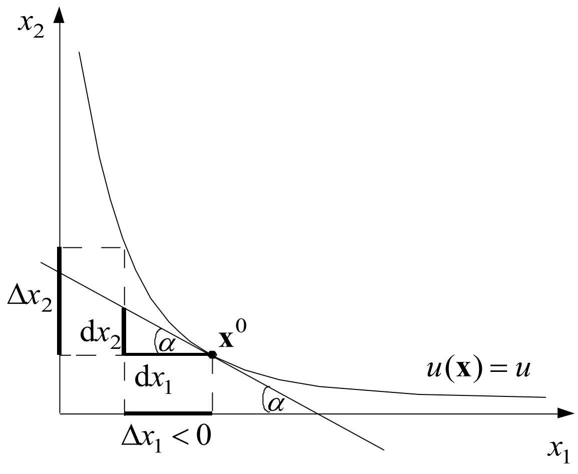


Fig. 2.10b Marginal rate of substitution of first good by second good in consumption bundle $\mathbf{x}^0 \in \mathbb{R}^2_+$ for power utility function

Note 2.16 The marginal rate of substitution of the second good by the first good is equal to a ratio of the marginal utility of the second good and the marginal utility of the first good in a consumption bundle $\mathbf{x} \in G \subset \mathbb{R}^2_+$.

Note 2.17 The marginal rate of substitution of the second good by the first good is equal to an inverse of the marginal rate of substitution of the first good by the second good in a consumption bundle $\mathbf{x} \in G \subset \mathbb{R}^2_+$.

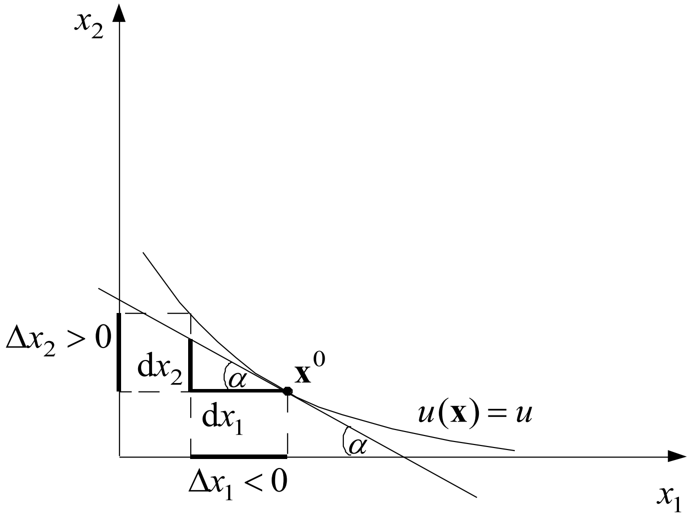


Fig. 2.10c Marginal rate of substitution of first good by second good in consumption bundle $\mathbf{x}^0 \in \mathbb{R}^2_+$ for logarithmic utility function

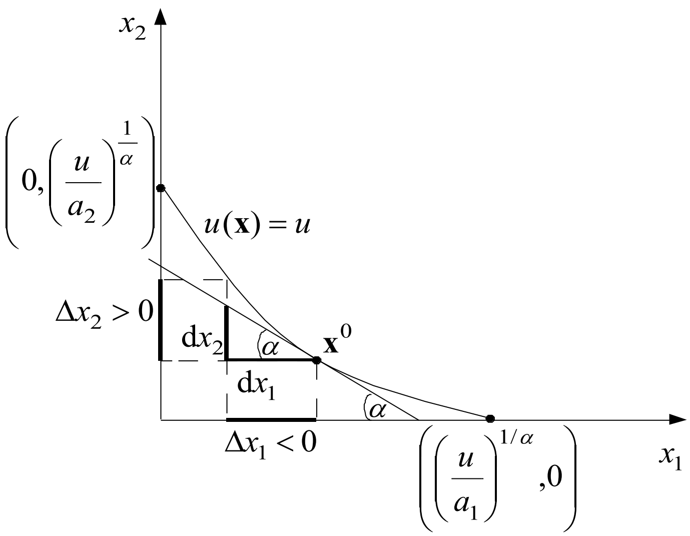


Fig. 2.10d Marginal rate of substitution of first good by second good in consumption bundle $\mathbf{x}^0 \in \mathbb{R}^2_+$ for subadditive utility function

Table 2.4 Formulas of the marginal rate of substitution and elasticity of substitution of consumer goods in consumption bundle $\mathbf{x} \in G \subset \mathbb{R}^2_+$, for selected utility functions

Type of a utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$	Marginal rate of substitution of goods	Elasticity of substitution of goods
Linear $u(\mathbf{x}) = a_1x_1 + a_2x_2$ $a_i > 0,\ i = 1, 2$	$s_{12}(\mathbf{x}) = \frac{a_1}{a_2}$ $s_{21}(\mathbf{x}) = \frac{a_2}{a_1}$	$\varepsilon_{12}(\mathbf{x}) = \frac{a_1x_1}{a_2x_2}$ $\varepsilon_{21}(\mathbf{x}) = \frac{a_2x_2}{a_1x_1}$
Power function $u(\mathbf{x}) = ax_1^{\alpha_1}x_2^{\alpha_2}$ $a, \alpha_i > 0,\ i = 1, 2$	$s_{12}(\mathbf{x}) = \frac{\alpha_1x_2}{\alpha_2x_1}$ $s_{21}(\mathbf{x}) = \frac{\alpha_2x_1}{\alpha_1x_2}$	$\varepsilon_{12}(\mathbf{x}) = \frac{\alpha_1}{\alpha_2}$ $\varepsilon_{21}(\mathbf{x}) = \frac{\alpha_2}{\alpha_1}$
Logarithmic $u(\mathbf{x}) = a_1 \ln x_1 + a_2 \ln x_2$ $a_i > 0, x_i \in \operatorname{int} \mathbb{R}_+,$ $i = 1, 2$	$s_{12}(\mathbf{x}) = \frac{a_1x_2}{a_2x_1}$ $s_{21}(\mathbf{x}) = \frac{a_2x_1}{a_1x_2}$	$\varepsilon_{12}(\mathbf{x}) = \frac{a_1}{a_2}$ $\varepsilon_{21}(\mathbf{x}) = \frac{a_2}{a_1}$
Subadditive $u(\mathbf{x}) = a_1x_1^{\alpha} + a_2x_2^{\alpha}$ $a_i, \alpha > 0,\ i = 1, 2$	$s_{12}(\mathbf{x}) = \frac{a_1}{a_2}\left(\frac{x_1}{x_2}\right)^{\alpha-1}$ $s_{21}(\mathbf{x}) = \frac{a_2}{a_1}\left(\frac{x_2}{x_1}\right)^{\alpha-1}$	$\varepsilon_{12}(\mathbf{x}) = \frac{a_1}{a_2}\left(\frac{x_1}{x_2}\right)^{\alpha}$ $\varepsilon_{21}(\mathbf{x}) = \frac{a_2}{a_1}\left(\frac{x_2}{x_1}\right)^{\alpha}$
CES $u(\mathbf{x}) = \left(a_1x_1^{\gamma} + a_2x_2^{\gamma}\right)^{\frac{\theta}{\gamma}}$ $\theta, a_i > 0, i = 1, 2$ $\gamma \in (-1; 0) \cap (0; +\infty)$	$s_{12}(\mathbf{x}) = \frac{a_1x_1^{\gamma-1}}{a_2x_2^{\gamma-1}}$ $s_{21}(\mathbf{x}) = \frac{a_2x_2^{\gamma-1}}{a_1x_1^{\gamma-1}}$	$\varepsilon_{12}(\mathbf{x}) = \frac{a_1x_1^{\gamma}}{a_2x_2^{\gamma}}$ $\varepsilon_{21}(\mathbf{x}) = \frac{a_2x_2^{\gamma}}{a_1x_1^{\gamma}}$

Definition 2.25 An **elasticity of substitution of the first good by the second good** in a consumption bundle $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ with a utility level $u(\mathbf{x}) = \text{const.} > 0$ is an expression:

$$(2.36) \qquad \varepsilon_{12}(x_1, x_2) = -\lim_{\substack{\Delta x_1 \to 0 \\ \Delta x_1 < 0}} \frac{\frac{\Delta x_2}{x_2}}{\frac{\Delta x_1}{x_1}} = -\lim_{\substack{\Delta x_1 \to 0 \\ \Delta x_1 < 0}} \frac{\Delta x_2}{\Delta x_1} \cdot \frac{x_1}{x_2} \cong -\frac{\mathrm{d}x_2}{\mathrm{d}x_1} \cdot \frac{x_1}{x_2},$$

which describes by approximately what % one should raise the quantity of the second good in a consumption bundle $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ when the quantity of the first good has been reduced by 1%, in order to keep the consumption bundle utility unchanged[21] (Table 2.4).

[21] The marginal rate of substitution and the elasticity of substitution are functions of quantities of consumer goods. Their economic interpretation concerns values of these functions for a given consumption bundle.

Note 2.18 The Koopmans-Leontief utility function is not differentiable, thus it is not possible to determine the marginal rate of substitution and the elasticity of substitution using Definitions 2.24 and 2.25.

Example 2.5 We are given:

1. a utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$,
2. an indifference curvea set $G(u) = \left\{\mathbf{x} \in \mathbb{R}^2_+ | u(\mathbf{x}) = u = \text{const.} > 0\right\}$ of all bundles with the same reference utility $u = \text{const.} > 0$.

Justify by geometric and analytical means that:

(a) a linear utility function $u(x_1, x_2) = a_1x_1 + a_2x_2, \quad a_i > 0, i = 1, 2$ describes consumer goods which are "perfect" substitutes and not complementary to each other,
(b) a Koopmans-Leontief utility function $u(x_1, x_2) = \min\{a_1x_1, a_2x_2\}, \quad a_i > 0, i = 1, 2$ describes consumer goods which are "perfect" complements and not substitute for each other.

Ad (a) Let us notice that $\mathrm{d}x_1 = \Delta x_1$ and $\mathrm{d}x_2 = \Delta x_2$. Since $s_{12}(\mathbf{x}) = -\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \mathrm{tg}\alpha = \text{const.}$ we see that the marginal rate of substitution of the first (second) good by the second (first) in a consumption bundle $\mathbf{x}^0 \in \mathbb{R}^2_+$, whose utility is described by the linear utility function, is constant. Thus, it does not depend on quantities of goods in the bundle. In this case the goods are called **perfect substitutes**. At the same time, we can notice that in order to rise (reduce) the utility level of any consumption bundle $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ it is enough to increase (decrease) the quantity of just one of the goods, not necessarily of both goods. This shows that the linear utility function describes goods that are not complementary to each other (Fig. 2.11a).

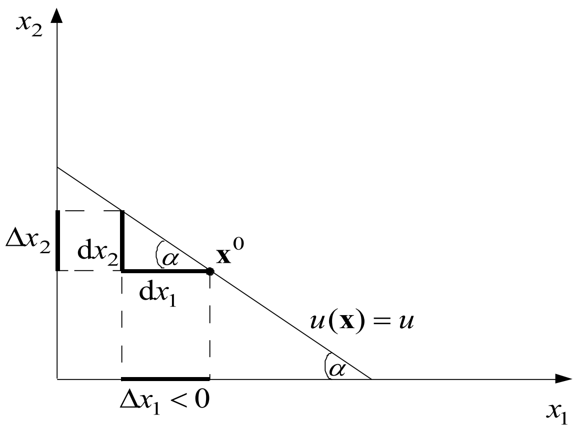


Fig. 2.11a Marginal rate of substitution of first good by second good in consumption bundle $\mathbf{x}^0 \in G \subset \mathbb{R}^2_+$ for linear utility function

Ad (b) Let us notice that $\mathrm{d}x_1 = \Delta x_1$ and $\mathrm{d}x_2 = \Delta x_2 = 0$. Since $s_{12}(\mathbf{x}) \cong -\frac{\mathrm{d}x_2}{\mathrm{d}x_1}$ we see that the marginal rate of substitution of the first good by the second in a consumption bundle $\mathbf{x}^0 \in \mathbb{R}^2_+$ is equal to $s_{12}(\mathbf{x}^0) = 0$, while that the marginal rate of substitution of the second good by the first good is undefined (its value is infinite). This shows that the Koopmans-Leontief utility function describes goods which are not substitute for each other. At the same time, we can notice that in order to rise (reduce) the utility level of any consumption bundle $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ it is necessary to increase (decrease) quantities of both goods in some fixed proportion, not just one of the goods. In this case the goods are called **perfect complements** (Fig. 2.11b).

Note 2.19 The substitutability of consumer goods takes place when the utility of a consumption bundle $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ is described by an increasing or by a decreasing utility function.

Note 2.20 The complementarity of consumer goods takes place when the utility of a consumption bundle $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ is described by a weakly increasing or by a weakly decreasing utility function.

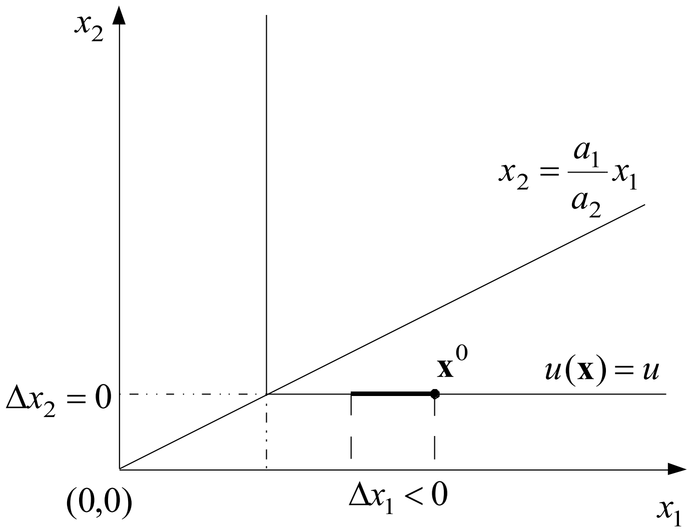


Fig. 2.11b Marginal rate of substitution of first good by second good in consumption bundle $\mathbf{x}^0 \in G \subset \mathbb{R}^2_+$ for Koopmans-Leontief utility function

2.4 Marshallian Demand Function

2.4.1 Static Approach

Let us consider a market for two consumer goods where:

$i = 1, 2$—consumer goods (products and services),
$X = \mathbb{R}^2_+$—a goods space,
$\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2_+$—a vector of prices of consumer goods,
$\mathbf{x} = (x_1, x_2) \in B \subset \mathbb{R}^2_+$—a bundle of goods that the consumer wants to purchase (a consumption bundle),
$B = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+ | x_1 \leq b_1, x_2 \leq b_2\}$—a supply set,
$b_i, i = 1, 2$—supply of i-th consumer good,[22]
$I \in \operatorname{int} \mathbb{R}_+$—a consumer's income,[23]
$u{:}\mathbb{R}^2_+ \to \mathbb{R}$—a utility function describing the preferences of a consumer (describing a relation of consumer preference).
$D(\mathbf{p}, I) = \left\{\mathbf{x} \in \mathbb{R}^2_+ | p_1x_1 + p_2x_2 \leq I\right\} \subset X = \mathbb{R}^2_+$—a set of all consumption bundles whose value is not greater than the consumer's income (a budget set),

Definition 2.26 A bundle $\mathbf{x}$ is called a limit of a sequence $\left\{\mathbf{x}^i\right\}_{i=1}^{+\infty}$ if a limit of sequence of metric values $\lim_{i \to +\infty} d(\mathbf{x}^i, \mathbf{x}) = 0$, which can be written as

$$\lim_{i \to +\infty} \mathbf{x}^i = \mathbf{x} \quad \text{or} \quad \mathbf{x}^i \to_{i \to +\infty} \mathbf{x}. \tag{2.37}$$

Definition 2.27 The budget set $D(\mathbf{p}, I) \subset X = \mathbb{R}^2_+$ is a **closed set** because:

$$\forall\, \mathbf{x}^i \in D(\mathbf{p}, I) \quad \lim_{i \to +\infty} \mathbf{x}^i = \mathbf{x} \Rightarrow \mathbf{x} \in D(\mathbf{p}, I). \tag{2.38}$$

Definition 2.28 The budget set $D(\mathbf{p}, I) \subset X = \mathbb{R}^2_+$ is a **bounded set** because:

$$\forall \mathbf{x}^1, \mathbf{x}^2 \in D(\mathbf{p}, I) \quad \exists \mathrm{N} > 0 \quad d\left(\mathbf{x}^1, \mathbf{x}^2\right) < N. \tag{2.39}$$

Definition 2.29 The budget set $D(\mathbf{p}, I) \subset X = \mathbb{R}^2_+$ is a **compact set** because is closed and bounded.

Definition 2.30 The budget set $D(\mathbf{p}, I) \subset X = \mathbb{R}^2_+$ is a **convex set** because:

$$\forall \mathbf{x}^1, \mathbf{x}^2 \in D(\mathbf{p}, I) \quad \forall \alpha, \beta \geq 0, \alpha + \beta = 1 \quad \alpha\mathbf{x}^1 + \beta\mathbf{x}^2 \in D(\mathbf{p}, I). \tag{2.40}$$

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[22] Later in this chapter we assume that the supply of each good is bounded but sufficiently big so that it is not a constraint for a consumer when he/she chooses his/her optimal consumption bundle.
[23] Here we do not focus on what the source of this income is.

Definition 2.31 A consumption bundle $\bar{\mathbf{x}} \in D(\mathbf{p}, I)$ is called an **optimal consumption bundle** in the budget set $D(\mathbf{p}, I) \subset X = \mathbb{R}^2_+$ if:

(2.41) $$\forall \mathbf{x} \in D(\mathbf{p}, I) \quad \bar{\mathbf{x}} \succsim \mathbf{x} \Leftrightarrow u(\bar{\mathbf{x}}) \geq u(\mathbf{x}),$$

which means that it is not worse than any other consumption bundle in a set $D(\mathbf{p}, I) \subset B \subset X$.

A consumer wants to choose an optimal consumption bundle in a set $D(\boldsymbol{p}, I) \subset B \subset X$. In view of a selected optimality criterion, which means the utility, this problem of choice of the optimal consumption bundle can be written as a consumption utility maximization problem of a form:

(2.42) $$(\mathbf{P1}) \qquad u(\mathbf{x}) \to \max$$

(2.43) $$\mathbf{x} \in D(\mathbf{p}, I)$$

or

(2.44) $$(\mathbf{P2}) \qquad u(x_1, x_2) \to \max$$

(2.45) $$p_1 x_1 + p_2 x_2 \leq I$$

(2.46) $$x_1, x_2 \geq 0.$$

Note 2.21 The problems (P1) and (P2) are equivalent problems of mathematical programming. If the utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ is a linear (concave) function then (P2) is the problem of linear programming. Whereas when the utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ is a nonlinear function (usually strictly concave, as we are interested in the utility maximization), then (P2) is a nonlinear programming problem. Conditions (2.45)–(2.46) define a set of feasible solutions in this problem.

Example 2.6 Use the geometric method to find the optimal solution to the problem (P2) when the utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ is[24]:

(a) linear $u(x_1, x_2) = a_1 x_1 + a_2 x_2,\ a_i > 0,\ i = 1, 2$
(increasing, differentiable, concave),
(b) power function $u(x_1, x_2) = a x_1^{\alpha_1} x_2^{\alpha_2},\ a, \alpha_i > 0,\ i = 1, 2,\ \alpha_1 + \alpha_2 < 1,$
(increasing, differentiable, strictly concave),

[24] Apply conclusions resulting from Example 2.4.

(c) logarithmic $u(x_1, x_2) = a_1 \ln x_1 + a_2 \ln x_2,\ a_i > 0,\ x_i \in \operatorname{int} \mathbb{R}_+,\ i = 1, 2$ (increasing, differentiable, strictly concave),
(d) subadditive $u(x_1, x_2) = a_1 x_1^\alpha + a_2 x_2^\alpha,\ a_1, a_2 > 0, \alpha \in (0, 1)$ (increasing, differentiable, strictly concave),
(e) Koopmans-Leontief function $u(x_1, x_2) = \min\{a_1 x_1, a_2 x_2\},\ a_i > 0,\ i = 1, 2$ (non-decreasing, non-differentiable, continuous, concave).

Justify that:

– for a linear utility function:
– if $\exists \lambda > 0\ \mathbf{a} = (a_1, a_2) = \lambda(p_1, p_2) = \lambda \mathbf{p}$, the problem (P2) has an infinite number of optimal solutions (Fig. 2.12a) belonging to a segment $\bar{\mathbf{x}} = \alpha \mathbf{x}^1 + \beta \mathbf{x}^2,\quad \forall \alpha, \beta \geq 0,\ \alpha + \beta = 1$, where

$$\mathbf{x}^1 = \left(\frac{I}{p_1}, 0\right),\ \mathbf{x}^2 = \left(0, \frac{I}{p_2}\right),\quad \text{thus}$$
$$\bar{\mathbf{x}} = \left(\alpha \frac{I}{p_1}, \beta \frac{I}{p_2}\right),\ \forall \alpha, \beta\ \geq 0,\ \alpha + \beta = 1,$$

– if $\mathbf{a} = (a_1, a_2) \neq \lambda(p_1, p_2) = \lambda \mathbf{p}$, then problem (P2) has exactly one optimal solution (Figs. 2.12b and 2.12c):

$$\bar{\mathbf{x}} = \left(\frac{I}{p_1}, 0\right) \quad \text{or} \quad \bar{\mathbf{x}} = \left(0, \frac{I}{p_2}\right),$$

– for the remaining utility functions the problem (P2) has exactly one optimal solution (Figs. 2.12d, 2.12e, 2.12f):

$$\exists_1\ \alpha, \beta > 0,\ \alpha + \beta = 1 \quad \bar{\mathbf{x}} = \alpha \mathbf{x}^1 + \beta \mathbf{x}^2,\ \text{where}$$
$$\mathbf{x}^1 = \left(\frac{I}{p_1}, 0\right),\ \mathbf{x}^2 = \left(0, \frac{I}{p_2}\right),\ \text{thus:}$$
$$\exists_1\ \alpha, \beta > 0,\ \alpha + \beta = 1 \quad \bar{\mathbf{x}} = \left(\alpha \frac{I}{p_1}, \beta \frac{I}{p_2}\right).$$

Ad (e) Let us present a geometric illustration of a solution to the problem of the consumption utility maximization:

$$u(x_1, x_2) = \min\{a_1 x_1, a_2 x_2\} \to \max \tag{2.47}$$

$$p_1 x_1 + p_2 x_2 \leq I, \tag{2.48}$$

$$x_1, x_2 \geq 0, \tag{2.49}$$

in the goods space $X = \mathbb{R}_+^2$.

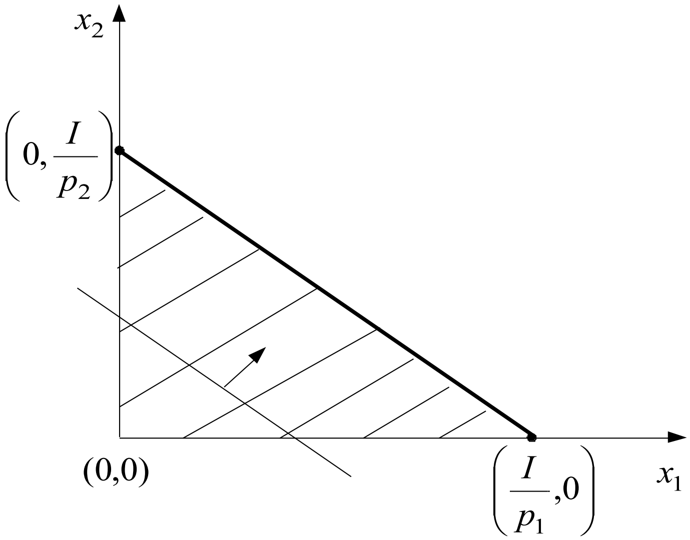


Fig. 2.12a Solution to consumption utility maximization problem with linear utility function when $a_i = \lambda p_i$, $i = 1, 2$

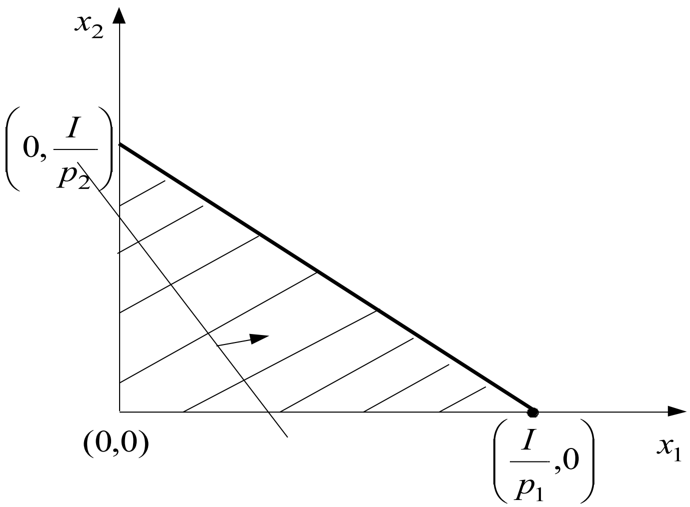


Fig. 2.12b Solution to consumption utility maximization problem with linear utility function when $a_i \neq \lambda p_i$, $i = 1, 2$ and $a_1 > a_2$

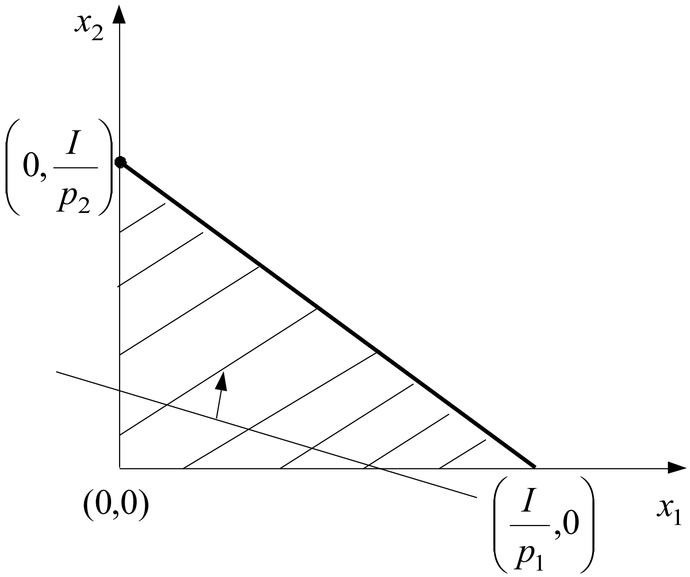


Fig. 2.12c Solution to consumption utility maximization problem with linear utility function when $a_i \neq \lambda p_i$, $i = 1, 2$ and $a_1 < a_2$

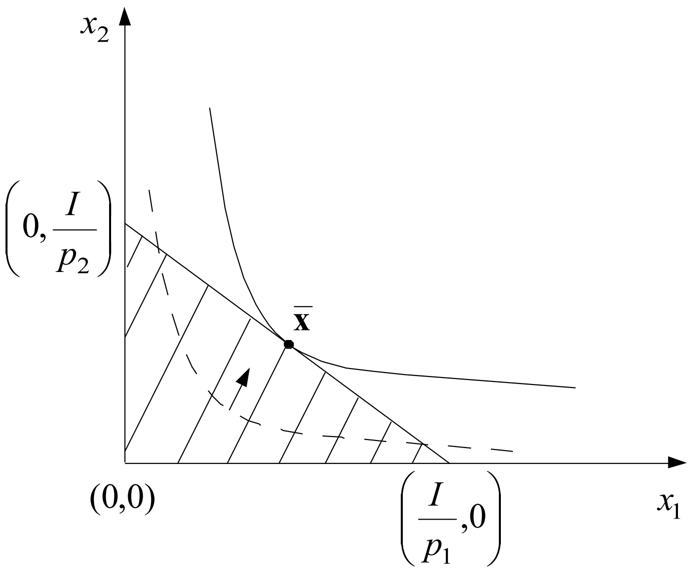


Fig. 2.12d Solution to consumption utility maximization problem with power utility function

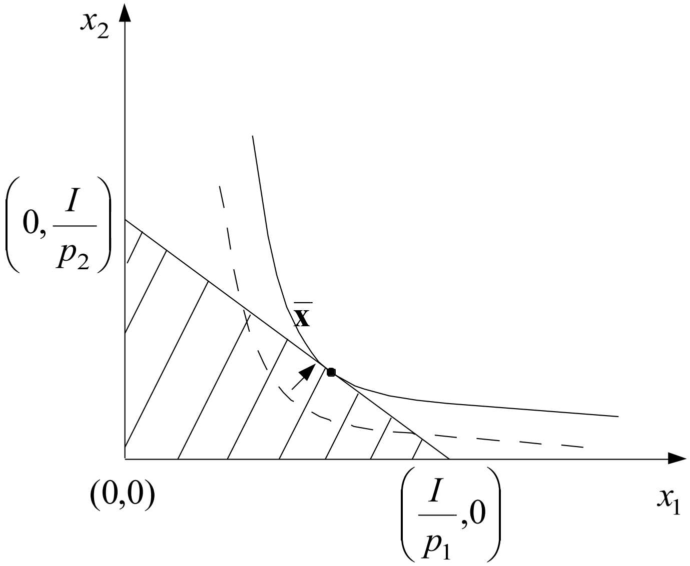


Fig. 2.12e Solution to consumption utility maximization problem with logarithmic utility function

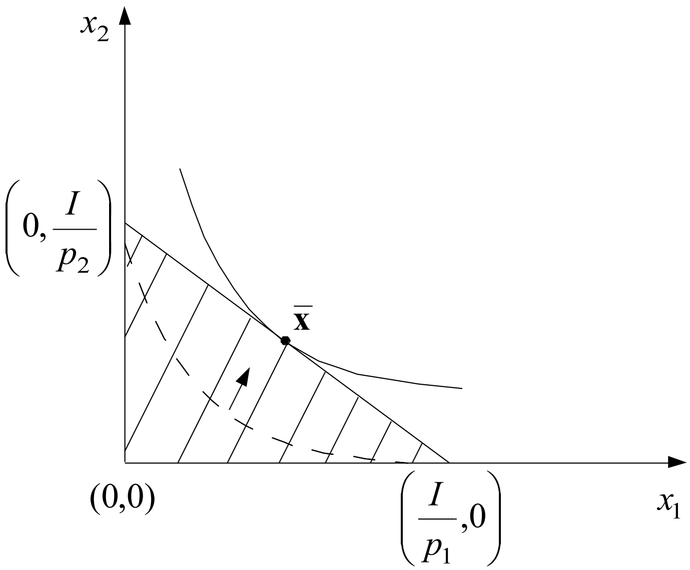


Fig. 2.12f Solution to consumption utility maximization problem with subadditive utility function

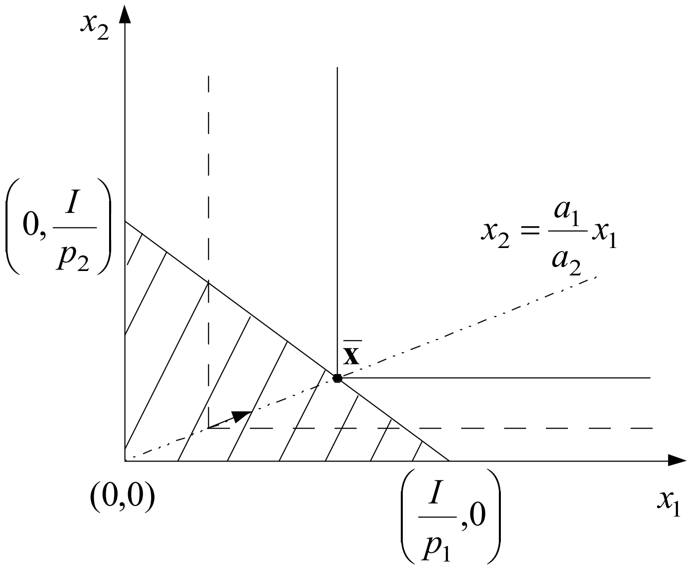


Fig. 2.12g Solution to consumption utility maximization problem with Koopmans-Leontief function

On the basis of Fig. 2.12g we can state that the optimal solution satisfies conditions:

$$p_1\bar{x}_1 + p_2\bar{x}_2 = I, \tag{2.50}$$

$$a_1\bar{x}_1 = a_2\bar{x}_2. \tag{2.51}$$

To find the optimal solution one needs to solve the equation system (2.50)–(2.51). The solution has a form:

$$\begin{aligned} \bar{\mathbf{x}} &= \left(\frac{a_2 I}{a_2 p_1 + a_1 p_2}, \frac{a_1 I}{a_2 p_1 + a_1 p_2}\right) \\ &= \left(\frac{p_1 a_2}{(p_1 a_2 + p_2 a_1)} \frac{I}{p_1}, \frac{p_2 a_1}{(p_1 a_2 + p_2 a_1)} \frac{I}{p_2}\right) = \left(\alpha \frac{I}{p_1}, \beta \frac{I}{p_2}\right), \end{aligned} \tag{2.52}$$

where $\alpha = \frac{p_1 a_2}{(p_1 a_2 + p_2 a_1)} > 0$, $\beta = \frac{p_2 a_1}{(p_1 a_2 + p_2 a_1)} > 0$, such that $\alpha + \beta = 1$.

Conclusions

C. 2.1 A set of feasible solutions $D(\mathbf{p}, I) \subset B \subset \mathbb{R}^2_+$ is a compact (closed and bounded) and convex set.

C. 2.2 The utility function (being the objective function of the optimization problem) is increasing or weakly increasing and concave or strictly concave.

C. 2.3 If the set of feasible solutions is compact and convex, and the objective function is strictly concave and increasing, then the problem (P2) has exactly one optimal solution.

C. 2.4 If the set of feasible solutions is compact and convex, and the objective function is concave and increasing or weakly increasing, then the problem (P2) has at least one optimal solution. If there are more than one, they form a compact and convex set.

Let us present analytical methods for finding the optimal solution to the consumption utility maximization problem (P2).

Method 2.1 If the set of feasible solutions $D(\mathbf{p}, I) \subset X = \mathbb{R}^2_+$ is compact and convex, and the utility function is increasing, concave or strictly concave, then the optimal solution $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ to the problem (P2) belongs to the budget line $L(\mathbf{p}, I)$.

It means that:

$$p_1 x_1 + p_2 x_2 = I \tag{2.53}$$

and hence:

$$x_2 = \frac{I - p_1 x_1}{p_2}. \tag{2.54}$$

Substituting (2.54) into the objective function in problem (P2), we get its equivalent form:

(P3)

$$h(x_1) = u\left(x_1, \frac{I - p_1 x_1}{p_2}\right) \to \max \tag{2.55}$$

$$0 \leq x_1 \leq \frac{I}{p_1}. \tag{2.56}$$

Case 1

If the objective function in problem (P3) is strictly concave in the set of feasible solutions, then problem (P3)condition has exactly one optimal solution $\bar{x}_1 \in \left(0; \frac{I}{P_1}\right)$ such that

$$\left.\frac{\mathrm{d}h(x_1)}{\mathrm{d}x_1}\right|_{x_1 = \bar{x}_1} = 0 \quad \text{necessary condition,} \tag{2.57}$$

$$\left.\frac{\mathrm{d}^2 h(x_1)}{\mathrm{d}x_1^2}\right|_{x_1 = \bar{x}_1} < 0 \quad \text{sufficient condition,} \tag{2.58}$$

for which the objective function has the maximum valuecondition.

In that case the optimal solution to problem (P2) is a consumption bundle:

(2.59) $$\bar{\mathbf{x}} = \left(\bar{x}_1, \frac{I - p_1\bar{x}_1}{p_2}\right) > (0, 0).$$

Moreover:

(2.60) $$u(\bar{\mathbf{x}}) = u\left(\bar{x}_1, \frac{I - p_1\bar{x}_1}{p_2}\right) = h(\bar{x}_1),$$

which means that maximum values of objectives functions in problems (P2) and (P3) are the same.

Case 2

If the objective function in problem (P3) is concave but not strictly in the set of feasible solutions, then there exists at least one quantity $\bar{x}_1 \in \left[0; \frac{I}{p_1}\right]$ for which the objective function has the maximum value.

If the objective function in problem (P3) has the maximum value for $\bar{x}_1 = \frac{I}{p_1}$ then the optimal solution to the maximization problem (P2) is a consumption bundle: $\bar{\mathbf{x}} = \left(\frac{I}{p_1}, 0\right)$.

If the objective function in problem (P3) has the maximum value for $\bar{x}_1 = 0$ then the optimal solution to the maximization problem (P2) is a consumption bundle: $\bar{\mathbf{x}} = \left(0, \frac{I}{p_2}\right)$.

If the objective function in problem (P3) has the maximum value for both $\bar{x}_1 = 0$ and $\bar{x}_1 = \frac{I}{p_1}$ then the maximization problem (P2) has infinitely many optimal solutions which form a compact and convex set of optimal consumption bundles:

(2.61) $$\bar{\mathbf{x}} = \alpha\mathbf{x}^1 + \beta\mathbf{x}^2, \quad \forall\alpha, \beta \geq 0, \ \alpha + \beta = 1,$$

which is the same as the budget line $L(\mathbf{p}, I)$.

Conclusions

C. 2.5 If a utility function $u{:}\mathbb{R}^2_+ \to \mathbb{R}$ in problem (P2) is increasing or weakly increasing then the optimal solution belongs to the budget line $L(\mathbf{p}, I)$.

C. 2.6 If a utility function $u{:}\mathbb{R}^2_+ \to \mathbb{R}$ in problem (P2) is increasing and strictly concave in a set $D(\mathbf{p}, I) \subset B \subset R^2_+$ then problem (P2) has exactly one optimal solution. This optimal solution belongs to the budget line $L(\mathbf{p}, I)$ and is of a form:

(2.62) $$\exists_1 \ \alpha, \beta > 0, \ \alpha + \beta = 1$$
$$\bar{\mathbf{x}} = \alpha\left(\frac{I}{p_1}, 0\right) + \beta\left(0, \frac{I}{p_2}\right) = \left(\alpha\frac{I}{p_1}, \beta\frac{I}{p_2}\right) > (0, 0)$$

C. 2.7 If a utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ in problem (P2) is increasing and concave but not strictly in a set $D(\mathbf{p}, I) \subset B \subset R^2_+$ then three cases are possible:

(a) problem (P2) has exactly one optimal solution. It belongs to the budget line and is of a form:

$$\bar{\mathbf{x}} = \left(\frac{I}{p_1}, 0\right), \tag{2.63}$$

(b) problem (P2) has exactly one optimal solution. It belongs to the budget line and is of a form:

$$\bar{\mathbf{x}} = \left(0, \frac{I}{p_2}\right), \tag{2.64}$$

(c) problem (P2) has infinitely many optimal solutions forming the budget line:

$$\begin{gathered} \forall\, \alpha, \beta > 0,\ \alpha + \beta = 1 \\ \bar{\mathbf{x}} = \alpha\left(\frac{I}{p_1}, 0\right) + \beta\left(0, \frac{I}{p_2}\right) = \left(\alpha\frac{I}{p_1}, \beta\frac{I}{p_2}\right). \end{gathered} \tag{2.65}$$

Method 2.2 We know that if a utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ is increasing and strictly concave, then problem (P2) has exactly one optimal solution. It belongs to the budget line and is of a form:

$$\begin{gathered} \exists_1\, \alpha, \beta >\ 0,\ \alpha + \beta = 1 \\ \bar{\mathbf{x}} = \alpha\left(\frac{I}{p_1}, 0\right) + \beta\left(0, \frac{I}{p_2}\right) = \left(\alpha\frac{I}{p_1}, \beta\frac{I}{p_2}\right) >\ (0, 0). \end{gathered} \tag{2.66}$$

At a point indicating the optimal consumption bundle $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ it is satisfied that the budget line is tangent to an indifference curve. Hence we get

$$s_{12}(\bar{x}_1, \bar{x}_2) = -\left.\frac{\mathrm{d}x_2}{\mathrm{d}x_1}\right|_{\mathbf{x} = \bar{\mathbf{x}}} = \frac{\left.\frac{\partial u(x_1, x_2)}{\partial x_1}\right|_{\mathbf{x} = \bar{\mathbf{x}}}}{\left.\frac{\partial u(x_1, x_2)}{\partial x_2}\right|_{\mathbf{x} = \bar{\mathbf{x}}}} = \frac{p_1}{p_2}, \tag{2.67}$$

which means that if a consumption bundle is optimal then a marginal rate of substitution of the first good by the second good in this bundle is constant and equal to ratio of prices of these two goods. This property, related to a linear budget constraint and to an increasing, differentiable, strictly concave utility function, is called **second Gossen's law**.[25]

We see that to find the optimal solution to problem (P2) one needs to solve a system of two equations with two unknowns:

$$s_{12}(\bar{x}_1, \bar{x}_2) = \frac{\left.\frac{\partial u(x_1,x_2)}{\partial x_1}\right|_{\mathbf{x}=\bar{\mathbf{x}}}}{\left.\frac{\partial u(x_1,x_2)}{\partial x_2}\right|_{\mathbf{x}=\bar{\mathbf{x}}}} = \frac{p_1}{p_2}, \tag{2.68}$$

$$p_1\bar{x}_1 + p_2\bar{x}_2 = I. \tag{2.69}$$

Method 2.3 Let us write problem (P2) in a form of a Lagrange function:

$$F(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda(I - p_1x_1 - p_2x_2), \tag{2.70}$$

where $\lambda \geq 0$ denotes a Lagrange multiplier.

If a utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ is increasing and strictly concave then problem (P2) has exactly one optimal solution. It belongs to the budget line and has a form:

$$\bar{\mathbf{x}} = \alpha\left(\frac{I}{p_1}, 0\right) + \beta\left(0, \frac{I}{p_2}\right) \underset{\exists_1\ \alpha,\beta>0,\ \alpha+\beta=1}{=} \left(\alpha\frac{I}{p_1}, \beta\frac{I}{p_2}\right) > (0,0), \tag{2.71}$$

and is a solution to a system of 3 equations with 3 unknowns:

$$\left.\frac{\partial F(x_1, x_2, \bar{\lambda})}{\partial x_1}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \left.\frac{\partial u(x_1, x_2)}{\partial x_1}\right|_{\mathbf{x}=\bar{\mathbf{x}}} - \bar{\lambda}p_1 = 0, \tag{2.72}$$

$$\left.\frac{\partial F(x_1, x_2, \bar{\lambda})}{\partial x_2}\right|_{x=\bar{\mathbf{x}}} = \left.\frac{\partial u(x_1, x_2)}{\partial x_2}\right|_{x=\bar{\mathbf{x}}} - \bar{\lambda}p_2 = 0, \tag{2.73}$$

$$\left.\frac{\partial F(\bar{x}_1, \bar{x}_2, \lambda)}{\partial \lambda}\right|_{\lambda=\bar{\lambda}} = I - p_1\bar{x}_1 - p_2\bar{x}_2 = 0, \tag{2.74}$$

or to an equivalent equation system of a form:

$$\left.\frac{\partial u(x_1, x_2)}{\partial x_1}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \bar{\lambda}p_1, \tag{2.75}$$

[25] First Gossen's law is presented in Sect. 2.2, Note 2.6.

(2.76) $$\left.\frac{\partial u(x_1, x_2)}{\partial x_2}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \bar{\lambda} p_2,$$

(2.77) $$p_1\bar{x}_1 + p_2\bar{x}_2 = I,$$

where:

$\bar{\lambda} = \left.\frac{\partial u(x_1,x_2)}{\partial I}\right|_{\mathbf{x}=\bar{\mathbf{x}}} > 0$ denotes an optimal Lagrange multiplier, which determines approximately how much a utility of the optimal consumption bundle changes (generally increases), when a consumer's income increases by one notional unit.

The equation system (2.75)–(2.77) has an interesting economic interpretation. If a bundle $\bar{\mathbf{x}} \in D(\mathbf{p}, I)$ is the optimal solution to problem (P2), then:

(a) the marginal utility of i-th good is proportional to the price of this good,
(b) money value of the optimal consumption bundle is equal to the consumer's income,
(c) a marginal utility of a money unit for the purchase of each of both goods $\frac{1}{p_1} \cdot \left.\frac{\partial u(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \frac{1}{p_2} \cdot \left.\frac{\partial u(\mathbf{x})}{\partial x_2}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \bar{\lambda} > 0$ is constant and equal to the optimal value of a Lagrange multiplier.

Let us notice that by solving the Eqs. (2.75)–(2.77), for example by dividing Eq. (2.75) into both sides by Eq. (2.76), we get a system:

(2.78) $$\frac{\left.\frac{\partial u(x_1,x_2)}{\partial x_1}\right|_{\mathbf{x}=\bar{\mathbf{x}}}}{\left.\frac{\partial u(x_1,x_2)}{\partial x_2}\right|_{\mathbf{x}=\bar{\mathbf{x}}}} = \frac{p_1}{p_2},$$

(2.79) $$p_1\bar{x}_1 + p_2\bar{x}_2 = I,$$

which is equivalent to system (2.68)–(2.69).

Equation (2.78) has an interesting economic interpretation too, since it shows that when a consumption bundle is optimal then a marginal rate of substitution of the first (second) good by the second (first) good in this bundle is constant and equal to the ratio of prices of these two goods.[26]

Example 2.7 Using methods 2.1–2.3 find the optimal solution to the consumption utility maximization problem:

(2.80) $$u(x_1, x_2) \to \max$$

[26] This is the case when the second Gossen's law is satisfied.

(2.81) $$(\mathbf{P2}) \quad p_1x_1 + p_2x_2 \le I$$

(2.82) $$x_1, x_2 \ge 0,$$

when:

(a) $u(x_1, x_2) = ax_1^{\alpha_1}x_2^{\alpha_2} \quad a, \alpha_i > 0, \ i = 1, 2 \quad \alpha_1 + \alpha_2 < 1,$
(b) $u(x_1, x_2) = a_1 \ln x_1 + a_2 \ln x_2 \quad a_i > 0, \quad x_i \in \operatorname{int} \mathbb{R}_+, \ i = 1, 2.$

Check that for:

(a) the power utility function:

(2.83) $$\bar{\mathbf{x}} = \left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\frac{I}{p_1}, \frac{\alpha_2}{\alpha_1 + \alpha_2}\frac{I}{p_2}\right) > (0, 0),$$

notice that if $\alpha = \frac{\alpha_1}{\alpha_1+\alpha_2}$ and $\beta = \frac{\alpha_2}{\alpha_1+\alpha_2}$, then $\alpha, \beta > 0, \alpha + \beta = 1$ and hence:

$$\bar{\mathbf{x}} = \left(\alpha\frac{I}{p_1}, \beta\frac{I}{p_2}\right) > (0, 0),$$

while

(2.84) $$u(\bar{\mathbf{x}}) = a\left(\frac{I}{\alpha_1 + \alpha_2}\right)^{\alpha_1+\alpha_2}\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2}\right)^{\alpha_2} > 0.$$

(b) the logarithmic utility function:

(2.85) $$\bar{\mathbf{x}} = \left(\frac{a_1}{a_1 + a_2}\frac{I}{p_1}, \frac{a_2}{a_1 + a_2}\frac{I}{p_2}\right) > (0, 0),$$

notice that if $\alpha = \frac{a_1}{a_1+a_2}$ and $\beta = \frac{a_2}{a_1+a_2}$, then $\alpha, \beta > 0, \alpha + \beta = 1$ and hence:

$$\bar{\mathbf{x}} = \left(\alpha\frac{I}{p_1}, \beta\frac{I}{p_2}\right) > (0, 0),$$

while

(2.86) $$u(\bar{\mathbf{x}}) = \ln\left[\left(\frac{I}{a_1 + a_2}\right)^{a_1+a_2}\left(\frac{a_1}{p_1}\right)^{a_1}\left(\frac{a_2}{p_2}\right)^{a_2}\right] > 0.$$

Definition 2.32 A mapping $\varphi\colon \operatorname{int}\mathbb{R}^3_+ \to \mathbb{R}^2_+$ which assigns the optimal solution of the maximization problem (P2) of consumption utility to any price vector $\mathbf{p} = (p_1, p_2) \in \operatorname{int}\mathbb{R}^2_+$ and any consumer's income $I > 0$ is called a **consumer demand function** and is of a form:

$$\varphi(\mathbf{p}, I) = (\varphi_1(p_1, p_2, I), \varphi_2(p_1, p_2, I)) = (\bar{x}_1, \bar{x}_2) = \bar{\mathbf{x}}. \tag{2.87}$$

Note 2.22 Since the function of consumer demand is a vector function then notation means:

$\bar{x}_1 = \varphi_1(p_1, p_2, I)$—the demand for the first consumer good expressed in physical units,
$\bar{x}_2 = \varphi_2(p_1, p_2, I)$—the demand for the second consumer good expressed in physical units.

Note 2.23 The vector function of consumer demand $\varphi(\mathbf{p}, I) = (\varphi_1(p_1, p_2, I), \varphi_2(p_1, p_2, I))$ is called a **Marshallian demand function.**

Definition 2.33 A mapping $\nu\colon \operatorname{int}\mathbb{R}^3_+ \to \operatorname{int}\mathbb{R}$ which assigns the maximum utility of the consumption bundle $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)$ to any price vector $\mathbf{p} = (p_1, p_2) \in \operatorname{int}\mathbb{R}^2_+$ and any consumer's income $I > 0$ is called an **indirect function of consumption utility** and is of a form:

$$\nu(\mathbf{p}, I) = u(\bar{\mathbf{x}}) = u(\bar{x}_1, \bar{x}_2) = u(\varphi_1(p_1, p_2, I), \varphi_2(p_1, p_2, I)). \tag{2.88}$$

Let us analyse important properties of the Marshallian demand function and the indirect utility function.

Definition 2.34 A consumer demand function is **homogeneous of degree 0** when:

$$\forall \lambda > 0 \quad \varphi(\lambda p_1, \lambda p_2, \lambda I) = \lambda^0 \varphi(p_1, p_2, I) = \varphi(p_1, p_2, I), \tag{2.89}$$

which means that:

(a) a proportionate change in prices of consumer goods and in a consumer's income does not change the demand for consumer goods,
(b) the demand for consumer goods' does not depend on absolute levels of goods prices and a consumer's income, but on relationships amongst them.

Definition 2.35 An indirect utility function is **homogeneous of degree 0** when:

$$\forall \lambda > 0 \quad \nu(\lambda p_1, \lambda p_2, \lambda I) = \lambda^0 \nu(p_1, p_2, I) = \nu(p_1, p_2, I), \tag{2.90}$$

which means that:

(a) a proportionate change in prices of consumer goods and in a consumer's income does not change a utility of the optimal consumption bundle,
(b) a utility of the optimal consumption bundle does not depend on absolute levels of goods' prices nor on a consumer's income, but on relationships amongst them.

Example 2.8 (continuation of Example 2.7 (a) We are given a consumer demand function:

$$\begin{aligned}\bar{\mathbf{x}} =&(\bar{x}_1, \bar{x}_2) = \left(\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{I}{p_1}, \frac{\alpha_2}{\alpha_1+\alpha_2}\frac{I}{p_2}\right)\\ =&(\varphi_1(p_1, p_2, I), \varphi_{12}(p_1, p_2, I)) = \varphi(\mathbf{p}, I).\end{aligned} \tag{2.91}$$

Let us notice that if $\alpha = \frac{\alpha_1}{\alpha_1+\alpha_2}$ and $\beta = \frac{\alpha_2}{\alpha_1+\alpha_2}$, then $\alpha, \beta > 0, \alpha + \beta = 1$ and hence:
$\bar{\mathbf{x}} = \left(\alpha\frac{I}{p_1}, \beta\frac{I}{p_2}\right) > (0, 0)$. The corresponding indirect utility function is of a form:

$$u(\bar{\mathbf{x}}) = \nu(p_1, p_2, I) = a\left(\frac{I}{\alpha_1+\alpha_2}\right)^{\alpha_1+\alpha_2}\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2}\right)^{\alpha_2} > 0. \tag{2.92}$$

1. Based on Definitions 2.34 and 2.35, justify that the consumer demand function and the indirect utility function are homogeneous of degree 0.
2. Justify this property for both functions by geometric means.

Ad 1 Homogeneity of degree 0 of the consumer demand function:

$$\begin{aligned}\forall\lambda > 0 \quad (\varphi_1(\lambda p_1, \lambda p_2, \lambda I), \varphi_2(\lambda p_1, \lambda p_2, \lambda I)) &= \left(\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{\lambda I}{\lambda p_1}, \frac{\alpha_2}{\alpha_1+\alpha_2}\frac{\lambda I}{\lambda p_2}\right)\\ &= (\varphi_1(p_1, p_2, I), \varphi_2(p_1, p_2, I))\end{aligned} \tag{2.93}$$

Homogeneity of degree 0 of the indirect utility function:

$$\begin{aligned}\forall\lambda > 0 \quad \nu(\lambda p_1, \lambda p_2, \lambda I) &= a\left(\frac{\lambda I}{\alpha_1+\alpha_2}\right)^{\alpha_1+\alpha_2}\left(\frac{\alpha_1}{\lambda p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{\lambda p_2}\right)^{\alpha_2}\\ &= \frac{\lambda^{\alpha_1+\alpha_2}}{\lambda^{\alpha_1}\lambda^{\alpha_2}}a\left(\frac{I}{\alpha_1+\alpha_2}\right)^{\alpha_1+\alpha_2}\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2}\right)^{\alpha_2}\\ &= \nu(p_1, p_2, I).\end{aligned} \tag{2.94}$$

Ad 2 Figure 2.13 shows that simultaneous and proportionate change in a consumer's income and consumer goods' prices does not change the demand for both goods, and thus a value of the intermediate utility function is unchanged too. We can interpret these dependencies in terms of a consumer's nominal and real income. If we consider a nominal income as $I > 0$, then we should consider a real income as: $\frac{I}{p_i} > 0,\ i = 1, 2$. When the prices of goods and the consumer's nominal income change

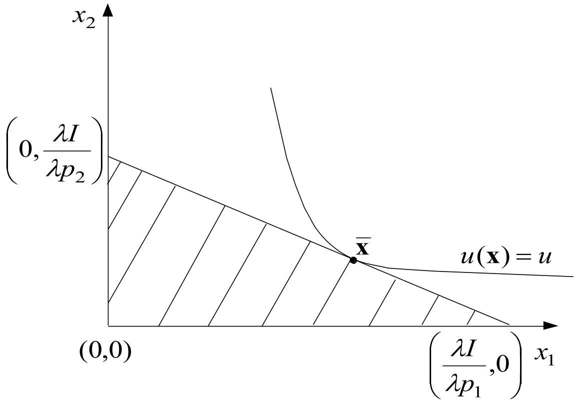


Fig. 2.13 Homogeneity of degree 0 of consumer demand function and of indirect utility function

proportionally (they increase when $\lambda > 1$ or decrease when $\lambda \in (0; 1)$) then the real income does not change. Thus, the value of the demand function for both goods and the value of the indirect utility function do not change as well.

Definition 2.36 A derivative of the indirect utility function with respect to a consumer's income is called a **marginal utility of income** and given as:

(2.95) $$\frac{\partial v(p_1, p_2, I)}{\partial I} = \lim_{\Delta I \to 0} \frac{v(p_1, p_2, I + \Delta I) - v(p_1, p_2, I)}{\Delta I},$$

determining approximately by how much a utility of the optimal consumption bundle changes (usually increases) when a consumer's income increases by a (notional) unit.

Theorem 2.3 If a utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ is increasing, twice differentiable and strictly concave, then $\forall p_1, p_2, I > 0$:

1. a consumer demand function $\varphi: \operatorname{int} \mathbb{R}^3_+ \to \mathbb{R}^2_+$ is differentiable in its domain,
2. an indirect utility function $v: \operatorname{int} \mathbb{R}^3_+ \to \mathbb{R}$ is differentiable in its domain,
3. $\frac{\partial v(p_1, p_2, I)}{\partial I} \neq 0 \quad \Rightarrow \quad \varphi_i(p_1, p_2, I) = -\frac{\frac{\partial v(p_1, p_2, I)}{\partial p_i}}{\frac{\partial v(p_1, p_2, I)}{\partial I}}, \quad i = 1, 2,$
   (it is the so-called **Roy's identity**, which allows to determine a consumer's demand function. for i-th product using analytical form of an indirect utility function),
4. an increase in a consumer's income results in an increase of the demand for at least one good:

$$\exists i \quad \frac{\partial \varphi_i(p_1, p_2, I)}{\partial I} > 0, \quad i = 1, 2,$$

5. an increase in the price of any good results in a decrease of the demand for at least one of the goods:

$$\forall i \ \exists j \quad \frac{\partial \varphi_j(p_1, p_2, I)}{\partial p_i} < 0, \quad i, j = 1, 2,$$

6. $\frac{\partial v(p_1, p_2, I)}{\partial I} = \frac{1}{p_i} \frac{\partial u(x_1, x_2)}{\partial x_i}\Bigg|_{\mathbf{x} = \bar{\mathbf{x}}} = \bar{\lambda} > 0, i = 1, 2,$

   which means that the marginal utility of a consumer's income is equal to the marginal utility of a money unit for the purchase of i-th good (which we know to be equal to the optimal Lagrange multiplier and therefore is the same for any consumer good).

Example 2.9 For a consumer demand function and the corresponding indirect utility function:

(a) $\varphi(p_1, p_2, I) = \left(\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{I}{p_1}, \frac{\alpha_2}{\alpha_1+\alpha_2}\frac{I}{p_2}\right),$

$$v(p_1, p_2, I) = a\left(\frac{I}{\alpha_1 + \alpha_2}\right)^{\alpha_1+\alpha_2}\left(\frac{\alpha_1}{p_1}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2}\right)^{\alpha_2} > 0$$

(in case of a power utility function in problem (P2))

(b) $\varphi(p_1, p_2, I) = \left(\frac{a_1}{a_1+a_2}\frac{I}{p_1}, \frac{a_2}{a_1+a_2}\frac{I}{p_2}\right),$

$$v(p_1, p_2, I) = \ln\left[\left(\frac{I}{a_1 + a_2}\right)^{a_1+a_2}\left(\frac{a_1}{p_1}\right)^{a_1}\left(\frac{a_2}{p_2}\right)^{a_2}\right] > 0$$

(in case of a logarithmic utility function in problem (P2)),

justify the properties that appear in statements 3–6 of Theorem 2.3.

Ad 3a Let us calculate partial derivatives of an indirect utility function with respect to the goods' prices and with respect to a consumer's income, having the power utility function:

(2.96) $$\frac{\partial v(p_1, p_2, I)}{\partial p_1} = -\alpha_1 a\left(\frac{I}{\alpha_1 + \alpha_2}\right)^{\alpha_1+\alpha_2}\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}p_1^{-\alpha_1-1}p_2^{-\alpha_2} < 0,$$

(2.97) $$\frac{\partial v(p_1, p_2, I)}{\partial p_2} = -\alpha_2 a\left(\frac{I}{\alpha_1 + \alpha_2}\right)^{\alpha_1+\alpha_2}\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}p_1^{-\alpha_1}p_2^{-\alpha_2-1} < 0,$$

(2.98) $$\frac{\partial v(p_1, p_2, I)}{\partial I} = a\left(\frac{I}{\alpha_1 + \alpha_2}\right)^{\alpha_1+\alpha_2-1}\alpha_1^{\alpha_1}\alpha_2^{\alpha_2}p_1^{-\alpha_1}p_2^{-\alpha_2} > 0,$$

which means that when a utility function is a power function then the indirect utility function is increasing in the consumer's income and decreasing in consumer goods' prices. Hence Roy's identities are satisfied:

$$\frac{\partial v(p_1, p_2, I)}{\partial I} \neq 0 \quad \Rightarrow \varphi_1(p_1, p_2, I) = -\frac{\frac{\partial v(p_1,p_2,I)}{\partial p_1}}{\frac{\partial v(p_1,p_2,I)}{\partial I}} = \frac{\alpha_1}{\alpha_1 + \alpha_2}\frac{I}{p_1},$$

$$\Rightarrow \varphi_2(p_1, p_2, I) = -\frac{\frac{\partial v(p_1,p_2,I)}{\partial p_2}}{\frac{\partial v(p_1,p_2,I)}{\partial I}} = \frac{\alpha_2}{\alpha_1 + \alpha_2}\frac{I}{p_2}. \tag{2.99}$$

Ad 3b Let us calculate partial derivatives of the indirect utility function with respect to the goods' prices and with respect to a consumer's income, having the logarithmic utility function:

$$\frac{\partial v(p_1, p_2, I)}{\partial p_1} = -\alpha_1 p_1^{-1} < 0, \tag{2.100}$$

$$\frac{\partial v(p_1, p_2, I)}{\partial p_2} = -\alpha_2 p_2^{-1} < 0, \tag{2.101}$$

$$\frac{\partial v(p_1, p_2, I)}{\partial I} = \left(\frac{I}{\alpha_1 + \alpha_2}\right)^{-1} > 0, \tag{2.102}$$

which means that when a utility function is logarithmic then the indirect utility function is increasing in a consumer' income and decreasing in consumer goods' prices. Hence Roy's identities are satisfied:

$$\frac{\partial v(p_1, p_2, I)}{\partial I} \neq 0 \quad \Rightarrow \quad \varphi_1(p_1, p_2, I) = -\frac{\frac{\partial v(p_1,p_2,I)}{\partial p_1}}{\frac{\partial v(p_1,p_2,I)}{\partial I}} = \frac{a_1}{a_1 + a_2}\frac{I}{p_1},$$

$$\wedge \quad \varphi_2(p_1, p_2, I) = -\frac{\frac{\partial v(p_1,p_2,I)}{\partial p_2}}{\frac{\partial v(p_1,p_2,I)}{\partial I}} = \frac{a_2}{a_1 + a_2}\frac{I}{p_2}. \tag{2.103}$$

Ad 4a Let us determine partial derivatives of the demand function for the i-th good with respect to the consumer's income, having the power utility function:

$$\frac{\partial \varphi_1(p_1, p_2, I)}{\partial I} = \frac{\alpha_1}{(\alpha_1 + \alpha_2)p_1} > 0, \tag{2.104}$$

$$\frac{\partial \varphi_2(p_1, p_2, I)}{\partial I} = \frac{\alpha_2}{(\alpha_1 + \alpha_2)p_2} > 0, \tag{2.105}$$

which means that in the case of the power utility function the demand function for the i-th good is increasing in a consumer's income.

Ad 4b Let us determine partial derivatives of the demand function for the i-th good with respect to the consumer's income, having the logarithmic utility function:

$$\frac{\partial \varphi_1(p_1, p_2, I)}{\partial I} = \frac{a_1}{(a_1 + a_2)p_1} > 0, \tag{2.106}$$

$$\frac{\partial \varphi_2(p_1, p_2, I)}{\partial I} = \frac{a_2}{(a_1 + a_2)p_2} > 0, \tag{2.107}$$

which means that in the case of the logarithmic utility function the demand function for the i-th good is increasing in a consumer's income.

Ad 5a Let us calculate partial derivatives of the demand function for the i-th good with respect to prices of both goods, having the power utility function:

$$\frac{\partial \varphi_1(p_1, p_2, I)}{\partial p_1} = -\frac{\alpha_1 I}{(\alpha_1 + \alpha_2)p_1^2} < 0, \tag{2.108}$$

$$\frac{\partial \varphi_2(p_1, p_2, I)}{\partial p_2} = -\frac{\alpha_2 I}{(\alpha_1 + \alpha_2)p_2^2} < 0, \tag{2.109}$$

$$\frac{\partial \varphi_1(p_1, p_2, I)}{\partial p_2} = \frac{\partial \varphi_2(p_1, p_2, I)}{\partial p_1} = 0, \tag{2.110}$$

which means that an increase in the price of i-th commodity results in a demand decrease for i-th good. On the other hand, an increase in the price of j-th good has no effect on the demand for i-th good, where $i \neq j, i, j = 1, 2$.

Ad 5b Let us calculate partial derivatives of the demand function for i-th good with respect to the prices of both goods, having the logarithmic utility function:

$$\frac{\partial \varphi_1(p_1, p_2, I)}{\partial p_1} = -\frac{a_1 I}{(a_1 + a_2)p_1^2} < 0, \tag{2.111}$$

$$\frac{\partial \varphi_1(p_1, p_2, I)}{\partial p_2} = -\frac{a_2 I}{(a_1 + a_2)p_2^2} < 0, \tag{2.112}$$

$$\frac{\partial \varphi_1(p_1, p_2, I)}{\partial p_2} = -\frac{\partial \varphi_2(p_1, p_2, I)}{\partial p_1} = 0, \tag{2.113}$$

which means that an increase in the price of i -th commodity results in a demand decrease for i -th good. On the other hand, an increase in the price of j -th good has no effect on the demand for i -th good, where $i \neq j, i, j = 1, 2$.

Ad 6a

$$\frac{\partial v(p_1, p_2, I)}{\partial I} = \frac{1}{p_1}\frac{\partial u(x_1, x_2)}{\partial x_1}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}} = \frac{1}{p_2}\frac{\partial u(x_1, x_2)}{\partial x_2}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}}$$

$$= a\left(\frac{I}{\alpha_1 + \alpha_2}\right)^{\alpha_1+\alpha_1-1} \alpha_1^{\alpha_1}\alpha_2^{\alpha_2} p_1^{-\alpha_1} p_2^{-\alpha_2} = \bar{\lambda} > 0. \tag{2.114}$$

Ad 6b

$$\frac{\partial v(p_1, p_2, I)}{\partial I} = \frac{1}{p_1}\frac{\partial u(x_1, x_2)}{\partial x_1}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}} = \frac{1}{p_2}\frac{\partial u(x_1, x_2)}{\partial x_2}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}}$$

$$= \left(\frac{I}{\alpha_1 + \alpha_2}\right)^{-1} = \bar{\lambda} > 0. \tag{2.115}$$

It follows from conditions (2.114) and (2.115) that in the case of power and logarithmic utility functions, the marginal utility of the consumer's income, the marginal utility of a money unit for the purchase of i-th good and the optimal Lagrange multiplier are equal to each other when the consumption bundle is optimal.

Let us define the criteria to classify consumption goods according to reactions of the consumer demand for goods to changes in goods' prices and in a consumer's income.

Definition 2.37 A derivative of a demand function φ_i with respect to the price p_j is called a **marginal demand for i-th good with respect to price of j-th good** and given as an expression:

$$P_{ij}(p_1, p_2, I) = \frac{\partial \varphi_i(p_1, p_2, I)}{\partial p_j}, \quad i, j = 1, 2, \tag{2.116}$$

which determines approximately by how many units the demand for i-th commodity changes when the price of j-th commodity increases by a (notional) unit, *ceteris paribus.*

Definition 2.38 A derivative of a demand function φ_i with respect to a consumer's income is called a **marginal demand for i-th good with respect to a consumer's income** and given as an expression:

$$P_i(p_1, p_2, I) = \frac{\partial \varphi_i(p_1, p_2, I)}{\partial I}, \quad i, j = 1, 2, \tag{2.117}$$

which determines approximately by how many units the demand for i-th good changes when a consumer's income increases by a (notional) unit, *ceteris paribus.*

Definition 2.39 An **elasticity of demand for *i*-th good with respect to the price of *j*-th good** is given as an expression:

(2.118) $$E_{ij}(p_1, p_2, I) = \frac{\partial \varphi_i(p_1, p_2, I)}{\partial p_j} \frac{p_j}{\varphi_i(p_1, p_2, I)}, \quad i, j = 1, 2,$$

which determines approximately by what % the demand for i-th good changes when the price of j-th good increases by 1%, *ceteris paribus*.

Note 2.24 If $i = j$, then an expression:

(2.119) $$E_{ii}(p_1, p_2, I) = \frac{\partial \varphi_i(p_1, p_2, I)}{\partial p_i} \frac{p_i}{\varphi_i(p_1, p_2, I)}, \quad i = 1, 2,$$

is called a simple price elasticity of demand for *i*-th good with respect to the price of *i*-th good.

If $i \neq j, i, j = 1, 2$ then an expression:

(2.120) $$E_{ij}(p_1, p_2, I) = \frac{\partial \varphi_i(p_1, p_2, I)}{\partial p_j} \frac{p_j}{\varphi_i(p_1, p_2, I)},$$

is called a cross elasticity[27] **of demand for *i*-th good with respect to the price of *j*-th good**.

Definition 2.40 An **elasticity of demand for *i*-th good with respect to a consumer's income** is given as an expression:

(2.121) $$E_i(p_1, p_2, I) = \frac{\partial \varphi_i(p_1, p_2, I)}{\partial I} \frac{I}{\varphi_i(p_1, p_2, I)}, \quad i = 1, 2,$$

which determines approximately by what % the demand for i-th good changes when a consumer's income increases by 1%, *ceteris paribus*.

Note 2.25 Since $i, j = 1, 2$, then for the demand function: $\varphi(p_1, p_2, I) = (\varphi_1(p_1, p_2, I), \varphi_2(p_1, p_2, I))$ we can determine four elasticities of demand with respect to commodity prices (two simple price elasticities of demand and two cross elasticities of demand) and two elasticities of demand with respect to a consumer's income.

[27] In the economic literature the cross elasticity is called also a mixed elasticity.

Criteria for classification of consumer goods

- simple demand price elasticities:
  $E_{ii}(p_1, p_2, I) > 0, \quad i = 1, 2$—Giffen goods or Veblen goods,[28] (an increase in the price of a given good results in an increase in the demand for this good).
  $E_{ii}(p_1, p_2, I) < 0, \quad i = 1, 2$—ordinary goods, (an increase in the price of a given good results in a decrease in the demand for this good).
- cross price elasticities of demand:
  $E_{ij}(p_1, p_2, I) > 0, \quad i, j = 1, 2, \ i \neq j$—substitute goods, (an increase in the price of j-th commodity results in an increase in the demand for i-th good),
  $E_{ij}(p_1, p_2, I) = 0, \quad i, j = 1, 2, \ i \neq j$—independent goods, (an increase in the price of j-th good does not affect the demand for i-th good),
  $E_{ij}(p_1, p_2, I) < 0, \quad i, j = 1, 2, \ i \neq j$—complementary goods, (an increase in the price of j-th good results in a decrease in the demand for i-th good).
- income elasticity of demand
  $E_i(p_1, p_2, I) > 0, \quad i = 1, 2$—normal goods, (an increase in a consumer's income results in an increase in the demand for i-th good).
  $E_i(p_1, p_2, I) < 0, \quad i = 1, 2$—inferior goods, (an increase in a consumer's income results in a decrease in the demand for i-th good).[29]

Note 2.26 If an increase in the price of an inferior good results in an increase in the demand for this good, then it is called a Giffen good. On the other hand, when an increase in the price of a normal good results in an increase in the demand for this good, then it is called a Veblen good.

Note 2.27 Let us notice that:

(2.122) $$E_{ij}(p_1, p_2, I) = P_{ij}(p_1, p_2, I)\frac{p_j}{\varphi_i(p_1, p_2, I)}, \quad i, j = 1, 2,$$

(2.123) $$E_i(p_1, p_2, I) = P_i(p_1, p_2, I)\frac{I}{\varphi_i(p_1, p_2, I)}, \quad i = 1, 2.$$

The price and income elasticities of demand have the same sign as the marginal demand for i-th good with respect to prices of goods or with respect to a consumer's income because prices and the demand for consumer goods are positive. Thus, to

[28] Whether a given good is so-called a Giffen good or a Veblen good is additionally determined by a reaction of the demand for this good to an increase in the consumer's income, along with an increase in the price of this good. See Note 2.26.
[29] Thus, for the demand functions corresponding to the power or logarithmic utility function, from Example 2.8 it follows that each of the consumer goods is ordinary and normal, and both are independent of each other.

determine the type of a consumer good, it is enough to determine the marginal demand for a given good (with respect to prices of goods or with respect to a consumer's income).

Example 2.10 A vector function $\varphi(p_1, p_2, I) = \left(\alpha \frac{I}{p_1}, \beta \frac{I}{p_2}\right)$ of consumer demand is given, which is the optimal solution of problem (P2) and can be written as a vector power function:

$$\begin{aligned} \varphi(p_1, p_2, I) &= (\varphi_1(p_1, p_2, I), \varphi_2(p_1, p_2, I)) \\ &= \left(\alpha p_1^{-1} p_2^0 I^1, \beta p_1^0 p_2^{-1} I^1\right), \end{aligned} \tag{2.124}$$

where $\alpha, \beta > 0,\ \alpha + \beta = 1$ are numbers.

Determine whether the first and the second goods are ordinary, normal and independent of each other.

In the case of a power function, an exponent of an independent variable is equal to the elasticity of the function with respect to this argument. On this basis, without making any calculations, we can conclude that:

$E_{11}(p_1, p_2, I) = -1$, therefore the first good is ordinary, which means that an increase in the price of the first commodity by 1% will reduce the demand for this good by 1%,
$E_{22}(p_1, p_2, I) = -1$, therefore the second good is ordinary, which means that an increase in the price of the second commodity by 1% will cause a decrease in the demand for this good by 1%,
$E_{12}(p_1, p_2, I) = E_{21}(p_1, p_2, I) = 0$, therefore the first and second goods are independent of each other, which means that 1% increase in the price of the first (second) good will not change the demand for the second (first) good,
$E_1(p_1, p_2, I) = 1$, therefore the first commodity is normal, which means that 1% increase in a consumer income will increase the demand for the first good by 1%,
$E_2(p_1, p_2, I) = 1$, therefore the second good is normal, which means that 1% increase in a consumer income will increase the demand for the second good by 1%.

Definition 2.41 A **path of price expansion of demand** is a sequence of optimal solutions to consumption utility maximization problems (P2), each of which corresponds to a change in the price of the first (second) good, as compared to the initial level of this price, with the price of the second (first) good unchanged and a consumer's income unchanged.

Definition 2.42 A **path of income expansion of demand** is a sequence of optimal solutions to consumption utility maximization problems (P2), each of which corresponds to a change in a consumer's income as compared to the initial level of income, with the prices of both goods unchanged.

Example 2.11 A vector function $\bar{\mathbf{x}} = \varphi(\mathbf{p}, I) = \left(\alpha \frac{I}{p_1}, \beta \frac{I}{p_2}\right)$ of consumer demand is given, being the optimal solution to the problem (P2), where $\alpha, \beta > 0,\ \alpha + \beta = 1$ are numbers.

Sketch graphs of hypothetical paths of price expansion and income expansion of demand for the given consumer demand function. Let us first consider the case where the price of the first good changes while a consumer's income and the price of the second good remain unchanged.

Figure 2.14a shows that the first good is ordinary and the second good is independent of the first good. Let us now consider the case where the price of the second good changes while a consumer's income and the price of the first good remain unchanged. Figure 2.14b shows that the second good is ordinary and the first good is independent of the second good.

Finally, let us consider the case where a consumer's income changes and the prices of both goods remain unchanged. It is illustrated in Fig. 2.14c, which clearly shows that the first and the second goods are normal.

It is worth emphasizing that the paths of income or price expansion depend on the goods prices, a consumer's income, and parameters of a utility function, by which a consumer chooses the optimal consumption bundles. Utility functions are, in

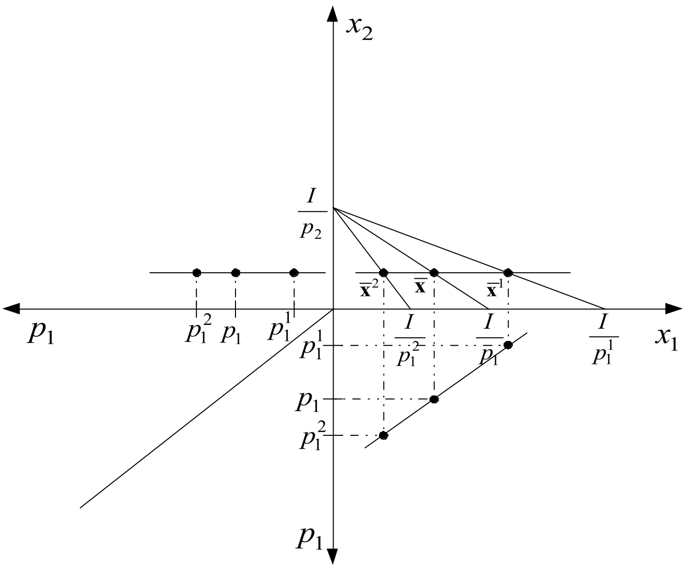


Fig. 2.14a Price expansion path of demand, when $p_1^1 < p_1 < p_1^2, \quad p_2, I = \text{const.} > 0.$

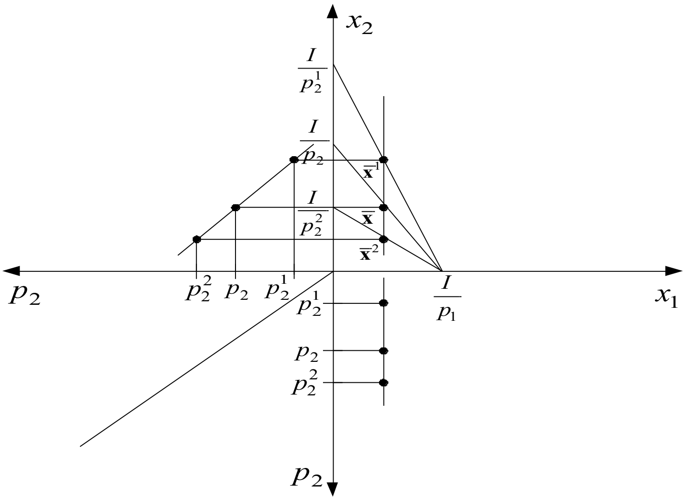


Fig. 2.14b Price expansion path of demand, when $p_2^1 < p_2 < p_2^2, \quad p_1, I = \text{const.} > 0$

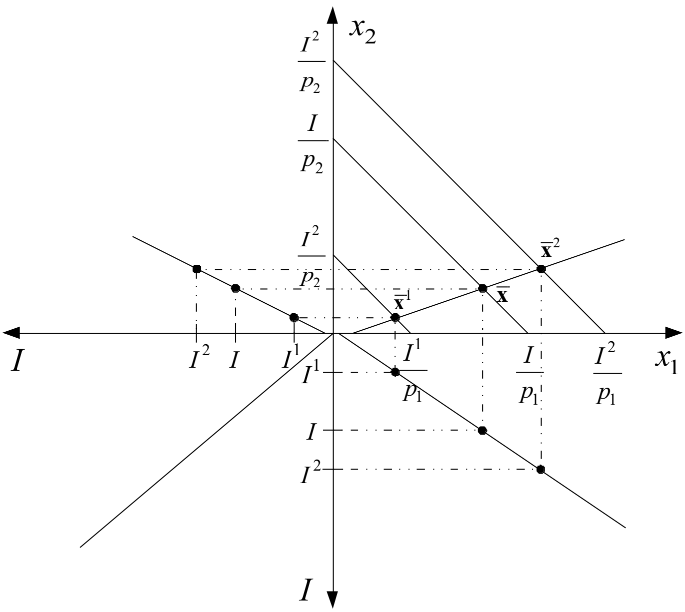


Fig. 2.14c Income expansion path of demand, when $I^1 < I < I^2, \quad p_1, p_2 = \text{const.} > 0$

particular, assumed to be increasing and strictly concave. Satisfying these conditions ensures an existence of exactly one optimal solution $\bar{\mathbf{x}} > 0$ to the consumption utility maximization problem, which by given fixed prices of goods, a consumer's income and parameters of the utility function, lies on the budget line.

2.4.2 Dynamic Approach

Let us introduce a notation t of the time. In a discrete version of a dynamic consumers problem the time changes in jumps, that is we consider the values of the analysed functions in subsequent periods $t = 0, 1, 2, \ldots, T$, where T means a time horizon. For example, assuming we treat periods as months, when T=30, this means the time horizon of 2.5 years. In a continuous version of a dynamic consumer's problem the time $t \in [0; T]$ changes continuously, that is we consider the values of the analysed functions at any moment of the considered time horizon. As in the static approach, we assume that we are interested in bundles composed of two consumer goods[30]:

$\mathbf{p}(t) = (p_1(t),\ p_2(t)) \geq 0$—a vector of time-varying prices of goods,
$I(t) \geq 0$—a consumer's income changing over time,
$\mathbf{x}(t) = (x_1(t), x_2(t)) \geq 0$—a bundle of goods that a consumer is willing to purchase at period/moment t at prices $\mathbf{p}(t)$.

When choosing a bundle $\mathbf{x}(t)$ the consumer takes into account her/his preferences towards bundles of goods, described by a utility function $u(\mathbf{x}(t))$. Over time, it is not the consumer's preferences that change, but only the bundle of goods that the consumer is willing to buy. This change occurs due to changes over time in the prices of goods and in the consumer's income.

The consumption utility maximization problem has a form:

$$u(\mathbf{x}(t)) \mapsto \max \tag{2.125}$$

$$p_1(t)x_1(t) + p_2(t)x_2(t) \leq I(t) \tag{2.126}$$

$$\mathbf{x}(t) \geq \mathbf{0}. \tag{2.127}$$

[30] For the discrete and continuous versions, we use the same denotation of dependence of a function value on time, for example income depending on time: $I(t)$. Whether the discrete or continuous version is used in a given formula will result from the context of the issue under consideration.

If we assume that an insatiability phenomenon occurs (utility functions are increasing in quantities of goods in a consumption bundle), then as a budget constraint, instead of the inequality of the budget set, we can use the budget line equation:

$$p_1(t)x_1(t) + p_2(t)x_2(t) = I(t). \tag{2.128}$$

If a utility function $u(\mathbf{x}(t))$ is increasing and strongly concave, then in each period/moment t the budget line (2.128) is tangent to the optimal bundle indifference curve, because a consumer wants to achieve the highest possible utility level of the consumption bundle whose value does not exceed a consumer's income. This results in an optimality condition of the consumption bundle:

$$s_{12}(\bar{\mathbf{x}}(t)) = \frac{p_1(t)}{p_2(t)} \qquad \forall t, \tag{2.129}$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0; T]$. This property is called Gossen's second law. The optimal bundle is a solution of the system of Eqs. (2.128) and (2.129).

From the consumption utility maximization problem we obtain the optimal bundle $\bar{\mathbf{x}}(t)$, a time-dependent consumer demand function:

$$\varphi(\mathbf{p}(t), I(t)) = \bar{\mathbf{x}}(t) \tag{2.130}$$

and a corresponding indirect utility function of consumption:

$$v(\mathbf{p}(t), I(t)) = u(\bar{\mathbf{x}}(t)). \tag{2.131}$$

The demand function, as well as the indirect utility function, in all periods/moments t has the same form, but at different periods/moments it may take different values depending on the prices of goods and income vary over time. Depending on a utility function describing consumer's preferences towards consumption bundles, the demand function $\varphi(\mathbf{p}(t), I(t))$ and the indirect utility function of consumption $v(\mathbf{p}(t), I(t))$ take the form accordingly:

(a) in case of a power utility function $u(\mathbf{x}(t)) = ax_1(t)^{\alpha_1}x_2(t)^{\alpha_2}$, $a > 0, \alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 < 1$

$$\varphi(\mathbf{p}(t), I(t)) = \left(\frac{\alpha_1 I(t)}{(\alpha_1 + \alpha_2)p_1(t)}, \frac{\alpha_2 I(t)}{(\alpha_1 + \alpha_2)p_2(t)}\right) \tag{2.132}$$

$$v(\mathbf{p}(t), I(t)) = a\left(\frac{I(t)}{\alpha_1 + \alpha_2}\right)^{\alpha_1+\alpha_2}\left(\frac{\alpha_1}{p_1(t)}\right)^{\alpha_1}\left(\frac{\alpha_2}{p_2(t)}\right)^{\alpha_2} \tag{2.133}$$

(b) in case of a logarithmic utility function $u(\mathbf{x}(t)) = a_1 \ln x_1(t) + a_2 \ln x_2(t)$, $a_1, a_2 > 0$, $x_1, x_2 > 0$

(2.134) $$\varphi(\mathbf{p}(t), I(t)) = \left(\frac{a_1 I(t)}{(a_1 + a_2)p_1(t)}, \frac{a_2 I(t)}{(a_1 + a_2)p_2(t)}\right)$$

(2.135) $$v(\mathbf{p}(t), I(t)) = \ln\left[\left(\frac{I(t)}{a_1 + a_2}\right)^{a_1+a_2}\left(\frac{a_1}{p_1(t)}\right)^{a_1}\left(\frac{a_2}{p_2(t)}\right)^{a_2}\right]$$

(c) in case of a Koopmans-Leontief utility function $u(\mathbf{x}(t)) = \min\{a_1 x_1(t), a_2 x_2(t)\}$, $a_1, a_2 > 0$

(2.136) $$\varphi(\mathbf{p}(t), I(t)) = \left(\frac{a_2 I(t)}{a_2 p_1(t) + a_1 p_2(t)}, \frac{a_1 I(t)}{a_2 p_1(t) + a_1 p_2(t)}\right)$$

(2.137) $$v(\mathbf{p}(t), I(t)) = \frac{a_1 a_2 I(t)}{a_2 p_1(t) + a_1 p_2(t)}.$$

Example 2.12 Let us assume that in every period a consumer's income and the prices of the first and of the second goods change according to equations:

$$I(t) = 10 \cdot 1.05^t,$$

$$p_1(t) = 4 \cdot 0.98^t,$$

$$p_2(t) = 0.006t^2 - 0.1t + 3,$$

$$t = 0, 1, 2, \ldots, 30,$$

which means that the income is initially 10 and increases by 5% in each subsequent period; the price of the first good is initially 4 and decreases by 2% in each period; the price of the second good is initially 3, decreases to around 2.6 in period 8 and then continues to rise. Price trajectories are shown in Fig. 2.15. Initially the first commodity is more expensive than the second one. From the 16th period it is the second one which is more expensive.

Consumer's preferences towards consumption bundles are described by a power utility function of a form $u(\mathbf{x}(t)) = x_1(t)^{0,5} x_2(t)^{0,5}$, for which a corresponding demand function is

$$\varphi(\mathbf{p}(t), I(t)) = \left(\frac{I(t)}{2p_1(t)}, \frac{I(t)}{2p_2(t)}\right)$$

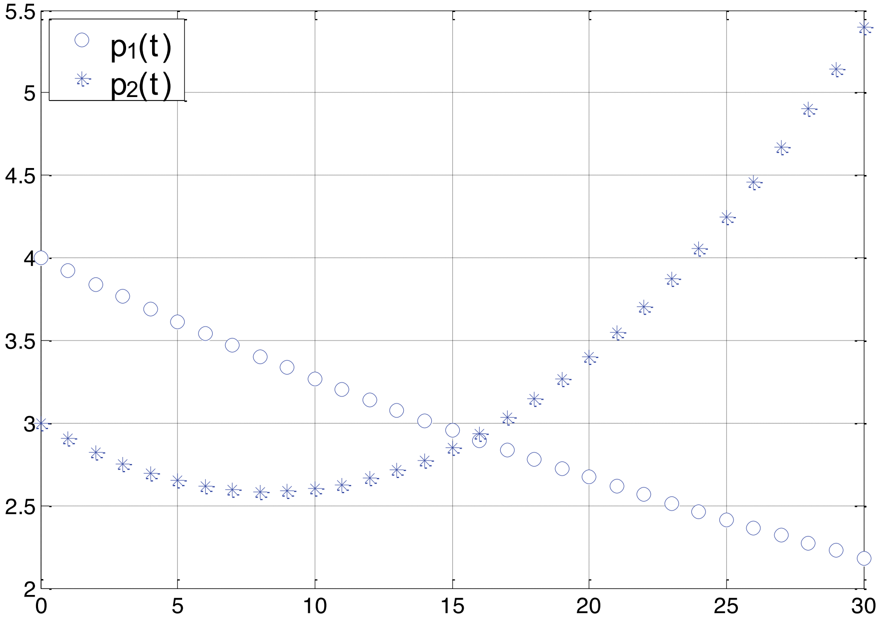


Fig. 2.15 Price trajectories

and an indirect utility function is

$$v(\mathbf{p}(t), I(t)) = \frac{I(t)}{2p_1(t)^{0.5} p_2(t)^{0.5}}.$$

In each period t, the budget line is tangent to the optimal bundle indifference curve. Figure 2.16 shows this relationship for periods: 0, 15, and 30. For each period $t = 0, 1, 2, \ldots, 30$ an analogous relationship can be presented. We can see from the figure that the budget set is getting bigger. This is due to the increase in the consumer's income over time. The slope of the budget line changes due to the relationship between the prices of both goods changing over time.

Trajectories of the demand for the first and the second goods are shown in Fig. 2.17. The demand for the first good increases from the level of 1.25 to the level of approximately 10 at the end of the considered time horizon. The demand for the second commodity grows from the level of 1.67, around the 20th period it reaches the level of 4 and remains at a similar level until the end of the time horizon. Initially, the consumer wants to have more of the second good in the consumption bundles he/she chooses due to the fact that this good is relatively cheaper. From period 16 onwards the consumer chooses the optimal bundle in which the quantity of the first good is greater, again because its price is relatively lower.

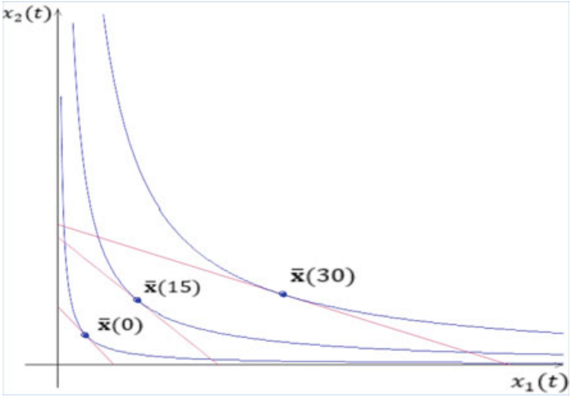


Fig. 2.16 Consumption utility maximization problem

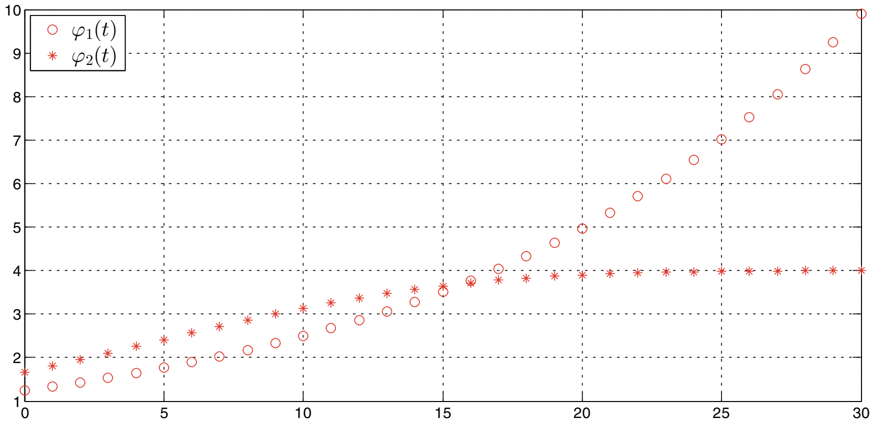


Fig. 2.17 Demand trajectories for first and second good

A trajectory of the indirect utility of consumption is shown in Fig. 2.18. The utility of the optimal bundle is constantly growing, because the relationship between the prices of goods and the consumer's income assumed in the example means the consumer can afford and consume more and more of both goods.

In each period Gossen's second law is obeyed, as shown in Fig. 2.19. This means that the trajectory of the marginal rate of substitution of the first good by the second good in the optimal bundle matches up with the trajectory of the quotient of the price of the first good by the price of the second good. This is one of the dependencies on the basis of which we have determined the optimal bundle.

In each period a consumer spends all of her/his income, as shown in Fig. 2.20. It results from the insatiability phenomenon, which manifests itself in the fact that the utility function increases in quantities of goods. As a consequence, the trajectory

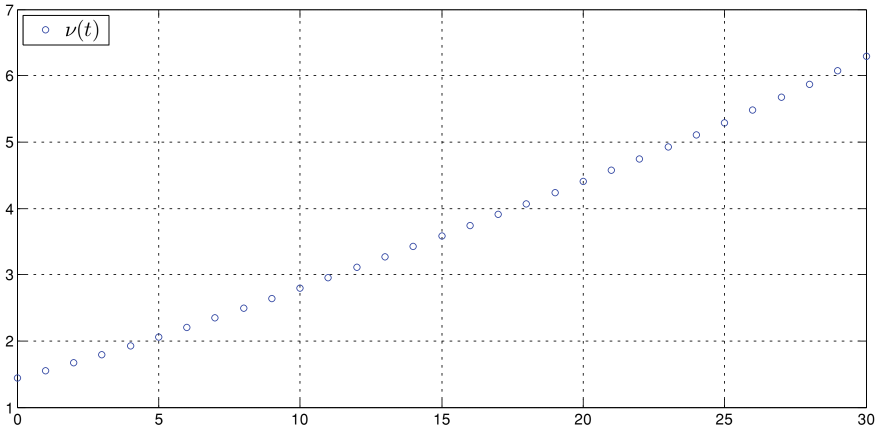


Fig. 2.18 Indirect utility trajectory

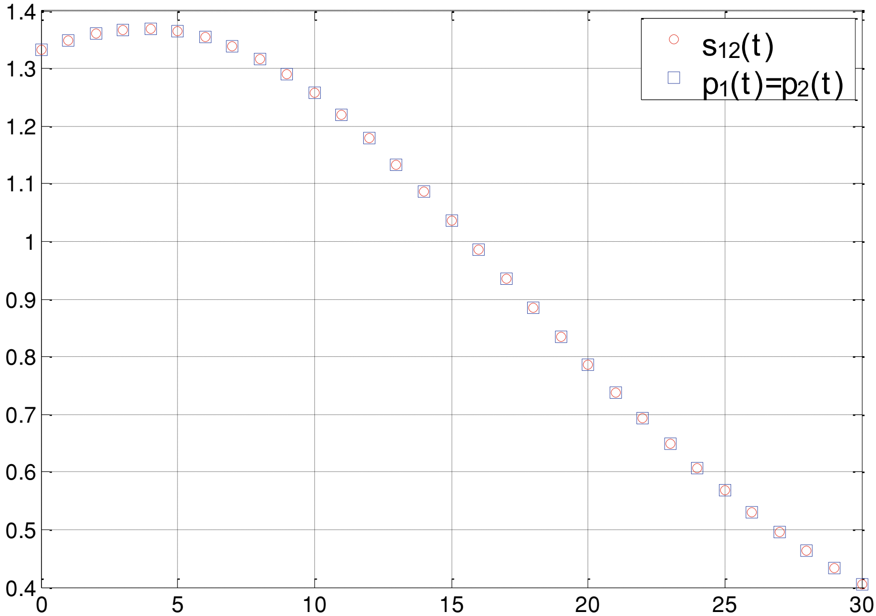


Fig. 2.19 Trajectory of marginal rate of substitution in optimal bundle

of the value of the purchased optimal bundle matches up with the trajectory of a consumer's income. This relationship, along with Gossen's second law, allows us to determine the optimal bundle. Figures 2.19 and 2.20 illustrate the fact that in each period of the considered time horizon, the demand function indicates optimal

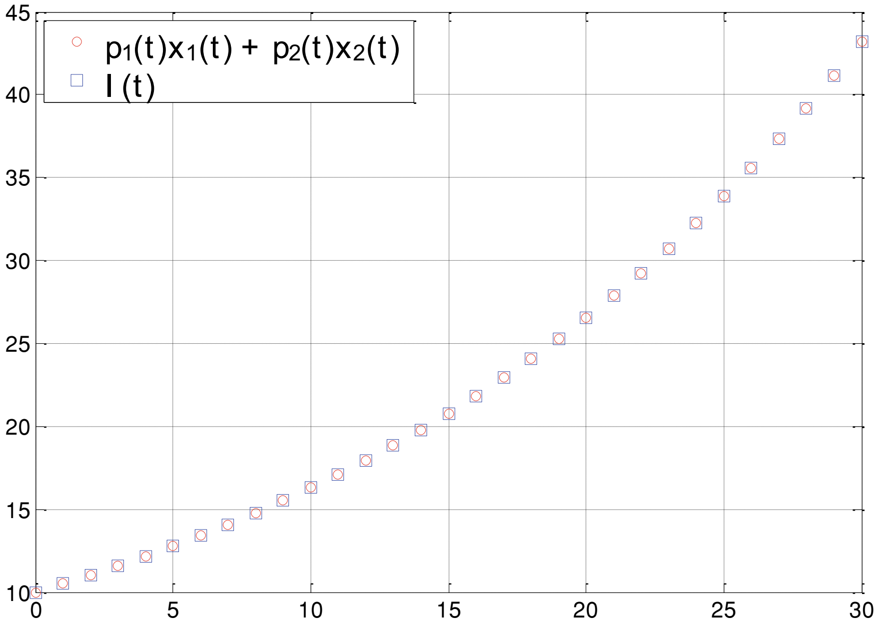


Fig. 2.20 Trajectory of purchased optimal bundles value

bundles, which results from the consumption utility maximization problem and form a definition of the demand function.

Let us now consider the trajectories of the marginal demand with respect to prices and to a consumer's income. The simple marginal demand trajectories shown in Fig. 2.21 illustrate the fact that both goods are ordinary. In each period the simple marginal demand for the first good and the one for the second good have a negative value, which results from the demand function form and from the definition of the simple marginal demand with respect to the price of a good:

$$P_{11}(\mathbf{p}(t), I(t)) = -\frac{I(t)}{2p_1(t)^2},$$

$$P_{22}(\mathbf{p}(t), I(t)) = -\frac{I(t)}{2p_2(t)^2}.$$

In Fig. 2.21 we can see that initially the demand for the second good responds more strongly to an increase in the price of the second good than the demand for the first good to an increase in the price of the first good. From period 16 on, this relationship is opposite the response of the demand for the first commodity becomes stronger. In both cases the strength of the demand response, to a change in the price of a given good by one money unit, results from relationships between the price of a good and a consumer's income ongoing in time. The cross marginal

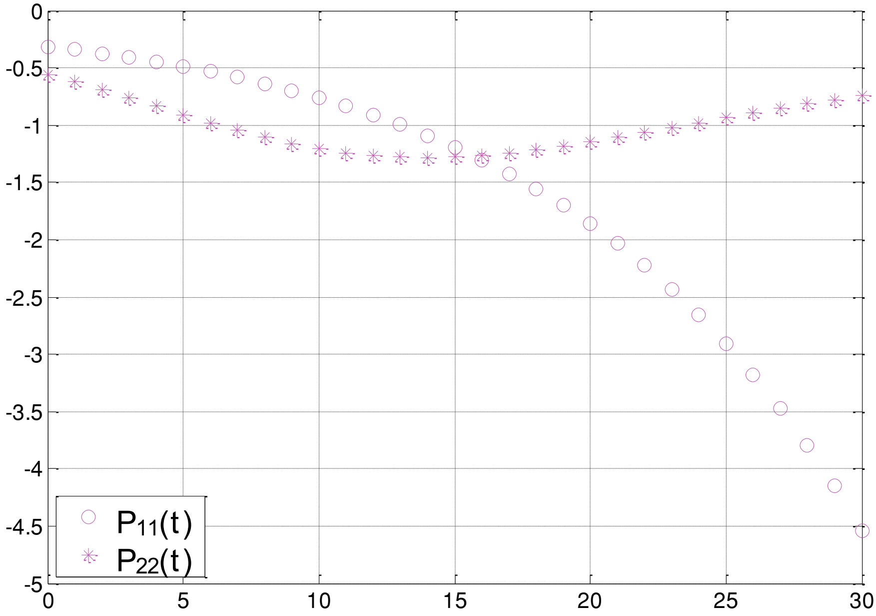


Fig. 2.21 Trajectories of simple marginal demand

demand for both goods is equal to 0 because a change in the price of a given good, *ceteris paribus*, does not affect the demand for the other good.

The marginal demand trajectories with respect to a consumer's income, as shown in Fig. 2.22, illustrate the fact that both goods are normal. In each period, the marginal demand for the first good and the one for the second good with respect to an income is positive, which results from the demand function form and from the definition of the marginal demand with respect to an income:

$$P_1(\mathbf{p}(t), I(t)) = \frac{1}{2p_1(t)},$$

$$P_2(\mathbf{p}(t), I(t)) = \frac{1}{2p_2(t)}.$$

In Fig. 2.22 we can see that initially the demand for the second good responds more strongly to the increase in a consumer's income than the demand for the first good. From period 16 on, this relationship is opposite the response of the demand for the first commodity becomes stronger. In both cases the strength of the demand response to a change in the income results from the price levels of goods in each period.

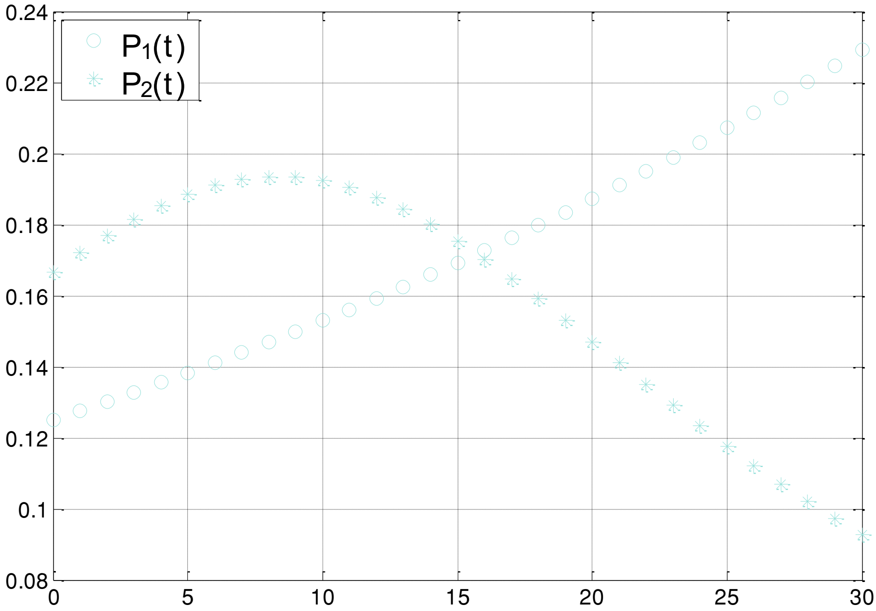


Fig. 2.22 Trajectories of marginal demand with respect to income

2.5 Hicksian Demand Function

2.5.1 Static Approach

Let us consider a market of two consumer goods where:

$i = 1, 2$—consumer goods (products and services),
$X = \mathbb{R}^2_+$—a goods space,
$\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2_+$—a vector of prices of consumer goods,
$u: \mathbb{R}^2_+ \to \mathbb{R}$—a utility function describing preferences of a consumer (describing a relation of consumer preferences).

Considering different consumption bundles with a fixed utility level $u(x_1, x_2) = u$, a consumer wants to choose such a bundle $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2)$ (the optimal one) for which the cost of purchase is minimal by prices determined by the market. In view of a selected optimality criterion, that is of the cost, this problem of choice of the optimal consumption bundle, can be written as a consumer's expenditure minimization problem of a form:

(2.138) **(P4)** $$w(x_1, x_2) = \{p_1 x_1 + p_2 x_2\} \to \min$$

(2.139) $$u(x_1, x_2) = u = \text{const.},$$

$$x_1, x_2 \geq 0. \tag{2.140}$$

Note 2.28 We assume in most cases that a utility function is increasing, strictly concave and differentiable.

Example 2.13 Use the geometric method to find the optimal solution to the expenditure minimization problem (P4) when the utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ is:

(a) linear: $u(\mathbf{x}) = a_1x_1 + a_2x_2 = u \Leftrightarrow x_2 = \frac{u-a_1x_1}{a_2}, \quad a_i > 0, \quad i = 1, 2,$
(b) power function:

$$u(\mathbf{x}) = ax_1^{\alpha_1}x_2^{\alpha_2} = u \Leftrightarrow x_2 = \left(\frac{u}{a_2}\right)^{\frac{1}{\alpha_2}} x_1^{-\frac{\alpha_1}{\alpha_2}},$$
$$a,\ \alpha_i\ > 0, i = 1, 2, \quad \alpha_1 + \alpha_2\ < 1$$

(c) logarithmic:

$$u(\mathbf{x}) = a_1 \ln x_1 + a_2 \ln x_2 = u \Leftrightarrow x_2 = e^{\frac{u}{\alpha_2}} x_1^{-\frac{\alpha_1}{\alpha_2}}, \quad a_i > 0, \quad x_i > 0 \quad i = 1, 2$$

(d) subadditive:

$$u(\mathbf{x}) = a_1x_1^{\alpha} + a_2x_2^{\alpha} = u \Leftrightarrow x_2 = \left(\frac{u - a_1x_1^{\alpha}}{a_2}\right)^{\frac{1}{\alpha}},$$
$$a_i\ > 0,\ i = 1, 2, \quad \alpha \in (0; 1)$$

(e) Koopmans-Leontief function:

$$u(\mathbf{x}) = \min\{a_1x_1, a_2x_2\} = u \Leftrightarrow x_1 = \frac{u}{a_1}, \quad x_2 = \frac{u}{a_2}, \quad a_i > 0, \quad i = 1, 2$$

(f) CES function: $u(\mathbf{x}) = \left(a_1x_1^{\gamma} + a_2x_2^{\gamma}\right)^{\frac{\theta}{\gamma}} = u \Leftrightarrow x_2 = \left(\frac{u^{\frac{\gamma}{\theta}} - a_1x_1^{\gamma}}{a_2}\right)^{\frac{1}{\gamma}},$

$$\theta, a_i > 0, \quad i = 1, 2, \quad \gamma \in (-1; 0) \cup (0; +\infty).$$

Justify that:

– for a linear utility function:

– if $\exists \lambda > 0$ $\mathbf{a} = (a_1, a_2) = \lambda(p_1, p_2) = \lambda\mathbf{p}$ then problem (P4) has an infinite number of optimal solutions belonging to a segment $\tilde{\mathbf{x}} = \alpha\mathbf{x}^1 + \beta\mathbf{x}^2, \quad \forall\alpha, \beta \geq 0, \ \alpha + \beta = 1$, where

$$\mathbf{x}^1 = \left(\frac{W}{p_1}, 0\right), \ \mathbf{x}^2 = \left(0, \frac{W}{p_2}\right), \ \text{thus}$$
$$\tilde{\mathbf{x}} = \left(\alpha\frac{W}{p_1}, \beta\frac{W}{p_2}\right), \ \exists_1 \ W > 0, \ \ \forall\alpha, \beta \geq 0, \ \alpha + \beta = 1,$$

– if $\mathbf{a} = (a_1, a_2) \neq \lambda(p_1, p_2) = \lambda\mathbf{p}$ then problem (P4) has exactly one optimal solution:

$$\tilde{\mathbf{x}} = \left(\frac{W}{p_1}, 0\right) \quad \text{or} \quad \tilde{\mathbf{x}} = \left(0, \frac{W}{p_2}\right),$$

where $W > 0$ means the minimal expenditure that a consumer needs to incur to purchase a consumption bundle with the utility $u > 0$.

– for the remaining utility functions problem (P4) has exactly one optimal solution:

$$\exists_1 \ W > 0, \quad \alpha, \beta > 0, \ \alpha + \beta = 1 \quad \tilde{\mathbf{x}} = \alpha\mathbf{x}^1 + \beta\mathbf{x}^2,$$

where

$$\mathbf{x}^1 = \left(\frac{W}{p_1}, 0\right), \ \mathbf{x}^2 = \left(0, \frac{W}{p_2}\right), \ \text{thus:}$$

$$\exists_1 \ W > 0, \quad \alpha, \beta > 0, \ \alpha + \beta = 1 \quad \tilde{\mathbf{x}} = \left(\alpha\frac{W}{p_1}, \beta\frac{W}{p_2}\right).$$

Ad (a) See Figs. 2.23a, 2.23b and 2.23c.

Ad (b) See Fig. 2.23d.

Ad (c) See Fig. 2.23e.

Ad (d) See Fig. 2.23f.

Ad (e) See Fig. 2.23g.

Ad (f) See Figs. 2.23h and 2.23i.

Let us present analytical methods of finding the optimal solution to the consumer's expenditure minimization problem (P4) in case of a strictly concave, differentiable, and increasing utility function.

Example 2.14 We are given a utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$:

1. power function $u(\mathbf{x}) = ax_1^{\alpha_1}x_2^{\alpha_2}, \quad a, \alpha_i > 0, \quad i = 1, 2, \quad \alpha_1 + \alpha_2 < 1$, for which an indifference curve with a fixed utility level $u > 0$ has a form:

(2.141) $$x_2 = \left(\frac{u}{a}\right)^{\frac{1}{\alpha_2}} x_1^{-\frac{\alpha_1}{\alpha_2}},$$

2. logarithmic $u(\mathbf{x}) = a_1 \ln x_1 + a_2 \ln x_2, \quad a_i > 0, \quad x_i > 0 \quad i = 1, 2$, for which an indifference curve with a fixed utility level $u > 0$ has a form:

(2.142) $$x_2 = e^{\frac{u}{a_2}} x_1^{-\frac{a}{a_2}}.$$

Solve the consumer's expenditure minimization problem in case of the power and in case of the logarithmic utility function.

Method 2.4 Ad 1 The optimal solution to problem (P4) belongs to an indifference curve $u(\mathbf{x}) = u > 0$. Hence we can substitute (2.141) into the objective function in problem (P4). We get then an equivalent problem of a form:

(2.143) (P5) $$g(x_1) = p_1x_1 + p_2\left(\frac{u}{a}\right)^{\frac{1}{\alpha_2}} x_1^{-\frac{\alpha_1}{\alpha_2}} \to \min$$

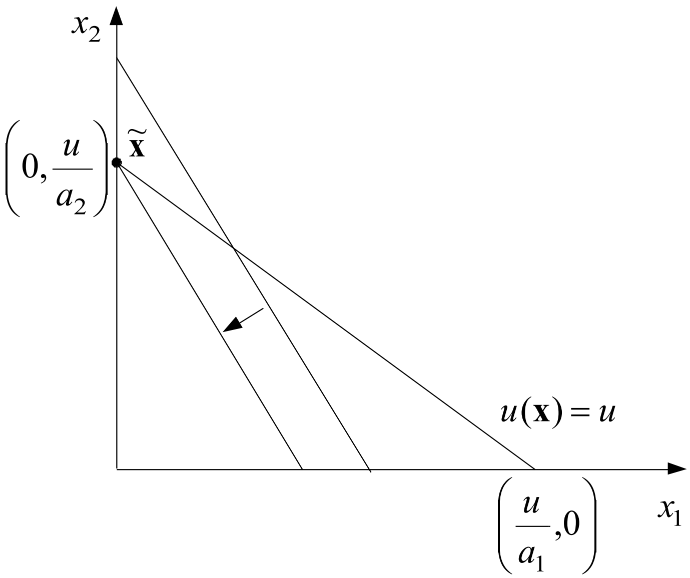


Fig. 2.23a Solution to consumer's expenditure minimization problem with linear utility function when $a_i \neq \lambda p_i, i = 1, 2$ and $a_1 > a_2$

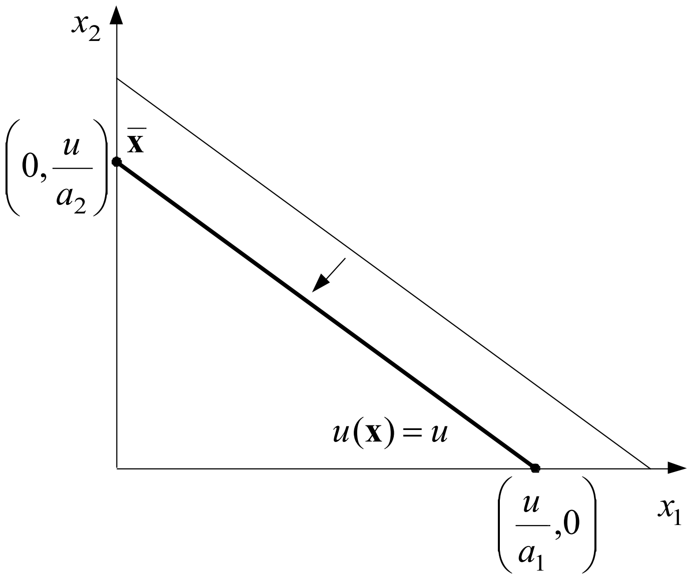


Fig. 2.23b Solution to consumer's expenditure minimization problem with linear utility function when $a_i = \lambda p_i, i = 1, 2$

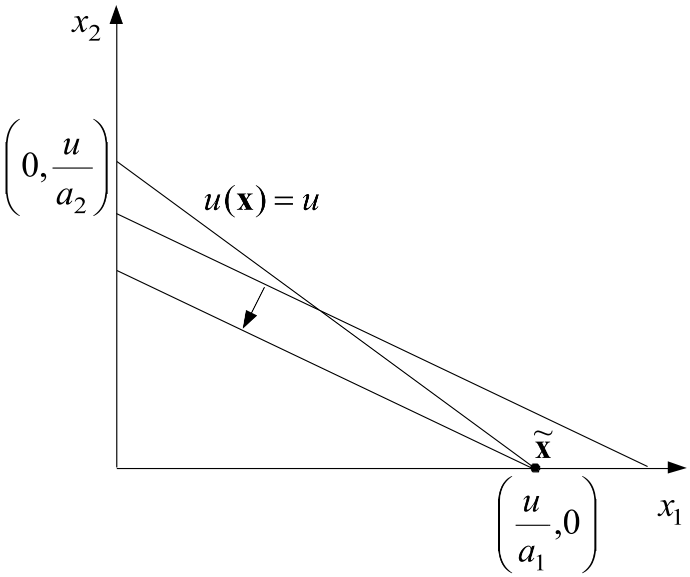


Fig. 2.23c Solution to consumer's expenditure minimization problem with linear utility function when $a_i \neq \lambda p_i, i = 1, 2$ and $a_1 < a_2$

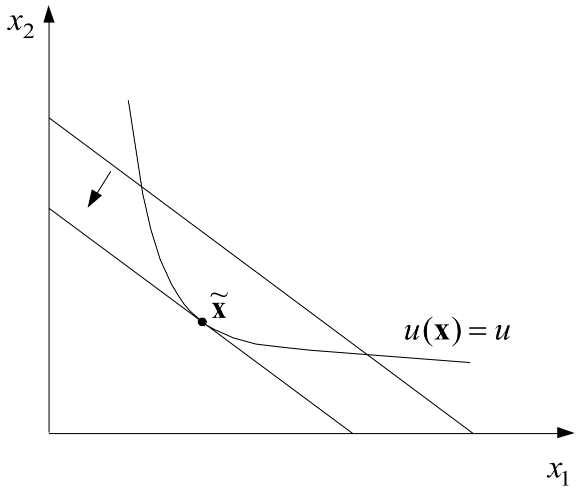


Fig. 2.23d Solution to consumer's expenditure minimization problem with power utility function

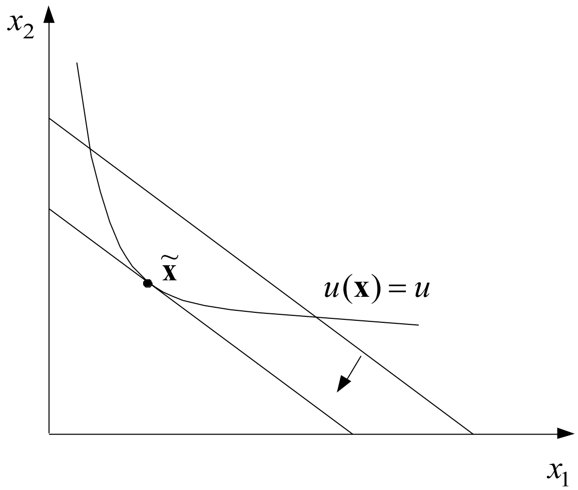


Fig. 2.23e Solution to consumer's expenditure minimization problem with logarithmic utility function

(2.144) $$x_1 \geq 0.$$

To find a solution to problem (P5) one needs to determine a stationary point in which the objective function has the minimal value. For this purpose one needs to use the necessary and the sufficient conditions of minimum existence for a function

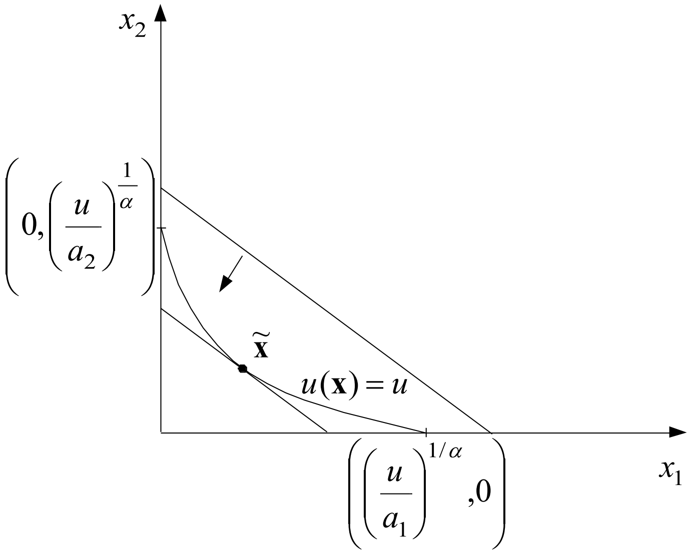


Fig. 2.23f Solution to consumer's expenditure minimization problem with subadditive utility function

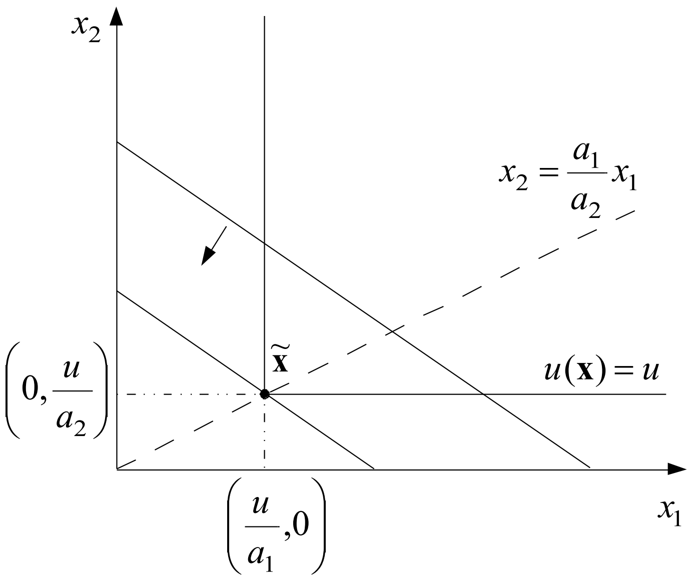


Fig. 2.23g Solution to consumer's expenditure minimization problem with Koopmans-Leontief utility function

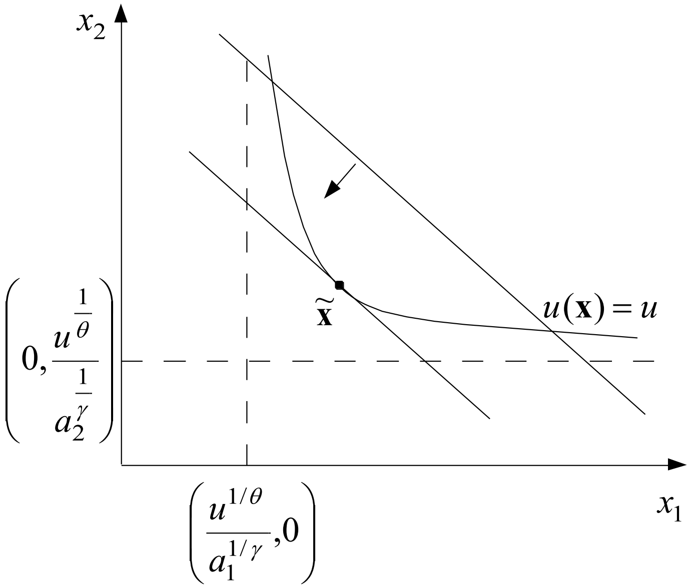


Fig. 2.23h Solution to consumer's expenditure minimization problem with CES utility function when $\gamma \in (0; +\infty)$

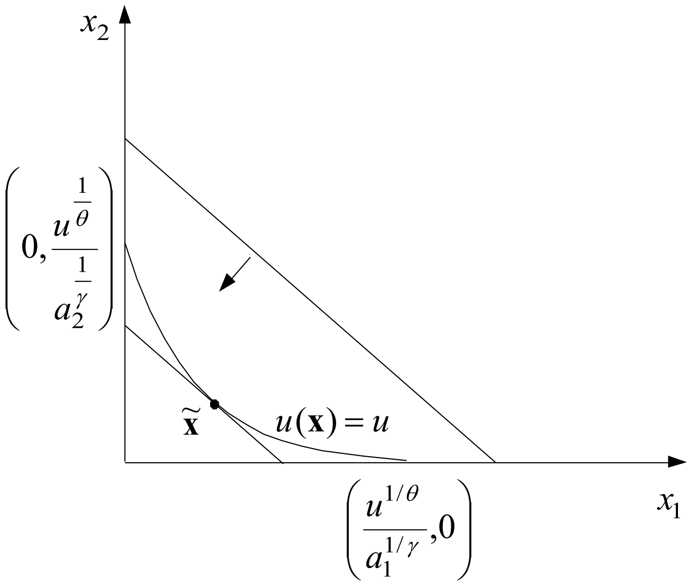


Fig. 2.23i Solution to consumer's expenditure minimization problem with CES utility function when $\gamma \in (-1; 0)$

$g: \mathbb{R}_+ \to \mathbb{R}_+$:

(2.145) $$\left.\frac{\mathrm{d}g(x_1)}{\mathrm{d}x_1}\right|_{x_1 = \tilde{x}_1} = 0 \quad \text{necessary condition,}$$

(2.146) $$\left.\frac{\mathrm{d}^2 g(x_1)}{\mathrm{d}x_1^2}\right|_{x_1 = \tilde{x}_1} > 0 \quad \text{sufficient condition.}$$

Let us make calculations:

(2.147) $$\left.\frac{\mathrm{d}g(x_1)}{\mathrm{d}x_1}\right|_{x_1 = \tilde{x}_1} = p_1 - \frac{\alpha_1}{\alpha_2} p_2 \left(\frac{u}{a}\right)^{\frac{1}{\alpha_2}} \tilde{x}_1^{-\frac{\alpha_1}{\alpha_2}-1} = 0,$$

and hence:

(2.148) $$\tilde{x}_1 = \left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\alpha_1 p_2}{\alpha_2 p_1}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} > 0.$$

We can notice that

(2.149) $$\left.\frac{\mathrm{d}^2 g(x_1)}{\mathrm{d}x_1^2}\right|_{x_1 = \tilde{x}_1} = \left(-\frac{\alpha_1}{\alpha_2} - 1\right)\left(-\frac{\alpha_1}{\alpha_2}\right) p_2 \left(\frac{u}{a}\right)^{\frac{1}{\alpha_2}} \tilde{x}_1^{-\frac{\alpha_1}{\alpha_2}-2} > 0,$$

thus for $\tilde{x}_1 = \left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\alpha_1 p_2}{\alpha_2 p_1}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} > 0$ the objective function in problem (P5) has the minimum value:

(2.150) $$W = g(\tilde{x}_1) = \frac{\alpha_1 + \alpha_2}{\alpha_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \alpha_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}}} \left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} p_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} p_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}}.$$

Substituting (2.148) into (2.141) we get

(2.151) $$\tilde{x}_2 = \left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\alpha_2 p_1}{\alpha_1 p_2}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} > 0,$$

hence:

(2.152) $$\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2) = \left(\left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\alpha_1 p_2}{\alpha_2 p_1}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}, \left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\alpha_2 p_1}{\alpha_1 p_2}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\right)$$

and

$$(2.153)\qquad W(\tilde{x}_1,\tilde{x}_2) = p_1\tilde{x}_1 + p_2\tilde{x}_2 = \frac{\alpha_1+\alpha_2}{\alpha_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\alpha_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}}}\left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} p_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} p_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}}.$$

Ad 2 The optimal solution to problem (P4) belongs to an indifference curve $u(\mathbf{x}) = u > 0$. Hence we can substitute (2.142) into the objective function in problem (P4). We get then an equivalent problem of a form:

$$(2.154)\qquad \textbf{(P6)}\qquad g(x_1) = p_1x_1 + p_2e^{\frac{u}{a_2}}x_1^{-\frac{a_1}{a_2}} \to \min$$

$$(2.155)\qquad x_1 \geq 0.$$

To find a solution to problem (P6) one needs to determine a stationary point in which the objective function has the minimal value. For this purpose one needs to use the necessary and the sufficient conditions of minimum existence for a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$:

$$(2.156)\qquad \left.\frac{\mathrm{d}g(x_1)}{\mathrm{d}x_1}\right|_{x_1=\tilde{x}_1} = 0 \quad \text{necessary condition,}$$

$$(2.157)\qquad \left.\frac{\mathrm{d}^2g(x_1)}{\mathrm{d}x_1^2}\right|_{x_1=\tilde{x}_1} > 0 \quad \text{sufficient condition.}$$

Let us make calculations:

$$(2.158)\qquad \left.\frac{\mathrm{d}g(x_1)}{\mathrm{d}x_1}\right|_{x_1=\tilde{x}_1} = p_1 - \frac{a_1}{a_2}p_2e^{\frac{u}{a_2}}x_1^{-\frac{a_1}{a_2}-1} = 0,$$

and hence:

$$(2.159)\qquad \tilde{x}_1 = e^{\frac{u}{a_1+a_2}}\left(\frac{a_1p_2}{a_2p_1}\right)^{\frac{a_2}{a_1+a_2}} > 0,$$

We can notice that

$$(2.160)\qquad \left.\frac{\mathrm{d}^2g(x_1)}{\mathrm{d}x_1^2}\right|_{x_1=\tilde{x}_1} = \left(-\frac{a_1}{a_2}-1\right)\left(-\frac{a_1}{a_2}\right)p_2e^{\frac{u}{a_2}}\tilde{x}_1^{-\frac{a_1}{a_2}-2} > 0,$$

thus for $\tilde{x}_1 = e^{\frac{u}{a_1+a_2}} \left(\frac{a_1 p_2}{a_2 p_1}\right)^{\frac{a_2}{a_1+a_2}} > 0$ the objective function in problem (P6) has the minimum value:

$$W = g(\tilde{x}_1) = \frac{a_1 + a_2}{a_1^{\frac{a_1}{a_1+a_2}} a_2^{\frac{a_2}{a_1+a_2}}} e^{\frac{u}{a_1+a_2}} p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}}. \tag{2.161}$$

Substituting (2.159) into (2.142) we get

$$\tilde{x}_2 = e^{\frac{u}{a_1+a_2}} \left(\frac{a_2 p_1}{a_1 p_2}\right)^{\frac{a_1}{a_1+a_2}} > 0, \tag{2.162}$$

hence:

$$\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2) = \left(e^{\frac{u}{a_1+a_2}} \left(\frac{a_1 p_2}{a_2 p_1}\right)^{\frac{a_2}{a_1+a_2}}, e^{\frac{u}{a_1+a_2}} \left(\frac{a_2 p_1}{a_1 p_2}\right)^{\frac{a_1}{a_1+a_2}}\right) \tag{2.163}$$

and

$$W(\tilde{x}_1, \tilde{x}_2) = p_1 \tilde{x}_1 + p_2 \tilde{x}_2 = \frac{a_1 + a_2}{a_1^{\frac{a_1}{a_1+a_2}} a_2^{\frac{a_2}{a_1+a_2}}} e^{\frac{u}{a_1+a_2}} p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}}. \tag{2.164}$$

Method 2.5 We know that if a utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ is increasing and strictly concave, then problem (P4) has exactly one optimal solution. It belongs to an indifference curve $u(\tilde{x}_1, \tilde{x}_2) = u$ and is of a form:

$$\exists_1 \; W > 0, \;\; \alpha, \beta > 0, \;\; \alpha + \beta = 1$$
$$\tilde{\mathbf{x}} = \alpha\left(\frac{W}{p_1}, 0\right) + \beta\left(0, \frac{W}{p_2}\right) = \left(\alpha\frac{W}{p_1}, \beta\frac{W}{p_2}\right) > (0, 0).$$

At a point indicating the optimal consumption bundle $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2)$ it is satisfied that an expenditure line W is tangent to the indifference curve. Hence we get

$$s_{12}(\tilde{x}_1, \tilde{x}_2) = -\left.\frac{\mathrm{d}x_2}{\mathrm{d}x_1}\right|_{\mathbf{x} = \tilde{\mathbf{x}}} = \frac{\left.\frac{\partial u(x_1, x_2)}{\partial x_1}\right|_{\mathbf{x} = \tilde{\mathbf{x}}}}{\left.\frac{\partial u(x_1, x_2)}{\partial x_2}\right|_{\mathbf{x} = \tilde{\mathbf{x}}}} = \frac{p_1}{p_2}. \tag{2.165}$$

We see that to find the optimal solution to problem (P4), when a utility function is increasing and strictly concave, one needs to solve a system of two equations with two unknowns:

(2.166)
$$s_{12}(\tilde{x}_1, \tilde{x}_2) = \frac{\left.\frac{\partial u(x_1,x_2)}{\partial x_1}\right|_{\mathbf{x} = \widetilde{\mathbf{x}}}}{\left.\frac{\partial u(x_1,x_2)}{\partial x_2}\right|_{\mathbf{x} = \widetilde{\mathbf{x}}}} = \frac{p_1}{p_2},$$

(2.167)
$$u(\tilde{x}_1, \tilde{x}_2) = u.$$

Method 2.6 Let us write problem (P4) in a form of a Lagrange function:

(2.168)
$$F(x_1, x_2, \lambda) = p_1x_1 + p_2x_2 + \lambda(u - u(x_1, x_2)),$$

where $\lambda \geq 0$ denotes a Lagrange multiplier.

If a utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ is increasing and strictly concave then problem (P4) has exactly one optimal solution. It belongs to an expenditure line W, and has a form:

$$\exists_1\ W\ >0,\ \alpha, \beta\ > 0,\ \alpha + \beta = 1$$
$$\widetilde{\mathbf{x}} = \alpha\left(\frac{W}{p_1}, 0\right) + \beta\left(0, \frac{W}{p_2}\right) = \left(\alpha\frac{W}{p_1}, \beta\frac{W}{p_2}\right) > \ (0, 0),$$

and is a solution to a system of 3 equations with 3 unknowns:

(2.169)
$$\left.\frac{\partial F\left(x_1, x_2, \tilde{\lambda}\right)}{\partial x_1}\right|_{\mathbf{x} = \tilde{\mathbf{x}}} = p_1 - \tilde{\lambda}\left.\frac{\partial u(x_1, x_2)}{\partial x_1}\right|_{\mathbf{x} = \tilde{\mathbf{x}}} = 0,$$

(2.170)
$$\left.\frac{\partial F\left(x_1, x_2, \tilde{\lambda}\right)}{\partial x_2}\right|_{\mathbf{x} = \tilde{\mathbf{x}}} = p_2 - \tilde{\lambda}\left.\frac{\partial u(x_1, x_2)}{\partial x_2}\right|_{\mathbf{x} = \tilde{\mathbf{x}}} = 0,$$

(2.171)
$$\left.\frac{\partial F(\tilde{x}_1, \tilde{x}_2, \lambda)}{\partial \lambda}\right|_{\lambda = \tilde{\lambda}} = u - u(\tilde{x}_1, \tilde{x}_2) = 0,$$

or to an equivalent equation system of a form:

(2.172)
$$\tilde{\lambda}\left.\frac{\partial u(x_1, x_2)}{\partial x_1}\right|_{\mathbf{x} = \tilde{\mathbf{x}}} = p_1,$$

(2.173)
$$\tilde{\lambda}\left.\frac{\partial u(x_1, x_2)}{\partial x_2}\right|_{\mathbf{x} = \tilde{\mathbf{x}}} = p_2,$$

$$u(\tilde{x}_1, \tilde{x}_2) = u, \tag{2.174}$$

where $\widetilde{\lambda} > 0$ denotes an optimal Lagrange multiplier.

Let us notice that solving the Eqs. (2.172)–(2.174), for example by dividing Eq. (2.172) on both sides by Eq. (2.173), we get a system:

$$\frac{\left.\frac{\partial u(x_1,x_2)}{\partial x_1}\right|_{\mathbf{x}=\tilde{\mathbf{x}}}}{\left.\frac{\partial u(x_1,x_2)}{\partial x_2}\right|_{\mathbf{x}=\tilde{\mathbf{x}}}} = \frac{p_1}{p_2}, \tag{2.175}$$

$$u(\tilde{x}_1, \tilde{x}_2) = u, \tag{2.176}$$

which is equivalent to system (2.166)–(2.167).

Definition 2.43 A **Hicksian demand function** or **compensated demand function** is a mapping $f\colon \operatorname{int}\mathbb{R}^2_+ \times \mathbb{R} \to \operatorname{int}\mathbb{R}^2_+$ which assigns the optimal solution of the minimization problem (P4) of consumer's expenditure to any price vector $\mathbf{p} = (p_1, p_2) \in \operatorname{int}\mathbb{R}^2_+$ and any utility level $u \in \mathbb{R}$ of a consumption bundle. It is given as

$$f(\mathbf{p}, u) = (f_1(p_1, p_2, u), f_2(p_1, p_2, I)) = (\tilde{x}_1, \tilde{x}_2) = \tilde{\mathbf{x}}. \tag{2.177}$$

Definition 2.44 A **consumer's expenditure function** is a mapping $e\colon \operatorname{int}\mathbb{R}^2_+ \times \mathbb{R} \to \operatorname{int}\mathbb{R}_+$ which assigns a minimal expenditure that a consumer incurs to purchase a consumption bundle with a utility level $u \in \mathbb{R}$ to any price vector $\mathbf{p} = (p_1, p_2) \in \operatorname{int}\mathbb{R}^2_+$ and the utility level u. It is given as

$$\begin{aligned} e(\mathbf{p}, u) &= w(\tilde{x}_1, \tilde{x}_2) = p_1\tilde{x}_1 + p_2\tilde{x}_2 \\ &= p_1 f_1(p_1, p_2, u) + p_2 f_2(p_1, p_2, I). \end{aligned} \tag{2.178}$$

Theorem 2.4 If a utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ is increasing, twice differentiable and strictly concave, then $\forall\, p_1, p_2 > 0, u > u(0)$:

1. the compensated demand function $f\colon \operatorname{int}\mathbb{R}^2_+ \times \mathbb{R} \to \operatorname{int}\mathbb{R}^2_+$ is differentiable in its domain,
2. the consumer's expenditure function $e\colon \operatorname{int}\mathbb{R}^2_+ \times \mathbb{R} \to \operatorname{int}\mathbb{R}_+$ is differentiable in its domain,
3. $\forall \lambda > 0 \quad e(\lambda p_1, \lambda p_2, u) = \lambda e(p_1, p_2, u)$,
   which means that a consumer's expenditure function is homogeneous of degree 1 with respect to prices of commodities (a proportionate change in prices of commodities results in a proportionate change in a consumer's expenditure),

4. $\forall \lambda > 0 \quad f(\lambda p_1, \lambda p_2, u) = f(p_1, p_2, u)$,
   which means that a compensated demand function is homogeneous of degree 0 with respect to prices of commodities (a proportionate change in prices of commodities income does not change the compensated demand for consumer goods),
5. $\dfrac{\partial e(\mathbf{p}, u)}{\partial p_i} = f_i(\mathbf{p}, u), \quad i = 1, 2,$

   which means that knowing an analytical form of a consumer's expenditure function one is able to determine an analytical form of a compensated demand function,
6. an increase in the price of i-th good results (always) in a decrease of the compensated demand for i-th good:
$$\frac{\partial f_i(\mathbf{p}, u)}{\partial p_i} < 0, \quad i = 1, 2,$$
7. $\frac{\partial f_1(\mathbf{p},u)}{\partial p_2} = \frac{\partial f_2(\mathbf{p},u)}{\partial p_1}$,
   which means that an effect of an increase in the second good price on the compensated demand for the first good is the same as an effect of an increase in the first good price on the compensated demand for the second good.

Example 2.15 For a compensated demand function and the corresponding consumer's expenditure function:

(a) in case of a power utility function in problem (P4):

$$f(\mathbf{p}, u) = \left(\left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\alpha_1 p_2}{\alpha_2 p_1}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}, \left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\alpha_2 p_1}{\alpha_1 p_2}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \right) \tag{2.179}$$

$$e(\mathbf{p}, u) = \frac{\alpha_1 + \alpha_2}{\alpha_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \alpha_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}}} \left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} p_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} p_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \tag{2.180}$$

(b) in case of a logarithmic utility function in problem (P4):

$$f(\mathbf{p}, u) = \left(e^{\frac{u}{a_1+a_2}} \left(\frac{a_1 p_2}{a_2 p_1}\right)^{\frac{a_2}{a_1+a_2}}, e^{\frac{u}{a_1+a_2}} \left(\frac{a_2 p_1}{a_1 p_2}\right)^{\frac{a_1}{a_1+a_2}} \right) \tag{2.181}$$

$$e(\mathbf{p}, u) = \frac{a_1 + a_2}{a_1^{\frac{a_1}{a_1+a_2}} a_2^{\frac{a_2}{a_1+a_2}}} e^{\frac{u}{a_1+a_2}} p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}} \tag{2.182}$$

justify the properties that appear in statements 3–7 of Theorem 2.4.

Let us first show that the given consumer's expenditure functions are homogenous of order 1 with respect to prices of commodities.

Ad 3a

$$\forall\lambda > 0 \quad e(\lambda\mathbf{p}, u) = \frac{\alpha_1 + \alpha_2}{\alpha_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \alpha_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}}} \left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} (\lambda p_1)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} (\lambda p_2)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} = \lambda e(\mathbf{p}, u)$$

Ad 3b

$$\forall\lambda > 0 \quad e(\lambda\mathbf{p}, u) = \frac{a_1 + a_2}{a_1^{\frac{a_1}{a_1+a_2}} a_2^{\frac{a_2}{a_1+a_2}}} e^{\frac{u}{a_1+a_2}} (\lambda p_1)^{\frac{a_1}{a_1+a_2}} (\lambda p_2)^{\frac{a_1}{a_1+a_2}} = \lambda e(\mathbf{p}, u)$$

Ad 4a Let us now present that the given Hicksian demand functions are homogenous of order 0 with respect to prices of commodities.

$$\forall\lambda > 0 \quad f(\lambda\mathbf{p}, u) = \left(\left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\alpha_1 p_2}{\alpha_2 p_1}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}, \left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\alpha_2 p_1}{\alpha_1 p_2}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\right) = f(\mathbf{p}, u)$$

Ad 4b

$$\forall\lambda > 0 \quad f(\lambda\mathbf{p}, u) = \left(e^{\frac{u}{a_1+a_2}} \left(\frac{a_1\lambda p_2}{a_2\lambda p_1}\right)^{\frac{a_2}{a_1+a_2}}, e^{\frac{u}{a_1+a_2}} \left(\frac{a_2\lambda p_1}{a_1\lambda p_2}\right)^{\frac{a_1}{a_1+a_2}}\right) = f(\mathbf{p}, u)$$

Ad 5a Let us justify that partial derivatives of each of the given consumer's expenditure functions with respect to price of i-th good are equal to the corresponding compensated demand functions.

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_1} = \frac{\alpha_1 + \alpha_2}{\alpha_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \alpha_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}}} \frac{\alpha_1}{\alpha_1 + \alpha_2} \left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} p_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}-1} p_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}} = f_1(\mathbf{p}, u)$$

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_2} = \frac{\alpha_1 + \alpha_2}{\alpha_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \alpha_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}}} \frac{\alpha_2}{\alpha_1 + \alpha_2} \left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} p_1^{\frac{\alpha_1}{\alpha_1+\alpha_2}} p_2^{\frac{\alpha_2}{\alpha_1+\alpha_2}-1} = f_2(\mathbf{p}, u)$$

Ad 5b

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_1} = \frac{a_1 + a_2}{a_1^{\frac{a_1}{a_1+a_2}} a_2^{\frac{a_2}{a_1+a_2}}} \frac{a_1}{a_1 + a_2} e^{\frac{u}{a_1+a_2}} p_1^{\frac{a_1}{a_1+a_2}-1} p_2^{\frac{a_2}{a_1+a_2}} = f_1(\mathbf{p}, u)$$

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_2} = \frac{a_1 + a_2}{a_1^{\frac{a_1}{a_1+a_2}} a_2^{\frac{a_2}{a_1+a_2}}} \frac{a_2}{a_1 + a_2} e^{\frac{u}{a_1+a_2}} p_1^{\frac{a_1}{a_1+a_2}} p_2^{\frac{a_2}{a_1+a_2}-1} = f_2(\mathbf{p}, u)$$

Ad 6a Let us verify if an increase in the price of i-th good results in a decrease of the compensated demand for i-th good.

$$\frac{\partial f(\mathbf{p},u)}{\partial p_1} = -\frac{\alpha_2}{\alpha_1+\alpha_2}\left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}}\left(\frac{\alpha_1 p_2}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} p_1^{-\frac{\alpha_2}{\alpha_1+\alpha_2}-1} < 0$$

$$\frac{\partial f(\mathbf{p},u)}{\partial p_2} = -\frac{\alpha_2}{\alpha_1+\alpha_2}\left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}}\left(\frac{\alpha_1 p_1}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} p_2^{-\frac{\alpha_1}{\alpha_1+\alpha_2}-1} < 0$$

Ad 6b

$$\frac{\partial f(\mathbf{p},u)}{\partial p_1} = -\frac{a_2}{a_1+a_2}e^{\frac{u}{a_1+a_2}}\left(\frac{a_1 p_2}{a_2}\right)^{\frac{a_2}{a_1+a_2}} p_1^{-\frac{a_2}{a_1+a_2}-1} < 0$$

$$\frac{\partial f(\mathbf{p},u)}{\partial p_2} = -\frac{a_1}{a_1+a_2}e^{\frac{u}{a_1+a_2}}\left(\frac{a_1 p_1}{a_2}\right)^{\frac{a_2}{a_1+a_2}} p_2^{-\frac{a_1}{a_1+a_2}-1} < 0$$

Ad 7a Let us verify if an effect of an increase in the second good price on the compensated demand for the first good is the same as an effect of an increase in the first good price on the compensated demand for the second good.

$$\left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}}\left(\frac{\alpha_1 p_2}{\alpha_2 p_1}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}, \left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}}\left(\frac{\alpha_2 p_1}{\alpha_1 p_2}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}}$$

$$\frac{\partial f_1(\mathbf{p},u)}{\partial p_2} = \frac{1}{\alpha_1+\alpha_2}\left(\frac{u}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \alpha_1^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \alpha_2^{\frac{\alpha_1}{\alpha_1+\alpha_2}} p_1^{\frac{-\alpha_2}{\alpha_1+\alpha_2}} p_2^{\frac{-\alpha_1}{\alpha_1+\alpha_2}} = \frac{\partial f_2(\mathbf{p},u)}{\partial p_1}$$

Ad 7b

$$\frac{\partial f_1(\mathbf{p},u)}{\partial p_2} = \frac{1}{a_1+a_2}e^{\frac{u}{a_1+a_2}} a_1^{\frac{a_2}{a_1+a_2}} a_2^{\frac{a_1}{a_1+a_2}} p_1^{\frac{-a_2}{a_1+a_2}} p_2^{\frac{-a_1}{a_1+a_2}} = \frac{\partial f_2(\mathbf{p},u)}{\partial p_1}$$

2.5.2 Dynamic Approach

A variable t means the discrete time: $t = 0, 1, 2, \ldots, T$ or the continuous time: $t \in [0; T]$. T means a time horizon. In the consumer's expenditure minimization problem we use the following notation:

$\mathbf{p}(t) = (p_1(t), p_2(t)) \geq 0$—a vector of time-varying prices of goods,
$u(t) \in \mathbb{R}$—a utility level that a consumer wants to achieve by purchasing a consumption bundle,
$\mathbf{x}(t) = (x_1(t), x_2(t)) \geq 0$—a bundle of goods that a consumer is willing to purchase at any period/moment t at prices $\mathbf{p}(t)$.

The consumer's expenditure minimization problem has a form:

(2.183) $$\{p_1(t)x_1(t) + p_2(t)x_2(t)\} \mapsto \min$$

(2.184) $$u(\mathbf{x}(t)) = u(t)$$

(2.185) $$\mathbf{x}(t) \geq 0.$$

If a utility function $u(\mathbf{x}(t))$ is increasing and strongly concave, then in each period/moment t a line indicating the minimal expenditure incurred for the optimal consumption bundle is tangent to an indifference curve resulting from Eq. (2.184), because a consumer wants to achieve a given utility level u by the minimal expenditure on purchasing the consumption bundle. This results in an optimality condition of the consumption bundle:

(2.186) $$s_{12}(\tilde{\mathbf{x}}(t)) = \frac{p_1(t)}{p_2(t)} \qquad \forall t,$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0; T]$. This property is called Gossen's second law. The optimal bundle is a solution of the system of Eqs. (2.184) and (2.186).

From the problem of the consumer's expenditure minimization, we obtain the optimal bundle $\widetilde{\mathbf{x}}(t)$, a time-dependent compensated (Hicksian) demand function:

(2.187) $$f(\mathbf{p}(t), u(t)) = \widetilde{\mathbf{x}}(t)$$

and a corresponding consumer's expenditure function:

(2.188) $$e(\mathbf{p}(t), u(t)) = p_1(t)\tilde{x}_1(t) + p_2(t)\tilde{x}_2(t).$$

The compensated demand function, as well as the consumer's expenditure function, in all periods/moments t has the same form, but at different periods/moments it may take different values depending on the prices of goods and on a utility level $u(t)$, which vary over time. Depending on a utility function describing consumer's preferences towards consumption bundles, the compensated demand function $f(\mathbf{p}(t), u(t))$ and the consumer's expenditure function $e(\mathbf{p}(t), u(t))$ take the form accordingly:

(a) in case of a power utility function $u(\mathbf{x}(t)) = ax_1(t)^{\alpha_1}x_2(t)^{\alpha_2}$, $a > 0$, $\alpha_1 + \alpha_2 < 1$

(2.189)
$$f(\mathbf{p}(t),\ u(t)) = \left(\left(\frac{u(t)}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\alpha_1 p_2(t)}{\alpha_2 p_1(t)}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}},\ \left(\frac{u(t)}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \left(\frac{\alpha_2 p_1(t)}{\alpha_1 p_2(t)}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \right)$$

(2.190) $$e(\mathbf{p}(t), u(t)) = \left(\frac{u(t)}{a}\right)^{\frac{1}{\alpha_1+\alpha_2}} \left[p_1(t)\left(\frac{\alpha_1 p_2(t)}{\alpha_2 p_1(t)}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} + p_2(t)\left(\frac{\alpha_2 p_1(t)}{\alpha_1 p_2(t)}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \right]$$

(b) in case of a logarithmic utility function $u(\mathbf{x}(t)) = a_1 \ln x_1(t) + a_2 \ln x_2(t)$, $a_1, a_2 > 0$, $x_1, x_2 > 0$

(2.191) $$f(\mathbf{p}(t), u(t)) = \left(e^{\frac{u(t)}{\alpha_1+\alpha_2}} \left(\frac{\alpha_1 p_2(t)}{\alpha_2 p_1(t)}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}}, e^{\frac{u(t)}{\alpha_1+\alpha_2}} \left(\frac{\alpha_2 p_1(t)}{\alpha_1 p_2(t)}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \right)$$

(2.192)
$$e(\mathbf{p}(t), u(t)) = e^{\frac{u(t)}{\alpha_1+\alpha_2}} \left[p_1(t)\left(\frac{\alpha_1 p_2(t)}{\alpha_2 p_1(t)}\right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} + p_2(t)\left(\frac{\alpha_2 p_1(t)}{\alpha_1 p_2(t)}\right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \right]$$

(c) in case of a Koopmans-Leontief utility function $u(\mathbf{x}(t)) = \min\{a_1 x_1(t), a_2 x_2(t)\}$, $a_1, a_2 > 0$

(2.193) $$f(\mathbf{p}(t), u(t)) = \left(\frac{u(t)}{a_1}, \frac{u(t)}{a_2}\right)$$

(2.194) $$e(\mathbf{p}(t), u(t)) = u(t)\left[\frac{p_1(t)}{a_1} + \frac{p_2(t)}{a_2}\right]$$

Example 2.16 Let us assume that at every moment a utility level that a consumer wants to achieve and the prices of the first and of the second goods change according to equations:

$$u(t) = 1 + 0.1 \ln(t+1),$$

$$p_1(t) = 4 \cdot 0.98^t,$$

$$p_2(t) = 0.006t^2 - 0.1t + 3,$$

$$t \in [0; 30],$$

which means that initially a consumer wants to have a utility level equal to 1 and after a gradual logarithmic increase the utility level equal to about 1.34 at the end of a time horizon. With time a consumer wants to have bigger and bigger utility of a consumption bundle he/she purchases. At the beginning this increase of utility is faster and from about moment $t = 10$ it is approximately a linear increase. In Fig. 2.24 we present a trajectory of the utility level a consumer wants to achieve by purchasing a consumption bundle at any given moment of the time horizon.

Price trajectories, shown in Fig. 2.25, are analogous to those presented in Example 2.12, with a difference that now they are considered in a continuous-time version.

Consumer's preferences towards consumption bundles are described by a power utility function of a form $u(\mathbf{x}(t)) = x_1(t)^{0,5} x_2(t)^{0,5}$, for which a corresponding compensated demand function is

$$f(\mathbf{p}(t), u(t)) = \left(u(t) \left(\frac{p_2(t)}{p_1(t)} \right)^{\frac{1}{2}}, u(t) \left(\frac{p_1(t)}{p_2(t)} \right)^{\frac{1}{2}} \right)$$

and a consumer's expenditure function is

$$e(\mathbf{p}(t), u(t)) = 2u(t) p_1(t)^{0.5} p_2(t)^{0.5}.$$

At each moment t, a line indicating the minimal expenditure incurred for the optimal consumption bundle is tangent to an indifference curve resulting from the consumer's preferences described by the given utility function and from the utility

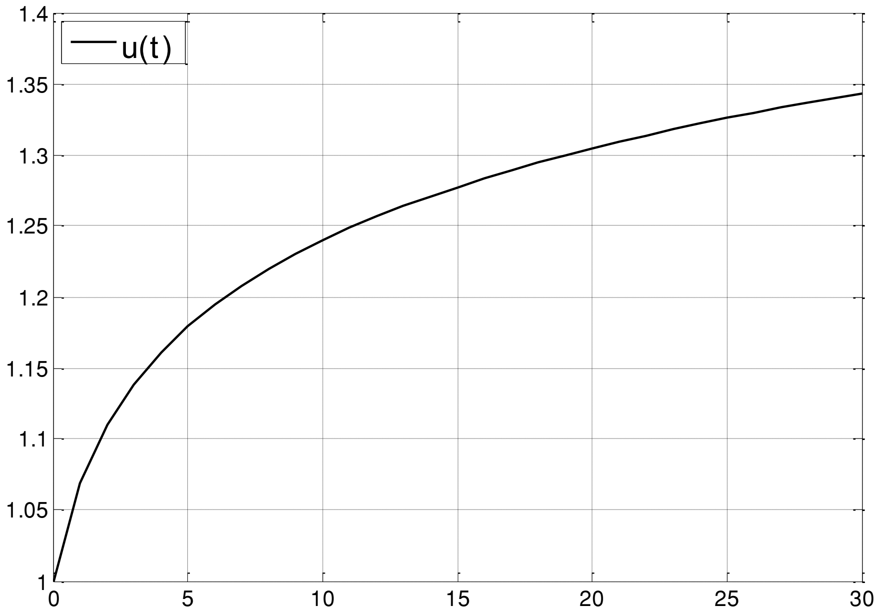


Fig. 2.24 Trajectory of utility level that consumer wants to achieve

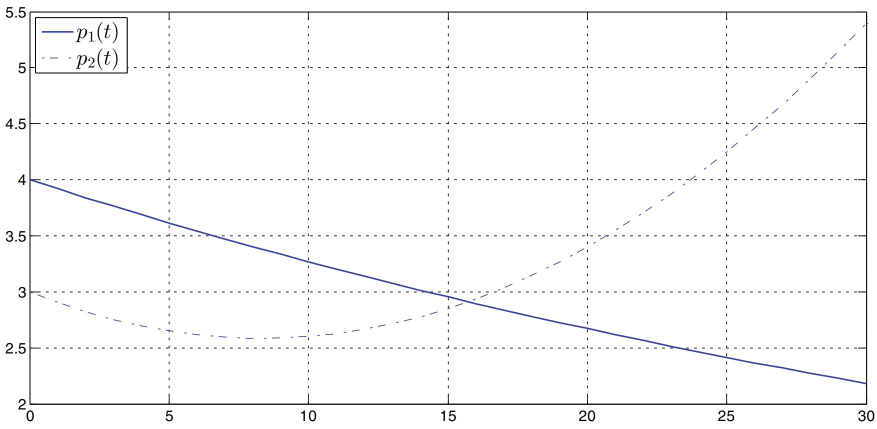


Fig. 2.25 Price trajectories

level he/she wants to achieve at a given moment. Figure 2.26 shows this relationship for moments: 0, 15 and 30. For each moment $t \in [0; 30]$ an analogous relationship can be presented. We can see from the figure that the line indicating the minimal expenditure changes its location and slope. This is due to the relationship between the prices of both goods changing over time and the utility level a consumer wants to achieve which also changes over time.

Trajectories of the compensated demand for the first and the second goods are shown in Fig. 2.27. The compensated demand for the first good increases in the whole considered time horizon, while the compensated demand for the second commodity initially grows, reaches its maximum equal to 4 at around moment $t = 8$ and then decreases until the end of the time horizon.

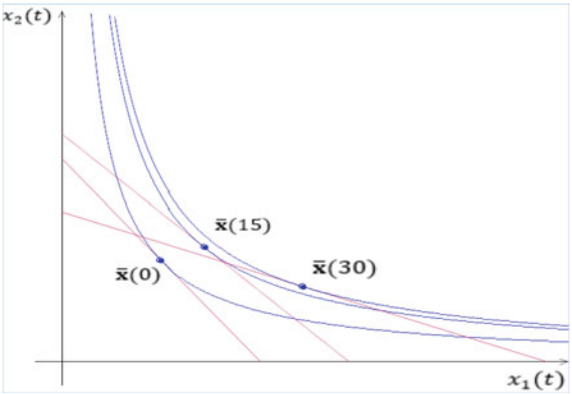


Fig. 2.26 Consumer's expenditure minimization problem

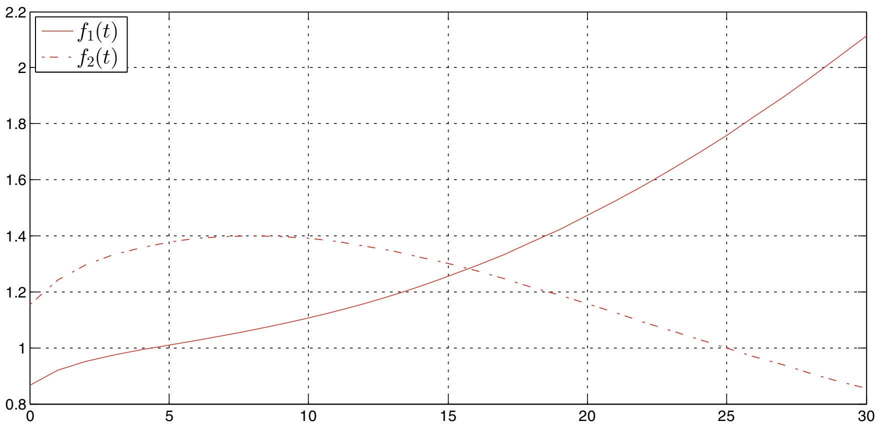


Fig. 2.27 Compensated demand trajectories for first and second good

A trajectory of the consumer's expenditure is shown in Fig. 2.28. The expenditure on purchasing a consumption bundle is growing, except time interval $(3.1; 9.3)$. But the decrease in expenditure is slight. Then, the expenditure is constantly growing until the end of the time horizon. The presented expenditure trajectory results from the evolution of prices of both goods and the evolution of the utility level a consumer wants to achieve, all of which change over time.

In each period Gossen's second law is obeyed, as shown in Fig. 2.29. This means that the trajectory of the marginal rate of substitution of the first good by the second good in the optimal bundle matches up with the trajectory of the quotient of the price of the first good by the price of the second good. On the basis of this dependency, together with the equation of the indifference curve with the utility level a consumer wants to achieve, we have determined the optimal bundle.

In Fig. 2.30 we see that at each moment a consumer achieves a given utility level he/she wants to have from purchasing the optimal consumption bundle. Hence a consumer achieves the given utility level and at the same time when purchasing the consumption bundle which gives this utility level he/she incurs the minimal expenditure. Figures 2.29 and 2.30 illustrate the fact that at each moment of the considered time horizon, the compensated demand function indicates optimal bundles, which results from the consumer's expenditure minimization problem and from a definition of the Hicksian demand function.

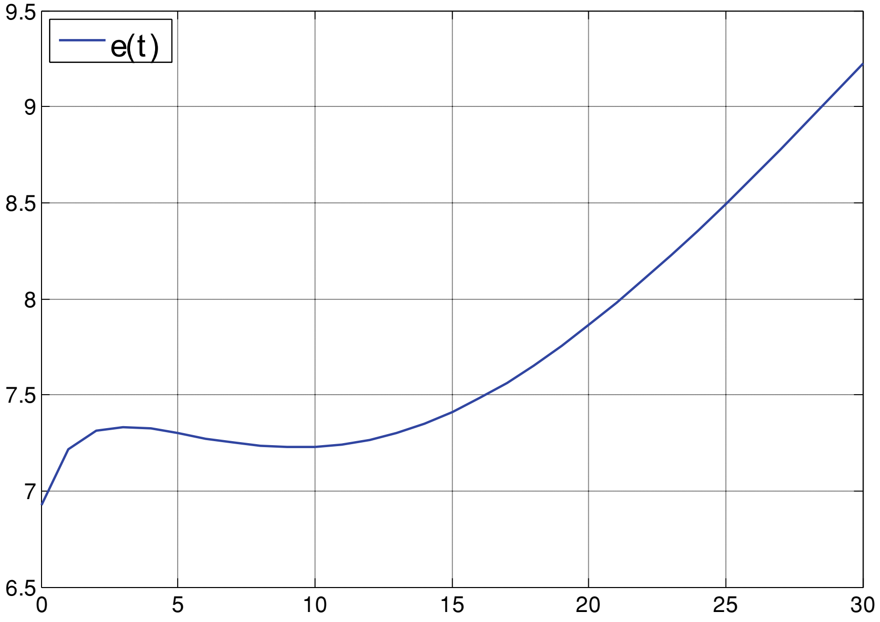


Fig. 2.28 Consumer's expenditure trajectory

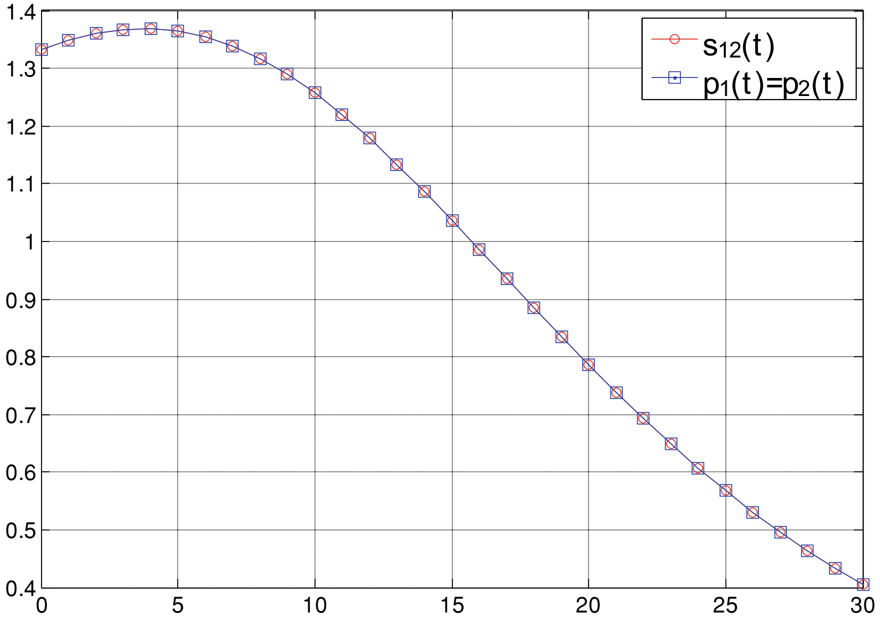


Fig. 2.29 Trajectory of marginal rate of substitution in optimal bundle

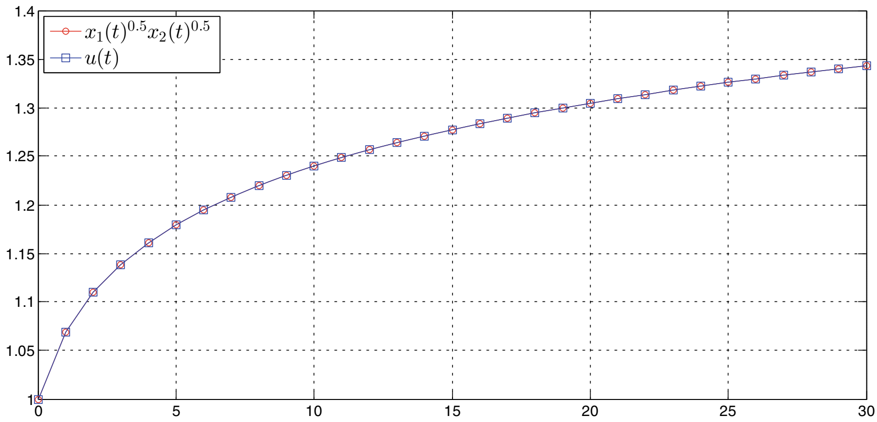


Fig. 2.30 Trajectory of purchased optimal bundle's utility

2.6 Substitution and Income Effects of Changes in Prices of Goods

2.6.1 Static Approach

Let us define links between the utility maximization problem (P2) and the consumer's expenditure minimization problem (P4). The question is when the optimal solutions to these problems are identical that means when values of Marshallian and Hicksian demand functions are the same:

$$\varphi(\mathbf{p}, I) = \bar{\mathbf{x}} = \widetilde{\mathbf{x}} = \boldsymbol{f}(\mathbf{p}, u). \tag{2.195}$$

Theorem 2.5 If a utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ is increasing, twice differentiable and strictly concave, then $\forall\, p_1, p_2 > 0, u > u(0)$:

1. $\varphi(\mathbf{p}, I) = f(\mathbf{p}, \nu(\mathbf{p}, I))$,
2. $f(\mathbf{p}, u) = \varphi(\mathbf{p}, e(\mathbf{p}, u))$,
3. $\nu(\mathbf{p}, e(\mathbf{p}, u)) = u$,
4. $e(\mathbf{p}, \nu(\mathbf{p}, I)) = I$.

From Theorem 2.5 it results that values of Marshallian and Hicksian demand functions are identical when a utility level $u > u(0)$ that a consumer wants to achieve from purchasing a consumption bundle is equal to the value of an indirect utility function $\nu(\mathbf{p}, I)$ by given market prices and a given consumer's income. At the same time for the equivalence of these two functions to be satisfied the consumer income $I > 0$ should be equal to minimal expenditure $e(\mathbf{p}, u)$ incurred by a consumer for purchasing a consumption bundle with the utility level $u > u(0)$.

Theorem 2.6 (the Slutsky equation) If a utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ is increasing, twice differentiable and strictly concave, then $\forall\, p_1, p_2 > 0, u = v(\mathbf{p}, I)$:

(2.196) $$\frac{\partial \varphi_i(\mathbf{p}, I)}{\partial p_j} = \frac{\partial f_i(\mathbf{p}, u)}{\partial p_j} - \frac{\partial \varphi_i(\mathbf{p}, I)}{\partial I}\varphi_j(\mathbf{p}, I) \;\; i, j = 1, 2, i \neq j$$

or

(2.197) $$\frac{\partial \varphi_i(\mathbf{p}, I)}{\partial p_i} = \frac{\partial f_i(\mathbf{p}, u)}{\partial p_i} - \frac{\partial \varphi_i(\mathbf{p}, I)}{\partial I}\varphi_i(\mathbf{p}, I) \;\; i = 1, 2.$$

Interpretation of the Slutsky equation in form (2.197):

$\frac{\partial \varphi_i(\mathbf{p}, I)}{\partial p_i}$—a price effect - how a change in the price of i-th good affects the demand for this good (*ceteris paribus*—the price of the other good and a consumer's income remain unchanged),
$\frac{\partial f_i(\mathbf{p}, u)}{\partial p_i} < 0$—a substitution effect - how a change in the price of i-th good affects the compensated demand for this good. An increase in the price of i-th good results in a decrease in utility of the optimal consumption bundle. But this utility's decrease is compensated by a hypothetical increase in a consumer's income such that a new optimal consumption bundle has the same utility level as the bundle before the increase in the price of i-th good,
$-\frac{\partial \varphi_i(\mathbf{p}, I)}{\partial I}\varphi_i(\mathbf{p}, I)$—an income effect - how the demand for i-th good is affected because of relatively smaller purchasing power of a consumer caused by an uncompensated increase in the price of i-th good purchased in quantity $\varphi_i(\mathbf{p}, I)$.

Note 2.29 The price effect is a result (a sum) of the substitution affect and the income effect.

There are three possible cases to consider:

Case 1

(2.198) $$\frac{\partial \varphi_i(\mathbf{p}, I)}{\partial I} > 0 \Leftrightarrow -\frac{\partial \varphi_i(\mathbf{p}, I)}{\partial I}\varphi_i(\mathbf{p}, I) < 0$$
$$\wedge \frac{\partial f_i(\mathbf{p}, u)}{\partial p_i} < 0 \Leftrightarrow \frac{\partial \varphi_i(\mathbf{p}, I)}{\partial p_i} < 0$$

which means that if i-th good is normal then it is also an ordinary good.

Case 2

(2.199) $$\frac{\partial \varphi_i(\mathbf{p}, I)}{\partial I} < 0 \Leftrightarrow -\frac{\partial \varphi_i(\mathbf{p}, I)}{\partial I}\varphi_i(\mathbf{p}, I) > 0 \wedge \frac{\partial f_i(\mathbf{p}, u)}{\partial p_i} < 0$$

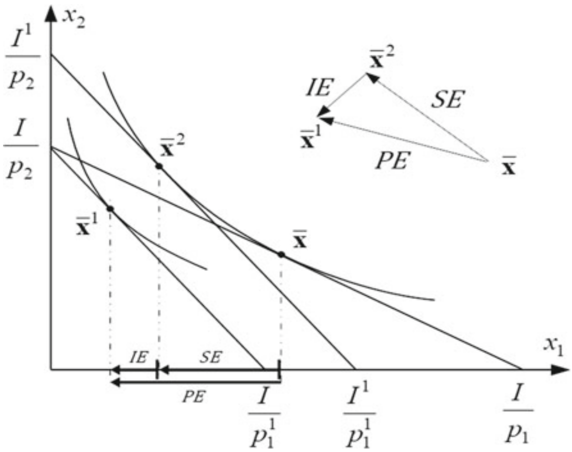


Fig. 2.31a First commodity as normal and ordinary good

If additionally the positive income effect is stronger than the negative substitution effect, then $\frac{\partial \varphi_i(\mathbf{p},I)}{\partial p_i} > 0$. It means that if i-th good is inferior then it is possible that it is also a Giffen good.

Case 3

(2.200)
$$\frac{\partial \varphi_i(\mathbf{p}, I)}{\partial I} < 0 \Leftrightarrow -\frac{\partial \varphi_i(\mathbf{p}, I)}{\partial I}\varphi_i(\mathbf{p}, I) > 0 \wedge \frac{\partial f_i(\mathbf{p}, u)}{\partial p_i} < 0$$

If additionally the positive income effect is weaker than the negative substitution effect, then $\frac{\partial \varphi_i(\mathbf{p},I)}{\partial p_i} < 0$. It means that if i-th good is inferior then it is possible that it is also an ordinary good.

Example 2.17 Present geometric illustrations together with a corresponding interpretation of three possible cases for the Slutsky (2.197) equation.

On the basis of Fig. 2.31a we can state that the income and substitution effects of an increase in the price of the first good are negative. The price effect, being a sum of the income and substitution effects, is negative as well. Thus, the first good is ordinary. Moreover, an increase of income results in the higher demand for the first good. Hence it is a normal good.

Let us also notice that an increase in the price of the first good results in the higher demand for the second good. It means that the first and the second goods are substitutes for each other. The second good is also normal, because an increase of income results in the higher demand for the second good.

On the basis of Fig. 2.31b we can state that the income effect is positive. At the same time it is weaker than the substitution effect. Both effects result from an increase in the price of the first good. The price effect, being a sum of the income and

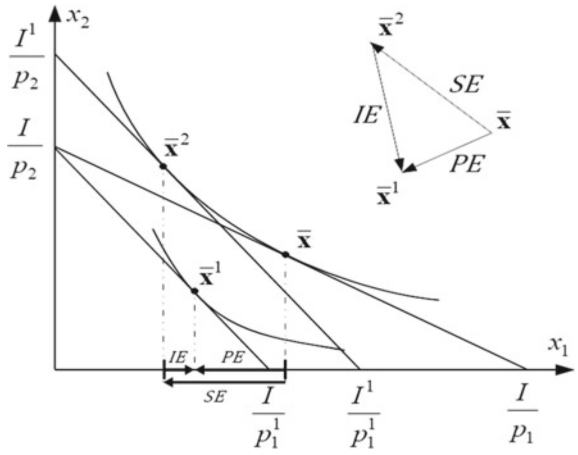


Fig. 2.31b First commodity as inferior and ordinary good

substitution effects, is negative then. Thus, the first good is ordinary. An increase of income results in the lower demand for the first good. Hence it is an inferior good.

Let us also notice that an increase in the price of the first good results in the lower demand for the second good. It means that the first and the second goods are complementary to each other. The second good is normal, because an increase of income results in the higher demand for the second good.

On the basis of Fig. 2.31c we can state that the income effect is positive. At the same time it is stronger than the substitution effect. Both effects result from an increase in the price of the first good. The price effect, being a sum of the income and substitution effects, is positive then. Thus, the first commodity is a Giffen good. An increase of income results in the lower demand for the first good. Hence it is an inferior good.

Let us also notice that an increase in the price of the first good results in the lower demand for the second good. It means that the first and the second goods are complementary to each other. The second good is normal, because an increase of income results in the higher demand for the second good.

Note 2.30 The substitution effect in the Slutsky equation shows an increment of the compensated demand for i-th good due to an increase in the price of i-th good. From Theorem 2.4 (statement 6) we know that this increment is always negative, regardless of the type of a good. Hence, it is not possible to consider Veblen goods in the Slutsky equation.

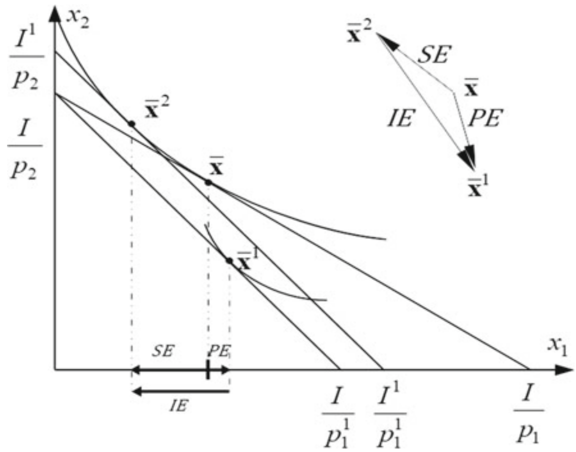


Fig. 2.31c First commodity as inferior and Giffen good

2.6.2 Dynamic Approach

From Sect. 2.6.1 we know that if by any prices and by any income a utility level that a consumer wants to achieve is equal to the value of an indirect utility function then Marshallian and Hicksian demand functions have the same values, thus the consumption utility maximization problem and the consumer's expenditure minimization problem are equivalent. In a dynamic approach we want to show that this dependency occurs also in time that is when prices change form period to period (discrete time version) or at any moment (continuous-time version).

The Slutsky equation presented in a static approach (Theorem 2.6) has its counterpart in a dynamic approach. A variable t means the discrete time: $t = 0, 1, 2, \ldots, T$ or the continuous time: $t \in [0; T]$. T means a time horizon. A utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ is assumed to be increasing, differentiable and strictly concave in each period/moment. We assume also that in every period/moment a utility level that a consumer wants to achieve is equal to value of an indirect utility function. Then for any positive prices and any positive income of a consumer it is satisfied:

$$\frac{\partial\varphi_i(\mathbf{p}(t),\ I(t))}{\partial p_j(t)} = \frac{\partial f_i(\mathbf{p}(t),\ u(t))}{\partial p_j(t)} - \frac{\partial\varphi_i(\mathbf{p}(t),\ I(t))}{\partial I(t)} \cdot \varphi_j(\mathbf{p}(t),\ I(t)), \quad i, j = 1, 2\ \forall t \tag{2.201}$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0; T]$. Equation (2.201) is called a Slutsky dynamic equation. One can consider its special case when changes in prices, in the Marshallian demand and in the compensated demand refer only to one of two goods in a consumption bundle:

$$\frac{\partial \varphi_i(\mathbf{p}(t), I(t))}{\partial p_i(t)} = \frac{\partial f_i(\mathbf{p}(t), u(t))}{\partial p_i(t)} - \frac{\partial \varphi_i(\boldsymbol{p}(t), I(t))}{\partial I(t)} \cdot \varphi_i(\boldsymbol{p}(t), I(t))$$

$$i = 1, 2 \ \forall t. \tag{2.202}$$

The left-hand side of Eq. (2.202), which means an impact of a change in the price of i-th good, *ceteris paribus*, on the demand for this good, is called a price effect. The first component of the right-hand side of the equation, which means an impact of a change in the price of i-th good, *ceteris paribus*, on the compensated demand for this good, is called a substitution effect. While the second component of the sum is called an income effect and presents how the demand for i-th good is affected when there is a relative decrease in purchasing power of a consumer caused by the increase in the price of a good purchased in quantity $\varphi_i(\mathbf{p}(t), I(t))$.

Example 2.18 In order to present the dynamic approach (discrete-time version) to the substitution and income effects of a change in good's prices we show that in every period of a considered time horizon the Slutsky equation is satisfied for each of two goods. That means we show that in every period the price effect can be presented as a sum of the substitution and income effects.

Let us exploit data given and obtained in Examples 2.12 and 2.16. Preferences of a consumer towards consumption bundles are described by a power utility function of a form $u(\mathbf{x}(\mathrm{t})) = x_1(t)^{0.5} x_2(t)^{0.5}$. We already know that a corresponding Marshallian demand function has a form:

$$\varphi(\mathbf{p}(t), I(t)) = \left(\frac{I(t)}{2p_1(t)}, \frac{I(t)}{2p_2(t)}\right),$$

an indirect utility function:

$$\nu(\mathbf{p}(t), I(t)) = \frac{I(t)}{2p_1(t)^{0.5} p_2(t)^{0.5}},$$

a compensated demand function:

$$f(\mathbf{p}(t), u(t)) = \left(u(t)\left(\frac{p_2(t)}{p_1(t)}\right)^{\frac{1}{2}}, u(t)\left(\frac{p_1(t)}{p_2(t)}\right)^{\frac{1}{2}}\right)$$

and a consumer's expenditure function:

$$e(\mathbf{p}(t), u(t)) = 2u(t) p_1(t)^{0.5} p_2(t)^{0.5}.$$

Let us assume, the same as in Example 2.12, that in periods $t = 0, 1, 2, \ldots, 30$ a consumer's income and prices of the first and of the second goods change according to equations:

$$I(t) = 10 \cdot 1.05^t,$$

$$p_1(t) = 4 \cdot 0.98^t,$$

$$p_2(t) = 0.006t^2 - 0.1t + 3,$$

while a utility level a consumer want to achieve from purchasing a consumption bundle evolves according to a formula of an indirect utility function:

$$u(t) = v(\mathbf{p}(t), I(t)) = \frac{I(t)}{2p_1(t)^{0.5}p_2(t)^{0.5}}.$$

For the first commodity the price effect is negative which means it is a normal good. In Fig. 2.32 we can notice moreover that this effect becomes stronger and stronger in subsequent periods. This results from a relationship between a consumer's income and the price of the first good that evolve over time.

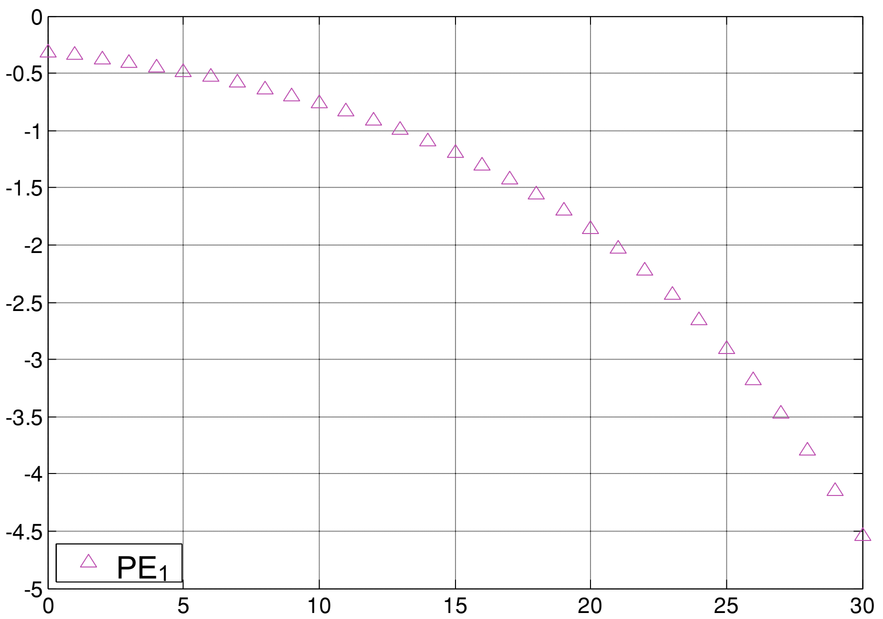


Fig. 2.32 Trajectory of price effect for first good

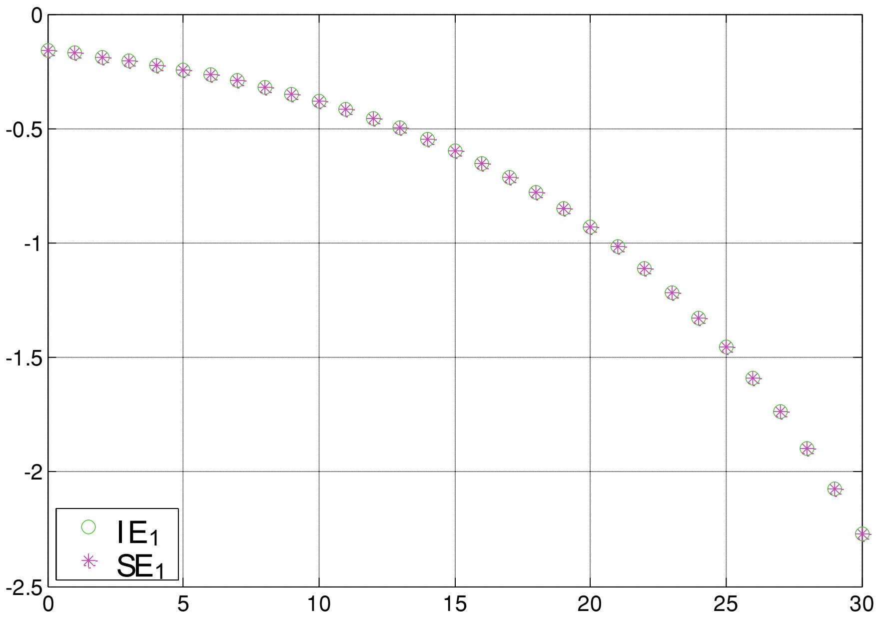


Fig. 2.33 Trajectories of income and substitution effects for the first good

In Fig. 2.33 we present trajectories of the income and substitution effects for the first good. In a given example these two effects have exactly the same strength in every period and this strength becomes bigger and bigger over time. The definition of the compensated demand shows that the substitution effect is always negative regardless of the type of a consumption bundle. The income effect can have positive or negative sign. In this example it is negative which means that the first commodity is a normal good. Comparing Figs. 2.32 and 2.33 we can notice that if we add trajectories of the income and substitution effect then we obtain exactly the same trajectory as of the price effect.

For the second commodity the price effect is also negative, as for the first commodity, which means the second good is normal. In Fig. 2.34 we can notice moreover that this effect becomes stronger and stronger in subsequent periods until period 14 and then it weakens until the end of time horizon. Let us recall that the price of the second good decreases in a first few periods, reaches its minimum in about period 8 and then increases until the end of time horizon.[31] This evolution of the price of the second good affects the trajectory of the price effect for this good. However, the exact course of the trajectory results from a relationship between a consumer's income and the price of the second good that evolve over time.

[31] This evolution of the price is presented in Fig. 2.15 in Sect. 2.4.2.

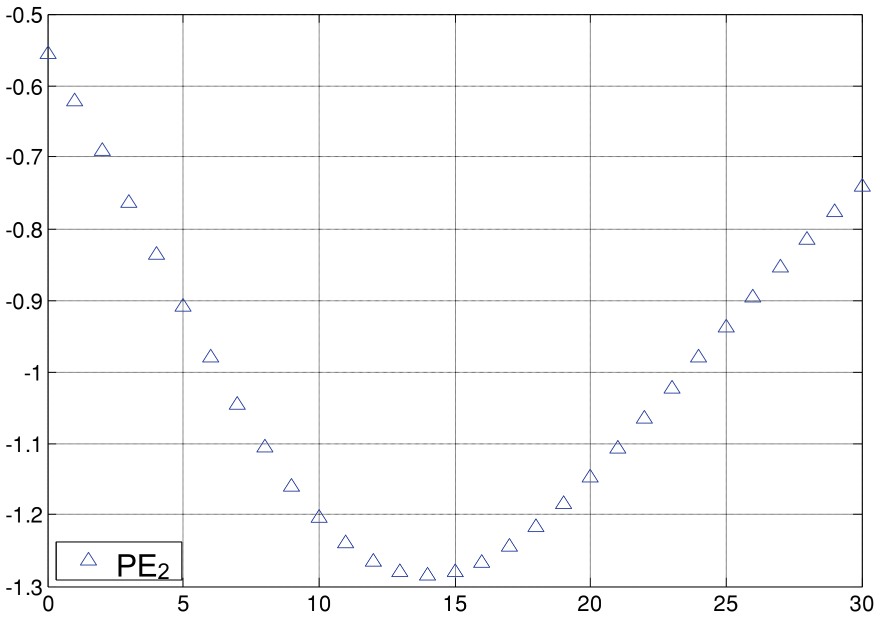


Fig. 2.34 Trajectory of price effect for the second good

In Fig. 2.35 we present trajectories of the income and substitution effects for the second good. Again, as for the first good, in a given example these two effects have exactly the same strength in every period. First this strength becomes bigger and bigger over time, reaching its maximum in period 14. Then, the strength of both these effects weakens until the end of time horizon. The definition of the compensated demand shows that the substitution effect is always negative regardless of the type of a consumption bundle. The income effect can have positive or negative sign. In this example it is negative which means that the second commodity is a normal good. Comparing Figs. 2.34 and 2.35 we can notice that if we add trajectories of the income and substitution effect then we obtain exactly the same trajectory as of the price effect.

2.7 Questions

1. What does it mean that a utility function is a numerical characteristics of a relation of consumer's preference?
2. What are first and second Gossen's laws? What properties are required for a utility function to have any of these laws satisfied?
3. Why does a linear utility function describe goods that are perfect substitutes and not complementary to each other? Why does a Koopmans-Leontief utility

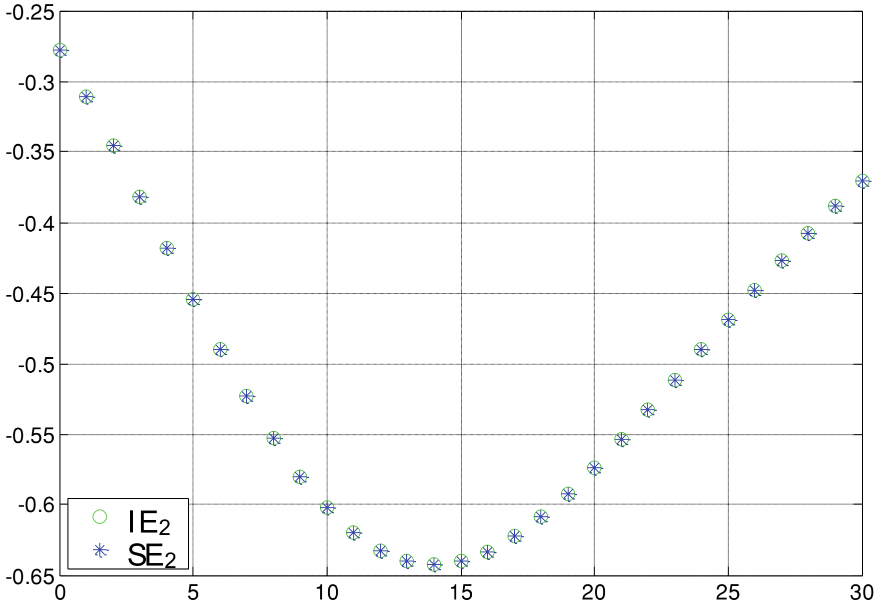


Fig. 2.35 Trajectories of income and substitution effects for the second good

function describe goods that are perfect complements and not substitute for each other?

4. What is a difference between a Giffen good and a Veblen good?
5. What are criteria to classify consumer goods and what is economic interpretation of these criteria?
6. What are basic properties of a Marshallian demand function and of an indirect utility function?
7. What are basic properties of a Hicksian demand function and of a consumer's expenditure function?
8. Why a Hicksian demand function is also called a compensated demand function?
9. Regarding a consumption utility maximization problem what assumptions are needed to have a marginal utility of a money unit for the purchase of i-th good equal to a marginal utility of a consumer's income and to an optimal Lagrange multiplier? How to interpret these economic terms?
10. What is Roy's identity in a consumption utility maximization problem? What is the counterpart of this identity in a consumer's expenditure minimization problem?
11. What conditions need to be satisfied to have a Hicksian demand function and a Marshallian demand function having the same values?
12. What assumptions should be satisfied to derive a Slutsky equation?
13. What conclusions can be drawn from a Slutsky equation?

14. Is it possible to consider Veblen goods in a Slutsky equation?
15. Regarding a Slutsky equation what is the economic interpretation of substitutive and income effects of changes in prices of goods?

2.8 Exercises

E1. There is given an increasing and twice differentiable utility function $u: \mathbb{R}_+^2 \to \mathbb{R}$ of a form:

(a) $u(\mathbf{x}) = a_1 e^{x_1} + a_2 e^{x_2} + a_3,\ a_i > 0, i = 1, 2, 3,$
(b) $u(\mathbf{x}) = a_1 e^{\frac{1}{x_1}} + a_2 e^{\frac{1}{x_2}} + a_3,\ a_i > 0, i = 1, 2, 3,$
(c) $u(\mathbf{x}) = \frac{x_1 x_2}{a_1 x_1 + a_2 x_2},\ a_i > 0, i = 1, 2,$
(d) $u(\mathbf{x}) = a_1 x_1 + a_2 x_2 + a x_1^{\alpha_1} x_2^{\alpha_2},\ a, a_i, \alpha_i > 0, \alpha_1 + \alpha_2 < 1, i = 1, 2,$
(e) $u(\mathbf{x}) = a_1(x_1 + \ln x_1) + a_2(x_2 + \ln x_2),\ a_i, x_i > 0, i = 1, 2,$
(f) $u(\mathbf{x}) = a_1 x_1 \left(1 + x_1^{\alpha-1}\right) + a_2 x_2 \left(1 + x_2^{\alpha-1}\right),\ \alpha \in (0; 1), a_i > 0, i = 1, 2.$

1. Calculate a value and give economic interpretation of:
   (a) a marginal utility of i-th good,
   (b) a growth rate of consumption bundle utility with respect to quantity of i-th good,
   (c) an elasticity of consumption bundle utility with respect to quantity of i-th good,
   (d) a marginal rate of substitution of the first (second) good by the second (first) good,
   (e) an elasticity of substitution of the first (second) good by the second (first) good,
   for consumption bundles: $\mathbf{x}^1 = (1, 1), \mathbf{x}^2 = (1, 2)$.
2. Check if the given function satisfies first Gossen's law.

E2. There are given utility functions $u: \mathbb{R}_+^2 \to \mathbb{R}$:

(a) $u_1(\mathbf{x}) = a_1 x_1 + a_2 x_2,\ a_i > 0, i = 1, 2,$
(b) $u_2(\mathbf{x}) = A^{a_1 x_1 + a_2 x_2},\ A \in (0; 1), a_i > 0, i = 1, 2,$
(c) $u_3(\mathbf{x}) = A^{a_1 x_1 + a_2 x_2},\ A > 1, a_i > 0, i = 1, 2,$
(d) $u_4(\mathbf{x}) = \frac{1}{\ln(a_1 x_1 + a_2 x_2)},\ a_i > 0, i = 1, 2,$
(e) $u_5(\mathbf{x}) = \ln(a_1 x_1 + a_2 x_2),\ a_i > 0, i = 1, 2,$
(f) $u_6(\mathbf{x}) = a x_1^{\alpha} x_2^{\beta},\ a > 0, \alpha, \beta \in (0; 1),$
(g) $u_7(\mathbf{x}) = \alpha \ln x_1 + \beta \ln x_2,\ \alpha, \beta \in (0; 1),$
(h) $u_8(\mathbf{x}) = A^{a x_1^{\alpha} x_2^{\beta}},\ a > 0, A, \alpha, \beta \in (0; 1),$
(i) $u_9(\mathbf{x}) = A^{a x_1^{\alpha} x_2^{\beta}},\ a > 0, A > 1, \alpha, \beta \in (0; 1),$
(j) $u_{10}(\mathbf{x}) = \min\{a_1 x_1, a_2 x_2\},\ a_i > 0,\ i = 1, 2,$
(k) $u_{11}(\mathbf{x}) = a_1 x_1^{\gamma} + a_2 x_2^{\gamma},\ a_i > 0, i = 1, 2,\ \gamma \in (0; 1),$

(l) $u_{12}(\mathbf{x}) = \left(a_1 x_1^{\gamma} + a_2 x_2^{\gamma}\right)^{\frac{\theta}{\gamma}}, \ \theta, \ a_i > 0, i = 1, 2, \ \gamma \in (0; 1).$
1. Which of these functions describes the same relation of consumer preference?
2. Which of them are positively homogeneous of degree $\theta > 0$?
3. Which of them have the same degree of homogeneity?
4. Which of them are (weakly) increasing, which are (weakly) decreasing?

E3. Determine if a given utility function $u: \mathbb{R}_+^2 \to \mathbb{R}$ of a form:

$$u(\mathbf{x}) = a_1 x_1 - a_2 x_2 + a_3, \ \ a_i > 0, i = 1, 2,$$

$$u(\mathbf{x}) = a_1 \ln x_1 - a_2 \ln x_2 + a_3, \ \ a_i > 0, i = 1, 2,$$

$$u(\mathbf{x}) = -a_1 x_1^{\frac{1}{2}} + a_2 x_2^{\frac{1}{2}} + a_3, \ \ a_i > 0, i = 1, 2,$$

is (weakly) increasing or (weakly) decreasing.

E4. There are given:

- a budget set $D(\mathbf{p}, I) = \left\{\mathbf{x} \in \mathbb{R}_+^2 \,|\, p_1 x_1 + p_2 x_2 \le I\right\} \subset X = \mathbb{R}_+^2$,
- a supply set $B = \{(x_1, x_2) \in \mathbb{R}_+^2 | x_1 \le b_1, x_2 \le b_2\} \subset X = \mathbb{R}_+^2$,

such that:

(a) $\forall \alpha, \beta \ge 0, \alpha + \beta = 1 \ \ 0 < b_1 < \frac{\alpha I}{p_1} \wedge 0 < b_2 < \frac{\beta I}{P_2}$,
(b) $0 < \frac{I}{p_1} \le b_1 \wedge 0 < b_2 < \frac{I}{P_2}$,
(c) $0 < b_1 < \frac{I}{p_1} \wedge 0 \le \frac{I}{P_2} < b_2$,
(d) $0 < \frac{I - p_2 b_2}{p_1} < b_1 < \frac{I}{p_1}$ and $0 < \frac{I - p_1 b_1}{P_2} < b_2 < \frac{I}{p_2}$.

1. Using a geometric method find a solution to a consumption utility Maximization Problem with a given utility function:
   A. linear: $u(\mathbf{x}) = a_1 x_1 + a_2 x_2, \ a_i > 0, \ i = 1, 2,$
   B. a power function: $u(\mathbf{x}) = a x_1^{\alpha_1} x_2^{\alpha_2}, \ a, \alpha_i > 0, \alpha_1 + \alpha_2 < 1, \ i = 1, 2,$
   C. logarithmic: $u(\mathbf{x}) = a_1 \ln x_1 + a_2 \ln x_2, \ a_i > 0, x_i > 0, \ i = 1, 2,$
   D. subadditive: $u(\mathbf{x}) = a_1 x_1^{\alpha} + a_2 x_2^{\alpha}, \ a_i > 0, \ i = 1, 2, \alpha \in (0; 1)$
   E. a Koopmans-Leontief function: $u(\mathbf{x}) = \min\{a_1 x_1, a_2 x_2\}, \ a_i > 0, \ i = 1, 2$
   F. a CES function: $u(\mathbf{x}) = \left(a_1 x_1^{\gamma} + a_2 x_2^{\gamma}\right)^{\frac{\theta}{\gamma}}, \ \theta, \ a_i > 0, i = 1, 2, \ \gamma \in (-1; 0) \cap (0; +\infty)$,

   knowing that a consumer chooses an optimal consumption bundle in a set $B \cap D(\mathbf{p}, I)$
2. For each of considered sets $B \cap D(\mathbf{p}, I)$ of feasible solutions by each of utility functions determine relationships between properties of the set (convex, bounded, closed, compact) and properties of the utility function (monotonicity, convexity

or strict convexity). Write conclusions about the number of optimal consumption bundles.

E5. Justify by geometric means that a linear utility function:

$$u(\mathbf{x}) = a_1x_1 + a_2x_2 + a_3, \quad a_i > 0, i = 1, 2, 3$$

describes consumer goods which are perfect substitutes and not complementary for each other, and that a Koopmans-Leontief utility function:

$$u(\mathbf{x}) = \min\{a_1x_1, a_2x_2\} + a_3, \quad a_i > 0, i = 1, 2, 3$$

describes consumer goods which are perfect complements and not substitute for each other.

E6. There are given a power utility function $u(\mathbf{x}) = ax_1^{\alpha_1}x_2^{\alpha_2}, \quad a, \alpha_i > 0, \quad \alpha_1 + \alpha_2 < 1, \; i = 1, 2$ and a utility function $u_2(\mathbf{x}) = \alpha_1 \ln x_1 + \alpha_2 \ln x_2 + \ln a, \quad x_i > 0, i = 1, 2$, which is a composition of the function u with an increasing logarithmic function. We know that u and u_2 describe the same relation of consumer preference. Solving consumption utility maximization problems with each of these functions check if they correspond to the same Marshallian demand function.

E7. Using the Kuhn-Tucker theorem[32] find an optimal solution to a consumption utility maximization problem with an additional constraint on the supply of both goods: $0 \le x_i \le b_i, i = 1, 2$, when a utility function is:

(a) a power function: $u(\mathbf{x}) = ax_1^{\alpha_1}x_2^{\alpha_2}, \quad a, \alpha_i > 0, \quad \alpha_1 + \alpha_2 < 1, \; i = 1, 2,$
(b) logarithmic: $u(\mathbf{x}) = a_1 \ln x_1 + a_2 \ln x_2, \quad a_i > 0, \quad x_i > 0, \; i = 1, 2,$
(c) subadditive: $u(\mathbf{x}) = a_1x_1^{\alpha} + a_2x_2^{\alpha}, \quad a_i > 0, \; i = 1, 2, \; \alpha \in (0; 1).$

E8. There is given a market of two consumer goods, where:
$\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$—a consumption bundle, $\mathbf{p} = (p_1, p_2) \in \operatorname{int} \mathbb{R}^2_+$—a vector of prices of goods, $I > 0$—a consumer's income and a utility function describing a relation of consumer preference of a form:

(a) $u(\mathbf{x}) = \min\{a_1x_1, a_2x_2\}, \quad a_i > 0, i = 1, 2,$
(b) $u(\mathbf{x}) = a_1 \ln x_1 + a_2 \ln x_2, \quad a_i > 0, \quad x_i > 0, \; i = 1, 2,$
(c) $u(\mathbf{x}) = ax_1^{\alpha}x_2^{\beta}, \;\; a > 0, \; \alpha, \beta \in (0; 1),$
   1. Write a form of a consumption utility maximization problem.
   2. Determine a Marshallian demand function and an indirect utility function.
   3. Write a form of a consumer expenditure minimization problem.

[32] See a Mathematical appendix.

4. Determine a Hicksian demand function and a consumer expenditure function.
5. Knowing that $I = e(\mathbf{p}, u)$ and that $u = v(p, I)$ check if the following Slutsky equations are true:

A. $\frac{\partial \varphi_1(\mathbf{p},I)}{\partial p_1} = \frac{\partial f_1(\mathbf{p},u)}{\partial p_1} - \frac{\partial \varphi_1(\mathbf{p},I)}{\partial I}\varphi_1(\mathbf{p}, I),$

B. $\frac{\partial \varphi_1(\mathbf{p},I)}{\partial p_2} = \frac{\partial f_1(\mathbf{p},u)}{\partial p_2} - \frac{\partial \varphi_1(\mathbf{p},I)}{\partial I}\varphi_2(\mathbf{p}, I),$

C. $\frac{\partial \varphi_2(\mathbf{p},I)}{\partial p_2} = \frac{\partial f_2(\mathbf{p},u)}{\partial p_2} - \frac{\partial \varphi_2(\mathbf{p},I)}{\partial I}\varphi_2(\mathbf{p}, I),$

D. $\frac{\partial \varphi_2(\mathbf{p},I)}{\partial p_1} = \frac{\partial f_2(\mathbf{p},u)}{\partial p_1} - \frac{\partial \varphi_2(\mathbf{p},I)}{\partial I}\varphi_1(\mathbf{p}, I),$

6. Present graphic illustrations of the Slutsky equations of point 5.

E9. Check properties of Marshallian demand functions which are optimal solutions to consumption utility maximization problems of Exercise 6. Check properties of the corresponding indirect utility functions.

E10. Using the Kuhn-Tucker theorem[33] find an optimal solution to a consumer expenditure minimization problem with an additional constraint on the supply of both goods: $0 \leq x_i \leq b_i, i = 1, 2$, when a utility function is:

(a) a power function: $u(\mathbf{x}) = ax_1^{\alpha_1}x_2^{\alpha_2}, \quad a, \alpha_i > 0,\ \alpha_1 + \alpha_2 < 1,\ i = 1, 2,$
(b) logarithmic: $u(\mathbf{x}) = a_1 \ln x_1 + a_2 \ln x_2, \quad a_i > 0, \quad x_i > 0,\ i = 1, 2,$
(c) subadditive: $u(\mathbf{x}) = a_1x_1^{\alpha} + a_2x_2^{\alpha}, \quad a_i > 0,\ i = 1, 2,\ \alpha \in (0; 1).$

E11. Check properties of Hicksian demand functions which are optimal solutions to consumer expenditure minimization problems of Exercise 8. Check properties of the corresponding consumer's expenditure functions.

E12. Knowing Hicksian and Marshallian demand functions which are optimal solutions to consumption utility maximization problems (E6) and to consumer expenditure minimization problems (E8), analyse income and substitutive effects of changes in prices of goods. For this purpose use a Slutsky equation for i-th good, $i = 1, 2$.

E13. At each moment $t \in [0, 20]$ a consumer's income, the price of the first good and the price of the second good change according to equations:

$$I(t) = 10 \cdot 1.05^{-t},$$

$$p_1(t) = 4 \cdot 0.98^{-t},$$

$$p_2(t) = -0.006t^2 + 0.1t + 3.$$

[33] See a Mathematical appendix.

Consumers' preferences towards consumption bundles are described by a Koopmans-Leontief function of a form: $u(\mathbf{x}(t)) = \min\{2x_1(t), x_2(t)\}$. Solve a consumption utility maximization problem in the dynamic approach. Present trajectories of the demand for the first and the second goods and a trajectory of indirect utility.

E14. In periods $t = 0, 1, 2, \ldots, 20$ a utility level a consumer wants to achieve, the price of the first good and the price of the second good change according to equations:

$$u(t) = 1 - 0.1\ln(t + 1),$$

$$p_1(t) = 4 \cdot 0.98^{-t},$$

$$p_2(t) = -0.006t^2 + 0.1t + 3.$$

Consumers' preferences towards consumption bundles are described by a Koopmans-Leontief function of a form: $u(\mathbf{x}(t)) = \min\{2x_1(t), x_2(t)\}$. Solve a consumer expenditure minimization problem in the dynamic approach. Present trajectories of the compensated demand for the first and the second goods and a trajectory of consumer's expenditure.

E15. At each moment $t \in [0, 20]$ a consumer's income, the price of the first good and the price

$$I(t) = 10 \cdot 1.05^{-t},$$

$$p_1(t) = 4 \cdot 0.98^{-t},$$

$$p_2(t) = -0.006t^2 + 0.1t + 3.$$

Consumers' preferences towards consumption bundles are described by a Koopmans-Leontief function of a form: $u(\mathbf{x}(t)) = \min\{2x_1(t), x_2(t)\}$. A utility level a consumer wants to achieve evolves at each moment according to an indirect utility function resulting from a consumption utility maximization problem. Present trajectories of price, income and substitutive effects for the first and the second goods.

3 Rationality of Choices Made by Group of Consumers

In this chapter you will learn:

– how to describe a direct exchange of consumer goods between traders
– how to present graphically a simple model of exchange and an Arrow-Hurwicz model
– what initial, feasible, accepted, blocked, Pareto optimal and Walrasian equilibrium allocations are
– what the equilibrium state in an Arrow-Hurwicz model is, how it is described and what the conditions necessary for its existence are
– what global supply, global demand and excess demand functions are
– what is stated in Walras's law
– what it means that a Walrasian equilibrium state is determined by a price structure
– what is described in an Arrow-Hurwicz model in a static approach and what in a dynamic approach
– what is the difference between discrete-time and continuous-time dynamic in Arrow-Hurwicz models
– what it means that a Walrasian equilibrium state is asymptotically globally stable.

In this chapter, we focus on rational behaviour of groups of consumers[1] and their choices and actions that can lead to a conflict of interests.

For the sake of simplicity, we consider two models of a competitive market with two commodities and two consumers, called traders, where the supply of

[1] In fact it is about rational and competitive behaviour of single consumers transferred to groups of consumers.

K. Malaga and K. Sobczak, *Microeconomics*, Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-031-10554-8_3

goods is constant and exogenously determined. In contrast to Chap. 2, we assume now that the supply of consumer goods is a binding constraint for the demand reported by consumers for two commodities. The first of considered competitive equilibrium models of two goods market is called a simple model of exchange. In this model, we determine rational behaviour of consumers that allows a market to achieve an equilibrium identified with a Pareto optimal allocation of the supply of each commodity.

The second competitive equilibrium model, commonly known in literature as the Arrow-Hurwicz model, is presented in a static and in a dynamic version. The static version serves defining a concept of a general equilibrium in a Walras sense. The Walrasian equilibrium state is determined by a Walrasian equilibrium price vector $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2) > (0, 0)$ for which the global demand expressed in physical units and the global supply of each good are equal to each other. In the same time, each consumer achieves her/his goal to the maximum extent. This goal for any trader is to purchase an optimal consumption bundle whose utility is maximum and not less than the utility of a bundle he/she comes to the market with and whose value does not exceed the income of the trader. As a result of reaching the Walrasian equilibrium it is possible to determine an optimum of the competitive market. It is identified with the Walrasian equilibrium allocation which by equilibrium price is a Pareto optimal allocation.

The dynamic Arrow-Hurwicz model, presented in a discrete and in a dynamic version, serves considering a mechanism of reaching the Walrasian equilibrium state in infinite time horizon. This state is described by the Walrasian equilibrium price vector and the Walrasian equilibrium allocation. We analyze also the issues of existence, uniqueness and asymptotical global stability of competitive equilibrium state in this model of consumer goods' market.

3.1 Simple Model of Exchange

Let us consider a market of two consumer goods and two traders where:

$i = 1, 2$—an index of consumer goods,
$k = 1, 2$—an index of consumers (traders),
$X = \mathbb{R}^2_+$—a goods space (a set of all bundles of goods available on the market),
$d: \mathbb{R}^4_+ \to \mathbb{R}_+$—a metric specified on the goods space (see Definition 2.2),
$u^k: \mathbb{R}^2_+ \to \mathbb{R}$—a utility function of k-th consumer describing his/her preferences (a relation of preference of k-th consumer),
$\mathbf{a}^k = (a_{k1}, a_{k2}) \in \mathbb{R}^2_+$—an initial consumption bundle the k-th consumer comes to the market with (k-th consumer's endowment),
$\mathbf{x}^k = (x_{k1}, x_{k2}) \in \mathbb{R}^2_+$—a consution bundle the k-th consumer wants to purchase.

The k-th consumer aims to purchase such a bundle of goods $\bar{\mathbf{x}}^k = (\bar{x}_{k1}, \bar{x}_{k2})$. whose utility would be maximum and at the same time not less than of the initial bundle $\mathbf{a}^k = (a_{k1}, a_{k2})$.

Definition 3.1 A vector $\mathbf{a} = \left(\mathbf{a}^1, \mathbf{a}^2\right) = (a_{11}, a_{12}, a_{21}, a_{22}) \in \mathbb{R}^4_+$, consisting of initial bundles $\mathbf{a}^k = (a_{k1}, a_{k2})$ that traders come to the market with, is called an **initial allocation** (also **endowment**).

Definition 3.2 A vector $\mathbf{x} = \left(\mathbf{x}^1, \mathbf{x}^2\right) = (x_{11}, x_{12}, x_{21}, x_{22}) \in \mathbb{R}^4_+$ is called an **allocation feasible with regard** to an initial allocation **a** when it meets the condition:

(3.1) $$\sum\nolimits_{k=1}^{2} \mathbf{x}^k = \sum\nolimits_{k=1}^{2} \mathbf{a}^k \Leftrightarrow \begin{pmatrix} x_{11} + x_{21} \\ x_{12} + x_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + a_{21} \\ a_{12} + a_{22} \end{pmatrix},$$

where:

$a_{11} + a_{21}$—total quantity of the first good available on the market,
$a_{12} + a_{22}$—total quantity of the second good available on the market,
$x_{11} + x_{21}$—total demand for the first good.
$x_{12} + x_{22}$—total dema for the second good.

Definition 3.3 A **set of allocations feasible** with regard to an initial allocation **a** is the set:

(3.2) $$F(\mathbf{a}) = \left\{\mathbf{x} \in \mathbb{R}^4_+ | \sum\nolimits_{k=1}^{2} \mathbf{x}^k = \sum\nolimits_{k=1}^{2} \mathbf{a}^k\right\} \subset \mathbb{R}^4_+.$$

Note 3.1 A geometric illustration of the set of allocations feasible with regard to the initial allocation **a** is called an **Edgeworth box**. (Fig. 3.1).

The Edgeworth box is created by overlapping two coordinate systems, each of which is associated with the first or the second trader. Each point inside the Edgeworth box is a vector with four non-negative coordinates: vector $\mathbf{x} = \left(\mathbf{x}^1, \mathbf{x}^2\right) = (x_{11}, x_{12}, x_{21}, x_{22}) \in \mathbb{R}^4_+$ where the coordinate $x_{ki} \geq 0$ describes the k-th trader's demand for the i-th consumer good. It is easy to notice that the vector defined in this way is an allocation feasible with regard to the initial allocation $\mathbf{a} = \left(\mathbf{a}^1, \mathbf{a}^2\right) = (a_{11}, a_{12}, a_{21}, a_{22}) \in \operatorname{int} \mathbb{R}^4_+$ (Fig. 3.1).

Definition 3.4 An allocation $\mathbf{x} \in F(\mathbf{a}) \subset \mathbb{R}^4_+$ is called an **allocation accepted by traders** when it satisfies:

(3.3) $$u^k\left(\mathbf{x}^k\right) \geq u^k\left(\mathbf{a}^k\right) \quad \forall k = 1, 2.$$

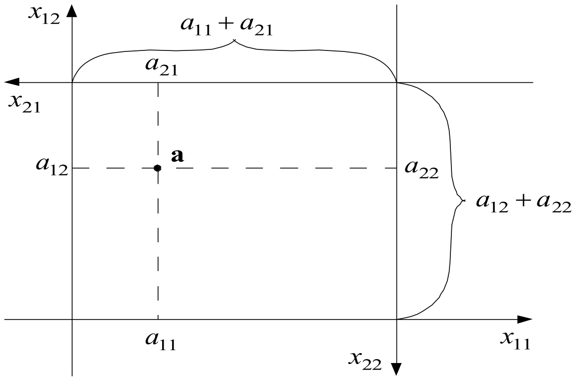


Fig. 3.1 Edgeworth box

Definition 3.5 A set of all allocations feasible with regard to an initial allocation **a**, in which the utility of a consumption bundle $\mathbf{x}^k \in \mathbb{R}_+^2$ is not less than the utility of an initial bundle $\mathbf{a}^k \in \mathbb{R}_+^2$, that is the set:

$$S(\mathbf{a}) = \left\{\mathbf{x} \in F(\mathbf{a}) | \, u^k\left(\mathbf{x}^k\right) \geq u^k\left(\mathbf{a}^k\right), \forall k = 1, 2\right\} \tag{3.4}$$

is called a **set of allocations accepted by traders**.

Definition 3.6 An allocation $\mathbf{x} \in S(\mathbf{a}) \subset \mathbb{R}_+^4$ accepted by traders is called an **allocation blocked by traders** if there is another allocation $\mathbf{y} \in S(\mathbf{a}) \subset \mathbb{R}_+^4$ they accept such that:

$$\forall k = 1, 2 \; u^k\left(\mathbf{y}^k\right) \geq u^k\left(\mathbf{x}^k\right), \tag{3.5}$$

$$\exists k \in \{1, 2\} \; u^k\left(\mathbf{y}^k\right) > u^k\left(\mathbf{x}^k\right). \tag{3.6}$$

Definition 3.7 An allocation $\mathbf{x} \in S(\mathbf{a}) \subset \mathbb{R}_+^4$ accepted by traders is called a **Pareto opt. (efficient) allocation**[2] if there is no other allocation accepted $\mathbf{y} \in S(\mathbf{a}) \subset \mathbb{R}_+^4$ such that:

$$\forall k = 1, 2 \; u^k\left(\mathbf{y}^k\right) \geq u^k\left(\mathbf{x}^k\right), \tag{3.7}$$

$$\exists k \in \{1, 2\} \; u^k\left(\mathbf{y}^k\right) > u^k\left(\mathbf{x}^k\right). \tag{3.8}$$

The set of Pareto-optimal allocations is denoted by a symbol $P(\mathbf{a})$ and called a **Pareto frontier** (also a **contract curve** when the Edgeworth box is used).

[2] Which means that any accepted and to optimal allocation is not a blocked allocation.

Definition 3.8 A set consisting of all allocations accepted by both traders and Pareto optimal at the same time, that is the set:

$$C(\mathbf{a}) = S(\mathbf{a}) \cap P(\mathbf{a}) \subset \mathbb{R}^4_+ \tag{3.9}$$

is called an **exchange core**.

From Definitions 3.1–3.8 it follows that:

$$C(\mathbf{a}) \subseteq S(\mathbf{a}) \subseteq F(\mathbf{a}) \subset \mathbb{R}^4_+, \tag{3.10}$$

which means that each Pareto optimal allocation belonging to the exchange core is an allocation accepted and feasible with regard to an initial allocation. On the other hand, not every feasible allocation is an allocation accepted by traders, not every allocation accepted by traders is a Pareto optimal allocation and not every Pareto optimal allocation is an allocation accepted by both traders.[3]

Example 3.1 There is a market of two consumer goods and two traders given, where:

$i = 1, 2$—an index of consumer goods,
$k = 1, 2$—an index of consumers (traders),
$X = \mathbb{R}^2_+$—a goods space (a set of all bundles of goods available on the market),
$d: \mathbb{R}^4_+ \to \mathbb{R}_+$—a metric specified on the goods space (see Definition 2.2),
$u^k: \mathbb{R}^2_+ \to \mathbb{R}$—a utility function of k-th consumer describing his/her preferences (a relation of preference of k-th consumer),
$\mathbf{a}^k = (a_{k1}, a_{k2}) \in \mathbb{R}^2_+$—an initial consumption bundle the k-th consumer comes to the market with (k-th consumer's endowment),
$\mathbf{x}^k = (x_{k1}, x_{k2}) \in \mathbb{R}^2_+$—a consumption bundle the k-th consumer wants to purchase.

The k-th consumer aims to purchase such a bundle of goods $\bar{\mathbf{x}}^k = (\bar{x}_{k1}, \bar{x}_{k2})$ whose utility would be maximum and at the same time not less than of the initial bundle$\mathbf{a}^k = (a_{k1}, a_{k2})$.

Let us consider a situation where the following utility functions are given:

(a) power functions: $u^k\left(\mathbf{x}^k\right) = a_k x_{k1}^{\alpha_{k1}} x_{k2}^{\alpha_{k2}} \to \max$,
(b) Koopmans-Leontief functions: $u^k\left(\mathbf{x}^k\right) = \min\{a_{k1}x_{k1}, a_{k2}x_{k2}\} \to \max$,

[3] There are infinitely many Pareto-optimal allocations in a set of allocations that are feasible with regard to an initial allocation. All Pareto optimal allocations create a so-called contract curve that consists of all tangency points of both traders' indifference curves. Only part of the Pareto optimal allocations that are accepted by both traders form the exchange core.

Using the Edgeworth box:

1. present geometric illustrations of allocation sets: feasible, accepted by traders and optimal in the Pareto sense,
2. indicate allocations blocked by traders,
3. justify that $C(\mathbf{a}) \subseteq S(\mathbf{a}) \subseteq F(\mathbf{a}) \subset \mathbb{R}^4_+$.

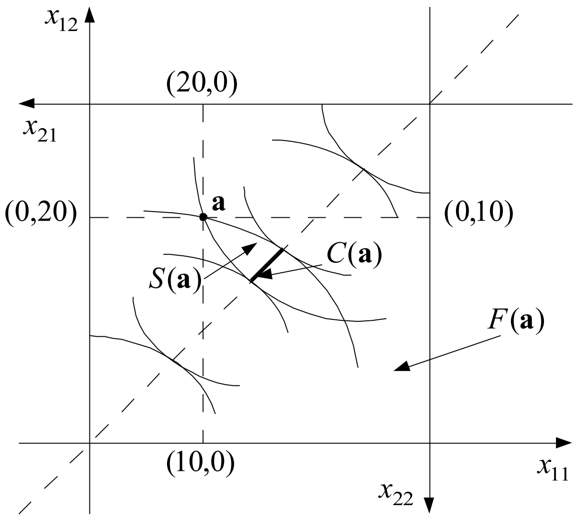
For the sake of simplicity, let us assume we are given:

– bundles of goods the traders come to the market with: $\mathbf{a}^1 = (10, 20)$, $\mathbf{a}^2 = (20, 10)$, which means the initial allocation is $\mathbf{a} = (10, 20, 20, 10)$,
– parameters of power utility functions: $a_k = 1,\ \alpha_{k1} = \alpha_{k2} = \frac{1}{4}$,
– parameters of Koopmans-Leontief utility functions: $a_{k1} = a_{k2} = 1$.

Ad (a) Figure 3.2a shows the initial allocation $\mathbf{a} = (10, 20, 20, 10)$, consisting of bundles of goods owned by both traders. Since the total amount of the first and second good brought to the market by both traders is the same and amounts to 30 units, the Edgeworth box in the considered example is a square. It is created by putting together two coordinate systems: of each of two traders (from her/his point of view). It is not hard to notice that any vector $\mathbf{x} = (x_{11}, x_{12}, x_{21}, x_{22}) \in \mathbb{R}^4_+$, belonging to the Edgeworth box is an allocation feasible with regard to the initial allocation because $\forall\, \mathbf{x} \in F(\mathbf{a})\ \ x_{11} + x_{21} = 30, x_{12} + x_{22} = 30$.

Since both traders use power utility functions when choosing the optimal bundles of goods, the corresponding indifference curves are strictly convex in the respective coordinate system of a given trader.

Fig. 3.2a Edgeworth box with power utility functions



Lens-shaped area $S(\mathbf{a})$ within the indifference curves of both traders containing the initial allocation consists of all the allocations accepted by both traders because the utility of the consumption bundles belonging to this area is not less than of the bundles of goods the traders entered the market with.

According to Definition 3.7, the allocations that are at the tangency points of the indifference curves of both traders are Pareto optimal allocations. There are infinitely many of them and they create a so-called **contract curve**, depicted in Fig. 3.2a as a dashed line connecting the origins of the coordinate systems of both traders. This part of the Pareto optimal allocations, which belongs also to the set of allocations accepted by traders, forms the exchange core denoted by a symbol $C(\mathbf{a})$.

From Fig. 3.2a, it follows that$C(\mathbf{a}) \subset S(\mathbf{a}) \subset F(\mathbf{a}) \subset \mathbb{R}^4_+$, which means that any Pareto optimal allocation accepted by traders is the feasible allocation with regard to the initial allocation. At the same time, not every Pareto optimal allocation is accepted by traders. Not every allocation accepted by traders is Pareto-optimal and not every feasible allocation is an allocation accepted by traders or Pareto-optimal.

Figure 3.2b shows the set of allocations accepted by both traders $S(\mathbf{a})$ and the exchange core $C(\mathbf{a})$. Using Fig. 3.2b one can explain the mechanism according to which both traders choose one of the accepted allocations, which is at the same time the Pareto optimal allocation.

Let us suppose the first trader proposes an allocation **b** that is better for him/her than the initial allocation **a** because it belongs to an indifference curve that is higher up than the indifference curve of the first trader containing the initial allocation.

For the second trader, the allocation **b** is just as good as the initial allocation **a** because both of these allocations belong to the same indifference curve. The

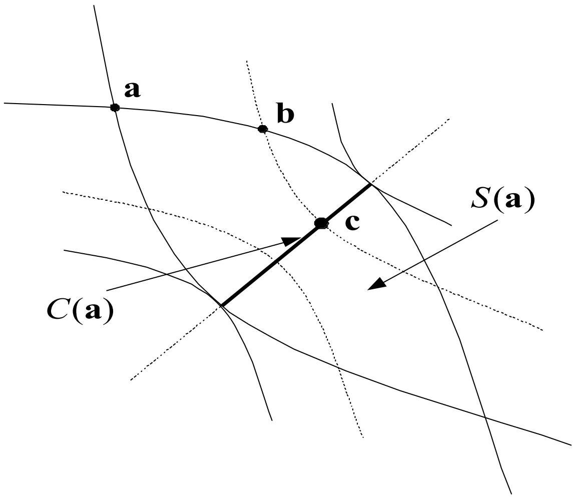


Fig. 3.2b Set of allocations accepted by both traders—the case of power utility functions

second trader, however, should block this allocation and propose a different allocation to the first trader, not worse for the first trader than the allocation **b** and at the same time better than the initial allocation **a** for the second trader. An example of such an allocation is an allocation **c** that is a Pareto optimal allocation too. The allocation **c** lies at the point where the indifference curves of both traders are tangent to each other. Neither of the traders can block it and propose a different allocation which, being better than the allocation **c** for one of them, would also not be worse than the allocation **c** for the other trader.

A reasonable question arises: since every allocation belonging to the exchange core is Pareto optimal, accepted by both traders and as such cannot be blocked by any trader, which one of them should eventually be chosen by traders?

In order to answer this question, it is necessary to define an additional criterion for choice of the optimal allocation in the set of Pareto optimal allocations. This would be the optimum of optima. There seems to be a need to define such an additional criterion since not all allocations Pareto optimal and accepted by traders are equally beneficial to them. For example, such an allocation optimal in the Pareto sense that belongs to the indifference curve of the first (second) trader bounding the set of allocations accepted by both traders from the bottom (top) is the most advantageous allocation among the Pareto optimal allocations for the second (first) trader and the least favourable for the first (second) trader.

The fairest choice would be a Pareto optimal allocation lying in the middle of a segment identified with the exchange core. A Pareto optimal allocation defined in such a way would guarantee an identical increase in the utility of the basket of goods purchased by each trader with regard to the bundle each of them entered the market with in order to exchange for another basket whose utility would be maximum and at the same time not less than the utility of the initial bundle.

Ad (b) Figures 3.3a and 3.3b show the Edgeworth box and the set of allocations accepted by both traders $S(\mathbf{a})$ when their preferences while choosing optimal consumption bundles are described by the Koopmans-Leontief utility function.

From Fig. 3.3a, it follows that $C(\mathbf{a}) \subset S(\mathbf{a}) \subset F(\mathbf{a}) \subset \mathbb{R}^4_+$, which means that any Pareto optimal allocation accepted by traders is the feasible allocation with regard to the initial allocation. At the same time, not every Pareto optimal allocation is accepted by traders. Not every allocation accepted by traders is Pareto-optimal and not every feasible allocation is an allocation accepted by traders or Pareto-optimal.

Figure 3.3b presents the set of allocations accepted by both traders $S(\mathbf{a})$ and the exchange core $C(\mathbf{a})$.

The mechanism according to which both traders choose the optimal allocation is the same as in the case (a) discussed above. As its result, both traders will agree to an allocation accepted and Pareto optimal at the same time.

If the first trader would propose an allocation **b** that is better for him/her than the initial allocation **a** then the second trader should block the allocation **b** because for her/him it is as good as the initial allocation **a** while he can propose for example an allocation **c** which is better for her/him than **b**. At the same time, the allocation **c** is not worse than **b** for the first trader.

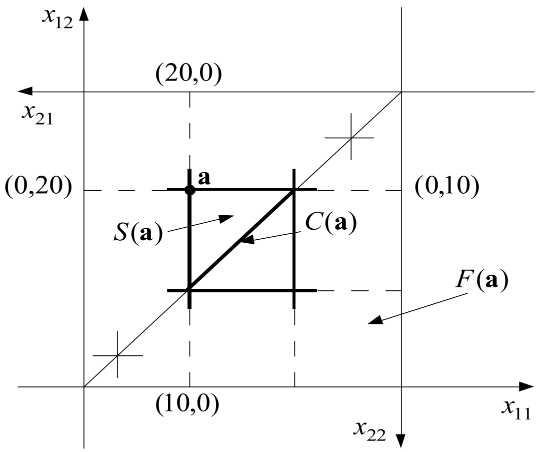


Fig. 3.3a Edgeworth box with Koopmans-Leontief utility functions

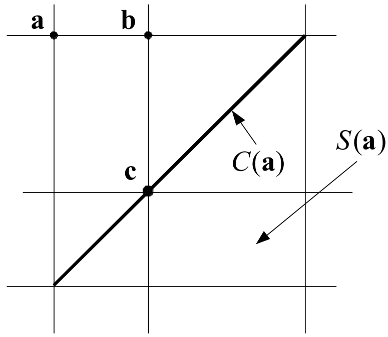


Fig. 3.3b Set of allocations accepted by both traders—case of Koopmans-Leontief utility functions

The allocation **c** cannot be blocked by any of traders because is optimal in the Pareto sense. This means that there exists no such allocation which would be better than **c** for one of the traders and at the same time not worse than **c** for the other trader.

The fairest choice would be a Pareto optimal allocation lying in the middle of a segment identified with the exchange core $C(\mathbf{a}) \subset S(\mathbf{a}) \subset F(\mathbf{a}) \subset \mathbb{R}^4_+$. A Pareto optimal allocation defined in such a way would guarantee an identical increase in the utility of the basket of goods purchased by each trader with regard to the bundle each of them entered the market with in order to exchange for another basket whose utility would be maximum and at the same time not less than the utility of the initial bundle.

But to have this fairest choice made by the traders, there would have to be an additional criterion how to choose an optimal allocation among allocations belonging to the exchange core. It would have the following form: among allocations of the exchange core choose such that guarantees the same greatest increase of consumption bundle utility for each trader in comparison to her/his initial bundle he/she came to the market with. The optimum of optima formulated in that way is not present explicitly in the simple model of exchange. The need to define it results from an observation that not every allocation accepted by both traders and optimal in the Pareto sense is equally beneficial for both of them.

3.2 Static Arrow-Hurwicz Model

It is a generalization of the simple model of exchange, in which one additionally takes into account the prices of consumer goods and the income of consumers.

There is a market of two consumer goods and two traders given, where:

$i = 1, 2$—an index of consumer goods,
$k = 1, 2$—an index of consumers (traders),
$X = \mathbb{R}^2_+$—a goods space (a set of all bundles of goods available on the market),
$d\colon \mathbb{R}^4_+ \to \mathbb{R}_+$—a metric specified on the goods space (see Definition 2.2),
$u^k\colon \mathbb{R}^2_+ \to \mathbb{R}$—a utility function of k-th consumer describing his/her preferences (a relation of preference of k-th consumer),
$\mathbf{a}^k = (a_{k1}, a_{k2}) \in \mathbb{R}^2_+$—an initial consumption bundle the k-th consumer comes to the market with (k-th consumer's endowment),
$\mathbf{x}^k = (x_{k1}, x_{k2}) \in \mathbb{R}^2_+$—a consumption bundle the k-th consumer wants to purchase.
$(p_1, p_2) \in \operatorname{int}\mathbb{R}^2_+$—a vector of prices of consumer goods,
$I^k(p_1, p_2) = p_1 a_{k1} + p_2 a_{k2} > 0$—an income of k -th consumer equal to a value of consumption bundle he/she came to the market with if only a purchase and sale transaction is made,
$B^k = \{\mathbf{x}^k \in \mathbb{R}^2_+ | x_{ki} \le a_{1i} + a_{2i} = b_i\, , i = 1, 2\}$—a supply set of k-th trader,
$D^k(\mathbf{p}, I^k(p)) = \{(x_{k1}, x_{k2}) \in \mathbb{R}^2_+ | p_1 x_{k1} + p_2 x_{k2} \le I^k(p_1, p_2)\} \subset B^k \subset \mathbb{R}^2_+$—a budget set of k-th consumer (a set of all consumption bundles of a value not exceeding her/his income).

The k-th ($k = 1, 2$) consumer aims to purchase such a bundle of goods $\bar{\mathbf{x}}^k = (\bar{x}_{k1}, \bar{x}_{k2})$, whose utility would be maximum and at the same time not less than of the initial bundle of goods $\mathbf{a}^k = (a_{k1}, a_{k2})$.

The problem of choosing the optimal bundle of goods by k-th consumer can be written as the consumption utility maximization problem, because:

$$(\mathbf{P1})\ \ u^k(x_{k1}, x_{k2}) \to \max \tag{3.11}$$

$$p_1 x_{k1} + p_2 x_{k2} \le p_1 a_{k1} + p_2 a_{k2}, \tag{3.12}$$

$$x_{k1}, x_{k2} \ge 0. \tag{3.13}$$

If the utility function $u\colon \mathbb{R}^2_+ \to \mathbb{R}$ is increasing, differentiable and strictly concave, then problem (P1) has exactly one optimal solution which lies on the budget line and is of the form:

$$\begin{aligned}\bar{\mathbf{x}}^k(\mathbf{p}) &= \varphi^k\Big(\mathbf{p}, I^k(\mathbf{p})\Big)\\ &= \left(\alpha^k \frac{I^k(\mathbf{p})}{p_1}, \beta^k \frac{I^k(\mathbf{p})}{p_2}\right) > 0,\ \forall \alpha^k, \beta^k \geq 0,\ \alpha^k + \beta^k = 1,\end{aligned} \tag{3.14}$$

where $k = 1, 2$ indicates the k-th consumer.

Definition 3.9 A **demand function of k -th consumer** is a mapping $\varphi^k\colon \operatorname{int}\mathbb{R}^2_+ \to \operatorname{int}\mathbb{R}^2_+$ that assigns the optimal solution of the consumption utility maximization problem (P1) to any price vector $\mathbf{p} = (p_1, p_2) \in \operatorname{int}\mathbb{R}^2_+$ and is of the form:

$$\begin{aligned}&\varphi^k(\mathbf{p}) = \varphi^k\Big(\mathbf{p}, I^k(\mathbf{p})\Big) = \Big(\varphi_{k1}\Big(\mathbf{p}, I^k(\mathbf{p})\Big), \varphi_{k2}\Big(\mathbf{p}, I^k(\mathbf{p})\Big)\Big) =\\ &= (\bar{x}_{k1}(\mathbf{p}), \bar{x}_{k2}(\mathbf{p})) = \bar{\mathbf{x}}^k(\mathbf{p}),\ k = 1, 2.\end{aligned} \tag{3.15}$$

Definition 3.10 A **function of global demand** is an expression:

$$\bar{\mathbf{x}}(\mathbf{p}) = \bar{\mathbf{x}}^1(\mathbf{p}) + \bar{\mathbf{x}}^2(\mathbf{p}) = \begin{pmatrix}\bar{x}_{11}(\mathbf{p}) + \bar{x}_{21}(\mathbf{p})\\ \bar{x}_{12}(\mathbf{p}) + \bar{x}_{22}(\mathbf{p})\end{pmatrix}, \tag{3.16}$$

describing the total demand of both traders for each good.

Definition 3.11 A total supply of each good provided by both traders is given with an expression:

$$\bar{\mathbf{a}} = \mathbf{a}^1 + \mathbf{a}^2 = \begin{pmatrix}a_{11} + a_{21}\\ a_{12} + a_{22}\end{pmatrix} \in \mathbb{R}^2_+ \tag{3.17}$$

and called **a function (vector) of global supply**. It is denoted by a symbol $\bar{\mathbf{a}}$ to distinguish it from the initial allocation $\mathbf{a} \in \operatorname{int}\mathbb{R}^2_+$.

Definition 3.12 A mapping $z\colon \operatorname{int}\mathbb{R}^2_+ \to \mathbb{R}^2_+$ given as:

$$\mathbf{z}(\mathbf{p}) = \bar{\mathbf{x}}(\mathbf{p}) - \bar{\mathbf{a}}, \tag{3.18}$$

or

$$\forall i = 1, 2 \quad z_i(\mathbf{p}) = \bar{x}_i(\mathbf{p}) - \bar{a}_i \tag{3.19}$$

is called a **function of excess demand**.

Definition 3.13 A **partial equilibrium** on the market of i-th consumer good is a state in which:

$$\exists\, i \quad \exists\, \mathbf{p} > 0 \quad z_i(\mathbf{p}) = \bar{x}_i(\mathbf{p}) - \bar{a}_i = 0 \Leftrightarrow \bar{x}_i(\mathbf{p}) = \bar{a}_i, \tag{3.20}$$

meaning there exists a positive price vector such that the global demand for i-th good, expressed in physical units, is equal to its global supply, expressed in the same physical units as the demand for the i-th consumer good.

We then say that there is a partial equilibrium on the market of i-th consumer good: the global demand for i-th good (expressed in physical units) is equal to global supply of i-th (expressed in physical units).

Definition 3.14 A **general equilibrium** (in the Walras sense) in the market of consumer goods is a state in which:

$$\forall\, i = 1, 2 \quad \exists\, \bar{\mathbf{p}} > 0 \quad z_i(\bar{\mathbf{p}}) = \bar{x}_i(\bar{\mathbf{p}}) - \bar{a}_i = 0 \Leftrightarrow \bar{x}_i\left(\bar{p}_1, \bar{p}_2\right) = \bar{a}_i, \tag{3.21}$$

meaning there exists a positive price vector, called an **equilibrium (Walrasian) price vector**, such that the global demand for any i-th good is equal to its global supply.

We then say that there is a general equilibrium on the consumer goods market: the global demand for each good (expressed in physical units) is equal to its global supply (expressed in physical units).

Theorem 3.1 If the utility function $u^k\colon \mathbb{R}^2_+ \to \mathbb{R}$, $k = 1, 2.$ is increasing, differentiable and strictly concave, then the excess demand function has the following properties:

1. is differentiable in int $\mathbb{R}^2_+$,
2. is homogeneous of degree 0:

$$\begin{gathered} \forall\, i = 1, 2 \;\; \forall \lambda > 0 \;\; \forall\, \mathbf{p} > 0 \\ z_i(\lambda \mathbf{p}) = \bar{x}_i(\lambda \mathbf{p}) - \bar{a}_i = \bar{x}_i(\mathbf{p}) - \bar{a}_i = z_i(\mathbf{p}), \end{gathered} \tag{3.22}$$

which means that a proportional in the prices of both consumer goods does not change the excess demand for any consumer good,

3. satisfies Walras's law:

$$\begin{gathered} \forall \mathbf{p} > 0 \quad \sum_{i=1}^{2} p_i z_i(\mathbf{p}) = \sum_{i=1}^{2} p_i(\bar{x}_i(\mathbf{p}) - \bar{a}_i) = 0 \\ \Leftrightarrow \sum_{i=1}^{2} p_i \bar{x}_i(\mathbf{p}) = \sum_{i=1}^{2} p_i \bar{a}_i, \end{gathered} \tag{3.23}$$

which means that for any price vector $\mathbf{p} = (p_1, p_2) > (0, 0)$ the value of the global demand for all goods is equal to the value of their global supply.

Note 3.2 The concept of the Walrasian equilibrium should be distinguished from Walras's law.

Note 3.3 The Walrasian equilibrium state described by the Walrasian equilibrium price vector may not exist, there may be exactly one such state or there may be more than one.

Note 3.4 The price vector of the Walrasian equilibrium (if it exists) is determined with an accuracy of a structure.

Let us suppose that $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2) > 0$ is the Walrasian equilibrium price vector. Then we can present it in a form:

(3.24) $$\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2) = \bar{p}_1\left(1, \frac{\bar{p}_2}{\bar{p}_1}\right) = \lambda\left(1, \frac{\bar{p}_2}{\bar{p}_1}\right), \quad \text{where } \lambda = \bar{p}_1 > 0,$$

or

(3.25) $$\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2) = \bar{p}_2\left(\frac{\bar{p}_1}{\bar{p}_1}, 1\right) = \lambda\left(\frac{\bar{p}_1}{\bar{p}_1}, 1\right), \quad \text{where } \lambda = \bar{p}_2 > 0.$$

In other words, if $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2) = \bar{p}_1\left(1, \frac{\bar{p}_2}{\bar{p}_1}\right)$ the equilibrium price vector then the vector $\bar{\mathbf{p}} = \lambda\left(1, \frac{\bar{p}_2}{\bar{p}_1}\right), \lambda > 0$ is the equilibrium vector too. This means that to one Walrasian equilibrium price vector, determined with an accuracy of a structure (accuracy of a multiplication by a positive number), infinitely many price vectors are related, each of them having the same structure the Walrasian equilibrium price vector, differing only in the absolute levels of consumer goods' prices.

Theorem 3.2 If the utility function $u: \mathbb{R}^2_+ \to \mathbb{R}, \quad k = 1, 2$ is increasing, differentiable and strictly concave, then in the Arrow-Hurwicz model there exists at least one price vector of the Walrasian equilibrium, determined with an accuracy of a structure.[4]

Note 3.5 The conditions, ensuring that in the Arrow-Hurwicz model exists exactly one Walrasian equilibrium price vector determined with an accuracy of a structure, are in the form of more complex assumptions. Therefore, we will not provide them as part of the basic lecture here.

Definition 3.15 A vector $\mathbf{a} = (a_{11}, a_{12}, a_{21}, a_{22}) \in \text{int } \mathbb{R}^4_+$, consisting of initial bundles $\mathbf{a}^k = (a_{k1}, a_{k2})$ that traders come to the market with is called an **initial allocation** (also **endowment**).

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[4] The conditions of existence and uniqueness of the Walrasian equilibrium price vector in the static Arrow-Hurwicz model have been discussed in more detail, inter alia, in work (Panek, 2003).

Definition 3.16 A vector $\bar{\mathbf{x}}(\mathbf{p}) = (\bar{x}_{11}(\mathbf{p}), \bar{x}_{12}(\mathbf{p}), \bar{x}_{21}(\mathbf{p}), \bar{x}_{22}(\mathbf{p})) \in \mathbb{R}^4_+$. is called an **allocation feasible with regard** to an initial allocation **a** when it meets a condition:

$$\sum_{k=1}^{2} \bar{\mathbf{x}}^k(\mathbf{p}) = \sum_{k=1}^{2} \mathbf{a}^k \Leftrightarrow \begin{pmatrix} \bar{x}_{11}(\mathbf{p}) + \bar{x}_{21}(\mathbf{p}) \\ \bar{x}_{21}(\mathbf{p}) + \bar{x}_{22}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} a_{11} + a_{12} \\ a_{21} + a_{22} \end{pmatrix}, \tag{3.26}$$

where:

$a_{11} + a_{21}$ total quantity of the first good available on a market,
$a_{12} + a_{22}$ total quantity of the second good available on a market.

Definition 3.17 A **set of allocations feasible** with regard to an initial allocation **a** is a s.t:

$$F(\mathbf{a}) = \left\{ \bar{\mathbf{x}}(\mathbf{p}) \in \mathbb{R}^4_+ | \sum_{k=1}^{2} \bar{\mathbf{x}}^k(\mathbf{p}) = \sum_{k=1}^{2} \mathbf{a}^k \right\}. \tag{3.27}$$

Note 3.6 A geometric illustration of the set of allocations feasible with regard to an initial allocation **a** is called an **Edgeworth box**.

Definition 3.18 An allocation $\bar{\mathbf{x}}(\mathbf{p}) \in F(\mathbf{a}) \subset \mathbb{R}^4_+$ is called an **allocation accepted by traders** when it satisfies a condition:

$$u^k\left(\bar{\mathbf{x}}^k(\mathbf{p})\right) \geq u^k\left(\mathbf{a}^k\right) \quad \forall k = 1, 2. \tag{3.28}$$

Definition 3.19 A **set of allocations accepted by traders** is a set:

$$S(\mathbf{a}) = \left\{ \bar{\mathbf{x}}(\mathbf{p}) \in F(\mathbf{a}) | \, u^k\left(\bar{\mathbf{x}}^k(\mathbf{p})\right) \geq u^k\left(\mathbf{a}^k\right), \; k = 1, 2 \right\}. \tag{3.29}$$

Definition 3.20 An allocation $\bar{\mathbf{x}}(\mathbf{p}) \in S(\mathbf{a}) \subset \mathbb{R}^4_+$. accepted by traders is called an **allocation blocked** by them if there is another allocation they accept $\bar{\mathbf{y}}(\mathbf{p}) \in S(\mathbf{a}) \subset \mathbb{R}^4_+$. such that:

$$\forall k = 1, 2 \quad u^k\left(\bar{\mathbf{y}}^k(\mathbf{p})\right) \geq u^k\left(\bar{\mathbf{x}}^k(\mathbf{p})\right), \tag{3.30}$$

$$\exists k \quad u^k\left(\bar{\mathbf{y}}^k(\mathbf{p})\right) \geq u^k\left(\bar{\mathbf{x}}^k(\mathbf{p})\right). \tag{3.31}$$

Definition 3.21 An allocation $\bar{\mathbf{x}}(\mathbf{p}) \in S(\mathbf{a}) \subset \mathbb{R}^4_+$ accepted by traders is called a **Pareto optimal (efficient) allation** if there is no other allocation accepted $\bar{\mathbf{y}}(\mathbf{p}) \in S(\mathbf{a}) \subset \mathbb{R}^4_+$ such that:

$$\forall k = 1, 2 \quad u^k\left(\bar{\mathbf{y}}^k(\mathbf{p})\right) \geq u^k\left(\bar{\mathbf{x}}^k(\mathbf{p})\right), \tag{3.32}$$

(3.33) $$\exists k \quad u^k\left(\bar{\mathbf{y}}^k(\mathbf{p})\right) > u^k\left(\bar{\mathbf{x}}^k(\mathbf{p})\right).$$

The set of Pareto-optimal allocations is denoted by the symbol $P(\mathbf{a})$. and called a **Pareto frontier** (also a **contract curve** when the Edgeworth box is used).

Definition 3.22 A set consisting of all allocations accepted by both traders and Pareto optimal at the same time, that is a set:

(3.34) $$C(\mathbf{a}) = S(\mathbf{a}) \cap P(\mathbf{a}) \subset \mathbb{R}^4_+$$

is called an **exchange core**.

Definition 3.23 A Pareto optimal allocation $\bar{\mathbf{x}}(\bar{\mathbf{p}}) \in C(\mathbf{a}) \subset \mathbb{R}^4_+$ is called a **Walrasian equilibrium allocation** when the price vector $\bar{\mathbf{p}} = \lambda\left(1, \frac{\bar{p}_2}{\bar{p}_1}\right) > (0, 0), \lambda > 0$**.** is the Walrasian equilibrium price vector.

Definition 3.24 A set consisting of all Walrasian equilibrium allocations, that is a set:

(3.35) $$W(\mathbf{a}) = \{\bar{\mathbf{x}}(\bar{\mathbf{p}}) \in C(\mathbf{a}) | \bar{\mathbf{x}}(\bar{\mathbf{p}}) = \bar{\mathbf{a}}\} \subset \mathbb{R}^4_+,$$

is called a **set of Walrasian equilibrium allocations**.

Note 3.7 From Definitions 3.15–3.24, it follows that:

(3.36) $$W(\mathbf{a}) \subseteq C(\mathbf{a}) \subseteq S(\mathbf{a}) \subseteq F(\mathbf{a}) \subset \mathbb{R}^4_+$$

which means that each Walrasian equilibrium allocation is an allocation: Pareto-optimal, accepted by traders and feasible with regard to an initial allocation.

Note 3.8 The reverse inclusion is not true, which means that not every feasible allocation is an accepted by traders, Pareto-optimal or Walrasian equilibrium allocation.

Example 3.2 There is a market of two consumer goods and two traders given, where:

$i = 1, 2$—an index of consumer goods,
$k = 1, 2$—an index of consumers (traders),
$X = \mathbb{R}^2_+$—a goods space,
$\mathbf{a}^1 = (10, 20),\ \mathbf{a}^2 = (20, 10)$—consumption bundles the consumer come to the market with,

$\mathbf{x}^k = (x_{k1}, x_{k2}) \in \mathbb{R}^2_+$—a consumion bundle the k-th consumer wants to purchase,
$\mathbf{p} = (p_1, p_2) \in \operatorname{int} \mathbb{R}^2_+$—a vectorf prices of consumer goods,
$I^1(p_1, p_2) = 10p_1 + 20p_2$, $I^2(p_1, p_2) = 20p_1 + 10p_2$—incomes of traders.

Utility functions of traders:

(a) $u^1(x_{11}, x_{12}) = x_{11}^{\frac{1}{4}} x_{12}^{\frac{1}{4}}$, $u^2(x_{21}, x_{22}) = x_{21}^{\frac{1}{4}} x_{22}^{\frac{1}{4}}$—power functions,
(b) $u^1(x_{11}, x_{12}) = \min\{x_{11}, x_{12}\}$, $u^2(x_{21}, x_{22}) = \min\{x_{21}, x_{22}\}$—Koopmans-Leontief functions.

1. Find optimal solutions to consumption utility maximization problems of both traders (demand functions of both traders).
2. Determine functions of global supply and global demand.
3. Determine an excess demand function and check if it is homogeneous of degree 0 and if it meets Walras's law.
4. Determine the Walrasian equilibrium price vector.
5. Explain what it means that the Walrasian equilibrium price vector is determined with an accuracy of a structure (accuracy of a multiplication by a positive number).
6. Determine the Walrasian equilibrium allocation.
7. Provide a geometric illustration of:
   - a set of allocations feasible with regard to an initial allocation,
   - a set of allocations accepted by traders,
   - a set of Pareto optimal allocations,
   - a set of Walrasian equilibrium allocations.
8. Justify by Geometric Means that: $W(\mathbf{a}) \subseteq C(\mathbf{a}) \subseteq S(\mathbf{a}) \subseteq F(\mathbf{a}) \subset \mathbb{R}^4_+$.

Ad (a) The consumption utility maximization problems for two traders are given:

(3.37) $$u^1(x_{11}, x_{12}) = x_{11}^{\frac{1}{4}} x_{12}^{\frac{1}{4}} \to \max$$

(3.38) $$p_1 x_{11} + p_2 x_{12} \leq 10p_1 + 20p_2,$$

(3.39) $$x_{11}, x_{12} \geq 0,$$

and

(3.40) $$u^2(x_{21}, x_{22}) = x_{21}^{\frac{1}{4}} x_{22}^{\frac{1}{4}} \to \max$$

(3.41) $$p_1 x_{21} + p_2 x_{22} \leq 20p_1 + 10p_2,$$

(3.42) $$x_{21}, x_{22} \geq 0.$$

We know that the optimal solution to the consumption utility maximization problem in case of the power utility function $u^k(x_{k1}, x_{k2}) = x_{k1}^{\alpha_1^k} x_{k2}^{\alpha_2^k}$ has a form:

$$\bar{\mathbf{x}}^k(\mathbf{p}) = \left(\frac{\alpha_1^k}{(\alpha_1^k + \alpha_2^k)} \frac{I^k(\mathbf{p})}{p_1} \frac{\alpha_2^k}{(\alpha_1^k + \alpha_2^k)} \frac{I^k(\mathbf{p})}{p_2} \right). \tag{3.43}$$

Substituting the parameters of the utility function of both traders into (2.243), we get the demand function of the first trader:

$$\bar{\mathbf{x}}^1(\mathbf{p}) = \frac{(10p_1 + 20p_2}{2p_1}, \frac{10p_1 + 20p_2)}{2p_2}, \tag{3.44}$$

and of the second trader:

$$\bar{\mathbf{x}}^2(\mathbf{p}) = \left(\frac{20p_1 + 10p_2}{2p_1}, \frac{20p_1 + 10p_2}{2p_2} \right). \tag{3.45}$$

Adding the demand functions of both traders, we get the global demand function for both consumer goods:

$$\bar{\mathbf{x}}(\bar{\mathbf{p}}) = \bar{\mathbf{x}}^1(\mathbf{p}) + \bar{\mathbf{x}}^2(\mathbf{p}) = \left(\frac{30p_1 + 30p_2}{2p_1}, \frac{30p_1 + 30p_2}{2p_2} \right). \tag{3.46}$$

The global supply function (vector) has a form:

$$\bar{\mathbf{a}} = \mathbf{a}^1 + \mathbf{a}^2 = (30; 30). \tag{3.47}$$

The excess demand function then has a form:

$$\begin{aligned} \mathbf{z}(\mathbf{p}) = \bar{\mathbf{x}}(\mathbf{p}) - \bar{\mathbf{a}} &= \left(\frac{30p_1 + 30p_2}{2p_1}, \frac{30p_1 + 30p_2}{2p_2} \right) - (30, 30) \\ &= \left(\frac{-30p_1 + 30p_2}{2p_1}, \frac{30p_1 - 30p_2}{2p_2} \right). \end{aligned} \tag{3.48}$$

Let us notice that the excess demand function is homogeneous of degree 0 because:

$$\forall \lambda > 0 \quad \mathbf{z}(\lambda \mathbf{p}) = \left(\frac{-30\lambda p_1 + 30\lambda p_2}{2\lambda p_1}, \frac{30\lambda p_1 - 30\lambda p_2}{2\lambda p_2} \right) = \mathbf{z}(\mathbf{p}), \tag{3.49}$$

which means that the excess demand for any commodity does not depend on the absolute price level of both goods, but on the rationship between the prices of goods.

Moreover, the function of excess demand satisfies Walras' law, because:

$$\forall \mathbf{p} > 0 \quad \langle \mathbf{p}, \mathbf{z}(\mathbf{p}) \rangle = \left(p_1 \frac{-30p_1 + 30p_2}{2p_1}, +p_2 \frac{30p_1 - 30p_2}{2p_2} \right) \tag{3.50}$$

which means that for any positive vector of consumer goods' prices, the value of global demand is equal to the value of global supply of both goods.

Let us determine the Walrasian equilibrium price vector $\bar{\mathbf{p}} > 0$ as a solution of the following system of equations:

$$\mathbf{z}(\mathbf{p}) = \left(\frac{-30\bar{p}_1 + 30\bar{p}_2}{2\bar{p}_1}, \frac{30\bar{p}_1 - 30\bar{p}_2}{2\bar{p}_2} \right) = (0, 0) \tag{3.51}$$

which can be written in an equivalent form:

$$\begin{aligned} -30\bar{p}_1 + 30\bar{p}_2 = 0 \\ 30\bar{p}_1 - 30\bar{p}_2 = 0 \end{aligned} \tag{3.52}$$

resulting in:

$$\bar{p}_1 = \bar{p}_2 = \lambda > 0, \tag{3.53}$$

therefore the Walrasian equilibrium price vector, determined with an accuracy of a structure, has the following form:

$$\bar{\mathbf{p}} \; = \lambda(1, 1). \tag{3.54}$$

Having this, after substituting (3.54) to (3.44) and (3.45), we can determine the values of the demand functions of both traders when prices are given by the Walrasian equilibrium price vector:

$$\bar{\mathbf{x}}^1(\bar{\mathbf{p}}) = (15, 15) \tag{3.55}$$

$$\bar{\mathbf{x}}^2(\bar{\mathbf{p}}) = (15, 15) \tag{3.56}$$

and the Walrasian equilibrium allocation:

$$\bar{\mathbf{x}}(\bar{\mathbf{p}}) = \left(\bar{\mathbf{x}}^1(\bar{\mathbf{p}}), \bar{\mathbf{x}}^2(\bar{\mathbf{p}}) \right) = (15, 15, 15, 15), \tag{3.57}$$

while the initial allocation is:

$$\mathbf{a} = \left(\mathbf{a}^1, \mathbf{a}^2 \right) = (10, 20, 20, 10). \tag{3.58}$$

Let us notice that in the Walrasian equilibrium state the relation between equilibrium prices is 1:1. This means that traders exchange goods in the relation 1 unit of the first good for 1 unit of the second good.

Since the utility functions of both traders are the same, their preferences to own each of the goods are also the same. Thus, in the Walrasian equilibrium state, as a result of the exchange made by the Walrasian equilibrium prices, both traders will have identical bundles of goods. It is also worth noticing that in the only state of the Walrasian equilibrium, an increase in the utility of bundles of goods purchased by traders with respect to utility of the initial consumption bundles will be the same for both traders.

Figure 3.4 presents geometrical illustrations of sets of allocations: of the Walrasian equilibrium $W(\mathbf{a})$, optimal in the Pareto sense and accepted in the same time $C(\mathbf{a})$, as well as the ones feasible with regard to the initial allocation: $\mathbf{a} = (10, 20, 20, 10)$.

Since the optimal solutions to both consumption utility maximization problems must lie on the budget lines respective to each of the traders, having the Walrasian equilibrium price vector we get:

(3.59) $$\bar{p}_1\bar{x}_{11} + \bar{p}_2\bar{x}_{12} = 10\bar{p}_1 + 20\bar{p}_2 \Leftrightarrow \bar{x}_{11} + \bar{x}_{12} = 30,$$

and

(3.60) $$\bar{p}_1\bar{x}_{21} + \bar{p}_2\bar{x}_{22} = 10\bar{p}_1 + 20\bar{p}_2 \Leftrightarrow \bar{x}_{21} + \bar{x}_{2} = 30.$$

This means that the budget lines of traders are not only parallel but also coincide because each of them includes the initial bundle of goods each of the traders came

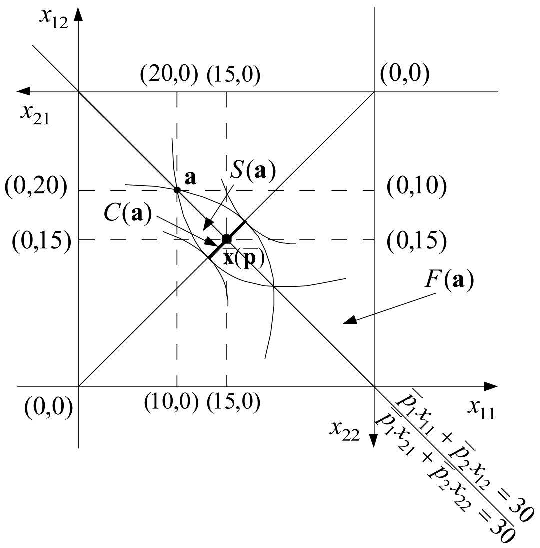


Fig. 3.4 Edgeworth box in case of power utility functions

to the market with. It is not difficult to notice that an angle of incline of both budget lines with respect to the horizontal axes is 45° in the coordinate system of each of the traders.

It is also easy to notice that: $W(\mathbf{a}) \subset C(\mathbf{a}) \subset S(\mathbf{a}) \subset F(\mathbf{a}) \subset \mathbb{R}^4_+$. This means that the only Walrasian equilibrium allocation corresponding to the only Walrasian equilibrium price vector, determined with an accuracy of a structure, is a Pareto optimal allocation, accepted and feasible with regard to an initial allocation.

Ad (b) The consumption utility maximization problems for two traders are given:

(3.61) $$u^1(x_{11}, x_{12}) = \min\{x_{11}, x_{12}\} \to \max$$

(3.62) $$p_1x_{11} + p_2x_{12} \le 10p_1 + p_2,$$

(3.63) $$x_{11}, x_{12} \ge 0,$$

and

(3.64) $$u^2(x_{21}, x_{22}) = \min\{x_{21}, x_{22}\} \to \max$$

(3.65) $$p_1x_{21} + p_2x_{22} \le 20p_1 + 10p_2,$$

(3.66) $$x_{21}, x_{22} \ge 0.$$

We know that the optimal solution to the consumption utility maximization problem in case of the power utility function $u^k(x_{k1}, x_{k2}) = \min\{a_{k1}x_{k1}, a_{k2}x_{k2}\}$ has a form:

(3.67) $$\bar{\mathbf{x}}^k(\mathbf{p}) = \left(\frac{a_{k2}p_1}{a_{k1}p_2 + a_{k2}p_1}\frac{I^k(\mathbf{p})}{p_1}, \frac{a_{k1}p_2}{a_{k1}p_2 + a_{k2}p_1}\frac{I^k(\mathbf{p})}{p_2}\right).$$

Substituting the parameters of the utility function of both traders into (3.67), we get the demand function of the first trader:

(3.68) $$\bar{\mathbf{x}}^1(\mathbf{p}) = \left(\frac{10p_1 + 20p_2}{p_1 + p_2}, \frac{10p_1 + 20p_2}{p_1 + p_2}\right),$$

of the second trader:

(3.69) $$\bar{\mathbf{x}}^2(\mathbf{p}) = \left(\frac{20p_1 + 10p_2}{p_1 + p_2}, \frac{20p_1 + 10p_2}{p_1 + p_2}\right).$$

Adding the demand functions of both traders, we get the global demand function for both consumer goods:

$$\bar{\mathbf{x}}(\mathbf{p}) = \bar{\mathbf{x}}^1(\mathbf{p}) + \bar{\mathbf{x}}^2(\mathbf{p}) = \left(\frac{30p_1 + 30p_2}{p_1 + p_2}, \frac{30p_1 + 30p_2}{p_1 + p_2}\right) = (30, 30). \tag{3.70}$$

The global supply function tor) has the form:

$$\bar{\mathbf{a}} = \mathbf{a}^1 + \mathbf{a}^2 = (30, 30). \tag{3.71}$$

The excess demand function then has the form:

$$\mathbf{z}(\mathbf{p}) = \bar{\mathbf{x}}(\mathbf{p}) - \bar{\mathbf{a}} = \left(\frac{30p_1 + 30p_2}{p_1 + p_2}, \frac{30p_1 + 30p_2}{p_1 + p_2}\right) - (30, 30) \; = (0, 0). \tag{3.72}$$

Let us notice that the excess demand function is homogeneous of degree 0 because:

$$\forall \lambda > 0 \quad \mathbf{z}(\lambda \mathbf{p}) = \mathbf{z}(\mathbf{p}) = (0, 0), \tag{3.73}$$

which means that the excess demand for any commodity does not depend on an absolute price level of both goods, but on the relationship between the prices of goods.

Moreover, the function of excess demand satisfies Walras's law, because:

$$\forall \, \mathbf{p} > 0 \quad \langle \mathbf{p}, \mathbf{z}(\mathbf{p}) \rangle = p_1 \cdot 0 + p_2 \cdot 0 = 0, \tag{3.74}$$

which means that for any positive vector of consumer goods prices, the value of global demand is equal to the value of global supply of both goods.

From condition (3.72), it llows that any positive vector of goods' prices is the Walrasian equilibrium price vector. Hence there exist infinitely many equilibrium price vectors, each of them determined with an accuracy of a structure:

$$\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2) = \bar{p}_1\left(1, \frac{\bar{p}_2}{\bar{p}_1}\right) = \bar{p}_2\left(\frac{\bar{p}_1}{\bar{p}_2}, 1\right) \quad \forall \, \bar{p}_1, \bar{p}_2 > 0. \tag{3.75}$$

Having this, after substituting (3.75) into (3.68) and (3.69), we can determine values of the demand functions of both traders when prices are given by any Walrasian equilibrium price vector. However, one should remember that no trader will accept a consumption bundle in which quantities of both goods would be smaller than quantities in the bundle he/she came to the market with.

We can notice that since:

$$\bar{\mathbf{x}}^1(\bar{\mathbf{p}}) = \left(\frac{10 + 20\frac{\bar{p}_2}{\bar{p}_1}}{1 + \frac{\bar{p}_2}{\bar{p}_1}}, \frac{10 + 20\frac{\bar{p}_2}{\bar{p}_1}}{1 + \frac{\bar{p}_2}{\bar{p}_1}}\right) = \left(\frac{10\frac{\bar{p}_1}{\bar{p}_2} + 20}{\frac{\bar{p}_1}{\bar{p}_2} + 1}, \frac{10\frac{\bar{p}_1}{\bar{p}_2} + 20}{\frac{\bar{p}_1}{\bar{p}_2} + 1}\right), \tag{3.76}$$

and

$$\bar{\mathbf{x}}^2(\bar{\mathbf{p}}) = \left(\frac{20 + 10\frac{\bar{p}_2}{\bar{p}_1}}{1 + \frac{\bar{p}_2}{\bar{p}_1}}, \frac{20 + 10\frac{\bar{p}_2}{\bar{p}_1}}{1 + \frac{\bar{p}_2}{\bar{p}_1}} \right) = \left(\frac{20\frac{\bar{p}_1}{\bar{p}_2} + 10}{\frac{\bar{p}_1}{\bar{p}_2} + 1}, \frac{20\frac{\bar{p}_1}{\bar{p}_2} + 10}{\frac{\bar{p}_1}{\bar{p}_2} + 1} \right), \tag{3.77}$$

hence if:

1. $\frac{\bar{p}_2}{\bar{p}_1} \to 0 \Rightarrow \bar{\mathbf{x}}^1(\bar{\mathbf{p}}) \to (10, 10) \wedge \bar{\mathbf{x}}^2(\bar{\mathbf{p}}) \to (20, 20)$ and $\bar{\mathbf{x}}(\bar{\mathbf{p}}) \to (10, 10, 20, 20)$,
   which means that if the first good price is infinitely high in comparison to the second good price, then such a price set is the most beneficial for the second trader and the least favourable for the first trader. As a result of the exchange by such equilibrium prices, the second trader will keep 20 units of the first good and will obtain additional 10 units of the second good from the first trader, while the first trader will keep 10 units of the first good but will lose 10 units of the second good. Consequently, the utility of a consumption bundle purchased by the second trader will increase nearly two times, while the utility of a consumption bundle purchased by the first trader will remain almost unchanged.
2. $\frac{\bar{p}_2}{\bar{p}_1} \to 1 \Rightarrow \bar{\mathbf{x}}^1(\bar{\mathbf{p}}) \to (15, 15) \wedge \bar{\mathbf{x}}^2(\bar{\mathbf{p}}) \to (15, 15)$ and $\bar{\mathbf{x}}(\bar{\mathbf{p}}) \to (10, 10, 15, 15)$,
   which means that if the prices of goods are close to each other, then such a price set is equally beneficial for both traders. As a result of the exchange by such equilibrium prices, both tradwill purchase identical consumption bundles. In terms of utility both will benefit the same since the utility of their consumption bundles will increase after the exchange to 15 in comparison to 10 before the exchange.
3. $\frac{\bar{p}_2}{\bar{p}_1} \to +\infty \Rightarrow \bar{\mathbf{x}}^1(\bar{\mathbf{p}}) \to (20, 20) \wedge \bar{\mathbf{x}}^2(\bar{\mathbf{p}}) \to (10, 10)$ and $\bar{\mathbf{x}}(\bar{\mathbf{p}}) \to (20, 20, 10, 10)$,
   which means that if the first good price is infinitely small in comparison to the second good price, then such a price set is the most beneficial for the first trader and the least favourable for the second trader. As a result of the exchange by such equilibrium prices the first trader will keep 20 units of the second good and will obtain additional 10 units of the first good from the second trader, while the second trader will keep 10 units of the second good but will lose 10 units of the first good. Consequently, the utility of a consumption bundle purchased by the first trader will increase nearly two times, while the utility of a consumption bundle purchased by the second trader will remain almost unchanged.

Figure 3.5 presents geometrical illustrations of sets of allocations: of the Walrasian equilibrium $W(\mathbf{a})$, optimal in the Pareto sense and accepted at the same time $C(\mathbf{a})$, as well as the ones feasible with regard to the initial allocation: $\mathbf{a} = (10, 20, 20, 10)$.

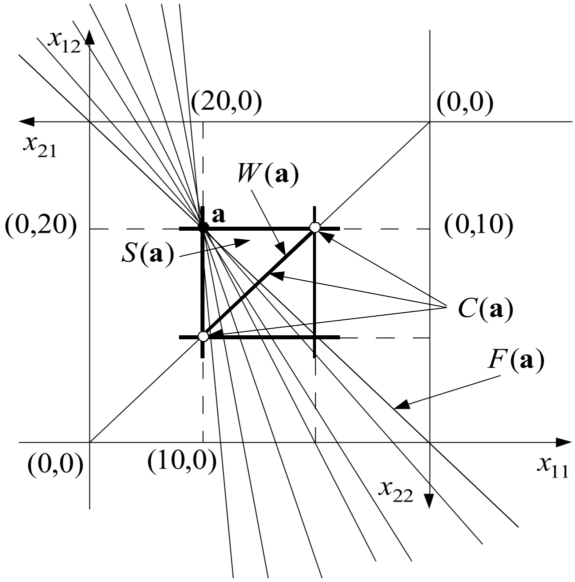


Fig. 3.5 Edgeworth box in case of Koopmans-Leontief functions

Since the optimal solutions to both consumption utility maximization problems must lie on the budget lines respective to each of the traders, we get:

(3.78) $$p_1\bar{x}_{11} + p_2\bar{x}_{12} = 10p_1 + 20p_2 \Leftrightarrow \bar{x}_{11} = -\frac{p_2}{p_1}\bar{x}_{12} + 10 + \frac{20p_2}{p_1},$$

and

(3.79) $$p_1\bar{x}_{21} + p_2\bar{x}_{22} = 10p_1 + 20p_2 \Leftrightarrow \bar{x}_{21} = -\frac{p_2}{p_1}\bar{x}_{22} + 20 + \frac{10p_2}{p_1}.$$

The budget lines of both traders are parallel because they have the same slope. Moreover, each of them contains the initial consumption bundle that each trader came to the market with and hence independently of the price set the budget lines coincide.

It is not difficult to notice that:

1. If $\frac{\bar{p}_2}{\bar{p}_1} \to 0$ then the budget lines of both traders are nearly perpendicular to the horizontal axes in the coordinate systems of both traders,
2. if $\frac{\bar{p}_2}{\bar{p}_1} \to +\infty$ then the budget lines of both traders are nearly perpendicular to the vertical axes in the coordinate systems of both traders,
3. if $\frac{\bar{p}_2}{\bar{p}_1} \to 1$ then the angle of incline with respect to the horizontal lines is 45° in the coordinate systems of both traders.

Let us also notice that: $W(\mathbf{a}) \subset C(\mathbf{a}) \subset S(\mathbf{a}) \subset F(\mathbf{a}) \subset \mathbb{R}^4_+$. This means that any of infinitely many Walrasian equilibrium allocations is optimal in the Pareto sense, accepted by traders and feasible with respect to the initial allocation. In fact a set $W(\mathbf{a})$ of Walrasian equilibrium allocations and a set $C(\mathbf{a})$ of allocations accepted and Pareto optimal (the exchange core) are almost identical. Actually, they differ only by two allocations, which means that a set $C(\mathbf{a})\backslash W(\mathbf{a})$, of allocations accepted and Pareto optimal that in the same time are not Walrasian equilibrium allocations, has only two elements:

$$C(\mathbf{a})\backslash W(\mathbf{a}) = \{(10, 10, 20, 20), (20, 20, 10, 10)\}.$$

3.3 Dynamic Arrow-Hurwicz Model

There is a market for two consumer goods, where:

$i = 1, 2$—an index of consumer goods,
$k = 1, 2$—an index of consumers (traders),
M—an index of a broker, that is a person responsible for price fixing in the consumer goods market, also called a Walrasian auctioneer,
$t = 1, 2, \ldots, T$—time as a discrete variable,
$t \in [0; T]$—time as a continuous variable,
T—time horizon, which can be finite or infinite,
$X(t) = \mathbb{R}^2_+$—a goods space in period/at moment t,
$\mathbf{a}^k = (a_{k1}, a_{k2}) \in \mathbb{R}^2_+$—an initial consumption basket k-th consumer comes to the market with (an endowment),
$\mathbf{p}(t) = (p_1(t), p_2(t)) \in \operatorname{int}\mathbb{R}^2_+$—a vector of consumer goods prices in period/at moment t,
$\mathbf{x}^k(t) = (x_{k1}(t), x_{k2}(t)) \in \mathbb{R}^2_+$—a consumption bundle the k-th consumer wants to purchase in period/at moment t,
$I^k(\mathbf{p}(t)) = p_1(t)a_{k1} + p_2(t)a_{k2} > 0$—consumer's income in period/at moment t,
$u^k\colon \mathbb{R}^2_+ \to \mathbb{R}$—a utility function of k-th consumer, describing his/her relation of preference,
$D^k(\mathbf{p}(t), I^k(\mathbf{p}(t)) = \left\{\mathbf{x}^k(\mathbf{p}(t)) \in \mathbb{R}^2_+ \middle| p_1(t)x_{k1}(t) + p_2(t)x_{k2}(t) \le I^k(\mathbf{p}(t))\right\} \subset X(t) = \mathbb{R}^2_+$—a budget set of k-th consumer, that is the set of all consumption bundles at time t of a value not exceeding the income of k-th consumer at time t.

The k-th consumer aims to purchase a bundle of goods $\bar{\mathbf{x}}^k(t) = (\bar{x}_{k1}(t), \bar{x}_{k2}(t))$ such that its value at time t does not exceed an income of k-th consumer and at the same time its utility is maximum and not less than of the initial bundle $\mathbf{a}^k = (a_{k1}, a_{k2})$.

The problem of choosing the optimal bundle of goods by k-th consumer at time t can be written as the consumption utility maximization problem:

$$\text{(P1)} \quad u^k(x_{k1}(t), x_{k2}(t)) \to \max \tag{3.80}$$

$$p_1(t)x_{k1}(t) + p_2(t)x_{k2}(t) \le p_1(t)a_{k1} + p_2(t)a_{k2}, \tag{3.81}$$

$$x_{k1}(t), x_{k2}(t) \ge 0, \tag{3.82}$$

$$t = 1, 2, ..., T \text{ or } t \in [0; T]. \tag{3.83}$$

If the utility function $u: \mathbb{R}^2_+ \to \mathbb{R}$ is increasing, differentiable and strictly concave, then at time t problem (p1) has exactly one optimal solution which lies on the budget line and is of a form:

$$\begin{aligned} \bar{\mathbf{x}}^k(\mathbf{p}(t)) &= \varphi^k\left(\mathbf{p}(t), I^k(\mathbf{p}(t))\right) \\ &= \left(\alpha^k(t)\frac{I^k(\mathbf{p}(t))}{p_1(t)}, \beta^k(t)\frac{I^k(\mathbf{p}(t))}{p_2(t)}\right) > (0, 0), \end{aligned} \tag{3.84}$$

$$\forall \alpha^k(t), \beta^k(t) \ge 0, \quad \alpha^k(t) + \beta^k(t) = 1, \ k = 1, 2, t = 1, 2, ..., T \vee t \in [0; T].$$

where k denotes the k-th consumer.

Definition 3.25 A **demand function of k-th consumer** at time t is a mapping $\varphi^k: \operatorname{int}\mathbb{R}^2_+ \to \operatorname{int}\mathbb{R}^2_+$ that assigns an optimal solution of the consumption utility maximization problem (P1) of k-th consumer to any price vector $\mathbf{p}(t) = (p_1(t), p_2(t)) \in \operatorname{int}\mathbb{R}^2_+$ and is of the form:

$$\begin{aligned} \varphi^k\left(\mathbf{p}(t), I^k(\mathbf{p}(t))\right) &= \varphi^k(\mathbf{p}(t)) \\ &= \left(\varphi_{k1}\left(\mathbf{p}(t), I^k(\mathbf{p}(t))\right), \varphi_{k2}\left(\mathbf{p}(t), I^k(\mathbf{p}(t))\right)\right) \\ &= (\bar{x}_{k1}(\mathbf{p}(t)), \bar{x}_{k2}(\mathbf{p}(t))) = \bar{\mathbf{x}}^k(\mathbf{p}(t)), \quad k = 1, 2. \end{aligned} \tag{3.85}$$

Definition 3.26 A **function of global demand** at time t is an expression:

$$\bar{\mathbf{x}}(\mathbf{p}(t)) = \bar{\mathbf{x}}^1(\mathbf{p}(t)) + \bar{\mathbf{x}}^2(\mathbf{p}(t)) = \begin{pmatrix} \bar{x}_{11}(\mathbf{p}(t)) + \bar{x}_{21}(\mathbf{p}(t)) \\ \bar{x}_{12}(\mathbf{p}(t)) + \bar{x}_{22}(\mathbf{p}(t)) \end{pmatrix}, \tag{3.86}$$

describing the total demand of both traders for each good.

Definition 3.27 A **function (vector) of global supply** at time t is an expression:

(3.87) $$\bar{\mathbf{a}} = \mathbf{a}^1 + \mathbf{a}^2 = \begin{pmatrix} a_{11} + a_{21} \\ a_{12} + a_{22} \end{pmatrix},$$

which describes the total supply of each good provided by both traders.

Definition 3.28 A mapping $\mathbf{z}\colon \operatorname{int} \mathbb{R}^2_+ \to \mathbb{R}^2$ given as:

(3.88) $$\mathbf{z}(\mathbf{p}(t))) = \bar{\mathbf{x}}(\mathbf{p}(t)) - \bar{\mathbf{a}}$$

or

(3.89) $$\forall i = 1, 2 \quad z_i(\mathbf{p}(\mathbf{t})) = \bar{x}_i(\mathbf{p}(t)) - \bar{a}_i$$

is called a **function of excess demand** at time t.

Definition 3.29 A **partial equilibrium** a market of i-th consumer good at time t. is a state in which:

(3.90) $$\exists i = 1, 2 \ \exists \mathbf{p}(t) > 0 \quad z_i(\mathbf{p}(t)) = \bar{x}_i(\mathbf{p}(t)) - \bar{a}_i = 0 \Leftrightarrow \bar{x}_i(\mathbf{p}(t)) = \bar{a}_i,$$

meaning that at time t there exists a positive price vector such that the global demand for i-th good is equal to its global supply.

We say then that at time t on the market of i-th consumer good there is a partial equilibrium: the global demand for i-th good (expressed in physical units) is equal to the global supply of i-th good (expressed in physical units).

Definition 3.30 A **general equilibrium** (in the Walras sense) on a market of consumer goods at time t is a state in which:

(3.91) $$\forall i = 1, 2 \ \exists \bar{\mathbf{p}}(t) > 0 \quad z_i(\bar{\mathbf{p}}(t)) = \bar{x}_i(\bar{\mathbf{p}}(t)) - \bar{a}_i = 0 \Leftrightarrow \bar{x}_i(\bar{\mathbf{p}}(t)) = \bar{a}_i,$$

meaning that at time t there exists a positive price vector, called the equilibrium (Walrasian) price vector such that the global demand for any good is equal to its global supply.

We then say that at time t there is a general equilibrium on the consumer goods market: the global demand for each good (expressed in physical units) is equal to its global supply (expressed in physical units).

Theorem 3.3 If the utility functions $u^k\colon \mathbb{R}^2_+ \to \mathbb{R}$, $k = 1, 2$ are increasing, differentiable and strictly concave, then at time t the excess demand function $\mathbf{z}(\mathbf{p}(t)))$ has the following properties:

1. is differentiable in $\operatorname{int}\mathbb{R}^2_+$,
2. is homogeneous of degree 0:

$$\forall i = 1, 2\ \ \forall\lambda > 0\ \forall\ \mathbf{p}(t) > 0$$
$$z_i(\lambda\mathbf{p}(t)) = \bar{x}_i(\lambda\mathbf{p}(t)) - \bar{a}_i = \bar{x}_i(\mathbf{p}(t)) - \bar{a}_i = z_i(\mathbf{p}(t)), \tag{3.92}$$

which means that a proportional change in the prices of all consumer goods at time t does not change the excess demand for any consumer good,
3. satisfies Walras's law:

$$\begin{aligned} &\forall\,\mathbf{p}(t) > 0\quad \sum\nolimits_{i=1}^{2} p_i z_i(\mathbf{p}(t)) = \sum\nolimits_{i=1}^{2} p_i(\bar{x}_i(\mathbf{p}(t)) - \bar{a}_i) = 0 \quad \Leftrightarrow \\ &\Leftrightarrow \quad \sum\nolimits_{i=1}^{2} p_i(\bar{x}_i(\mathbf{p}(t))) = \sum\nolimits_{i=1}^{2} p_i\bar{a}_i, \end{aligned} \tag{3.93}$$

which means that for any price vector $\mathbf{p}(t) = (p_1(t), p_2(t)) > (0, 0)$ at time t the value of global demand for all goods is equal to the value of their global supply.

Note 3.9 The concept of Walrasian equilibrium should be distinguished from Walras' law.

Note 3.10 The Walrasian equilibrium state described by the Walrasian equilibrium price vector may not exist, there may be exactly one such state or there may be more than one.

Note 3.11 The price vector of the Walrasian equilibrium (if it exists) is determined with an accuracy of a structure.

Let us assume that $\bar{\mathbf{p}}(t) = \left(\bar{p}_1(t), \bar{p}_2(t)\right) > 0$ is a price vector of the Walrasian equilibrium. Then we can present it in a form:

$$\begin{aligned} \bar{\mathbf{p}}(t) &= \left(\bar{p}_1(t), \bar{p}_2(t)\right) \\ &= \bar{p}_1(t)\left(1, \frac{\bar{p}_2(t)}{\bar{p}_1(t)}\right) = \lambda\left(1, \frac{\bar{p}_2(t)}{\bar{p}_1(t)}\right),\ \text{where } \lambda\ =\ \bar{p}_1(t) > 0, \end{aligned} \tag{3.94}$$

or

$$\begin{aligned} \bar{\mathbf{p}}(t) &= \left(\bar{p}_1(t), \bar{p}_2(t)\right) \\ &= \bar{p}_2(t)\left(\frac{\bar{p}_1(t)}{\bar{p}_2(t)}, 1\right) = \lambda\left(\frac{\bar{p}_1(t)}{\bar{p}_2(t)}, 1\right),\ \text{where } \lambda\ =\ \bar{p}_2(t) > 0. \end{aligned} \tag{3.95}$$

This shows that if $\bar{\mathbf{p}}(t) = \bar{p}_1(t)\left(1, \frac{\bar{p}_2(t)}{\bar{p}_1(t)}\right)$ is an equilibrium price vector then $\bar{\mathbf{p}}(t) = \lambda\left(1, \frac{\bar{p}_2(t)}{\bar{p}_1(t)}\right), \lambda > 0$ is an equilibrium price vector too. This means that to one Walrasian equilibrium price vector, determined with an accuracy of a structure, infinitely many price vectors are related differing only in the absolute levels of consumer goods' prices but having an identical structure.

Theorem 3.4 If the utility function $u^k : \mathbb{R}^2_+ \to \mathbb{R},\ k = 1, 2$ is increasing, differentiable and strictly concave, then in the Arrow-Hurwicz model there exists at least one price vector of the Walrasian equilibrium, determined with an accuracy of a structure.

Note 3.12 The conditions, ensuring that in the Arrow-Hurwicz model exists exactly one Walrasian equilibrium price vector determined with an accuracy of a structure, are in the form of more complex assumptions. Therefore, we will not provide them as part of the basic lecture[5] here.

In the description of the dynamic Arrow-Hurwicz model presented so far we have used a time variable to denote moments in which particular events happen on a market of consumer goods. We have not distinguished discrete and continuous time yet.

Before we define discrete-time and continuous-time versions of the dynamic Arrow-Hurwicz model let use present in a descriptive way the rules according to which a market of consumer goods acts in this model. Let us consider a market with two traders and with two goods, where a distinct economic agent operates—a broker[6] who each time determines prices of all goods. The aim of every trader is to exchange a bundle of goods he/she entered the market with for a bundle of goods of a value not exceeding an income of a given trader and whose utility is maximum and at the same time not less than the utility of the initial consumption bundle. The value of a bundle of goods the k-th trader entered the market with, equal to an income of this trader, and the value of a bundle that the k-th trader wants to purchase depend on the prices of consumer goods set by the broker.

Rules according to which the market operates are as follows. Each trader coming to the market lets the broker know about quantities of goods he/she brings to the market. On this basis, the broker obtains information about the global supply of each commodity. The broker's task is to provide such a system of prices of goods, by which each trader will decide to purchase a bundle of goods whose value by prices proposed by the broker will not exceed the income of a given trader and whose utility will be maximum and at the same time no less than the utility of the bundle of goods a given trader came to the market with. An exchange transaction

[5] More complete knowledge on this issue is presented, among others, in work (Panek, 2003).

[6] In microeconomic literature such a person, or in general entity, is also often called a Walrasian auctioneer. However, it should be emphasized that in the Arrow-Hurwicz model a market of consumer goods is not an auction market and that an auctioneer (broker) in this context does not aim to set the highest prices.

between traders will take place only when the broker proposes such a price vector by which the global supply and the global demand for all goods, expressed in physical units, are identical. In other words, the goal of the broker is to determine a price vector of the Walrasian equilibrium. This task is not easy. To make decision about prices the broker can use intuition, experience or some strict rules.

The process of reaching equilibrium prices can be written as the following iterative procedure. At an initial moment $t = 0$, the broker knows the global supply of each commodity and proposes some initial price system $\mathbf{p}(0) = \mathbf{p}_0 > 0$. He/she anticipates the global demand for each commodity and proposes such a price system $\mathbf{p}(t) > 0$, which will equalize the global supply and the global demand for goods and at the same time will allow each trader to undertake the transaction that he/she judges as the best. Having the knowledge about the prices of consumer goods, each trader determines his/her demand function for each commodity, which is the solution to the consumption utility maximization problem of the k-th trader ($k = 1, 2$). Knowing values of the demand functions of all traders, the broker can determine the value of the global demand function for each good by a given set of prices. Comparing it with the global supply of a given good he/she determines the value of an excess demand function. If the initial prices proposed by the broker were the Walrasian equilibrium price vector, then all traders would exchange their bundles among themselves. Otherwise, the broker should propose new prices of all goods that would establish the equilibrium. Knowing the value of the excess demand for each commodity at any time t and goods' prices at any time$t$, in order to set the prices the broker should proceed in the following way:

- if $\forall i \;\; z_i(\mathbf{p}(t) = \bar{x}_i(\mathbf{p}(t)) - \bar{a}_i > 0$, then $\bar{x}_i(\mathbf{p}(t)) > \bar{a}_i \Leftrightarrow p_i(t+1) > p_i(t)$, which means that if the global demand for i-th good is greater than the global supply of this good when its price at time t is $p_i(t) > 0$, then in order to adjust the demand for this good to its supply, the price of i-th good at time $t + 1$ should be higher than at time t.
- if $\forall i \;\; z_i(\mathbf{p}(t) = \bar{x}_i(\mathbf{p}(t)) - \bar{a}_i < 0$, then $\bar{x}_i(\mathbf{p}(t)) < \bar{a}_i \Leftrightarrow p_i(t+1) < p_i(t)$, which means that if the global demand rh good is smaller than the global supply of this good when its price at tim t is $p_i(t) > 0$, then in order to adjust the demand for this good to its supply, the price of i-th good at time $t + 1$ should be lower than at time t.
- if $\forall i \;\; z_i(\mathbf{p}(t) = \bar{x}_i(\mathbf{p}(t)) - \bar{a}_i = 0$, then $\bar{x}_i(\mathbf{p}(t)) = \bar{a}_i \Leftrightarrow p_i(t+1) = p_i(t)$, which means that if on a market of i-th good at time t the global supply and the global demand are equal, then the price of this good should not be changed at time $t + 1$. In this case, when the price of i-th good is an equilibrium price for this good, we say that on a markeof all commodities there is a **partial equilibrium** with respect to i-th good.

If by a price system proposed by a broker, there is an equilibrium of global supply and global demand for all goods, we say then that a global equilibrium has been reached on the consumer goods market—the equilibrium defined by the Walrasian equilibrium price vector $\bar{\mathbf{p}} > 0$.

The main questions regarding a market described by the dynamic Arrow-Hurwicz model are:

- does a state of the Walrasian equilibrium exist on a consumer goods market?
- is there exactly one or at least one state of the Walrasian equilibrium?
- whether and in what time horizon is it possible to reach the state of the Walrasian equilibrium?

To answer these questions, one needs to determine in what way a broker sets prices of consumer goods.

Definition 3.31 A **dynamic discrete-time Arrow-Hurwicz model** is a system of difference equations of a form:

(3.96) $$\forall i \quad p_i(t+1) = p_i(t) + \sigma_i z_i(\mathbf{p}(t)),$$

with an initial condition:

(3.97) $$\forall i \quad p_i(0) = p_i^0 > 0,$$

(3.98) $$t = 0, 1, 2, \ldots$$

where $\sigma_i > 0$ denotes a measure of broker 's sensitivity to an imbalance on i-th good's market, which for the sake of simplicity is assumed to be the same for markets of all goods: $\forall i \ \sigma_i = \sigma > 0$.

Condition (3.96) can be written in an equivalent form:

(3.99) $$\forall i \quad p_i(t+1) - p_i(t) = \sigma_i z_i(\mathbf{p}(t)).$$

On the basis of conditions (3.96) and (3.99) it can be concluded that $\forall t = 0, 1, 2, \ldots$ and $\forall i = 1, 2$:

$$z_i(\mathbf{p}(t)) > 0 \Rightarrow p_i(t+1) - p_i(t) > 0 \Rightarrow p_i(t+1) > p_i(t),$$
$$z_i(\mathbf{p}(t)) < 0 \Rightarrow p_i(t+1) - p_i(t) < 0 \Rightarrow p_i(t+1) < p_i(t),$$
$$z_i(\mathbf{p}(t)) = 0 \Rightarrow p_i(t+1) - p_i(t) = 0 \Rightarrow p_i(t+1) = p_i(t).$$

Equivalent conditions (3.96) and (3.99) lead to a simple recursive rule for determining prices of all goods in subsequent periods of time. However, this rule does not ensure that the resulting price systems will make economic sense. We are not interested in situations where a price of any good is negative. Therefore, our attention should be focused only on such solutions to systems of difference equations (3.96) or (3.99), in which the vectors of consumer goods' prices determined on the basis of these solutions are positive: $\forall i \quad p_i(t+1) > 0$.

Definition 3.32 A **feasible price trajectory** in the dynamic discrete-time Arrow-Hurwicz model is an infinite sequence of solutions to the difference equations' system (3.99) with an initial condition $\mathbf{p}(0) = \mathbf{p}^0 > 0$ such that $\forall\, t = 0, 1, 2, \ldots\ \mathbf{p}(t+1) > 0$.

Assuming there exists a feasible price trajectory in the dynamic discrete-time Arrow-Hurwicz model, one is interested in the conditions of existence, uniqueness and stability of the Walrasian equilibrium state.

Definition 3.33 A Walrasian equilibrium state $\bar{\mathbf{p}} > 0$ is called **asymptotically globally stable** when a feasible trajectory of goods' prices satisfies a condition:

$$\lim_{t \to +\infty} \mathbf{p}(t+1) = \bar{\mathbf{p}}. \tag{3.100}$$

Global stability means that any feasible trajectory of goods' prices, starting from any initial price system $\mathbf{p}(0) = \mathbf{p}^0 > 0$, after reaching a state of the Walrasian equilibrium will remain in this state. The stability is also asymptotic one, because the state of the Walrasian equilibrium is a target state which, if exists, can be achieved in an infinite time horizon.

Definition 3.34 A **dynamic continuous-time Arrow-Hurwicz model** is a system of differential equations of a form:

$$\forall\, i \quad \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = \sigma_i z_i(\mathbf{p}(t)), \tag{3.101}$$

with an initial condition:

$$\forall\, i \quad p_i(0) = p_i^0 > 0, \tag{3.102}$$

$$t \in [0; +\infty). \tag{3.103}$$

where:

where $\sigma_i > 0$ denotes a measure of broker 's sensitivity to an imbalance on i-th good's market, which for the sake of simplicity is assumed to be the same for markets of all goods: $\forall\, i \quad \sigma_i = \sigma > 0$.

On the basis of condition (3.101), it can be concluded that $\forall\, t \in [0; +\infty)$ and $\forall\, i = 1, 2$:

$$z_i(\mathbf{p}(t)) > 0 \;\Rightarrow\; \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} > 0 \;\Rightarrow\; p_i(t+1) > p_i(t),$$
$$z_i(\mathbf{p}(t)) < 0 \;\Rightarrow\; \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} < 0 \;\Rightarrow\; p_i(t+1) < p_i(t),$$
$$z_i(\mathbf{p}(t)) = 0 \;\Rightarrow\; \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = 0 \;\Rightarrow\; p_i(t+1) = p_i(t).$$

This simple recursive rule, described by conditions (3.101)–(3.102), shows how to determine prices of all goods in subsequent moments. However, it does not ensure that the resulting price systems will make economic sense. Therefore, our attention should be focused only on such solutions, to systems of differential equations (3.101), in which the vectors of consumer goods' prices determined on the basis of these solutions are positive: $\forall\, i \quad p_i(t + \Delta t) > 0, \quad \Delta t \to 0.$

Definition 3.35 A **feasible price trajectory** in the dynamic continuous-time Arrow-Hurwicz model is an infinite sequence of solutions to the differential equations system (3.101) with an initial condition $\mathbf{p}(0) = \mathbf{p}^0 > 0$ such that $\forall\, t \in [0; +\infty) \quad \mathbf{p}(t + \Delta t) > 0$.

Assuming that there exists a feasible price trajectory in the dynamic continuous-time Arrow-Hurwicz model, one is interested in conditions of existence, uniqueness and stability of theV Walrasian equilibrium state.

Definition 3.36 A Walrasian equilibrium state $\bar{\mathbf{p}} > \mathbf{0}$ is called **asymptotically globally stable** when a feasible trajectory of goods' prices satisfies a condition:

$$\lim_{\substack{t \to +\infty \\ \Delta t \to 0}} \mathbf{p}(t + \Delta t) = \bar{\mathbf{p}}. \tag{3.104}$$

Global stability means that any feasible trajectory of goods' prices, starting from any initial price system $\mathbf{p}(0) = \mathbf{p}^0 > 0$, after reaching a state of the Walrasian equilibrium will remain in this state. The stability is also asymptotic one, because the state of the Walrasian equilibrium is a target state which, if exists, can be achieved in an infinite time horizon.

In Figs. 3.6 and 3.7, a graph of a feasible price trajectory in a dynamic discrete-time Arrow-Hurwicz model is presented. A state space is $\mathbb{R}^2_+$, while a phase state is $\mathbb{R}^3_+$. The phase state can be seen as an extension of the state space by adding time as the third dimension. Let us notice that the graph of the feasible trajectory of prices of both goods presented in the state space is obtained from the feasible trajectory in the phase space projected to a plane.

Both figures have purely hypothetic character. They do not relate to the dynamic discrete-time Arrow-Hurwicz model in a particular form which would indicate undoubtedly that in this model there exists exactly one Walrasian equilibrium price vector determined with an accuracy of a structure.

Nevertheless, both figures illustrate that if there exists exactly one Walrasian equilibrium price vector, determined with an accuracy of a structure then this vector is a limit for a feasible price trajectory. A process of reaching the Walrasian equilibrium price vector does not need to be done in a finite number of periods. This means that it has an asymptotic character. If, however, a feasible price trajectory reaches the Walrasian equilibrium state, described by the Walrasian equilibrium price vector, then it will remain in this state. In this sense, the Walrasian equilibrium state is globally asymptotically stable.

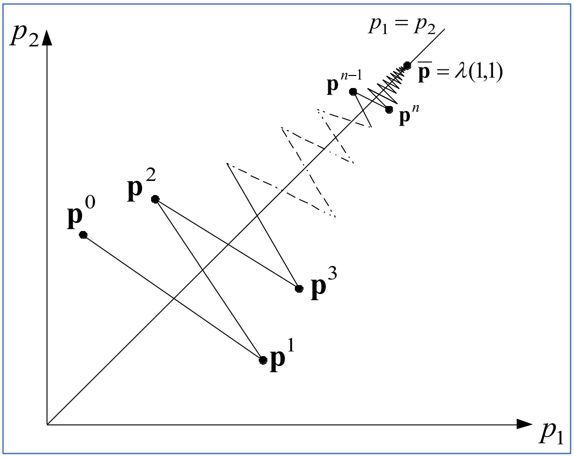


Fig. 3.6 Feasible price trajectory in state space in dynamic Arrow-Hurwicz model

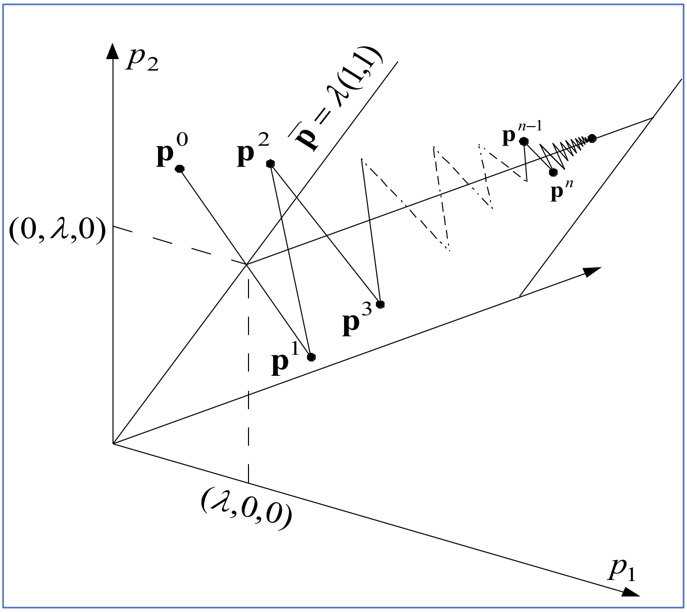


Fig. 3.7 Feasible price trajectory in phase space in dynamic Arrow-Hurwicz model

Example 3.3 Two traders come to a market with bundles of goods: $\mathbf{a}^1 = (10, 20)$, $\mathbf{a}^2 = (20, 10)$. Utility functions of traders are: $u^1(x_{11}, x_{12}) = x_{11}^{1/4} x_{12}^{1/4}$, $u^2(x_{21}, x_{22}) = x_{21}^{1/4} x_{22}^{1/3}$. We know from Example 3.2 that in the static Arrow-Hurwicz model for a given initial allocation and given utility functions, the excess demand function takes the form:

$$\mathbf{z}(\mathbf{p}) = \left(15\frac{p_2}{p_1} - 15, 15\frac{p_1}{p_2} - 15\right),$$

and the Walrasian equilibrium price vector has a structure:

$$\bar{\mathbf{p}} = \lambda(1, 1), \quad \lambda > 0.$$

Let us first consider a discrete-time version of the dynamic Arrow-Hurwicz model. A broker announces initial prices:

$$\mathbf{p}(0) = (2, 4).$$

1. Determine trajectories of a price vector satisfying a system of equations of the dynamic discrete-time Arrow-Hurwicz taking a proportionality coefficient σ equal to 0.25, 0.35 and 1.25. Calculate price ratios $\frac{p_2(t)}{p_1(t)}$ and compare them with the equilibrium price ratio $\frac{\bar{p}_2}{\bar{p}_1}$.
2. State which trajectories determined in point 1 are feasible.
3. State if and when (in which period) a structure of prices stabilizes around the equilibrium price structure and whether it reaches this structure in time horizon $T = 15$.
4. Present graphs of the price trajectories in the state space.
5. Present graphs of the price trajectories as functions of time.

Ad 1 The price trajectories of the first and of the second good are determined, respectively, from formulas:

$$p_1(t+1) - p_1(t) = \sigma\left(15\frac{p_2(t)}{p_1(t)} - 15\right),$$

$$p_2(t+1) - p_2(t) = \sigma\left(15\frac{p_1(t)}{p_2(t)} - 15\right).$$

Ad 2 The data in Table 3.3 show that the price trajectories are not feasible when the measure σ of the broker's sensitivity to the imbalance on the markets of both goods is equal to 1.25. The price of the second commodity has a negative value just right in period 2.

Table 3.1 Price trajectories when $\sigma = 0.25$

t	p_1	p_2	$\frac{p_2(t)}{p_1(t)}$	$\left\lvert \frac{\bar{p}_2}{\bar{p}_1} - \frac{p_2(t)}{p_1(t)} \right\rvert$
0	2	4	2	1
1	2.75	3.625	1.318182	0.318181818
2	2.988636	3.443966	1.152353	0.152353481
3	3.102901	3.344807	1.077961	0.077961226
4	3.161372	3.290565	1.040866	0.040866096
5	3.192022	3.261119	1.021647	0.02164683
6	3.208257	3.245228	1.011524	0.011523673
7	3.2169	3.236684	1.00615	0.006149976
8	3.221512	3.232099	1.003286	0.003286371
9	3.223977	3.229643	1.001757	0.001757333
10	3.225295	3.228327	1.00094	0.000940043
11	3.226	3.227623	1.000503	0.000502949
12	3.226377	3.227246	1.000269	0.000269119
13	3.226579	3.227044	1.000144	0/000144009
14	3.226687	3.226936	1.000077	7.70629E-05
15	3.226745	3.226878	1.000041	4.12391E-05
16	3.226776	3.226847	1.000022	2.20687E-05
17	3.226792	3.226831	1.000012	1.18099E-05
18	3.226801	3.226822	1.000006	6.31997E-06
19	3.226806	3.226817	1.000003	3.3821E-06
20	3.226809	3.226814	1.000002	1.80991E-06

Ad 3 The data in Tables 3.1, 3.2 and 3.3 show that the price structure stabilizes around the equilibrium structure when $\sigma = 0.25$ or $\sigma = 0.35$. In the first case, it reaches this structure in period around $t = 14$, in the second case in period around $t = 25$.

Ad 4 Figures 3.8, 3.9 and 3.10 presenting the graphs of price trajectories in the state space show also the equilibrium price vector, which in this example is $\bar{\mathbf{p}} = \lambda(1, 1)$.

Ad 5 Graphs of price trajectories as functions of time,[7] presented in Figs. 3.11, 3.12 and 3.13, allow for easy observation whether for given price trajectories the price

[7] Time is considered here as discrete, which means we are interested in values of functions only at points $t = 0, 1, 2, \ldots$ which denote subsequent periods. However, to make the figures clear and easier to observe changes of prices over time we present linear plots instead of scatter ones.

Table 3.2 Price trajectories when $\sigma = 0.35$

t	p_1	p_2	$\frac{p_2(t)}{p_1(t)}$	$\left\lvert \frac{\bar{p}_2}{\bar{p}_1} - \frac{p_2(t)}{p_1(t)} \right\rvert$
0	2	4	2	1
1	7.25	1.375	0.189655	0.810344828
2	299,569	23.80682	7.947024	6.947024199
3	39.46757	19.21744	0.486917	0.513082654
4	36.77388	24.74956	0.67302	0.326979955
5	35.05724	27.30022	0.778733	0.221267248
6	33.89558	28.79194	0.84943	0.150569556
7	33.10509	29.72255	0.897824	0.102175844
8	32.56867	30.32002	0.930957	0.069043271
9	32.20619	30.70938	0.953524	0.04647586
10	31.9622	30.96527	0.968809	0.031190638
11	31.79845	31.1343	0.979114	0.020886168
12	31.68879	31.24629	0.986036	0.013964048
13	31.61548	31.32064	0.990674	0.009325909
14	31.56652	31.37006	0.993776	0.006223687
15	31.53385	31.40294	0.995849	0.004151311
16	31.51205	31.42482	0.997232	0.002768058
17	31.49752	31.4394	0.998155	0.001845296
18	31.48783	31.4491	0.99877	0.001229958
19	31.48137	31.45557	0.99918	0.000819729
20	31.47707	31.45988	0.999454	0.000546287
21	31.4742	31.46274	0.999636	0.000364042
22	31.47229	31.46466	0.999757	0.000242588
23	31.47102	31.46593	0.999838	0.000161651
24	31.47017	31.46678	0.999892	0.000107716
25	31.4696	31.46735	0.999928	7.17761E-05

structure converges in time to the equilibrium price structure, thus reaching the state of Walrasian equilibrium, and whether it remains in this state.

Example 3.4 Let us now consider a continuous-time version of the Arrow-Hurwicz dynamic model for the same data as in Example 3.3.

1. Determine trajectories of a price vector satisfying a system of equations of the dynamic continuous-time Arrow-Hurwicz model taking a proportionality coefficient σ equal to 0.25, 0.35, 1.25 and determine whether these trajectories are feasible.

Table 3.3 Price trajectories when $\sigma = 1.25$

t	p_1	p_2	$\frac{p_2(t)}{p_1(t)}$	$\left\| \frac{\bar{p}_2}{\bar{p}_1} - \frac{p_2(t)}{p_1(t)} \right\|$
0	2	4	2	1
1	20.75	−5.375	−0.259036	1.259036145
2	−2.85693	−96.5087	33.780596	32.78059605
3	611.7792	−114.704	−0.187492	1.187491924
4	589.5138	−233.458	−0.396018	1.396017847
5	563.3384	−299.554	−0.531748	1.531748422
6	534.6182	−353.565	−0.661342	1.661341852
7	503.468	−400.667	−0.795814	1.795813863
8	469.7965	−442.978	−0.942914	1.942913813
9	433.3669	−481.613	−1.111328	2.11132811
10	393.7795	−517.234	−1.313513	2.313513108

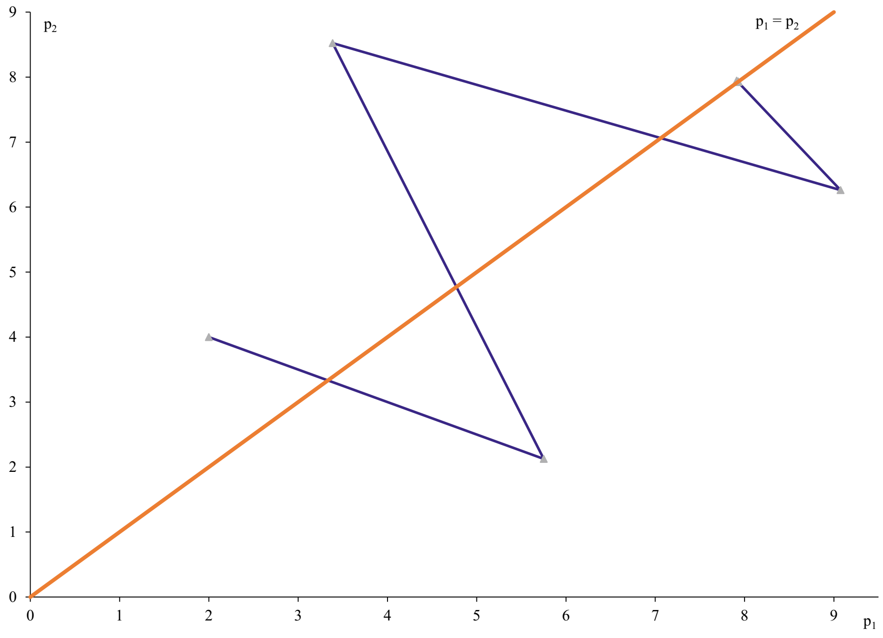


Fig. 3.8 Price trajectories in state space when $\sigma = 0.25$

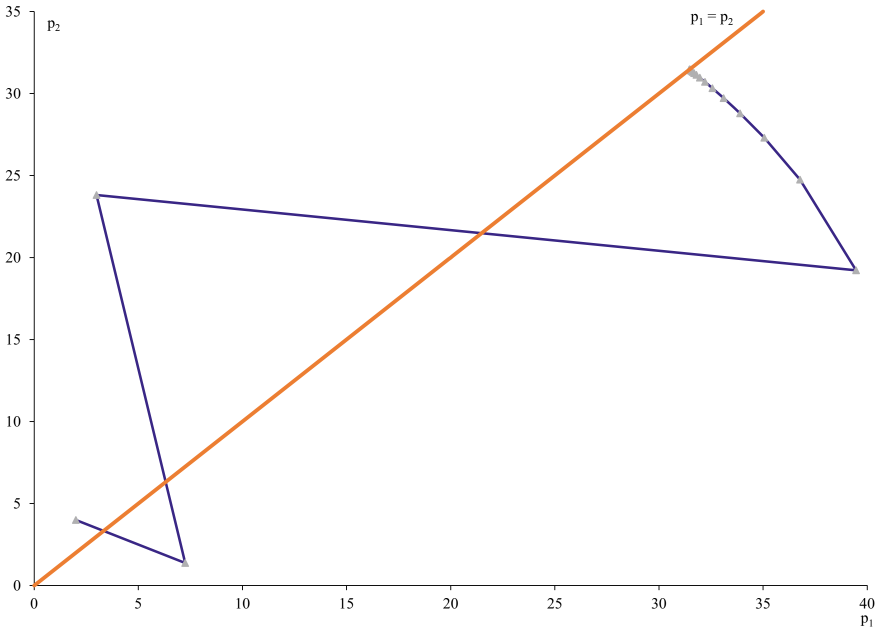


Fig. 3.9 Price trajectories in state space when $\sigma = 0.35$

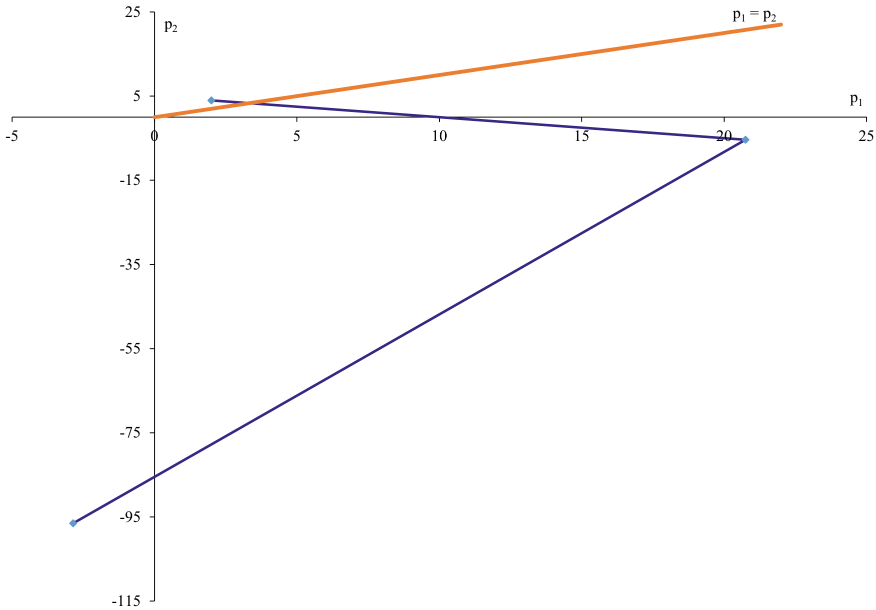


Fig. 3.10 Price trajectories in state space when $\sigma = 1.25$

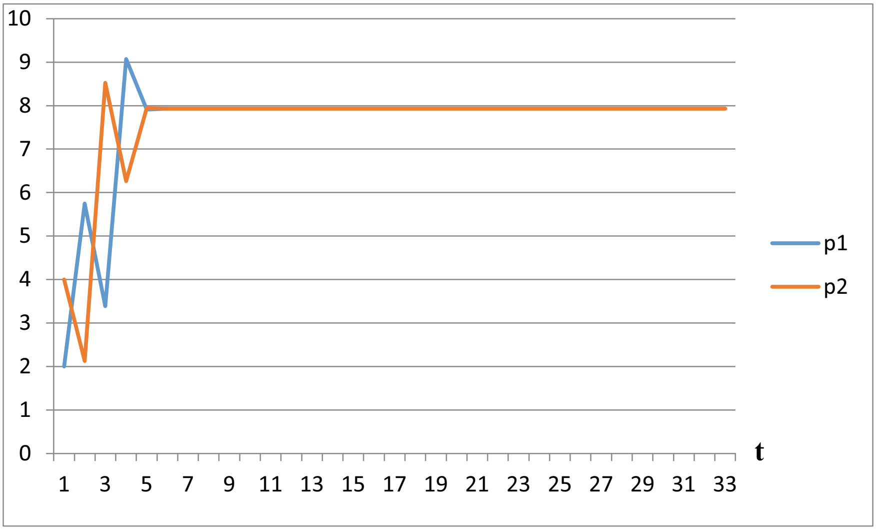


Fig. 3.11 Price trajectories as functions of time when $\sigma = 0.25$

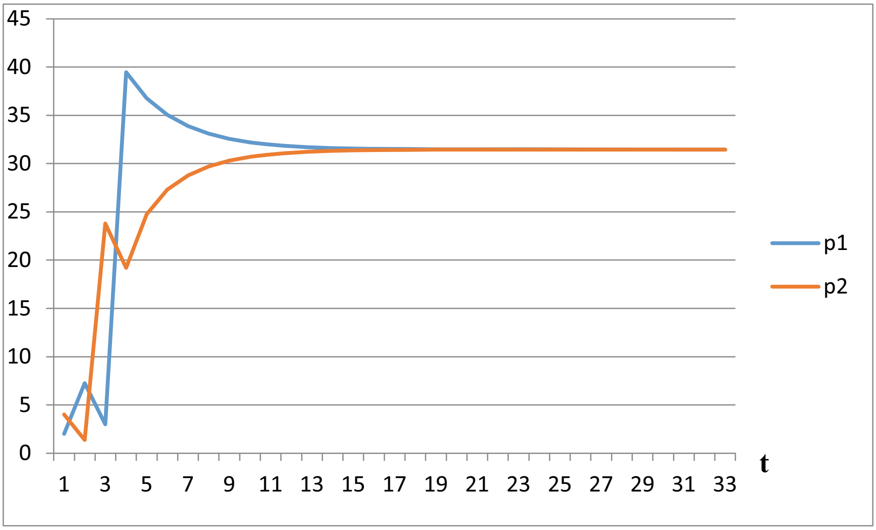


Fig. 3.12 Price trajectories as functions of time when $\sigma = 0.35$

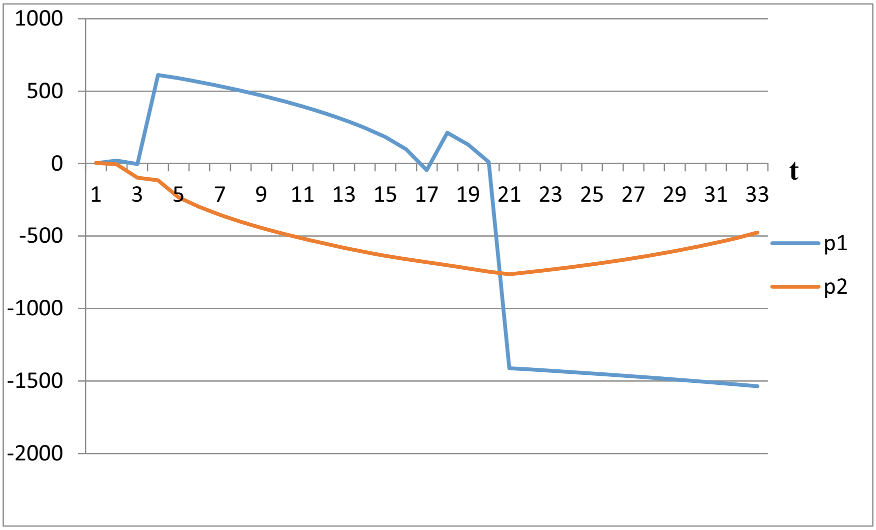


Fig. 3.13 Price trajectories as functions of time when $\sigma = 1.25$

2. Determine if and when (at what moment) a structure of prices stabilizes around the equilibrium price structure.
3. Present graphs of price trajectories as functions of time.

Ad 1 The price trajectories of the first and of the second good are determined, respectively, from formulas:

$$\frac{\mathrm{d}p_1(t)}{\mathrm{d}t} = \sigma\left(15\frac{p_2(t)}{p_1(t)} - 15\right),$$

$$\frac{\mathrm{d}p_2(t)}{\mathrm{d}t} = \sigma\left(15\frac{p_1(t)}{p_2(t)} - 15\right).$$

To determine approximate price trajectories, we can use the Euler method in which one approximates differential equations with difference equations. For example, for the price of the first good, its trajectory can be determined from a formula:

$$p_1(t+1) - p_1(t) = \sigma\left(15\frac{p_2(t)}{p_1(t)} - 15\right) \cdot \Delta t,$$

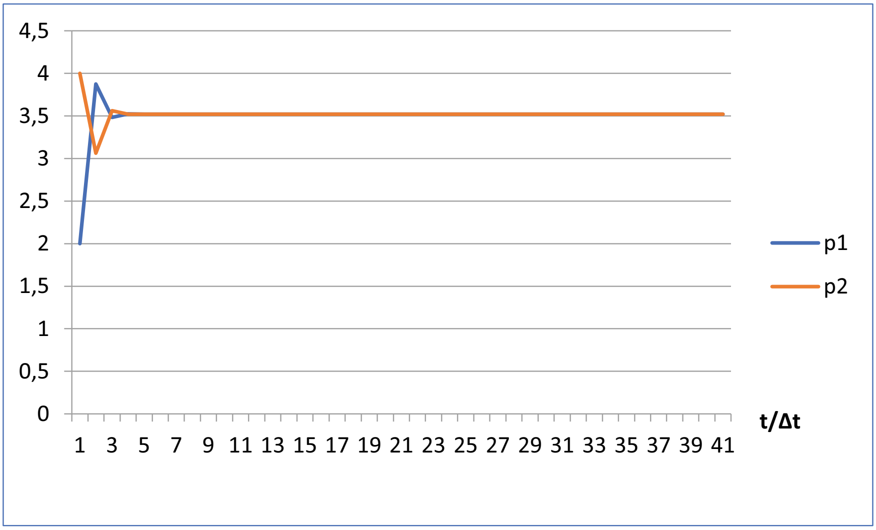


Fig. 3.14 Price trajectories as functions of time when $\sigma = 0.25$

where Δt denotes a time increment.[8] Let us assume that $\Delta t = 0.5$. Then, we obtain trajectories that are not feasible when the propoionality coefficient σ is equal to 1.25, as shown in Fig. 3.16.

Ad 2 With the proportionaly coefficient $\sigma = 0.25$, the price structure stabilizes around the equilibrium structure at time $t = 2$. With $\sigma = 0.35$, the convergence is achieved at time $t = 4.5$. When $\sigma = 1.35$, the price of the second good has a negative value just right at $t = 0.5$.

Ad 3 In order to determine trajectories of prices satisfying a given system of differential equations with an initial condition, we used the Euler method to obtain an approximate solution this way. Graphs, presented in Figs. 3.14, 3.15 and 3.16, show price trajectories for each moment by every 0.5 time unit, where the fortieth step $(t/\Delta t)$ of the calculation means the twentieth period ($t = 20$).

[8] Depending on the value of parameter Δt meaning the time increment we observe a lack of convergence, faster or slower convergence to the equilibrium state determined as a structure of prices.

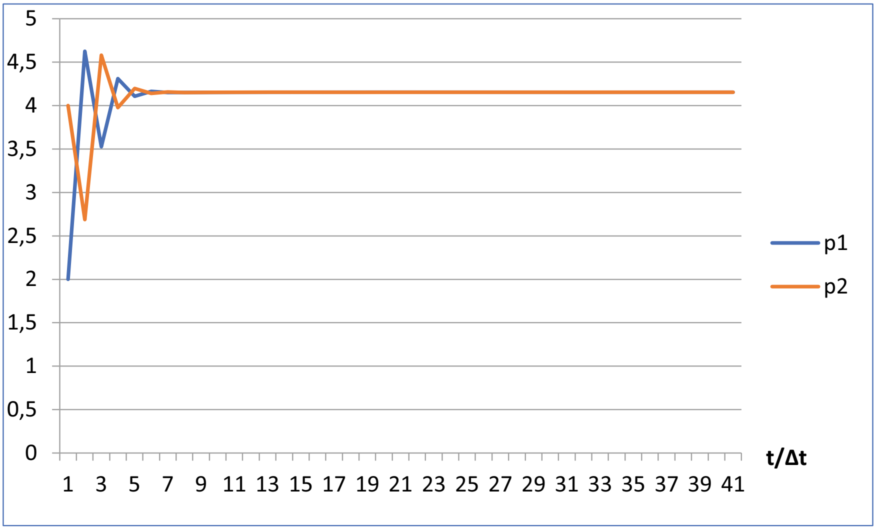


Fig. 3.15 Price trajectories as functions of time when $\sigma = 0.35$

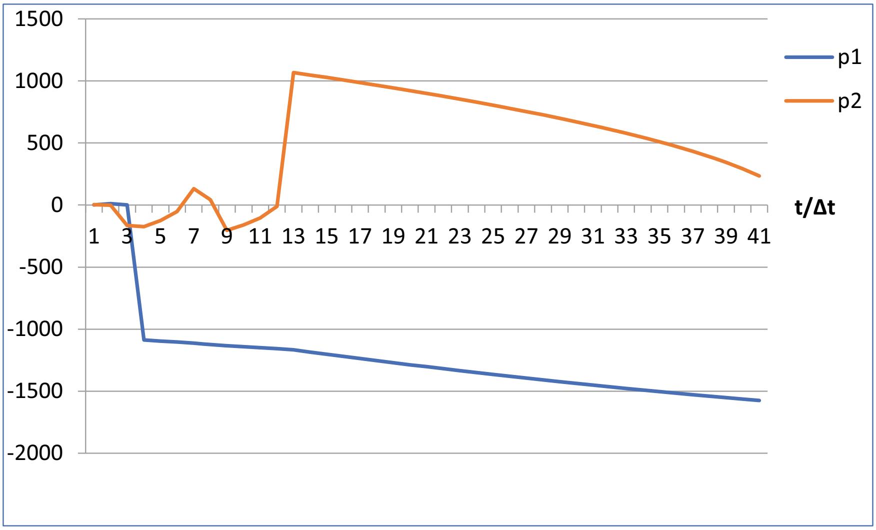


Fig. 3.16 Price trajectories as functions of time when $\sigma = 1.25$

3.4 Questions

1. Why the simple model of exchange is a special case of the static Arrow-Hurwicz model?
2. What is the Edgeworth box used for in the simple model of exchange and in the static Arrow-Hurwicz model?
3. What kind of allocation accepted by traders and feasible with regard to an initial allocation is called Pareto optimal (efficient) in the simple model of exchange?
4. Why in the simple model of exchange every allocation which is Pareto optimal and accepted by traders is feasible with regard to an initial allocation? Use the Edgeworth box to explain this.
5. Is it possible in the simple model of exchange or in the static Arrow-Hurwicz model that an allocation not accepted by traders is Pareto optimal?
6. Until when is it worth for a trader to block allocations accepted by other traders?
7. Why are budget lines of the first and of the second traders in static Arrow-Hurwicz model identical (coincide)?
8. What is the difference between Walras's law and the Walrasian equilibrium state in the static Arrow-Hurwicz model?
9. How an excess demand function in the static Arrow-Hurwicz model is defined and what are its properties?
10. What does it mean that the Walrasian equilibrium price vector is determined to accuracy of a structure (multiplication by a positive number)? What does it result from? What does it result in?
11. Why, in the static Arrow-Hurwicz model, is every allocation which is Pareto optimal and accepted by traders feasible with regard to an initial allocation? Use the Edgeworth box to explain this.
12. Why, in the static Arrow-Hurwicz model, is every Walrasian equilibrium allocation: Pareto optimal, accepted by traders and feasible with regard to an initial allocation? Use the Edgeworth box to explain this.
13. In what way is the dynamic Arrow-Hurwicz model defined in its discrete-time and continuous-time versions?
14. What is the role of a broker (Walrasian auctioneer) in the dynamic Arrow-Hurwicz model?
15. Why are only feasible price trajectories taken into consideration? How are they defined in both versions of the dynamic Arrow-Hurwicz model?
16. What does it mean that the Walrasian equilibrium price vector in the dynamic Arrow-Hurwicz model is asymptotically globally stable?
17. What is the difference between the state space and the phase space in the dynamic Arrow-Hurwicz model?

3.5 Exercises

E1. There is given a market of two traders and two goods described by the simple model of exchange, in which:

$i = 1, 2$—an index of consumer goods,
$k = 1, 2$—an index of traders (consumers),
$X^k = \mathbb{R}^2_+$—a goods space of k-th trader,
$u^k : \mathbb{R}^2_+ \to \mathbb{R}$—a utility function of k-th trader describing her/his preferences (relation of k-th consumer's preference),
$\mathbf{a}^k = (a_{k1}, a_{k2}) \in \mathbb{R}^2_+$—an initial bundle the k-th consumer comes to the market with,
$\mathbf{x}^k = (x_{k1}, x_{k2}) \in \mathbb{R}^2_+$—a consumption bundle the k-th consumer wants to purchase.

The k-th ($k = 1, 2$) consumer aims to purchase such a bundle of goods $\bar{\mathbf{x}}^k = (\bar{x}_{k1}, \bar{x}_{k2})$, whose utility would be maximum and at the same time not less than of the initial bundle of goods $\mathbf{a}^k = (a_{k1}, a_{k2})$.

Let us consider cases when:

(a) $u^k(\mathbf{x}^k) = a_{k1} \ln x_{k1} + a_{k2} \ln x_{k2} \to \max$,
(b) $u^k(\mathbf{x}^k) = a_{k1} x_{k1}^{\alpha} + a_{k2} x_{k2}^{\alpha} \to \max$.

Using the Edgeworth box:

1. present geometric illustration of sets of allocations: feasible with regard to an initial allocation, accepted by traders, Pareto optimal,
2. determine which allocations are blocked by traders,
3. justify that $C(\mathbf{a}) \subseteq S(\mathbf{a}) \subseteq F(\mathbf{a}) \subset \mathbb{R}^4_+$.

Let us assume that there are given:

- initial bundles both consumers come to the market with: $\mathbf{a}^1 = (10,\ 20)$, $\mathbf{a}^2 = (20,\ 10)$, which means that the initial allocation has a form: $\mathbf{a} = (10,\ 20, 20, 10)$,
- values of parameters in the logarithmic and in the subadditive utility functions: $a_{k1} = a_{k2} = 1$,
- values of parameters in the subadditive utility function: $\alpha = \frac{1}{4}$.

E2. There is given a market of two traders and two goods described by the static Arrow-Hurwicz model, in which:

$i = 1, 2$—an index of consumer goods,
$k = 1, 2$—an index of traders (consumers),
$X = \mathbb{R}^2_+$—a goods space,

$\mathbf{a}^1 = (10, 20)$, $\mathbf{a}^2 = (20, 10)$—initial bundles the consumers come to the market with,
$\mathbf{p} = (p_1, p_2) \in \operatorname{int} \mathbb{R}^2_+$—a vector of goods' prices,
$\mathbf{x}^k = (x_{k1}, x_{k2}) \in \mathbb{R}^2_+$—a consumption bundle the k-th consumer wants to purchase,
$I^1(p_1, p_2) = 10p_1 + 20p_2$, $I^2(p_1, p_2) = 20p_1 + 10p_2$—incomes of traders.

Utility functions of traders have forms:

(a) $u^1(x_{11}, x_{12}) = \ln x_{11} + \ln x_{12}$, $u^2(x_{21}, x_{22}) = \ln x_{21} + \ln x_{22}$—logarithmic,
(b) $u^1(x_{11}, x_{12}) = x_{11}^{\frac{1}{4}} + x_{11}^{\frac{1}{4}}$, $u^2(x_{21}, x_{22}) = x_{21}^{\frac{1}{4}} + x_{21}^{\frac{1}{4}}$—subadditive.

1. Find solution of the utility maximization problem of each trader—a demand function of each trader.
2. Determine functions of global supply and global demand.
3. Determine an excess demand function and justify that it is homogenous of degree 0 and that it satisfies Walras's law.
4. Determine a Walrasian equilibrium price vector.
5. Explain what it means that the Walrasian equilibrium price vector is determined with an accuracy of a structure (multiplication by a positive number).
6. Determine a Walrasian equilibrium allocation.
7. Present geometric illustrations of:
   – a set of allocations feasible with regard to the initial allocation,
   – a set of Pareto optimal allocations,
   – a set of Walrasian equilibrium allocations.
8. Justify by geometric means that: $W(\mathbf{a}) \subseteq C(\mathbf{a}) \subseteq S(\mathbf{a}) \subseteq F(\mathbf{a}) \subset \mathbb{R}^4_+$.

E3. Using the Edgeworth box for the static Arrow-Hurwicz model of a market with two traders and two goods present a geometric illustration of a case when there exists no Walrasian equilibrium price vector in this model.

E4. Define the dynamic Arrow-Hurwicz model in a discrete-time and in a continuous-time versions, taking as the basis the static Arrow-Hurwicz model from Exercise 2 with logarithmic and subadditive utility functions of traders.

E5. For the dynamic Arrow-Hurwicz model in a discrete-time version from Exercise 4 with logarithmic utility functions of traders and the Walrasian equilibrium price vector known from Exercise 2 determine a feasible price trajectory for a few subsequent periods taking some value of parameter $\sigma_i = \sigma > 0$. Present a geometric illustration of this trajectory in the state space and in the phase space.

E6. For the dynamic Arrow-Hurwicz model in a discrete-time version from Exercise 4 with subadditive utility functions of traders and the Walrasian equilibrium price vector known from Exercise 2 determine a feasible price trajectory for a few subsequent periods taking some value of parameter $\sigma_i = \sigma > 0$. Present a geometric illustration of this trajectory in the state space and in the phase space.

E7. Two traders come to a market with bundles of goods: $\mathbf{a}^1 = (10, 20)$, $\mathbf{a}^2 = (20, 10)$. Their utility functions are following:

(a) $u^1(x_{11}, x_{12}) = \ln x_{11} + \ln x_{12}, \quad u^2(x_{21}, x_{22}) = \ln x_{21} + \ln x_{22}$,
(b) $u^1(x_{11}, x_{12}) = x_{11}^{1/4} + x_{12}^{1/4}, \quad u^2(x_{21}, x_{22}) = x_{21}^{1/4} + x_{22}^{1/4}$,.

Consider a discrete-time version of dynamic discrete-time Arrow-Hurwicz model. A broker announces initial prices:

$$\mathbf{p}(0) = (2, 4).$$

Using a form of the excess demand function and a structure of the Walrasian equilibrium price vector found in Exercise 2 for the static Arrow-Hurwicz model:

1. Determine trajectories of a price vector satisfying a system of equations of the dynamic discrete-time Arrow-Hurwicz model, taking a proportionality coefficient σ equal to 0.25, 0.35 and 1.25. Calculate price ratios $\frac{p_2(t)}{p_1(t)}$ and compare them with the equilibrium price ratio $\frac{\bar{p}_2}{\bar{p}_1}$.
2. State which trajectories determined in point 1 are feasible.
3. State if and when (in which period) a structure of prices stabilizes around the equilibrium price structure and whether it reaches this structure in time horizon $T = 15$.
4. Present graphs of the price trajectories in the state space.
5. Present graphs of the price trajectories as functions of time.

E8. Consider a continuous-time version of the dynamic Arrow-Hurwicz model for the same data given as in Exercise 7.

1. Determine trajectories of a price vector satisfying a system of equations of the dynamic continuous-time Arrow-Hurwicz model taking a proportionality coefficient σ equal to 0.25, 0.35, 1.25 and determine whether these trajectories are feasible.
2. Determine if and when (at what moment) a structure of prices stabilizes around the equilibrium price structure.
3. Present graphs of price trajectories as functions of time.

4 Rationality of Choices Made by Individual Producers

In this chapter you will learn the following points:

- what the production processes are, when they are called technologically effective and how to describe them
- what the standard properties of a production function are
- what relationships between inputs of production factors and the output are described by characteristics of a production function
- what it means that a production function is positively homogenous of degree $\theta > 0$ and what characteristics of a production function are related to this property
- what is a set of assumptions to describe a firm acting in perfect competition and to describe a monopoly
- what the price of a product used by a firm acting in perfect competition is and what the price set by a monopoly is
- what it means that a firm decides on a long-term strategy or a short-term strategy of its activity
- on the basis of what criteria a producer chooses optimal inputs of production factors or the optimal supply and how he/she makes this choice
- how to formulate a profit maximization problem and a production cost minimization problem
- what the conditions guaranteeing a firm that it chooses optimal inputs or the optimal supply are
- how to justify that problems of maximization with regard to inputs of production factors and with regard to the output are equivalent
- how to define functions of demand for prediction factors, conditional demand for production factors and of product supply

K. Malaga and K. Sobczak, *Microeconomics*, Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-031-10554-8_4

– what the differences between problems of a given type in a long-term strategy and in a short-term strategy are and what the importance of constraints on resources of production factors is when optimal inputs or the optimal output are chosen.

In this chapter, we focus on production processes. Our attention is paid particularly to the description of technological and financial characteristics of production processes.

Production technology is identified with technologically feasible processes. We are interested most in technologically effective processes, described by a production function. One of the most important aspects of production processes to be considered is substitutability of production factors. That is why the subject of our considerations is CES production function[1] which has special cases when it comes to substitutability of production factors: linear function (perfect substitutability), power function and Koopmans-Leontief function (lack of substitutability).

Financial aspects of production processes refer to such economic terms as profit, selling revenue and total costs of production which are a sum of variable costs and fixed costs. Each of these financial categories is described by a function whose arguments are the output of production or inputs of production factors.

Our essential aim is to describe rational behaviour of an individual producer who acts in perfect competition or monopoly conditions and when resources of production factors owned by the producer are limited or unlimited.

An important distinguishing characteristic of considerations presented in this chapter is an assumption that a rationally behaving producer (maximizing profit and/or minimizing production costs) does not encounter any constraints related to the demand for a product he/she produces. This is a similar situation to the one described in Chap. 2 in which we took into consideration a supply constraint, however we conducted the analysis assuming that the supply constraint is not binding for an individual consumer.

In this chapter, we do not discard the fact that demand constraint is very important when it comes to the manufacturing activity. However, for the sake of simplicity and clarity of our analyses we assume explicitly that the demand constraint is not binding for a producer. This assumption will be gradually released in subsequent chapters.

[1] CES stands for constant elasticity of substitution. This kind of function was used in Chap. 2 to describe substitutability of consumer goods regarding the utility of a consumer derived from a consumption bundle.

4.1 Production Space and Production Function

If we consider production as a process of transforming a bundle of inputs (of production factors) into a bundle of outputs (of products) then we can describe any production process by means of production function as a mapping $f: \mathbb{R}_+^n \to \mathbb{R}_+^m$ that assigns at most one vector $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}_+^m$ of production outputs (of final products) to any vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ of inputs of production factors.

For the sake of simplicity, we assume that $n = 2,\ m = 1$. This way we limit ourselves to production processes in which one product is produced by using two production factors.

Definition 4.1 A **production process** is a vector $\mathbf{z} = (\mathbf{x}, y) \in \mathbb{R}_+^3$, consisting of a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}_+^2$ of inputs of production factors and of a variable $y \in \mathbb{R}_+$ which describes a quantity of a product that can be produced by a given vector of inputs of production factors.

Note 4.1 When by a given technology it is possible to obtain any (not necessarily maximum) quantity of a product using a given vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}_+^2$ of inputs of production factors then we relate such situations to technologically feasible productions processes.

Definition 4.2 A **production space** is a set $Z = \left\{z = (\mathbf{x}, y) \in \mathbb{R}_+^3 \,\middle|\, \mathbf{x} \in \mathbb{R}_+^2, y \in \mathbb{R}_+\right\} \subseteq \mathbb{R}_+^3$ of all technologically feasible production processes with a norm[2] $\|\mathbf{z}\| = \max\{|y|, |x_1|, |x_2|\} = \max\{y, x_1, x_2\}$ defined on this set.

When by a given technology it is possible to obtain maximum quantity $y \in \mathbb{R}_+$ of a product using a given vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}_+^2$ of inputs of production factors then we say that it is a **technologically feasible production process**.

Definition 4.3 A **production function**[3] is a mapping $f: \mathbb{R}_+^2 \to \mathbb{R}_+$ which assigns maximum quantity $y \in \mathbb{R}_+$ of a product that can be produced when using a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}_+^2$ of inputs of production factors.

[2] In Chap. 2, we use the term of a non-Euclidean metric which is treated there as a measure of distance (similarity) between two consumption bundles. Here we use the term of a norm as a measure of length of a vector describing technologically feasible production process. The production space is assumed to be a metric space with a non-Euclidean metric defined on this space: $\forall \mathbf{z}^1, \mathbf{z}^2 \in \mathbb{R}_+^3\ d\left(\mathbf{z}^1, \mathbf{z}^2\right) = \max\left\{\left|\mathbf{z}^1\right|, \left|\mathbf{z}^2\right|\right\} = \max\left\{\mathbf{z}^1, \mathbf{z}^2\right\}$ since production processes are described by non-negative vectors of output and inputs of production factors.

[3] It is a scalar production function. A vector production function describes technologically feasible production processes in which one obtains at most one vector $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}_+^m$ of maximum production outputs using a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ of inputs of production factors.

Let us take into account the following standard system of assumptions[4] about a scalar production function $f:\mathbb{R}^2_+ \to \mathbb{R}_+$ of two variables:

(F1) $f:\mathbb{R}^2_+ \to \mathbb{R}_+$ is continuous and twice differentiable in its domain.

(F2) $f(0,0) = 0$, which means that zero inputs of production factors give zero production output.

(F3) $\forall \mathbf{x^1}, \mathbf{x^2} \in \mathbb{R}^2_+, \quad \mathbf{x^1} \geq \mathbf{x^2} \wedge \mathbf{x^1} \neq \mathbf{x^2} \Rightarrow f\left(\mathbf{x^1}\right) > f\left(\mathbf{x^2}\right)$, which means that the production function is increasing. From (F1) it is twice differentiable, hence

$$\forall \mathbf{x} \in \mathbb{R}^2_+ \quad \frac{\partial f(x_1, x_2)}{\partial x_i} > 0, \quad i = 1, 2. \tag{4.1}$$

(F4) $f:\mathbb{R}^2_+ \to \mathbb{R}_+$ is concave:

$$\begin{aligned} &\forall \mathbf{x^1}, \mathbf{x^2} \in \mathbb{R}^2_+ \ \ \forall \alpha, \beta \geq 0 \ \ \alpha + \beta = 1 \\ &f(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2) \geq \alpha f(\mathrm{x}^1) + \beta f(\mathbf{x}^2), \end{aligned} \tag{4.2a}$$

or strictly concave:

$$\begin{aligned} &\forall \mathbf{x^1}, \mathbf{x^2} \in \mathbb{R}^2_+ \ \ \mathbf{x}^1 \neq \mathbf{x}^2, \ \ \forall \alpha, \beta > 0 \ \ \alpha + \beta = 1 \\ &f(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2) > \alpha f(\mathbf{x}^1) + \beta f(\mathbf{x}^2). \end{aligned} \tag{4.2b}$$

(F5) $f:\mathbb{R}^2_+ \to \mathbb{R}_+$ is positively homogeneous of degree $\theta > 0$:

$$\forall \lambda > 0 \ \ \forall \mathbf{x} \in \mathbb{R}^2_+ \quad f(\lambda \mathbf{x}) = f(\lambda x_1, \lambda x_2) = \lambda^\theta f(x_1, x_2) = \lambda^\theta f(\mathbf{x}), \tag{4.3}$$

where $\theta > 0$ means a degree of homogeneity of a production function.

Note 4.2 Let us notice that:

(1) If $\lambda \in (0; 1)$, then $\lambda \mathbf{x}$ means a vector of inputs in which quantities of all production factors are **decreased** to the same extent in comparison to a vector $\mathbf{x}$.
(2) If $\lambda > 1$, then $\lambda \mathbf{x}$ means a vector of inputs in which quantities of all production factors are **increased** to the same extent in comparison to a vector $\mathbf{x}$.

Assumption (F5) refers to terms: constant, decreasing and increasing returns to scale.

Let us consider three cases:

[4] The standard assumptions mean that we treat them as basic ones. Which assumptions are seen as standard depends on which properties of production function we need in further analyses.

1. If $\theta = 1, \forall \lambda > 0 \;\; \forall \mathbf{x} \in \mathbb{R}^2_+ \;\; f(\lambda \mathbf{x}) = f(\lambda x_1, \lambda x_2) = \lambda f(x_1, x_2) = \lambda f(\mathbf{x})$, then this is a case of **constant returns**, called also **proportional revenues**. It means that when inputs of all production factors are increased/decreased λ times then the output level increases/decreases proportionally, thus also λ times.
2. If $\theta \in (0; 1), \forall \lambda > 1 \;\; \forall \mathbf{x} \in \mathbb{R}^2_+ \;\; f(\lambda \mathbf{x}) = \lambda^\theta f(\mathbf{x}) < \lambda f(\mathbf{x})$, then this is a case of **decreasing returns to scale**, called also **decreasing revenues**. It means that when inputs of all production factors are increased/decreased λ times then the output level increases/decreases less than proportionally, thus less than λ times.
3. If $\theta > 1, \;\; \forall \lambda > 1 \;\; \forall \mathbf{x} \in \mathbb{R}^2_+ \;\; f(\lambda \mathbf{x}) = \lambda^\theta f(\mathbf{x}) > \lambda f(\mathbf{x})$, then this is a case of **increasing returns to scale**, called also **increasing revenues**. It means that when inputs of all production factors are increased/decreased λ times then the output level increases/decreases more than proportionally, thus more than λ times.

Note 4.3 Assumptions (F4) and (F5) are not contradictory with each other when the homogeneity degree of a production function $\theta \in (0; 1]$, since if $\theta > 1$ then a production function is neither concave nor strictly concave.

Let us now present some characteristics of a production function $f: \mathbb{R}^2_+ \to \mathbb{R}_+$ which have an important application in the theory of production.

Definition 4.4 A **growth speed of production** with respect to i-th production factor is

$$T_i(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_i}, \quad i = 1, 2, \tag{4.4}$$

which describes by approximately how many units the output level $\mathbf{x}$ changes (increases, decreases or remains unchanged) when input of i-th production factor increases by one (notional) unit and the input of the other factor does not change.

Note 4.4 In the theory of production, the growth speed of production with respect to i-th production factor is called usually a **marginal productivity** (or **marginal effectiveness**) of i-th production factor.

Definition 4.5 A **growth rate of production** with respect to i-th production factor is

$$S_i(\mathbf{x}) = \frac{\partial f(x_1, x_2)}{\partial x_i} \frac{1}{f(\mathbf{x})} = \frac{T_i(\mathbf{x})}{f(\mathbf{x})}, \tag{4.5}$$

which describes by approximately what % the output level $f(\mathbf{x})$ changes (increases, decreases or remains unchanged) when the input of i-th production factor increases by one (notional) unit and the input of the other factor does not change.

Definition 4.6 An **elasticity of production with respect to i-th production factor** is

$$E_i(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_i}\frac{x_i}{f(\mathbf{x})} = S_i(\mathbf{x}) \cdot x_i, \tag{4.6}$$

which describes by approximately what % the output level $f(\mathbf{x})$ changes (increases, decreases or remains unchanged) when the input of i-th production factor increases by 1% and the input of the other factor does not change.

Definition 4.7 An **elasticity of production with respect to scale of inputs** is

$$\begin{aligned} E_\lambda(x_1, x_2) &= \lim_{\Delta\lambda \to 0} \frac{\frac{f((\lambda+\Delta\lambda)\mathbf{x}) - f(\mathbf{x})}{f(\mathbf{x})}}{\frac{\Delta\lambda}{\lambda}} \\ &= \lim_{\lambda \to 1} \lim_{\Delta\lambda \to 0} \frac{\frac{f((\lambda+\Delta\lambda)\mathbf{x}) - f(\lambda\mathbf{x})}{f(\mathbf{x})}}{\frac{\Delta\lambda}{\lambda}} \\ &= \lim_{\lambda \to 1} \lim_{\Delta\lambda \to 0} \left(\frac{f((\lambda+\Delta\lambda)\mathbf{x}) - f(\lambda\mathbf{x})}{\Delta\lambda} \cdot \frac{\lambda}{f(\lambda\mathbf{x})} \right) \\ &= \lim_{\lambda \to 1} \left(\frac{\partial f(\lambda\mathbf{x})}{\partial\lambda} \frac{\lambda}{f(\lambda\mathbf{x})} \right), \end{aligned} \tag{4.7}$$

which describes by approximately what % the output level $f(\mathbf{x})$ changes (increases, decreases or remains unchanged) when the input of each production factor increases by 1%.

Theorem 4.1 If a production function $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ is positively homogeneous of degree $\theta > 0$ then:

$$E_\lambda(x_1, x_2) = \theta, \tag{4.8}$$

which means that an elasticity of production with respect to scale of inputs is equal to a degree of homogeneity of a production function.

Let us notice that if a production function is positively homogenous of degree $\theta > 0$ then it satisfies a condition:

$$\forall \lambda > 0 \ \ \forall \mathbf{x} \in \mathbb{R}^2_+, \quad f(\lambda\mathbf{x}) = \lambda^\theta f(\mathbf{x}) \tag{4.9}$$

resulting in:

$$\frac{\partial f(\lambda\mathbf{x})}{\partial\lambda} = \theta\lambda^{\theta-1} f(\mathbf{x}). \tag{4.10}$$

Substituting (4.10) into a definition in Eq. (4.7), we get

$$E_\lambda(x_1, x_2) = \lim_{\lambda \to 1} \left(\frac{\partial f(\lambda \mathbf{x})}{\partial \lambda} \frac{\lambda}{f(\lambda \mathbf{x})} \right)$$

$$(4.11) \qquad = \lim_{\lambda \to 1} \left(\theta \lambda^{\theta - 1} f(\mathbf{x}) \frac{\lambda}{\lambda^\theta f(\mathbf{x})} \right) = \lim_{\lambda \to 1} \theta = \theta.$$

4.2 Substitutability and Complementarity of Production Factors

Let us use our knowledge of the topic of substitutability of consumer goods to discuss the substitutability of production factors.[5]

Definition 4.8 Production isoquant is a set:

$$(4.12) \qquad G = \left\{ \mathbf{x} \in \mathbb{R}^2_+ \,|\, f(\mathbf{x}) = y = \text{const.} \geq 0 \right\} \subset \mathbb{R}^2_+,$$

consisting of all vectors of production factors' inputs by which the output can be produced at a fixed level $y = \text{const.} \geq 0$.

Definition 4.9 A **marginal rate of substitution of the first production factor by the second production factor** in a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ of inputs by which the output can be produced at level $y_0 = \text{const.} > 0$ is an expression:

$$(4.13) \qquad \sigma_{12}(x_1, x_2) = -\lim_{\substack{\Delta x_1 \to 0 \\ \Delta x_1 < 0}} \frac{\Delta x_2}{\Delta x_1} \cong -\frac{\mathrm{d}x_2}{\mathrm{d}x_1},$$

which describes by approximately how many units one should raise the quantity of the second production factor in a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ of inputs when input of the first production factor has been reduced by one (notional) unit, in order to keep the output level $y_0 = \text{const.} > 0$ unchanged.

Note 4.5 The minus sign in Definition 4.9 results from the fact that the input of the first production factor has been reduced.

A total differential of the differentiable production function by a fixed output level $f(x_1, x_2) = y$ is of a form:

[5] The issue of complementarity of production factors is of minor significance in our further considerations. Hence, we focus on substitutability of production factors.

$$\text{(4.14)} \qquad \mathrm{d}y = \frac{\partial f(x_1, x_2)}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f(x_1, x_2)}{\partial x_2}\mathrm{d}x_2.$$

Since we are interested in vectors of inputs by which the output level is unchanged, then:

$$\mathrm{d}y = 0 \Leftrightarrow \frac{\partial f(x_1, x_2)}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f(x_1, x_2)}{\partial x_2}\mathrm{d}x_2 = 0$$

$$\text{(4.15)} \qquad \Leftrightarrow -\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{\frac{\partial f(x_1,x_2)}{\partial x_1}}{\frac{\partial f(x_1,x_2)}{\partial x_2}} = \sigma_{12}(x_1, x_2).$$

Note 4.6 The marginal rate of substitution of the first production factor by the second production factor is equal to the ratio of marginal productivity of the first production factor and the marginal productivity of the second production factor.

Note 4.7 The marginal rate of substitution of the second production factor by the first production factor is equal to an inverse of the marginal rate of substitution of the first production factor by the second production factor which is given as

$$-\frac{\mathrm{d}x_1}{\mathrm{d}x_2} = \frac{\frac{\partial f(x_1,x_2)}{\partial x_2}}{\frac{\partial f(x_1,x_2)}{\partial x_1}} = \frac{1}{\sigma_{12}(x_1, x_2)} = \sigma_{21}(x_1, x_2).$$

Definition 4.10 An **elasticity of substitution of the first production factor by the second production factor** in a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ of inputs by which the output can be produced at level $y_0 = \text{const.} > 0$ is an expression:

$$\text{(4.16)} \qquad \varepsilon_{12}(x_1, x_2) = -\lim_{\substack{\Delta x_1 \to 0 \\ \Delta x_1 < 0}} \frac{\frac{\Delta x_2}{x_2}}{\frac{\Delta x_1}{x_1}} = -\lim_{\substack{\Delta x_1 \to 0 \\ \Delta x_1 < 0}} \frac{\Delta x_2}{\Delta x_1} \cdot \frac{x_1}{x_2} \cong -\frac{\mathrm{d}x_2}{\mathrm{d}x_1} \cdot \frac{x_1}{x_2},$$

which describes by approximately what % one should raise the quantity of the second production factor in a vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ of inputs when input of the first production factor has been reduced by 1%, in order to keep the output level $y_0 = \text{const.} > 0$ unchanged.

Note 4.8 The Koopmans-Leontief production function (presented in Table 4.1) is not differentiable. Thus on the basis of Definition 4.4 it is not possible to determine the value of a marginal productivity. Hence, it is not possible to determine the value of a marginal rate of substitution[6] of i-th production factor by j-th production factor using Definition 4.9.

[6] In the case of the Koopmans-Leontief production function, it can be noticed that substitution of one production factor by the other is not possible (when reducing input of one production factor

Table 4.1 Exemplary production functions and their selected characteristics

Type of a production function	Marginal productivity of ***i***-th production factor	Marginal rate of substitution
Linear $f(x_1, x_2) = a_1 x_1 + a_2 x_2$ $a_i > 0,\ i = 1, 2$	$\frac{\partial f(x_1,x_2)}{\partial x_1} = a_1$ $\frac{\partial f(x_1,x_2)}{\partial x_2} = a_2$	$\sigma_{12}(x_1, x_2) = \frac{a_1}{a_2}$ $\sigma_{21}(x_1, x_2) = \frac{a_2}{a_1}$
Power function $f(x_1, x_2) = a x_1^{\alpha_1} x_2^{\alpha_2}$ $a, \alpha_i > 0,\ i = 1, 2$	$\frac{\partial f(x_1,x_2)}{\partial x_1} = \alpha_1 a x_1^{\alpha_1 - 1} x_2^{\alpha_2}$ $\frac{\partial f(x_1,x_2)}{\partial x_2} = \alpha_2 a x_1^{\alpha_1} x_2^{\alpha_2 - 1}$	$\sigma_{12}(x_1, x_2) = \frac{\alpha_1 x_2}{\alpha_2 x_1}$ $\sigma_{12}(x_1, x_2) = \frac{\alpha_1 x_1}{\alpha_2 x_2}$
Cobb-Douglas function $f(x_1, x_2) = a x_1^{\alpha_1} x_2^{\alpha_2}$ $a, \alpha_1, \alpha_2 > 0$ $\alpha_1 + \alpha_2 = 1$	$\frac{\partial f(x_1,x_2)}{\partial x_1} = \alpha_1 a x_1^{\alpha_1 - 1} x_2^{\alpha_2}$ $\frac{\partial f(x_1,x_2)}{\partial x_2} = \alpha_2 a x_1^{\alpha_1} x_2^{\alpha_2 - 1}$	$\sigma_{12}(x_1, x_2) = \frac{\alpha_1 x_2}{\alpha_2 x_1}$ $\sigma_{21}(x_1, x_2) = \frac{\alpha_2 x_1}{\alpha_1 x_2}$
Subadditive $f(x_1, x_2) = a_1 x_1^{\alpha} + a_2 x_2^{\alpha}$ $a_i, \alpha > 0, i = 1, 2$	$\frac{\partial f(x_1,x_2)}{\partial x_1} = \alpha a_1 x_1^{\alpha - 1}$ $\frac{\partial f(x_1,x_2)}{\partial x_2} = \alpha a_2 x_2^{\alpha - 1}$	$\sigma_{12}(x_1, x_2) = \frac{a_1}{a_2}\left(\frac{x_1}{x_2}\right)^{\alpha - 1}$ $\sigma_{21}(x_1, x_2) = \frac{a_2}{a_1}\left(\frac{x_2}{x_1}\right)^{\alpha - 1}$
CES (constant elasticity of substitution) function $f(x_1, x_2) = \left(a_1 x_1^{\gamma} + a_2 x_2^{\gamma}\right)^{\frac{\theta}{\gamma}}$ $\theta, a_i > 0,\ i = 1, 2$ $\gamma \in (-\infty; 0) \cup (0; 1)$	$\frac{\partial f(x_1,x_2)}{\partial x_1} = \frac{\theta}{\gamma}\left(a_1 x_1^{\gamma} + a_2 x_2^{\gamma}\right)^{\frac{\theta}{\gamma} - 1} \gamma a_1 x_1^{\gamma - 1}$ $\frac{\partial f(x_1,x_2)}{\partial x_2} = \frac{\theta}{\gamma}\left(a_1 x_1^{\gamma} + a_2 x_2^{\gamma}\right)^{\frac{\theta}{\gamma} - 1} \gamma a_2 x_2^{\gamma - 1}$	$\sigma_{12}(x_1, x_2) = \frac{a_1}{a_2}\left(\frac{x_1}{x_2}\right)^{\gamma - 1}$ $\sigma_{21}(x_1, x_2) = \frac{a_2}{a_1}\left(\frac{x_2}{x_1}\right)^{\gamma - 1}$

Note 4.9 The Cobb-Douglas production function (presented in Table 4.1) is a special case of power production function[7] with a homogeneity degree $\theta = \alpha_1 + \alpha_2 = 1$.

Note 4.10 All the characteristics of a production function $f: \mathbb{R}_+^2 \to \mathbb{R}_+$ are scalar and two-variable functions of inputs of production factors

Note 4.11 Let us consider three limit cases of marginal rates of substitution of production factors when a production function is a CES function.

cannot be compensated by increasing input of the other factor) or needless (when reducing input of one production factor does not change a given initial output level). Here, similar to complementarity of consumer goods discussed in Chap. 2, we talk about complementarity of production factors.

[7] Identifying the Cobb-Douglas function with a power production function in its special case when it is positively homogenous of degree 1 matters when we use a concept of a neoclassical production function. One of the properties of such a function is the positive homogeneity of first degree, which shows that returns to scale are constant (revenues are proportional). This way the Cobb-Douglas production function is the only one example of power function being the neoclassical production function.

Case 1

$$\lim_{\gamma\to 1^-} \sigma_{12}(x_1, x_2) = \lim_{\gamma\to 1^-} \left(\frac{a_1}{a_2} \left(\frac{x_1}{x_2} \right)^{\gamma-1} \right) = \frac{a_1}{a_2} \tag{4.17}$$

and

$$\lim_{\gamma\to 1^-} \sigma_{21}(x_1, x_2) = \lim_{\gamma\to 1^-} \left(\frac{a_2}{a_1} \left(\frac{x_2}{x_1} \right)^{\gamma-1} \right) = \frac{a_2}{a_1}, \tag{4.18}$$

which means that when $\gamma \to 1^-$ then the CES production function is convergent to a linear production function, thus a function which describes perfect substitute and not complementary production factors.

Case 2

$$\lim_{\gamma\to 0} \sigma_{12}(x_1, x_2) = \lim_{\gamma\to 0} \left(\frac{a_1}{a_2} \left(\frac{x_1}{x_2} \right)^{\gamma-1} \right) = \frac{\alpha_1 x_2}{\alpha_2 x_1} \tag{4.19}$$

and

$$\lim_{\gamma\to 0} \sigma_{21}(x_1, x_2) = \lim_{\gamma\to 0} \left(\frac{a_2}{a_1} \left(\frac{x_2}{x_1} \right)^{\gamma-1} \right) = \frac{\alpha_2 x_1}{\alpha_1 x_2}, \tag{4.20}$$

which means that when $\gamma \to 0$ then the CES production function is convergent to a power production function.

Case 3

$$\lim_{\gamma\to -\infty} \sigma_{12}(x_1, x_2) = \lim_{\gamma\to -\infty} \left(\frac{a_1}{a_2} \left(\frac{x_1}{x_2} \right)^{\gamma-1} \right) = \begin{cases} 0 & \text{if } x_1 > x_2 \\ +\infty & \text{if } x_1 < x_2 \end{cases} \tag{4.21}$$

and

$$\lim_{\gamma\to -\infty} \sigma_{21}(x_1, x_2) = \lim_{\gamma\to -\infty} \left(\frac{a_2}{a_1} \left(\frac{x_2}{x_1} \right)^{\gamma-1} \right) = \begin{cases} 0 & if\ x_1 < x_2 \\ +\infty & if\ x_1 > x_2 \end{cases} \tag{4.22}$$

which means that when $\gamma \to -\infty$ then the CES production function is convergent to the Koopmans-Leontief production function, thus a function which describes perfect complementary and not substitute production factors.

Note 4.12 Production functions: linear, power (including Cobb-Douglas one), Koopmans-Leontief one are limit (special) cases of a CES production function with respect to values of a parameter $\gamma \in (-\infty; 0) \cup (0; 1)$.

Note 4.13 A linear production function describes production factors which are perfect substitutes and not complements to each other. A power production function describes production factors which are substitute and complementary. While the Koopmans-Leontief production function describes production factors which are perfect complements and not substitutes for each other.

4.3 Financial and Technological Aspects of a Firm's Activity

So far we have focused on technical issues of production processes. For economists, the financial aspects are at least as important. Thus, let us define selected financial characteristics of production processes. As before, we consider technologically effective production processes described by scalar and two-variable production functions $f: \mathbb{R}^2_+ \to \mathbb{R}_+$.

Let us denote:

$p \in \operatorname{int} \mathbb{R}_+$—a price of a product manufactured by a firm,
$\mathbf{c} = (c_1, c_2) \in \operatorname{int} \mathbb{R}^2_+$—a vector of prices of production factors,
$\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$—a vector of inputs of production factors,
$y = f(x_1, x_2)$—an output level (quantity of a manufactured product).

Definition 4.11 Revenue (turnover) from sales of a manufactured product as a function of output level is an expression:

$$r(y) = py. \tag{4.23}$$

Definition 4.12 Revenue (turnover) from sales of a manufactured product as a function of inputs of production factors is an expression:

$$r(x_1, x_2) = pf(x_1, x_2). \tag{4.24}$$

Definition 4.13 Total cost of production as a function of output level is an expression:

$$c^{tot}(y) = c^v(y) + c^f(y), \tag{4.25}$$

where:

$c^v(y)$—variable cost of production, dependent on production level,

$c^f(y) = d = \text{const.} \geq 0$—fixed cost of production, independent on production level.

According to needs of use, we describe the variable costs of production by various elementary functions. Mostly we use the following functions:

(1) linear function of cost:

$$c^v(y) = ay, \ a > 0, \tag{4.26}$$

(2) polynomial of degree 2:

$$c^v(y) = ay^2 + by, \ a, b > 0, \tag{4.27}$$

(3) polynomial of degree 3:

$$c^v(y) = ay^3 + by^2 + cy, \ a, b, c > 0. \tag{4.28}$$

If we add a free term d to variable costs, then resultant cost function can be seen as a function of the total cost of production. General approach when deciding about an analytical form of the function of total costs of production is to state if we need it to be convex, strictly convex or at least convex/strictly convex in intervals.

Definition 4.14 Total cost of production as a function of inputs of production factors is an expression:

$$c^{tot}(x_1, x_2) = c^v(x_1, x_2) + c^f(x_1, x_2), \tag{4.29}$$

where:

$c^v(x_1, x_2)$—variable cost of production, dependent on production level,
$c^f(x_1, x_2) = d = \text{const.} \geq 0$—fixed cost of production, independent on production level.

Let us remember that we are interested in technologically effective production processes described by production functions of a form $y = f(x_1, x_2)$. Then, substituting it into Eq. (4.25), we get Eq. (4.29). Similarly, substituting the production function into conditions (4.26)–(4.28), we get analytical forms of functions of variable costs depending on inputs of production factors.

Definition 4.15 Profit of a producer as a function of output level is an expression:

$$\pi(y) = r(y) - c^{tot}(y) = py - \left(c^v(y) + c^f(y)\right), \tag{4.30}$$

which is a difference between revenue from sales of a manufactured product and total cost of production, both being functions of output level y.

Definition 4.16 Profit of a producer as a function of inputs of production factors is an expression:

$$\begin{aligned} \pi(x_1, x_2) &= r(x_1, x_2) - c^{tot}(x_1, x_2) \\ &= pf(x_1, x_2) - \left(c^v(x_1, x_2) + c^f(x_1, x_2)\right), \end{aligned} \tag{4.31}$$

which is a difference between revenue from sales of a manufactured product and total cost of production, both being functions of inputs (x_1, x_2) of production factors.

Functions of: production, profit and costs are assumed to be differentiable. Hence, one can calculate their derivatives and give an economic interpretation to the values of these derivatives.

Definition 4.17 Marginal revenue (turnover) with respect to output level is an expression:

$$\frac{\mathrm{d}r(y)}{\mathrm{d}y}, \tag{4.32}$$

which describes by approximately how many money units the revenue from sales of a manufactured product changes when the output level increases by one (notional) unit.

Definition 4.18 Marginal revenue (turnover) with respect to input of i-th production factor is an expression:

$$\frac{\partial r(x_1, x_2)}{\partial x_i} \quad i = 1, 2, \tag{4.33}$$

which describes by approximately how many money units the revenue from sales of a manufactured product changes when the input of i-th production factor increases by one (notional) unit and the input of the other production factor does not change.

Definition 4.19 Marginal total cost of production with respect to output level is an expression:

$$\frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y} = \frac{\mathrm{d}c^{v}(y)}{\mathrm{d}y} + \frac{\mathrm{d}c^{f}(y)}{\mathrm{d}y} = \frac{\mathrm{d}c^{v}(y)}{\mathrm{d}y}, \tag{4.34}$$

which describes by approximately how many money units the total cost changes when the output level increases by one (notional) unit.

Definition 4.20 Marginal total cost of production with respect to input of i-th production factor is an expression:

$$\frac{\partial c^{tot}(x_1, x_2)}{\partial x_i} = \frac{\partial c^{v}(x_1, x_2)}{\partial x_i} + \frac{\partial c^{f}(x_1, x_2)}{\partial x_i} = \frac{\partial c^{v}(x_1, x_2)}{\partial x_i}, \quad i = 1, 2, \tag{4.35}$$

which describes by approximately how many money units the total cost changes when the input of i-th production factor increases by one (notional) unit and the input of the other production factor does not change.

Note 4.14 Definitions 4.13, 4.19, 4.20 show that the marginal cost of production is always equal to marginal variable cost of production, regardless of the fact whether it is a function with of one variable or two variables.[8]

Definition 4.21 Marginal profit of a producer with respect to output level is an expression:

$$\frac{\mathrm{d}\pi(y)}{\mathrm{d}y} = \frac{\mathrm{d}r(y)}{\mathrm{d}y} - \frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y} = \frac{dr(y)}{\mathrm{d}y} - \frac{dc^{v}(y)}{dy}, \tag{4.36}$$

which describes by approximately how many money units the profit of a producer changes when the output level increases by one (notional) unit.

Definition 4.22 Marginal profit of a producer with respect to input of i-th production factor is an expression:

$$\begin{aligned} \frac{\partial \pi(x_1, x_2)}{\partial x_i} &= \frac{\partial r(x_1, x_2)}{\partial x_i} - \frac{\partial c^{tot}(x_1, x_2)}{\partial x_i} \\ &= \frac{\partial r(x_1, x_2)}{\partial x_i} - \frac{\partial c^{v}(x_1, x_2)}{\partial x_i}, \quad i = 1,\ 2, \end{aligned} \tag{4.37}$$

[8] Thus, in the whole textbook we use the name "marginal cost of production", omitting the word "variable".

which describes by approximately how many money units the profit of a producer changes when the input of i-th production factor increases by one (notional) unit and the input of the other production factor does not change.

Note 4.15 Definitions 4.21 and 4.22 show that the marginal profit of a producer is equal to a difference between the marginal revenue and marginal cost of production. These functions are functions of output level and then scalar and one-variable functions. Or they are functions of inputs of production factors and then scalar and two-variable functions.

Having defined basic terms related to technical and financial aspects of production processes we can now proceed to a description of rational behaviour of an individual producer which we identify with a firm.

Let us take the following set of assumptions determining framework of rational activity of a firm or rational behaviour of a producer.

(F1) A firm produces one product using two production factors.

(F2) Production processes are described by production functions $f:\mathbb{R}^2_+ \to \mathbb{R}_+$ that are assumed to be increasing, strictly concave, twice differentiable and having value 0 for zero inputs of production factors.

(F3) The price of a product of a firm and prices of production factors are determined by

(a) a market—then the firm has no impact on the prices and treats them as parameters
or
(b) firms—then the price of a product is described by a decreasing function of its supply and prices of production factors are described by increasing functions of demand reported for them:

$$p(y) > 0, \ \frac{\mathrm{d}p(y)}{\mathrm{d}y} < 0, \tag{4.38}$$

$$\frac{\mathrm{d}c_i(x_i)}{\mathrm{d}x_i} > 0, \ i = 1, 2. \tag{4.39}$$

(F4) Resources of production factors are

(a) unlimited:

$$0 \le x_i, \ i = 1, 2, \tag{4.40}$$

or

(b) limited:

(4.41) $$0 \le x_i \le b_i,\ i = 1, 2,$$

where $b_i > 0$ means the resource of i-th production factor owned by a given firm.

(F5) The demand for a product is big enough to sell any quantity of the manufactured good.[9]

(F6) A firm in its activity is driven by the criterion of

(a) profit maximization
or
(b) minimization of cost of manufacturing a product in specified quantity.

If we take a set of assumptions: (F1), (F2), (F3a), (F4a)/(F4b), (F5), (F6) it means that we consider a firm that acts in perfect competition and decides on a long-term /short-term strategy.

The perfect competition occurs when firms, acting as individuals, have no impact on the price of a product they manufacture nor on the prices of production factors they use in their production processes.

Distinction between long-term and short-term strategies of a firm's activity relate to resources of production factors they own. It is commonly assumed that in long term a firm can obtain any quantities of production factors. Hence, lack of constraints in resources of production factors is identified with the firm activity in the long term. While in the short-term resources of production factors are assumed to be constrained.

This way of reasoning (distinction between long term and short term of the firm's activity) is not completely correct form methodological point of view. It would be if time, as continuous or discrete variable, was present explicitly among variables describing the firm's activity.

If we take a set of assumptions: (F1), (F2), (F3b), (F4a)/(F4b), (F5), (F6) it means that we consider a firm that acts as a monopoly and decides on a long-term /short-term strategy.

Let us now analyse in detail each of these four models describing the activity of a firm.

[9] In further chapters of the textbook, we use a demand function according to which a market has its capacity and there is some upper limit of demand even if the product is offered for free.

4.4 Firm Acting in Perfect Competition—Long-Term Strategy

4.4.1 Static Approach

Let us use the following notation:

$p > 0$—a price of a product manufactured by a firm,
$\mathbf{c} = (c_1, c_2) > (0, 0)$—a vector of prices of production factors,
$\mathbf{x} = (x_1, x_2) \geq (0, 0)$—a vector of inputs of production factors,
$y = f(x_1, x_2)$—an output level,
$r(y) = py$—revenue (turnover) from sales of a manufactured product as a function of output level,
$r(x_1, x_2) = pf(x_1, x_2)$—revenue (turnover) from sales of a manufactured product as a function of inputs of production factors,
$c^{tot}(x_1, x_2) = c_1x_1 + c_2x_2 + d$—total cost of production,
$c^v(x_1, x_2) = c_1x_1 + c_2x_2$—variable cost of production,
$c^f(x_1, x_2) = d$—fixed cost of production,
$c(y)$—minimum cost of producing y output units, derived as an objective function corresponding to an optimal solution to problem (P2c),
$\pi(y) = r(y) - c(y) = py - c(y)$—firm's profit as a function of output level,
$\pi(x_1, x_2) = r(x_1, x_2) - c^{tot}(x_1, x_2)$—firm's profit as a function of inputs of production factors.

Problem of profit maximization with regard to inputs of production factors (P1c)

The aim of a firm is to maximize its profit expressed as a function of inputs of production factors, which can be written as a problem to solve in the following way:

$$\begin{aligned} \pi(x_1, x_2) &= r(x_1, x_2) - c^{tot}(x_1, x_2) \\ &= \{pf(x_1, x_2) - (c_1x_1 + c_2x_2 + d)\} \rightarrow \max \end{aligned} \tag{4.42}$$

$$x_1, x_2 \geq 0. \tag{4.43}$$

Since a production function from assumption (F2) is strictly concave while a production total cost is linear, then a profit function is strictly concave. Moreover, we are interested in an optimal solution $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2) > (0, 0)$.

Necessary and sufficient conditions for the existence of an optimal solution to problem (P1c) are given in the following theorem.

Theorem 4.2 If a firm's profit function is strictly concave and satisfies the following condition:

$$\lim_{x_i \to 0^+} \frac{\partial \pi(x_1, x_2)}{\partial x_i} > 0 \quad \wedge \lim_{x_i \to +\infty} \frac{\partial \pi(x_1, x_2)}{\partial x_i} < 0$$

(4.44) $$\Leftrightarrow \lim_{x_i \to +\infty} \frac{\partial f(x_1, x_2)}{\partial x_i} < c_i < \lim_{x_i \to 0^+} \frac{\partial f(x_1, x_2)}{\partial x_i}, \quad i = 1, 2$$

then:

(1) $\exists_1 \overline{\mathbf{x}} > 0$ such that $\left.\frac{\partial \pi(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = 0 \quad i = 1, 2,$

(2) a necessary and sufficient condition for $\overline{\mathbf{x}} > 0$ being an optimal solution to problem (P1c) is

(4.45) $$\left.\frac{\partial \pi(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = 0 \quad \Leftrightarrow \left.\frac{\partial r(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = \left.\frac{\partial c^v(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}}$$
$$\Leftrightarrow p \left.\frac{\partial f(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = c_i \quad i = 1, 2$$

which means that there exists exactly one solution $\overline{\mathbf{x}} > 0$ for which:

- marginal profit equals zero,
- marginal revenue is equal to marginal cost of production,
- i-th production factor's productivity expressed in money units is equal to the price of this production factor.

Definition 4.23 A **function of demand for production factors** is a mapping $\psi\colon \text{int } \mathbb{R}^3_+ \to \text{int } \mathbb{R}^2_+$ which assigns an optimal solution of problem (P1c) to any price p of a product and any prices $\mathbf{c} = (c_1, c_2)$ of production factors in the following way:

(4.46) $$\psi(p, \mathbf{c}) = (\psi_1(p, \mathbf{c}), \psi_2(p, \mathbf{c})) = \overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2).$$

Definition 4.24 A **firm's maximal profit function** is a mapping $\Pi\colon \text{int } \mathbb{R}^4_+ \to \text{int } \mathbb{R}_+$ which assigns maximum profit to any price p of a product, any prices $\mathbf{c} = (c_1, c_2)$ of production factors and any fixed cost d in the following way:

(4.47) $$\Pi(p, \mathbf{c}, d) = \pi(\overline{\mathbf{x}}).$$

Theorem 4.3 If assumptions of Theorem 4.2 are satisfied then:

(1) $\forall \lambda > 0 \quad \psi(\lambda p, \lambda \mathbf{c}) = \psi(p, \mathbf{c}),$
which means that a function of demand for production factors is positively homogenous of degree 0,

(2) $\forall \lambda > 0 \quad \Pi(\lambda p, \lambda \mathbf{c}, \lambda d) = \lambda \Pi(p, \mathbf{c}, d),$
which means that a firm's maximal profit function is positively homogenous of degree 1 with respect to the price of a product, prices of production factors and the fixed cost of production.

Problem of cost minimization when producing the output at a fixed level (P2c)

The aim of a firm is to produce $y > 0$ units of output at minimum total cost, which can be written as a problem to solve in the following way:

$$c^{tot}(x_1, x_2) = \{c_1x_1 + c_2x_2 + d\} \to \min \tag{4.48}$$

$$f(x_1, x_2) = y = \text{const.} > 0, \tag{4.49}$$

$$x_1, x_2 \geq 0. \tag{4.50}$$

One can express problem (P2c) using Lagrange function:

$$F(x_1, x_2, \lambda) = \{c_1x_1 + c_2x_2 + d + \lambda(y - f(x_1, x_2))\} \to \min. \tag{4.51}$$

Theorem 4.4 If a production function satisfies assumption (F2) then $\widetilde{\mathbf{x}} > 0$ is an optimal soultion to problem (P2c) if and only if a pair $\left(\widetilde{\mathbf{x}}, \widetilde{\lambda}\right) > \mathbf{0}$ is a solution to the following system of equations:

$$\begin{aligned} \left.\frac{\partial F\left(\mathbf{x}, \widetilde{\lambda}\right)}{\partial x_i}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} = 0 \quad &\Leftrightarrow \quad \widetilde{\lambda}\left.\frac{\partial f(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} = c_i, \quad i = 1, 2, \\ \left.\frac{\partial F(\widetilde{\mathbf{x}}, \lambda)}{\partial \lambda}\right|_{\lambda=\widetilde{\lambda}} = 0 \quad &\Leftrightarrow \quad f(\tilde{x}_1, \tilde{x}_2) = y. \end{aligned} \tag{4.52}$$

Necessary condition: if $\widetilde{\mathbf{x}} > 0$ is an optimal solution to problem (P2c) then a pair $\left(\widetilde{\mathbf{x}}, \widetilde{\lambda}\right) > \mathbf{0}$ is a solution to equation system (4.52).

Sufficient condition: if a pair $\left(\widetilde{\mathbf{x}}, \widetilde{\lambda}\right) > \mathbf{0}$ is a solution to equation system (3.52) then $\widetilde{\mathbf{x}} > 0$ is an optimal solution to problem (P2c).

Definition 4.25 A **function of conditional demand for production factors** is a mapping $\xi\colon \operatorname{int} \mathbb{R}^3_+ \to \operatorname{int} \mathbb{R}^2_+$ which assigns an optimal solution of problem (P2c) to any output level y and any prices $\mathbf{c} = (c_1, c_2)$ of production factors in the following way:

$$\xi(\mathbf{c}, y) = (\xi_1(\mathbf{c}, y), \xi_2(\mathbf{c}, y)) = \widetilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2). \tag{4.53}$$

Definition 4.26 A **firm minimal cost function** is a mapping $\mu\colon \operatorname{int} \mathbb{R}^4_+ \to \operatorname{int} \mathbb{R}_+$ which assigns minimum cost of producing y output units to any output level y, any prices $\mathbf{c} = (c_1, c_2)$ of production factors and any fixed cost d in the following way:

$$\mu(\mathbf{c}, d, y) = c^{tot}(\widetilde{\mathbf{x}}) = c_1\tilde{x}_1 + c_2\tilde{x}_2 + d = c_1\xi_1(\mathbf{c}, y) + c_2\xi_2(\mathbf{c}, y) + d. \tag{4.54}$$

If prices of production factors and the fixed cost of production are known then one can express the firm minimal cost function of producing y output units as a function of output level:

$$\mu(\mathbf{c}, d, y) = c(y). \tag{4.55}$$

Theorem 4.5 If assumptions of Theorem 4.2 are satisfied then:

(1) $\forall\lambda > 0 \quad \xi(\lambda\mathbf{c}, y) = \xi(\mathbf{c}, y)$,
which means that a function of conditional demand for production factors is positively homogenous of degree 0 with respect to prices of production factors,
(2) $\forall\lambda > 0 \quad \mu(\lambda\mathbf{c}, \lambda d, y) = \lambda\mu(\mathbf{c}, d, y)$,
which means that a firm's minimal cost function of producing y output units is positively homogenous of degree 1 with respect to prices of production factors and the fixed cost of production,
(3) $\forall\lambda > 0 \quad \mu(\mathbf{c}, d, \lambda y) = \lambda^{\frac{1}{\theta}}\mu(\mathbf{c}, d, y)$,
which means that a firm minimal cost function of producing y output units is positively homogenous of degree $\frac{1}{\theta}$ with respect to output level, where $\theta > 0$ is a degree of homogeneity of a production function.

Problem of profit maximization with regard to output level (P3c)

The aim of a firm is to maximize its profit expressed as a function of output level, which can be written as a problem to solve in the following way:

$$\pi(y) = r(y) - c(y) = \{py - c(y)\} \to \max \tag{4.56}$$

$$y \geq 0. \tag{4.57}$$

Since a revenue function is linear (thus concave) while a firm's minimal cost function of producing y output units is strictly convex then a firm's profit function is strictly concave. Moreover, we are interested in an optimal solution $\overline{y} > 0$.

Necessary and sufficient conditions for the existence of an optimal solution to problem (P3c) are given in the following theorem.

Theorem 4.6 If a firm's profit function is strictly concave and the following condition is satisfied:

$$\lim_{y\to 0^+}\frac{\mathrm{d}\pi(y)}{\mathrm{d}y} > 0 \quad \wedge \lim_{y\to +\infty}\frac{\mathrm{d}\pi(y)}{\mathrm{d}y} < 0$$
$$\Leftrightarrow \lim_{y\to 0^+}\frac{\mathrm{d}c(y)}{\mathrm{d}y} < p < \lim_{y\to +\infty}\frac{\mathrm{d}c(y)}{\mathrm{d}y} \tag{4.58}$$

then:

(1) $\exists_1\, \overline{y} > 0$ such that $\left.\frac{d\pi(y)}{dy}\right|_{y=\overline{y}} = 0.$

(2) A necessary and sufficient condition for $\overline{y} > 0$ being an optimal solution to problem (P3c) is

$$\left.\frac{d\pi(y)}{dy}\right|_{y=\overline{y}} = 0 \;\Leftrightarrow\; \left.\frac{dr(y)}{dy}\right|_{y=\overline{y}} = \left.\frac{dc(y)}{dy}\right|_{y=\overline{y}} \Leftrightarrow p = \left.\frac{dc(y)}{dy}\right|_{y=\overline{y}} \tag{4.59}$$

which means that there exists exactly one solution $\overline{y} > 0$ for which:

- marginal profit equals zero,
- marginal revenue is equal to marginal minimal cost of producing y output units,
- price of a product is equal to marginal minimal cost of producing y output units.

Definition 4.27 A **function of product supply** is a mapping $\eta\colon \text{int}\ \mathbb{R}^3_+ \to \text{int}\ \mathbb{R}_+$ which assigns an optimal solution of problem (P3c) to any price p of a product and any prices $\mathbf{c} = (c_1, c_2)$ of production factors in the following way:

$$\eta(p, \mathbf{c}) = \overline{y}. \tag{4.60}$$

Definition 4.28 A **firm's maximal profit function** is a mapping $\Pi\colon \text{int}\ \mathbb{R}^4_+ \to \text{int}\ \mathbb{R}_+$ which assigns maximum profit to any price p of a product, any prices $\mathbf{c} = (c_1, c_2)$ of production factors and any fixed cost d in the following way:

$$\Pi(p, \mathbf{c}, d) = \pi(\overline{y}). \tag{4.61}$$

Theorem 4.7 If assumptions of Theorem 4.6 are satisfied then:

(1) $\forall \lambda > 0 \quad \eta(\lambda p, \lambda \mathbf{c}) = \eta(p, \mathbf{c},),$
which means that a function of product supply is positively homogenous of degree 0 with respect to prices of production factors and the price of a product,

(2) $\forall \lambda > 0 \quad \Pi(\lambda p, \lambda \mathbf{c}, \lambda d) = \lambda \Pi(p, \mathbf{c}, d),$
which means that a firm's maximal profit function is positively homogenous of degree 1 with respect to the price of a product, prices of production factors and the fixed cost of production.

Theorem 4.8 If assumptions of Theorem 4.2 are satisfied then problems (P1c) and (P3c) are equivalent.

This means that:

- knowing an optimal solution to problem (P1c) one can determine an optimal solution to problem (P3c): $\overline{y} = f(\overline{\mathbf{x}}),$

- knowing an optimal solution to problems (P3c) and (P2c) one can determine an optimal solution to problem (P1c): $\widetilde{\mathbf{x}} = \xi(\mathbf{c}, \overline{y}) = \psi(p, \mathbf{c}) = \overline{\mathbf{x}}$,
- $\pi(\overline{\mathbf{x}}) = \Pi(p, \mathbf{c}, d) = \pi(\overline{y})$.

Example 4.1 Let us take the following notation:

p—a price of a product manufactured by a firm,
$c_1 > 0$—a price of a production factor,
$x \geq 0$—an input of a production factor,
$y = f(x) = ax^{\frac{1}{2}}$—an output level as a nonlinear function of a production factor input,
$r(y) = py$—revenue (turnover) from sales of a manufactured product as a linear function of output level,
$r(x) = pf(x) = pax^{\frac{1}{2}}$—revenue (turnover) from sales of a manufactured product as a nonlinear function of a production factor input,
$c^{tot}(x) = c_1 x + d$—total cost of production as a linear function of a production factor input,
$c^v(x) = c_1 x$—variable cost of production,[10]
$c^f(x) = d$—fixed cost of production,
$\pi(y) = r(y) - c(y) = py - c(y)$—firm's profit as a function of output level,
$\pi(x) = r(x) - c^{tot}(x) = pf(x) - (c_1 x + d) = pax^{\frac{1}{2}} - (c_1 x + d)$—firm's profit as a function of a production factor input,
$c(y)$—minimum cost of producing y output units, derived as an objective function corresponding to an optimal solution to problem (P2c).

Tasks

1. Solve the profit maximization problem (P1c).
2. Present a geometric illustration of the profit maximization problem (P1c).
3. Give an economic interpretation of necessary and sufficient conditions of the existence of an optimal solution to problem (P1c).
4. Justify that the function of demand for a production factor is homogeneous of degree 0 with respect to the price of a product and the price of a production factor. Justify that a firm's maximal profit function is homogenous of degree 1 with respect to the price of a product, the price of a production factor and the fixed cost of production.
5. Analyse sensitivity of the demand for a production factor and of the firm's maximum profit to changes in the price of a product and changes in values of parameters of the cost function and of the production function.
6. Solve the cost minimization problem (P2c).
7. Present a geometric illustration of the cost minimization problem (P2c).

[10] The price of a production factor is equal to a marginal cost of production.

8. Give an economic interpretation of necessary and sufficient conditions of the existence of an optimal solution to problem (P2c).
9. Check if the function of conditional demand for a production factor is homogenous of degree 0. Check if the function of a firm's minimal cost of producing y output units is homogenous of degree 1 with respect to the price of a production factor. If not, determine the degrees of homogeneity of both functions with respect to output level.
10. Analyse sensitivity of the conditional demand for a production factor and of the firm's minimum cost to changes in the price of a product and changes in values of parameters of the cost function and of the production function.
11. Solve the profit maximization problem (P3c).
12. Present a geometric illustration of the profit maximization problem (P3c).
13. Give an economic interpretation of necessary and sufficient conditions of the existence of optimal solution to problem (P3c).
14. Justify that the product supply function is homogeneous of degree 0 with respect to the price of a product and the price of a production factor. Justify that a firm's maximal profit function is homogenous of degree 1 with respect to the price of a product, the price of a production factor and the fixed cost of production.
15. Analyse sensitivity of the product supply and of the firm's maximum profit to changes in the price of a product and changes in values of parameters of the cost function and of the production function.
16. Justify that the profit maximizations problems (P1c) and (P3c) are equivalent.

Ad 1 The profit maximization problem (P1c) has a form:

(4.62) $$\pi(x) = \left\{pax^{\frac{1}{2}} - (c_1x + d)\right\} \to \max$$

(4.63) $$x \geq 0.$$

Since the production function from the assumption is strictly concave while the total cost of production is a linear (thus concave and convex) function, then the profit function is strictly concave.

We know that if a profit function is strictly concave then problem (P1c) can have:

- No optimal solution when revenue from sales of a product is lower than the total cost of production.
- Exactly one optimal solution $\overline{x} = 0$ which, due to the positive fixed cost of production, corresponds to a loss equal to the fixed cost.
- Exactly one optimal solution $\overline{x} > 0$ which, with the sufficiently low fixed cost of production, corresponds to the positive profit.

A condition ensuring the existence of a unique and positive optimal solution to problem (P1c) has a form:

$$\lim_{x\to 0^+} \frac{\mathrm{d}\pi(x)}{\mathrm{d}x} > 0 \quad \wedge \lim_{x\to +\infty} \frac{\mathrm{d}\pi(x)}{\mathrm{d}x} < 0 \Leftrightarrow \lim_{x\to +\infty} p\frac{\mathrm{d}f(x)}{\mathrm{d}x} < c_1 < p \lim_{x\to 0^+} \frac{\mathrm{d}f(x)}{\mathrm{d}x}, \tag{4.64}$$

which means that from the strict concavity of the firm's profit function it results that by a relatively big production factor input the marginal revenue is lower than the marginal production cost, while by a relatively small production factor input the marginal revenue is higher than the marginal production cost.[11]

Let us determine a marginal profit function in problem (P1c) and check if it satisfies condition (4.64):

$$\frac{\mathrm{d}\pi(x)}{\mathrm{d}x} = \frac{\mathrm{d}r(x)}{\mathrm{d}x} - \frac{\mathrm{d}c^{tot}(x)}{\mathrm{d}x} = \frac{1}{2}pax^{-\frac{1}{2}} - c_1 = \frac{ap}{2x^{\frac{1}{2}}} - c_1. \tag{4.65}$$

Let us notice that:

$$\lim_{x\to +\infty} \frac{\mathrm{d}\pi(x)}{\mathrm{d}x} = \lim_{x\to +\infty}\left(p\frac{\mathrm{d}f(x)}{\mathrm{d}x} - c_1\right) = \lim_{x\to +\infty}\left(\frac{ap}{2x^{\frac{1}{2}}} - c_1\right) = -c_1 < 0 \tag{4.66}$$

and

$$\lim_{x\to 0^+} \frac{\mathrm{d}\pi(x)}{\mathrm{d}x} = \lim_{x\to 0^+}\left(p\frac{\mathrm{d}f(x)}{\mathrm{d}x} - c_1\right) = \lim_{x\to 0^+}\left(\frac{ap}{2x^{\frac{1}{2}}} - c_1\right) = +\infty > 0. \tag{4.67}$$

Since condition (4.64) is satisfied then we can determine an optimal solution to problem (P1c) from the following equation:

$$\exists_1\, \overline{x} > 0 \quad \left.\frac{\mathrm{d}\pi(x)}{\mathrm{d}x}\right|_{x=\overline{x}} = 0, \tag{4.68}$$

which means that there exists a production factor input such that the marginal revenue is equal to marginal production cost.

Hence:

$$\left.\frac{\mathrm{d}\pi(x)}{\mathrm{d}x}\right|_{x=\overline{x}} = \frac{1}{2}pa\overline{x}^{-\frac{1}{2}} - c_1 = 0, \tag{4.69}$$

[11] The marginal revenue is equal to the marginal productivity of a production factor expressed in money units. The marginal production cost is equal to the price of a production factor.

and after some transformations we get:

$$\overline{x} = \left(\frac{ap}{2c_1}\right)^2 > 0. \tag{4.70}$$

Let us substitute the optimal solution obtained above into the profit function:

$$\pi(\overline{x}) = pa\overline{x}^{\frac{1}{2}} - (c_1\overline{x} + d) = pa\left(\frac{ap}{2c_1}\right) - c_1\left(\frac{ap}{2c_1}\right)^2 - d. \tag{4.71}$$

After transformations, we get

$$\pi(\overline{x}) = \frac{a^2p^2}{4c_1} - d. \tag{4.72}$$

Whether the maximum profit is positive depends on how high the fixed cost of production is. If the fixed cost satisfies the following condition:

$$0 \leq d < \frac{(ap)^2}{4c_1}, \tag{4.73}$$

then the maximum profit that a firm can obtain is positive.

Ad 2 See Figs. 4.1a, 4.1b and 4.1c.

Ad 3 Let us determine the value of a second derivative of the profit function using the optimal solution to problem (P1c) as its argument:

$$\left.\frac{\mathrm{d}^2\pi(x)}{\mathrm{d}x^2}\right|_{x=\overline{x}} = -\frac{1}{4}ap\overline{x}^{-\frac{3}{2}} < 0, \text{ since } \overline{x} = \left(\frac{ap}{2c_1}\right)^2 > 0 \tag{4.74}$$

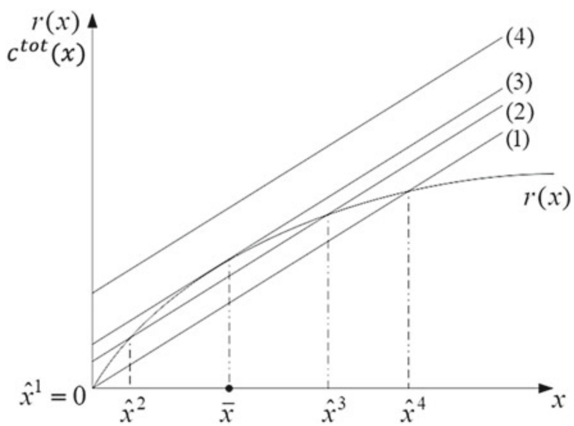


Fig. 4.1a Graphs of revenue function and production total cost function

Fig. 4.1b Graphs of profit function

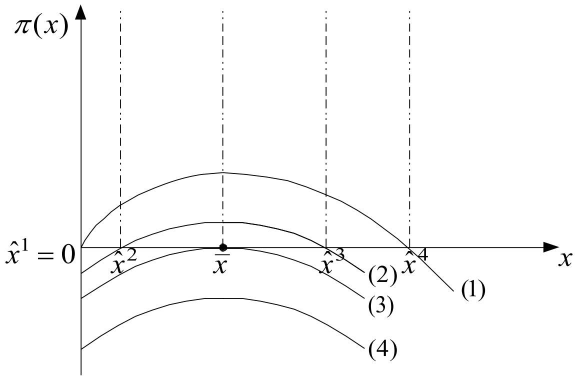
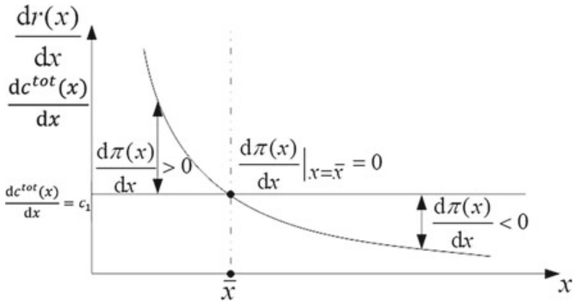


Fig. 4.1c Graphs of marginal revenue function and marginal production cost function



Hence, we can see that using the optimal input $\bar{x} = \left(\frac{ap}{2c_1}\right)^2$ of a production factor a firm obtains the maximum profit. From the analysis we have conducted above it results that since the profit function is strictly concave then condition (4.68) is necessary and sufficient for the existence of the optimal solution to problem (P1c). Condition (4.64) in turn ensures that $\bar{x} > 0$.

Ad 4 The optimal solution to problem (P1c) determines a function of demand for a production factor, about which we know that it is positively homogenous of degree 0, because

(4.75) $$\forall \lambda > 0 \quad \psi(\lambda p, \lambda c_1) = \left(\frac{\lambda a p}{2\lambda c_1}\right)^2 = \left(\frac{ap}{2c_1}\right)^2 = \psi(p, c_1),$$

which means that a proportional change in the price of a product and in the price of a production factor does not impact the demand for a production factor.

The firm's maximal profit function in turn is positively homogenous of degree 1 because

(4.76) $$\forall \lambda > 0 \quad \Pi(\lambda p, \lambda c_1, \lambda d) = \frac{\lambda^2 a^2 p^2}{4\lambda c_1} - \lambda d = \lambda \Pi(p, c_1, d),$$

Table 4.2 Values of reaction measures of demand for production factor and of firm's maximum profit to changes in values of parameters

Function of demand for production factor $\overline{x} = \left(\frac{ap}{2c_1}\right)^2 > 0$	$\frac{\partial \overline{x}}{\partial p} = \frac{pa^2}{2c_1^2} > 0$	$\frac{\partial \overline{x}}{\partial a} = \frac{ap^2}{2c_1^2} > 0$	$\frac{\partial \overline{x}}{\partial c_1} = \frac{-a^2p^2}{2c_1^3} < 0$	$\frac{\partial \overline{x}}{\partial d} = 0$
Firm's maximal profit function $\pi(\overline{x}) = \frac{a^2p^2}{4c_1} - d$	$\frac{\partial \pi(\overline{x})}{\partial p} = \frac{pa^2}{2c_1} > 0$	$\frac{\partial \pi(\overline{x})}{\partial a} = \frac{ap^2}{2c_1} > 0$	$\frac{\partial \pi(\overline{x})}{\partial c_1} = \frac{-a^2p^2}{4c_1^2} < 0$	$\frac{\partial \pi(\overline{x})}{\partial d} = -1$

which means that a proportional change in the price of product, in the price of a production factor and in the fixed cost induces proportional change in the firm's maximum profit.

Ad 5 In Table 4.2, we present values of reaction measures of the demand for a production factor and of the firm's maximum profit to changes in the price of a product and in values of parameters of the production function and of the cost function.

When the production factor productivity a or a price p of a product increases by one unit then the demand for a production factor and the firm's maximum profit increase. When the marginal production cost equal to a price c_1 of production factor increases by one unit then the demand for a production factor and the firm's maximum profit decrease. When the fixed cost of production d increases by one unit then it does not affect the demand for a production factor and induces one unit decrease in the firm's maximum profit.

Ad 6 The cost minimization problem (P2c) when producing y output units has a form:

$$c^{tot}(x) = (c_1 x + d) \rightarrow \min \tag{4.77}$$

$$ax^{\frac{1}{2}} = y, \tag{4.78}$$

$$x \geq 0. \tag{4.79}$$

Since a set of feasible solutions to this problem has only one element, then a production factor input resulting from (4.78) is the optimal solution to this problem:

$$\tilde{x} = \left(\frac{y}{a}\right)^2, \tag{4.80}$$

and is positive by the positive output level.

A firm's minimal cost function of producing y output units corresponds to this solution:

$$c^{tot}(\tilde{x}) = \mu(c_1, d, y) = c_1\left(\frac{y}{a}\right)^2 + d = c(y), \tag{4.81}$$

and is nonlinear and strictly convex function of the output level.

Ad 7 See Figs. 4.2a and 4.2b.

Ad 8 In problem (P2c) exactly one production factor input corresponds to exactly one fixed output level. This production factor input is at the same time the only one

Fig. 4.2a Illustration of problem (P2c)

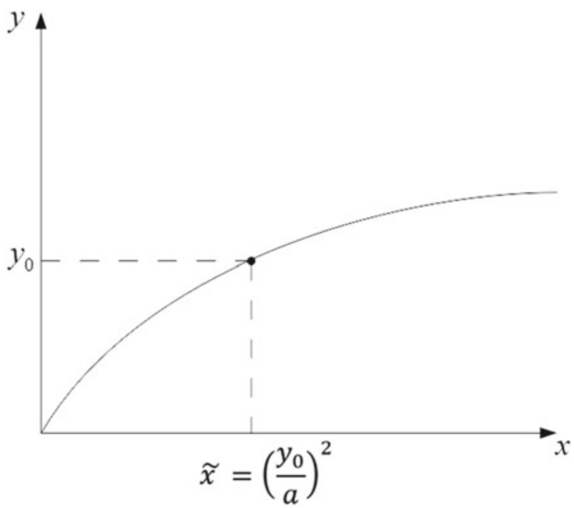
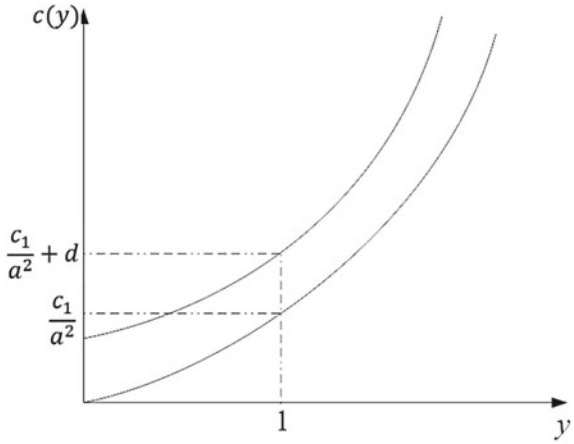


Fig. 4.2b Graphs of the firm's minimal cost function of producing y output units



solution to problem (P2c). As a consequence, a set of feasible solutions has only one element. In this case, independently of an optimality criterion, the only one feasible solution to the problem is at the same time its only one optimal solution.

Ad 9 Let us notice that the function of conditional demand for a production factor does not depend on a price of a production factor, thus is not homogenous of degree 0 with respect to the price of a production factor.[12]

Determining a degree of homogeneity of this function with respect to the output level:

$$\forall\lambda > 0 \quad \xi(\lambda y) = \left(\frac{\lambda y}{a}\right)^2 = \lambda^2\left(\frac{y}{a}\right)^2 = \lambda^2\xi(y), \tag{4.82}$$

we notice that it is

$$\theta = 2 > 0. \tag{4.83}$$

A function of variable cost of production is positively homogenous of degree 1 with respect to the price of a production factor, since:

$$\forall\lambda > 0 \quad c^v(\lambda c_1, y) = \lambda c_1\left(\frac{y}{a}\right)^2 = \lambda c^v(c_1, y). \tag{4.84}$$

A function of total cost of production is positively homogenous of degree 1 with respect to the price of a production factor and the fixed production cost, since:

(4.85)

$$\forall\lambda > 0 \quad c^{tot}(\lambda c_1, \lambda d, y) = \lambda c_1\left(\frac{y}{a}\right)^2 + \lambda d = \lambda\left(c_1\left(\frac{y}{a}\right)^2 + d\right) = \lambda c^{tot}(c_1, d, y).$$

The function of variable cost of production is positively homogenous of degree 2 with respect to the output level, since:

$$\forall\lambda > 0 \quad c^v(c_1, \lambda y) = c_1\left(\frac{\lambda y}{a}\right)^2 = \lambda^2 c_1\left(\frac{y}{a}\right)^2 = \lambda^2 c^v(c_1, y). \tag{4.86}$$

Ad 10 In Table 4.3, we present values of reaction measures of the conditional demand for a production factor and of the firm's minimum production cost to changes in the output level and in values of parameters of the production function and of the cost function.

[12] This results from the fact that here in the example we consider one-variable production function, meaning there is one production factor and hence one input. As a consequence, a set of feasible solutions has only one element not depending on (4.77) nor on the price of a production function. In case of two production factors, their prices matter for the optimal solution, thus the conditional demand for production factors depend on these prices and Theorem 3.5 works exactly in the way it is given.

Table 4.3 Values of reaction measures of conditional demand for production factor and of firm's minimum production cost to changes in values of parameters

Function of conditional demand for production factor $\widetilde{x} = \left(\frac{y}{a}\right)^2 > 0$	$\frac{\partial \widetilde{x}}{\partial y} = \frac{2y}{a^2} > 0$	$\frac{\partial \widetilde{x}}{\partial a} = \frac{-y^2}{a^2} < 0$	$\frac{\partial \widetilde{x}}{\partial c_1} = 0$	$\frac{\partial \widetilde{x}}{\partial d} = 0$
Function of firm minimal production cost $c(\widetilde{x}) = c_1\left(\frac{y}{a}\right)^2 + d > 0$	$\frac{\partial c(\widetilde{x})}{\partial y} = \frac{2yc_1}{a^2} > 0$	$\frac{\partial c(\widetilde{x})}{\partial a} = \frac{-c_1 y^2}{a^2} < 0$	$\frac{\partial c(\widetilde{x})}{\partial c_1} = \frac{y^2}{a^2} > 0$	$\frac{\partial c(\widetilde{x})}{\partial d} = 1$

When the output level y increases by one unit then the conditional demand for a production factor and the firm minimum production cost increase. When the production factor productivity a increases by one unit then the conditional demand for a production factor and the firm minimum production cost decrease. When the marginal production cost equal to a price c_1 of production factor increases by one unit then it does not affect the conditional demand for a production factor and induces an increase in the firm's minimum production cost. When the fixed cost of production d increases by one unit then it does not affect the conditional demand for a production factor and induces one unit increase in the firm's minimum production cost.

Ad 11 The profit maximization problem (P3c) has a form:

$$\pi(y) = py - c(y) = \left\{py - \left(c_1\left(\frac{y}{a}\right)^2 + d\right)\right\} \rightarrow \max, \tag{4.87}$$

$$y \geq 0. \tag{4.88}$$

The revenue function is linear and hence convex. The firm's minimal cost function of producing y output units is nonlinear and strictly convex. Thus, the profit function is a strictly concave function of the output level.

It is known that when a profit function is strictly concave then problem (P3c) can have:

- no optimal solution when revenue from sales of a product is lower than the firm's minimum cost of producing y output units,
- exactly one optimal solution $\overline{y} = 0$ which, due to the positive fixed cost of production, corresponds to a loss equal to the fixed cost,
- exactly one optimal solution $\overline{y} > 0$ which, by the sufficiently low fixed cost of production, corresponds to the positive profit.

A condition ensuring the existence of a unique and positive optimal solution to problem (P3c) has a form:

(4.89)
$$\lim_{y\to 0^+}\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}>0 \quad \wedge \lim_{y\to+\infty}\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}<0 \Leftrightarrow \lim_{y\to 0^+}\frac{\mathrm{d}c(y)}{\mathrm{d}y}<p<\lim_{y\to+\infty}\frac{\mathrm{d}c(y)}{\mathrm{d}y},$$

which means that from the strict concavity of the firm's profit function it results that by a relatively small output level the marginal minimal cost of producing y output units is lower than the price of a product, while by a relatively big output level the marginal minimal cost of producing y output units is higher than the price of a product. Since the revenue function from sales of a product is a linear function of the output level then the marginal revenue from sales of a product is equal to the price of a product.

Let us determine a marginal profit function in problem (P3c) and check if it satisfies condition (4.89):

$$\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}=\frac{\mathrm{d}r(y)}{\mathrm{d}y}-\frac{\mathrm{d}c(y)}{\mathrm{d}y}=p-2\frac{yc_1}{a^2}. \tag{4.90}$$

Let us notice that:

$$\lim_{y\to 0^+}\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}=\lim_{y\to 0^+}\left(p-2\frac{yc_1}{a^2}\right)=p>0 \tag{4.91}$$

and

$$\lim_{y\to+\infty}\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}=\lim_{y\to+\infty}\left(p-2\frac{yc_1}{a^2}\right)=-\infty<0. \tag{4.92}$$

Since condition (4.89) is satisfied then we can determine an optimal solution to problem (P3c) from the following equation:

$$\exists_1\,\bar{y}>0 \quad \left.\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}\right|_{y=\bar{y}}=0, \tag{4.93}$$

which means that there exists an output level such that the marginal revenue from sales of a product is equal to the marginal minimal cost of producing y output units.

Hence:

$$\left.\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}\right|_{y=\bar{y}}=p-2\frac{c_1\bar{y}}{a^2}=0 \tag{4.94}$$

and after some transformations, we get

(4.95)
$$\overline{y} = \frac{a^2 p}{2c_1} > 0.$$

Let us substitute the optimal solution to problem (P3c) obtained above into the profit function:

(4.96)
$$\pi(\overline{y}) = p\overline{y} - c(\overline{y}) = \frac{a^2 p^2}{2c_1} - c_1\left(\frac{ap}{2c_1}\right)^2 - d.$$

After transformations, we get

(4.97)
$$\pi(\overline{y}) = \frac{a^2 p^2}{4c_1} - d.$$

Whether the maximum profit is positive depends on how high the fixed cost of production is. If a condition $0 \le d < \frac{a^2 p^2}{4c_1}$ is satisfied then the maximum profit that a firm can obtain is positive.

Ad 12 See Figs. 4.3a, 4.3b and 4.3c.

Ad 13 Let us determine a value of a second derivative of the profit function using the optimal solution to problem (P3c) as its argument:

(4.3.98)
$$\left.\frac{\mathrm{d}^2\pi(y)}{\mathrm{d}y^2}\right|_{y=\overline{y}} = -2\frac{c_1}{a^2} < 0.$$

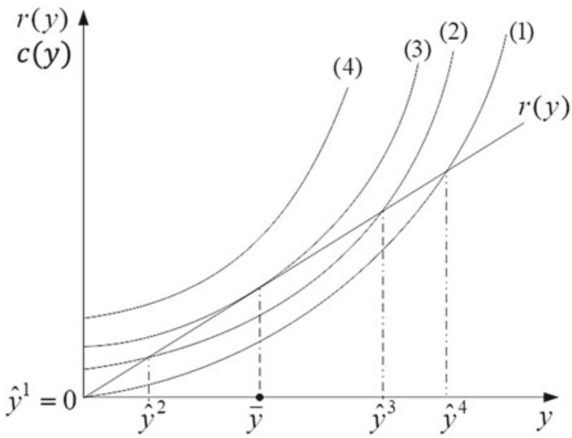


Fig. 4.3a Graphs of revenue function and firm's minimal cost function of producing y output units

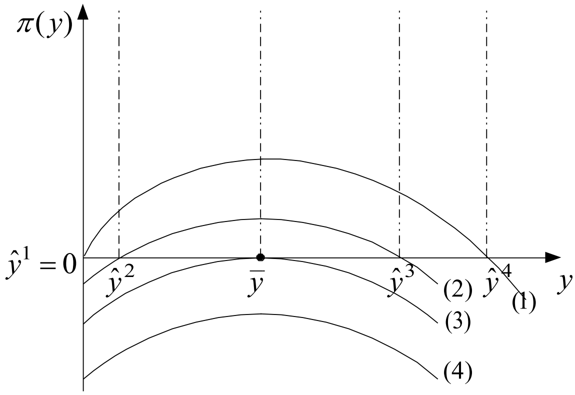


Fig. 4.3b Graphs of profit function

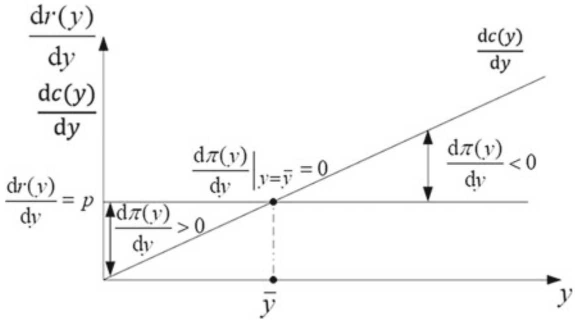


Fig. 4.3c Graphs of marginal revenue function and marginal minimal cost function of producing y output units

Hence, we can see that having the optimal output level $\overline{y} = \frac{pa^2}{2c_1}$ a firm obtains the maximum profit. The analysis we have conducted above shows that since the profit function is strictly concave then condition (4.93) is necessary and sufficient for the existence of the optimal solution to problem (P3c). Condition (4.89) in turn ensures that $\overline{y} > 0$.

Ad 14 The optimal solution to problem (P3c) determines a product supply function, about which we know that it is positively homogenous of degree 0, because

$$\forall \lambda > 0 \quad \eta(\lambda p, \lambda c_1) = \frac{a^2 \lambda p}{2\lambda c_1} = \frac{a^2 p}{2c_1} = \eta(p, c_1), \tag{4.99}$$

which means that a proportional change in the price of a product and in the price of a production factor does not impact the supply of a product.

The firm's maximal profit function in turn is positively homogenous of degree 1 because

$$\forall \lambda > 0 \quad \Pi(\lambda p, \lambda c_1, \lambda d) = \frac{a^2 \lambda^2 p^2}{4\lambda c_1} - \lambda d = \lambda \Pi(p, c_1, d), \tag{4.100}$$

Table 4.4 Values of reaction measures of product supply and of firm's maximum profit to changes in values of parameters

Product supply function $\overline{y} = \frac{pa}{2c_1}^2 > 0$	$\frac{\partial \overline{y}}{\partial p} = \frac{a^2}{2c_1} > 0$	$\frac{\partial \overline{y}}{\partial a} = \frac{p}{c_1} > 0$	$\frac{\partial \overline{y}}{\partial c_1} = \frac{-pa^2}{2c_1^2} < 0$	$\frac{\partial \overline{y}}{\partial d} = 0$
Firm's maximal profit function $\pi(\overline{y}) = \frac{p^2a^2}{4c_1} - d$	$\frac{\partial \pi(\overline{y})}{\partial p} = \frac{pa^2}{2c_1} > 0$	$\frac{\partial \pi(\overline{y})}{\partial a} = \frac{ap^2}{2c_1} > 0$	$\frac{\partial \pi(\overline{y})}{\partial c_1} = \frac{-a^2p^2}{4c_1^2} < 0$	$\frac{\partial \pi(\overline{y})}{\partial d} = -1$

which means that a proportional change in the price of product, in the price of a production factor and in the fixed cost induces the proportional change in the firm's maximum profit.

Ad 15 In Table 4.4, we present values of reaction measures of the product supply and of the firm's maximum profit to changes in the price of a product and in values of parameters of the production function and of the cost function.

When the production factor productivity a or a price p of a product increases by one unit then the product supply and the firm's maximum profit increase. When the marginal production cost equal to a price c_1 of production factor increases by one unit then the product supply and the firm's maximum profit decrease. When the fixed cost of production d increases by one unit then it does not affect the product supply and induces one unit decrease in the firm's maximum profit.

Ad 16 To show that problems (P1c) and (P3c) are equivalent let us notice that:

(1) for $\overline{x} = \left(\frac{ap}{2c_1}\right)^2 > 0$ and $\overline{y} = \frac{a^2p}{2c_1} > 0$ we have

$$\pi(\overline{x}) = \Pi(p, c_1, d) = \frac{a^2p^2}{4c_1} - d = \pi(\overline{y}). \tag{4.101}$$

(2) Knowing the optimal solution to problem (P1c) and substituting it into the production function, we get the optimal solution to problem (P3c)

$$\overline{y} = f(\overline{x}) = a\overline{x}^{\frac{1}{2}} = \frac{a^2p}{2c_1}. \tag{4.102}$$

(3) Knowing the optimal solution to problem (P3c) and substituting it into the optimal solution to problem (P2c) we get the optimal solution to problem (P1c):

$$\widetilde{x} = \left(\frac{\overline{y}}{a}\right)^2 = \left(\frac{ap}{2c_1}\right)^2 = \overline{x}, \tag{4.103}$$

which means that profit maximization problems (P1c) and (P3c) are equivalent.

4.4.2 Dynamic Approach

When we use the dynamic approach to present the profit maximization and the cost minimization problems we assume that part of quantities and levels taken into account by a producer changes over time. In the time horizon considered, the production technology described by a production function is assumed to be invariant. The price of a product, prices of production factors and the fixed cost of production (independent of the output level) can change over time due to different reasons. Still a firm acting in the perfect competition has no impact on the price of a product and prices of production factors in any period or at any moment of time. Let us introduce the following notation:

t—time as discrete ($t = 0, 1, 2, \ldots, T$) or as continuous[13] variable ($t \in [0; T]$),
T—end of the time horizon,
$p(t) > 0$—a time-variant price of a product manufactured by a firm,
$\mathbf{x}(t) = (x_1(t), x_2(t)) \geq \mathbf{0}$—a vector of inputs of production factors that a producer uses in the production process in period/at moment t,
$\mathbf{c}(t) = (c_1(t), c_2(t)) > \mathbf{0}$—a vector of time-variant prices of production factors,
$y = f(\mathbf{x}(t))$—a production function,[14]
$d(t) \geq 0$—time-variant fixed cost of production, that is, the cost not depending on the output level nor on inputs of production factors.

A producer, who aims to maximize the firm's profit, in every period/at any moment t determines what the optimal inputs of production factors are. When deciding about the vector $\mathbf{x}(t)$ of production factors' inputs he/she relies on the relation between the revenue from sales of a product and the production total cost by given time variant: prices of production factors, price of a product and fixed production cost. The profit maximization problem with regard to inputs of production factors has a form:

$$\begin{aligned}\pi(\mathbf{x}(t)) &= r(\mathbf{x}(t)) - c^{tot}(\mathbf{x}(t)) \\ &= \{p(t)f(\mathbf{x}(t)) - (c_1(t)x_1(t) + c_2(t)x_2(t) + d(t))\} \mapsto \max\end{aligned} \tag{4.104}$$

$$\mathbf{x}(t) \geq \mathbf{0}. \tag{4.105}$$

[13] For the discrete and continuous versions, we use the same denotation of the dependence of the function value on time, for example, the fixed production cost on time: $d(t)$. Whether the discrete or continuous version is used in a given formula will result from the context of the issue under consideration.

[14] The value of a production function changes over time because inputs of production factors change over time due to changes in their prices. However, the production process itself does not change, thus the production function does not change its form.

The production function is assumed to be strictly concave and increasing with respect to inputs of production factors. The function of production total cost is linear and increasing with respect to inputs of production factors. As a consequence, the profit function $\pi(\mathbf{x}(t))$ is strictly concave and in every period/at any moment t Problem (4.104)–(4.105) has a solution $\overline{\mathbf{x}}(t) > (0, 0)$. The necessary condition for the existence of maximum profit is

$$\left.\frac{\partial \pi(\mathbf{x}(t))}{\partial x_i(t)}\right|_{\mathbf{x}(t)=\overline{\mathbf{x}}(t)} = 0 \Leftrightarrow \left.\frac{\partial r(\mathbf{x}(t))}{\partial x_i(t)}\right|_{\mathbf{x}(t)=\overline{\mathbf{x}}(t)} = \left.\frac{\partial c^{tot}(\mathbf{x}(t))}{\partial x_i(t)}\right|_{\mathbf{x}(t)=\overline{\mathbf{x}}(t)}$$

$$(4.106) \qquad \Leftrightarrow p(t)\left.\frac{\partial f(\mathbf{x}(t))}{\partial x_i(t)}\right|_{\mathbf{x}(t)=\overline{\mathbf{x}}(t)} = c_i(t) \quad i = 1, 2, \quad \forall t,$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0; T]$.

From the profit maximization problem, one gets a vector $\overline{\mathbf{x}}(t)$ of optimal inputs of production factors, a time-variant function of demand for production factors[15]:

$$(4.107) \qquad \psi(p(t), \mathbf{c}(t)) = \overline{\mathbf{x}}(t)$$

and a firm's maximal profit function:

$$(4.108) \qquad \Pi(p(t), \mathbf{c}(t), d(t)) = \pi(\overline{\mathbf{x}}(t)).$$

The function of demand for production factors as well as the firm's maximal profit function, they have both time-invariant forms but in different periods/at different moments t they can have different values, depending on the time variant: prices of production factors, the price of a product and the fixed production cost.

A producer, who aims to minimize the cost of production, in every period/at any moment t determines what the optimal inputs of production factors which guarantee some fixed output level at minimal cost are. The fixed output level is time variant. "Fixed" means here that it is not any output level but always a level that a producer fixes. The production cost minimization problem has a form:

$$(4.109) \qquad c^{tot}(\mathbf{x}(t)) = \{c_1(t)x_1(t) + c_2(t)x_2(t) + d(t)\} \mapsto \min$$

$$(4.110) \qquad f(\mathbf{x}(t)) = y(t)$$

$$(4.111) \qquad \mathbf{x}(t) \geq \mathbf{0}.$$

[15] This function depends on time in the sense that its values depend on time, while its form is time invariant.

If the production function $f(\mathbf{x}(t))$ is increasing and strictly concave then in every period/at any moment t a line indicating minimal cost of producing $y(t)$ output units is tangent to a production isoquant resulting from Eq. (4.110), since a producer wants to bear the minimal cost when producing the fixed output level. As a consequence a vector of optimal inputs of production factors satisfies the following condition[16]:

$$\sigma_{12}(\widetilde{\mathbf{x}}(t)) = \left.\frac{\partial f(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}(t)=\widetilde{\mathbf{x}}(t)} : \left.\frac{\partial f(\mathbf{x})}{\partial x_2}\right|_{\mathbf{x}(t)=\widetilde{\mathbf{x}}(t)} = \frac{c_1(t)}{c_2(t)} \quad \forall t, \tag{4.112}$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0; T]$ and $\sigma_{12}(\widetilde{\mathbf{x}}(t))$ means the marginal rate of substitution of first production factor by the second production factor in a vector of optimal inputs of production factors. The vector of optimal inputs is a solution to a system of Eqs. (4.110) and (4.112).

From the problem of production cost minimization one gets a vector $\widetilde{\mathbf{x}}(t)$ of optimal inputs of production factors, a time-variant function of conditional demand for production factors:

$$\xi(\mathbf{c}(t), y(t)) = \widetilde{\mathbf{x}}(t) \tag{4.113}$$

and a firm minimal cost function of producing $y(t)$ output units:

$$\begin{aligned} \mu(\mathbf{c}(t), y(t), d(t)) &= c^{tot}(\widetilde{\mathbf{x}}(t)) \\ &= c_1(t)\tilde{x}_1(t) + c_2(t)\tilde{x}_2(t) + d(t) = c(y(t)). \end{aligned} \tag{4.114}$$

The function of conditional demand for production factors as well as the firm's minimal production cost function, they have both time-invariant forms but in different periods/at different moments t they can have different values, depending on the time variant: prices of production factors, the output level fixed by a producer and the fixed production cost.

If a producer considers the profit maximization problem with regard to the output level, then in every period/at any moment t he/she determines what the optimum supply of a product manufactured in her/his firm is. When deciding about the output level $y(t)$ he/she relies on the relation between the revenue from sales of a product and the firm's minimal cost of producing y output units by a given time-variant price of a product. The profit maximization problem with regard to the output level has a form:

$$\pi(y(t)) = r(y(t)) - c(y(t)) = \{p(t)y(t) - c(y(t))\} \mapsto \max \tag{4.115}$$

[16] This condition results from a method of solving a conditional minimization problem for the production total cost function. Then a necessary condition for the existence of unconditional extremum has a form of equation system with partial derivatives of a Lagrange function. This method is presented in Sect. 4.4.1.

$$y(t) \geq 0. \tag{4.116}$$

The revenue function is linear and increasing with respect to the output level. The firm's minimal cost function of producing $y(t)$ output units is strictly convex and increasing with respect to the output level. As a consequence, the profit function $\pi(y(t))$ is strictly concave and in every period/at any moment t Problem (4.115)–(4.116) has a solution $\overline{y}(t) > 0$. The necessary condition for the existence of maximum profit is

$$\left.\frac{\mathrm{d}\pi(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\overline{y}(t)} = 0 \Leftrightarrow \left.\frac{\mathrm{d}r(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\overline{y}(t)} = \left.\frac{\mathrm{d}c(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\overline{y}(t)}$$
$$\Leftrightarrow p(t) = \left.\frac{\mathrm{d}c(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\overline{y}(t)} \quad \forall t, \tag{4.117}$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0; T]$.

From the profit maximization problem, one gets the optimal output level $\overline{y}(t)$, a time-variant function of product supply:

$$\eta(p(t), \mathbf{c}(t)) = \overline{y}(t) \tag{4.118}$$

and a firm's maximal profit function:

$$\Pi(p(t), \mathbf{c}(t), d(t)) = \pi(\overline{y}(t)). \tag{4.119}$$

The product supply as well as the firm's maximal profit function, they have both time-invariant forms but in different periods/at different moments t they can have different values, depending on the time variant: prices of production factors, a price of a product and the fixed production cost.

Example 4.2 A production process in a firm acting in the perfect competition is described by a one-variable production function of a form:

$$f(x(t)) = x(t)^{0.5}.$$

This production function has the same form as the function of Example 4.1, when taking $a = 1$, but now the production factor input changes over time.

Let us assume that at any moment $t \in [0; 30]$ the price of a product, a price of a production factor and the fixed production cost change according to equations:

$$c(t) = 4 \cdot 0.98^t,$$

$$p(t) = 0.006t^2 - 0.1t + 3,$$

$$d(t) = \frac{\left(0.006t^2 - 0.1t + 3\right)^2 t}{480 \cdot 0.98^t} - \frac{t}{30} + 1.$$

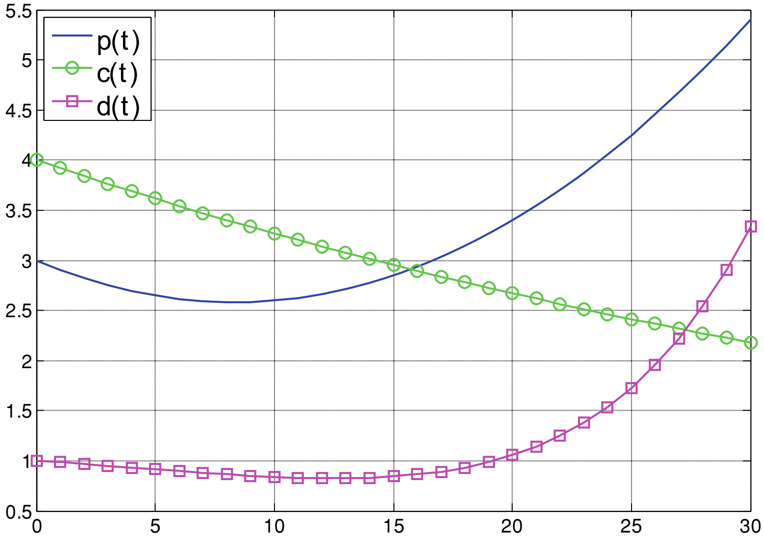


Fig. 4.4 Trajectories of prices and fixed cost

Trajectories of prices and the fixed cost are presented in Fig. 4.4. The price of a production factor constantly decreases. The production factor's price decreases at the beginning of the considered time horizon, reaches its minimum and then increases until the end of the horizon. The fixed cost decreases slightly until moment $t \approx 13$ and then increases until the end of time horizon.

From Example 4.1, we know that for the given form of the production function as an optimal solution to the profit maximization problem with regard to production factor input one obtains a function of demand for a production factor of a form:

$$\psi(p(t), c(t)) = \overline{x}(t) = \left(\frac{p(t)}{2c(t)}\right)^2$$

and a firm's maximal profit function:

$$\Pi(p(t), c(t), d(t)) = \pi(\overline{x}(t)) = \frac{p(t)^2}{4c(t)} - d(t).$$

A firm that aims to maximize its profit determines at any moment t a production factor input such that the marginal revenue equals the marginal production cost equal to the time-variant price $c(t)$ of a production factor. As a result one obtains a value of the function of demand for a production factor by prices of a product and of a production factor given at moment t. A trajectory of the demand is presented in Fig. 4.5. Looking again at Fig. 4.4 we can notice that about moment $t = 16$ the

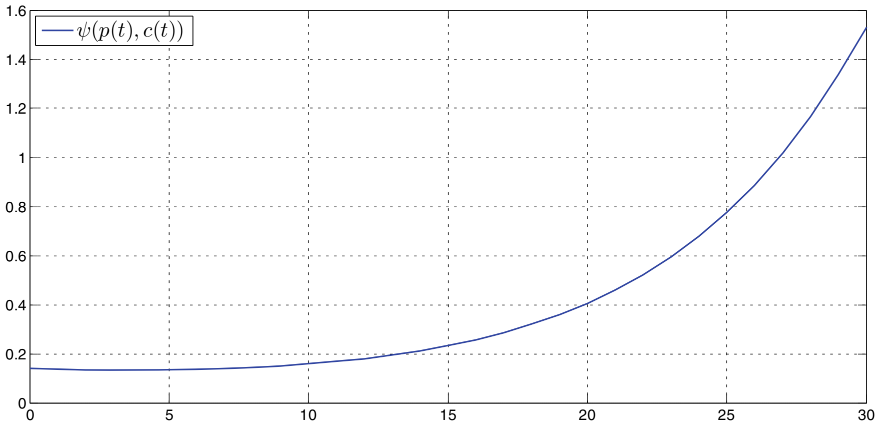


Fig. 4.5 Trajectory of demand for production factor

product price is higher than the production factor price. In Fig. 4.5, it is reflected by higher speed of growth of the demand for a production factor since the production factor price decreases and the product price increases which make the difference between them bigger.

From Fig. 4.6 presenting a trajectory of the firm's maximal profit, it results that until moment $t \approx 19$ the maximum profit that a firm can obtain is negative. Then the maximal profit increases, reaches its maximum equal to around 0.15 and from a moment $t \approx 26$ decreases and equals 0 at the end of the time horizon.

In the production cost minimization problem, a firm fixes at any moment t the output level $y(t)$ it wants to achieve. Let us assume that at any moment $t \in [0; 30]$ this level is determined by a firm according to an equation:

$$y(t) = -0.0035(t - 15)^2 + 1.25.$$

A trajectory of the fixed output level is presented in Fig. 4.7. It can be seen that a firm wants to raise the output from level 0.4625 at the beginning of the time horizon to level 1.25 in the middle of the horizon and then it wants to reduce the output again to level 0.4625.

From Example 4.1, we know that for the given form of the production function as an optimal solution to the cost minimization problem, one obtains a function of conditional demand for a production factor of a form:

$$\xi(y(t), c(t)) = \widetilde{x}(t) = y(t)^2$$

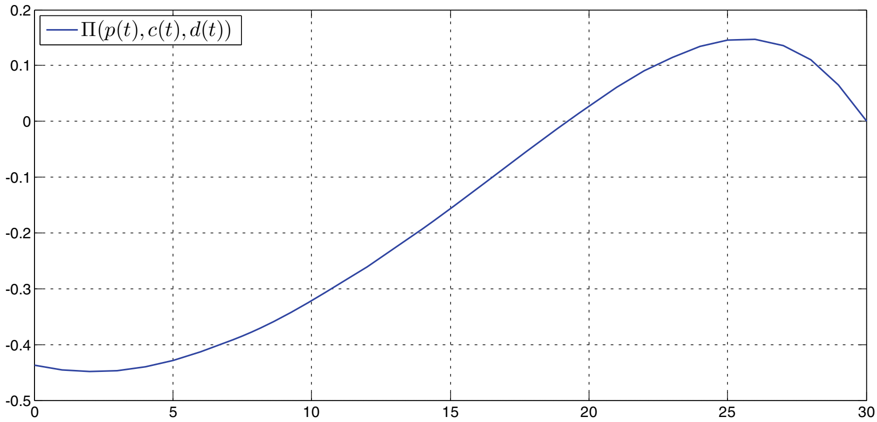


Fig. 4.6 Trajectory of firm's maximum profit

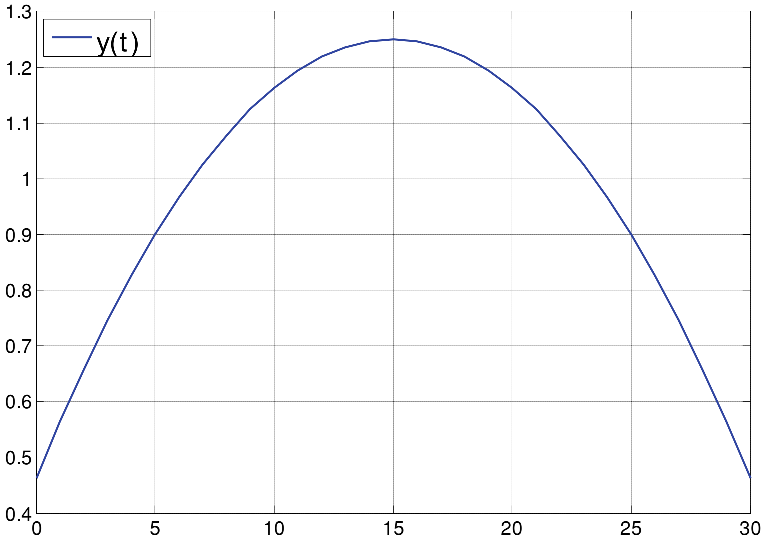


Fig. 4.7 Trajectory of fixed output level

and the firm minimal cost function of producing $y(t)$ output unit:

$$\mu(y(t), \mathrm{c}(t), d(t)) = c^{tot}(\widetilde{x}(t)) = c(t)\widetilde{x}(t) + d(t)$$
$$= c(t)y(t)^2 + d(t) = k(y(t)).$$

One can notice, as in Example 4.1, that when a production function is one variable then the function of conditional demand for a production factor does not depend on the price of a production factor, but only on the fixed output level. It results from the fact that when only one production factor is used in a production process then there is no possibility to substitute this factor by some other factor which could have a different price. Hence, a firm deciding about the input of a production factor does condition its choice on a price of this only one production factor. A trajectory of the conditional demand for a production factor, presented in Fig. 4.8, has a shape similar to the trajectory of the fixed output level $y(t)$, presented in Fig. 4.7, in line with a formula of the conditional demand function. The biggest conditional demand for a production factor occurs in the middle of the time horizon, which means at the time when a firm wants to produce at the biggest output level. At the beginning and at the end of the time horizon, the conditional demand for a production factor is the smallest since then a firm fixes the smallest output level to produce.

The minimal cost of producing $y(t)$ output units, presented in Fig. 4.9, changes over time. This cost is the highest at moment $t \approx 13$, then it decreases and reaches its local minimum at moment $t \approx 27$. This time-variant value depends not only on the fixed output level but also on the price of a production factor and on the fixed production cost.

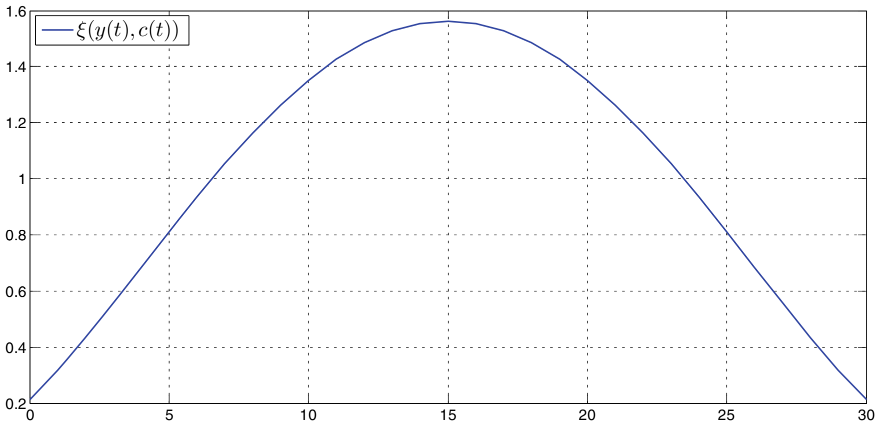


Fig. 4.8 Trajectory of conditional demand for a production factor

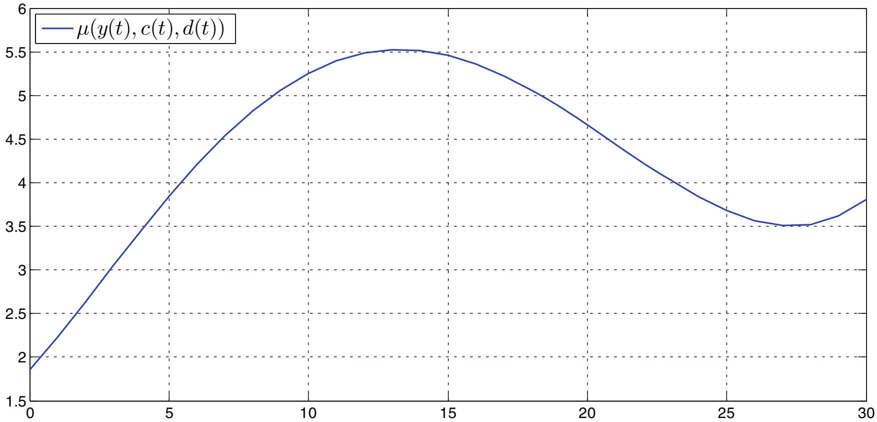


Fig. 4.9 Trajectory of minimum production cost

From the profit maximization problem with regard to the output level, one obtains a product supply function:

$$\eta(p(t), c(t)) = \overline{y}(t) = \frac{p(t)}{2c(t)}$$

and a firm's maximal profit function:

$$\Pi(p(t), c(t), d(t)) = \pi(\overline{y}(t)) = \frac{p(t)^2}{4c(t)} - d(t).$$

Let us notice that the form of the firm's maximal profit function is the same as the form obtained as a solution to the profit maximization problem with regard to production factor inputs. Thus, the resulting trajectory of the firm's maximum profit is the same as in Fig. 4.6.

A firm that aims to maximize its profit determines at any moment t the product supply such that the marginal production cost equals the marginal revenue equal to the time-variant price $p(t)$ of a product. As a result, one obtains the value of the product supply function by prices of a product and of a production factor given at moment t. A trajectory of the optimal product supply is presented in Fig. 4.10.

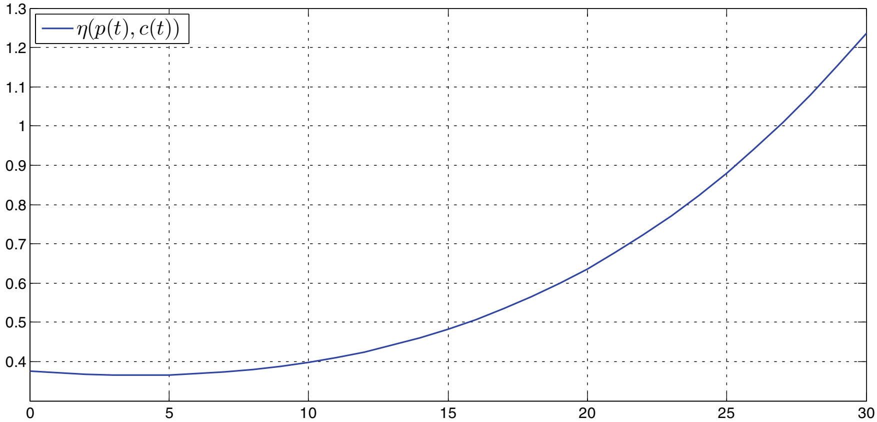


Fig. 4.10 Trajectory of optimal product supply

4.5 Firm Acting in Perfect Competition—Short-Term Strategy

4.5.1 Static Approach

Let us recall the notation:

$p > 0$—a price of a product manufactured by a firm,
$\mathbf{c} = (c_1, c_2) > (0, 0)$—a vector of prices of production factors,
$\mathbf{x} = (x_1, x_2) \geq (0, 0)$—a vector of inputs of production factors,
$\mathbf{b} = (b_1, b_2) > (0, 0)$—a vector of resources of production factors,
$B = [0; b_1] \times [0; b_2] \subset X = \mathbb{R}^2_+$—a set of constraints on resources of production factors,
$w = f(b_1, b_2)$—an output level constrained due to production factors' limitation,
$W = [0; f(b_1, b_2)] = [0; w]$—a set constraining the output level,
$y = f(x_1, x_2)$—an output level,
$r(y) = py$—revenue (turnover) from sales of a manufactured product as a function of output level,
$r(x_1, x_2) = pf(x_1, x_2)$—revenue (turnover) from sales of a manufactured product as a function of inputs of production factors,
$c^{tot}(x_1, x_2) = c_1x_1 + c_2x_2 + d$—total cost of production,
$c^v(x_1, x_2) = c_1x_1 + c_2x_2$—variable cost of production,
$c^f(x_1, x_2) = d$—fixed cost of production,
$c(y)$—minimum cost of producing y output units,
$\pi(y) = r(y) - c(y) = py - c(y)$—firm's profit as a function of output level,
$\pi(x_1, x_2) = r(x_1, x_2) - c^{tot}(x_1, x_2)$—firm's profit as a function of inputs of production factors.

Problem of profit maximization with regard to inputs of production factors whose resources are limited (P1c-s)

The aim of a firm is to maximize its profit expressed as a function of inputs of production factors whose resources are limited, which can be written as a problem to solve in the following way:

$$\begin{aligned}\pi(x_1, x_2) &= r(x_1, x_2) - c^{tot}(x_1, x_2)\\ &= \{pf(x_1, x_2) - (c_1x_1 + c_2x_2 + d)\} \mapsto \max\end{aligned} \tag{4.120}$$

$$x_i \le b_i,\ i = 1, 2 \tag{4.121}$$

$$x_1, x_2 \ge 0. \tag{4.122}$$

Since a production function from Assumption (F2) is strictly concave while the total cost of production is a linear (thus concave) function, then a profit function is strictly concave. Moreover, we are interested in an optimal solution $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2) > (0, 0)$.

Necessary and sufficient conditions for the existence of optimal solution to problem (P1c-s) are given in the following theorem.

Theorem 4.9 If a firm's profit function is strictly concave, twice differentiable and satisfies the following condition:

$$\begin{aligned}&\lim_{x_i \to 0^+} \frac{\partial \pi(x_1, x_2)}{\partial x_i} > 0 \quad \wedge \lim_{x_i \to +\infty} \frac{\partial \pi(x_1, x_2)}{\partial x_i} < 0\\ \Leftrightarrow &\lim_{x_i \to +\infty} \frac{\partial f(x_1, x_2)}{\partial x_i} < c_i < \lim_{x_i \to 0^+} \frac{\partial f(x_1, x_2)}{\partial x_i}, \quad i = 1, 2\end{aligned} \tag{4.123}$$

then a vector $\overline{\mathbf{x}} > 0$ is an optimal solution to problem (P1c-s) if and only if a pair $\overline{\mathbf{x}} > \mathbf{0}$, $\overline{\boldsymbol{\lambda}} \ge \mathbf{0}$ is a solution to the following system of equations:

$$\overline{x}_1\left(p\left.\frac{\partial f(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\overline{\mathbf{x}}} - c_1 - \overline{\lambda}_1\right) + \overline{x}_2\left(p\left.\frac{\partial f(\mathbf{x})}{\partial x_2}\right|_{\mathbf{x}=\overline{\mathbf{x}}} - c_2 - \overline{\lambda}_2\right) = 0, \tag{4.124}$$

$$\overline{\lambda}_1(b_1 - \overline{x}_1) + \overline{\lambda}_2(b_2 - \overline{x}_2) = 0, \tag{4.125}$$

where $\overline{\lambda}_i = \frac{\partial \pi(\mathbf{x})}{\partial b_i}\Big|_{\mathbf{x}=\overline{\mathbf{x}}} \geq 0,\ i = 1, 2$ means an optimal Lagrange multiplier which determines by how much the maximum value of the profit function $\pi: \mathbb{R}^2_+ \to \mathbb{R}$ approximately increases when a value of parameter b_i increases by one notional unit.

If $\overline{\lambda}_i > 0$ then the i-th constraint on resources is binding. When $\overline{\lambda}_i = 0$ then the i-th constraint is not binding.

If we are interested only in a positive optimal solution $\overline{\mathbf{x}} > \mathbf{0}$ to problem (P1c-s) then condition (4.124) is satisfied if and only if:

(4.126) $$p \frac{\partial f(\mathbf{x})}{\partial x_i}\Big|_{\mathbf{x}=\overline{\mathbf{x}}} - c_i - \overline{\lambda}_i = 0,\ i = 1, 2.$$

If $\forall i = 1, 2\ \ \overline{\lambda}_i = 0$ then condition (4.126) takes the form:

(4.127) $$p \frac{\partial f(\mathbf{x})}{\partial x_i}\Big|_{\mathbf{x}=\overline{\mathbf{x}}} = c_i,\ i = 1, 2,$$

which means that i-th production factor's marginal productivity expressed in money units is equal to a price of this production factor. This takes place when any constraint on resources is not binding, which means that:

(4.128) $$\overline{\mathbf{x}} = \overline{\mathbf{x}}^G \leq \mathbf{b}.$$

In the case when any constraint on resources is not binding then an optimal solution $\overline{\mathbf{x}} = \overline{\mathbf{x}}^G$ to problem (P1c-s) is identical to a global maximum that a strictly concave function $\pi: \mathbb{R}^2_+ \to \mathbb{R}$ reaches in space $X = \mathbb{R}^2_+$. Then a necessary and sufficient condition for $\overline{\mathbf{x}} > \mathbf{0}$ being an optimal solution to problem (P1c-) is

(4.129) $$\frac{\partial \pi(x_1, x_2)}{\partial x_i}\Big|_{\mathbf{x}=\overline{\mathbf{x}}} = 0 \quad i = 1, 2$$

and the conditional maximization problem is the same as the unconditional maximization problem.

In the case when $\forall i\ \overline{\lambda}_i > 0$ then each constraint on resources is binding and condition (4.124) is satisfied in the initial form. At the same time from condition (4.125), we get that

(4.130) $$\overline{x}_i = b_i, \quad i = 1, 2.$$

In this case, an optimal solution to problem (P1c-s) is a vector $\overline{\mathbf{x}} = \overline{\mathbf{x}}^L = \mathbf{b}$ such that $\overline{\mathbf{x}}^G \gneqq \overline{\mathbf{x}}^L$. Then a stationary point $\overline{\mathbf{x}} = \overline{\mathbf{x}}^L$ is called a local maximum of a function $\pi: \mathbb{R}^2_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}^2_+$. It satisfies the following condition:

(4.131) $$\frac{\partial \pi(x_1, x_2)}{\partial b_i}\Big|_{\mathbf{x}=\overline{\mathbf{x}}} = \overline{\lambda}_i, \quad i = 1, 2.$$

If $\overline{\lambda}_1 > 0$ while $\overline{\lambda}_2 = 0$ then from conditions (4.124), (4.125) it results that:

(4.132) $$\overline{x}_1 = b_1,$$

and a value of $\overline{x}_2$ can be obtained from a condition:

(4.133) $$\left.\frac{\partial \pi(b_1, x_2)}{\partial x_2}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = 0.$$

In this case, an optimal solution to problem (P1c-s) is a vector $\overline{\mathbf{x}} = \overline{\mathbf{x}}^L \lneqq \mathbf{b}$ such that[17] $\overline{\mathbf{x}}^G \gneqq \overline{\mathbf{x}}^L$. Then a stationary point $\overline{\mathbf{x}} = \overline{\mathbf{x}}^L$ is called a local maximum of a function $\pi\colon \mathbb{R}^2_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}^2_+$.

If $\overline{\lambda}_1 = 0$ while $\overline{\lambda}_2 > 0$ then from conditions (3.124), (3.125) it results that:

(4.134) $$\overline{x}_2 = b_2,$$

and a value of $\overline{x}_1$ can be obtained from a condition:

(4.135) $$\left.\frac{\partial \pi(x_1, b_2)}{\partial x_1}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = 0.$$

In this case, an optimal solution to problem (P1c-s) is a vector $\overline{\mathbf{x}} = \overline{\mathbf{x}}^L \lneqq \mathbf{b}$such that $\overline{\mathbf{x}}^G \gneqq \overline{\mathbf{x}}^L$. Then a stationary point $\mathbf{x} = \overline{\mathbf{x}}^L$ is called a local maximum of a function $\pi\colon \mathbb{R}^2_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}^2_+$.

Definition 4.29 A **function of demand for production factors** is a mapping $\psi\colon \mathrm{int}\ \mathbb{R}^3_+ \to \mathrm{int}\ \mathbb{R}^2_+$ which assigns an optimal solution of problem (P1c-s) to any price p of a product and any prices $\mathbf{c} = (c_1, c_2)$ of production factors in the following way:

(4.136) $$\psi(p, \mathbf{c}) = (\psi_1(p, \mathbf{c}), \psi_2(p, \mathbf{c})) = \overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2).$$

Definition 4.30 A **firm's maximal profit function** is a mapping $\Pi\colon \mathrm{int}\ \mathbb{R}^4_+ \to \mathrm{int}\ \mathbb{R}_+$ which assigns maximum profit to any price p of a product, any prices $\mathbf{c} = (c_1, c_2)$ of production factors and any fixed cost d in the following way:

(4.137) $$\Pi(p, \mathbf{c}, d) = \pi(\overline{\mathbf{x}}).$$

Theorem 4.10 If assumptions of Theorem 4.9 are satisfied then:

[17] The notation $\mathbf{x} \gneqq \mathbf{y}$ means that at least one of the coordinates of a vector $\mathbf{x}$ is bigger than the corresponding coordinate of a vector $\mathbf{y}$ while the other corresponding coordinates are equal to each other.

(1) $\forall \lambda > 0 \quad \psi(\lambda p, \lambda \mathbf{c}) = \psi(p, \mathbf{c})$,
which means that a function of demand for production factors is positively homogenous of degree 0,

(2) $\forall \lambda > 0 \quad \Pi(\lambda p, \lambda \mathbf{c}, \lambda d) = \lambda \Pi(p, \mathbf{c}, d)$,
which means that a firm's maximal profit function is positively homogenous of degree 1 with respect to the price of a product, prices of production factors and the fixed cost of production.

Problem of cost minimization when producing the output at a fixed level with limited resources of production factors (P2c-s)

The aim of a firm is to produce $y \geq 0$ units of output at minimum total cost when resources of production factors are limited. This problem can be written in the following way:

$$c^{tot}(x_1, x_2) = \{c_1 x_1 + c_2 x_2 + d\} \mapsto \min \tag{4.138}$$

$$f(x_1, x_2) = y = \text{const.} > 0, \tag{4.139}$$

$$x_i \leq b_i, \; i = 1, 2, \tag{4.140}$$

$$x_1, x_2 \geq 0. \tag{4.141}$$

One can express problem (P2c-s) using a Lagrange function:

$$\begin{aligned} F(x_1, x_2, \boldsymbol{\lambda}) = &\{c_1 x_1 + c_2 x_2 + d + \lambda_1 (b_1 - x_1) \\ &+ \lambda_2 (b_2 - x_2) + \lambda (y - f(x_1, x_2))\} \mapsto \min. \end{aligned} \tag{4.142}$$

Theorem 4.11 If a production function satisfies assumption (F2) then $\widetilde{\mathbf{x}} > 0$ is an optimal soultion to problem (P2c-s) if and only if a pair $\widetilde{\mathbf{x}} > 0$, $\widetilde{\boldsymbol{\lambda}} \geq \mathbf{0}$ is a solution to the following system of equations:

$$\begin{aligned} &\tilde{x}_1 \left(c_1 - \tilde{\lambda}_1 - \tilde{\lambda} \frac{\partial f(x_1, x_2)}{\partial x_1} \bigg|_{\mathbf{x} = \tilde{\mathbf{x}}} \right) \\ &+ \tilde{x}_2 \left(c_2 - \tilde{\lambda}_2 - \tilde{\lambda} \frac{\partial f(x_1, x_2)}{\partial x_2} \bigg|_{\mathbf{x} = \tilde{\mathbf{x}}} \right) = 0, \end{aligned} \tag{4.143}$$

$$\widetilde{\lambda}_1 (b_1 - \tilde{x}_1) + \widetilde{\lambda}_2 (b_2 - \tilde{x}_2) + \widetilde{\lambda} (y - f(\tilde{x}_1, \tilde{x}_2)) = 0, \tag{4.144}$$

where $\widetilde{\lambda}_i = \frac{\partial c^{tot}(\mathbf{x})}{\partial b_j}\Big|_{\mathbf{x} = \widetilde{\mathbf{x}}} \geq 0, \; i = 1, 2$ and $\widetilde{\lambda} = \frac{\partial c^{tot}(\mathbf{x})}{\partial y}\Big|_{\mathbf{x} = \widetilde{\mathbf{x}}} \geq 0$ mean optimal Lagrange multipliers which determine by how much the minimum value of the production total cost function $c^{tot} : \mathbb{R}^2_+ \to \mathbb{R}$ approximately increases when a value of parameter b_i or the fixed output level increases by one notional unit.

If $\widetilde{\lambda}_i > 0$ then the i-th constraint on resources is binding. When $\widetilde{\lambda}_i = 0$ then the i-th constraint is not binding.

If we are interested only in a positive optimal solution $\widetilde{\mathbf{x}} > \mathbf{0}$ to problem (P2c-s) then condition (4.143) is satisfied if and only if:

(4.145) $$c_i - \widetilde{\lambda}_i - \widetilde{\lambda}\left.\frac{\partial f(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} = 0,\ i = 1, 2.$$

If $\forall i = 1, 2$ $\widetilde{\lambda}_i = 0$ and $\widetilde{\lambda} > 0$ then condition (4.145) takes the form:

(4.146) $$\widetilde{\lambda}\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} = \frac{c_i}{\widetilde{\lambda}_i},\ i = 1, 2.$$

This takes place when any constraint on resources is not binding, which means that:

(4.147) $$\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}^{\mathrm{G}} \le \mathbf{b}.$$

In the case when any constraint on resources is not binding then an optimal solution $\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}^G$ to problem (P2c-s) is identical to a global minimum that a strictly convex function $c^{tot}: \mathbb{R}^2_+ \to \mathbb{R}$ reaches in space $X = \mathbb{R}^2_+$. Then the conditional minimization problem is the same as the unconditional minimization problem. This gives us a useful conclusion. If we want to determine the optimal solution to problem (P2c-s), we should find a global minimum of a function $c^{tot}: \mathbb{R}^2_+ \to \mathbb{R}$ in its domain $X = \mathbb{R}^2_+$. If $\widetilde{\mathbf{x}}^G \in B$ then the optimal solution to problem (P2c-s) is $\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}^G \le \mathbf{b}$.

In the case when $\forall i = 1, 2$ $\widetilde{\lambda}_i > 0$ then each constraint on resources is binding and condition (4.145) is satisfied. At the same time from condition (3.144), having $\widetilde{\lambda} > 0$, we get that

(4.148) $$\widetilde{x}_i = b_i, \quad i = 1, 2,$$

and

(4.149) $$f(\tilde{x}_1, \tilde{x}_2) = y.$$

In this case an optimal solution to problem (P2c-s) is a vector $\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}^L = \mathbf{b}$ such that $\widetilde{\mathbf{x}}^G \gneqq \widetilde{\mathbf{x}}^L$. Then a stationary point $\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}^L$ is called a local minimum of a function $c^{tot}: \mathbb{R}^2_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}^2_+$.

In each of remaining six cases when one or two Lagrange multipliers are positive one proceeds in a similar way. As a result on the basis of conditions (4.143) and (4.144) one gets the optimal solution $\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}^L$ such that $\widetilde{\mathbf{x}}^G \lneqq \widetilde{\mathbf{x}}^L$. Then a stationary point $\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}^L$ is called a local minimum of a function $c^{tot}: \mathbb{R}^2_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}^2_+$.

Definition 4.31 A **function of conditional demand for production factors** is a mapping $\xi\colon \text{int}\,\mathbb{R}^3_+ \to \text{int}\,\mathbb{R}^2_+$ which assigns an optimal solution of problem (P2c-s) to any output level y and any price $\mathbf{c} = (c_1, c_2)$ of production factors in the following way:

(4.150) $$\xi(\mathbf{c}, y) = (\xi_1(\mathbf{c}, y), \xi_2(\mathbf{c}, y)) = \widetilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2).$$

Definition 4.32 A **firm's minimal cost function** is a mapping $\mu\colon \text{int}\,\mathbb{R}^4_+ \to \text{int}\,\mathbb{R}_+$ which assigns minimum cost of producing y output units to any output level y, any price $\mathbf{c} = (c_1, c_2)$ of production factors and any fixed cost d in the following way:

(4.151) $$\mu(\mathbf{c}, d, y) = c^{tot}(\widetilde{\mathbf{x}}) = c_1\tilde{x}_1 + c_2\tilde{x}_2 + d = c_1\xi_1(\mathbf{c}, y) + c_2\xi_2(\mathbf{c}, y) + d.$$

If prices of production factors and fixed cost of production are known then one can express the firm minimal cost function of producing y output units as a function of output level:

(4.152) $$\mu(\mathbf{c}, d, y) = k(y).$$

Theorem 4.12 If assumptions of Theorem 4.9 are satisfied then:

(1) $\forall \lambda > 0 \quad \xi(\lambda\mathbf{c}, y) = \xi(\mathbf{c}, y)$,
which means that a function of conditional demand for production factors is positively homogenous of degree 0 with respect to prices of production factors,
(2) $\forall \lambda > 0 \quad \mu(\lambda\mathbf{c}, \lambda d, y) = \lambda\mu(\mathbf{c}, d, y)$,
which means that a firm's minimal cost function of producing y output units is positively homogenous of degree 1 with respect to prices of production factors and the fixed cost of production,
(3) $\forall \lambda > 0 \quad \mu(\mathbf{c}, d, \lambda y) = \lambda^{\frac{1}{\theta}}\mu(\mathbf{c}, d, y)$,
which means that a firm's minimal cost function of producing y output units is positively homogenous of degree $\frac{1}{\theta}$ with respect to output level, where $\theta > 0$ is a degree of homogeneity of a production function.

Problem of profit maximization with regard to output level with limited resources of production factors (P3c-s)

The aim of a firm is to maximize its profit expressed as a function of output level when resources of production factors are limited. This problem can be written in the following way:

(4.153) $$\pi(y) = r(y) - c(y) = \{py - c(y)\} \mapsto \max$$

(4.154) $$y \le f(b_1, b_2),$$

(4.155) $$y \geq 0,$$

which can be expressed using a Lagrange function:

(4.156) $$L(y, \lambda) = \{\pi(y) + \lambda(f(b_1, b_2) - y)\} \mapsto \max.$$

Since a revenue function is linear (thus concave) while a firm's minimal cost function of producing y output units is strictly convex then a firm's profit function is strictly concave. Moreover, we are interested in an optimal solution $\overline{y} > 0$.

Necessary and sufficient conditions for the existence of optimal solution to problem (P3c-s) are given in the following theorem.

Theorem 4.13 If a firm's profit function is strictly concave and the following condition is satisfied:

(4.157) $$\lim_{y \to 0^+} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} > 0 \quad \wedge \lim_{y \to +\infty} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} < 0 \Leftrightarrow \Leftrightarrow \lim_{y \to 0^+} \frac{\mathrm{d}c(y)}{\mathrm{d}y} < p < \lim_{y \to +\infty} \frac{\mathrm{d}c(y)}{\mathrm{d}y}$$

then $\overline{y} > 0$ is an optimal solution to problem (P3c-s) if and only if a pair $\overline{y} > 0$, $\overline{\lambda} \geq 0$ is a solution to the following system of equations:

(4.158) $$\overline{y}\left(p - \left.\frac{\mathrm{d}c(y)}{\mathrm{d}y}\right|_{y=\overline{y}} - \overline{\lambda}\right) = 0,$$

(4.159) $$\overline{\lambda}(f(b_1, b_2) - \overline{y}) = 0,$$

where $\overline{\lambda} = \left.\frac{\mathrm{d}\pi(y)}{\mathrm{d}f(b_1,b_2)}\right|_{y=\overline{y}} \geq 0$ means an optimal Lagrange multiplier which determines by how much the maximum value of the profit function $\pi: \mathbb{R}_+ \to \mathbb{R}$ approximately increases when the constrained output level resulting from constraints on resources of production factors increases by one notional unit.

If $\overline{\lambda} > 0$ then the constraint on output level is binding. When $\overline{\lambda} = 0$ then the constraint is not binding.

If we are interested only in a positive optimal solution $\overline{y} > 0$ to problem (P3c-s) then condition (4.158) is satisfied if and only if:

(4.160) $$p - \left.\frac{\mathrm{d}c(y)}{\mathrm{d}y}\right|_{y=\overline{y}} - \overline{\lambda} = 0.$$

If $\overline{\lambda} = 0$ then condition (3.160) takes the form:

(4.161) $$\left.\frac{\mathrm{d}c(y)}{\mathrm{d}y}\right|_{y=\overline{y}} = p$$

which means that the marginal production cost is equal to the price of a product. This takes place when the constraint on output level is not binding, which means that:

(4.162) $$\overline{y} = \overline{y}^G \leq f(b_1, b_2).$$

In the case when the constraint on output level is not binding then an optimal solution $\overline{y} = \overline{y}^G$ to problem (P3c-s) is identical to a global maximum that a strictly concave function $\pi:\mathbb{R}_+ \to \mathbb{R}$ reaches in space $\mathbb{R}_+$. Then a necessary and sufficient condition for $\overline{y} > 0$ being an optimal solution to problem (P3c-) is

(4.163) $$\left.\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}\right|_{y=\overline{y}} = 0$$

and the conditional maximization problem is the same as the unconditional maximization problem.

In the case when $\overline{\lambda} > 0$ then the constraint on output level is binding and condition (4.160) is satisfied in the initial form. At the same time from condition (4.159), we get that

(4.164) $$\overline{y} = f(b_1, b_2).$$

In this case, an optimal solution to problem (P3c-s) is the product supply $\overline{y} = \overline{y}^L$ such that $\overline{y}^G > \overline{y}^L$. Then a stationary point $\overline{y}^L$ is called a local maximum of a function $\pi: \mathbb{R}_+ \to \mathbb{R}$ in a set $W = [0; f(b_1, b_2)] \subset \mathbb{R}_+$.

Definition 4.33 A **function of product supply** is a mapping η:int $\mathbb{R}^3_+ \to$ int $\mathbb{R}_+$ which assigns an optimal solution of problem (P3c-s) to any price p of a product and any price $\mathbf{c} = (c_1, c_2)$ of production factors in the following way:

(4.165) $$\eta(p, \mathbf{c}) = \overline{y}.$$

Definition 4.34 A **firm's maximal profit function** is a mapping Π: int $\mathbb{R}^4_+ \to$ int $\mathbb{R}_+$ which assigns maximum profit to any price p of a product, any price $\mathbf{c} = (c_1, c_2)$ of production factors and any fixed cost d in the following way:

(4.166) $$\Pi(p, \mathbf{c}, d) = \pi(\overline{y}).$$

Theorem 4.14 If assumptions of Theorem 4.13 are satisfied then:

(1) $\forall \lambda > 0 \quad \eta(\lambda p, \lambda \mathbf{c}) = \eta(p, \mathbf{c},)$,
which means that a function of product supply is positively homogenous of degree 0 with respect to prices of production factors and the price of a product,
(2) $\forall \lambda > 0 \quad \Pi(\lambda p, \lambda \mathbf{c}, \lambda d) = \lambda \Pi(p, \mathbf{c}, d)$,
which means that a firm's maximal profit function is positively homogenous of degree 1 with respect to a price of a product, prices of production factors and the fixed cost of production.

Theorem 4.15 If assumptions of Theorem 4.13 are satisfied then problems (P1c-s) and (P3c-s) are equivalent.

This means that:

- knowing an optimal solution to problem (P1c-s) one can determine an optimal solution to problem (P3c-s): $\overline{y} = f(\overline{\mathbf{x}})$,
- knowing an optimal solution to problems (P3c-s) and (P2c-s) one can determine an optimal solution to problem (P1c-s): $\widetilde{\mathbf{x}} = \xi(\mathbf{c}, \overline{y}) = \psi(p, \mathbf{c}) = \overline{\mathbf{x}}$,
- $\pi(\overline{\mathbf{x}}) = \Pi(p, \mathbf{c}, d) = \pi(\overline{y})$.

Example 4.3 The following data is given:

p—a price of a product manufactured by a firm,
$c_1 > 0$—a price of a production factor,
$x \geq 0$—an input of a production factor,
$b > 0$—a resource of a production factor,
$y = f(x) = ax^{\frac{1}{2}}$—an output level as a nonlinear function of a production factor input,
$w = f(b) = ab^{\frac{1}{2}}$—an output level constrained due to the production factor limitation,
$r(y) = py$—revenue (turnover) from sales of a manufactured product as a linear function of output level,
$r(x) = pf(x) = pax^{\frac{1}{2}}$—revenue (turnover) from sales of a manufactured product as a nonlinear function of a production factor input,
$c^{tot}(x) = c_1 x + d$—total cost of production as a linear function of a production factor input,
$c^v(x) = c_1 x$—variable cost of production,[18]
$c^f(x) = d$—fixed cost of production,
$\pi(y) = r(y) - c(y) = py - c(y)$—firm's profit as a function of output level,
$\pi(x) = r(x) - c^{tot}(x) = pf(x) - (c_1 x + d) = pax^{\frac{1}{2}} - (c_1 x + d)$—firm's profit as a function of a production factor input,
$c(y)$—minimal cost of producing y output units as an optimal solution to problem (P2c-s).

[18] The price of a production factor is equal to a marginal cost of production.

Tasks

1. Solve the profit maximization problem (P1c-s).
2. Present a geometric illustration of the profit maximization problem (P1c-s).
3. Give an economic interpretation of necessary and sufficient conditions of the existence of optimal solution to problem (P1c-s).
4. Justify that the function of demand for a production factor is homogeneous of degree 0 with respect to the price of a product and the price of a production factor. Justify that a firm's maximal profit function is homogenous of degree 1 with respect to the price of a product, the price of a production factor and the fixed cost of production.
5. Solve the cost minimization problem (P2c-s).
6. Present a geometric illustration of the cost minimization problem (P2c-s).
7. Give an economic interpretation of necessary and sufficient conditions of the existence of optimal solution to problem (P2c-s).
8. Check if the function of conditional demand for a production factor is homogenous of degree 0. Check if the function of firm's minimal cost of producing y output units is homogenous of degree 1 with respect to a price of a production factor. If not, determine the degrees of homogeneity of both functions with respect to output level.
9. Solve the profit maximization problem (P3c-s).
10. Present a geometric illustration of the profit maximization problem (P3c-s).
11. Give an economic interpretation of necessary and sufficient conditions of existence of optimal solution to problem (P3c-s).
12. Justify that the product supply function is homogeneous of degree 0 with respect to the price of a product and the price of a production factor. Justify that a firm's maximal profit function is homogenous of degree 1 with respect to the price of a product, the price of a production factor and the fixed cost of production.
13. Justify that the profit maximization problems (P1-sc) and (P3c-s) are equivalent.

Ad 1 The profit maximization problem (P1c-s) when the production factor resource is limited takes the form:

(4.167) $$\pi(x) = \left\{pax^{\frac{1}{2}} - (c_1x + d)\right\} \mapsto \max$$

(4.168) $$0 \le x \le b.$$

Since the production function from the assumption is strictly concave while the total cost of production is a linear (thus concave and convex) function, then the profit function is strictly concave.

A condition ensuring the existence of a unique and positive optimal solution to problem (P1c-s) has a form:

$$\lim_{x\to 0^+}\frac{\mathrm{d}\pi(x)}{\mathrm{d}x}>0\wedge\lim_{x\to+\infty}\frac{\mathrm{d}\pi(x)}{\mathrm{d}x}<0 \quad (4.169)$$
$$\Leftrightarrow\lim_{x\to+\infty}p\frac{\mathrm{d}f(x)}{\mathrm{d}x}<c_1<p\lim_{x\to 0^+}\frac{\mathrm{d}f(x)}{\mathrm{d}x},$$

which means that from the strict concavity of the firm's profit function it results that by a relatively big production factor input the marginal revenue is lower than the marginal production cost, while by a relatively small production factor input the marginal revenue is higher than the marginal production cost.

Let us determine a marginal profit function in problem (P1c-s) and check if it satisfies condition (4.169):

$$\frac{\mathrm{d}\pi(x)}{\mathrm{d}x}=\frac{\mathrm{d}r(x)}{\mathrm{d}x}-\frac{\mathrm{d}c^{tot}(x)}{\mathrm{d}x}=\frac{1}{2}pax^{-\frac{1}{2}}-c_1=\frac{ap}{2x^{\frac{1}{2}}}-c_1. \quad (4.170)$$

Let us notice that:

$$\lim_{x\to+\infty}\frac{\mathrm{d}\pi(x)}{\mathrm{d}x}=\lim_{x\to+\infty}\left(p\frac{\mathrm{d}f(x)}{\mathrm{d}x}-c_1\right) \quad (4.171)$$
$$=\lim_{x\to+\infty}\left(\frac{ap}{2x^{\frac{1}{2}}}-c_1\right)=-c_1<0$$

and

$$\lim_{x\to 0^+}\frac{\mathrm{d}\pi(x)}{\mathrm{d}x}=\lim_{x\to 0^+}p\frac{\mathrm{d}f(x)}{\mathrm{d}x}-c_1=\lim_{x\to 0^+}\left(\frac{ap}{2x^{\frac{1}{2}}}-c_1\right)=+\infty>0. \quad (4.172)$$

Since condition (4.169) is satisfied then we can determine an optimal solution to problem (P1c-s) from the Kuhn-Tucker theorem. Let us express the problem (P1c-s) using a Lagrange function:

$$L(x,\lambda)=\pi(x)+\lambda(b-x)=pf(x)-(c_1x+d)+\lambda(b-x). \quad (4.173)$$

One gets the optimal solution to problem (P1c-s) from the following equation system:

$$\overline{x}\left(p\left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=\overline{x}}-c_1-\overline{\lambda}\right)=0, \quad (4.174)$$

$$\overline{\lambda}(b-\overline{x})=0, \quad (4.175)$$

where $\overline{\lambda} = \frac{d\pi(x)}{db}\Big|_{x=\overline{x}} \geq 0$ means an optimal Lagrange multiplier which determines by how much the maximum value of the profit function $\pi:\mathbb{R}_+ \to \mathbb{R}$ approximately increases when a value of parameter b increases by one notional unit.

If $\overline{\lambda} > 0$ then the resource constraint is binding. When $\overline{\lambda} = 0$ then the constraint is not binding.

If we are interested only in a positive optimal solution $\overline{x} > 0$ to problem (P1c-s) then condition (4.174) is satisfied if and only if:

(4.176) $$p\frac{df(x)}{dx}\Big|_{x=\overline{x}} - c_1 - \overline{\lambda} = 0.$$

If $\overline{\lambda} = 0$ then condition (3.176) takes the form:

(4.177) $$p\frac{df(x)}{dx}\Big|_{x=\overline{x}} = c_1,$$

which means that a production factor's marginal productivity expressed in money units is equal to a price of this production factor.

After some transformations, one gets

(4.178) $$\frac{1}{2}pa\overline{x}^{-\frac{1}{2}} = c_1 \Leftrightarrow \overline{x} = \left(\frac{ap}{2c_1}\right)^2.$$

This takes place when the resource constraint is not binding, which means that an optimal solution $\overline{x} = \overline{x}^G$ to problem (P1c-s) is identical to a global maximum that a strictly concave function $\pi : \mathbb{R}_+ \to \mathbb{R}$ reaches in space $X = \mathbb{R}_+$. Then the conditional maximization problem is the same as the unconditional maximization problem.

In the case when $\overline{\lambda} > 0$ then the resource constraint is binding and condition (4.174) is satisfied in the initial form. At the same time from condition (4.175) we get that

(4.179) $$\overline{x} = b.$$

In this case an optimal solution to problem (P1c-s) is an input $\overline{x} = \overline{x}^L$ such that $\overline{x}^G > \overline{x}^L$. Then a stationary point $\overline{x} = \overline{x}^L$ is called a local maximum of a function $\pi : \mathbb{R}_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}_+$.

Let us substitute the optimal solutions obtained above into the profit function. Then the firm's maximum profit is

(4.180)
$$\pi(\overline{x}) = pa\overline{x}^{\frac{1}{2}} - (c_1\overline{x} + d) = pa\left(\frac{ap}{2c_1}\right) - c_1\left(\frac{ap}{2c_1}\right)^2 - d = \frac{a^2p^2 - 4c_1d}{4c_1}$$

or

$$\pi(\overline{x}) = pab^{\frac{1}{2}} - c_1 b - d. \quad (4.181)$$

Ad 2 See Figs. 4.11a, 4.11b and 4.11c.

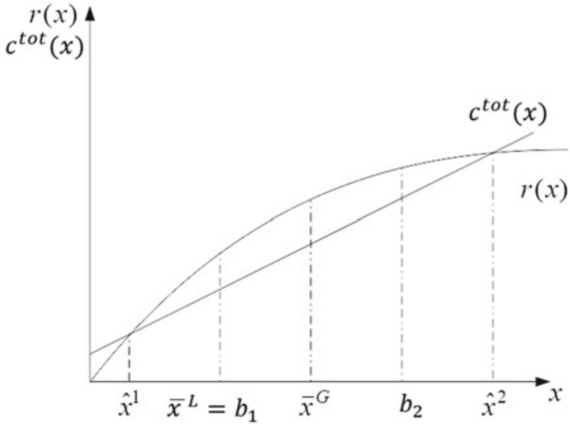


Fig. 4.11a Graphs of revenue function and function of total cost of production

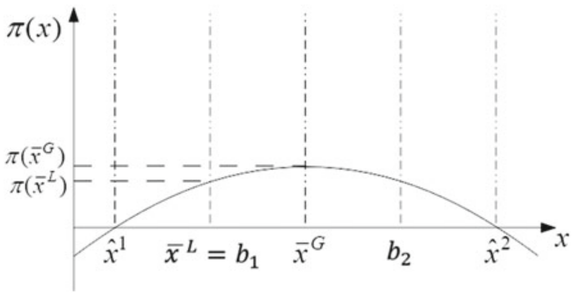


Fig. 4.11b Graph of profit function

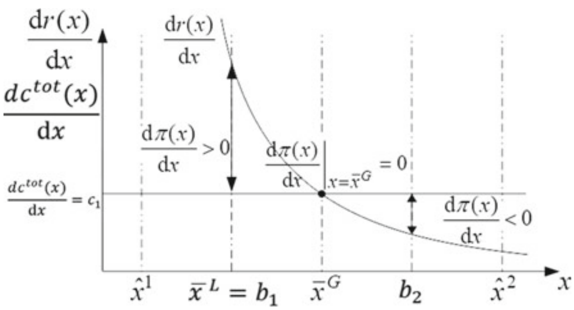


Fig. 4.11c Graphs of marginal revenue function and marginal production cost function

Ad 3 Necessary and sufficient conditions for the existence of the optimal solution to problem (P1c-s) have the following form:

$$\overline{x}\left(p\frac{\mathrm{d}f(x)}{\mathrm{d}x}\bigg|_{x=\overline{x}}-c_1-\overline{\lambda}\right)=0, \tag{4.182}$$

$$\overline{\lambda}(b-\overline{x})=0. \tag{4.183}$$

If $\overline{\lambda}>0$ $\overline{x}=\overline{x}^L=b$ which means that the resource constraint is binding for a firm. The optimal solution to problem (P1c-s) corresponds then with full exploitation of the production factor resource but the maximum profit obtained in that case is lower than the maximum profit a firm would obtain if it owned the production factor resource $b\geq\overline{x}^G$. If the production factor resource is that big then from condition (4.182), having $\overline{\lambda}=0$, one gets $\overline{x}=\overline{x}^G=\left(\frac{ap}{2c_1}\right)^2$.

Ad 4 The optimal solution to problem (P1c-s) determines a function of demand for a production factor. If the solution is $\overline{x}=\overline{x}^L=b$ then $\psi(p,c_1)=b$ is still the function of demand for a production factor but it depends only on the production factor resource and is positively homogenous of degree 1 with respect to the resource.

If the optimal solution to problem (P1c-s) is $\overline{x}=\overline{x}^G=\left(\frac{ap}{2c_1}\right)^2$ then the function of demand for a production factor is positively homogenous of degree 0 with respect to the price of a product and the price of a production factor, because:

$$\forall\lambda>0\quad \overline{x}^G=\psi(\lambda p,\lambda c_1)=\left(\frac{\lambda ap}{2\lambda c_1}\right)^2=\left(\frac{ap}{2c_1}\right)^2=\psi(p,c_1), \tag{4.184}$$

which means that a proportional change in the price of a product and in the price of a production factor does not impact the demand for a production factor.

The firm's maximal profit function in turn is positively homogenous of degree 1 because

$$\forall\lambda>0\quad \pi\left(\overline{x}^G\right)=\Pi(\lambda p,\lambda c_1,\lambda d)=\frac{\lambda^2a^2p^2}{4\lambda c_1}-\lambda d=\lambda\Pi(p,c_1,d), \tag{4.185}$$

which means that a proportional change in the price of product, in the price of a production factor and in the fixed cost induces the proportional change in the firm's maximum profit.[19]

[19] The price of a production factor equals the marginal production cost.

Ad 5 The cost minimization problem (P2c-s) when producing y output units and when the production factor resource is limited takes the form:

(4.186) $$c^{tot}(x) = (c_1 x + d) \mapsto \min$$

(4.187) $$a x^{\frac{1}{2}} = y = \text{const.},$$

(4.188) $$0 \leq x \leq b.$$

Since a set of feasible solutions to this problem has only one element, then a production factor input resulting from (4.187) is the optimal solution to this problem:

(4.189) $$\widetilde{x} = \left(\frac{y}{a}\right)^2 \leq b,$$

and is positive by the positive output level $0 < y \leq ab^{\frac{1}{2}}$.

A firm's minimal cost function of producing y output units corresponds to this solution:

(4.190) $$c^{tot}(\widetilde{x}) = \mu(c_1, d, y) = c_1\left(\frac{y}{a}\right)^2 + d = c(y),$$

for

(4.191) $$0 < y \leq ab^{\frac{1}{2}}$$

and is nonlinear and strictly convex function of the output level.

Ad 6 See Figs. 4.12a and 4.12b.

Ad 7 In problem (P2c-s) exactly one production factor input corresponds to exactly one fixed output level. This production factor input is at the same time the only one solution to problem (P2c-s). As a consequence, a set of feasible solutions has only one element. In this case, independently of an optimality criterion, the only one feasible solution to the problem is at the same time its only one optimal solution.

Ad 8 Let us notice that the function of conditional demand for a production factor does not depend on a price of a production factor, thus is not homogenous of degree 0 with respect to a price of a production factor.

Determining a degree of homogeneity of this function with respect to the output level:

(4.192) $$\forall \lambda > 0 \quad \xi(\lambda y) = \left(\frac{\lambda y}{a}\right)^2 = \lambda^2\left(\frac{y}{a}\right)^2 = \lambda^2 \xi(y),$$

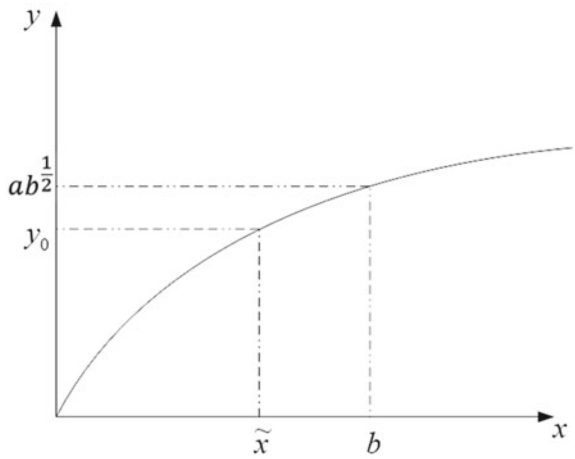


Fig. 4.12a Illustration of problem (P2c-s)

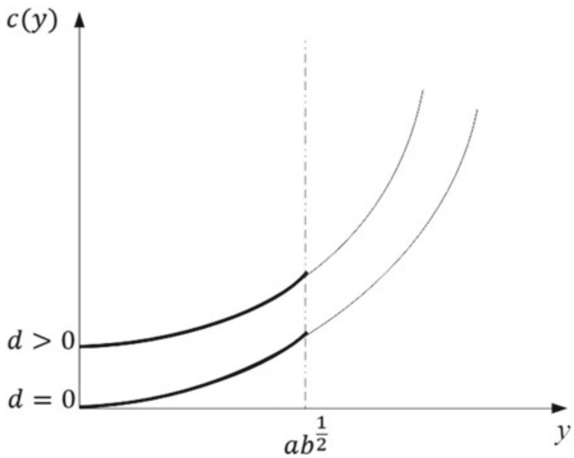


Fig. 4.12b Graphs of firm minimal cost function of producing *y* output units

we notice that it is

$$\theta = 2 > 0. \tag{4.193}$$

A function of variable cost of production is positively homogenous of degree 1 with respect to the price of a production factor, since

$$\forall \lambda > 0 \quad c^v(\lambda c_1, y) = \lambda c_1 \left(\frac{y}{a}\right)^2 = \lambda c^v(c_1, y). \tag{4.194}$$

A function of total cost of production is positively homogenous of degree 1 with respect to the price of a production factor and the fixed production cost, since

$$\forall\lambda > 0 \quad c^{tot}(\lambda c_1, \lambda d, y) = \lambda c_1\left(\frac{y}{a}\right)^2 + \lambda d = \lambda\left(c_1\left(\frac{y}{a}\right)^2 + d\right) = \lambda c^{tot}(c_1, d, y). \tag{4.195}$$

The function of variable cost of production is positively homogenous of degree 2 with respect to the output level, since

$$\forall\lambda > 0 \quad c^v(c_1, \lambda y) = c_1\left(\frac{\lambda y}{a}\right)^2 = \lambda^2 c_1\left(\frac{y}{a}\right)^2 = \lambda^2 c^v(c_1, y). \tag{4.196}$$

Ad 9 The profit maximization problem (P3c-s) when the output level is constrained (due to the production factor resource limitation) takes the form:

$$\pi(y) = py - c(y) = \left\{py - \left(c_1\left(\frac{y}{a}\right)^2 + d\right)\right\} \mapsto \max, \tag{4.197}$$

$$0 \le y \le f(b). \tag{4.198}$$

The revenue function is linear and hence concave. The firm's minimal cost function of producing y output units is nonlinear and strictly convex. Thus, the profit function is a strictly concave function of the output level.

A condition ensuring the existence of a unique and positive optimal solution to problem (P3c-s) has a form:

$$\lim_{y\to 0^+}\frac{d\pi(y)}{dy} > 0 \ \wedge \lim_{y\to+\infty}\frac{d\pi(y)}{dy} < 0 \Leftrightarrow \lim_{y\to 0^+}\frac{dc(y)}{dy} < p < \lim_{y\to+\infty}\frac{dc(y)}{dy}, \tag{4.199}$$

which means that from the strict concavity of the firm's profit function it results that by a relatively small output level the marginal minimal cost of producing y output units is lower than the price of a product, while by a relatively big output level the marginal minimal cost of producing y output units is higher than the price of a product. Since the revenue function from sales of a product is a linear function of the output level then the marginal revenue from sales of a product is equal to the price of a product.

Let us determine a marginal profit function in problem (P3c-s) and check if it satisfies condition (4.199):

$$\frac{d\pi(y)}{dy} = \frac{dr(y)}{dy} - \frac{dc(y)}{dy} = p - 2\frac{yc_1}{a^2}. \tag{4.200}$$

Let us notice that:

$$\lim_{y\to 0^+} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} = \lim_{y\to 0^+} \left(p - 2\frac{yc_1}{a^2}\right) = p > 0 \tag{4.201}$$

and

$$\lim_{y\to +\infty} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} = \lim_{y\to +\infty} \left(p - 2\frac{yc_1}{a^2}\right) = -\infty < 0. \tag{4.202}$$

Since condition (4.199) is satisfied then we can determine an optimal solution to problem (P3c-s) from the Kuhn-Tucker theorem. Let us express the problem (P3c-s) using a Lagrange function:

$$L(y,\lambda) = \pi(y) + \lambda\left(ab^{\frac{1}{2}} - y\right) = py - \left(c_1\left(\frac{y}{a}\right)^2 + d\right) + \lambda\left(ab^{\frac{1}{2}} - y\right). \tag{4.203}$$

One gets the optimal solution to problem (P1c-s) from the following equation system:

$$\overline{y}\left(p - \frac{2c_1\overline{y}}{a^2} - \overline{\lambda}\right) = 0, \tag{4.204}$$

$$\overline{\lambda}\left(ab^{\frac{1}{2}} - \overline{y}\right) = 0, \tag{4.205}$$

where $\overline{\lambda} = \left.\frac{\mathrm{d}\pi(y)}{\mathrm{d}ab^{\frac{1}{2}}}\right|_{y=\overline{y}} \geq 0$ means an optimal Lagrange multiplier which determines by how much the maximum value of the profit function $\pi : \mathbb{R}_+ \to \mathbb{R}$ approximately increases when the constrained output level $ab^{\frac{1}{2}}$ resulting from the production factor resource increases by one notional unit.

If $\overline{\lambda} > 0$ then the constraint on output level is binding. When $\overline{\lambda} = 0$ then the constraint is not binding.

If we are interested only in a positive optimal solution $\overline{y} > 0$ to problem (P3c-s) then condition (3.204) is satisfied if and only if:

$$p - \frac{2c_1\overline{y}}{a^2} - \overline{\lambda} = 0. \tag{4.206}$$

If $\overline{\lambda} = 0$ then condition (4.206) takes the form:

$$p - \frac{2c_1\overline{y}}{a^2} = 0 \tag{4.207}$$

and hence:

$$\overline{y} = \frac{a^2 p}{2c_1}. \tag{4.208}$$

This takes place when the constraint on output level is not binding, which means that an optimal solution $\overline{y} = \overline{y}^G$ to problem (P3c-s) is identical to a global maximum that a strictly concave function $\pi : \mathbb{R}_+ \to \mathbb{R}$ reaches in space $X = \mathbb{R}_+$. Then the conditional maximization problem is the same as the unconditional maximization problem.

In the case when $\overline{\lambda} > 0$ then the constraint on output level is binding and condition (4.204) is satisfied in the initial form. At the same time from condition (4.205) we get that

$$\overline{y} = ab^{\frac{1}{2}}. \tag{4.209}$$

In this case, an optimal solution to problem (P3c-s) is the product supply $\overline{y} = \overline{y}^L$ such that $\overline{y}^G > \overline{y}^L$. Then a stationary point $\overline{y} = \overline{y}^L$ is called a local maximum of a function $\pi : \mathbb{R}_+ \to \mathbb{R}$ in a set $W \subset X = \mathbb{R}_+$.

Let us substitute the optimal solutions obtained above into the profit function. Then the firm's maximum profit is

$$\pi(\overline{y}) = p\overline{y} + c(\overline{y}) = \frac{p^2 a^2}{2c_1} - c_1\left(\frac{pa}{2c_1}\right)^2 - d \tag{4.210}$$

or

$$\pi(\overline{y}) = pab^{\frac{1}{2}} - c_1 b - d. \tag{4.211}$$

Ad 10 See Figs. 4.13a, 4.13b and 4.13c.

Ad 11, 12 The optimal solution to problem (P3c-s) determines a product supply function. If the solution is $\overline{y} = \overline{y}^L = w$ then $\eta(p, c_1) = w$ is still the product supply function but it depends only on the output level constrained due to the production factor limitation. It is positively homogenous of degree 1 with respect to the constrained output level.

If the optimal solution to problem (P3c-s) is $\overline{y} = \overline{y}^G = \frac{a^2 p}{2c_1}$ then the product supply function is positively homogenous of degree 0 with respect to the price of a product and the price of a production factor, because

$$\forall \lambda > 0 \quad \eta(\lambda p, \lambda c_1) = \frac{a^2 \lambda p}{2\lambda c_1} = \frac{a^2 p}{2c_1} = \eta(p, c_1), \tag{4.212}$$

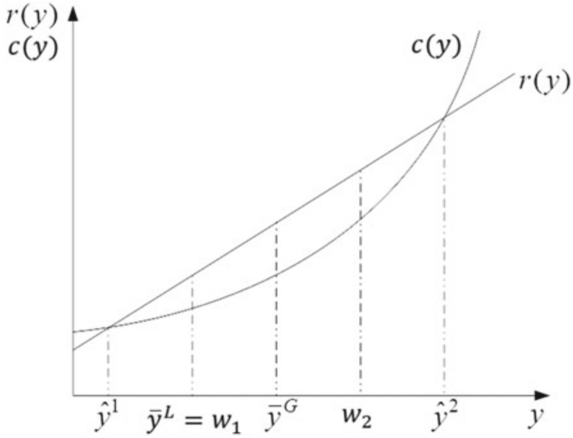


Fig. 4.13a Graphs of revenue function and firm's minimal cost function of producing y output units

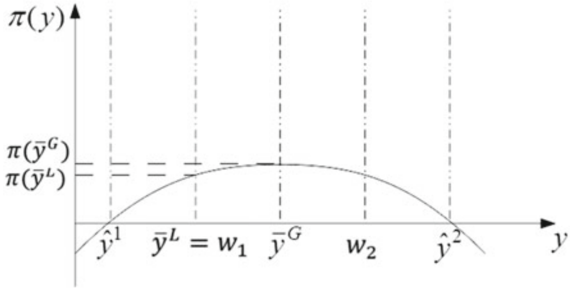


Fig. 4.13b Graph of profit function

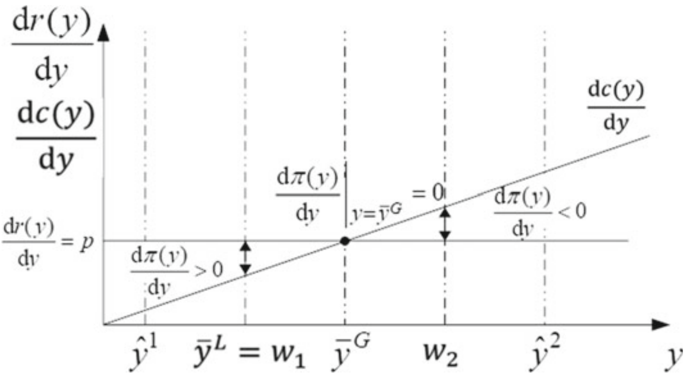


Fig. 4.13c Graphs of marginal revenue function and marginal minimal cost function of producing y output units

which means that a proportional change in the price of product and in the price of a production factor does not impact the supply of a product.

The firm's maximal profit function in turn is positively homogenous of degree 1 because

$$\forall \lambda > 0 \quad \Pi(\lambda p, \lambda c_1, \lambda d) = \frac{a^2 \lambda^2 p^2}{4\lambda c_1} - \lambda d = \lambda \Pi(p, c_1, d), \tag{4.213}$$

which means that a proportional change in the price of a product, in the price of a production factor and in the fixed cost induces the proportional change in the firm's maximum profit.

Ad 13 To show that problems (P1c-s) and (P3c-s) are equivalent let us notice that.[20]

(1) For $\overline{x} = \left(\frac{ap}{2c_1}\right)^2 > 0$ and $\overline{y} = \frac{a^2 p}{2c_1} > 0$ we have

$$\pi(\overline{x}) = \Pi(p, c_1, d) = \frac{a^2 p^2}{4c_1} - d = \pi(\overline{y}). \tag{4.214}$$

(2) Knowing the optimal solution to problem (P1c-s) and substituting it into the production function we get the optimal solution to problem (P3c-w):

$$\overline{y} = f(\overline{x}) = a\overline{x}^{\frac{1}{2}} = \frac{a^2 p}{2c_1}. \tag{4.215}$$

(3) Knowing the optimal solution to problem (P3c-s) and substituting it into the optimal solution to problem (P2c-s), we get the optimal solution to problem (P1c-s):

$$\widetilde{x} = \left(\frac{\overline{y}}{a}\right)^2 = \left(\frac{ap}{2c_1}\right)^2 = \overline{x}, \tag{4.216}$$

which means that profit maximization problems (P1c) and (P3c) are equivalent.

[20] The way of justifying the equivalence of the maximization problems (P1c-s) and (P2c-s) is identical when optimal solutions to both problems are the local maxima which indicate full exploitation of the production factor resource.

4.5.2 Dynamic Approach

The profit maximization problems and the production cost minimization problem in the short-term firm strategy have similar forms to versions of these problems presented in long-term strategy. The difference is that considering the short-term strategy a firm has to take into account additional constraints on resources of production factors.[21] They can turn out to be binding when an optimal production factor input resulting from one of the optimization problems considered without constraints on resources is bigger than the actual resource of this production factor. Then a firm determining the optimal input has to use this quantity constrained by the production factor resource. In the case of deciding about the optimal supply, the constraint results from the constrained output level when using limited inputs of production factors equal to their resources.

We use the same notation as in Sect. 4.4.2 discussing the dynamic approach in long-term strategy. Let us also introduce additional notation:

$\mathbf{b}(t) = (b_1(t), b_2(t)) > \mathbf{0}$—a vector of time-variant resources of production factors,
$w(t) = f(b_1(t), b_2(t))$—a time-variant output level constrained due to the production factors' limitation,
$\overline{\mathbf{x}}^G(t)$—an optimal solution to the profit maximization problem with regard to inputs of production factors whose resources are unlimited,
$\widetilde{\mathbf{x}}^G(t)$—an optimal solution to the production cost minimization problem when resources of production factors are unlimited,
$\overline{y}^G(t)$—an optimal solution to the profit maximization problem with regard to output level with unlimited resources of production factors.

In the short-term strategy, the profit maximization problem with regard to inputs of production factors takes the form:

$$\begin{aligned}\pi(\mathbf{x}(t)) &= r(\mathbf{x}(t)) - c^{tot}(\mathbf{x}(t)) \\ &= \{p(t)f(\mathbf{x}(t)) - (c_1(t)x_1(t) + c_2(t)x_2(t) + d(t))\} \mapsto \max\end{aligned} \tag{4.217}$$

$$x_i(t) \leq b_i(t) \quad i = 1, 2 \tag{4.218}$$

$$\mathbf{x}(t) \geq \mathbf{0}. \tag{4.219}$$

Initially, one solves problem (4.217)–(4.219) in the same way as the analogical problem in the long-term strategy. After determining the optimal solution $\overline{\mathbf{x}}^G(t)$,

[21] Let us recall that the distinction between the long and short terms of a firm's activity does not involve the time dimension but resources of production factors. It is assumed that in the short term the resources are limited while in the long term they are unlimited.

we compare it in each period/at any moment t of the considered time horizon with a vector $\mathbf{b}(t)$ of resources of production factors. As the solution to the whole problem in the short-term strategy, one gets a vector of optimal inputs of production factors:

(4.220) $$\overline{\mathbf{x}}(t) = \left(\min\left\{\overline{x}_1^G(t), b_1(t)\right\}, \min\left\{\overline{x}_2^G(t), b_2(t)\right\}\right),$$

a time-variant function of demand for production factors:

(4.221) $$\psi(p(t), \mathbf{c}(t)) = \overline{\mathbf{x}}(t)$$

and a firm's maximal profit function:

(4.222) $$\Pi(p(t), \mathbf{c}(t), d(t)) = \pi(\overline{\mathbf{x}}(t)).$$

The production cost minimization problem in the short-term strategy has a similar form to the analogical problem in the long-term strategy. The difference is accounting additionally for the constraints on the resources of production factors:

(4.223) $$c^{tot}(\mathbf{x}(t)) = \{c_1(t)x_1(t) + c_2(t)x_2(t) + d(t)\} \mapsto \min$$

(4.224) $$f(\mathbf{x}(t)) = y(t)$$

(4.225) $$x_i(t) \leq b_i(t) \quad i = 1, 2$$

(4.226) $$\mathbf{x}(t) \geq \mathbf{0}.$$

Initially one solves Problem (4.223)–(4.226) in the same way as the analogical problem in the long-term strategy. After determining the optimal solution $\widetilde{\mathbf{x}}^G(t)$, we compare it in each period/at any moment t of the considered time horizon with a vector $\mathbf{b}(t)$ of resources of production factors. As the solution to the whole problem in the short-term strategy one gets a vector of optimal inputs of production factors:

(4.227) $$\widetilde{\mathbf{x}}(t) = \left(\min\left\{\tilde{x}_1^G(t), b_1(t)\right\}, \min\left\{\tilde{x}_2^G(t), b_2(t)\right\}\right),$$

a time-variant function of conditional demand for production factors:

(4.228) $$\xi(\mathbf{c}(t), y(t)) = \tilde{\mathbf{x}}(t)$$

and a firm's minimal cost function of producing $y(t)$ output units:

$$\begin{aligned} \mu(\mathbf{c}(t), y(t), d(t)) =& c^{tot}(\tilde{\mathbf{x}}(t)) \\ =& c_1(t)\tilde{x}_1(t) + c_2(t)\tilde{x}_2(t) + d(t) = c(y(t)). \end{aligned} \tag{4.229}$$

The profit maximization problem with regard to output level in the short-term strategy has a similar form to the analogical problem in the long-term strategy. The difference is accounting additionally for the constraint on the output level due to the limitations of the resources of production factors. In short-term strategy, the profit maximization problem with regard to output level takes the form:

$$\pi(y(t)) = r(y(t)) - c(y(t)) = \{p(t)y(t) - c(y(t))\} \mapsto \max \tag{4.230}$$

$$y(t) \le f(\mathbf{b}(t)) \tag{4.231}$$

$$y(t) \ge 0. \tag{4.232}$$

For Problem (4.230)–(4.232), one determines first a solution $\overline{y}^G(t)$, that is, a solution to the analogical problem in the long-term strategy. Then we compare it in each period/at any moment t with the constrained output level $f(\mathbf{b}(t))$ resulting from the limitations of resources of production factors. As the solution to the whole problem in the short-term strategy, one gets an optimal output level:

$$\overline{y}(t) = \min\left\{\overline{y}^G(t), f(\mathbf{b}(t))\right\}, \tag{4.233}$$

a time-variant function of product supply:

$$\eta(p(t), \mathbf{c}(t)) = \overline{y}(t) \tag{4.234}$$

and a firm's maximal profit function:

$$\Pi(p(t), \mathbf{c}(t), d(t)) = \pi(\overline{y}(t)). \tag{4.235}$$

Example 4.4 Let us take the same assumptions as in Example 4.2, introducing additionally a constraint of production factor resource. A firm acts in the prefect competition and considers the short-term strategy. The production process is described by a one-variable production function of a form:

$$f(x(t)) = x(t)^{0.5}.$$

At any moment[22] $t \in [0; 30]$ the price of a product, the price of a production factor and the fixed production cost change according to equations:

$$c(t) = 4 \cdot 0.98^t,$$

$$p(t) = 0.006t^2 - 0.1t + 3,$$

$$d(t) = \frac{\left(0.006t^2 - 0.1t + 3\right)^2 t}{480 \cdot 0.98^t} - \frac{t}{30} + 1.$$

Their trajectories are presented in Sect. 4.4.2, in Example 4.2 in Fig. 4.4.

Additionally, unlike the long-term strategy from Example 4.2, now we assume that a production factor input is limited by its resource which changes over time according to an equation:

$$b(t) = -0.01t + 1.$$

Figure 4.14 presents a trajectory of the production factor resource and a trajectory of an optimal solution to the monopoly profit maximization problem with regard to production factor input in the long-term strategy. Up to a moment $t \approx 25$ the production factor constraint is not binding because a value $\overline{x}^G(t)$ does not exceed the resource $b(t)$. From the moment $t \approx 25$, the resource constraint is binding until the end of the time horizon. Thus, a trajectory of an optimal production factor input resulting from the profit maximization problem in the short-term strategy has a form as in Fig. 4.15.

Figure 4.16 presents a comparison of the firm's maximum profit in the case when the production factor input is constrained by its resource and in the case when such limitation does not exist. The difference of the maximum profit in both cases is visible from a moment $t \approx 25$.

Fixing the output level at any given moment t, a firm which uses the short-term strategy is constrained by the production factor resource. Thus, the optimal production factor input resulting from the production cost minimization problem in the long-term strategy has to be compared with the production factor input, as it is illustrated in Fig. 4.17. The resource constrained turns out to be binding in the period between moments $t \approx 7$ and $t \approx 26$. Hence, a trajectory of optimal production factor input resulting from the cost minimization problem in the short-term strategy takes the form as presented in Fig. 4.18.

[22] The fact that we regard the same length of the time horizon as in Example 4.2 discussing the long-term strategy does not mean that a firm's activity is considered in the long term. We can assume that a time unit is shorter than the one used in Example 4.2, e.g. a week instead of a month. However, this kind of an assumption is not necessary. It should be noticed that a firm from period to period (or at any moment t) can be constrained by the production factor resource, thus it needs to consider the short-term strategy. Hence, the length of the time horizon should not be seen as the basis when deciding the kind of a strategy: the long term or the short term.

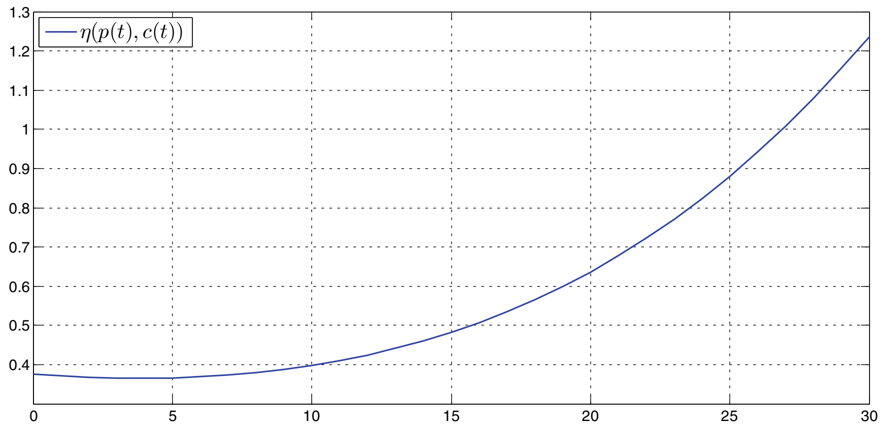


Fig. 4.14 Trajectories of resource and of optimal production factor input in long-term strategy—profit maximization problem with regard to production factor input

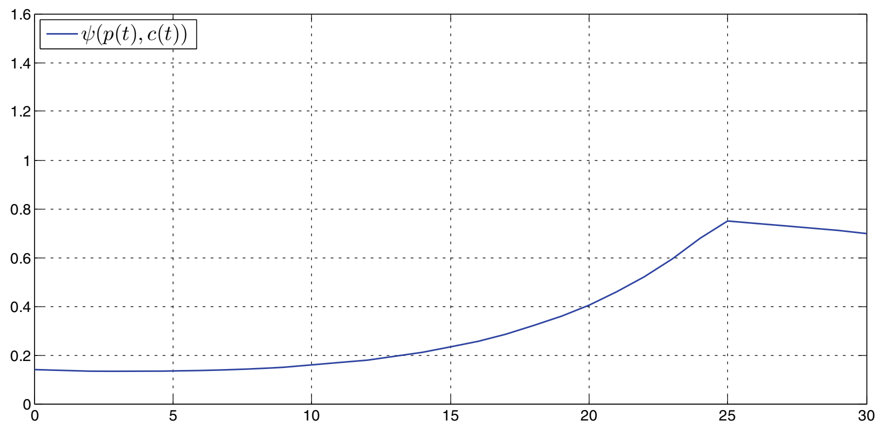


Fig. 4.15 Trajectory of demand for production factor in short-term strategy

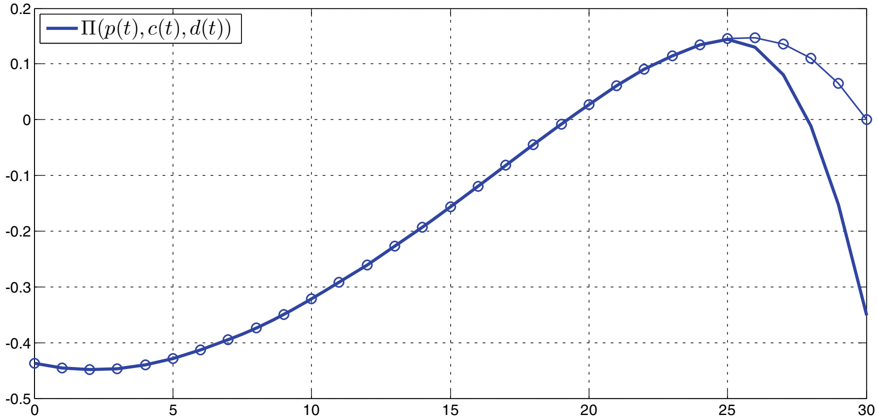


Fig. 4.16 Trajectory of firm's maximum profit in short-term strategy

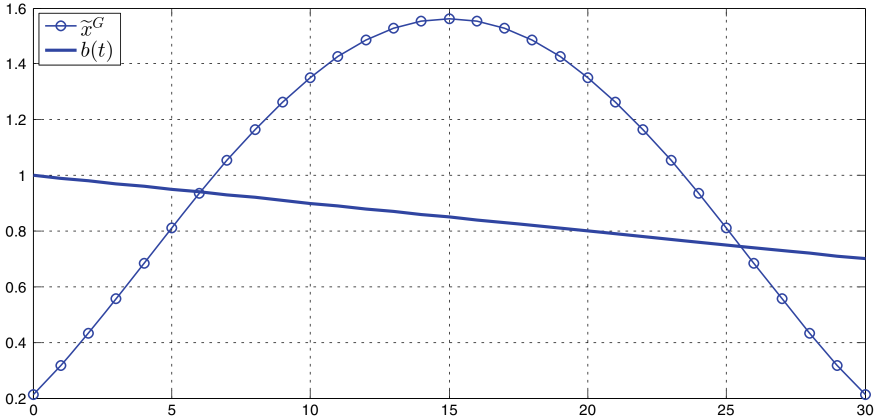


Fig. 4.17 Trajectories of resource and of optimal production factor input in long-term strategy—cost minimization problem

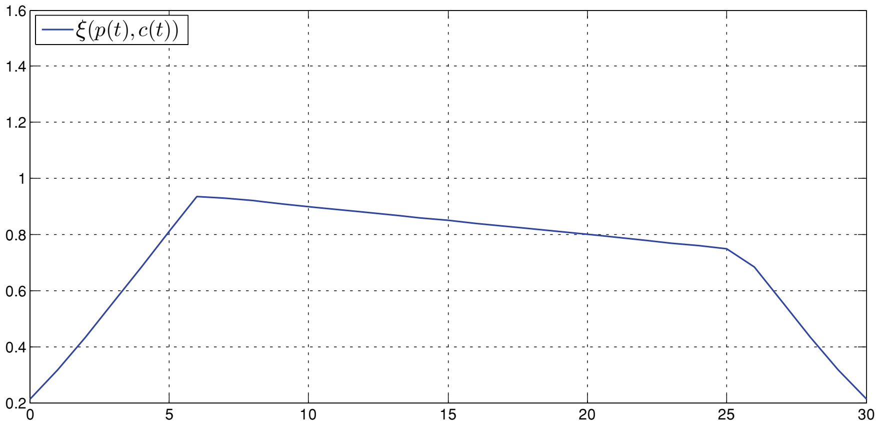


Fig. 4.18 Trajectory of conditional demand for production factor in short-term strategy

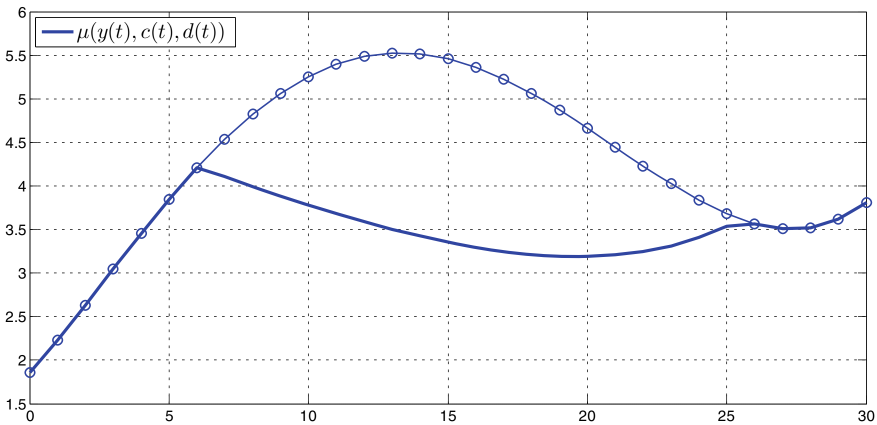


Fig. 4.19 Trajectory of minimum production cost in short-term strategy

Figure 4.19 presents a comparison of the firm minimum cost of producing $y(t)$ output units in the case when the production factor input is constrained by its resource and in the case when such limitation does not exist. Since the resource constraint is binding in period between moments $t \approx 7$ and $t \approx 26$ in this time interval one can observe a difference in the firm's minimum costs in the short-term and long-term strategies. Then, that is in this time interval, the resource constraint involves the usage of smaller production factor input than it results from the long-term strategy and hence also lower production cost than in the long-term strategy.

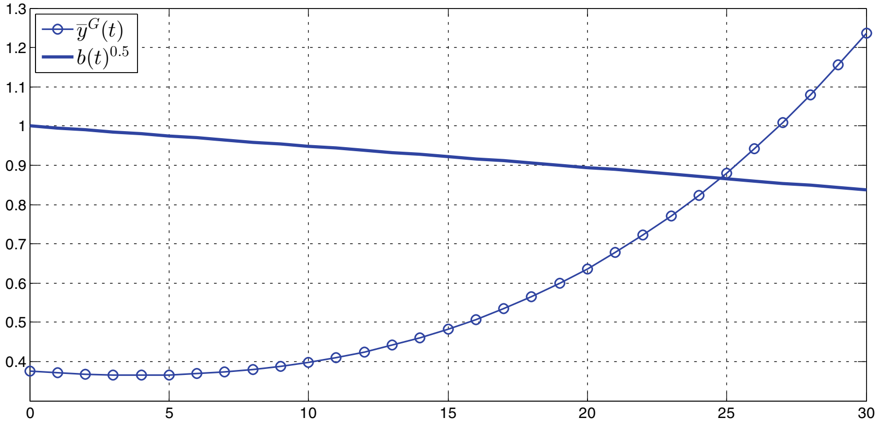


Fig. 4.20 Trajectories of output level constrained by production factor resource and of optimal output levelin long-term strategy—profit maximization problem with regard to output level

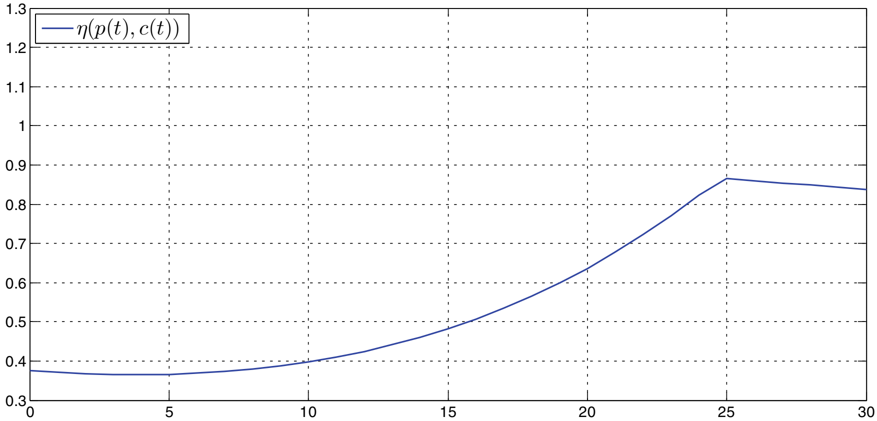


Fig. 4.21 Trajectory of optimal product supply in short-term strategy

Figure 4.20 presents a trajectory of output level constrained due to the limitation of the production factor resource and a trajectory of the optimal solution to the profit maximization problem with regard to output level in the long-term strategy. Until a moment $t \approx 25$ the rsource constraint and the output level constraint are not binding because output level $\overline{y}^G(t)$ does not exceed the constrained output level equal to $f(b(t)) = b(t)^{0,5}$. From the moment $t \approx 25$, the resource constraint is binding up to the end of the considered time horizon. Hence, a trajectory of the optimal output level resulting from the profit maximization problem in the short-term strategy takes the form as presented in Fig. 4.21.

The maximum profits resulting from the profit maximization problems with regard to output level and with regard to production factor input evolve in the same way, thus a trajectory of the former is the same as the one presented in Fig. 4.16.

4.6 Monopoly—Long-Term Strategy

4.6.1 Static Approach

Let us use the following notation:

$\mathbf{x} = (x_1, x_2) \geq (0, 0)$—a vector of inputs of production factors,
$y = f(x_1, x_2)$—an output level,
$p(y) > 0$, $\frac{\mathrm{d}p(y)}{\mathrm{d}y} < 0$—a price of a product manufactured by a monopoly as a decreasing function of product supply, set by a monopoly,
$\mathbf{c}(\mathbf{x}) = (c_1(x_1), c_2(x_2)) > (0, 0)$, $\frac{\mathrm{d}c_i(x_i)}{\mathrm{d}x_i} > 0, \quad i = 1, 2$—a vector of prices of production factors, each of whom is an increasing function of demand reported by a monopoly for a given production factor,
$r(y) = p(y)y$—revenue (turnover) from sales of a manufactured product as a function of product supply,
$r(x_1, x_2) = p(f(x_1, x_2))f(x_1, x_2)$—revenue (turnover) from sales of a manufactured product as a function of inputs of production factors,
$c^{tot}(x_1, x_2) = c_1(x_1)x_1 + c_2(x_2)x_2 + d$—total cost of production as a nonlinear function of inputs of production factors,
$c^v(x_1, x_2) = c_1(x_1)x_1 + c_2(x_2)x_2$—variable cost of production as a function of inputs of production factors,
$c^f(x_1, x_2) = d$—fixed cost of production,
$c(y)$—minimum cost of producing y output units, derived as an objective function corresponding to an optimal solution to problem (P2m),
$\pi(y) = r(y) - c(y) = p(y)y - c(y)$—firm's profit as a function of output level,
$\pi(x_1, x_2) = r(x_1, x_2) - c^{tot}(x_1, x_2)$—firm's profit as a function of inputs of production factors.

Problem of profit maximization with regard to inputs of production factors (P1m)

The aim of a monopoly is to maximize its profit expressed as a function of inputs of production factors, which can be written as a problem to solve in the following way:

$$\pi(x_1, x_2) = r(x_1, x_2) - c^{tot}(x_1, x_2)$$
$$= \{p(f(x_1, x_2)) f(x_1, x_2) - (c_1(x_1)x_1 + c_2(x_2)x_2 + d)\} \mapsto \max \tag{4.236}$$

$$x_1, x_2 \geq 0. \tag{4.237}$$

Since a production function from assumption (F2) is strictly concave then a revenue function is strictly concave too. At the same time, a production total cost function is strictly convex. As a result, a firm's profit function is strictly concave. Moreover, we are interested in an optimal solution $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2) > (0, 0)$ for which the profit $\pi(\overline{x}_1, \overline{x}_2)$ is the maximum.

Necessary and sufficient conditions for the existence of an optimal solution to problem (P1m) are given in the following theorem.

Theorem 4.16 If a firm's profit function is strictly concave and satisfies the following condition:

$$\forall i = 1,\ 2 \quad \lim_{x_i \to 0^+} \frac{\partial \pi(x_1, x_2)}{\partial x_i} > 0 \quad \wedge \lim_{x_i \to +\infty} \frac{\partial \pi(x_1, x_2)}{\partial x_i} < 0 \Leftrightarrow$$
$$\Leftrightarrow \lim_{x_i \to 0^+} \frac{\partial r(x_1, x_2)}{\partial x_i} > \lim_{x_i \to 0^+} \frac{\partial c^{tot}(x_1, x_2)}{\partial x_i} \tag{4.238}$$
$$\wedge \lim_{x_i \to +\infty} \frac{\partial r(x_1, x_2)}{\partial x_i} < \lim_{x_i \to +\infty} \frac{\partial c^{tot}(x_1, x_2)}{\partial x_i}$$

then:

(1) $\exists_1 \, \overline{\mathbf{x}} > 0$ such that $\left.\frac{\partial \pi(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = 0 \quad i = 1, 2,$

(2) a necessary and sufficient condition for $\overline{\mathbf{x}} > 0$ being an optimal solution to problem (P1m) is

$$\left.\frac{\partial \pi(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = 0 \quad \Leftrightarrow \left.\frac{\partial r(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = \left.\frac{\partial c^{tot}(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}}$$
$$\Leftrightarrow \left.\frac{\partial r(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = \left.\frac{\partial c^{v}(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} \quad i = 1, 2, \tag{4.239}$$

which means that there exists exactly one solution $\overline{\mathbf{x}} > 0$ for which:

- marginal profit equals zero,
- marginal revenue is equal to marginal total cost of production,
- marginal revenue is equal to marginal variable cost of production,

Definition 4.35 A **function of demand for production factors** is a mapping ψ : $\text{int}\,\mathbb{R}^2_+ \to \text{int}\,\mathbb{R}^2_+$ which assigns an optimal solution of problem (P1m) to any price $p(f(x_1, x_2))$ of a product and any price $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ in the following way:

$$\psi(p(f(\mathbf{x})), \mathbf{c}(\mathbf{x})) = \overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2). \tag{4.240}$$

Definition 4.36 A **monopoly maximal profit function** is a mapping Π : $\text{int}\,\mathbb{R}^2_+ \times \mathbb{R}_+ \to \text{int}\,\mathbb{R}_+$ which assigns maximum profit[23] to any price $p(f(x_1, x_2))$ of a product, any prices $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ of production factors and any fixed cost d in the following way:

$$\Pi(p(f(\mathbf{x})), \mathbf{c}(\mathbf{x}), d) = \pi(\overline{\mathbf{x}}). \tag{4.241}$$

Problem of cost minimization when producing the output at a fixed level (P2m)

The aim of a monopoly is to produce $y > 0$ units of output at minimum total cost, which can be written as a problem to solve in the following way:

$$c^{tot}(x_1, x_2) = \{c_1(x_1)x_1 + c_2(x_2)x_2 + d\} \mapsto \min \tag{4.242}$$

$$f(x_1, x_2) = y = \text{const.} > 0, \tag{4.243}$$

$$x_1, x_2 \geq 0. \tag{4.244}$$

One can express problem (P2m) using a Lagrange function:

$$F(x_1, x_2, \lambda) = \{c_1(x_1)x_1 + c_2(x_2)x_2 + d + \lambda(y - f(x_1, x_2))\} \mapsto \min. \tag{4.245}$$

Theorem 4.17 If a production function satisfies assumption (F2) then $\widetilde{\mathbf{x}} > 0$ is an optimal solution to problem (P2c) if and only if a pair $\left(\widetilde{\mathbf{x}}, \widetilde{\lambda}\right) > \mathbf{0}$ is a solution to the following system of equations:

[23] One should remember that if condition (4.238) is satisfied then the profit maximization problem (P1m) has a positive optimal solution which, depending on value of the fixed production cost d, corresponds to the positive, zero or negative maximum profit.

$$\left.\frac{\partial F(\mathbf{x}, \widetilde{\lambda})}{\partial x_1}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} = 0 \quad \Leftrightarrow \quad \left.\widetilde{\lambda}\frac{\partial f(x_1, x_2)}{\partial x_1}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} = \left.\frac{\mathrm{d}c_1(x_1)}{\mathrm{d}x_1}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} \tilde{x}_1 + c_1(\tilde{x}_1),$$

$$\left.\frac{\partial F(\mathbf{x}, \widetilde{\lambda})}{\partial x_2}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} = 0 \quad \Leftrightarrow \quad \left.\widetilde{\lambda}\frac{\partial f(x_1, x_2)}{\partial x_2}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} = \left.\frac{\mathrm{d}c_2(x_2)}{\mathrm{d}x_2}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} \tilde{x}_2 + c_2(\tilde{x}_2),$$

(4.246) $$\left.\frac{\partial F(\widetilde{\mathbf{x}}, \lambda)}{\partial \lambda}\right|_{\lambda=\widetilde{\lambda}} = 0 \quad \Leftrightarrow \quad f(\tilde{x}_1, \tilde{x}_2) = y.$$

Necessary condition: if $\widetilde{\mathbf{x}} > 0$ is an optimal solution to problem (P2m) then a pair $(\widetilde{\mathbf{x}}, \widetilde{\lambda}) > \mathbf{0}$ is a solution to equation system (3.246).

Sufficient condition: if a pair $(\widetilde{\mathbf{x}}, \widetilde{\lambda}) > \mathbf{0}$ is a solution to equation system (3.246) then $\widetilde{\mathbf{x}} > 0$ is an optimal solution to problem (P2m).

Definition 4.37 A **function of conditional demand for production factors** is a mapping $\xi : \text{int}\, \mathbb{R}^3_+ \to \text{int}\, \mathbb{R}^2_+$ which assigns an optimal solution of problem (P2m) to any output level y and any prices $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ factors in the following way:

(4.247) $$\xi(\mathbf{c}(\mathbf{x}), y) = \widetilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2).$$

Definition 4.38 A **firm minimal cost function** is a mapping $\mu : \text{int}\, \mathbb{R}^4_+ \to \text{int}\, \mathbb{R}_+$ which assigns minimum cost of producing y output units to any output level y, any prices $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ of production factors and any fixed cost d in the following way:

(4.248) $$\mu(\mathbf{c}(\mathbf{x}), d, y) = c^{tot}(\widetilde{\mathbf{x}}) = c_1(\tilde{x}_1)\tilde{x}_1 + c_2(\tilde{x}_2)\tilde{x}_2 + d.$$

If prices of production factors are determined and the fixed cost of production is known then one can express the firm's minimal cost function of producing y output units as a function of output level:

(4.249) $$\mu(\mathbf{c}(\mathbf{x}), d, y) = c(y).$$

Problem of profit maximization with regard to output level (P3m)

The aim of a monopoly is to maximize its profit expressed as a function of output level, which can be written as a problem to solve in the following way:

(4.250) $$\pi(y) = r(y) - c(y) = \{p(y)y - c(y)\} \mapsto \max$$

(4.251) $$y \geq 0.$$

Since a revenue function is strictly concave while a firm's minimal cost function of producing y output units is strictly convex then a firm's profit function is strictly concave. Moreover, we are interested in an optimal solution $\overline{y} > 0$.

Necessary and sufficient conditions for the existence of an optimal solution to problem (P3m) are given in the following theorem.

Theorem 4.18 If a firm's profit function is strictly concave, differentiable and the following condition is satisfied:

$$\lim_{y\to 0^+} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} > 0 \quad \wedge \lim_{y\to +\infty} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} < 0$$

$$\Leftrightarrow \lim_{y\to 0^+} \frac{\mathrm{d}r(y)}{\mathrm{d}y} > \lim_{y\to 0^+} \frac{\mathrm{d}c(y)}{\mathrm{d}y} \wedge \lim_{y\to +\infty} \frac{\mathrm{d}r(y)}{\mathrm{d}y} < \lim_{y\to +\infty} \frac{\mathrm{d}c(y)}{\mathrm{d}y} \tag{4.252}$$

then:

(1) $\exists_1 \overline{y} > 0$ such that $\left.\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}\right|_{y=\overline{y}} = 0$,

(2) a necessary and sufficient condition for $\overline{y} > 0$ being an optimal solution to problem (P3m) is

$$\left.\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}\right|_{y=\overline{y}} = 0 \quad \Leftrightarrow \quad \left.\frac{\mathrm{d}r(y)}{\mathrm{d}y}\right|_{y=\overline{y}} = \left.\frac{\mathrm{d}c(y)}{\mathrm{d}y}\right|_{y=\overline{y}} \tag{4.253}$$

which means that there exists exactly one solution $\overline{y} > 0$ for which:

- marginal profit equals zero,
- marginal revenue is equal to marginal minimal cost of producing y output units,
- price of a product is equal to marginal minimal cost of producing y output units.

Definition 4.39 A **function of product supply** is a mapping $\eta : \mathrm{int}\, \mathbb{R}^2_+ \to \mathrm{int}\, \mathbb{R}_+$ which assigns an optimal solution of problem (P3m) to any price $p(f(x_1, x_2))$ of a product and any price $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ of production factors in a following way:

$$\eta(p(f(\mathbf{x})), \mathbf{c}(\mathbf{x})) = \overline{y}. \tag{4.254}$$

Definition 4.40 A **monopoly maximal profit function** is a mapping $\Pi : \mathrm{int}\, \mathbb{R}^3_+ \to \mathrm{int}\, \mathbb{R}_+$ which assigns maximum profit[24] to any price $p(f(x_1, x_2)$ of a product, any

[24] As in the problem (P1m) the fact that there exists the optimal positive supply guaranteeing the monopoly maximum profit does not mean that this profit is positive. This depends on the fixed production cost d. If the fixed cost is big enough the maximum profit can be negative. Then this maximum profit can be seen as the minimum loss that a monopoly incurs.

price $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ of production factors and any fixed cost d in the following way:

$$\Pi(p(f(\mathbf{x})), \mathbf{c}(\mathbf{x}), d) = \pi(\overline{y}). \tag{4.255}$$

Definition 4.41 A **monopoly optimal price function** is a mapping $\overline{p} : \text{int } \mathbb{R}^2_+ \to \text{int } \mathbb{R}_+$ which assigns the price guaranteeing the maximum profit to any production function $f(\mathbf{x})$ and any price $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ of production factors in the following way:

$$\overline{p}(f(\mathbf{x}), \mathbf{c}(\mathbf{x})) = p(\overline{y}). \tag{4.256}$$

Theorem 4.19 If assumptions of Theorem 4.16 are satisfied then problems (P1m) and (P3m) are equivalent.

This means that:

- knowing an optimal solution to problem (P1m) one can determine an optimal solution to problem (P3m): $\overline{y} = f(\overline{\mathbf{x}})$,
- knowing an optimal solution to problems (P3m) and (P2m) one can determine an optimal solution to problem (P1m): $\widetilde{\mathbf{x}} = \xi(\mathbf{c}(\widetilde{\mathbf{x}}), \overline{y}) = \psi(p(f(\widetilde{\mathbf{x}})), \mathbf{c}(\widetilde{\mathbf{x}})) = \overline{\mathbf{x}}$,
- $\pi(\overline{\mathbf{x}}) = \Pi(p(f(\mathbf{x})), \mathbf{c}(\mathbf{x}), d) = \pi(\overline{y})$.

Example 4.5 The following data is given:

$x \geq 0$—an input of a production factor,
$y = f(x) = ax^{\frac{1}{2}}$—an output level as a nonlinear function of a production factor input,
$p(y) = \left(\frac{a}{y}\right)^{\frac{1}{2}}$—a price of a product manufactured by a monopoly,
$c_1(x) = ax$—a price of a production factor as a linear function of demand reported by a monopoly for this factor,
$r(y) = p(y)y = a^{\frac{1}{2}}y^{\frac{1}{2}}$—revenue (turnover) from sales of a manufactured product as a nonlinear function of output level,
$r(x) = p(f(x))f(x) = x^{-\frac{1}{4}}ax^{\frac{1}{2}} = ax^{\frac{1}{4}}$—revenue (turnover) from sales of a manufactured product as a nonlinear function of a production factor input,
$c^{tot}(x) = c_1(x)x + d = ax^2 + d$—total cost of production as a nonlinear function of a production factor input,
$c^v(x) = ax^2$—variable cost of production,
$c^f(x) = d \geq 0$—fixed cost of production,
$c(y)$—minimum cost of producing y output units, derived as an objective function corresponding to an optimal solution to problem (P2m),
$\pi(y) = r(y) - c(y)$—firm's profit as a function of output level,
$\pi(x) = r(x) - c^{tot}(x) = ax^{\frac{1}{4}} - ax^2 - d$—firm's profit as a function of a production factor input.

Tasks

1. Solve the profit maximization problem (P1m).
2. Present a geometric illustration of the profit maximization problem (P1m).
3. Give an economic interpretation of necessary and sufficient conditions of the existence of an optimal solution to problem (P1m).
4. Solve the cost minimization problem (P2m).
5. Present a geometric illustration of the cost minimization problem (P2m).
6. Give an economic interpretation of necessary and sufficient conditions of the existence of an optimal solution to problem (P2m).
7. Solve the profit maximization problem (P3m).
8. Present a geometric illustration of the profit maximization problem (P3m).
9. Give an economic interpretation of necessary and sufficient conditions of the existence of an optimal solution to problem (P3m).
10. Justify that the profit maximizations problems (P1m) and (P3m) are equivalent.
11. Determine the optimal price by which a monopoly obtains the maximum profit.

Ad 1 The profit maximization problem (P1m) has a form:

(4.257) $$\pi(x) = \left\{ax^{\frac{1}{4}} - \left(ax^2 + d\right)\right\} \mapsto \max$$

(4.258) $$x \geq 0.$$

Since the production function from assumption (F2) is strictly concave then a revenue function is strictly concave too. At the same time, a production total cost function is strictly convex. As a result, a monopoly profit function is strictly concave.

It is known that when a profit function is strictly concave then problem (P1m) can have:

- no optimal solution when revenue from sales of a product is lower than the total cost of production,
- exactly one optimal solution $\overline{x} = 0$ which, due to the positive fixed cost of production, corresponds to a loss equal to the fixed cost,
- exactly one optimal solution $\overline{x} > 0$ which, with the sufficiently low fixed cost of production, corresponds to the positive profit.

A condition ensuring the existence of a unique and positive optimal solution to problem (P1m) has a form:

$$\lim_{x\to 0^+} \frac{\mathrm{d}\pi(x)}{\mathrm{d}x} > 0 \quad \wedge \lim_{x\to +\infty} \frac{\mathrm{d}\pi(x)}{\mathrm{d}x} < 0$$

(4.259) $$\Leftrightarrow \lim_{x\to 0^+} \frac{\mathrm{d}r(x)}{\mathrm{d}x} > \lim_{x\to 0^+} \frac{\mathrm{d}c^{tot}(x)}{\mathrm{d}x} \wedge \lim_{x\to +\infty} \frac{\mathrm{d}r(x)}{\mathrm{d}x} < \lim_{x\to +\infty} \frac{\mathrm{d}c^{tot}(x)}{\mathrm{d}x},$$

which means that from the strict concavity of the monopoly profit function it results that by a relatively big production factor input the marginal revenue is lower than the marginal production cost, while by a relatively small production factor input the marginal revenue is higher than the marginal production cost.

Let us determine a marginal profit function in problem (P1m) and check if it satisfies condition (4.259):

$$\frac{d\pi(x)}{dx} = \frac{dr(x)}{dx} - \frac{dc^{tot}(x)}{dx} = \frac{1}{4}ax^{-\frac{3}{4}} - 2ax. \tag{4.260}$$

Let us notice that:

$$\lim_{x\to+\infty} \frac{d\pi(x)}{dx} = \lim_{x\to+\infty}\left(\frac{1}{4}ax^{-\frac{3}{4}} - 2ax\right) = -\infty < 0 \tag{4.261}$$

and

$$\lim_{x\to 0^+} \frac{d\pi(x)}{dx} = \lim_{x\to 0^+}\left(\frac{1}{4}ax^{-\frac{3}{4}} - 2ax\right) = +\infty > 0. \tag{4.262}$$

Since condition (4.259) is satisfied then we can determine an optimal solution to problem (P1m) from the following equation:

$$\exists_1 \overline{x} > 0 \quad \left.\frac{d\pi(x)}{dx}\right|_{x=\overline{x}} = 0, \tag{4.263}$$

which means that there exists a production factor input such that the marginal revenue is equal to marginal production cost.

Hence:

$$\left.\frac{d\pi(x)}{dx}\right|_{x=\overline{x}} = \frac{1}{4}a\overline{x}^{-\frac{3}{4}} - 2a\overline{x} = 0, \tag{4.264}$$

and after some transformations, we get

$$\overline{x} = 2^{-\frac{12}{7}} > 0. \tag{4.265}$$

Let us substitute the optimal solution obtained above into the profit function:

$$\pi(\overline{x}) = a\overline{x}^{\frac{1}{4}} - (a\overline{x} + d). \tag{4.266}$$

After transformations, we get

$$\pi(\overline{x}) = 7a2^{-\frac{24}{7}} - d. \tag{4.267}$$

If the fixed cost satisfies a condition $0 \le d < 7a2^{-\frac{24}{7}}$ then the maximum profit that a monopoly can obtain is positive.

Ad 2 It is worth noticing that inputs denoted in Fig. 4.22a as $\hat{x}^j$, $j = 1, 2, 3, 4$ are **break-even points (profitability thresholds)** of a monopoly determined by level of the production total cost, depending especially on the fixed production cost. They correspond to graphs of the cost functions (1) and (2) when the fixed cost is sufficiently low to ensure positive profit for some inputs of a production factor. A condition to obtain the positive profit in both these cases is that the production factor input passes the first ($\hat{x}^1$ or $\hat{x}^2$) break-even point and does not pass the second ($\hat{x}^3$ or $\hat{x}^4$) break-even point. In cases when graphs of the cost functions are curves (3) or (4) there is no break-even point and the maximum profit of a monopoly is zero or negative and equal to minimal loss incurred by a monopoly performing production activity.

Ad 3 On the basis of Figs. 4.22a, 4.22b and 4.22c, one can state that a necessary and sufficient condition for the existence of the optimal solution $\bar{x} > 0$ to problem (P1m) is satsifed when for the optimal production factor input the marginal profit

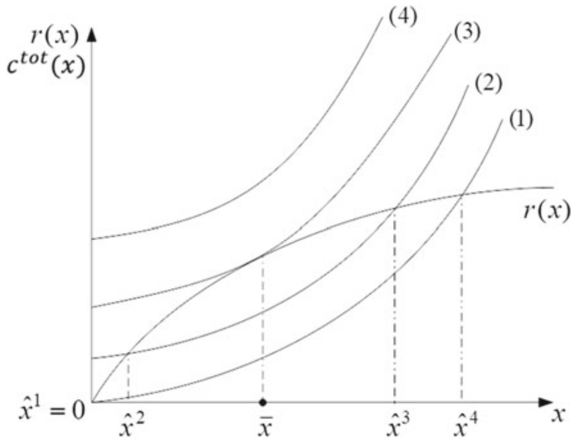


Fig. 4.22a Graphs of revenue function and production total cost function

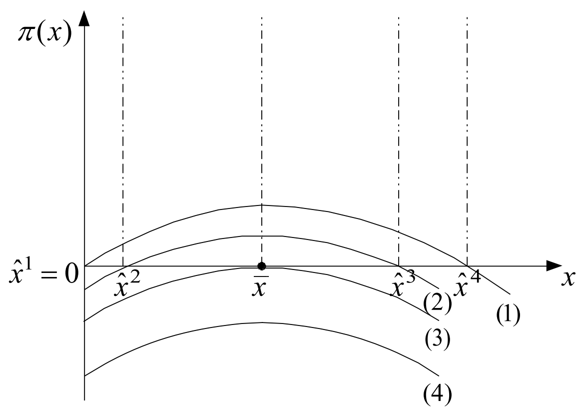


Fig. 4.22b Graphs of profit function

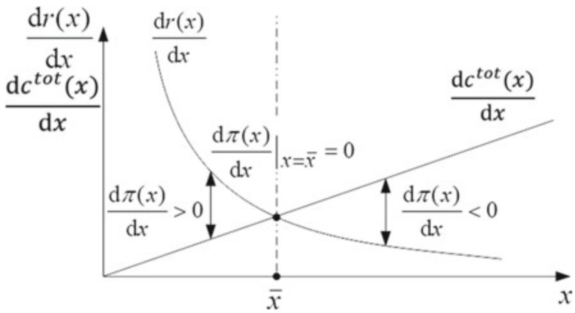


Fig. 4.22c Graphs of marginal revenue function and marginal production cost function

equals 0 or equivalently when the marginal revenue from sales of a product equals the marginal production cost.

Let us also determine the value of a second derivative of the profit function using the optimal solution to problem (P1m) as its argument:

(4.268) $$\left.\frac{d^2\pi(x)}{dx^2}\right|_{x=\overline{x}} = -\frac{3}{16}a\overline{x}^{-\frac{7}{4}} - 2a < 0, \text{ since } \overline{x} = 2^{-\frac{12}{7}} > 0.$$

Hence, we can see that using the optimal input $\overline{x} = 2^{-\frac{12}{7}}$ of a production factor a monopoly obtains the maximum profit. From the analysis which is conducted above it results that since the profit function is strictly concave then condition (4.263) is necessary and sufficient for the existence of the optimal solution to problem (P1m). Condition (4.259) in turn ensures that $\overline{x} > 0$.

Ad 4 The cost minimization problem (P2m) when producing y output units has a form:

(4.269) $$c^{tot}(x) = \left(ax^2 + d\right) \mapsto \min$$

(4.270) $$ax^{\frac{1}{2}} = y = \text{const.},$$

(4.271) $$x \geq 0.$$

Since a set of feasible solutions to this problem has only one element, then a production factor input resulting from (4.270) is the optimal solution to this problem:

(4.272) $$\widetilde{x} = \left(\frac{y}{a}\right)^2,$$

and is positive by the positive output level.

A monopoly minimal cost function of producing y output units corresponds to this solution:

(4.273) $$c^{tot}(\widetilde{x}) = a\left(\frac{y}{a}\right)^4 + d = a^{-3}y^4 + d = c(y),$$

and is nonlinear and strictly convex function of the output level.

Ad 5 See Figs. 4.23a and 4.23b.

Ad 6 In problem (P2m) exactly one production factor input corresponds to exactly one fixed output level. This production factor input is at the same time the only one

Fig. 4.23a Illustration of problem (P2m)

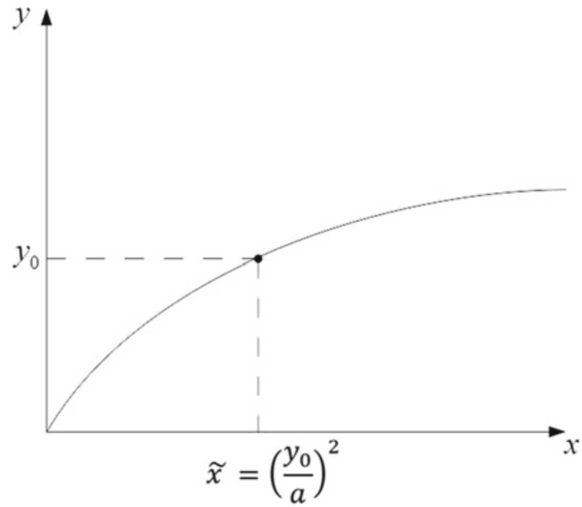
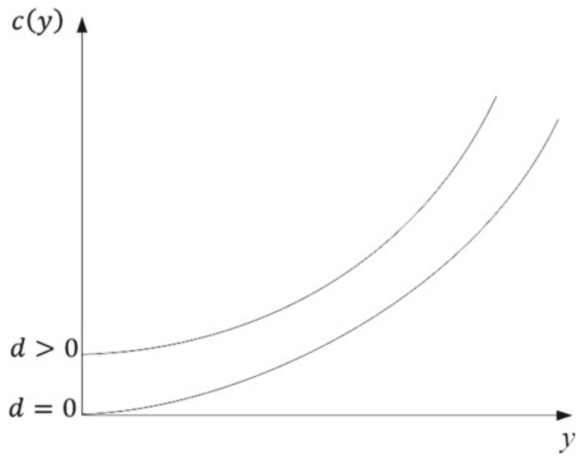


Fig. 4.23b Graphs of firm minimal cost function of producing y output units



solution to problem (P2m). As a consequence, a set of feasible solutions has only one element. In this case, independently of an optimality criterion, the only one feasible solution to the problem is at the same time its only one optimal solution.

Ad 7 The profit maximization problem (P3c) has a form:

$$\pi(y) = r(y) - c(y) = \left\{a^{\frac{1}{2}} y^{\frac{1}{2}} - \left(a^{-3} y^4 + d\right)\right\} \mapsto \max, \tag{4.274}$$

$$y \geq 0. \tag{4.275}$$

The revenue function is nonlinear and strictly concave. The monopoly minimal cost function of producing y output units is nonlinear and strictly convex. Thus, the profit function is a strictly concave function of the output level.

It is known that when a profit function is strictly concave then problem (P3m) can have:

- no optimal solution when revenue from sales of a product is lower than the firm minimum cost of producing y output units,
- exactly one optimal solution $\overline{y} = 0$ which, due to the positive fixed cost of production, corresponds to a loss equal to the fixed cost,
- exactly one optimal solution $\overline{y} > 0$ which, by the sufficiently low fixed cost of production, corresponds to the positive profit.

A condition ensuring the existence of a unique and positive optimal solution to problem (P3m) has a form:

$$\begin{aligned} &\lim_{y \to 0^+} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} > 0 \quad \wedge \lim_{y \to +\infty} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} < 0 \Leftrightarrow \\ &\Leftrightarrow \lim_{y \to 0^+} \frac{\mathrm{d}r(y)}{\mathrm{d}y} > \lim_{y \to 0^+} \frac{\mathrm{d}c(y)}{\mathrm{d}y} \wedge \lim_{y \to +\infty} \frac{\mathrm{d}r(y)}{\mathrm{d}y} < \lim_{y \to +\infty} \frac{\mathrm{d}c(y)}{\mathrm{d}y}, \end{aligned} \tag{4.276}$$

which means that from the strict concavity of the firm's profit function it results that by a relatively small output level the marginal revenue is higher than the marginal production cost, while by a relatively big output level the marginal revenue is lower than the marginal production cost.

Let us determine a marginal profit function in problem (P3m) and check if it satisfies condition (4.276):

$$\frac{\mathrm{d}\pi(y)}{\mathrm{d}y} = \frac{\mathrm{d}r(y)}{\mathrm{d}y} - \frac{\mathrm{d}c(y)}{\mathrm{d}y} = \frac{1}{2} a^{\frac{1}{2}} y^{-\frac{1}{2}} - 4a^{-3} y^3. \tag{4.277}$$

Let us notice that:

(4.278) $$\lim_{y\to 0^+} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} = \lim_{y\to 0^+}\left(\frac{1}{2}a^{\frac{1}{2}}y^{-\frac{1}{2}} - 4a^{-3}y^3\right) = +\infty > 0$$

and

(4.279) $$\lim_{y\to +\infty} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} = \lim_{y\to +\infty}\left(\frac{1}{2}a^{\frac{1}{2}}y^{-\frac{1}{2}} - 4a^{-3}y^3\right) = -\infty < 0.$$

Since condition (4.276) is satisfied then we can determine an optimal solution to problem (P3m) from the following equation:

(4.280) $$\exists_1 \overline{y} > 0 \quad \left.\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}\right|_{y=\overline{y}} = 0,$$

which means that there exists an output level such that the marginal revenue from sales of a product is equal to the marginal minimal cost of producing y output units.

Hence:

(4.281) $$\left.\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}\right|_{y=\overline{y}} = \frac{1}{2}a^{\frac{1}{2}}y^{-\frac{1}{2}} - 4a^{-3}y^3 = 0$$

and after some transformations, we get

(4.282) $$\overline{y} = a2^{-\frac{6}{7}} > 0.$$

Let us substitute the optimal solution to problem (P3m) obtained above into the profit function:

(4.283) $$\pi(\overline{y}) = p(\overline{y})\overline{y} - c(\overline{y}) = a\left(2^{-\frac{3}{7}} - 2^{-\frac{24}{7}}\right) - d.$$

After transformations, we get

(4.284) $$\pi(\overline{y}) = 7a2^{-\frac{24}{7}} - d.$$

If the fixed cost satisfies a condition $0 \le d < 7a2^{-\frac{24}{7}}$ then the maximum profit that a firm can obtain is positive.

Ad 8 See Figs. 4.24a, 4.24b and 4.24c.

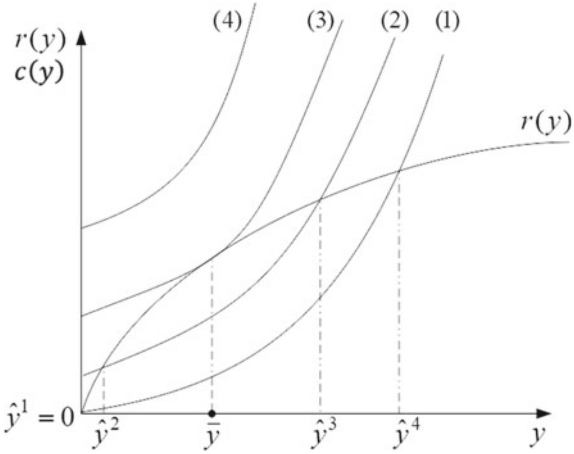


Fig. 4.24a Graphs of revenue function and firm minimal cost function of producing y output units

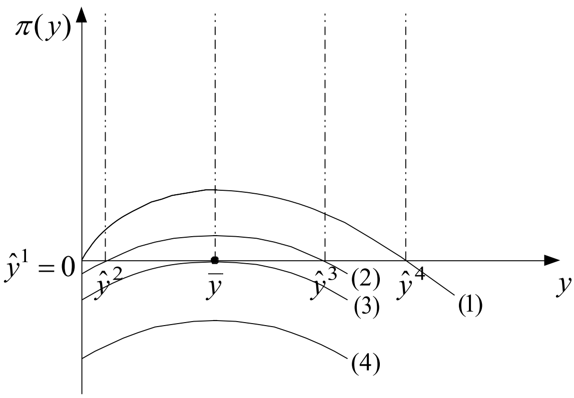


Fig. 4.24b Graphs of profit function

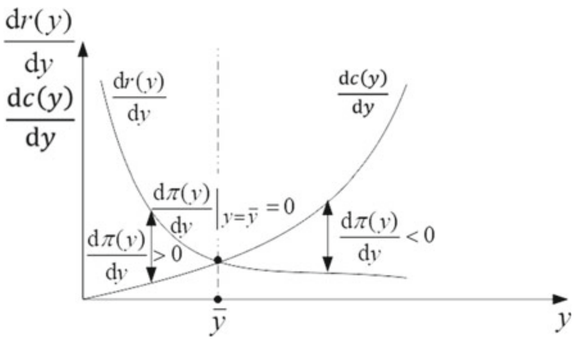


Fig. 4.24c Graphs of marginal revenue function and marginal minimal cost function of producing y output units

Ad 9 On the basis of Figs. 4.24a, 4.24b and 4.24c one can state that a necessary and sufficient condition for the existence of the optimal solution $\overline{y} > 0$ to problem (P3m) is satisfied when for the optimal output level the marginal profit equals 0 or equivalently when the marginal revenue from sales of a product equals the marginal minimal production cost of producing y output units.

Let us also determine a value of a second derivative of the profit function using the optimal solution to problem (P3m) as its argument: $\frac{1}{2}a^{\frac{1}{2}}y^{-\frac{1}{2}} - 4a^{-3}y^3$

(4.285) $$\left.\frac{d^2\pi(y)}{dy^2}\right|_{y=\overline{y}} = -\frac{1}{4}a^{\frac{1}{2}}\overline{y}^{-\frac{3}{2}} - 12a^{-3}\overline{y}^2 < 0, \text{ since } \overline{y} = a2^{-\frac{6}{7}} > 0.$$

Hence we can see that having the optimal output level $\overline{y} = a2^{-\frac{6}{7}}$ a monopoly obtains the maximum profit. From the analysis we have conducted above, it results that since the profit function is strictly concave then condition (4.280) is necessary and sufficient for existence of the optimal solution to problem (P3m). Condition (4.276) in turn ensures that $\overline{y} > 0$.

Ad 10 To show that problems (P1m) and (P3m) are equivalent let us notice that:

(1) For $\overline{x} = 2^{-\frac{12}{7}} > 0$ and $\overline{y} = a2^{-\frac{6}{7}} > 0$, we have

(4.286) $$\pi(\overline{x}) = 7a2^{-\frac{24}{7}} - d = \pi(\overline{y}).$$

(2) Knowing the optimal solution to problem (P1m) and substituting it into the production function, we get the optimal solution to problem (P3m):

(4.287) $$\overline{y} = f(\overline{x}) = a\overline{x}^{\frac{1}{2}} = a2^{-\frac{6}{7}}.$$

(3) Knowing the optimal solution to problem (P3m) and substituting it into the optimal solution to problem (P2m), we get the optimal solution to problem (P1m):

(4.288) $$\widetilde{x} = \left(\frac{\overline{y}}{a}\right)^2 = \left(2^{-\frac{6}{7}}\right)^2 = 2^{-\frac{12}{7}} = \overline{x},$$

which means that profit maximization problems (P1m) and (P3m) are equivalent.

Ad 11 The price of a product manufactured by a monopoly is assumed to have a form: $p(y) = \left(\frac{a}{y}\right)^{\frac{1}{2}}$. Thus, the optimal price by which a monopoly can obtain the maximum profit is

(4.289) $$p(\overline{y}) = \left(\frac{a}{\overline{y}}\right)^{\frac{1}{2}} = 2^{\frac{3}{7}} > 0.$$

4.6.2 Dynamic Approach

A monopoly, as the only one company manufacturing a given product, sets its price itself. Hence the product price depends on the monopoly supply of a product. Since a monopoly reports big demand for production factors it influences also their prices set by suppliers of production factors. In the static approach presented in Sect. 4.5.2, we assume that a price of a product is a decreasing function of product supply (output level) and a price of each production factor is an increasing function of production factor input. But these functions are time invariant in the static approach. Now, in the dynamic approach, we assume that forms of these functions can change over time which reflects the fact that a monopoly takes into account changes in demand for its product and suppliers of production factors can change a way they set prices of their products. Let us use the following notation:

t—time as discrete ($t = 0, 1, 2, \ldots, T$) or as continuous[25] variable ($t \in [0; T]$),
T—end of the time horizon,
$p(y(t)) > 0$—a time-variant price of a product manufactured by a firm as a decreasing function of product supply,
$\mathbf{x}(t) = (x_1(t), x_2(t)) \geq \mathbf{0}$—a vector of inputs of production factors that a monopolist uses in the production process in period/at moment t,
$\mathbf{c}((\mathbf{x}(t)) = (c_1(x_1(t)), c_2(x_2(t))) > \mathbf{0}$—a vector of time-variant prices of production factors, each of whom is an increasing function of production factor input,
$y = f(\mathbf{x}(t))$ —a production function,
$d(t) \geq 0$—time-variant fixed cost of production, that is, the cost not depending on the output level nor on inputs of production factors.

The monopoly profit maximization problem with regard to inputs of production factors has a form:

$$\begin{aligned} \pi(\mathbf{x}(t)) &= r(\mathbf{x}(t)) - c^{tot}(\mathbf{x}(t)) \\ &= \{p(f(\mathbf{x}(t))) \cdot f(\mathbf{x}(t)) - (c_1(x_1(t)) \cdot x_1(t) + c_2(x_2(t)) \cdot x_2(t) + d(t))\} \\ &\mapsto \max \end{aligned} \tag{4.290}$$

$$\mathbf{x}(t) \geq \mathbf{0}. \tag{4.291}$$

The production function is assumed to be strictly concave and increasing with respect to inputs of production factors. The function of production total cost is strictly convex and increasing with respect to inputs of production factors. As a consequence, the profit function $\pi(\mathbf{x}(t))$ is strictly concave and in every period/at

[25] For the discrete and continuous versions, we use the same denotation of the dependence of the function value on time, for example, the fixed production cost on time: $d(t)$. Whether the discrete or continuous version is used in a given formula will result from the context of the issue under consideration.

any moment t Problem (4.290)–(4.291) has a solution $\bar{\mathbf{x}}(t) > (0, 0)$. The necessary condition for the existence of maximum profit is

$$\left.\frac{\partial \pi(\mathbf{x}(t))}{\partial x_i(t)}\right|_{\mathbf{x}(t)=\bar{\mathbf{x}}(t)} = 0 \Leftrightarrow \left.\frac{\partial r(\mathbf{x}(t))}{\partial x_i(t)}\right|_{\mathbf{x}(t)=\bar{\mathbf{x}}(t)} = \left.\frac{\partial c^{tot}(\mathbf{x}(t))}{\partial x_i(t)}\right|_{\mathbf{x}(t)=\bar{\mathbf{x}}(t)} \quad i = 1, 2, \quad \forall t, \tag{4.292}$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0; T]$.

From the profit maximization problem, one gets a vector $\bar{\mathbf{x}}(t)$ optimal inputs of production factors which change over time due to the fact that a form of price product function and a vector of prices of production factors change over time.

In the dynamic approach, the output level fixed by a monopoly as desired to produce is also time variant. Hence the production cost minimization problem for a monopoly has a form:

$$c^{tot}(\mathbf{x}(t)) = \{c_1(x_1(t)) \cdot x_1(t) + c_2(x_2(t)) \cdot x_2(t) + d(t)\} \mapsto \min \tag{4.293}$$

$$f(\mathbf{x}(t)) = y(t) \tag{4.294}$$

$$\mathbf{x}(t) \geq \mathbf{0}. \tag{4.295}$$

In the case of a monopoly, the optimality condition for a vector of inputs of production factors takes the form[26]:

$$\begin{aligned} \sigma_{12}(\tilde{\mathbf{x}}(t)) &= \left.\frac{\partial f(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}(t)=\tilde{\mathbf{x}}(t)} : \left.\frac{\partial f(\mathbf{x})}{\partial x_2}\right|_{\mathbf{x}(t)=\tilde{\mathbf{x}}(t)} \\ &= \frac{\tilde{x}_1(t) \cdot \left.\frac{\mathrm{d}c_1(x_1(t))}{\mathrm{d}x_1(t)}\right|_{\mathbf{x}(t)=\tilde{\mathbf{x}}(t)} + c_1(\tilde{x}_1(t))}{\tilde{x}_2(t) \cdot \left.\frac{\mathrm{d}c_2(x_2(t))}{\mathrm{d}x_2(t)}\right|_{\mathbf{x}(t)=\tilde{\mathbf{x}}(t)} + c_2(\tilde{x}_2(t))} \quad \forall t, \end{aligned} \tag{4.296}$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0; T]$ and $\sigma_{12}(\tilde{\mathbf{x}}(t))$ means the marginal rate of substitution of first production factor by the second production factor in a vector of optimal inputs of production factors. The vector of optimal inputs is a solution to a system of Eqs. (4.294) and (4.296).

From the problem of production cost minimization, one gets a vector $\tilde{\mathbf{x}}(t)$ of optimal inputs of production factors which change over time due to the fact that a vector of prices of production factors, the fixed cost and the fixed output level $y(t)$ change over time. The optimal solution depends on value of $y(t)$ which is time

[26] This condition results from the method of solving the problem of conditional minimization problem for a production total cost function. In the method, the necessary condition of minimum existence takes the form of a system of equations presenting partial derivatives of a Lagrange function. The method is presented in Sect. 4.6.1.

variant in the dynamic approach. Substituting this solution into the production total cost function, we get a monopoly minimal cost function of producing $y(t)$ output units, depending on value of $y(t)$:

$$\begin{aligned} \min c^{tot}(\mathbf{x}(t)) &= c^{tot}(\widetilde{\mathbf{x}}(t)) \\ &= c_1(\tilde{x}_1(t)) \cdot \tilde{x}_1(t) + c_2(\tilde{x}_2(t)) \cdot \tilde{x}_2(t) + d(t) = c(y(t)). \end{aligned} \tag{4.297}$$

The profit maximization problem with regard to the output level for a monopoly has a form:

$$\pi(y(t)) = r(y(t)) - c(y(t)) = \{p(y(t)) \cdot y(t) - c(y(t))\} \mapsto \max \tag{4.298}$$

$$y(t) \geq 0. \tag{4.299}$$

The revenue function is strictly concave and increasing with respect to the output level. The firm minimal cost function of producing $y(t)$ output units is strictly convex and increasing with respect to the output level. As a consequence, the profit function $\pi(y(t))$ is strictly concave and in every period/at any moment t Problem (4.298)–(4.299) has a solution $\overline{y}(t) > 0$. The necessary condition for the existence of maximum profit is

$$\left.\frac{\mathrm{d}\pi(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\overline{y}(t)} = 0 \Leftrightarrow \left.\frac{\mathrm{d}r(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\overline{y}(t)} = \left.\frac{\mathrm{d}c(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\overline{y}(t)} \quad \forall t, \tag{4.300}$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0; T]$.

From the profit maximization problem, one gets the optimal output level $\overline{y}(t)$ which changes over time due to the fact that a form of price product function and a vector of prices of production factors change over time. The optimal price of a product manufactured by a monopoly is $p(\overline{y}(t))$ and is time variant in the dynamic approach.

Example 4.6 A production process in a firm acting as a monopoly is described by a one-variable production function of a form[27]:

$$f(x(t)) = x(t)^{0.5}.$$

[27] One can find analogies of this example to Example 4.5 with a monopoly in the static approach and to Example 4.2 with the dynamic approach and a firm acting in perfect competition.

A price of a product manufactured by this monopoly changes according to a function of a form:

$$p(y(t)) = \left(\frac{a(t)}{y(t)}\right)^{0.5}, \text{ where } a(t) > 0 \quad \forall t,$$

and a production factor price changes in the following way:

$$c(x(t)) = C(t)x(t), \text{ where } C(t) > 0 \quad \forall t.$$

One can notice that a product price decreases when the product supply increases. How fast it decreases depends on a value of $a(t)$. Hence, this value reflects changes that occur in the demand reported by consumers for a product manufactured by a monopoly. At the same time, the price of a production factor set by its supplier increases when the demand reported by a monopoly for this production factor increases. How fast it increases depends on a value of $C(t)$. Hence, this value reflects changes that occur in the way the supplier of a production factor reacts to the demand reported by a monopoly.

Let us assume that at any moment $t \in [0; 30]$, a value of $a(t)$, a value of $C(t)$ and the fixed production cost change according to equations:

$$a(t) = 2^{0.1t} + 5,$$

$$C(t) = 2^{-0.1t},$$

$$d(t) = \frac{\left(0.006t^2 - 0.1t + 3\right)^2 t}{480 \cdot 0.98^t} - \frac{t}{30} + 1.$$

Trajectories of these values are presented in Fig. 4.25. The way the values of $a(t)$ and $C(t)$ change influences the results of optimization problems, that is, the production factor optimal input, the product optimal supply as well as the product optimal price.

For the data given in this example, the monopoly profit maximization problem with regard to production factor input takes the form:

$$\pi(x(t)) = r(x(t)) - c^{tot}(x(t)) = \left\{a(t)^{0,5}x(t)^{0,25} - C(t)x(t)^2 - d(t)\right\} \mapsto \max$$

$$x(t) \geq 0.$$

The solution of the profit maximization problem is the optimal input of a production factor:

$$\overline{x}(t) = 2^{-\frac{12}{7}} a(t)^{\frac{2}{7}} C(t)^{-\frac{4}{7}},$$

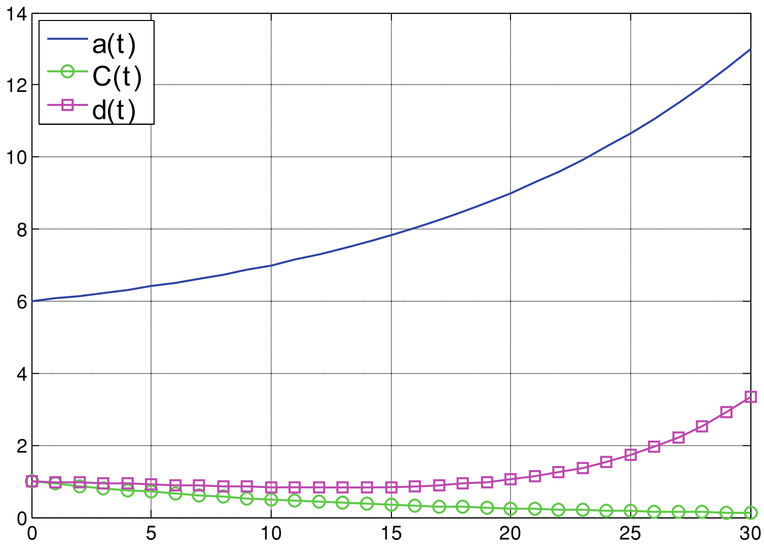


Fig. 4.25 Trajectories of $a(t)$, $C(t)$ and fixed cost

and from its form it can be noticed that the production factor optimal input is the bigger the higher the value of $a(t)$ is and the lower the value of $C(t)$ is. Eventually the input depends on the interrelation of these two values. This is reflected in Fig. 4.26.

A trajectory of the monopoly maximal profit is presented in Fig. 4.27. The bigger production factor input involves not only the bigger output and hence the higher revenue from sales of a product but also the higher production cost related to a higher price of a production factor due to the bigger demand for this factor reported by a monopoly. The revenue depends on $a(t)$, while the production cost on $C(t)$. The profit level, besides depending on $a(t)$ and $C(t)$, is influenced also by a level of the fixed production cost $d(t)$.

In the production cost minimization problem, a monopoly fixes at any moment t what the output level $y(t)$ it wants to achieve is. Let us assume that at any moment $t \in [0; 30]$ this level is determined by a monopoly according to an equation[28]:

$$y(t) = -0.0035(t - 15)^2 + 1.25.$$

For the data given in this example, the production cost minimization problem for a monopoly takes the form:

$$c^{tot}(x(t)) = C(t)x(t)^2 + d(t) \mapsto \min$$

[28] The same equation for the output level fixed by a firm is assumed in examples related to the perfect competition, that is, Examples 4.2 and 4.4. A trajectory of this fixed output level is presented in Fig. 4.7 in Example 4.2.

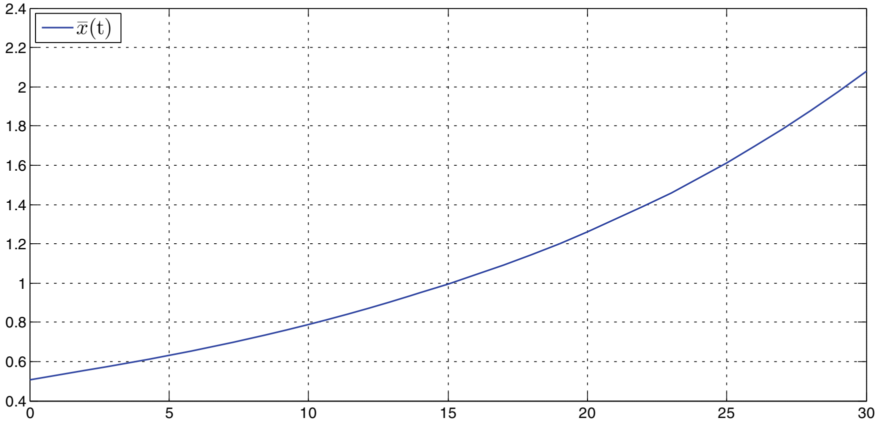


Fig. 4.26 Trajectory of demand for production factor—case of monopoly

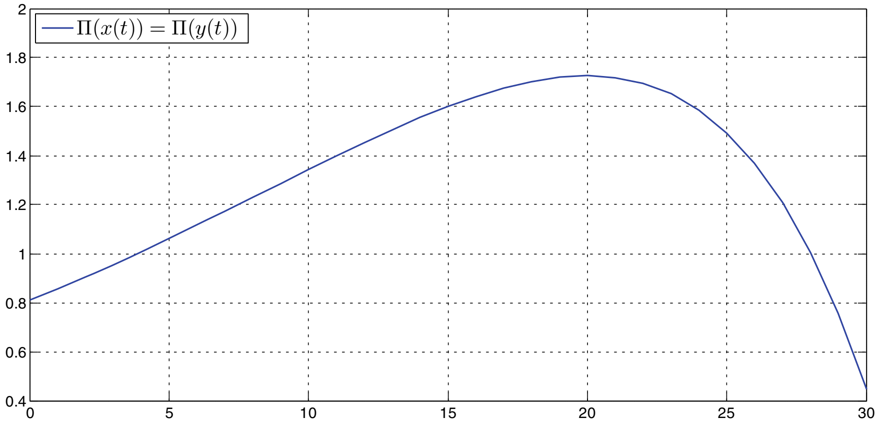


Fig. 4.27 Trajectory of monopoly maximum profit

$$x(t)^{0.5} = y(t)$$

$$x(t) \geq 0,$$

and its solution is the production factor optimal:

$$\widetilde{x}(t) = y(t)^2,$$

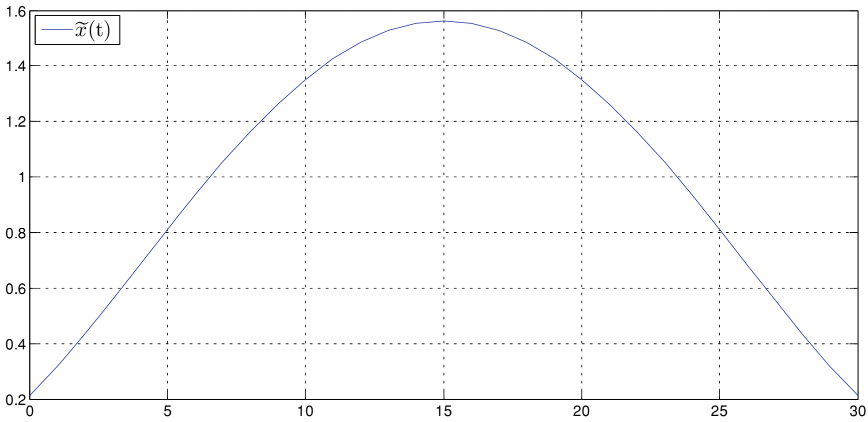


Fig. 4.28 Trajectory of conditional demand for a production factor—case of monopoly

whose trajectory, presented in Fig. 4.28, is exactly the same as the trajectory of the conditional demand for a production factor in Example 4.2 when a firm is considered to be acting in the prefect competition.

The fact that the production cost minimization problem actually differs from the analogical problem in the case of a firm acting in the perfect competition is revealed in a form of the minimal cost of producing $y(t)$ output units which for the data given in this example takes the form:

$$c(y(t)) = C(t)y(t)^4 + d(t),$$

and its trajectory is presented in Fig. 4.29. In the case of the perfect competition firm, we get a result[29]:

$$c(y(t)) = c(t)y(t)^2 + d(t).$$

For the data given in the example, the monopoly profit maximization problem with regard to output level takes the form:

$$\pi(y(t)) = r(y(t)) - c(y(t)) = \left\{a(t)^{0,5} \cdot y(t)^{0,5} - C(t)y(t)^4 - d(t)\right\} \mapsto \max$$

$$y(t) \geq 0.$$

[29] See Example 4.2 in Sect. 4.4.2.

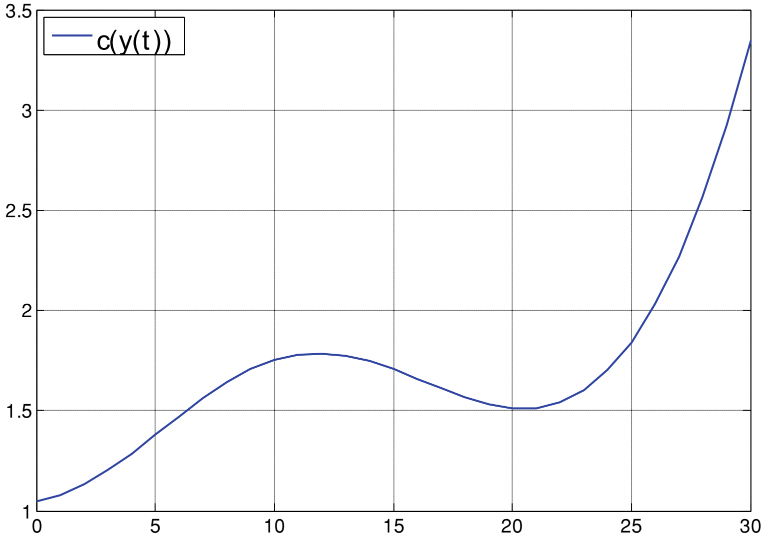


Fig. 4.29 Trajectory of minimum production cost—case of monopoly

The solution of the profit maximization problem is the optimal supply of a product:

$$\overline{y}(t) = 2^{-\frac{6}{7}} a(t)^{\frac{1}{7}} C(t)^{-\frac{2}{7}},$$

and from its form it can be noticed that the product optimal supply is the bigger the higher the value of $a(t)$ is and the lower the value of $C(t)$ is. Eventually the supply depends on the interrelation of these two values. This is reflected in Fig. 4.30.

The monopoly maximum profit resulting from the profit maximization problem with regard to output level has the same form as the monopoly maximum profit in the profit maximization problem with regard to production factor input. Its trajectory is presented in Fig. 4.27.

The product optimal price as a value of the product price function by the optimal product supply takes the form:

$$p(\overline{y}(t)) = \left(\frac{a(t)}{\overline{y}(t)}\right)^{0.5} = 2^{\frac{3}{7}} a(t)^{\frac{3}{7}} C(t)^{\frac{1}{7}}.$$

From the form of this function, it can be noticed that the product optimal price is the higher the higher the value of $a(t)$ is and the lower the value of $C(t)$ is. Hence the price of a monopoly product depends on changes that occur in the demand reported by consumers for this product and on changes that a supplier of a production factor introduces when setting a price of this factor. A trajectory of the optimal price is presented in Fig. 4.31. In the considered time horizon, it reaches its minimum at moment $t \approx 13$.

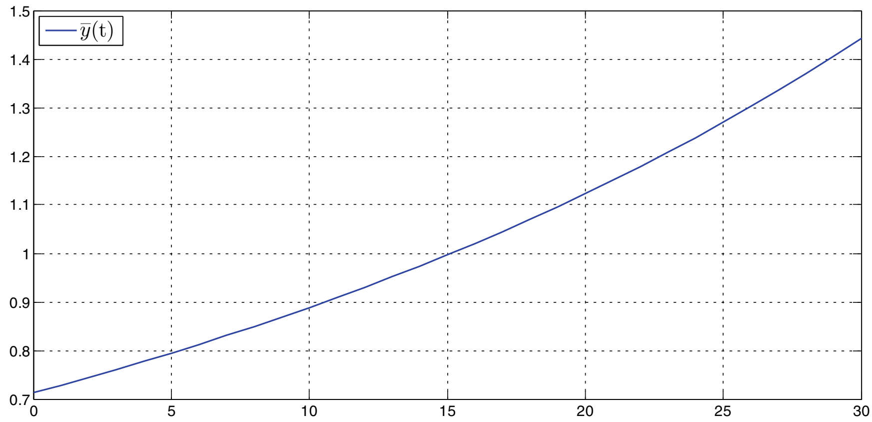


Fig. 4.30 Trajectory of optimal product supply—case of monopoly

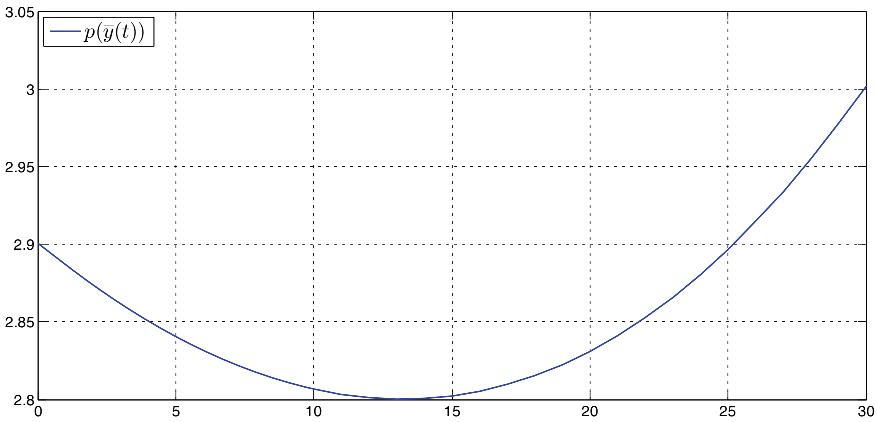


Fig. 4.31 Trajectory of optimal product supply—case of monopoly

4.7 Monopoly—Short-Term Strategy

4.7.1 Static Approach

Let us use the following notation:

$p(y) > 0$, $\frac{\mathrm{d}p(y)}{\mathrm{d}y} < 0$—a price of a product manufactured by a monopoly as a decreasing function of product supply, set by a monopoly,
$\mathbf{c} = (c_1(x_1), c_2(x_2)) > (0, 0)$, $\frac{\mathrm{d}c_i(x_i)}{\mathrm{d}x_i} > 0$, $i = 1, 2$—a vector of prices of production factors, each of whom is an increasing function of demand reported by a monopoly for a given production factor,

$\mathbf{x} = (x_1, x_2) \geq (0, 0)$—a vector of inputs of production factors,
$\mathbf{b} = (b_1, b_2) > (0, 0)$—a vector of resources of production factors,
$B = [0; b_1] \times [0; b_2] \subset X = \mathbb{R}^2_+$—a set of constraints on resources of production factors,
$w = f(b_1, b_2)$—an output level constrained due to production factors' limitation,
$W = [0; f(b_1, b_2)] = [0; w]$—a set constraining the output level,
$y = f(x_1, x_2)$—an output level,
$r(y) = p(y)y$—revenue (turnover) from sales of a manufactured product as a function of product supply,
$r(x_1, x_2) = p(f(x_1, x_2))f(x_1, x_2)$—revenue (turnover) from sales of a manufactured product as a function of inputs of production factors,
$c^{tot}(x_1, x_2) = c_1(x_1)x_1 + c_2(x_2)x_2 + d$—total cost of production as a nonlinear function of inputs of production factors,
$c^v(x_1, x_2) = c_1(x_1)x_1 + c_2(x_2)x_2$—variable cost of production as a function of inputs of production factors,
$c^f(x_1, x_2) = d$—fixed cost of production,
$c(y)$—minimum cost of producing y output units,
$\pi(y) = r(y) - c(y) = p(y)y - c(y)$—firm's profit as a function of output level,
$\pi(x_1, x_2) = r(x_1, x_2) - c^{tot}(x_1, x_2)$—firm's profit as a function of inputs of production factors.

Problem of profit maximization with regard to inputs of production factors whose resources are limited (P1m-s)

The aim of a monopoly is to maximize its profit expressed as a function of inputs of production factors whose resources are limited, which can be written as a problem to solve in the following way:

$$\begin{aligned}\pi(x_1, x_2) &= r(x_1, x_2) - c^{tot}(x_1, x_2)\\ &= \{p(f(x_1, x_2))f(x_1, x_2) - (c_1(x_1)x_1 + c_2(x_2)x_2 + d)\} \mapsto \max \end{aligned} \tag{4.301}$$

$$x_i \leq b_i,\ i = 1, 2, \tag{4.302}$$

$$x_1, x_2 \geq 0. \tag{4.303}$$

Since a production function from assumption (F2) is strictly concave then a revenue function is strictly concave too. At the same time, a production total cost function is strictly convex. As a result, a firm's profit function is strictly concave. Moreover, we are interested in an optimal solution $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2) > (0, 0)$ for which the profit $\pi(\overline{x}_1, \overline{x}_2)$ is the maximum. As a consequence, we have.

$$
\begin{aligned}
&\forall i = 1,2 \lim_{x_i \to 0^+} \frac{\partial \pi(x_1, x_2)}{\partial x_i} > 0 \quad \wedge \lim_{x_i \to +\infty} \frac{\partial \pi(x_1, x_2)}{\partial x_i} < 0 \\
&\Leftrightarrow \lim_{x_i \to 0^+} \frac{\partial r(x_1, x_2)}{\partial x_i} > \lim_{x_i \to 0^+} \frac{\partial c^{tot}(x_1, x_2)}{\partial x_i} \\
&\wedge \lim_{x_i \to +\infty} \frac{\partial r(x_1, x_2)}{\partial x_i} < \lim_{x_i \to +\infty} \frac{\partial c^{tot}(x_1, x_2)}{\partial x_i} \\
&\exists_1 \overline{\mathbf{x}} > 0 \left.\frac{\partial \pi(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = 0 \quad \Leftrightarrow \quad \left.\frac{\partial r(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = \left.\frac{\partial c^{tot}(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} \quad i = 1,2
\end{aligned}
\tag{4.304}
$$

If a vector $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2) > (0,0)$ satisfies Constraint (4.302) then this vector is an optimal solution to the problem (P1m-s). Otherwise, to solve the problem (P1m-s) one needs to use Kuhn-Tucker theorem. Let us write the problem using a Lagrange function:

$$
L(\mathbf{x}, \boldsymbol{\lambda}) = \{\pi(\mathbf{x}) + \lambda_1(b_1 - x_1) + \lambda_2(b_2 - x_2)\} \to \max. \tag{4.305}
$$

Then necessary and sufficient conditions for the existence of optimal solution to problem (P1m-s) have a form:

$$
\begin{aligned}
&\bar{x}_1 \left(\left.\frac{\partial r(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\overline{\mathbf{x}}} - \left.\frac{\partial c^{tot}(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\overline{\mathbf{x}}} - \bar{\lambda}_1 \right) \\
&+\bar{x}_2 \left(\left.\frac{\partial r(\mathbf{x})}{\partial x_2}\right|_{\mathbf{x}=\overline{\mathbf{x}}} - \left.\frac{\partial c^{tot}(\mathbf{x})}{\partial x_2}\right|_{\mathbf{x}=\overline{\mathbf{x}}} - \bar{\lambda}_2 \right) = 0,
\end{aligned}
\tag{4.306}
$$

$$
\overline{\lambda}_1(b_1 - \overline{x}_1) + \overline{\lambda}_2(b_2 - \overline{x}_2) = 0, \tag{4.307}
$$

where $\overline{\lambda}_i = \left.\frac{\partial \pi(\mathbf{x})}{\partial b_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} \geq 0,\ i = 1,2$ means an optimal Lagrange multiplier which determines by how much the maximum value of the profit function $\pi : \mathbb{R}^2_+ \to \mathbb{R}$ approximately increases when a value of parameter b_i increases by one notional unit.

If $\overline{\lambda}_i > 0$ then the i-th constraint on resources is binding. When $\overline{\lambda}_i = 0$ then the i-th constraint is not binding.

If we are interested only in a positive optimal solution $\overline{\mathbf{x}} > \mathbf{0}$ to problem (P1m-s) then Condition (4.306) is satisfied if and only if:

$$
\left.\frac{\partial r(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} - \left.\frac{\partial c\wedge tot(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} - \overline{\lambda}_i = 0,\ i = 1,2. \tag{4.308}
$$

If $\forall i = 1,2\ \overline{\lambda}_i = 0$ then condition (4.308) takes the form:

$$
\left.\frac{\partial r(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} = \left.\frac{\partial c^{tot}(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}},\ i = 1,2 \tag{4.309}
$$

This takes place when any constraint on resources is not binding, which means that:

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}^G \leq \mathbf{b}. \tag{4.310}$$

In the case when no constraint on resources is binding then an optimal solution $\bar{\mathbf{x}} = \bar{\mathbf{x}}^G$ to problem (P1m-s) is identical to a global maximum that a strictly concave function $\pi : \mathbb{R}^2_+ \to \mathbb{R}$ reaches in space $X = \mathbb{R}^2_+$. Then the conditional maximization problem is the same as the unconditional maximization problem.

In the case when $\forall i \ \bar{\lambda}_i > 0$ then each constraint on resources is binding and condition (4.306) is satisfied in the initial form. At the same time, from Condition (4.307) we get that

$$\bar{x}_i = b_i, \quad i = 1, 2. \tag{4.311}$$

In this case, an optimal solution to problem (P1m-s) is a vector $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L = \mathbf{b}$ such that $\bar{\mathbf{x}}^G > \bar{\mathbf{x}}^L$. Then a stationary point $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L$ is called a local maximum of a function $\pi : \mathbb{R}^2_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}^2_+$.

Two other cases should be also considered. If $\bar{\lambda}_1 > 0$ while $\bar{\lambda}_2 = 0$ then one obtains the optimal solution to problem (P1m-s) from the following equation system:

$$\bar{x}_1 = b_1, \tag{4.312}$$

$$\left.\frac{\partial r(\mathbf{x})}{\partial x_2}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \left.\frac{\partial c^{tot}(\mathbf{x})}{\partial x_2}\right|_{\mathbf{x}=\bar{\mathbf{x}}}. \tag{4.313}$$

If $\bar{\lambda}_1 = 0$ while $\bar{\lambda}_2 > 0$ then one obtains an optimal solution to problem (P1m-s) from the following equation system:

$$\left.\frac{\partial r(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \left.\frac{\partial c^{tot}(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\bar{\mathbf{x}}}, \tag{4.314}$$

$$\bar{x}_2 = b_2. \tag{4.315}$$

In both cases, an optimal solution to problem (P1m-s) is a vector $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L \lneqq \mathbf{b}$ such that[30] $\bar{\mathbf{x}}^G \gneqq \bar{\mathbf{x}}^L$. Then a stationary point $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L$ is called a local maximum of a function $\pi : \mathbb{R}^2_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}^2_+$.

[30] The notation $\mathbf{x} \gneqq \mathbf{y}$ means that at least one of coordinates of a vector $\mathbf{x}$ is bigger than the corresponding coordinate of a vector $\mathbf{y}$ while the other corresponding coordinates are equal to each other.

Definition 4.41 A **function of demand for production factors** is a mapping ψ : int $\mathbb{R}_+^2 \to$ int $\mathbb{R}_+^2$ which assigns an optimal solution of problem (P1m-s) to any price $p(f(x_1, x_2))$ of a product and any price $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ in the following way:

$$\psi(p(f(\mathbf{x})), \mathbf{c}(\mathbf{x})) = \bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2). \tag{4.316}$$

Definition 4.42 A **monopoly maximal profit function** is a mapping Π : int $\mathbb{R}_+^2 \times \mathbb{R}_+ \to$ int $\mathbb{R}_+$ which assigns maximum profit to any price $p(f(x_1, x_2))$ of a product, any price $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ of production factors and any fixed cost d in the following way:

$$\Pi(p(f(\mathbf{x})), \mathbf{c}(\mathbf{x}), d) = \pi(\bar{\mathbf{x}}). \tag{4.317}$$

Problem of cost minimization when producing the output at a fixed level with limited resources of production factors (P2m-s)

The aim of a monopoly is to produce $y \geq 0$ units of output at minimum total cost when resources of production factors are limited. This problem can be written in the following way:

$$c^{tot}(x_1, x_2) = \{c_1(x_1)x_1 + c_2(x_2)x_2 + d\} \mapsto \min \tag{4.318}$$

$$f(x_1, x_2) = y = \text{const.} > 0, \tag{4.319}$$

$$x_i \leq b_i, \ i = 1, 2, \tag{4.320}$$

$$x_1, x_2 \geq 0. \tag{4.321}$$

One can express problem (P2m-s) using a Lagrange function:

$$\begin{aligned} L(x_1, x_2, \ \boldsymbol{\lambda}) = \{&c_1(x_1)x_1 + c_2(x_2)x_2 + d + \lambda_1(b_1 - x_1) \\ &+\lambda_2(b_2 - x_2) + \lambda(y - f(x_1, x_2))\} \mapsto \min. \end{aligned} \tag{4.322}$$

Then necessary and sufficient conditions for the existence of an optimal solution to problem (P2m-s) have a form:

$$\tilde{x}_1 \left.\frac{\partial L(\mathbf{x}, \tilde{\boldsymbol{\lambda}})}{\partial x_1}\right|_{\mathbf{x}=\tilde{\mathbf{x}}} + \tilde{x}_2 \left.\frac{\partial L(\mathbf{x}, \tilde{\boldsymbol{\lambda}})}{\partial x_2}\right|_{\mathbf{x}=\tilde{\mathbf{x}}} = 0, \tag{4.323}$$

$$\tilde{\lambda}_1 \left.\frac{\partial L(\tilde{\mathbf{x}}, \boldsymbol{\lambda})}{\partial \lambda_1}\right|_{\boldsymbol{\lambda}=\tilde{\boldsymbol{\lambda}}} + \tilde{\lambda}_2 \left.\frac{\partial L(\tilde{\mathbf{x}}, \boldsymbol{\lambda})}{\partial \lambda_2}\right|_{\boldsymbol{\lambda}=\tilde{\boldsymbol{\lambda}}} + \tilde{\lambda} \left.\frac{\partial L(\tilde{\mathbf{x}}, \boldsymbol{\lambda})}{\partial \lambda}\right|_{\boldsymbol{\lambda}=\tilde{\boldsymbol{\lambda}}} = 0. \tag{4.324}$$

If we are interested only in a positive optimal solution $\widetilde{\mathbf{x}} > \mathbf{0}$ to problem (P2m-s) then condition (4.323) is satisfied if and only if:

$$\left.\frac{\partial c_i(x_i)}{\partial x_i}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} \tilde{x}_i + c_i(x_i) - \widetilde{\lambda}_i - \widetilde{\lambda} \left.\frac{\partial f(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} = 0, \quad i = 1, 2, \tag{4.325}$$

If $\forall i = 1, 2\ \widetilde{\lambda}_i = 0$ and $\widetilde{\lambda} > 0$ then one obtains the optimal solution to problem (P2m-s) from the following equation system:

$$\left.\frac{\partial c_i(x_i)}{\partial x_i}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} \tilde{x}_i + c_i(x_i) = \widetilde{\lambda} \left.\frac{\partial f(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}}, \quad i = 1, 2, \tag{4.326}$$

$$f(\tilde{x}_1, \tilde{x}_2) = y. \tag{4.327}$$

In this case, no constraint on resources is binding then an optimal solution $\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}^G$ to problem (P2m-s) is identical to a global minimum that a strictly convex function $c^{tot} : \mathbb{R}^2_+ \to \mathbb{R}$ reaches in space $X = \mathbb{R}^2_+$.

In the case when $\forall i = 1, 2\ \widetilde{\lambda}_i > 0$ then each constraint on resources is binding and one obtains the optimal solution to problem (P2m-s) from the following equation system:

$$\widetilde{x}_i = b_i, \quad i = 1, 2, \tag{4.328}$$

$$f(\tilde{x}_1, \tilde{x}_2) = y. \tag{4.329}$$

In the case when $\widetilde{\lambda}_1 > 0$ while $\widetilde{\lambda}_2 = 0$ then a constraint on resource of the first production factor is binding and a constraint on resource of the second production factor is not binding. Then one obtains the optimal solution to problem (P2m-s) from the following equation system:

$$\widetilde{x}_1 = b_1, \tag{4.330}$$

$$\left.\frac{\partial c_2(x_2)}{\partial x_2}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} \tilde{x}_2 + c_2(x_2) = \widetilde{\lambda} \left.\frac{\partial f(x_1, x_2)}{\partial x_2}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}}, \tag{4.331}$$

$$f(\tilde{x}_1, \tilde{x}_2) = y. \tag{4.332}$$

In the case when $\widetilde{\lambda}_1 = 0$ while $\widetilde{\lambda}_2 > 0$ then a constraint on resource of the first production factor is not binding and a constraint on resource of the second production factor is binding. Then one obtains the optimal solution to problem (P2m-s) from the following equation system:

$$\left.\frac{\partial c_1(x_2)}{\partial x_1}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}} \tilde{x}_1 + c_1(x_2) = \widetilde{\lambda} \left.\frac{\partial f(x_1, x_2)}{\partial x_1}\right|_{\mathbf{x}=\widetilde{\mathbf{x}}}, \tag{4.333}$$

(4.334) $$\widetilde{x}_2 = b_2,$$

(4.335) $$f(\tilde{x}_1, \tilde{x}_2) = y.$$

When one or both of $\widetilde{\lambda}_1, \widetilde{\lambda}_2$ Lagrange multipliers are positive one gets the optimal solution $\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}^L$ such that $\widetilde{\mathbf{x}}^G \gneq \widetilde{\mathbf{x}}^L$. Then a stationary point $\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}^L$ is called a local minimum of a function $c^{tot} : \mathbb{R}^2_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}^2_+$.

Definition 4.43 A **function of conditional demand for production factors** is a mapping ξ : int $\mathbb{R}^3_+ \to$ int $\mathbb{R}^2_+$ which assigns an optimal solution of problem (P2m-s) to any output level y and any price $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ factors in the following way:

(4.336) $$\xi(\mathbf{c}(\mathbf{x}), y) = \widetilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2).$$

Definition 4.44 A **monopoly minimal cost function** is a mapping μ : int $\mathbb{R}^4_+ \to$ int $\mathbb{R}_+$ which assigns minimum cost of producing y output units to any output level y, any price $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ of production factors and any fixed cost d in the following way:

(4.337) $$\mu(\mathbf{c}(\mathbf{x}), d, y) = c^{tot}(\widetilde{\mathbf{x}}) = c_1(\tilde{x}_1)\tilde{x}_1 + c_2(\tilde{x}_2)\tilde{x}_2 + d.$$

If prices of production factors are determined and the fixed cost of production is known then one can express the firm minimal cost function of producing y output units as a function of output level:

(4.338) $$\mu(\mathbf{c}(\mathbf{x}), d, y) = c(y).$$

Problem of profit maximization with regard to output level with limited resources of production factors (P3m-s)

The aim of a monopoly is to maximize its profit expressed as a function of output level when resources of production factors are limited. This problem can be written in the following way:

(4.339) $$\pi(y) = r(y) - c(y) = \{p(y)y - c(y)\} \mapsto \max$$

(4.340) $$y \leq f(b_1, b_2),$$

(4.341) $$y \geq 0.$$

Since a revenue function is a strictly concave while a monopoly minimal cost function of producing y output units is strictly convex then a firm's profit function is strictly concave. Moreover, we are interested in an optimal solution $\overline{y} > 0$ for which the profit $\pi(\overline{y})$ is the maximum. As a consequence, we have

$$\lim_{y\to 0^+} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} > 0 \quad \wedge \lim_{y\to +\infty} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} < 0$$

$$\Leftrightarrow \lim_{y\to 0^+} \frac{\mathrm{d}r(y)}{\mathrm{d}y} > \lim_{y\to 0^+} \frac{\mathrm{d}c(y)}{\mathrm{d}y} \ \wedge \lim_{y\to +\infty} \frac{\mathrm{d}r(y)}{\mathrm{d}y} < \lim_{y\to +\infty} \frac{\mathrm{d}c(y)}{\mathrm{d}y}$$

(4.342) $$\exists_1\, \overline{y} > 0 \quad \left.\frac{\mathrm{d}\pi(y)}{\mathrm{d}y}\right|_{y=\overline{y}} = 0 \Leftrightarrow \left.\frac{\mathrm{d}r(y)}{\mathrm{d}y}\right|_{y=\overline{y}} = \left.\frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y}\right|_{y=\overline{y}}$$

If a product optimal supply $\overline{y} > 0$ satisfies Constraint (3.340) then this supply is a solution to the problem (P3m-s). Otherwise, to solve the problem (P3m-s) one needs to use Kuhn-Tucker theorem. Let us write the problem using a Lagrange function:

(4.343) $$L(y, \lambda) = \{\pi(y) + \lambda(f(b_1, b_2) - y)\} \mapsto \max.$$

Then necessary and sufficient conditions for the existence of optimal solution to problem (P3m-s) have a form:

(4.344) $$\overline{y} \left.\frac{\mathrm{d}L\left(y, \overline{\lambda}\right)}{\mathrm{d}y}\right|_{y=\overline{y}} = 0,$$

(4.345) $$\overline{\lambda} \left.\frac{\mathrm{d}L(\overline{y}, \lambda)}{\mathrm{d}\lambda}\right|_{\lambda=\overline{\lambda}} = 0.$$

Condition (4.344) can be also written in equivalent forms:

(4.346) $$\overline{y}\left(\left.\frac{\mathrm{d}r(y)}{\mathrm{d}y}\right|_{y=\overline{y}} - \left.\frac{\mathrm{d}c(y)}{\mathrm{d}y}\right|_{y=\overline{y}} - \overline{\lambda} \right) = 0$$

or

(4.347) $$\overline{y}\left(\overline{y} \left.\frac{\mathrm{d}p(y)}{\mathrm{d}y}\right|_{y=\overline{y}} + p(\overline{y}) - \left.\frac{\mathrm{d}c(y)}{\mathrm{d}y}\right|_{y=\overline{y}} - \overline{\lambda} \right) = 0.$$

Condition (4.345) in turn can be written in the following way:

(4.348) $$\overline{\lambda}(f(b_1, b_2) - \overline{y}) = 0,$$

where $\overline{\lambda} = \left.\frac{\mathrm{d}\pi(y)}{\mathrm{d}f(b_1,b_2)}\right|_{y=\overline{y}} \geq 0$ means an optimal Lagrange multiplier which determines by how much the maximum value of the profit function $\pi : \mathbb{R}_+ \to$

$\mathbb{R}$ approximately increases when the constrained output level resulting from constraints on resources of production factors increases by one notional unit.

If $\overline{\lambda} > 0$ then the constraint on output level is binding and one gets an optimal solution to problem (P3m-s) from Eq. (4.348):

$$\overline{y} = \overline{y}^L = f(b_1, b_2). \tag{4.349}$$

If $\overline{\lambda} = 0$ then the constraint on output level is not binding and one gets an optimal solution to problem (P3m-s) from Eq. (4.347):

$$\overline{y} \frac{\mathrm{d}p(y)}{\mathrm{d}y}\bigg|_{y=\overline{y}} + p(\overline{y}) = \frac{\mathrm{d}c(y)}{\mathrm{d}y}\bigg|_{y=\overline{y}}. \tag{4.350}$$

In this case, $\overline{y} = \overline{y}^G$ which means that then an optimal solution $\overline{y} = \overline{y}^G$ to problem (P3m-s) is identical to a global maximum that a strictly concave function $\pi : \mathbb{R}_+ \to \mathbb{R}$ reaches in space $\mathbb{R}_+$. Then the conditional maximization problem is the same as the unconditional maximization problem.

Definition 4.45 A **function of product supply** is a mapping $\eta : \text{int } \mathbb{R}^2_+ \to \text{int } \mathbb{R}_+$ which assigns an optimal solution of problem (P3m-s) to any price $p(f(x_1, x_2))$ of a product and any price $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ of production factors in the following way:

$$\eta(p(f(\mathbf{x})), \mathbf{c}(\mathbf{x})) = \overline{y}. \tag{4.351}$$

Definition 4.46 A **monopoly maximal profit function** is a mapping $\Pi : \text{int } \mathbb{R}^3_+ \to \text{int } \mathbb{R}_+$ which assigns maximum profit to any price $p(f(x_1, x_2)$ of a product, any price $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ of production factors and any fixed cost d in the following way:

$$\Pi(p(f(\mathbf{x})), \mathbf{c}(\mathbf{x}), d) = \pi(\overline{y}). \tag{4.352}$$

Definition 4.47 A **monopoly optimal price function** is a mapping $\overline{p} : \text{int } \mathbb{R}^2_+ \to \text{int } \mathbb{R}_+$ which assigns the price guaranteeing the maximum profit to any production function $f(\mathbf{x})$ and any price $\mathbf{c}(x_1, x_2) = (c_1(x_1), c_2(x_2))$ of production factors in the following way:

$$\overline{p}(f(\mathbf{x}), \mathbf{c}(\mathbf{x})) = p(\overline{y}). \tag{4.353}$$

Theorem 4.20 If assumptions of Theorem 4.16 are satisfied then problems (P1m-s) and (P3m-s) are equivalent.

This means that:

- Knowing an optimal solution to problem (P1m-s) one can determine an optimal solution to problem (P3m-s): $\overline{y} = f(\overline{\mathbf{x}})$.

- Knowing an optimal solution to problems (P3m-s) and (P2m-s) one can determine an optimal solution to problem (P1m): $\widetilde{\mathbf{x}} = \xi(\mathbf{c}(\widetilde{\mathbf{x}}), \overline{y}) = \psi(p(f(\widetilde{\mathbf{x}})), \mathbf{c}(\widetilde{\mathbf{x}})) = \overline{\mathbf{x}}$,
- $\pi(\overline{\mathbf{x}}) = \Pi(p(f(\mathbf{x})), \mathbf{c}(\mathbf{x}), d) = \pi(\overline{y})$.

Example 4.7 The following data is given:

$y = f(x) = ax^{\frac{1}{2}}$—an output level as a nonlinear function of a production factor input,
$p(y) = \left(\frac{a}{y}\right)^{\frac{1}{2}}$—a price of a product manufactured by a monopoly,
$x \geq 0$—an input of a production factor,
$c_1(x) = ax$—a price of a production factor as a linear function of demand reported by a monopoly for this factor,
$b > 0$—a resource of a production factor,
$w = f(b) = ab^{\frac{1}{2}}$—an output level constrained due to the production factor limitation,
$r(y) = p(y)y = a^{\frac{1}{2}}y^{\frac{1}{2}}$—revenue (turnover) from sales of a manufactured product as a nonlinear function of output level,
$r(x) = p(f(x))f(x) = x^{-\frac{1}{4}}ax^{\frac{1}{2}} = ax^{\frac{1}{4}}$—revenue (turnover) from sales of a manufactured product as a nonlinear function of a production factor input,
$c^{tot}(x) = c_1(x)x + d = ax^2 + d$—total cost of production as a nonlinear function of a production factor input,
$c^v(x) = ax^2$—variable cost of production,
$c^f(x) = d \geq 0$—fixed cost of production,
$\pi(x) = r(x) - c^{tot}(x) = ax^{\frac{1}{4}} - ax^2 - d$—firm's profit as a function of a production factor input,
$\pi(y) = r(y) - c(y)$—firm's profit as a function of output level,
$c(y)$—minimal cost of producing y output units as an optimal solution to problem (P2m-s).

Tasks

1. Solve the profit maximization problem (P1m-s).
2. Present a geometric illustration of the profit maximization problem (P1m-s).
3. Give an economic interpretation of necessary and sufficient conditions of the existence of an optimal solution to problem (P1m-s).
4. Solve the cost minimization problem (P2m-s).
5. Present a geometric illustration of the cost minimization problem (P2m-s).
6. Give an economic interpretation of necessary and sufficient conditions of the existence of an optimal solution to problem (P2m-s).
7. Solve the profit maximization problem (P3m-s).
8. Present a geometric illustration of the profit maximization problem (P3m-s).
9. Give an economic interpretation of necessary and sufficient conditions of the existence of an optimal solution to problem (P3m-s).

10. Justify that the profit maximization problems (P1m-s) and (P3m-s) are equivalent.
11. Determine the optimal price by which a monopoly obtains the maximum profit.

Ad 1 The profit maximization problem (P1m-s) has a form:

$$\pi(x) = \left\{ax^{\frac{1}{4}} - \left(ax^2 + d\right)\right\} \mapsto \max \tag{4.354}$$

$$0 \le x \le b. \tag{4.355}$$

Since the production function from assumption (F2) is strictly concave then a revenue function is strictly concave too. At the same time, a production total cost function is strictly convex. As a result, a monopoly profit function is strictly concave.

It is known that when a profit function is strictly concave then problem (P1m-s) can have:

- no optimal solution when revenue from sales of a product is lower than the total cost of production,
- exactly one optimal solution $\overline{x} = 0$ which, due to the positive fixed cost of production, corresponds to a loss equal to the fixed cost,
- exactly one optimal solution $\overline{x} > 0$ which, with the sufficiently low fixed cost of production, corresponds to the positive profit.

A condition ensuring the existence of a unique and positive optimal solution to problem (P1m-s) has a form:

$$\begin{gathered}\lim_{x\to 0^+}\frac{\mathrm{d}\pi(x)}{\mathrm{d}x} > 0 \quad\wedge \lim_{x\to +\infty}\frac{\mathrm{d}\pi(x)}{\mathrm{d}x} < 0\\ \Leftrightarrow \lim_{x\to 0^+}\frac{\mathrm{d}r(x)}{\mathrm{d}x} > \lim_{x\to 0^+}\frac{\mathrm{d}c^{tot}(x)}{\mathrm{d}x} \;\wedge\; \lim_{x\to +\infty}\frac{\mathrm{d}r(x)}{\mathrm{d}x} < \lim_{x\to +\infty}\frac{\mathrm{d}c^{tot}(x)}{\mathrm{d}x},\end{gathered} \tag{4.356}$$

which means that from the strict concavity of the monopoly profit function it results that by a relatively big production factor input the marginal revenue is lower than the marginal production cost, while by a relatively small production factor input the marginal revenue is higher than the marginal production cost.

Let us determine a marginal profit function in problem (P1m-s) and check if it satisfies condition (4.356):

$$\frac{\mathrm{d}\pi(x)}{\mathrm{d}x} = \frac{\mathrm{d}r(x)}{\mathrm{d}x} - \frac{\mathrm{d}c^{tot}(x)}{\mathrm{d}x} = \frac{1}{4}ax^{-\frac{3}{4}} - 2ax. \tag{4.357}$$

Let us notice that:

(4.358) $$\lim_{x\to+\infty}\frac{\mathrm{d}\pi(x)}{\mathrm{d}x}=\lim_{x\to+\infty}\left(\frac{1}{4}ax^{-\frac{3}{4}}-2ax\right)=-\infty<0$$

and

(4.359) $$\lim_{x\to0^+}\frac{\mathrm{d}\pi(x)}{\mathrm{d}x}=\lim_{x\to0^+}\left(\frac{1}{4}ax^{-\frac{3}{4}}-2ax\right)=+\infty>0.$$

Since condition (4.356) is satisfied then we can determine an optimal solution to problem (P1m-s) from the following equation system:

(4.360) $$\overline{x}\left(\frac{\mathrm{d}r(x)}{\mathrm{d}x}\bigg|_{x=\overline{x}}-\frac{\mathrm{d}c^{tot}(x)}{\mathrm{d}x}\bigg|_{x=\overline{x}}-\overline{\lambda}\right)=0,$$

(4.361) $$\overline{\lambda}(b-\overline{x})=0.$$

If $\overline{x}>0,\overline{\lambda}=0$ then the resource constraint is not binding and $\overline{x}=\overline{x}^G=2^{-\frac{12}{7}}$. If $\overline{x}>0,\overline{\lambda}>0$ then the resource constraint is binding and $\overline{x}=\overline{x}^L=b$. Let us substitute this optimal solution into the profit function:

(4.362) $$\pi(\overline{x})=7a2^{-\frac{24}{7}}-d$$

or

(4.363) $$\pi(\overline{x})=ab^{\frac{1}{4}}-\left(ab^2+d\right).$$

If the fixed cost satisfies a condition $0\le d<7a2^{-\frac{24}{7}}$ or equivalently a condition $0\le d<a\left(b^{\frac{1}{4}}-b^2\right)$, where $b\in(0,1)$, then the maximum profit that a monopoly can obtain is positive.

Ad 2 See Figs. 4.32a, 4.32b and 4.32c.

Ad 3 Conditions (4.360) and (4.361) are necessary and sufficient conditions for the existence of an optimal solution to problem (P1m-s). If the production factor resource equals $b_1>0$ then the optimal solution, by which a monopoly obtains its maximum profit, is $\overline{x}=\overline{x}^L=b_1$ equal to the production factor resource. If the production factor resource equals $b_2>0$ then the optimal solution, by which a monopoly obtains its maximum profit, is $\overline{x}=\overline{x}^G>0$ and the production factor resource is not entirely exploited. It is worth noticing that $\pi\left(\overline{x}^L\right)<\pi\left(\overline{x}^G\right)$, which means that $\overline{x}=\overline{x}^G$ is the global maximum of the profit function obtained due to the fact that the production factor resource is large enough.

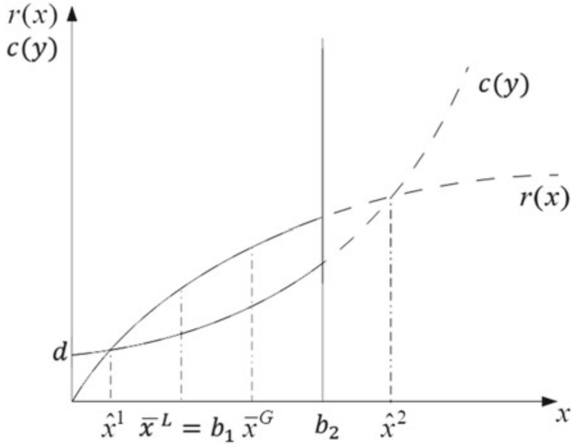


Fig. 4.32a Graphs of revenue function and production total cost function

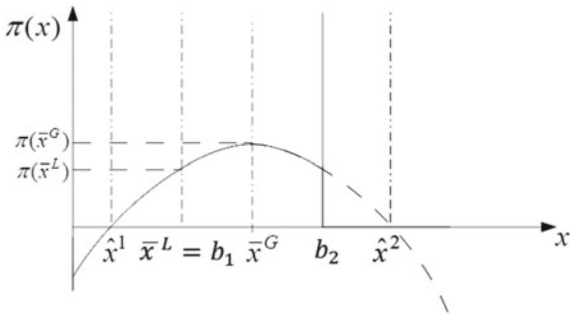


Fig. 4.32b Graph of profit function

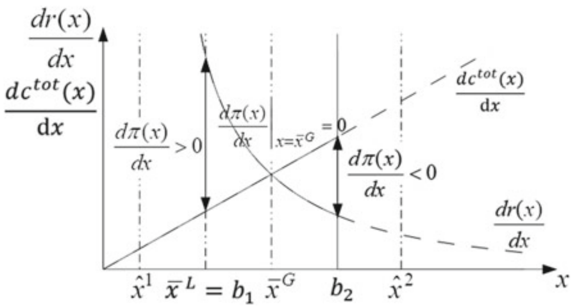


Fig. 4.32c Graphs of marginal revenue function and marginal production cost function

Ad 4 The cost minimization problem (P2m-s) when producing y output units and when the production factor resource is limited takes the form:

(4.364) $$c^{tot}(x) = \left(ax^2 + d\right) \mapsto \min$$

(4.365) $$ax^{\frac{1}{2}} = y = \text{const.},$$

$$0 \leq x \leq b. \tag{4.366}$$

Since a set of feasible solutions to this problem has only one element, then a production factor input resulting from (3.365) is the optimal solution to this problem:

$$\widetilde{x} = \left(\frac{y}{a}\right)^2 \leq b, \tag{4.367}$$

and is positive by the positive output level $0 < y \leq ab^{\frac{1}{2}}$.

A monopoly minimal cost function of producing y output units corresponds to this solution:

$$c^{tot}(\widetilde{x}) = a\left(\frac{y}{a}\right)^4 + d = a^{-3}y^4 + d = c(y), \tag{4.368}$$

and is nonlinear and strictly convex function of the output level.

Ad 5 See Figs. 4.33a and 4.33b.

Ad 6 In problem (P2m-s) exactly one production factor input corresponds to exactly one fixed output level. This production factor input is at the same time the only one solution to problem (P2m-s). As a consequence, a set of feasible solutions has only one element. In this case, independently of an optimality criterion, the only one feasible solution to the problem is at the same time its only one optimal solution.

Ad 7 The profit maximization problem (P3c-s) when the output level is constrained (due to the production factor resource limitation) takes the form:

$$\pi(y) = r(y) - c(y) = \left\{a^{\frac{1}{2}}y^{\frac{1}{2}} - \left(a^{-3}y^4 + d\right)\right\} \mapsto \max, \tag{4.369}$$

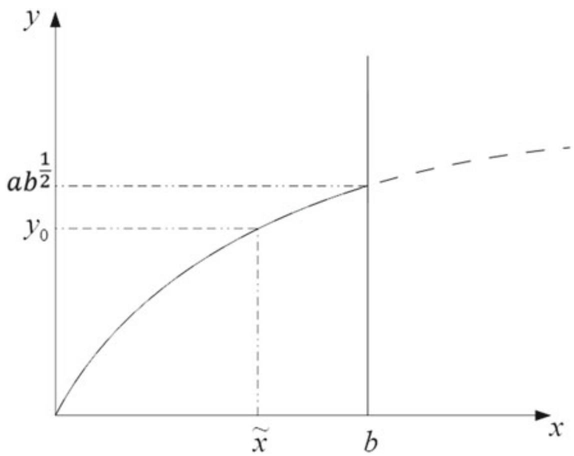


Fig. 4.33a Illustration of problem (P2m-s)

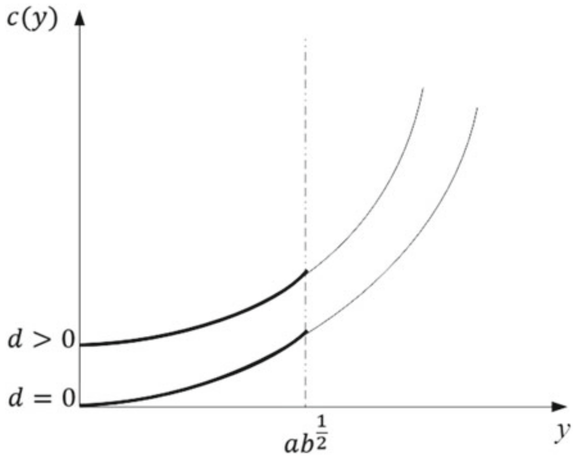


Fig. 4.33b Graphs of firm's minimal cost function of producing y output units

$$0 \leq y \leq f(b). \tag{4.370}$$

The revenue function is nonlinear and strictly concave. The monopoly minimal cost function of producing y output units is nonlinear and strictly convex. Thus, the profit function is a strictly concave function of the output level.

It is known that when a profit function is strictly concave then problem (P3m-s) can have:

- no optimal solution when revenue from sales of a product is lower than the firm minimum cost of producing y output units,
- exactly one optimal solution $\overline{y} = 0$ which, due to the positive fixed cost of production, corresponds to a loss equal to the fixed cost,
- exactly one optimal solution $\overline{y} > 0$ which, by the sufficiently low fixed cost of production, corresponds to the positive profit.

  A condition ensuring the existence of a unique and positive optimal solution to problem (P3m) has a form:

$$\lim_{y \to 0^+} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} > 0 \quad \wedge \lim_{y \to +\infty} \frac{\mathrm{d}\pi(y)}{\mathrm{d}y} < 0$$
$$\Leftrightarrow \lim_{y \to 0^+} \frac{\mathrm{d}r(y)}{\mathrm{d}y} > \lim_{y \to 0^+} \frac{\mathrm{d}c(y)}{\mathrm{d}y} \wedge \lim_{y \to +\infty} \frac{\mathrm{d}r(y)}{\mathrm{d}y} < \lim_{y \to +\infty} \frac{\mathrm{d}c(y)}{\mathrm{d}y}, \tag{4.371}$$

which means that from the strict concavity of the firm's profit function it results that by a relatively small output level the marginal revenue is higher than the marginal production cost, while by a relatively big output level the marginal revenue is lower than the marginal production cost.

Let us determine an optimal solution to problem (P3m-s) from the following equation system:

$$\overline{y}\left(\left.\frac{\mathrm{d}r(y)}{\mathrm{d}y}\right|_{y=\overline{y}} - \left.\frac{\mathrm{d}c(y)}{\mathrm{d}y}\right|_{y=\overline{y}} - \overline{\lambda}\right) = 0, \tag{4.372}$$

$$\overline{\lambda}\left(ab^{\frac{1}{2}} - \overline{y}\right) = 0. \tag{4.373}$$

If $\overline{y} > 0, \overline{\lambda} = 0$ then the constraint on output level is not binding and $\overline{y} = \overline{y}^G = a2^{-\frac{6}{7}}$. When $\overline{y} > 0, \overline{\lambda} > 0$ then the constraint is binding and $\overline{y} = \overline{y}^L = ab^{\frac{1}{2}}$.

Let us substitute the optimal solutions obtained above into the profit function. Then the firm's maximum profit is

$$\pi(\overline{y}) = 7a2^{-\frac{24}{7}} - d \tag{4.374}$$

or

$$\pi(\overline{y}) = ab^{\frac{1}{4}} - \left(ab^2 + d\right). \tag{4.375}$$

If the fixed cost satisfies a condition $0 \leq d < 7a2^{-\frac{24}{7}}$ or equivalently a condition $0 \leq d < a\left(b^{\frac{1}{4}} - b^2\right)$ then the maximum profit that a monopoly can obtain is positive.

Ad 8 See Figs. 4.34a, 4.34b and 4.34c.

Ad 9 Conditions (4.372) and (4.373) are necessary and sufficient conditions for the existence of an optimal solution to problem (P3m-s). If the product supply resulting from production factor limitation b_1 equals $w_1 = ab_1^{\frac{1}{2}} > 0$ then the optimal solution,

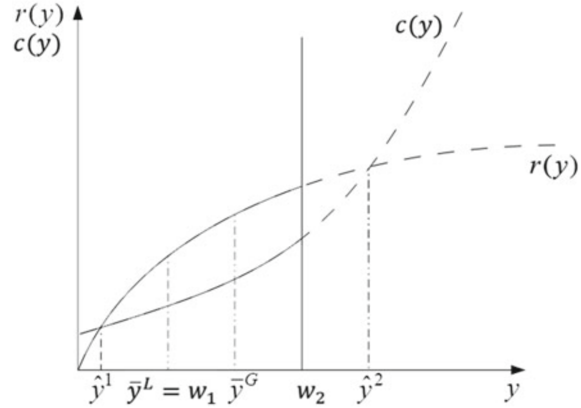


Fig. 4.34a Graphs of revenue function and firm's minimal cost function of producing y output units

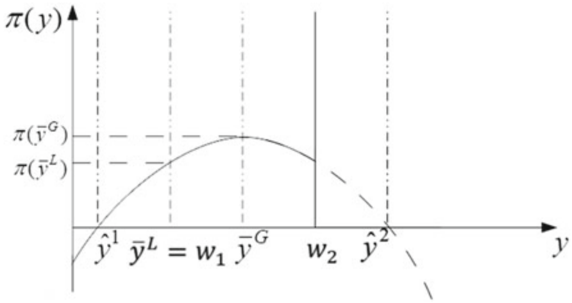


Fig. 4.34b Graph of profit function

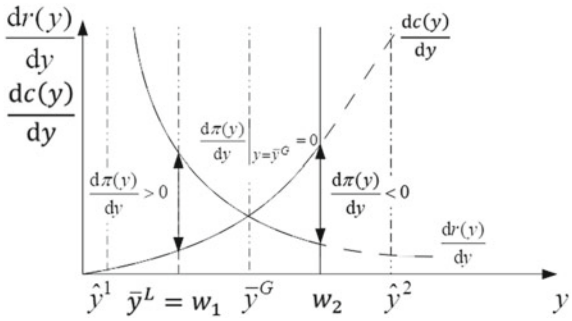


Fig. 4.34c Graphs of marginal revenue function and marginal minimal cost function of producing y output units

by which a monopoly obtains its maximum profit, is $\overline{y} = \overline{y}^L = w_1$. If the feasible product supply resulting from production factor resource b_2 equals $w_2 = ab_2^{\frac{1}{2}} > 0$ then the optimal solution, by which a monopoly obtains its maximum profit, is $\overline{y} = \overline{y}^G > 0$ and the production factor resource is not entirely exploited. It is worth noticing that $\pi\left(\overline{y}^L\right) < \pi\left(\overline{y}^G\right)$, which means that $\overline{y} = \overline{y}^G$ is the global maximum of the profit function obtained due to the fact that the production factor resource is large enough.

Ad 10 To show that problems (P1m-s) and (P3m-s) are equivalent let us notice that:

(1) for $\overline{x} = 2^{-\frac{12}{7}} > 0$ and $\overline{y} = a2^{-\frac{6}{7}} > 0$ we have

(4.376) $$\pi(\overline{x}) = 7a2^{-\frac{24}{7}} - d = \pi(\overline{y}).$$

(2) Knowing the optimal solution to problem (P1m-s) and substituting it into the production function we get the optimal solution to problem (P3m-s):

(4.377) $$\overline{y} = f(\overline{x}) = a\overline{x}^{\frac{1}{2}} = a2^{-\frac{6}{7}}.$$

(3) Knowing the optimal solution to problem (P3m-s) and substituting it into the optimal solution to problem (P2m-s) we get the optimal solution to problem (P1m-s):

(4.378)
$$\widetilde{x} = \left(\frac{\overline{y}}{a}\right)^2 = \left(2^{-\frac{6}{7}}\right)^2 = 2^{-\frac{12}{7}} = \overline{x},$$

which means that profit maximization problems (P1m) and (P3m) are equivalent.

Ad 11 A price of a product manufactured by a monopoly is assumed to have a form $p(y) = \left(\frac{a}{y}\right)^{\frac{1}{2}}$. Thus, the optimal price by which a monopoly can obtain the maximum profit is

(4.379)
$$p(\overline{y}) = \left(\frac{a}{\overline{y}}\right)^{\frac{1}{2}} = 2^{\frac{3}{7}} > 0 \quad \text{or} \quad p(\overline{y}) = \left(\frac{a}{w}\right)^{\frac{1}{2}} = b^{-\frac{1}{4}} > 0.$$

4.7.2 Dynamic Approach

In the short-term strategy, a monopoly determining optimal inputs of production factors, an optimal output and an optimal price of a monopoly product takes into account the limitations resulting from available resources of production factors. They are binding if optimal inputs of production factors exceed their resources. Then solutions to the profit maximizations problems and a solution to the production cost minimization problem differ from analogical problems in the long-term strategy. Let us use the same notation as in Sect. 4.6.2. Moreover, let us introduce additional notation:

$\mathbf{b}(t) = (b_1(t), b_2(t)) > \mathbf{0}$—a vector of time-variant resources of production factors,
$w(t) = f(b_1(t), b_2(t))$—a time-variant output level constrained due to the production factors' limitation,
$\overline{\mathbf{x}}^G(t)$—an optimal solution to the profit maximization problem with regard to inputs of production factors whose resources are unlimited,
$\widetilde{\mathbf{x}}^G(t)$—an optimal solution to the production cost minimization problem when resources of production factors are unlimited,
$\overline{y}^G(t)$—an optimal solution to the profit maximization problem with regard to output level with unlimited resources of production factors.

In short-term strategy,the monopoly profit maximization problem with regard to inputs of production factors takes the form:

$$\begin{aligned} &\pi(\mathbf{x}(t)) = r(\mathbf{x}(t)) - c^{tot}(\mathbf{x}(t)) \\ &= \{p(f(\mathbf{x}(t))) \cdot f(\mathbf{x}(t)) - (c_1(x_1(t)) \cdot x_1(t) + c_2(x_2(t)) \cdot x_2(t) + d(t))\} \\ &\mapsto \max \end{aligned} \tag{4.380}$$

$$x_i(t) \le b_i(t) \quad i = 1, 2 \tag{4.381}$$

$$\mathbf{x}(t) \ge \mathbf{0}. \tag{4.382}$$

Initially one solves problems (4.380)–(4.382) in the same way as the analogical problem in the long-term strategy. After determining the optimal solution $\overline{\mathbf{x}}^G(t)$ we compare it in each period/at any moment t of the considered time horizon with a vector $\mathbf{b}(t)$ of resources of production factors. As a solution to the whole problem in the short-term strategy one gets a vector of optimal inputs of production factors:

$$\overline{\mathbf{x}}(t) = \left(\min\left\{\overline{x}_1^G(t), b_1(t)\right\}, \min\left\{\overline{x}_2^G(t), b_2(t)\right\}\right). \tag{4.383}$$

The production cost minimization problem in the short-term strategy has a similar form to the analogical problem in the long-term strategy. The difference is accounting additionally for the constraints on the resources of production factors:

$$c^{tot}(\mathbf{x}(t)) = \{c_1(x_1(t)) \cdot x_1(t) + c_2(x_2(t)) \cdot x_2(t) + d(t)\} \mapsto \min \tag{4.384}$$

$$f(\mathbf{x}(t)) = y(t) \tag{4.385}$$

$$x_i(t) \le b_i(t) \quad i = 1, 2 \tag{4.386}$$

$$\mathbf{x}(t) \ge \mathbf{0}\,. \tag{4.387}$$

Initially one solves problems (4.384)–(4.387) in the same way as the analogical problem in the long-term strategy. After determining the optimal solution $\widetilde{\mathbf{x}}^G(t)$, we compare it in each period/at any moment t of the considered time horizon with a vector $\mathbf{b}(t)$ of resources of production factors. As a solution to the whole problem in the short-term strategy one gets a vector of optimal inputs of production factors:

$$\tilde{\mathbf{x}}(t) = \left(\min\left\{\tilde{x}_1^G(t), b_1(t)\right\}, \min\left\{\tilde{x}_2^G(t), b_2(t)\right\}\right). \tag{4.388}$$

Substituting this solution into the production total cost function, we get a monopoly minimal cost function of producing $y(t)$ output units, depending on value of $y(t)$:

$$\min c^{tot}(\mathbf{x}(t)) = c^{tot}(\tilde{\mathbf{x}}(t)) =$$

$$= c_1(\tilde{x}_1(t)) \cdot \tilde{x}_1(t) + c_2(\tilde{x}_2(t)) \cdot \tilde{x}_2(t) + d(t)$$
$$= c(y(t)). \quad (4.389)$$

The profit maximization problem with regard to output level in the short-term strategy has a similar form to the analogical problem in the long-term strategy. The difference is accounting additionally for the constraint on the output level due to the limitations of the resources of production factors. In short-term strategy, the monopoly profit maximization problem with regard to output level takes the form:

$$\pi(y(t)) = r(y(t)) - c(y(t)) = \{p(y(t)) \cdot y(t) - c(y(t))\} \mapsto \max \quad (4.390)$$

$$y(t) \le f(\mathbf{b}(t)) \quad (4.391)$$

$$y(t) \ge 0. \quad (4.392)$$

For problems (4.390)–(4.392), one determines first a solution $\overline{y}^G(t)$, that is, a solution to the analogical problem in the long-term strategy. Then we compare it in each period/at any moment t with the constrained output level $f(\mathbf{b}(t))$ resulting from the limitations of resources of production factors. As a solution to the whole problem in the short-term strategy one gets an optimal output level:

$$\overline{y}(t) = \min\left\{\overline{y}^G(t),\ f(\mathbf{b}(t))\right\}. \quad (4.393)$$

The optimal price of a product manufactured by a monopoly is $p(\overline{y}(t))$ and is time variant in the dynamic approach.

Example 4.8 Let us take the same assumptions as in Example 4.6, introducing additionally a constraint of production factor resource. A production process in a firm acting as a monopoly is described by a one-variable production function of a form[31]:

$$f(x(t)) = x(t)^{0.5}.$$

A price of product manufactured by this monopoly changes according to a function of a form:

$$p(y(t)) = \left(\frac{a(t)}{y(t)}\right)^{0.5}, \text{ where } \ a(t) > 0 \quad \forall t,$$

[31] One can find analogies of this example to Example 4.5 with a monopoly in the static approach and to Example 4.4 with the dynamic approach and a firm acting in perfect competition in the short-term strategy.

and a production factor price changes in the following way:

$$c(x(t)) = C(t)x(t), \text{ where } C(t) > 0 \quad \forall t.$$

Let us assume that at any moment $t \in [0; 30]$, value of $a(t)$, a value of $C(t)$ and the fixed production cost change according to equations:

$$a(t) = 2^{0.1t} + 5,$$

$$C(t) = 2^{-0.1t},$$

$$d(t) = \frac{\left(0.006t^2 - 0.1t + 3\right)^2 t}{480 \cdot 0.98^t} - \frac{t}{30} + 1.$$

Their trajectories are presented in Sect. 4.6.2, in Example 4.6 in Fig. 4.25.

Additionally, unlike the long-term strategy from Example 3.6, now we assume that a production factor input is limited by its resource which changes over time according to the equation:

$$b(t) = -0.01t + 1.$$

Figure 4.35 presents a trajectory of the production factor resource and a trajectory of an optimal solution to the monopoly profit maximization problem with regard to production factor input in the long-term strategy. Up to a moment $t \approx 13$ the production factor constraint is not binding because a value $\overline{x}^G(t)$ does not exceed the resource $b(t)$. From the moment $t \approx 13$, the resource constraint is binding until the end of the time horizon. Thus, a trajectory of an optimal production factor input resulting from the profit maximization problem in the short-term strategy has a form as shown in Fig. 4.36.

Figure 4.37 presents a comparison of the monopoly maximum profit in the case when the production factor input is constrained by its resource and in the case when such limitation does not exist. The difference of the maximum profit in both cases is visible from a moment $t \approx 13$.

From the production cost minimization problem in the short-term strategy, one obtains[32] an optimal production factor input $\widetilde{x}(t)$ accounting also for the constraint

[32] In this section, we do not present the trajectory of optimal production factor input nor its comparison with the optimal input in the long-term strategy for the case of a monopoly. This is due to the fact that up to this part the production cost minimization problem does not differ from the analogical problem for a firm acting in the perfect competition. Trajectories of these values are presented in Sect. 4.5.2, in Example 4.4 in Figs. 4.17 and 4.18. The reason for the lack of difference is that in both cases we consider examples in which a production process relies only on one production factor and that is why a set of feasible solutions has only one element (in given period/at given moment t). The problems start to be different from each other when we consider a form of the minimal cost function of producing $y(t)$ output units.

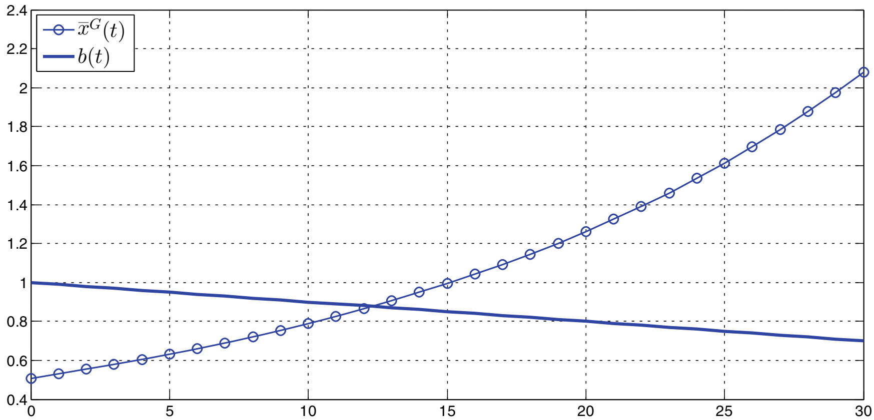


Fig. 4.35 Trajectories of resource and of optimal production factor input in long-term strategy—monopoly profit maximization problem with regard to production factor input

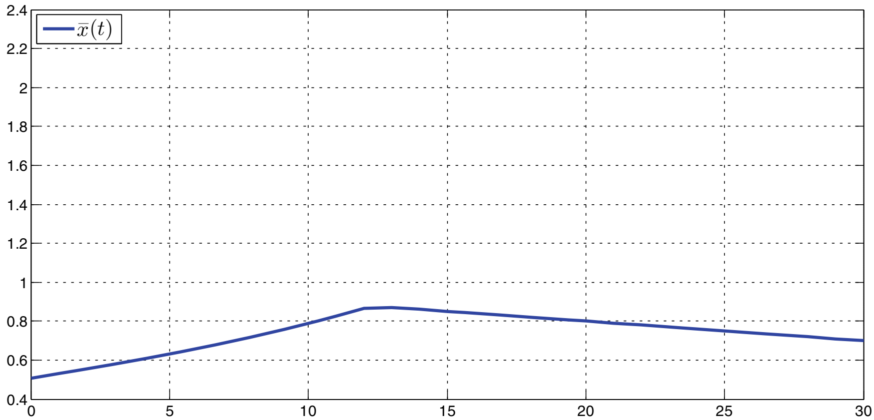


Fig. 4.36 Trajectory of demand for production factor in short-term strategy—case of monopoly

on the production factor resource $b(t)$, in accordance with an equation analogical to Eq. (4.388):

$$\tilde{x}(t) = \min\left\{\tilde{x}^G(t), b(t)\right\}.$$

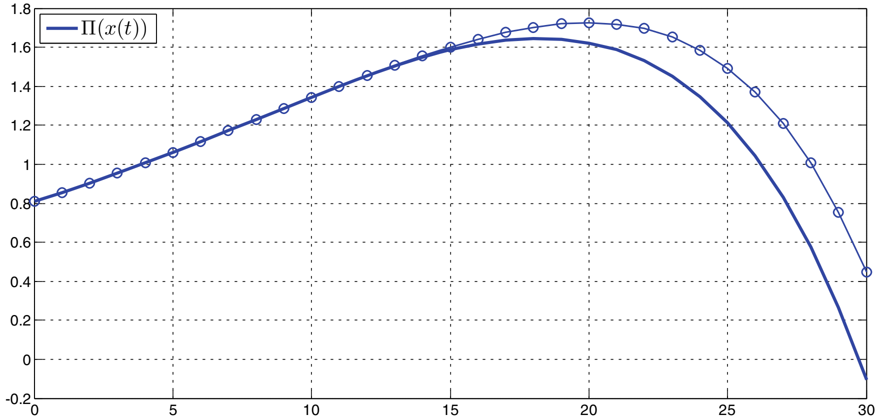


Fig. 4.37 Trajectory of monopoly maximum profit in short-term strategy

One can substitute this result into the production total cost function obtaining the minimal cost function of producing $y(t)$ output units which depends on value of $y(t)$:

$$\min c^{tot}(x(t)) = c^{tot}(\widetilde{x}(t)) = C(t)\widetilde{x}(t)^2 + d(t) = c(y(t)).$$

Figure 4.38 presents a comparison of the monopoly minimum cost of producing $y(t)$ output units in the case when the production factor input is constrained by its resource and in the case when such limitation does not exist. Since the resource constraint is binding in period between moments $t \approx 7$ and $t \approx 26$ in this time interval one can observe a difference in monopoly minimum costs in the short-term and long-term strategies. Then, that is, in this time interval, the resource constraint involves usage of smaller production factor input than it results from the long-term strategy and hence also lower production cost than in the long-term strategy.

Figure 4.39 presents a trajectory of output level constrained due to the limitation of the production factor resource and a trajectory of the optimal solution to the monopoly profit maximization problem with regard to output level in the long-term strategy. Until a moment $t \approx 13$ the resource constraint and the output level constraint are not binding because output level $\overline{y}^G(t)$ does not exceed the constrained output level equal to $f(b(t)) = b(t)^{0.5}$. From the moment $t \approx 13$, the resource constraint is binding up to the end of the considered time horizon. Hence, a trajectory of the optimal output level resulting from the profit maximization problem in the short-term strategy takes the form as presented in Fig. 4.40. The maximum profits resulting from the profit maximization problems with regard to output level and with regard to production factor input evolve the same, thus a trajectory of the former is the same as the one presented in Fig. 4.37.

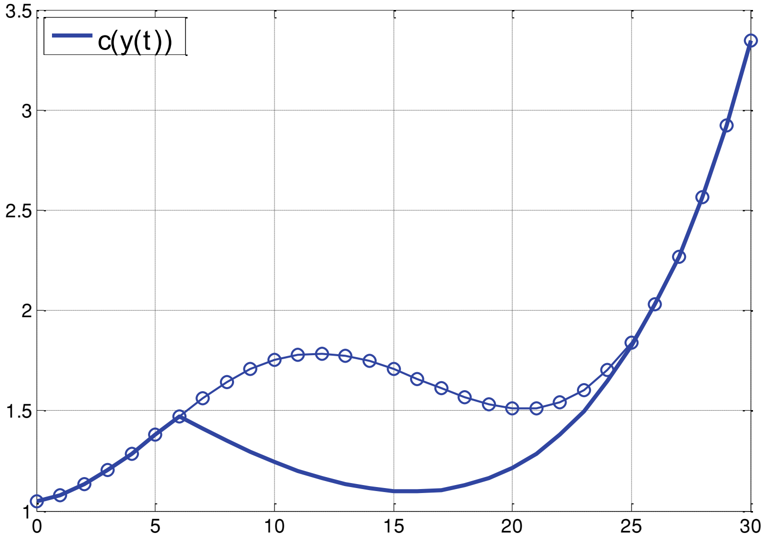


Fig. 4.38 Trajectory of monopoly minimum production cost in a short-term strategy

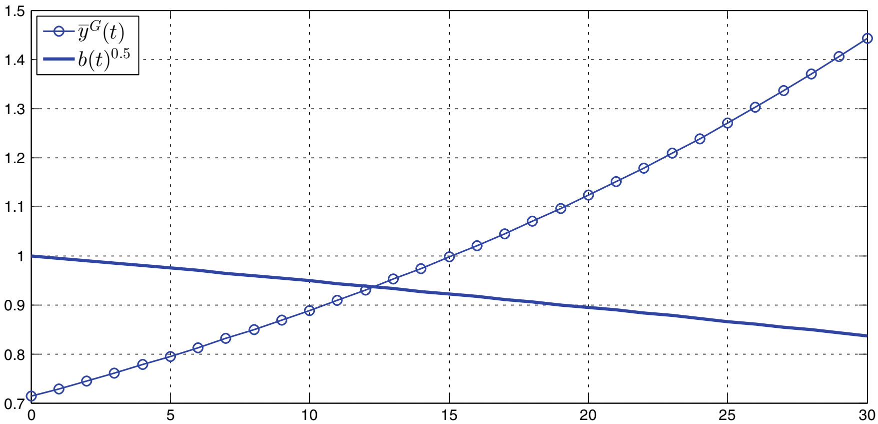


Fig. 4.39 Trajectories of output level constrained by production factor resource and of optimal output levelin long-term strategy—monopoly profit maximization problemwith regard to output level

The product optimal price as a value of the product price function by the optimal product supply takes the form:

$$p(\overline{y}(t)) = \left(\frac{a(t)}{\overline{y}(t)}\right)^{0.5}.$$

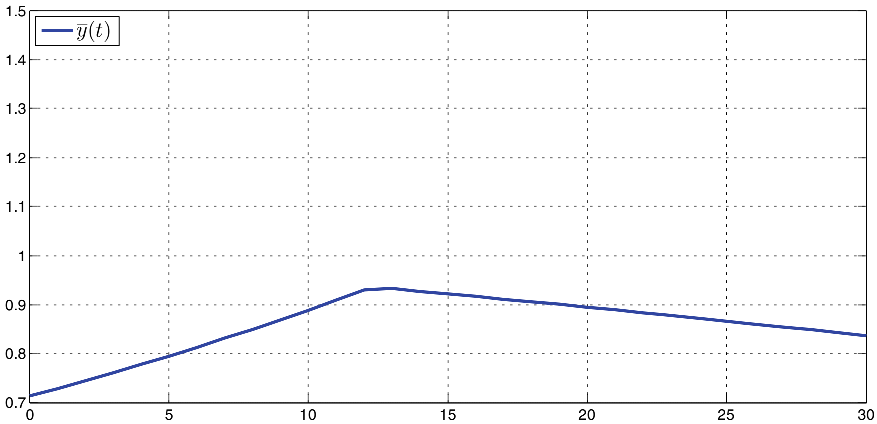


Fig. 4.40 Trajectory of optimal product supply in short-term strategy—case of monopoly

A trajectory of the optimal price is presented in Fig. 4.41. Due to the constraint on the production factor resource, which is binding from a moment $t \approx 13$, the optimal price of a monopoly product in the short-time strategy exceeds the price resulting from the long-term strategy. At any moment t, a monopoly has to take into account the constraint on the production factor resource, and hence it reduces the product supply to the level resulting from the constraint. As a consequence, the optimal price is higher since a monopoly wants to generate high revenues and maximize the profit by the product optimal supply.

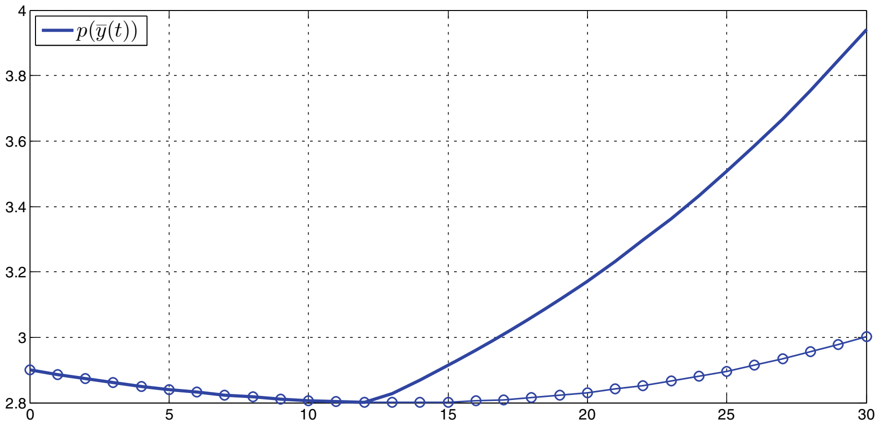


Fig. 4.41 Trajectory of optimal product supply—case of monopoly

4.8 Questions

1. What does it mean that a production function describes a set of technologically effective production processes?
2. What are the relationships and differences in definitions and interpretation of: a marginal productivity of i-th production factor, a growth speed of production, a growth rate of production, an elasticity of production with respect to i-th production factor?
3. What is the difference between an elasticity of production with respect to i-th production factor and an elasticity of production with respect to scale of inputs?
4. Determine relationships of the: concavity, strict concavity, convexity, strict convexity with a degree of the homogeneity of a production function referring to constant, decreasing or increasing returns to scale.
5. What is the relationship between positive homogeneity of a production function and an elasticity of production with respect to scale of inputs?
6. What is the difference between a power production function and a Cobb-Douglas production function?
7. What are the relationships and differences in definitions and interpretation between a marginal rate of substitution and an elasticity of substitution of the first (second) production factor by the second (first) production factor in a vector $\mathbf{x} = (x_1, x_2) \in G$ of production factors' inputs where $G = \{\mathbf{x} \in \mathbb{R}^2_+ | f(\mathbf{x}) = y_0 = \text{const.} \geq 0\} \subset \mathbb{R}^2_+$ means a production isoquant for a fixed output level $y_0 = \text{const.} \geq 0$?
8. What are the assumptions about a marginal rate of substation of the first (second) production factor by the second (first) production factors in a vector $\mathbf{x} = (x_1, x_2) \in G$ of production factors' inputs where $G = \{\mathbf{x} \in \mathbb{R}^2_+ | f(\mathbf{x}) = y_0 = \text{const.} \geq 0\} \subset \mathbb{R}^2_+$ by which a CES production function $f(x_1, x_2) = \left(a_1 x_1^\gamma + a_2 x_2^\gamma\right)^{\frac{1}{\gamma}}$, $a_i > 0$, $i = 1, 2, \gamma \in (-\infty; 0) \cup (0; 1)$ positively homogenous of first degree is convergent to a: linear, Cobb-Douglas, Koopmans-Leontief function?
9. Is it proper to distinguish between a short- and a long-term strategy in view of limited or unlimited resources of production factors?
10. What is the core of a sensitivity analysis of optimal solutions to profit maximization and cost minimization problems for a perfect competition firm, when resources of production factors are limited /unlimited?
11. How to justify that profit maximization problems with regard to production factors' inputs and with regard to output level are equivalent for a perfect competition firm/monopoly when resources of production factors are limited /unlimited?

4.9 Exercises

E1. There is given a CES production function of a form: $f(x_1, x_2) = \left(a_1 x_1^\gamma + a_2 x_2^\gamma\right)^{\frac{\theta}{\gamma}}$, $\theta, a_i > 0$, $i = 1, 2$, $\gamma \in (-\infty; 0) \cup (0; 1)$. Let us assume that a variable $u = \frac{x_2}{x_1}$ describes a quantity of the second production factor per one unit of the first production factor.

1. Justify that:
   (a) an elasticity of marginal rate of substitution of the first production factor by the second production factor is constant and equal to $E_{\sigma_{12}}(u) = 1 - \gamma$,
   (b) an elasticity of marginal rate of substitution of the second production factor by the first production factor is constant and equal to $E_{\sigma_{21}}(u) = \gamma - 1$,
2. Determine a range of variability of constant elasticities $E_{\sigma_{12}}(u)$ and $E_{\sigma_{21}}(u)$ of marginal rates of substitution.
3. Determine an elasticity of production with respect to scale of inputs and justify that it is equal to a degree $E_\lambda(\mathbf{x}) = \theta$ of positive homogeneity of the production function.

E2. For solutions of optimization problems from Examples 3.3, 3.5 and 3.7 analyse sensitivity of.

1. the demand for a production factor and the maximum profit to changes in a product price and to changes in values of parameters of a production function and of a production cost function,
2. the conditional demand for a production factor and the minimum cost of producing y output units to changes in an output level and to changes in values of parameters of a production function and of a production cost function,
3. the product supply and the maximum profit to changes in a product price and to changes in values of parameters of a production function and of a production cost function.

E3. There are given:

$p > 0$—a price of a product manufactured by a firm,
$\mathbf{c}(\mathbf{x}) = (c_1(x_1), c_2(x_2)) > (0, 0)$—a vector of prices of production factors,
$c_i(x_i) = ax_i$—a price of i-th production factor, proportional to the demand for i-th factor,
$\mathbf{x} = (x_1, x_2) \geq (0, 0)$—a vector of inputs of production factors,
$y = f(\mathbf{x})$—an output level described by an increasing, strictly concave and twice differentiable production function,
$r(y) = py$—revenue (turnover) from sales of a manufactured product as a function of output level,
$r(\mathbf{x}) = pf(\mathbf{x})$—revenue (turnover) from sales of a manufactured product as a function of inputs of production factors,

$c^{tot}(\mathbf{x}) = c_1(x_1)x_1 + c_2(x_2)x_2 + d$—total cost of production,
$c^{v}(\mathbf{x}) = c_1(x_1)x_1 + c_2(x_2)x_2$—variable cost of production,
$c^{f}(\mathbf{x}) = d$—fixed cost of production,
$c(y)$—minimum cost of producing y output units, derived as an objective function corresponding to an optimal solution to problem (P2c),
$\pi(y) = r(y) - c(y) = py - c(y)$—firm's profit as a function of output level,
$\pi(\mathbf{x}) = r(\mathbf{x}) - c^{tot}(\mathbf{x})$—firm's profit as a function of inputs of production factors.

For a production function:

(a) power: $y = f(\mathbf{x}) = ax_1^{\alpha_1}x_2^{\alpha_2}, \quad a > 0, \ \alpha_i \in (0; 1), \ \alpha_1 + \alpha_2 < 1, i = 1, 2,$
(b) logarithmic: $y = f(\mathbf{x}) = a_1 \ln x_1 + a_2 \ln x_2, \quad a_i > 0, \ i = 1, 2,$
(c) subadditive: $y = f(\mathbf{x}) = a_1x_1^{\alpha} + a_2x_2^{\alpha}, \quad a_i > 0, \ \alpha \in (0; 1), \ i = 1, 2,$

1. Solve the profit maximization problem (P1c).
2. Present a geometric illustration of the profit maximization problem (P1c).
3. Give an economic interpretation of necessary and sufficient conditions of the existence of an optimal solution to problem (P1c).
4. Analyse sensitivity of the demand for a production factor and of the firm's maximum profit to changes in a price of a product and changes in values of parameters of the cost function and of the production function.
5. Solve the cost minimization problem (P2c).
6. Present a geometric illustration of the cost minimization problem (P2c).
7. Give an economic interpretation of the necessary and sufficient conditions of the existence of an optimal solution to problem (P2c).
8. Analyse sensitivity of the conditional demand for a production factor and of the firm's minimum cost to changes in the price of a product and changes in values of parameters of the cost function and of the production function.
9. Solve the profit maximization problem (P3c).
10. Present a geometric illustration of the profit maximization problem (P3c).
11. Give an economic interpretation of the necessary and sufficient conditions of the existence of an optimal solution to problem (P3c).
12. Analyse sensitivity of the product supply and of the firm's maximum profit to changes in the price of a product and changes in values of parameters of the cost function and of the production function.
13. Justify that the profit maximizations problems (P1c) and (P3c) are equivalent

E4. Solve Exercise E3, when the production total cost function has a form:

$$c^{tot}(x_1, x_2) = \alpha(f(x_1, x_2))^2 + \beta f(x_1, x_2) + \gamma,$$

or equivalently:

$$c^{tot}(y) = \alpha y^2 + \beta y + \gamma,$$

where $y = f(x_1, x_2)$ is an increasing, strictly concave and twice differentiable production function and $\Delta = \beta^2 - 4\alpha\gamma = 0,\ \alpha, \beta, \gamma > 0$.

E5. Solve Exercise E3 assuming additionally that resources of production factors are limited: $\forall i = 1, 2\quad 0 \le x_i \le b_i$, where $\forall i = 1, 2\quad b_i > 0$ means the constrained resource of i-th production factor.

E6. Solve Exercise E3 taking simultaneously into account the data from Exercises E4 and E5.

E7. There are given:

$\mathbf{x} = (x_1, x_2) \ge (0, 0)$—a vector of inputs of production factors,
$y = f(\mathbf{x})$—an output level described by an increasing, strictly concave and twice differentiable production function,
$p(y) = \left(\frac{a}{y}\right)^{\alpha} > 0$—a price of a product manufactured by a monopoly as a function of product supply, set by a monopoly,
$p(f(\mathbf{x})) = \left(\frac{a}{f(\mathbf{x})}\right)^{\alpha} > 0$—a price of a product manufactured by a monopoly as a function of production factors' inputs,
$\mathbf{c}(\mathbf{x}) = (c_1(x_1), c_2(x_2)) > (0, 0)$—a vector of prices of production factors, each of whom is a function of demand reported by a monopoly for a given production factor,
$c_i(x_i) = ax_i$—a price of i-th production factor is proportional to the demand for i-th factor,
$r(y) = p(y)y$—revenue (turnover) from sales of a manufactured product as a function of product supply,
$r(\mathbf{x}) = p(f(\mathbf{x}))f(\mathbf{x})$—revenue (turnover) from sales of a manufactured product as a function of inputs of production factors,
$c^{tot}(\mathbf{x}) = c_1(x_1)x_1 + c_2(x_2)x_2 + d = a\left(x_1^2 + x_2^2\right) + d$—total cost of production,
$c^{v}(\mathbf{x}) = c_1(x_1)x_1 + c_2(x_2)x_2 = a\left(x_1^2 + x_2^2\right)$—variable cost of production,
$c^{f}(\mathbf{x}) = d$—fixed cost of production,
$c(y)$—minimum cost of producing y output units, derived as an objective function corresponding to an optimal solution to problem (P2m),
$\pi(y) = r(y) - c(y) = p(y)y - c(y)$—firm's profit as a function of output level,
$\pi(\mathbf{x}) = r(\mathbf{x}) - c^{tot}(\mathbf{x})$—firm's profit as a function of inputs of production factors.

For a production function:

(a) power: $y = f(\mathbf{x}) = ax_1^{\alpha_1}x_2^{\alpha_2},\ \ a > 0,\ \alpha_i \in (0; 1),\ \alpha_1 + \alpha_2 < 1, i = 1, 2,$
(b) logarithmic: $y = f(\mathbf{x}) = a_1 \ln x_1 + a_2 \ln x_2,\ \ a_i > 0,\ i = 1, 2,$
(c) subadditive: $y = f(\mathbf{x}) = a_1x_1^{\alpha} + a_2x_2^{\alpha},\ \ a_i > 0,\ \alpha \in (0; 1),\ i = 1, 2,$

1. Solve the profit maximization problem (P1m).
2. Present a geometric illustration of the profit maximization problem (P1m).

3. Give an economic interpretation of the necessary and sufficient conditions of the existence of an optimal solution to problem (P1m).
4. Solve the cost minimization problem (P2m).
5. Present a geometric illustration of the cost minimization problem (P2m).
6. Give an economic interpretation of the necessary and sufficient conditions of the existence of an optimal solution to problem (P2m).
7. Solve the profit maximization problem (P3m).
8. Present a geometric illustration of the profit maximization problem (P3m).
9. Give an economic interpretation of the necessary and sufficient conditions of the existence of an optimal solution to problem (P3m).
10. Justify that the profit maximization problems (P1m) and (P3m) are equivalent.
11. Determine the optimal price by which a monopoly obtains the maximum profit.

E8. Solve Exercise E7, when the production total cost function has a form:

$$c^{tot}(x_1, x_2) = \alpha(f(x_1, x_2))^2 + \beta f(x_1, x_2) + \gamma,$$

or equivalently:

$$c^{tot}(y) = \alpha y^2 + \beta y + \gamma,$$

where $y = f(x_1, x_2)$ is an increasing, strictly concave and twice differentiable production function and $\Delta = \beta^2 - 4\alpha\gamma = 0,\ \alpha, \beta, \gamma > 0$.

E9. Solve Exercise E7 assuming additionally that resources of production factors are limited: $\forall i = 1, 2 \quad 0 \le x_i \le b_i$, where $\forall i = 1, 2 \quad b_i > 0$ means the constrained resource of i-th production factor.

E10. Solve Exercise E7 taking simultaneously into account the data from Exercises E8 and E9.

E11. A production process in some firm acting in the perfect competition is described by a one-variable production function of a form:

$$f(x(t)) = x(t)^{0.25}.$$

In periods $t = 0, 1, 2, \ldots, 20$ the price of a production factor, the product price and the production fixed cost evolve according to the following equations:

$$c(t) = 4 \cdot 0.98^{-t},$$

$$p(t) = -0.006t^2 + 0.1t + 3,$$

$$d(t) = \frac{\left(-0.006t^2 + 0.1t + 3\right)^2 t}{480 \cdot 0.98^{-t}} + \frac{t}{30} + 1.$$

Using the dynamic approach:

1. Solve the profit maximization problem with regard to an input of a production factor.
2. Present a trajectory of the demand for a production factor and a trajectory of the firm's maximum profit.
3. Solve the production cost minimization problem assuming that the fixed output level $y(t)$ that the firm wants to achieve in subsequent periods is given by a formula:

$$y(t) = 0.0035(t + 15)^2 + 1.25.$$

4. Present a trajectory of the conditional demand for a production factor and a trajectory of the production minimum cost.
5. Solve the profit maximization problem with regard to output level.
6. Present a trajectory of the product optimal supply and a trajectory of the firm's maximum profit.

E12. At any moment $t \in [0; 20]$ a firm and conditions in which it acts are described as in Exercise E11, except additional constraint in a form of a production factor resource:

$$b(t) = 0.01t + 1.$$

Using the dynamic approach:

1. Solve the profit maximization problem with regard to an input of a production factor and determine time intervals in which the constraint on the production factor resource is binding.
2. Present a trajectory of the demand for a production factor and a trajectory of the firm's maximum profit.
3. Solve the production cost minimization problem assuming that the fixed output level $y(t)$ that the firm wants to achieve in subsequent periods is given by a formula:

$$y(t) = 0.0035(t + 15)^2 + 1.25.$$

   Determine time intervals in which the constraint on the production factor resource is binding

4. Present a trajectory of the conditional demand for a production factor and a trajectory of the production minimum cost.

5. Solve the profit maximization problem with regard to output level and determine time intervals in which a constraint on output level resulting from the constraint on the production factor resource is binding.
6. Present a trajectory of the product optimal supply and a trajectory of the firm's maximum profit.

E13. A production process in a firm acting as a monopoly is described by a one-variable production function of a form:

$$f(x(t)) = x(t)^{0.25}.$$

The price of a product manufactured by this monopoly changes according to a function of a form:

$$p(y(t)) = \left(\frac{a(t)}{y(t)}\right)^{0.5}, \text{ where } a(t) > 0 \quad \forall t,$$

and a production factor price changes in the following way:

$$c(x(t)) = C(t)x(t), \text{ where } C(t) > 0 \quad \forall t.$$

In periods $t = 0, 1, 2, .., 20$ a value of $a(t)$, a value of $C(t)$ and the fixed production cost change according to following equations:

$$a(t) = 3^{0.1t} + 3,$$

$$C(t) = 3^{-0.1t},$$

$$d(t) = \frac{\left(-0.006t^2 + 0.1t + 3\right)^2 t}{480 \cdot 0.98^{-t}} + \frac{t}{30} + 1.$$

Using the dynamic approach:

1. Solve the monopoly profit maximization problem with regard to an input of a production factor.
2. Present a trajectory of the demand for a production factor and a trajectory of the monopoly maximum profit.
3. Solve the production cost minimization problem assuming that the fixed output level $y(t)$ that the monopoly wants to achieve in subsequent periods is given by a formula:

$$y(t) = 0.0035(t + 15)^2 + 1.25.$$

4. Present a trajectory of the conditional demand for a production factor and a trajectory of the production minimum cost.
5. Solve the monopoly profit maximization problem with regard to output level.
6. Present a trajectory of the product optimal supply and a trajectory of the monopoly maximum profit.
7. Determine the product optimal price and present its trajectory.

E14. At any moment $t \in [0; 20]$ a monopoly and conditions in which it acts are described as in Exercise E13, except for an additional constraint in a form of a production factor resource:

$$b(t) = 0.01t + 1.$$

Using the dynamic approach:

1. Solve the monopoly profit maximization problem with regard to an input of a production factor and determine time intervals in which the constraint on the production factor resource is binding.
2. Present a trajectory of the demand for a production factor and a trajectory of the monopoly maximum profit.
3. Solve the production cost minimization problem assuming that the fixed output level $y(t)$ that the monopoly wants to achieve in subsequent periods is given by a formula:

$$y(t) = 0.0035(t + 15)^2 + 1.25.$$

   Determine time intervals in which the constraint on the production factor resource is binding
4. Present a trajectory of the conditional demand for a production factor and a trajectory of the production minimum cost.
5. Solve the monopoly profit maximization problem with regard to output level and determine time intervals in which a constraint on output level resulting from the constraint on the production factor resource is binding.
6. Present a trajectory of the product optimal supply and a trajectory of the monopoly maximum profit.
7. Determine the product optimal price and present its trajectory in comparison to a trajectory of a price which would be set by the monopoly if there was no constraint on the production factor resource.

Rationality of Choices Made by a Group of Producers by Exogenously Determined Function of Demand for a Product

5

In this chapter, you will learn:

- what it means that the demand for a product is described by a function determined exogenously;
- what parameters are used in an exogenous function of demand and what their economic interpretation is;
- what the importance of price elasticity of demand for a product is when setting an optimal price by a monopoly;
- what discriminatory pricing is and how it is practised by a monopoly;
- what type of competition is used by firms in a Cournot, Stackelberg or Bertrand duopoly;
- what it means that firms have equal positions on a market in a duopoly and what results from this;
- what a leader position and a follower position in a duopoly are and how it matters for the shares of duopolists on a market of a product;
- what assumptions in a Bertrand duopoly model are needed to describe a situation when one of the producers is a leader and the other one is a follower.

In this chapter, we proceed with further analysis of the issues presented in Chap. 4, that is, rational choices made by producers acting on a market of one product in perfect competition or as a monopoly when resources of production factors are unlimited or limited. Our attention was paid to technological and financial aspects of the rational behaviour of producers assuming that there is no binding constraint resulting from the demand for a product they manufacture.

Now, we want to focus on financial and market aspects of the rational behaviour of producers. For this purpose, we assume that on a market of a given product

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its supply matches the demand described by an exogenously determined demand function.

In Sect. 5.1, we analyse the rational behaviour of producers in perfect competition who are interested in the product optimal supply that enables them to reach maximum profits by an exogenously determined function of demand for a product. An important part of the considerations taken in this chapter is an attempt to determine what conditions need to be satisfied in order to have the price of a product established by a market in perfect competition being a Walrasian equilibrium price.

In Sect. 5.2, we consider the rational behaviour of a monopolist who sets the optimal supply of a product he/she manufactures as well as the optimal price to maximize profit by an exogenously determined function of demand for a product.[1]

In Sect. 5.3, we regard the issue of price discrimination practised by a monopolist who can sell her/his product at different prices on different markets where exogenously determined functions of demand for a product differ too.

In Sects. 5.4 and 5.5, we discuss the issues of quantity competition in a duopoly described by the Cournot and Stackelberg models. Attention is paid especially to equilibrium states in these models determined by the product optimal supply by both producers and by equilibrium prices when a function of demand for a product is determined exogenously.

In Sect. 5.6, we regard the topic of price competition in a duopoly described by the Bertrand model. We focus on a mechanism of setting prices of two substitute products manufactured by two producers when functions of demand for each product are determined exogenously.

For each of discussed topics, we conduct a sensitivity analysis of equilibrium states to changes in parameters that determine these states.

5.1 Firm Acting in Perfect Competition—Determining Optimal Output Level

Conditions of the perfect competition are to be understood as a description of a market of a given product based on four assumptions:

- atomization: number of economic agents reporting the demand for a given product (consumers) or the supply of this product (producers) is large enough that each of them has no crucial impact on a price level of the product or conditions of its exchange. Each firm treats a product price as given by the market, thus as a parameter and adjusts the level of its output to the price.
- homogeneity of a product: products manufactured by firms are not differentiated,

[1] For further discussion of topics presented in this chapter, we recommend examining the work (Tokarski, 2011a).

- transparency: each economic agent has perfect knowledge about the supply and a price of product available on the market,
- liquidity: there are no barriers making it difficult to enter the market or to leave the market since every such decision does not involve any additional costs.

In economic reality fulfilling all these four principles is highly unlikely. Hence, every model of a product market that does not satisfy at least one of these four principles is the model of imperfect competition.

Let us analyse the rational behaviour of a producer (of a firm) acting in the perfect competition on a market of a homogenous product where two producers offer their product.[2]

Definition 5.1 A **function of consumer demand** for a product manufactured by two producers is a mapping $h:\mathbb{R}_+ \to \mathbb{R}_+$ given in a form:

$$y^d = h(p), \tag{5.1}$$

where y^d means a level of demand for a given product and p means the price of this product.

This function is assumed to be twice differentiable and decreasing:

$$\frac{\mathrm{d}y^d}{\mathrm{d}p} = \frac{\mathrm{d}h(p)}{\mathrm{d}p} < 0, \tag{5.2}$$

meaning that, when the price of the product increases, the demand for this product decreases. A graph of such a function is called a **demand curve**.

An output level (supply) of a product produced by both producers must be equal to the demand reported by consumers at the price of the product given by a market:

$$y^s = y^d = h(p). \tag{5.3}$$

It is a very strong assumption, and in case of exogenously determined functions of the demand and of the product supply being functions of a price of a homogenous product, it means that the product price is an equilibrium price.

For the sake of simplicity, let us assume that the functions of the demand and the supply are linear.

[2] We present the analysis for just two producers for the sake of simplicity. More general approach is to consider $r \in \mathbb{N}$ producers but an essential assumption in the perfect competition model is that each of them has no market power to set a price of her/his product and each adjusts the product supply to the price given by a market. Thus, regarding r producers is a simple extension of a case when just two producers are considered.

A demand function is decreasing with respect to a product price[3]:

(5.4) $$y^d = y_1^d + y_2^d = \sum\nolimits_{k=1}^{2} (-a_k p + b_k) = -ap + b, \quad a_k, b_k > 0,\ k = 1, 2,$$

where $a = a_1 + a_2 > 0,\ b = b_1 + b_2 > 0.$

Parameter a can be interpreted as a **measure of the consumers' reaction strength** to a unit increase in the price of a product:

(5.5) $$\frac{\mathrm{d}y^d}{\mathrm{d}p} = -a < 0,$$

then **price elasticity of demand for a product** is negative and takes the form:

(5.6) $$E(y^d) = \frac{\mathrm{d}y^d}{\mathrm{d}p}\frac{p}{y^d} = \frac{-ap}{-ap + b} < 0,$$

since, from condition (5.4), the following expression is positive[4]:

(5.7) $$y^d = -ap + b > 0.$$

Parameter b can be interpreted as a **measure of a market capacity**, which corresponds to the maximum demand that consumers can report for this product at a product price equal to zero.

A supply function is increasing with respect to a product price

(5.8) $$y^s = y_1^s + y_2^s = \sum\nolimits_{j=1}^{2} \left(c_j p + d_j\right) = cp + d, \quad c_j, d_j > 0,\ j = 1, 2,$$

where $c = c_1 + c_2 > 0,\ d = d_1 + d_2 > 0,$

and

(5.9) $$b > d.$$

Parameter c can be interpreted as a **measure of the producers' reaction strength** to a unit increase in the price of a product:

(5.10) $$\frac{\mathrm{d}y^s}{\mathrm{d}p} = c > 0,$$

[3] In general, the number of consumers is $k = 1, 2, \ldots, m$ and the number of producers is $j = 1, 2, \ldots, r$. However, without loss of generality, for the sake of simplicity, we take that $m = r = 2$.
[4] The negative value of the price elasticity of the demand results also from condition (5.5) and from the fact that the price of a product and the demand for a product should be positive.

then price elasticity of the product supply is negative and takes the form:

(5.11) $$E(y^s) = \frac{dy^s}{dp}\frac{p}{y^s} = \frac{cp}{cp+d} > 0,$$

since, from condition (5.8), the following expression is positive[5]:

(5.12) $$cp + d > 0.$$

Parameter d can be interpreted as a **level of stocks** owned by producers without undertaking the production process. It determines a minimum level of the product supply.

From definitions of the demand function and of the supply function, it results that

(5.13) $$y^d \in [0; b],$$

(5.14) $$p \in \left[0; \frac{b}{a}\right],$$

while an output level satisfies a condition:

(5.15) $$y^s \in [d; b].$$

Let us notice that the range of a product price is determined by a function of the demand for a product. If one knows a demand function and a supply function, then one can determine an equilibrium price $\bar{p} \in \left(0; \frac{b}{a}\right)$ which equalizes the demand for a product and the product supply both expressed in the same physical units. In order to do this one needs to solve an equation:

(5.16) $$y^s = y^d \Leftrightarrow \quad c\bar{p} + d = -a\bar{p} + b,$$

from which, after some transformations, we get

(5.17) $$\bar{p} = \frac{b-d}{a+c} \in \left(0; \frac{b}{a}\right),$$

when $b > d$.

In Fig. 5.1, we present graphs of the functions of the demand and the product supply. It can be seen from this figure that there exists an exactly one positive

[5] The negative value of the price elasticity of the product supply results also from condition (5.10) and from the fact that the price of a product and the demand for a product should be positive.

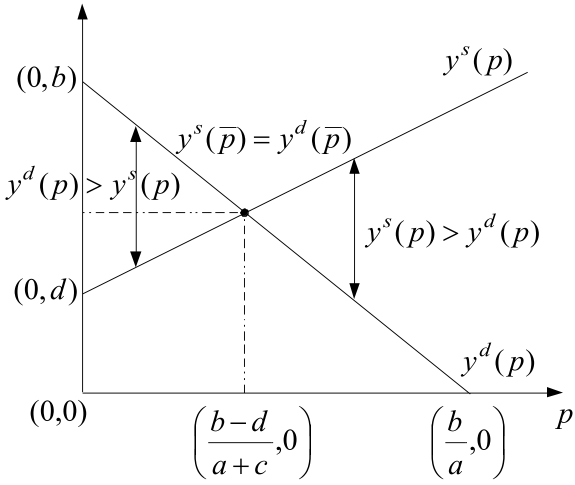


Fig. 5.1 Functions of demand for product and of product supply

equilibrium price by which the demand for a product and the product supply expressed in the physical units are equal to each other. Moreover, the equilibrium state described this way is beneficial for producers as well as for consumers. When producers need to manufacture a product then the equilibrium price enables them to increase the product supply without creating stocks that mean unmade sales of the manufactured product. From the consumers' point of view, the equilibrium price is also beneficial because it enables them to purchase a product maximum quantity that producers are willing to manufacture. The equilibrium price is thus a synonym for the rational behaviour of consumers and producers willing to respect consumer preferences. Consumers are interested in purchasing a product's maximum quantity at the lowest price, while producers are interested in manufacturing a product's maximum quantity at the highest price of their product.

Both consumers and producers have to take into account various constraints related to the way a market is organized, conditions of purchase and sales, as well as management and technology of manufacturing a product. In this book, we concentrate essentially on constraints resulting from financial criteria of production profitability. Such a criterion for each producer is the profit as a difference between revenue from sales of a manufactured product and production total cost consisting of variable cost (dependent on an output level) and fixed cost (independent of an output level).

Here, it is worth asking two substantial questions concerning the equilibrium price. What are the conditions for the existence of the equilibrium price? And what is a mechanism that enables establishing the equilibrium price on a market?

In the market model considered in this subchapter, thus assuming linear functions of the demand for a product and of the product supply, a condition for the existence of the positive equilibrium price takes the form: $b > d > 0$. This means that the market capacity has to be bigger than existing product stocks.

The mechanism of determining the equilibrium price in the market model considered here is simple. Let us recall that consumers and producers know the functions of the demand and the product supply. Thus, it is enough to determine a price by which consumers and producers accomplish their goals to the maximum extent possible which, however, includes existing and known constraints from the supply side and the demand side. In order to determine this price, one needs to just solve Eq. (5.16). Then one needs to state what the rules of market functioning which would be respected by producers as well as by consumers are and also organizational principles of establishing the equilibrium price.[6]

In reality, such a simple, and at the same time desirable, situation is very rare to happen because functions of the demand of the product supply are known not in advance but only *ex post*. That is, why both consumers and producers, who try to make rational purchase and sale decisions on the basis of their understanding of a market; in fact, they have only rough knowledge. This knowledge is extrapolated from data about what happened on a market in the past and from mutual contacts serving recognition what the situation of each agent which is important in exchange transactions taken on market of a given product is.

The price of a product at which it is purchased and sold is then most of the time not the equilibrium price. This means that part of the effectively existing demand is not satisfied or part of the supply does not find recipients because producers overestimate the demand by an actual price level.

Let us consider the rational behaviour of producers when we are given an exogenously determined function of the demand for a product they manufacture. On this basis, one can determine a feasible price range. Let us also assume that a product supply function is not known and that it is a result of the rational choices of producers. The rational choices made by producers are to be understood as their decisions enabling each of them to obtain maximum profit.

Let us start the analysis of rational decisions of producers with a case when, motivated by profit maximization, they determine an output level by a price given by the market.[7]

[6] An example of such a mechanism can be found in static and dynamic Arrow-Hurwicz models discussed in ths chapter. However, it is worth emphasizing that, in both these models, producers are not present and more generally speaking the production side of the economy is beyond the scope of interest. In this chapter, we discuss such mechanisms with reference to static and dynamic market models with exogenous or endogenous functions of the demand and of the supply of a product or of products.

[7] In Sect. 5.2, we regard a case when a monopolist as the only one producer of a given product decides on a price level and an output level guaranteeing maximum profit by an exogenously determined function of the demand for a product.

Let us assume that each producer aims to maximize profit by a price given by a market at which levels of the global supply and of the global demand are equal to each other[8]:

$$y^s = y^d = y. \tag{5.18}$$

We conduct our considerations assuming that a product price takes values in an interval $\left(0; \frac{b}{a}\right)$. The price given by a market is treated by producers as a parameter that is a quantity that they do not affect directly. Since a precise product supply level is not known, a product price is set in advance in conditions of incomplete knowledge about the demand and the product supply. Hence, this price can, but does not, have to be the equilibrium price at a moment when producers decide on the output levels and consumers do not know everything about the product supply. The *ex post* price set on a market is the equilibrium price in reference to this product quantity that has been effectively sold and purchased. Equation (5.18) is interpreted in two ways which means it determines the conditions of the equilibrium as the *ex ante* or the *ex post*. Here, we allow the possibility that a product price given by a market and treated by producers as a parameter is not necessarily the equilibrium price. However, producers relying on this price make decisions on output levels and this way they determine what product supplies are the most profitable from their perspective. When the transaction of product exchange takes place on a market, a price given by the market equalizes the actual demand (which can be satisfied) and the actual supply (the product output which can be sold due to the existing demand).

Let us now consider in more detail the rational behaviour of producers acting in the perfect competition by an exogenously determined function of the demand for a product and by a product price given by a market and treated by producers as a parameter.

Definition 5.2 An **inverse function of consumer demand** for a product supplied to a market by producers is a mapping $g:\mathbb{R}_+ \to \mathbb{R}_+$ given in a form:

$$p(y^s) = g\left(y^d\right), \tag{5.19}$$

which is an identity transformation of a demand function:

$$y^d = h(p(y^s)) = -ap(y^s) + b \iff p(y^s) = \frac{b - y^d}{a}. \tag{5.20}$$

It is worth explaining more precisely whether the inverse demand function is the inverse of the demand function.

[8] When it is reasonable and useful, we write y as an output level instead of distinguishing between the product supply y^s and the demand y^d for a product.

Definition 5.3 Functions $y = h(p)$ and $p = g(y)$ are called **inverse to each other** if $\forall a, b \in \mathbb{R}_+$ if the following conditions are satisfied:

$$b = h(a) \Rightarrow a = g(b) \tag{5.21}$$

and at the same time:

$$a = h(b) \Rightarrow b = g(a). \tag{5.22}$$

If h function is given, then g function is called an inverse[9] (or an inverse function) to h. Since the choice of the variable is arbitrary, we can write the inverse $p = g(y)$ also as $y = g(p)$. Then graphs of functions $y = h(p)$ and $y = g(p)$ inverse to each other are symmetric with respect to the line $y = p$.

Example 5.1 Let us determine an inverse to a linear demand function:

$$y^d = h(p) = -ap + b, \quad a, b > 0. \tag{5.23}$$

From Eq. (5.23), one determines functional relationships of a product price depending on a demand level:

$$y^d = -ap + b \Leftrightarrow p = \frac{b - y^d}{a} \quad a, b > 0, \tag{5.24}$$

which is an inverse function of consumer demand.

In the form of this function, one changes the denotation so that a dependent variable is the demand level and an independent variable is the product price:

$$y^d = g(p) = \frac{b - p}{a} \quad a, b > 0. \tag{5.25}$$

The function derived this way is an inverse function to the demand function. Its graph is symmetric to the graph of a function $y^d = h(p)$ with respect to the line $y = p$ which is presented[10] in Fig. 5.2.

[9] From Definitions 5.2 to 5.3, it results that an inverse function of consumer demand is not inverse to a demand function. An inverse function is just a mathematical concept and it does not matter what the interpretation of variables is, thus if we write $p = g(y)$, p depends on y, or if $y = p(y)$, y depends on p. While in the concept of an inverse function of consumer demand, it is very important that y^d means the demand, p means the product price and y^s means the supply and what depends on what.

[10] The graph of the inverse function of consumer demand coincides with the graph of the demand function.

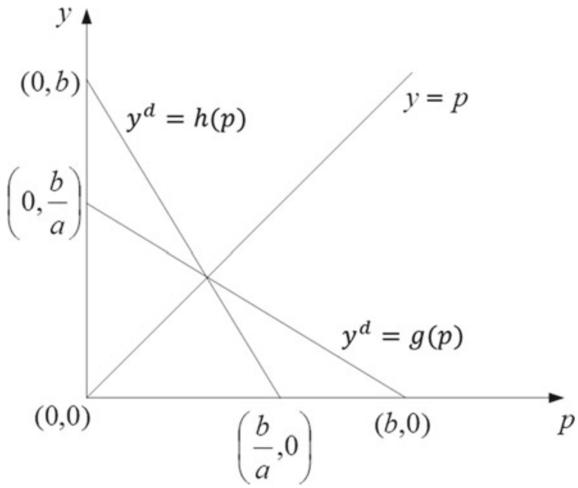


Fig. 5.2 Graphs of linear function of consumer demand, inverse function of consumer demand and function inverse to demand function

Theorem 5.1 If functions $y = h(p)$ and $p = g(y)$ are inverse to each other, then the following condition is satisfied:

(5.26) $$\frac{\mathrm{d}y}{\mathrm{d}p} = \frac{\mathrm{d}h(p)}{\mathrm{d}p} = \frac{1}{\frac{\mathrm{d}g(y)}{\mathrm{d}y}} = \frac{1}{\frac{\mathrm{d}p}{\mathrm{d}y}}.$$

On the basis of Eqs. (5.23) and (5.24), one can notice that

(5.27) $$\frac{\mathrm{d}y^d}{\mathrm{d}p(y^s)} = -a$$

and

(5.28) $$\frac{\mathrm{d}p(y^s)}{\mathrm{d}y^d} = -\frac{1}{a},$$

which means that condition (5.26) is satisfied in case of the linear demand function and its inverse.

Definition 5.4 Revenue (turnover) from sales of a product manufactured by j-th producer ($j = 1, 2$) is an expression:

(5.29) $$r_j\left(y_j^s\right) = py_j^s,$$

where $p > 0$ is the product price given by a market that can, but does not, have to be the equilibrium price. Then

(5.30)
$$\frac{\mathrm{d}r_j(y_j^s)}{\mathrm{d}y_j^s} = p > 0,$$

which means that marginal revenue from sales of a product is equal to the price of a product.

Definition 5.5 A function of production total cost for j-th producer ($j = 1, 2$) is an expression:

(5.31)
$$c_j^{tot}\left(y_j^s\right) = c_j^v\left(y_j^s\right) + c_j^f\left(y_j^s\right),$$

where

$c_j^v(y_j^s)$—variable cost of production, dependent on production level,
$c_j^f(y_j^s) = \text{const.}$—fixed cost of production, independent of production level.

Note 5.1 From definitions of functions of production total, variable and fixed cost it results that

(5.32)
$$\frac{\mathrm{d}c_j^{tot}\left(y_j^s\right)}{\mathrm{d}y_j^s} = \frac{\mathrm{d}c_j^v\left(y_j^s\right)}{\mathrm{d}y_j^s}, \quad j = 1, 2,$$

which means that the marginal total cost of production for j-th producer is equal to the marginal variable cost of production.

For the sake of simplicity, let us assume that functions of production total cost for both producers are linear and of a form:

(5.33)
$$c_j^{tot}\left(y_j^s\right) = \gamma_j y_j^s + \delta_j, \quad j = 1, 2,$$

where:

$\gamma_j > 0$—the production marginal cost for j-th producer,
$\delta_j \geq 0$—the production fixed cost for j-th producer.

Let us consider a profit maximization problem[11] for j-th producer:

(5.34) $$\Pi_j(y) = r_j\left(y_j^s\right) - c_j^{tot}\left(y_j^s\right) = \left\{py_j^s - c_j^{tot}\left(y_j^s\right)\right\} \mapsto \max$$

(5.35) $$y_j^s \geq 0, \;\; j = 1, 2.$$

Knowing that $y^s = y^d = y$, we substitute a product price with an expression from condition (5.20):

(5.36) $$p = \frac{b - y^d}{a} = \frac{b - y^s}{a} = \frac{b - (y_1^s + y_2^s)}{a} = \alpha - \beta\left(y_1^s + y_2^s\right),$$

which, for the sake of simplicity, we write as

(5.37) $$p = \alpha - \beta(y_1 + y_2),$$

where: $\alpha = \frac{b}{a}, \;\; \beta = \frac{1}{a}$.

Let us also assume that the first (second) producer wants to determine an output level $\bar{y}_1 > 0$ ($\bar{y}_2 > 0$) by which he/she obtains the maximum profit by a fixed output level of the second (first) producer.

Then profit maximization problems of the first and the second producers take the form:

(5.38) $$\Pi_1(y) = r_1\left(y_1^s\right) - c_1^{tot}\left(y_1^s\right) = \left\{py_1^s - c_1^{to}\left(y_1^s\right)\right\} \mapsto \max$$

(5.39) $$y_1 \geq 0,$$

(5.40) $$\Pi_2(y) = r_2\left(y_2^s\right) - c_1^{tot}\left(y_2^s\right) = \left\{py_2^s - c_2^{tot}\left(y_2^s\right)\right\} \mapsto \max$$

(5.41) $$y_2 \geq 0.$$

Substituting a product price determined as in Condition (5.37) into the profit functions, we get the profit maximization problems in an equivalent form:

(5.42) $$\Pi_1(y_1) = \left\{-\beta y_1^2 - \beta y_1 \bar{y}_2 + (\alpha - \gamma_1)y_1 - \delta_1\right\} \mapsto \max$$

(5.43) $$y_1 \geq 0, \bar{y}_2 = \text{const.} > 0$$

(5.44) $$\Pi_2(y_2) = \left\{-\beta y_2^2 - \beta \bar{y}_1 y_2 + (\alpha - \gamma_2)y_2 - \delta_2\right\} \mapsto \max$$

[11] It is the profit maximization problem of type (P3c) discussed in Chap. 4.

(5.45) $$y_2 \geq 0, \bar{y}_1 = \text{const.} > 0.$$

From Theorem 4.6, we know if a profit function of j-th producer is strictly concave and the following condition is satisfied:

(5.46)
$$\begin{aligned} &\forall j = 1,2 \lim_{y_j \to 0^+} \frac{\mathrm{d}\Pi_j(y_j)}{\mathrm{d}y_j} > 0 \quad \wedge \quad \lim_{y_j \to +\infty} \frac{\mathrm{d}\Pi_j(y_j)}{dy_j} < 0 \\ &\Leftrightarrow \lim_{y_j \to 0^+} \frac{\mathrm{d}r_j(y_j)}{\mathrm{d}y_j} > \lim_{y_j \to 0^+} \frac{\mathrm{d}c_j^{tot}(y_j)}{\mathrm{d}y_j} \\ &\wedge \lim_{y_j \to +\infty} \frac{\mathrm{d}r(y_j)}{\mathrm{d}y_j} < \lim_{y_j \to +\infty} \frac{\mathrm{d}c_j^{tot}(y_j)}{\mathrm{d}y_j} \end{aligned}$$

then:

(1) $\exists_1 \bar{y}_j > 0$ such that $\left.\frac{\mathrm{d}\Pi_j(y_j)}{\mathrm{d}y_j}\right|_{y_j=\bar{y}_j} = 0,$

(2) a necessary and sufficient condition for $\bar{y}_j > 0$ being an optimal solution to the profit maximization problem for the first producer is

(5.47)
$$\begin{aligned} &\left.\frac{\mathrm{d}\Pi_1(y_1)}{\mathrm{d}y_1}\right|_{y_1=\bar{y}_1, \bar{y}_2=\text{const.}>0} = 0 \quad \Leftrightarrow \\ &\Leftrightarrow \left.\frac{\mathrm{d}r_1(y_1)}{\mathrm{d}y_1}\right|_{y_1=\bar{y}_1, \bar{y}_2=\text{const.}>0} = \left.\frac{\mathrm{d}c_1^{tot}(y_1)}{\mathrm{d}y_1}\right|_{y_1=\bar{y}_1, \bar{y}_2=\text{const.}>0} \end{aligned}$$

and for the second producer:

(5.48)
$$\begin{aligned} &\left.\frac{\mathrm{d}\Pi_2(y_2)}{\mathrm{d}y_2}\right|_{y_2=\bar{y}_2, \bar{y}_1=\text{const.}>0} = 0 \quad \Leftrightarrow \\ &\Leftrightarrow \left.\frac{\mathrm{d}r_2(y_2)}{\mathrm{d}y_2}\right|_{y_2=\bar{y}_2, \bar{y}_1=\text{const.}>0} = \left.\frac{\mathrm{d}c_2^{tot}(y_2)}{\mathrm{d}y_2}\right|_{y_2=\bar{y}_2, \bar{y}_1=\text{const.}>0}. \end{aligned}$$

which means that by $\bar{y}_j$ the marginal profit of j-th producer equals zero. This happens if and only if the marginal revenue from sales of a product by j-th producer is equal to the production marginal cost for j-th producer.

Conditions (5.47) and (5.48) can be transformed into an equation system:

(5.49) $$\left.\frac{\mathrm{d}\Pi_1(y_1)}{\mathrm{d}y_1}\right|_{y_1 = \bar{y}_1, \bar{y}_2 = \text{const.} > 0} = -2\beta\bar{y}_1 - \beta\bar{y}_2 + (\alpha - \gamma_1) = 0,$$

(5.50) $$\left.\frac{\mathrm{d}\Pi_2(y_2)}{\mathrm{d}y_2}\right|_{y_2=\bar{y}_2,\bar{y}_1=\text{const.}>0} = -2\beta\bar{y}_2 - \beta\bar{y}_1 + (\alpha - \gamma_2) = 0,$$

from which one determines an optimal output level maximizing the profit of the first producer:

(5.51) $$\bar{y}_1 = \frac{\alpha - 2\gamma_1 + \gamma_2}{3\beta}$$

or having $\alpha = \frac{b}{a},\ \beta = \frac{1}{a}$:

(5.52) $$\bar{y}_1 = \frac{b - a(2\gamma_1 - \gamma_2)}{3},$$

an optimal output level maximizing the profit of the first producer:

(5.53) $$\bar{y}_2 = \frac{\alpha + \gamma_1 - 2\gamma_2}{3\beta}$$

or having $\alpha = \frac{b}{a},\ \ \beta = \frac{1}{a}$:

(5.54) $$\bar{y}_2 = \frac{b - a(2\gamma_2 - \gamma_1)}{3}.$$

Then the product global supply is

(5.55) $$\bar{y} = \bar{y}_1 + \bar{y}_2 = \frac{2\alpha - (\gamma_1 + \gamma_2)}{3\beta}$$

or having $\alpha = \frac{b}{a},\ \beta = \frac{1}{a}$,

(5.56) $$\bar{y} = \frac{2b - a(\gamma_1 + \gamma_2)}{3}.$$

From conditions (5.52), (5.54) and (5.56), it results that the optimal supply by each producer and the optimal global supply, taking a product price as given by a market, they all depend on a market capacity $b > 0$, the strength $a > 0$ of consumers' reaction to changes in the product price, production marginal costs $\gamma_1, \gamma_2 > 0$ for both producers.

Let us analyse the sensitivity of the optimal supply of each producer and the optimal global supply to changes in values of parameters that describe these optimal levels. Values of adequate measures are presented in Tables 5.1a, 5.1b and 5.1c.

Table 5.1a Measures of response of product supply by the first producer to changes in parameters' values

Characteristic	$\frac{\partial \bar{y}_1}{\partial a}$	$\frac{\partial \bar{y}_1}{\partial b}$	$\frac{\partial \bar{y}_1}{\partial \gamma_1}$	$\frac{\partial \bar{y}_1}{\partial \gamma_2}$
Value	$-\frac{2\gamma_1-\gamma_2}{3}$	$\frac{1}{3}$	$-\frac{2a}{3}$	$\frac{a}{3}$

Table 5.1b Measures of response of product supply by the second producer to changes in parameters' values

Characteristic	$\frac{\partial \bar{y}_2}{\partial a}$	$\frac{\partial \bar{y}_2}{\partial b}$	$\frac{\partial \bar{y}_2}{\partial \gamma_1}$	$\frac{\partial \bar{y}_2}{\partial \gamma_2}$
Value	$-\frac{2\gamma_2-\gamma_1}{3}$	$\frac{1}{3}$	$\frac{a}{3}$	$-\frac{2a}{3}$

Table 5.1c Measures of response of the product global supply to changes in parameters' values

Characteristic	$\frac{\partial \bar{y}}{\partial a}$	$\frac{\partial \bar{y}}{\partial b}$	$\frac{\partial \bar{y}}{\partial \gamma_1}$	$\frac{\partial \bar{y}}{\partial \gamma_2}$
Value	$-\frac{\gamma_1+\gamma_2}{3}$	$\frac{2}{3}$	$-\frac{a}{3}$	$-\frac{a}{3}$

Assuming $\gamma_1 > \frac{1}{2}\gamma_2$, the optimal supply by the first producer decreases when the strength of consumers' reaction or marginal cost for her/him increases (*ceteris paribus*). It increases when a market capacity or marginal cost for the second producer increases (*ceteris paribus*).

Assuming $\gamma_2 > \frac{1}{2}\gamma_1$, the optimal supply by the second producer decreases when the strength of consumers' reaction or marginal cost for her/him increases (*ceteris paribus*). It increases when a market capacity or marginal cost for the first producer increases (*ceteris paribus*).

The optimal global supply of a product decreases when the strength of consumers' reaction increases (*ceteris paribus*) or marginal cost for the first or the second producer increases. It increases only when the market capacity increases. A unit increase in the market capacity affects the optimal output level for each producer in the same way causing an increase by 1/3 of a physical unit, regardless of the fact that marginal costs for two producers can differ. This is due to the fact that producers act in perfect competition, thus changes in the market capacity affect each producer the same way.[12]

[12] Conducting similar calculations for three producers, one can check, and this is also one of the exercises given at the end of this chapter, that the impact of a unit increase in a market capacity on the optimal total supply is equal to 3/4. In general, assuming linear functions of costs and of the consumer demand, when there is r producers ($r \in \mathbb{N}, r \geq 2$) the impact of a unit increase in market capacity on the optimal total supply is equal to $r/(r+1)$ and on the optimal supply of j-th producer is equal to $1/(r+1)$. Thus, the larger the number of producers in the perfect competition market, the bigger the positive impact of an increase in market capacity on the optimal total supply is, but at the same time, it is smaller when it comes to the optimal supply of a single producer.

Let us now determine the maximum profit for each producer, substituting the optimal output levels into the profit functions. After some not complex but yet quite time-consuming transformations, one gets the maximum profit for the first producer:

$$\Pi_1(\bar{y}_1) = \frac{\alpha^2 - 4\alpha\gamma_1 + 2\alpha\gamma_2 + 4\gamma_1^2 - 4\gamma_1\gamma_2 + \gamma_2^2 - 9\beta\delta_1}{9\beta}, \tag{5.57}$$

or having $\alpha = \frac{b}{a},\ \beta = \frac{1}{a}$:

$$\Pi_1(\bar{y}_1) = \frac{b^2 - 4ab\gamma_1 + 2ab\gamma_2 + 4a^2\gamma_1^2 - 4a^2\gamma_1\gamma_2 + a^2\gamma_2^2 - 9a\delta_1}{9a} \tag{5.58}$$

and the maximum profit for the first producer:

$$\Pi_2(\bar{y}_2) = \frac{\alpha^2 + 2\alpha\gamma_1 - 4\alpha\gamma_2 + \gamma_1^2 - 4\gamma_1\gamma_2 + 4\gamma_2^2 - 9\beta\delta_2}{9\beta}, \tag{5.59}$$

or having $\alpha = \frac{b}{a},\ \beta = \frac{1}{a}$:

$$\Pi_2(\bar{y}_2) = \frac{b^2 + 2ab\gamma_1 - 4ab\gamma_2 + a^2\gamma_1^2 - 4a^2\gamma_1\gamma_2 + 4a^2\gamma_2^2 - 9a\delta_2}{9a}. \tag{5.60}$$

From Eqs. (5.42) and (5.44), it results that the maximum profit of each producer depends on the optimal supply by her/him and the optimal supply by the other producer. As a consequence, the maximum profit of j-th producer depends on market capacity $b > 0$, the strength $a > 0$ of consumers' reaction to changes in a product price, production marginal costs $\gamma_1, \gamma_2 > 0$ for both producers and on the production fixed cost δ_j for j-th producer. Whether this maximum profit is positive, equal to zero or negative (minimum loss) depends on the values of mentioned parameters, especially on a level of the fixed cost for j-th producer.

At the end, let us determine if the price by which both producers decide on their optimal output levels is the equilibrium price which would become established on a market with the consumer demand function (5.4) and the product supply function (5.8).

From condition (5.37), one can determine a price level by which both producers decide on their optimal output levels guaranteeing maximum profits:

$$\begin{aligned} \bar{p} &= \alpha - \beta(\bar{y}_1 + \bar{y}_2) = \alpha - \beta\left(\frac{2\alpha - (\gamma_1 + \gamma_2)}{3\beta}\right) \\ &= \frac{\alpha + (\gamma_1 + \gamma_2)}{3} = \frac{b + a(\gamma_1 + \gamma_2)}{3a} > 0. \end{aligned} \tag{5.61}$$

At the same time, from the definition of the product supply given in Eq. (5.8), we know that a positive product price should satisfy a condition:

$$\bar{p} = \frac{\bar{y} - d}{c} = \frac{2\alpha - (\gamma_1 + \gamma_2) - 3\beta d}{3\beta c} = \frac{2b - a(\gamma_1 + \gamma_2) - 3d}{3c}. \tag{5.62}$$

Since from (5.17) $\bar{p} = \frac{b-d}{a+c} > 0$ then from conditions (5.61) and (5.62) we get that

$$\frac{b + a(\gamma_1 + \gamma_2)}{3a} = \frac{2b - a(\gamma_1 + \gamma_2) - 3d}{3c}, \tag{5.63}$$

which results in condition:

$$\gamma_1 + \gamma_2 = \frac{2ab - 3ad - bc}{a(a + c)} \tag{5.64}$$

that determines relationships between values of parameters of a product supply function, a consumer demand function and a production variable cost function and more generally speaking relationships between these functions.[13]

If condition (5.64) is satisfied, then the price by which producers decide on the product optimal supply guaranteeing the maximum profits is the equilibrium price.

5.2 Monopoly—Determining Optimal Price and Optimal Output Level

5.2.1 Static Approach

A firm is called a **monopoly** and its owner is called a **monopolist** in reference to a certain product (good or service) when it is the sole supplier of this product. A market of a given product in which there is only one supplier of this product is also called a **monopoly**.

A monopolistic firm, that aims to obtain maximum profit, sets an optimal output level, as well as an optimal price of its product. Conventionally, two types of a monopoly can be distinguished: a technological one when a monopolist is the only producer who owns the technology and infrastructure needed to manufacture and sell a certain product; and an institutional one when a monopolist is the only producer who holds the exclusive license for the manufacturing and selling a certain product. This distinction has no particular significance for our discussion

[13] Let us recall that condition (5.64) is derived by an assumption that all these functions are linear. That is why it has this quite simple form. In general, when the functions are nonlinear, it is also possible to derive a condition determining relationships between functions but of more complex form.

since we are interested in the quantitative analysis of an equilibrium state for any monopolistic company.

Let us consider a market of some product that is supplied by a monopolistic firm.

Definition 5.6 A **function of consumer demand** for a product manufactured by a monopolist is a mapping $h:\mathbb{R}_+ \to \mathbb{R}_+$ given in a form:

(5.65) $$y^d = h(p),$$

which is assumed to be twice differentiable and decreasing:

(5.66) $$\frac{\mathrm{d}y^d}{\mathrm{d}p} = \frac{\mathrm{d}h(p)}{\mathrm{d}p} < 0,$$

meaning that, when the price of the product increases, the demand for this product decreases. A graph of such a function is called a **demand curve**.

An output level (the supply) of a product manufactured by a monopolist has to be equal to the demand reported by consumers by a product price set by a monopolist:

(5.67) $$y^s = y^d = h(p).$$

To emphasize how strong this assumption is, let us illustrate it by assuming, for the sake of simplicity, that functions of the demand and the product supply are, as before, linear (Fig. 5.3):

(5.68) $$y^d = -ap + b, \quad a, b > 0,$$

(5.69) $$y^s = cp + d, \quad c > 0, d \geq 0.$$

When the product supply matches the demand reported for this product, then from definitions of functions of the demand and of the product supply, it results that

(5.70) $$y^d \in [0; b],$$

(5.71) $$p \in \left[0; \frac{b}{a}\right],$$

(5.72) $$y^s \in \left[d; y^s(\bar{p})\right] \subset [d; b], \quad b > d$$

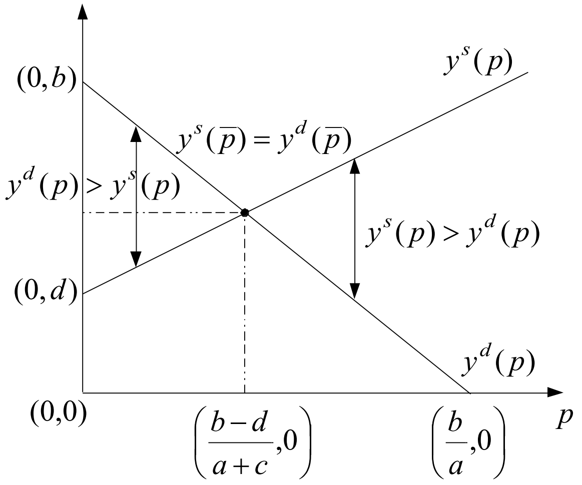


Fig. 5.3 Functions of demand and of product supply

where $y^s(\bar{p})$ means an output level which becomes established by the equilibrium price $\bar{p} \in \left(0; \frac{b}{a}\right)$ which equalizes the supply of a product and the demand reported for it.

One derives the equilibrium price from an equation:

$$y^s = y^d \Leftrightarrow \quad c\bar{p} + d = -a\bar{p} + b, \tag{5.73}$$

from which, after some transformations, one gets

$$\bar{p} = \frac{b-d}{a+c}. \tag{5.74}$$

Eventually, due to the necessary adjustment of the product supply to the demand, three cases should be considered:

(a) when the demand for a product exceeds its supply:

(5.75)
$$p \in \left[0; \frac{b-d}{a+c}\right) \Rightarrow y^d \in \left[\frac{ad+bc}{a+c}; b\right) \wedge y^s \in \left[d; \frac{ad+bc}{a+c}\right) \iff y^d > y^s.$$

A monopolist raising a price of her/his product causes a decline in the demand for it, but still, some part of the demand will be not satisfied. Assuming that consumers do not have the opportunity to purchase the product from another producer nor to substitute it with some product, some part of their demand will not be satisfied by a given product price set by a monopolist.

(b) when levels of the demand for a product and of its supply are the same:

$$p = \frac{b-d}{a+c} \Rightarrow y^d = \frac{ad+bc}{a+c} \wedge y^s = \frac{ad+bc}{a+c} \iff y^d = y^s. \tag{5.76}$$

A product price set on this level equalizes the demand and the product supply.

(c) when the supply of a product exceeds the demand for it:

$$\begin{gathered} p \in \left(\frac{b-d}{a+c}; \frac{b}{a} \right] \Rightarrow \\ y^d \in \left[0; \frac{ad+bc}{a+c} \right) \wedge y^s \in \left(\frac{ad+bc}{a+c}; \frac{ad+bc}{a} \right] \Leftrightarrow y^s > y^d. \end{gathered} \tag{5.77}$$

If a monopolist would set a price higher than the equilibrium price, then as a consequence, the product supply would exceed the demand for a product. As a result, he/she would create stocks of a product that could not be sold on a market with a strictly determined demand function.

Let us assume that a monopolist aiming at profit maximization sets a price level which is the equilibrium price that is a level equalizing the demand and the product supply[14]:

$$y^s = y^d = y. \tag{5.78}$$

At the same time, a product price depends on its supply which should be equal to the demand reported by consumers for a product.

For the sake of simplicity, let us assume that a function of demand for a product is linear and decreasing with respect to a product price:

$$y^d = h\big(p\big(y^s\big)\big) = -ap\big(y^s\big) + b, \quad a, b > 0, \tag{5.79}$$

where $p(y^s)$ denotes a price set by a monopolist dependent on the product supply.

A monopolist is interested only in the nonnegative demand, then, by a product price equal to zero, the demand is $b > 0$, while by a price at level $p(y^s) = \frac{b}{a}$, the demand for a product equals zero. On this basis, we can state that, on a monopolistic market with the demand described by a linear function of consumer demand, the following conditions are satisfied:

$$y^d \in [0; b] \tag{5.80}$$

[14] When it is reasonable and useful, we write y as an output level instead of distinguishing between the product supply y^s and the demand y^d for a product.

and

(5.81) $$p\left(y^s\right) \in \left[0; \frac{b}{a}\right].$$

Let us notice that

(5.82) $$\frac{\mathrm{d}y^d}{\mathrm{d}p(y^s)} = -a < 0,$$

which means that, when a monopolist raises a price of its product by one notional money unit, the demand for a product decreases by a physical units. Hence, parameter a in the linear demand function can be interpreted as a **measure of the consumers' reaction strength** to a unit increase in the price of a product.

Price elasticity of demand for a product is a function of the product price and takes the form:

(5.83) $$E\left(y^d\right) = \frac{\mathrm{d}y^d}{\mathrm{d}p(y^s)} \frac{p(y^s)}{y^d} = \frac{-ap(y^s)}{-ap(y^s) + b}.$$

If $p(y^s) \in \left(0; \frac{b}{a}\right)$, then the following condition is satisfied:

(5.84) $$-ap(y^s) + b > 0.$$

Then the price elasticity of demand for a product is negative. It describes by approximately what % the demand for a product decreases when a monopolist raises a product price by 1%.

Equations (5.82) and (5.83) define measures of the consumers' reaction strength to, respectively, a unit or 1% increase in the price of a product.

Parameter b can be interpreted as a **measure of a market capacity**, which corresponds to the maximum demand that consumers can report for this product at a product price equal to zero.

Definition 5.7 An **inverse function of consumer demand** for a product supplied to a market by producers is a mapping $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ given in a form:

(5.85) $$p\left(y^s\right) = g\left(y^d\right),$$

which is an identity transformation of a demand function:

(5.86) $$y^d = h\left(p\left(y^s\right)\right) = -ap\left(y^s\right) + b \iff p\left(y^s\right) = \frac{b - y^d}{a}.$$

It is worth explaining more precisely whether the inverse demand function is the inverse of the demand function.

Definition 5.8 Functions $y = h(p)$ and $p = g(y)$ are called **inverse to each other** if $\forall a, b \in \mathbb{R}_+$ if the following conditions are satisfied:

$$b = h(a) \Rightarrow a = g(b), \tag{5.87}$$

and at the same time,

$$a = h(b) \Rightarrow b = g(a). \tag{5.88}$$

If h function is given, then g function is called an inverse (or an inverse function) to h. Since the choice of the variable is arbitrary, we can write the inverse $p = g(y)$ also as $y = g(p)$. Then graphs of functions $y = h(p)$ and $y = g(p)$ inverse to each other are symmetric with respect to the line $y = p$.

Example 5.2 Let us determine an inverse to a power demand function:

$$y^d = h\big(p(y^s)\big) = -ap(y^s)^\alpha + b, \quad a, b > 0, \alpha \in (0, 1). \tag{5.89}$$

From Eq. (5.89), one determines a functional relationship of a product price depending on a demand level:

$$y^d = -ap(y^s)^\alpha + b \Leftrightarrow p\big(y^s\big) = \left(\frac{b - y^d}{a}\right)^{\frac{1}{\alpha}} \quad a, b > 0, \tag{5.90}$$

which is an inverse function of consumer demand.

In the form of this function, one changes the denotation so that a dependent variable is the demand level and an independent variable is the product price:

$$y^d = g(p) = \left(\frac{b - p(y^s)}{a}\right)^{\frac{1}{\alpha}} \quad a, b > 0. \tag{5.91}$$

The function derived this way is an inverse function to the demand function. Its graph is symmetric to the graph of a function $y^d = h(p)$ with respect to the line $y = p$ which is presented[15] in Fig. 5.4.

Theorem 5.2 If functions $y = h(p)$ and $p = g(y)$ are inverse to each other, then the following condition is satisfied:

$$\frac{\mathrm{d}y}{\mathrm{d}p} = \frac{\mathrm{d}h(p)}{\mathrm{d}p} = \frac{1}{\frac{\mathrm{d}g(y)}{\mathrm{d}y}} = \frac{1}{\frac{\mathrm{d}p}{\mathrm{d}y}}. \tag{5.92}$$

[15] The graph of the inverse function of consumer demand coincides with the graph of the demand function.

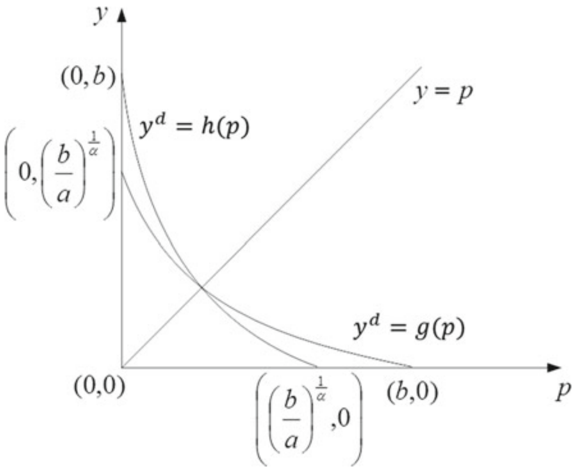


Fig. 5.4 Graphs of power function of consumer demand, inverse function of consumer demand and function inverse to demand function

On the basis of Eqs. (5.89) and (5.90), one can notice that

(5.93) $$\frac{\mathrm{d}y^d}{\mathrm{d}p(y^s)} = -\alpha a p(y^s)^{\alpha-1}$$

and

(5.94) $$\frac{\mathrm{d}p(y^s)}{\mathrm{d}y^d} = -\frac{1}{\alpha a p(y^s)^{\alpha-1}},$$

which means that condition (5.92) is satisfied in case of the power demand function and its inverse.

Definition 5.9 Revenue (turnover) from sales of a product manufactured by a monopolist is an expression:

(5.95) $$r\left(y^s\right) = p\left(y^s\right)y^s.$$

The supply of a product is presumed to match the demand for this product $y^s = y^d$; thus, marginal revenue from sales of a product manufactured by a monopolist is lower than a product price set by a monopolist:

(5.96) $$\frac{\mathrm{d}r(y^s)}{\mathrm{d}y^s} = \frac{\mathrm{d}p(y^s)}{\mathrm{d}y^s}y^s + p\left(y^s\right) < p\left(y^s\right),$$

since:

(5.97) $$y^s > 0, \quad p(y^s) > 0, \quad \frac{\mathrm{d}p(y^s)}{\mathrm{d}y^s} < 0.$$

Let us recall that, in case of a producer acting in the perfect competition, the marginal revenue from sales of a product is described by a linear function since a price is given by a market, thus treated as a parameter by a producer:

(5.98) $$r(y) = py,$$

which results in:

(5.99) $$\frac{\mathrm{d}r(y)}{\mathrm{d}y} = p > 0,$$

meaning that the marginal revenue is equal to the product price.

From conditions (5.92) and (5.96), it results that, in case of a monopolistic company, we have

(5.100) $$\begin{aligned}\frac{\mathrm{d}r(y^s)}{\mathrm{d}y^s} &= \frac{\mathrm{d}p(y^s)}{\mathrm{d}y^s}y^s + p(y^s)\\ &= p(y^s)\left(\frac{\mathrm{d}p(y^s)}{\mathrm{d}y^s}\frac{y^s}{p(y^s)} + 1\right) = p(y^s)\left(\frac{1}{\frac{\mathrm{d}y^s}{\mathrm{d}p(y^s)}\frac{p(y^s)}{y^s}} + 1\right).\end{aligned}$$

Since we assume that $y = y^s = y^d$, **price elasticity of product supply** is equal to the price elasticity of demand for a product:

(5.101) $$E(y) = \frac{\mathrm{d}y}{\mathrm{d}p(y)}\frac{p(y)}{y} = \frac{\mathrm{d}y^s}{\mathrm{d}p(y^s)}\frac{p(y^s)}{y^s} = \frac{\mathrm{d}y^d}{\mathrm{d}p(y^d)}\frac{p(y^d)}{y^d} < 0,$$

which is negative for a decreasing demand function of demand.

Condition (5.100) can be written also in an equivalent form:

(5.102) $$\frac{\mathrm{d}r(y^s)}{\mathrm{d}py^s} = p(y)\left(\frac{1}{E(y^d)} + 1\right) = p(y)\frac{1 + E(y^d)}{E(y^d)},$$

which means that the monopolist's marginal revenue (turnover) from sales depends on the price he/she sets and on the price elasticity of consumer demand for a product.

Definition 5.10 A **function of production total cost** for a monopolist is an expression:

(5.103) $$c^{tot}\left(y^s\right) = c^v\left(y^s\right) + c^f\left(y^s\right),$$

where:

$c^v(y^s)$—a function of production variable cost, dependent on production level,
$c^f(y^s) = c = \text{const.} \geq 0$—a function of production fixed cost, independent of production level.

Definition 5.11 **A function of average production total cost** is an expression:

(5.104) $$\frac{c^{tot}(y^s)}{y^s} = \frac{c^v(y^s)}{y^s} + \frac{c^f(y^s)}{y^s},$$

where:

$\frac{c^v(y^s)}{y^s}$—a function of production average variable cost,
$\frac{c^f(y^s)}{y^s} = \frac{c}{y}$—a function of production average fixed cost.

Note 5.2 From definitions of functions of production total, variable and fixed costs, it results that

(5.105) $$\frac{\mathrm{d}c^{tot}(y^s)}{\mathrm{d}y^s} = \frac{\mathrm{d}c^v(y^s)}{\mathrm{d}y^s},$$

which means that the marginal total cost of production is equal to the marginal variable cost of production.

Let us consider a profit maximization problem[16] for a monopolist:

(5.106) $$\begin{aligned} &\Pi(y) = r(y) - c^{tot}(y) = \left\{p(y)y - c^{tot}(y)\right\} \mapsto \max \\ &y \geq 0, \end{aligned}$$

for which we assume that $y^s = y^d = y$.

From Theorem 4.18, we know if a profit function of a monopolist is strictly concave and the following condition is satisfied:

[16] It is the profit maximization problem of type (P3m) discussed in Chap. 4.

$$\lim_{y\to 0^+} \frac{\mathrm{d}\Pi(y)}{\mathrm{d}y} > 0 \quad \wedge \quad \lim_{y\to +\infty} \frac{\mathrm{d}\Pi(y)}{\mathrm{d}y} < 0 \Leftrightarrow$$

(5.107) $$\Leftrightarrow \lim_{y\to 0^+} \frac{\mathrm{d}r(y)}{\mathrm{d}y} > \lim_{y\to 0^+} \frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y} \quad \wedge \quad \lim_{y\to +\infty} \frac{\mathrm{d}r(y)}{\mathrm{d}y} < \lim_{y\to +\infty} \frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y}$$

(5.108) $$\Leftrightarrow \lim_{y\to 0^+} \frac{\mathrm{d}r(y)}{\mathrm{d}y} > \lim_{y\to 0^+} \frac{\mathrm{d}c^{v}(y)}{\mathrm{d}y} \quad \wedge \quad \lim_{y\to +\infty} \frac{\mathrm{d}r(y)}{\mathrm{d}y} < \lim_{y\to +\infty} \frac{\mathrm{d}c^{v}(y)}{\mathrm{d}y},$$

then:

(1) $\exists_1 \bar{y} > 0$,
(2) a necessary and sufficient condition for $\bar{y} > 0$ being an optimal solution to the profit maximization problem for a monopolist is

(5.109) $$\left.\frac{\mathrm{d}\Pi(y)}{\mathrm{d}y}\right|_{y=\bar{y}} = 0 \quad \Leftrightarrow \quad \left.\frac{\mathrm{d}r(y)}{\mathrm{d}y}\right|_{y=\bar{y}} = \left.\frac{\mathrm{d}c^{v}(y)}{\mathrm{d}y}\right|_{y=\bar{y}},$$

which means that, by $\bar{y} > 0$,
– the marginal profit equals zero;
– the marginal revenue from sales of a product is equal to the marginal total (variable) of production.

From conditions (5.102) and (5.109), it results that

(5.110) $$\left.\frac{\mathrm{d}r(y)}{\mathrm{d}y}\right|_{y=\bar{y}} = p(\bar{y})\frac{1 + E(\bar{y}^d)}{E(\bar{y}^d)} = \left.\frac{\mathrm{d}c^{v}(y)}{\mathrm{d}y}\right|_{y=\bar{y}},$$

which gives

(5.111) $$p(\bar{y}) = \frac{E(\bar{y}^d)}{E(\bar{y}^d) + 1} \cdot \left.\frac{\mathrm{d}c^{v}(y)}{\mathrm{d}y}\right|_{y=\bar{y}}.$$

From condition (5.111), it follows that a product price set by a monopolist depends on the marginal variable cost and the price elasticity of demand reported by consumers for a product. Since the marginal variable cost and a product price set by a monopolist are positive, while the elasticity of demand for a product is negative, condition (4.111) is met when:

(5.112) $$E\left(\bar{y}^d\right) + 1 < 0 \Leftrightarrow E\left(\bar{y}^d\right) < -1,$$

which means that in the described market, the price elasticities of demand reported by consumers for a product manufactured by a monopolist must be lower than -1. Thus, an increase in the product price by 1% should result in a decrease in the demand for the product by more than 1%.

Let us consider the dependence of the price level of a product on the price elasticity of demand for this product. Since:

$$E\left(\bar{y}^d\right) \in (-\infty; -1), \tag{5.113}$$

it is enough to consider two limiting cases.

Case 1

(5.114)

$$\lim_{E(\bar{y}^d)\to-\infty} p\left(\bar{y}^s\right) = \lim_{E(\bar{y}^d)\to-\infty}\left\{\frac{1}{1+\frac{1}{E\left(\bar{y}^d\right)}}\cdot\left.\frac{\mathrm{d}c^v(y^s)}{\mathrm{d}y^s}\right|_{y^s=\bar{y}}\right\} = \left.\frac{\mathrm{d}c^v(y^s)}{\mathrm{d}y^s}\right|_{y^s=\bar{y}},$$

meaning that, if the reaction of consumers to changes in a price level is very strong (high in absolute value), then a product price set by a monopolist at a level ensuring maximum profit for her/him will converge to the marginal variable cost of production. We can say then that this situation is similar to that of the perfect competition: a product price will be equal to the marginal variable (total) cost of production (Fig. 5.5).

Case 2

$$\lim_{E(\bar{y}^d)\to-1^-} p\left(\bar{y}^s\right) = \lim_{E(\bar{y}^d)\to-1^-}\left\{\frac{1}{1+\frac{1}{E\left(\bar{y}^d\right)}}\cdot\left.\frac{\mathrm{d}c^v(y^s)}{\mathrm{d}y^s}\right|_{y^s=\bar{y}}\right\} = +\infty, \tag{5.115}$$

meaning that, if the reaction of consumers to changes in a price level is very weak (low in absolute value), then a product price set by a monopolist at a level ensuring maximum profit for her/him will converge to infinity. When a reaction of the consumer demand to changes in a product price is weak, a monopolist may set the price at any high level (Fig. 5.5).

Knowing an output level $\bar{y}^s > 0$ that guarantees the maximum profit for a monopolist, on the basis of the inverse demand function (4.86), we can determine a price level of a product:

$$\bar{p}\left(\bar{y}^s\right) = g\left(\bar{y}^s\right) = \frac{b-\bar{y}^s}{a}. \tag{5.116}$$

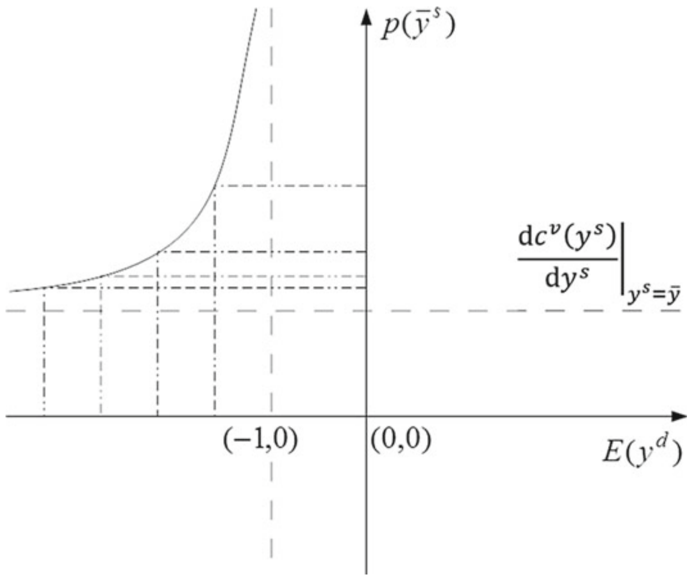


Fig. 5.5 Dependence of product price set by monopolist on price elasticity of demand

Example 5.3 The following functions are given:

– a linear function of demand for a product:

(5.117) $$y^d = h(p) = -ap + b, \quad a, b > 0,$$

– a linear inverse function of demand for a product:

(5.118) $$p\left(y^d\right) = g\left(y^d\right) = \frac{b - y^d}{a} = \alpha - \beta y^d, \quad a, b, \alpha = \frac{b}{a}, \beta = \frac{1}{a} > 0,$$

– a linear function of total cost of production:

(5.119) $$c^{tot}\left(y^s(p)\right) = \gamma y^s + \delta, \quad \gamma, \delta > 0$$

where:

$c^v(y^s(p)) = \gamma y^s$—a linear function of the variable cost,
$c^f(y^s(p)) = \delta$—a function of the fixed cost,
$r(y^s) = p(y^s)y^s$—revenue (turnover) from sales of a product.

We assume that the supply of a product is equal to the demand reported by consumers:

$$y^s = y^d = y. \tag{5.120}$$

Then the profit maximization problem for a monopolist takes the form:

$$\begin{aligned} \Pi(y) &= r(y) - c^{tot}(y) = p(y)y - c^{tot}(y) = (\alpha - \beta y)y - \gamma y - \delta \\ &= \left\{-\beta y^2 + (\alpha - \gamma)y - \delta\right\} \mapsto \max \\ & y \geq 0. \end{aligned} \tag{5.121}$$

If assumptions of Theorem 4.18 are satisfied, then an optimal solution to problem (5.121) is an output level determined from a condition:

$$\begin{aligned} &\left.\frac{\mathrm{d}\Pi(y)}{\mathrm{d}y}\right|_{y=\bar{y}} = -2\beta\bar{y} + (\alpha - \gamma) = 0 \Leftrightarrow \\ &\Leftrightarrow \bar{y} = \frac{\alpha - \gamma}{2\beta} = \frac{\frac{b}{a} - \gamma}{\frac{2}{a}} = \frac{b - a\gamma}{2} > 0, \end{aligned} \tag{5.122}$$

which means that an output level guaranteeing the maximum profit for a monopolist is positive when:

$$\gamma < \frac{b}{a}, \tag{5.123}$$

thus when the total (variable) marginal cost of production is below the maximum price level that a monopolist can set. Otherwise, a monopolist will not engage in production, since, at best, he/she would make zero profit.

Then the level of a product price set by a monopolist is

$$\begin{aligned} p(\bar{y}) &= g(\bar{y}) = \alpha - \beta\bar{y} = \alpha - \beta\frac{\alpha - \gamma}{2\beta} = \frac{\alpha + \gamma}{2} = \frac{\frac{b}{a} + \gamma}{2} = \frac{b + a\gamma}{2a} \\ &= \frac{1}{2} \cdot \frac{b}{a} + \frac{1}{2}\gamma > 0. \end{aligned} \tag{5.124}$$

Let us notice that the price elasticity of demand reported by consumers for a product meets the condition:

$$E(y^d) = E(\bar{y}^s) = -\frac{b + a\gamma}{b - a\gamma} < -1. \tag{5.125}$$

Table 5.2a Measures of response of the product supply by a monopolist to changes in parameters' values

Characteristic	$\frac{\partial \bar{y}}{\partial a}$	$\frac{\partial \bar{y}}{\partial b}$	$\frac{\partial \bar{y}}{\partial \gamma}$
Value	$\frac{-\gamma}{2}$	$\frac{1}{2}$	$-\frac{a}{2}$

Table 5.2b Measures of response of the product price set by a monopolist to changes in parameters' values

Characteristic	$\frac{\partial p(\bar{y})}{\partial a}$	$\frac{\partial p(\bar{y})}{\partial b}$	$\frac{\partial p(\bar{y})}{\partial \gamma}$
Value	$-\frac{b}{2a^2}$	$\frac{1}{2a}$	$\frac{1}{2}$

The monopolist's maximum profit at the equilibrium price is

$$\Pi(\bar{y}) = \frac{(\alpha - \gamma)^2}{4\beta} - \delta = \frac{(b - a\gamma)^2}{4a} - \delta. \tag{5.126}$$

If, in addition:

$$\frac{(b - a\gamma)^2}{4a} > \delta, \tag{5.127}$$

where $\gamma < \frac{b}{a}$, then the monopolist's maximum profit is positive.

Let us notice that the product price and the product supply set by a monopolist, which by given demand for a product guarantee the maximum profit, depend on a market capacity$b$, the strength a of consumers' reaction to changes in a product price and the production marginal total (variable) costγ. The maximum profit of a monopolist depends also on the fixed cost of production.

Let us analyse the sensitivity of the product optimal supply by a monopolist and of the product price to changes in values of parameters that describe the considered model of a monopolistic market. Values of adequate measures are presented in Tables 5.2a and 5.2b.

From Tables 5.2a and 5.2b, it follows that a unit increase in market capacity b results in an increase in the optimal supply and the product price. A unit increase in the marginal variable cost of production γ results in a decrease in the product optimal supply and an increase in the product optimal price. A unit increase in the strength a of consumers' reaction to changes in the price of a product set by monopolist results in a decrease in the product price and the product optimal supply.

5.2.2 Dynamic Approach

Determining the optimal level of a product price a monopoly decides on the basis of what an output level will ensure the maximum profit by a given demand reported by consumers. Knowing a demand function for the product a monopolist can determine the dependence of a product price on the product, that is, he/she can determine the inverse demand function. In the static approach, in Sect. 5.2.1, we assume that consumers' behaviour is time-invariant in the sense they always respond to changes in a product price in the same way determined by the demand function. Total cost of production is also time-invariant. In the dynamic approach, we assume that a form of a demand function, as well as a form of a production cost function, may change over time. For example, consumers may become more or less sensitive to changes in the product price. A market capacity of a product manufactured by a monopolistic company may also change over time, which shows that the extent of consumers' interest in a product is not constant. If a form of a production cost function changes over time, this reflects the fact that the cost of manufacturing a unit of a product may be higher or lower over the considered time horizon. The reasons for these changes may be changes in the production technology, as well as an increase or decrease in prices of production factors.

Let us take the following notation:

t—time, in a discrete version ($t = 0, 1, 2, \ldots, T$) or in a continuous version[17] ($t \in [0; T]$),
T—time horizon,
$y^d(t) = h(p(t))$—a consumer demand function with a time-variant form,
$p(y^s(t)) = g(y^d(t))$—an inverse function of demand for a product,
$c^{tot}(y^s(t)) = c^v(y^s(t)) + \delta(t)$—a function of production total cost,
$c^v(y^s(t))$—a function of production variable cost, that is, the cost depending on an output level,
$\delta(t) \geq 0$—the time-variant fixed cost, that is, the cost that does not depend on the output level,
$E(y^d(t))$—price elasticity of consumer demand for a product.

In the monopolist's profit maximization problem with an exogenously determined function of demand for a product, we make a simplifying assumption that, in each period/at any moment, a monopolist adjusts an output level to the demand that consumers report by a given price. This affects the choice of the optimal supply and the optimal price by a monopolist but allows her/him to avoid the problem of dealing with stocks of unsold units of the output. This assumption takes the form:

$$y^s(t) = y^d(t) = y(t). \tag{5.128}$$

[17] For the discrete and continuous versions, we use the same denotation of dependency of a function value on time, for example, the fixed cost depending on time: $\delta(t)$. Whether the discrete or continuous version is used in a given formula will result from the context of the issue under consideration.

Then the monopolist's profit maximization problem can be written as

$$\Pi(y(t)) = r(y(t)) - c^{tot}(y(t)) = \{p(y(t)) \cdot y(t) - c^{tot}(y(t))\} \mapsto \max \tag{5.129}$$

$$y(t) \geq 0. \tag{5.130}$$

If the profit function $\Pi(y(t))$ is strictly concave, then in every period/at any moment t, Problem (5.129)–(5.130) has a solution $\bar{y}(t) > 0$, derived from the first-order condition of the profit maximization:

$$\left.\frac{\mathrm{d}\Pi(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\bar{y}(t)} = 0 \quad \Leftrightarrow \quad \left.\frac{\mathrm{dr}(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\bar{y}(t)} = \left.\frac{\mathrm{d}c^{v}(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\bar{y}(t)} \qquad \forall t, \tag{5.131}$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0; T]$.

From the profit maximization problem, one derives the optimal supply $\bar{y}(t)$ of a product, which changes over time, since the form of the demand function and the form of the production total cost function vary in time. The optimal price level of a product manufactured by a monopoly is $p(\bar{y}(t))$.

Example 5.4 Let us assume that the demand for a product of a monopoly is given in a linear function form:

$$y^d(t) = -a(t)p(t) + b(t), \qquad a(t), b(t) > 0, \qquad \forall t \in [0; 30],$$

where $a(t)$ is a function determining how strong the reaction of consumers to changes in a product price is, while $b(t)$ is a function of the market capacity. In order to evaluate how the formation of the consumers' sensitivity to changes in a product price affects the optimal supply and the optimal price level, we will compare results by an assumption that value of $a(t)$ is constant over time and equal to 2 with those by an assumption that value of $a(t)$ varies in time according to the formula:

$$a(t) = \frac{1}{t+1} + 1.$$

We will compare as well the results by constant market capacity equal to 10 with those by an assumption that the market capacity varies in time according to the formula:

$$b(t) = -0.025t^2 + 0.75t + 10.$$

Trajectories of $a(t)$ and $b(t)$ are shown in Fig. 5.6. The consumers' sensitivity to changes in a price is constant over the whole time horizon and equals to 2 or it decreases over time (in the second of assumed versions), starting at 2 at the beginning of the horizon, ending at approximately 1.03 at the end of the horizon.

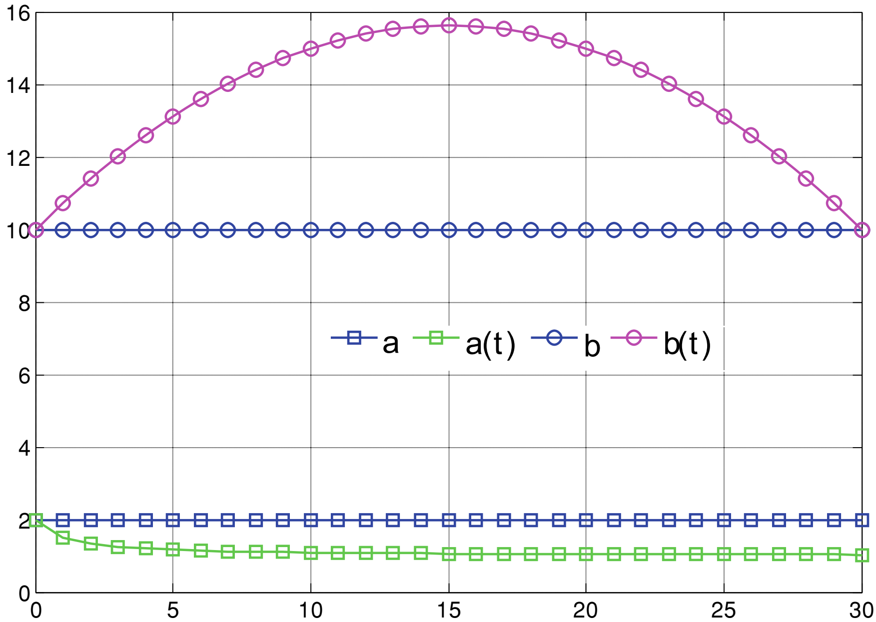


Fig. 5.6 Trajectories of strength of consumers' reaction to changes in price and trajectories of market capacity

The market capacity is constant over time at 10 or rising (in the second of assumed versions) from 10 at the beginning of the horizon, reaching its maximum at 15.62 at time $t = 1 = 15$, and then decreasing to the end of the horizon, again taking a value of 10.

Let us assume that a production total cost function is linear too

$$c^{tot}\left(y^s(t)\right) = \gamma(t)y^s(t) + \delta(t), \qquad \gamma(t), \delta(t) > 0, \qquad t \in [0; 30]),$$

where $\gamma(t)$ denotes the marginal cost of production and $\delta(t)$ is a function of the fixed cost of production. Since the production fixed cost is irrelevant for a company when choosing an output level or when determining a price level of a product but only affects the amount of profit, we will assume, for the sake of simplicity, that it is constant over time and amounts to 0.5. In order to assess how formation of the marginal cost affects a supply optimal level and an optimal level of a product price, we will compare results obtained assuming that the value of $\gamma(t)$ is constant over time and equals to 1 with results obtained by an assumption that $\gamma(t)$ varies in time according to the formula:

$$\gamma(t) = -\frac{1}{t+1} + 2.$$

Trajectories of $\gamma(t)$ and $\delta(t)$ are shown in Fig. 5.7. The fixed cost of production does not change over time and is always equal to 0.5. The marginal cost of production is constant over the time horizon and equal to 1 or increases over time

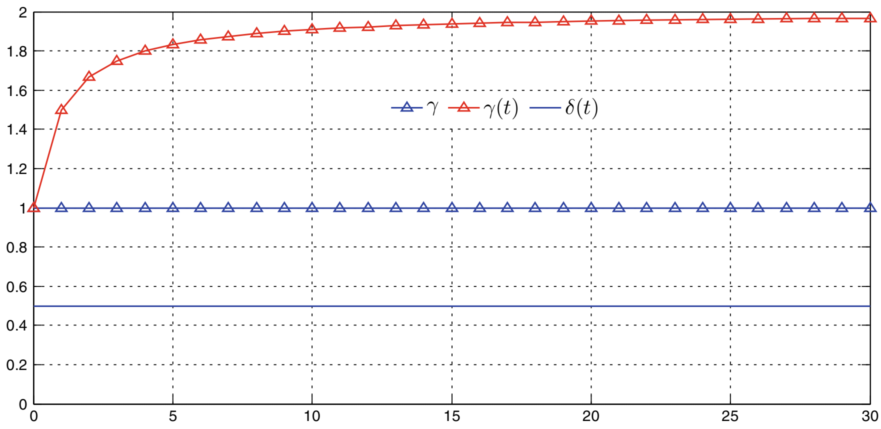


Fig. 5.7 Trajectories of production marginal cost and of production fixed cost

(in the second of assumed versions), starting at 1 at the beginning of the horizon, ending at approximately 1.97 at the end of the horizon.

From the demand function, assuming that the supply matches the demand, that is, $y^s(t) = y^d(t) = y(t)$, one derives an inverse function of demand which takes the form:

$$p(y(t)) = -\frac{1}{a(t)}y(t) + \frac{b(t)}{a(t)}.$$

Hence, the monopolist's profit maximization problem can be written as

$$\begin{aligned} \Pi(y(t)) = r(y(t)) - c^{tot}(y(t)) &= \left(-\frac{1}{a}y(t) + \frac{b(t)}{a(t)}\right)y(t) - \{\gamma(t) + \delta(t)\} \\ &= \left\{-\frac{1}{a}y(t)^2 + \left(\frac{b(t)}{a(t)} - \gamma(t)\right)y(t) - \delta(t)\right\} \mapsto \max \end{aligned}$$

$$y(t) \geq 0.$$

A solution to the problem is the optimal supply given as

$$\bar{y}(t) = \frac{b(t) - a(t)\gamma(t)}{2},$$

$$\bar{y}(t) > 0 \quad \Leftrightarrow \quad \gamma(t) < \frac{b(t)}{a(t)} \qquad \forall t \in [0; 30],$$

where we can notice that the supply increases when the market capacity grows and decreases when the consumers' sensitivity to price changes or the marginal cost of production rises.

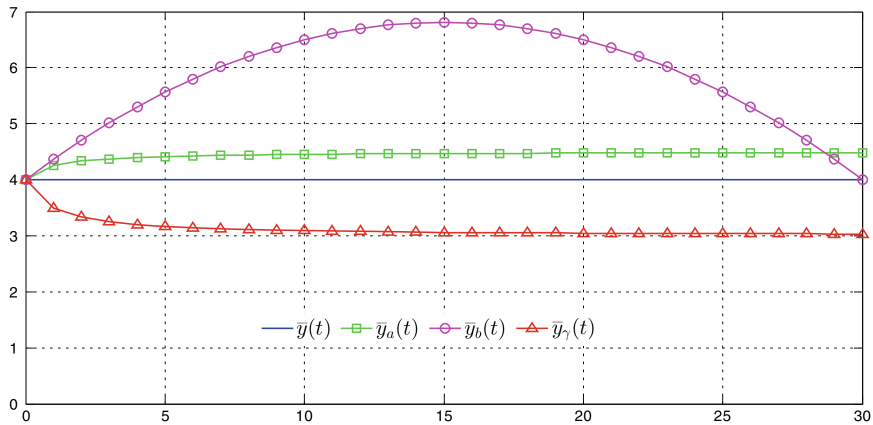


Fig. 5.8 Trajectories of product optimal supply

In Fig. 5.8, we present a comparison of a constant optimal level of supply $\bar{y}(t)$, when all the functions: $a(t), b(t), \gamma(t)$ are constant over time with optimal levels of the supply when one of these functions $(a(t), b(t)$ or $\gamma(t))$ changes over time. The figure demonstrates the same observation one made considering the optimal output formula. For example, the fact that, when the consumers' sensitivity to changes in a price declines over time, it leads to an increase in the optimal supply $\bar{y}_a(t)$. When what changes over time is the market capacity, the optimal supply $\bar{y}_b(t)$ has the biggest value when the market capacity is the greatest, that is, at moment, $t = 15$. If the marginal cost of production increases over time, it leads to a decline in the optimal supply $\bar{y}_\gamma(t)$. The sign of a difference between the level $\bar{y}(t)$ and levels $\bar{y}_a(t)$, $\bar{y}_b(t)$ or $\bar{y}_\gamma(t)$ (determining if the reaction is positive or negative) complies with the formula for the optimal output level, whereas the size of this difference depends on how big the changes in values of $a(t), b(t)$ or $\gamma(t)$ in time are.

On the basis of the inverse function of demand, a monopolist determines the optimal price level for her/his product dependent on the optimal output level:

$$p(\bar{y}(t)) = -\frac{1}{a}\bar{y}(t) + \frac{b(t)}{a(t)} = \frac{b(t) + a(t)\gamma(t)}{2a(t)}.$$

The price level depends positively on the market capacity $b(t)$ and the marginal cost of production $\gamma(t)$, and negatively on the consumers' sensitivity $a(t)$ to price changes. The higher market capacity means increased consumers' interest in the product which enables the producer to increase the price. With an increase in the marginal cost of production, a monopolist gives his/her product a higher price to compensate for the loss of some profit from the higher cost of production. The increase in the strength of consumers' reaction to changes in the product price means that it is necessary for a monopolist to lower the price, since he/she is driven by profit maximization and such a reduction will allow her/him to obtain an increased demand for the product, and thus an increase in the income from sales of the product.

In Fig. 5.9, we compare a constant optimal level $p(\bar{y}(t))$ of a product price when all functions,$a(t), b(t), \gamma(t)$, are constant over time with optimal price levels when one of these functions $(a(t), b(t)$ or $\gamma(t))$ changes over time. The figure demonstrates the same one can observe on the basis of the optimal price formula. If the consumers' sensitivity to changes in a price declines in time, it leads to an increase in the optimal price $p(\bar{y}_a(t))$. If it is the market capacity that changes over time, the optimal price $p(\bar{y}_b(t))$ has the biggest value when the market capacity is the greatest, that is, at moment $t = 15$. If the marginal cost of production increases over time, it leads to an increase in the optimal supply $p(\bar{y}_\gamma(t))$. The sign of a difference between the level $p(\bar{y}(t))$ and levels $p(\bar{y}_a(t))$, $p(\bar{y}_b(t))$ or $p(\bar{y}_\gamma(t))$ (determining if the reaction is positive or negative) complies with the formula for the optimal price level, whereas the size of this difference depends on how big the changes in values of $a(t), b(t)$ or $\gamma(t)$ in time are.

Let us also analyse the maximum profit formation over time. Let us recall that we took an assumption that the fixed cost of production is constant over time. Therefore, its amount does not matter for the obtained differences in the results of the maximum profit value. They are determined by the shaping of the values of functions $a(t), b(t)$ and $\gamma(t)$. The profit value is positively influenced by an increase in the market capacity, which can be seen on the basis of trajectories shown in Figs. 5.6 and 5.10. An increase in the marginal cost of production affects the maximum profit negatively, while a decrease in the consumers' sensitivity to price changes has a positive effect, thus an increase in this value has a negative impact on the maximum profit.

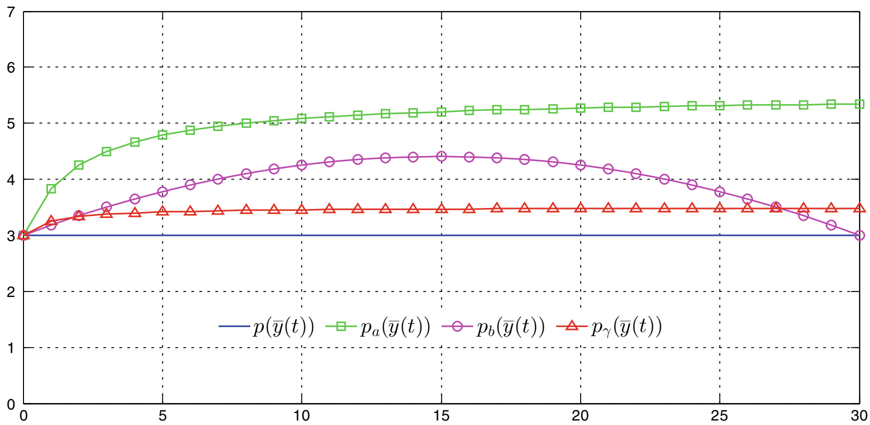


Fig. 5.9 Trajectories of optimal price

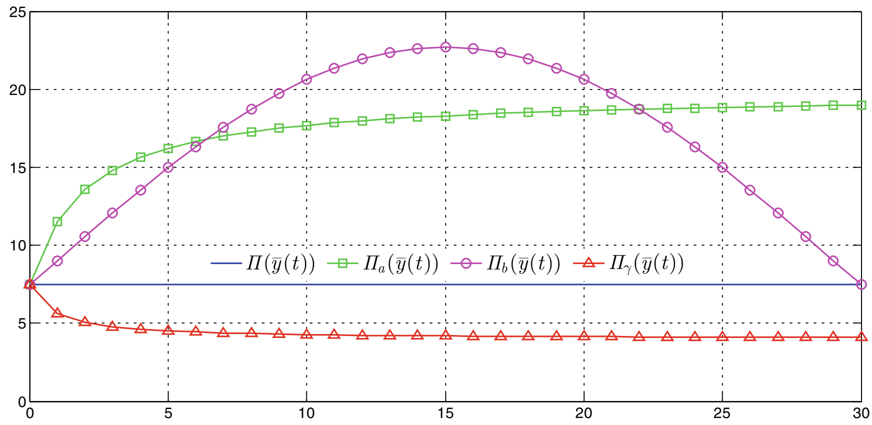


Fig. 5.10 Trajectories of maximum profit of a monopolist

Based on Fig. 5.11, which presents the formation of price elasticity of demand, and comparing it with Fig. 5.9, we can observe the relationship between the product price set by a monopolist and the price elasticity of demand, which determines the strength of consumers' reaction to price changes. We can also notice that the value of the price elasticity of demand; therefore, how strong the reaction of consumers to price increases is, depends on the shaping of values of functions: $a(t)$, the market capacity $b(t)$ and the marginal cost of production $\gamma(t)$. This last dependence results from the fact that a monopolist determines the price of her/his product on the basis of the optimal level of supply, which he/she makes dependent, among others, on the marginal cost.

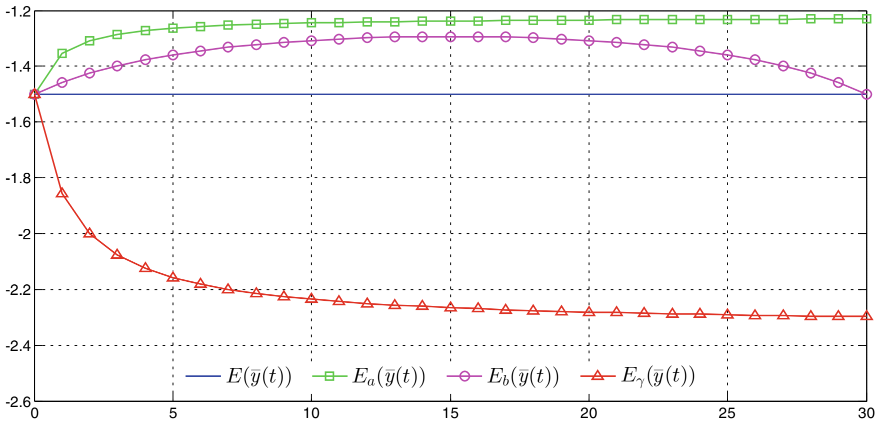


Fig. 5.11 Trajectories of price elasticity of demand reported for a product by consumers

5.3 Monopolistic Discriminatory Pricing on Two Different Markets

5.3.1 Static Approach

Let us apply the following set of assumptions:

(D1) A monopolistic company manufactures one product and supplies it to two different markets on which it can set two different prices.[18] We assume that it is not possible to resell the product between the markets.

(D2) Total quantity of a product manufactured by a monopolist supplied to both markets is[19]:

(5.132) $$y = \xi(y_1, y_2) = y_1 + y_2,$$

where:

y_1 - an output level intended for the first market,
y_2 - an output level intended for the second market.

Hence, we have

(5.133) $$\frac{\partial y}{\partial y_1} = \frac{\partial \xi(y_1, y_2)}{\partial y_1} = 1 \quad \text{and} \quad \frac{\partial y}{\partial y_2} = \frac{\partial \xi(y_1, y_2)}{\partial y_2} = 1.$$

(D3) A production total cost function is an increasing function of output level:

(5.134) $$c^{tot}(y) = c^v(y) + c^f(y), \text{ such that}$$

(5.135) $$\frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y} = \frac{\mathrm{d}c^v(y)}{\mathrm{d}y} > 0.$$

[18] Not only monopolistic firms practice the discriminatory pricing. However, it is easier to discriminate prices when there are no or just few competitors, since discriminatory pricing with competition of other producers can discourage consumers from the producer who practices it and what is more it requires gathering a lot of information about sensitivity of consumers demand in various groups of consumers. For discriminatory pricing survey, to present the topic by means of the monopolistic company model is the simplest way to do it, since the only one factor one focuses on is the price discrimination, not competition by prices or by quantities.

[19] From condition (5.132), it results that the total quantity of a product manufactured by a monopolist supplied to both markets is a scalar linear function of output levels intended for the first and the second markets. Having in mind that $y = \xi(y_1, y_2)$, we will write just y in order to make the notation simpler and shorter.

Moreover,

(5.136)
$$\frac{\partial c^{tot}(y)}{\partial y_1} = \frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y}\frac{\partial y}{\partial y_1} = \frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y} > 0,$$

and

(5.137)
$$\frac{\partial c^{tot}(y)}{\partial y_2} = \frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y}\frac{\partial y}{\partial y_2} = \frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y} > 0,$$

which means that regardless of where the output is intended, for the first or the second markets, production marginal total (variable) cost is the same.

(D4) Each function of the demand reported by consumers on i-th market is known:

(5.138)
$$y_1^d = h_1(p_1), \quad \frac{\mathrm{d}y_1^d}{\mathrm{d}p_1} < 0,$$

(5.139)
$$y_2^d = h_2(p_2), \quad \frac{\mathrm{d}y_2^d}{\mathrm{d}p_2} < 0,$$

and is decreasing in a product price set on i-th market by a monopolist. As a result, we have an assumption that a change in a product price on one of the markets does not affect the price of the product set on the other market.

(D5) For the demand reported for a product on i-th market, there exists an inversed function of demand:

(5.140)
$$p_1\left(y_1^d\right) = g_1\left(y^d\right), \quad \frac{\mathrm{d}p_1}{\mathrm{d}y_1^d} = \frac{1}{\frac{\mathrm{d}y_1^d}{\mathrm{d}p_1}} < 0,$$

(5.141)
$$p_2\left(y_2^d\right) = g_2\left(y_2^d\right), \quad \frac{\mathrm{d}p_2}{\mathrm{d}y_2^d} = \frac{1}{\frac{\mathrm{d}y_2^d}{\mathrm{d}p_2}} < 0,$$

and it is decreasing in the demand reported by consumers for a product on i-the market.

(D6) Each price elasticity of demand reported by consumers on i-th market is negative:

(5.142)
$$E_1(p_1) = \frac{\mathrm{d}y_1^d}{\mathrm{d}p_1}\frac{p_1}{y_1^d} < 0,$$

(5.143)
$$E_2(p_2) = \frac{\mathrm{d}y_2^d}{\mathrm{d}p_2}\frac{p_2}{y_2^d} < 0.$$

(D7) Total quantity of a product (the total supply) manufactured by a monopolist is equal to the total demand reported by consumers for the product on both markets by the prices of this product set by the monopolist on each market:

(5.144) $$y^s = y^d,$$

and at the same time, the product supply on each market matches the demand reported for the product on this market:

(5.145) $$\forall i = 1, 2 \quad y_i^s = y_i^d = y_i.$$

(D8) The aim of a monopolist is to maximize total profit from both markets of a product:

(5.146) $$\Pi(y_1, y_2) = r_1(y_1) + r_2(y_2) - c^{tot}(y) = \left\{p_1(y_1)y_1 + p_2(y_2)y_2 - c^{tot}(y)\right\} \mapsto \max$$

(5.147) $$y_1, y_2 \geq 0,$$

where:

$r_1(y_1) = p_1(y_1)y_1$—revenue (turnover) from sales of a product on the first market,
$r_2(y_2) = p_2(y_2)y_2$—revenue (turnover) from sales of a product on the second market,
$c^{tot}(y)$—total cost of overall production intended for both markets.

Necessary conditions for the existence of an optimal solution to the profit maximization problem (5.146)–(5.147) take the form:

(5.148) $$\left.\frac{\partial \Pi(y)}{\partial y_1}\right|_{y=\bar{y}} = 0,$$

(5.149) $$\left.\frac{\partial \Pi(y)}{\partial y_2}\right|_{y=\bar{y}} = 0.$$

Sufficient conditions for the existence of the optimal solution to the profit maximization problem (5.146)–(5.147) take the form:

(5.150) $$\left.\frac{\partial^2 \Pi(y)}{\partial y_1^2}\right|_{y=\bar{y}} = 0,$$

(5.151) $$\left.\frac{\partial^2 \Pi(y)}{\partial y_2^2}\right|_{y=\bar{y}} = 0,$$

(5.152) $$\left.\frac{\partial^2 \Pi(y)}{\partial y_1^2}\right|_{y=\bar{y}} \cdot \left.\frac{\partial^2 \Pi(y)}{\partial y_2^2}\right|_{y=\bar{y}} - \left.\frac{\partial^2 \Pi(y)}{\partial y_1 \partial y_2}\right|_{y=\bar{y}} \cdot \left.\frac{\partial^2 \Pi(y)}{\partial y_2 \partial y_1}\right|_{y=\bar{y}} > 0,$$

where:

(5.153) $$\bar{y} = \bar{y}_1 + \bar{y}_{\mathbf{2}},$$

(5.154) $$\left.\frac{\partial^2 \Pi(y)}{\partial y_1 \partial y_2}\right|_{y=\bar{y}} = \left.\frac{\partial^2 \Pi(y)}{\partial y_2 \partial y_1}\right|_{y=\bar{y}},$$

since a matrix of second-order partial derivatives (the Hessian) for the profit function:

(5.155) $$H(y) = \begin{bmatrix} \frac{\partial^2 \Pi(y)}{\partial y_1^2} & \frac{\partial^2 \Pi(y)}{\partial y_2 \partial y_1} \\ \frac{\partial^2 \Pi(y)}{\partial y_1 \partial y_2} & \frac{\partial^2 \Pi(y)}{\partial y_2^2} \end{bmatrix}$$

is symmetric with respect to its main diagonal which means that its elements on both sides of the main diagonal are equal to each other.

Let us determine the necessary conditions for the existence of an optimal solution to the profit maximization problem (5.146)–(5.147):

- for the first market:

(5.156) $$\left.\frac{\partial \Pi(y)}{\partial y_1}\right|_{y=\bar{y}} = \left.\frac{\mathrm{d}r_1(y_1)}{\mathrm{d}y_1}\right|_{y=\bar{y}} - \left.\frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y}\right|_{y=\bar{y}} = 0$$
$$\Leftrightarrow \left.\frac{\mathrm{d}r_1(y_1)}{\mathrm{d}y_1}\right|_{y=\bar{y}} = \left.\frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y}\right|_{y=\bar{y}};$$

- for the second market:

(5.157) $$\left.\frac{\partial \Pi(y)}{\partial y_2}\right|_{y=\bar{y}} = \left.\frac{\mathrm{d}r_2(y_2)}{\mathrm{d}y_2}\right|_{y=\bar{y}} - \left.\frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y}\right|_{y=\bar{y}} = 0$$
$$\Leftrightarrow \left.\frac{\mathrm{d}r_2(y_2)}{\mathrm{d}y_2}\right|_{y=\bar{y}} = \left.\frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y}\right|_{y=\bar{y}},$$

from which it follows that

(5.158) $$\left.\frac{\mathrm{d}r_1(y_1)}{\mathrm{d}y_1}\right|_{y=\bar{y}} = \left.\frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y}\right|_{y=\bar{y}} = \left.\frac{\mathrm{d}r_2(y_2)}{\mathrm{d}y_2}\right|_{y=\bar{y}},$$

thus, for $\bar{y} = \bar{y}_1 + \bar{y}_2$, the marginal revenue from sales of a product on the first market is equal to the marginal revenue from sales of the product on the second market, both equal to the marginal total cost of production intended for both markets.

Let us determine the sufficient conditions for the existence of an optimal solution to the profit maximization problem (5.146)–(5.147):

- for the first market:

$$\begin{aligned}&\left.\frac{\partial^2\Pi(y)}{\partial y_1^2}\right|_{y=\bar{y}}=\left.\frac{\mathrm{d}^2 r_1(y_1)}{\mathrm{d}y_1^2}\right|_{y=\bar{y}}-\left.\frac{\mathrm{d}^2 c^{tot}(y)}{\mathrm{d}y^2}\right|_{y=\bar{y}}<0\\&\Leftrightarrow\left.\frac{\mathrm{d}^2 r_1(y_1)}{\mathrm{d}y_1^2}\right|_{y=\bar{y}}<\left.\frac{\mathrm{d}^2 c^{tot}(y)}{\mathrm{d}y^2}\right|_{y=\bar{y}}\end{aligned}\tag{5.159}$$

- for the second market:

$$\begin{aligned}&\left.\frac{\partial^2\Pi(y)}{\partial y_2^2}\right|_{y=\bar{y}}=\left.\frac{\mathrm{d}^2 r_2(y_2)}{\mathrm{d}y_2^2}\right|_{y=\bar{y}}-\left.\frac{\mathrm{d}^2 c^{tot}(y)}{\mathrm{d}y^2}\right|_{y=\bar{y}}<0\\&\Leftrightarrow\left.\frac{\mathrm{d}^2 r_2(y_2)}{\mathrm{d}y_2^2}\right|_{y=\bar{y}}<\left.\frac{\mathrm{d}^2 c^{tot}(y)}{\mathrm{d}y^2}\right|_{y=\bar{y}}.\end{aligned}\tag{5.160}$$

Let us recall that functions of revenue from sales of a product are defined as

$$r_1(y_1)=p_1(y_1)y_1,\tag{5.161}$$

$$r_2(y_2)=p_2(y_2)y_2.\tag{5.162}$$

Let us determine functions of marginal revenue from sales of a product:

$$\begin{aligned}&\left.\frac{\mathrm{d}r_1(y_1)}{\mathrm{d}y_1}\right|_{y_1=\bar{y}_1}=\left.\frac{\mathrm{d}p_1(y_1)}{\mathrm{d}y_1}\right|_{y_1=\bar{y}_1}\bar{y}_1+p_1(\bar{y}_1)=\\&=p_1(\bar{y}_1)\left(\left.\frac{\mathrm{d}p_1(y_1)}{\mathrm{d}y_1}\right|_{y_1=\bar{y}_1}\cdot\frac{\bar{y}_1}{p_1(\bar{y}_1)}+1\right)\\&=p_1(\bar{y}_1)\left(\frac{1}{\left.\frac{\mathrm{d}y_1}{\mathrm{d}p_1\ (y_1)}\right|_{y_1=\bar{y}_1}\cdot\frac{p_1(\bar{y}_1)}{\bar{y}_1}}+1\right)\\&=p_1(\bar{y}_1)\frac{1+E_1(\bar{y}_1)}{E_1(\bar{y}_1)},\end{aligned}\tag{5.163}$$

$$\left.\frac{\mathrm{d}r_2(y_2)}{\mathrm{d}y_2}\right|_{y_2=\bar{y}_2} = \left.\frac{\mathrm{d}p_2(y_2)}{\mathrm{d}y_2}\right|_{y_2=\bar{y}_2} \bar{y}_2 + p_2(\bar{y}_2) =$$

(5.164)
$$= p_2(\bar{y}_2)\left(\left.\frac{\mathrm{d}p_2(y_2)}{\mathrm{d}y_2}\right|_{y_2=\bar{y}_2} \cdot \frac{\bar{y}_2}{p_2(\bar{y}_2)} + 1\right)$$
$$= p_2(\bar{y}_2)\left(\frac{1}{\left.\frac{\mathrm{d}y_2}{\mathrm{d}p_2\ (y_2)}\right|_{y_2=\bar{y}_2} \cdot \frac{p_2(\bar{y}_2)}{\bar{y}_2}} + 1\right)$$
$$= p_2(\bar{y}_2)\frac{1 + E_2(\bar{y}_2)}{E_2(\bar{y}_2)},$$

where:

(5.165)
$$E_i(\bar{y}_i) = \left.\frac{\mathrm{d}y_i}{\mathrm{d}p_i(y_i)}\right|_{y_i=\bar{y}_i} \cdot \frac{p_i(\bar{y}_i)}{\bar{y}_i}, \quad i = 1, 2.$$

means the price elasticity of demand reported by consumers for a product on i-th market, negative from assumption (D4).

From now on to make notation simpler and shorter, we use the following abbreviated denotation:

(5.166)
$$E_i(\bar{y}_i) = \left.\frac{\mathrm{d}y_i}{\mathrm{d}p_i}\right|_{y_i=\bar{y}_i} \cdot \frac{p_i}{\bar{y}_i}, \quad i = 1, 2.$$

We know that

(5.167)
$$\left.\frac{\mathrm{d}r_1(y_1)}{\mathrm{d}y_1}\right|_{y=\bar{y}} = \left.\frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y}\right|_{y=\bar{y}} = \left.\frac{\mathrm{d}r_2(y_2)}{\mathrm{d}y_2}\right|_{y=\bar{y}}.$$

Let us then take a notation:

(5.168)
$$\left.\frac{\mathrm{d}c^{tot}(y)}{\mathrm{d}y}\right|_{y=\bar{y}} = c,$$

which gives

(5.169)
$$\left.\frac{\mathrm{d}r_1(y_1)}{\mathrm{d}y_1}\right|_{y=\bar{y}} = c = p_1(\bar{y}_1)\frac{1 + E_1}{E_1},$$

and hence

(5.170)
$$p_1(\bar{y}_1) = \frac{E_1}{E_1 + 1}c,$$

as well as analogically for the second market:

$$\left.\frac{\mathrm{d}r_2(y_2)}{\mathrm{d}y_2}\right|_{y=\bar{y}} = c = p_2(\bar{y}_2)\frac{1+E_2}{E_2}, \tag{5.171}$$

hence:

$$p_2(\bar{y}_2) = \frac{E_2}{E_2+1}c. \tag{5.172}$$

We know that

$$i = 1,2 \quad p_i(\bar{y}_i) > 0, \quad c > 0, \quad E_i < 0 \Rightarrow 1 + E_i < 0 \Leftrightarrow E_i < -1. \tag{5.173}$$

From conditions (5.170) and (5.172), it results that a product price set on i-the market by an output level intended for this market guaranteeing the maximum profit for a monopolist depends on the price elasticity of demand reported for a product on i-th market and on the production marginal total (variable) cost which is the same regardless of the market. Thus, a question arises: when is the product price set on the first market higher (lower) than or equal to the product price set on the second market? To answer this question, one needs to subtract on both sides Eq. (5.172) from Eq. (5.170). It gives us

$$p_1(\bar{y}_1) - p_2(\bar{y}_2) = \left(\frac{E_1}{E_1+1} - \frac{E_2}{E_2+1}\right)c = \frac{E_1 - E_2}{(E_1+1)(E_2+1)}c. \tag{5.174}$$

On the basis of condition (5.174), it can be stated that.

- the product price on the first market is the same as on the second market when

$$E_1 = E_2, \tag{5.175}$$

  that is, when values of price elasticities of demand for a product on both markets are equal to each other,
- the product price on the first market is higher than on the second market when

$$E_1 > E_2, \tag{5.176}$$

  but one needs to remember that the price elasticities of demand for a product on both markets are negative and what is more they are smaller than -1. Hence, condition (5.176) is satisfied when the price elasticity of demand for a product on the second market is bigger in the absolute value than the price elasticity of demand for a product on the first market. This means that consumers on the second market react stronger to changes in the price of a product than consumers on the first market.

- the product price on the first market is lower than on the second market when

$$E_1 < E_2, \tag{5.177}$$

  but again one needs to remember that the price elasticities of demand for a product on both markets are smaller than -1. Hence, condition (5.177) is satisfied when the price elasticity of demand for a product on the first market is bigger in the absolute value than the price elasticity of demand for a product on the second market. This means that consumers on the first market react stronger to changes in the price of a product than consumers on the first market.

Example 5.5 A monopolistic company manufactures one product and supplies it to two markets. The monopolist is interested in discriminatory pricing of her/his product intended for these two markets. For the first (domestic) and the second (foreign) markets, he/she determines output levels and sets prices of the product so that to maximize the profit from production intended for both markets.

Let us take the following notation:

y_1—an output level intended for the first market,
y_2—an output level intended for the second market,
$y = \xi(y_1, y_2) = y_1 + y_2$—total quantity of the product manufactured by the monopolist supplied to both markets,
$c^{tot}(y) = c^v(y) + c^f(y)$—production total cost,
$c^v(y)$—production variable cost,
$c^f(y)$—production fixed cost,
$p_1(y_1) > 0$—a product price set by the monopolist on the first market,
$p_2(y_2) > 0$—a product price set by the monopolist on the second market,
$y_1^d = y_1^d(p_1)$—the demand reported by consumers for a product on the first market,
$y_2^d = y_2^d(p_2)$—the demand reported by consumers for a product on the second market,
$y_1^s = y_1^s(p_1)$—the supply of the product intended for the first market,
$y_2^s = y_2^s(p_2)$—the supply of the product intended for the second market,
$r_1(y_1) = p_1(y_1)y_1$—revenue from sales of the product on the first market,
$r_2(y_2) = p_2(y_2)y_2$—revenue from sales of the product on the second market.

(D9) From now on, it is assumed that

$$\forall i = 1, 2 \quad y_i^d = y_i^d(p_i) = y_i^s(p_i) = y_i, \tag{5.178}$$

which means that the output level on each market matches the demand reported for the product on this market. We assume also that it is not possible to resell the product between the markets.

(D10) There are given:

A production total cost function:

(5.179) $$c^{tot}(y) = c^{v}(y) + c^{f}(y) = \gamma y + \delta = \gamma(y_1 + y_2) + \delta, \quad \gamma, \delta > 0,$$

according to which the production total cost depends on the total output, regardless of the fact whether it is intended for the first or the second market.

A function of demand reported for a product on the first market:

(5.180) $$y_1^d(p_1) = -a_1 p_1 + b_1,$$

such that

(5.181) $$\frac{\mathrm{d}y_1^d(p_1)}{\mathrm{d}p_1} = -a_1 < 0,$$

and is decreasing in a price of a product price set on the first market.

An inversed function of demand reported for a product on the first market:

(5.182) $$p_1\left(y_1^d\right) = \frac{b_1}{a_1} - \frac{y_1^d}{a_1} = \alpha_1 - \beta_1 y_1^d, \quad a_1, b_1 > 0 \Rightarrow \alpha_1, \beta_1 > 0.$$

A function of revenue from sales of a product one the first market:

(5.183) $$r_1(y_1) = p_1(y_1)y_1 = (\alpha_1 - \beta_1 y_1)y_1 = \alpha_1 y_1 - \beta_1 y_1^2.$$

A function of demand reported for a product on the second market:

(5.184) $$y_2^d(p_2) = -a_2 p_2 + b_2,$$

such that

(5.185) $$\frac{\mathrm{d}y_2^d(p_2)}{\mathrm{d}p_2} = -a_2 < 0$$

and is decreasing in a price of a product price set on the second t market.

An inversed function of demand reported for a product on the second market:

(5.186) $$p_2\left(y_2^d\right) = \frac{b_2}{a_2} - \frac{y_2^d}{a_2} = \alpha_2 - \beta_2 y_2^d, \quad a_2, b_2 > 0 \Rightarrow \alpha_2, \beta_2 > 0.$$

A function of revenue from sales of a product one the second market:

(5.187) $$r_2(y_2) = p_2(y_2)y_2 = (\alpha_2 - \beta_2 y_2)y_2 = \alpha_2 y_2 - \beta_2 y_2^2.$$

Price elasticity of demand reported for a product on the first market:

(5.188) $$E_1 = \frac{\mathrm{d}y_1}{\mathrm{d}p_1}\frac{p_1}{y_1} = -a_1\frac{p_1}{y_1} < 0, \quad \text{since} \quad a_1, p_1, y_1 > 0.$$

Price elasticity of demand reported for a product on the second market:

(5.189) $$E_2 = \frac{\mathrm{d}y_2}{\mathrm{d}p_2}\frac{p_2}{y_2} = -a_2\frac{p_2}{y_2} < 0, \quad \text{since} \quad a_2, p_2, y_2 > 0.$$

A monopolist's profit function of a form:

(5.190) $$\begin{aligned} \Pi(y_1, y_2) &= r_1(y_1) + r_2(y_2) - c^{tot}(y_1, y_2) \\ &= \alpha_1 y_1 - \beta_1 y_1^2 + \alpha_2 y_2 - \beta_2 y_2^2 - \{\gamma(y_1 + y_2) + \delta\}, \end{aligned}$$

(5.191) $$y_1, y_2 \geq 0.$$

Tasks

Knowing that the aim of a monopolist is to maximize the profit determine:

1. an optimal level of output intended for:
   (a) the first market,
   (b) the second market,
   (c) both markets overall;
2. the maximum profit that a monopolist can obtain on:
   (a) the first market,
   (b) the second market,
   (c) both markets overall;
3. an optimal price level of a product set on:
   (a) the first market,
   (b) the second market;
4. analyse sensitivity:
   (a) separately for each market of the optimal output level and the optimal price level which guarantee the maximum profit for a monopolist,
   (b) of the optimal total supply of a product intended for both markets, to changes of values of parameters of the demand functions and of the production cost function.

Ad 1 Necessary conditions for the existence of an optimal solution to the monopolist's profit maximization problem have a form:

$$\left.\frac{\partial \Pi(y_1, y_2)}{\partial y_1}\right|_{y=\bar{y}} = \alpha_1 - 2\beta_1 \bar{y}_1 - \gamma = 0, \tag{5.192}$$

$$\left.\frac{\partial \Pi(y_1, y_2)}{\partial y_2}\right|_{y=\bar{y}} = \alpha_2 - 2\beta_2 \bar{y}_2 - \gamma = 0, \tag{5.193}$$

which gives.

- for the first market:

$$\bar{y}_1 = \frac{\alpha_1 - \gamma}{2\beta_1} = \frac{b_1 - a_1\gamma}{2} > 0, \quad \text{where} \quad \alpha_1 = \frac{b_1}{a_1}, \beta_1 = \frac{1}{a_1}, \tag{5.194}$$

  when:

$$\gamma < \frac{b_1}{a_1}, \tag{5.195}$$

  that is when the production marginal total cost is lower that a maximal price level that a monopolist can set on the first market;
- for the second market:

$$\bar{y}_2 = \frac{\alpha_2 - \gamma}{2\beta_2} = \frac{b_2 - a_2\gamma}{2} > 0, \quad \text{where} \quad \alpha_2 = \frac{b_2}{a_2}, \beta_2 = \frac{1}{a_2}, \tag{5.196}$$

  when:

$$\gamma < \frac{b_2}{a_2}, \tag{5.197}$$

  that is when the production marginal total cost is lower that a maximal price level that a monopolist can set on the second market.

Sufficient conditions for the existence of an optimal solution to the monopolist's profit maximization problem have a form:

$$\left.\frac{\partial^2 \Pi(y_1, y_2)}{\partial y_1^2}\right|_{y=\bar{y}} = -2\beta_1 < 0, \tag{5.198}$$

$$\left.\frac{\partial^2 \Pi(y_1, y_2)}{\partial y_2^2}\right|_{y=\bar{y}} = -2\beta_2 < 0, \tag{5.199}$$

(5.200)

$$\left.\frac{\partial^2 \Pi(y_1, y_2)}{\partial y_1^2}\right|_{y=\bar{y}} \cdot \left.\frac{\partial^2 \Pi(y_1, y_2)}{\partial y_2^2}\right|_{y=\bar{y}} - \left.\frac{\partial^2 \Pi(y_1, y_2)}{\partial y_1 \partial y_2}\right|_{y=\bar{y}} \cdot \left.\frac{\partial^2 \Pi(y_1, y_2)}{\partial y_2 \partial y_1}\right|_{y=\bar{y}} > 0.$$

The Hessian of the profit function has a form:

(5.201)
$$H(y_1, y_2) = \begin{bmatrix} -2\beta_1 & 0 \\ 0 & -2\beta_2 \end{bmatrix}.$$

Since $\det H(y_1, y_2) = 4\beta_1\beta_2 > 0$ and conditions (5.198)–(5.200) are satisfied, for the output levels given by conditions (5.194) and (5.196), the monopolist obtains the maximum profit.

Let us notice that the optimal supply of a product intended for each of the markets depends on i-th's market capacity $b_i > 0$ (the demand for a product by zero price), the strength $a_i > 0$ of consumers' reaction to changes in a product price on i-th market and on the production marginal total (variable) cost $\gamma > 0$.

The optimal total supply of a product supplied for both markets is

(5.202)
$$\begin{aligned} \bar{y} &= \bar{y}_1 + \bar{y}_2 = \frac{\alpha_1 - \gamma}{2\beta_1} + \frac{\alpha_2 - \gamma}{2\beta_2} \\ &= \frac{b_1 - a_1\gamma}{2} + \frac{b_2 - a_2\gamma}{2} = \frac{b_1 + b_2 - \gamma(a_1 + a_2)}{2} > 0. \end{aligned}$$

Let us notice that the optimal total supply of a product intended for both markets depends on market capacities $b_1, b_2 > 0$, the strength $a_1, a_2 > 0$ of consumers' reaction to changes in product prices on both markets and the production marginal total cost $\gamma > 0$.

Ad 2

(a) the monopolist's maximum profit on the first market:

(5.203)
$$\begin{aligned} \Pi_1(\bar{y}_1, \bar{y}_2) &= r_1(\bar{y}_1, \bar{y}_2) - \frac{\bar{y}_1}{\bar{y}} c^{tot}(\bar{y}_1, \bar{y}_2) \\ &= \frac{b_1^2 - \gamma^2 a_1^2}{4a_1} - \frac{(b_1 - \gamma a_1)[\gamma(b_1 + b_2 - \gamma(a_1 + a_2)) + 2\delta]}{2[b_1 + b_2 - \gamma(a_1 + a_2)]}, \end{aligned}$$

(b) the monopolist's maximum profit on the second market:

(5.204)
$$\begin{aligned} \Pi_2(\bar{y}_1, \bar{y}_2) &= r_2(\bar{y}_1, \bar{y}_2) - \frac{\bar{y}_2}{\bar{y}} c^{tot}(\bar{y}_1, \bar{y}_2) \\ &= \frac{b_2^2 - \gamma^2 a_2^2}{4a_2} - \frac{(b_2 - \gamma a_2)[\gamma(b_1 + b_2 - \gamma(a_1 + a_2)) + 2\delta]}{2[b_1 + b_2 - \gamma(a_1 + a_2)]}, \end{aligned}$$

(c) the monopolist's maximum profit overall on both markets:

(5.205)
$$\Pi(\bar{y}_1, \bar{y}_2) = \frac{b_1^2 - \gamma^2 a_1^2}{4a_1} + \frac{b_2^2 - \gamma^2 a_2^2}{4a_2} - \frac{\gamma(b_1 + b_2 - \gamma(a_1 + a_2)) + 2\delta}{2}.$$

Ad 3 From assumption (D9), it is known that the output level on i-th market ($i = 1, 2$), including the optimal level, matches the demand reported for the product on this market. On the basis of conditions (5.182) and (5.186) describing the inverse demand functions, one can determine the optimal levels of a product price on two markets that a monopolist wants to set to maximize her/his profit.

The product price on the first market is

(5.206)
$$p_1(\bar{y}_1) = \frac{\alpha_1 + \gamma}{2} = \frac{b_1 + a_1\gamma}{2a_1},$$

and on the second market:

(5.207)
$$p_2(\bar{y}_2) = \frac{\alpha_2 + \gamma}{2} = \frac{b_2 + a_2\gamma}{2a_2}.$$

Let us notice that the optimal price of a product on i-th market ($i = 1, 2$) depends on i-th's market capacity $b_i > 0$, the strength $a_i > 0$ of consumers' reaction to changes in a product price on i-th market and on the production marginal total (variable) cost $\gamma > 0$.

Price elasticities of demand are:

- for the first market (Fig. 5.12):

(5.208)
$$E_1(\bar{y}_1) = -\frac{b_1 + a_1\gamma}{b_1 - a_1\gamma} < -1,$$

- for the second market (Fig. 5.12):

(5.209)
$$E_2(\bar{y}_2) = -\frac{b_2 + a_2\gamma}{b_2 - a_2\gamma} < -1.$$

Ad 4 Tables 5.3a, 5.3b and 5.3c present measures of sensitivity of the optimal supply on each market and of the optimal total supply to changes of values of parameters that describe these optimal levels.

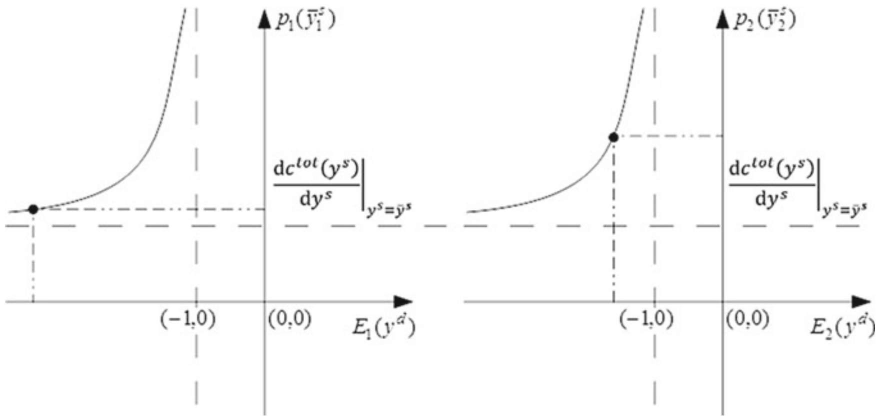


Fig. 5.12 Dependence of product prices set by a monopolist on price elasticities of demand reported by consumers for product

Table 5.3a Measures of response of the first market optimal supply to changes in parameters' values

Characteristic	$\frac{\partial \bar{y}_1}{\partial \gamma}$	$\frac{\partial \bar{y}_1}{\partial a_1}$	$\frac{\partial \bar{y}_1}{\partial b_1}$	$\frac{\partial \bar{y}_1}{\partial a_2}$	$\frac{\partial \bar{y}_1}{\partial b_2}$
Value	$-\frac{a_1}{2}$	$-\frac{\gamma}{2}$	$\frac{1}{2}$	0	0

Table 5.3b Measures of response of the second market optimal supply to changes in parameters' values

Characteristic	$\frac{\partial \bar{y}_2}{\partial \gamma}$	$\frac{\partial \bar{y}_2}{\partial a_1}$	$\frac{\partial \bar{y}_2}{\partial b_1}$	$\frac{\partial \bar{y}_1}{\partial a_2}$	$\frac{\partial \bar{y}_1}{\partial b_2}$
Value	$-\frac{a_2}{2}$	0	0	$-\frac{\gamma}{2}$	$\frac{1}{2}$

Table 5.3c Measures of response of both markets' total optimal supply to changes in parameters' values

Characteristic	$\frac{\partial \bar{y}}{\partial \gamma}$	$\frac{\partial \bar{y}}{\partial a_1}$	$\frac{\partial \bar{y}}{\partial b_1}$	$\frac{\partial \bar{y}}{\partial a_2}$	$\frac{\partial \bar{y}}{\partial b_2}$
Value	$-\frac{a_1+a_2}{2}$	$-\frac{\gamma}{2}$	$\frac{1}{2}$	$-\frac{\gamma}{2}$	$\frac{1}{2}$

Conclusions

1. Strength of the response of both markets' total optimal supply to changes in the value of any parameter is equal to a sum of the strength of the responses of the optimal supply on each market separately.
2. When the production marginal cost γ increases by one money unit then i-th's market optimal supply (total optimal supply) decreases by a number of physical units equal to half of a_i (a half of $a_1 + a_2$).

Table 5.4a Measures of response of first market optimal price to changes in parameters' values

Characteristic	$\frac{\partial p_1(\bar{y}_1)}{\partial \gamma}$	$\frac{\partial p_1(\bar{y}_1)}{\partial a_1}$	$\frac{\partial p_1(\bar{y}_1)}{\partial b_1}$	$\frac{\partial p_1(\bar{y}_1)}{\partial a_2}$	$\frac{\partial p_1(\bar{y}_1)}{\partial b_2}$
Value	$\frac{1}{2}$	$-\frac{b_1}{2a_1^2}$	$\frac{1}{2a_1}$	0	0

Table 5.4b Measures of response of second market optimal price to changes in parameters' values

Characteristic	$\frac{\partial p_2(\bar{y}_2)}{\partial \gamma}$	$\frac{\partial p_2(\bar{y}_2)}{\partial a_1}$	$\frac{\partial p_2(\bar{y}_2)}{\partial b_1}$	$\frac{\partial p_2(\bar{y}_2)}{\partial a_2}$	$\frac{\partial p_2(\bar{y}_2)}{\partial b_2}$
Value	$\frac{1}{2}$	0	0	$-\frac{b_2}{2a_2^2}$	$\frac{1}{2a_2}$

3. When consumers' sensitivity on i-th market a_i to changes in a product price increases by one unit then i-th's market optimal supply, as well as the total optimal supply, decreases by a number of physical units equal to half of the production marginal cost.
4. When i-th market capacity b_i increases by one unit then i-th's market optimal supply, as well as the total optimal supply, decreases by half of one physical unit.
5. Changes in values of parameters of the function of demand on the first (second) market do not affect the second (first) market optimal supply.

Tables 5.4a and 5.4b present measures of sensitivity of the optimal levels of a product price on each market to changes in values of parameters describing these optimal levels.

Conclusions

1. When the production marginal cost γ increases by one money unit then the optimal price on each market increases by half of one money unit.
2. When consumers' sensitivity on i-th market a_i to changes in a product price increases by one unit then i-th's market optimal price decreases by a number of money units equal to $\frac{b_i}{2a_i^2}$.
3. When i-th market capacity b_i increases by one unit then i-th's market optimal price increases by a number of money units equal to $\frac{1}{2a_i}$.
4. Changes in values of parameters of the function of demand on the first (second) market do not affect the second (first) market optimal price.

5.3.2 Dynamic Approach

A monopolist may decide that to maximize the profit it is optimal to discriminate prices of her/his product between two or more markets if he/she notices disparities in the demand reported by consumers on these markets. From the point of view of the monopolist, a criterion for discriminatory pricing is the existence of differences in price elasticities of demand on the markets which result from different forms of functions of demand reported on the markets. From the point of view of a consumer, the criterion for discriminatory pricing has to be clear, unambiguous and reasonable to be well accepted by them.[20] Such a criterion can be, for example, the age of a consumer, student status or location distinguishing one group of consumers from the other.

In the static approach presented in Sect. 5.3.1, we assume that the dependency of the demand on a product price has a form which does not change over time. In the dynamic approach, we allow the possibility that properties characterizing the demand reported by a group of consumers on a given market can vary over time. For example, a given group over time can be more or less interested in some product, or the group can become more or less sensitive to changes in a price of the product.

Let us use the following notation:

t—time, in a discrete version ($t = 0, 1, 2, \ldots, T$) or in a continuous version[21] ($t \in [0; T]$),
T—time horizon,
$y_i^d(t) = h_i(p_i(t))$—a function of consumer demand reported for a product on i-th market, with a time-variant form,
$p_i\left(y_i^s(t)\right) = g_i(y_i^d(t))$—an inverse function of consumer demand reported for a product on i-th market, with a time-variant form,
$y^s(t) = y_1^s(t) + y_2^s(t)$—the total supply intended for both markets,
$c^{tot}(y^s(t)) = c^v(y^s(t)) + \delta(t)$—a function of production total cost,
$c^v(y^s(t))$—a function of production variable cost, that is, the cost depending on an output level,
$\delta(t) \geq 0$—the time-variant fixed cost, that is, the cost that does not depend on the output level,
$E(y_i^d(t))$—price elasticity of consumer demand reported for a product on i-th market.

[20] There exist situations in which consumers do not have all information about prices, even if there is only one supplier of a product or of a service. It happens for example when a seller or a service provider does not use any official price list. Then such a supplier of a product can discriminate prices without consumers' knowledge.

[21] For the discrete and continuous versions, we use the same denotation of dependency of a function value on time, for example, the fixed cost depending on time: $\delta(t)$. Whether the discrete or continuous version is used in a given formula will result from the context of the issue under consideration.

In the profit maximization problem for a monopolist who considers the discriminatory pricing, we make a simplifying assumption that, in every period/at any moment$t$, the monopolist adjusts the output intended for i-th market to a level of the demand reported by consumers on i-th market by a price p_i(t) set on this market:

(5.210) $$y_i^s(t) = y_i^d(t) = y_i(t) \qquad i = 1, 2.$$

As a consequence, the total output intended for both markets overall is adjusted by a monopolist to the total demand reported by consumers on both markets:

(5.211) $$y^s(t) = y^d(y) = y(t).$$

Production marginal cost is assumed to be identical regardless of the fact where the output is intended, for the first or for the second market:

(5.212) $$\frac{\partial c^{tot}(y(t))}{\partial y_i(t)} = \frac{\partial c^{tot}(y(t))}{\partial y(t)} \cdot \frac{\partial y(t)}{\partial y_i(t)} = \frac{\mathrm{d}c^{tot}(y(t))}{\mathrm{d}y(t)} \qquad i = 1, 2.$$

This means that, regardless of the fact where the product is to be supplied, the cost of its production forms in the same way. We do not consider here the issues related to additional costs of discriminatory pricing, for example, transport costs of a product intended for export.

The profit maximization problem for a monopolist discriminating prices on two markets has the form:

(5.213) $$\begin{aligned} \Pi(y_1(t), y_1(t)) &= r_1(y_1(t)) + r_2(y_2(t)) - c^{tot}(y(t)) \\ &= \left\{p_1(y_1(t)) \cdot y_1(t) + p_2(y_2(t)) \cdot y_2(t) - c^{to}(y(t))\right\} \mapsto \max \end{aligned}$$

(5.214) $$y_1(t), \quad y_2(t) \geq 0.$$

If the profit function $\Pi(y_1(t), y_2(t))$ is strictly concave, then in every period/at any moment t, Problem (5.213)–(5.214) has a solution $\bar{\boldsymbol{y}}(t) = \left(\bar{y}_1(t), \bar{y}_1(t)\right) > (0, 0)$, derived from the first-order condition of the profit maximization:

(5.215) $$\begin{cases} \left.\frac{\partial \Pi(y(t))}{\partial y_1(t)}\right|_{y(t)=\bar{y}(t)} = 0 \\ \left.\frac{\partial \Pi(y(t))}{\partial y_2(t)}\right|_{y(t)=\bar{y}(t)} = 0 \end{cases}, \qquad \forall t,$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0; T]$. Using the profit function form and Eq. (5.212), the system above can be written also in an equivalent form as

(5.216) $$\begin{cases} \left.\frac{\mathrm{d}r_1(y_1(t))}{\mathrm{d}y_1(t)}\right|_{y_1(t)=\bar{y}_1(t)} = \left.\frac{\mathrm{d}c^{tot}(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\bar{y}(t)} \\ \left.\frac{\mathrm{d}r_2(y_2(t))}{\mathrm{d}y_2(t)}\right|_{y_2(t)=\bar{y}_2(t)} = \left.\frac{\mathrm{d}c^{tot}(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\bar{y}(t)} \end{cases}, \qquad \forall t,$$

Hence, one derives a condition:

$$\left.\frac{\mathrm{d}r_i(y_i(t))}{\mathrm{d}y_i(t)}\right|_{y_i(t)=\bar{y}_i(t)} = \left.\frac{\mathrm{d}c^{tot}(y(t))}{\mathrm{d}y(t)}\right|_{y(t)=\bar{y}(t)} \qquad i = 1, 2, \tag{5.217}$$

which shows that a monopolist who discriminates prices of her/his product in order to maximize the profit should aim at equalizing values of marginal profits on both markets with a value of the production marginal cost.

From the profit maximization problem, one derives the optimal supply $\bar{y}(t)$ of a product $\bar{y}_i(t)$ intended for i-th market ($i = 1, 2$) and the optimal total supply $\bar{y}(t) = \bar{y}_1(t) + \bar{y}_2(t)$. The optimal level of a product price $p_i(\bar{y}_i(t))$ to be set on i-th market is derived on the basis of the inverse function of demand on i-th market.

Example 5.6 Let us assume that the demand reported for a product of a monopoly on i-th market is given in a linear function form:

$$y_i^d(t) = -a_i(t)p_i(t) + b_i(t), \qquad a_i(t), b_i(t) > 0, \qquad i = 1, 2, \qquad \forall t \in [0; 30],$$

where $a_i(t)$ is the function determining how strong the reaction of consumers to changes in a product price on i-th market is, while $b_i(t)$ is a function of i-th market capacity. In order to evaluate how the formation of the consumers' sensitivity to changes in a product price and the formation of the market capacity affect the discriminatory pricing practised by a monopolist, we will compare results derived by different assumptions. Particularly, we will compare results when two markets do not differ at all from each other when they differ by forms of functions $a_i(t)$ or $b_i(t)$ and when the markets differ by forms of both of these functions.

Let us assume that the strength of the reaction of consumers to changes in a product price on i-th market is described by the formula:

$$a_I(t) = 4 \cdot 0.98^t$$

or

$$a_{II}(t) = 0.006t^2 - 0.1t + 3,$$

where subscript I or II denotes the version of the formula. The i-th market capacity varies in time according to the formula:

$$d_I(t) = 0.025t^2 - 0.75t + 20$$

or

$$d_{II}(t) = -0.025t^2 + 0.75t + 15.$$

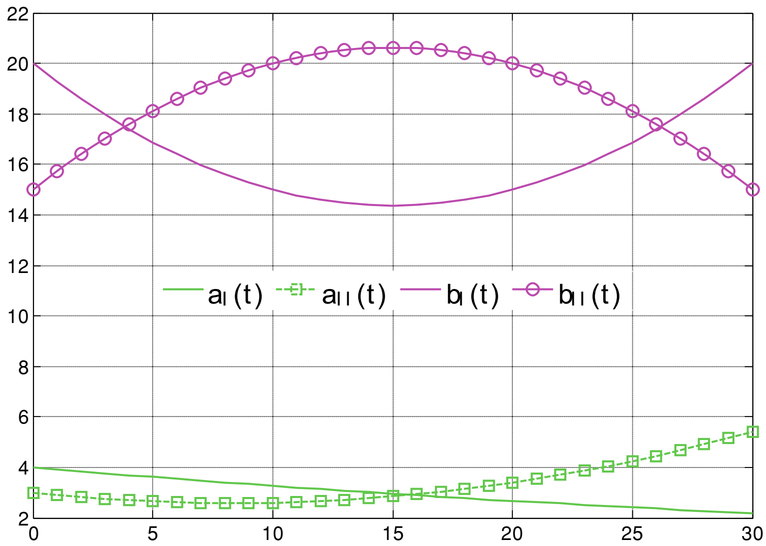


Fig. 5.13 Trajectories of the strength of consumers' reaction to changes in price and trajectories of market capacity

Trajectories of these values in each of two versions of the formulas are presented in Fig. 5.13. In version I, the strength of the reaction of consumers to changes in a product price declines in time, while in version II, it rises. The market capacity in version I reaches its minimum at moment $t = 15$, while in version II, it reaches its maximum at this moment.

Let us assume that a production total cost function is linear too:

$$c^{tot}\left(y^s(t)\right) = \gamma(t)y^s(t) + \delta(t), \qquad \gamma(t), \delta(t) > 0, \qquad t \in [0; 30],$$

where $\gamma(t)$ denotes the marginal cost of production, and $\delta(t)$ is a function of the fixed cost of production. For the sake of simplicity, we assume that the production fixed cost is constant over time and amounts to 1 at any moment. The production marginal cost is assumed to vary over time according to the formula:

$$\gamma(t) = -\frac{1}{t+1} + 2.$$

Trajectories of $\gamma(t)$ and $\delta(t)$ are shown in Fig. 5.14. The fixed cost of production does not change over time and is always equal to 1. The marginal cost of production increases over time starting at 1 at the beginning of the horizon and ending at approximately 1.97 at the end of the horizon.

From the function of demand on i-th market ($i = 1, 2$), assuming that the supply intended for this market matches the demand, that is, $y_i^s(t) = y_i^d(t) = y_i(t)$, we

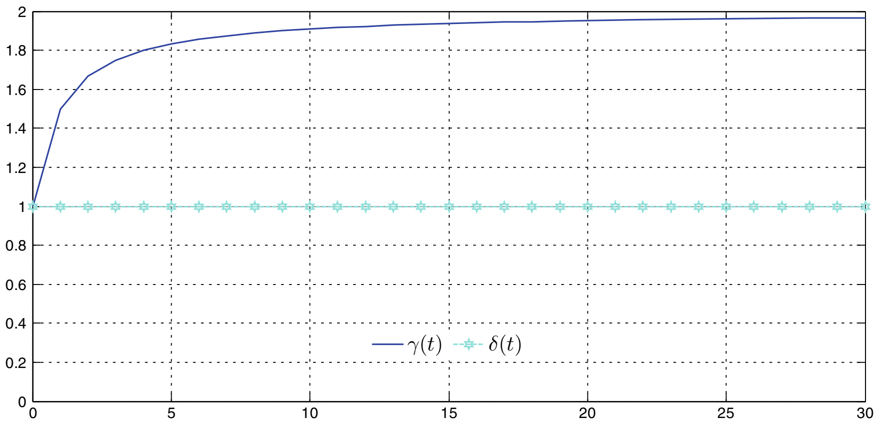


Fig. 5.14 Trajectories of production marginal cost and of production fixed cost

derive an inverse function of demand on i-th market which takes the form:

$$p_i(y_i(t)) = -\frac{1}{a_i(t)}y_i(t) + \frac{b_i(t)}{a_i(t)}.$$

Hence, the profit function of a monopolist who discriminates prices of her/his product can be written as

$$\begin{aligned}
&\Pi(y_1(t), y_1(t)) = r_1(y_1(t)) + r_2(y_2(t)) - c^{tot}(y(t)) \\
&= \left(-\frac{1}{a_1(t)}y_1(t) + \frac{b_1(t)}{a_1(t)}\right)y_1(t) + \left(-\frac{1}{a_2(t)}y_2(t) + \frac{b_2(t)}{a_2(t)}\right) \\
&y_2(t) - \{\gamma(t)y(t) - \delta(t)\} \\
&= -\frac{1}{a_1(t)}y_1(t)^2 + \frac{b_1(t)}{a_1(t)}y_1(t) - \frac{1}{a_2(t)}y_2(t)^2 + \frac{b(t)}{a_2(t)}y_2(t) \\
&-\gamma(t)(y_1(t) + y_2(t)) - \delta(t).
\end{aligned}$$

The solution to the profit maximization problem is the optimal supply intended for i-th market ($i = 1, 2$) given as

$$\bar{y}_i(t) = \frac{b_i(t) - a_i(t)\gamma(t)}{2},$$

$$\bar{y}_i(t) > 0 \quad \Leftrightarrow \quad \gamma(t) < \frac{b_i(t)}{a_i(t)} \qquad \forall t \in [0; 30]$$

where we can notice that the supply intended for i-th market increases when i-th market capacity $b_i(t)$ grows and when the consumers' sensitivity $a_i(t)$ to price changes on i-th market or the marginal cost of production declines.

On the basis of the inverse function of demand on i-th market, a monopolist determines the optimal price level for her/his product on i-th market. This level depends on the optimal supply intended for i-th market:

$$p_i(\bar{y}_i(t)) = -\frac{1}{a_i}\bar{y}(t) + \frac{b_i(t)}{a_i(t)} = \frac{b_i(t) + a_i(t)\gamma(t)}{2a_i(t)}, \qquad i = 1, 2.$$

The price level on i-th market increases when i-th market capacity $b_i(t)$ or the production marginal cost $\gamma(t)$ rise and when the consumers' sensitivity $a_i(t)$ to price changes on i-th market declines.

Let us now analyse three scenarios of differences in the demand between the two markets. In the first scenario, we assume that the markets have the same capacities (the demand by zero price), but they differ by the consumers' sensitivity to price changes which, for the first market, evolves according to formula $a_I(t)$ and, for the second market, according to formula $a_{II}(t)$. In the second scenario, we assume that both markets are characterized by the same consumers' sensitivity to price changes, but they differ by the market capacity which, for the first market, evolves according to formula $b_I(t)$ and, for the second market, according to formula $b_{II}(t)$. In the third scenario, the markets differ in both characteristics:

$$y_1^d(t) = -c_I(t)p_1(t) + d_I(t),$$

$$y_2^d(t) = -c_{II}(t)p_2(t) + d_{II}(t).$$

To capture the difference between two markets relevant for further analysis, we compare the price elasticities of demand on two markets in each scenario and present this comparison in Fig. 5.15.

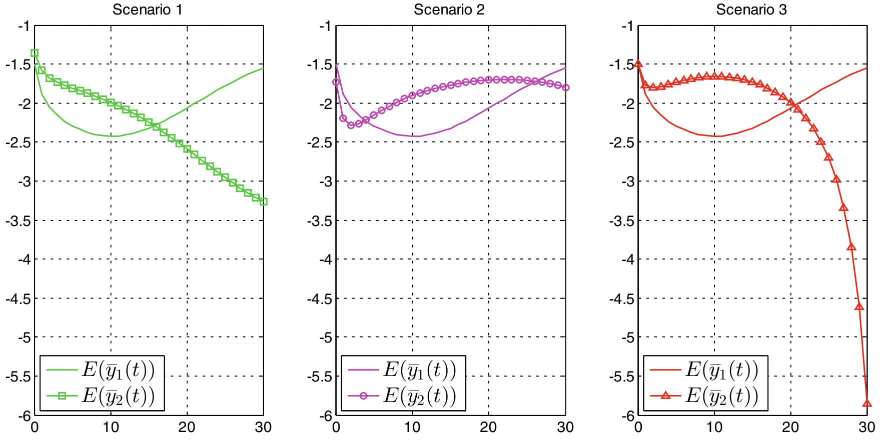


Fig. 5.15 Trajectories of price elasticities of demand reported for a product on two markets

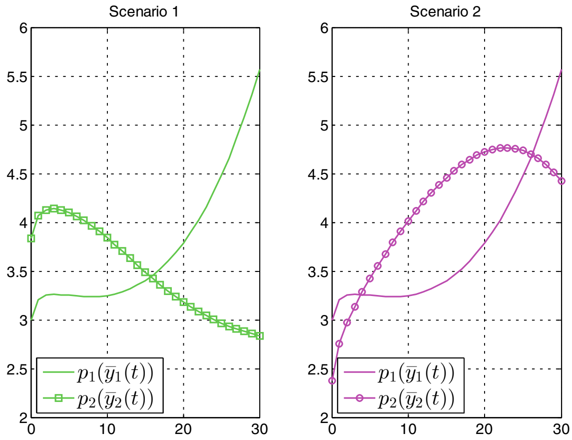


Fig. 5.16 Trajectories of optimal prices of product on two markets

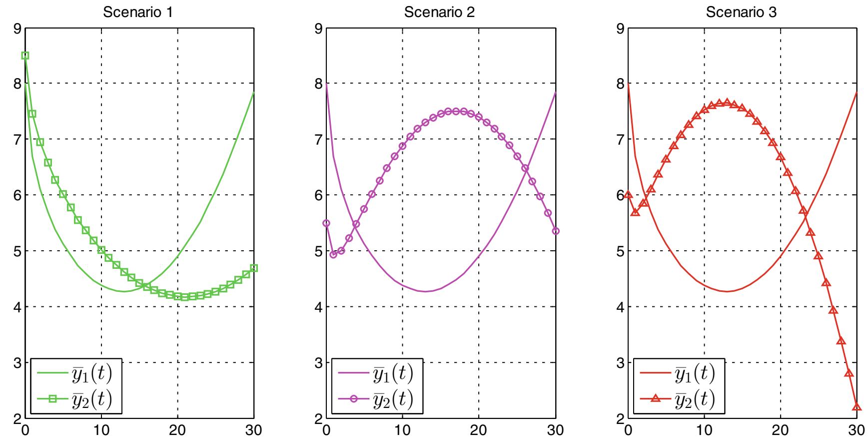


Fig. 5.17 Trajectories of product optimal supplies on two markets

From Sect. 5.3.1, describing the topic of discriminatory pricing in the static approach, we know that the smaller in absolute value the price elasticity of demand on a given market is then the higher price of a product can be set by a monopolist on this market. Hence, from Fig. 5.15, we can derive in what time intervals in a given scenario which price is to be set higher: on the first or on the second market. Formation of the price elasticities of demand on two markets in the first scenario indicates that a monopoly sets the higher price on the second market in time interval [0; 16) and in the rest of the time horizon on the first market. In the second scenario, the price on the first market is higher than on the second market in time intervals [0; 3) and [27; 30] and lower in [3; 27). In the third scenario, in

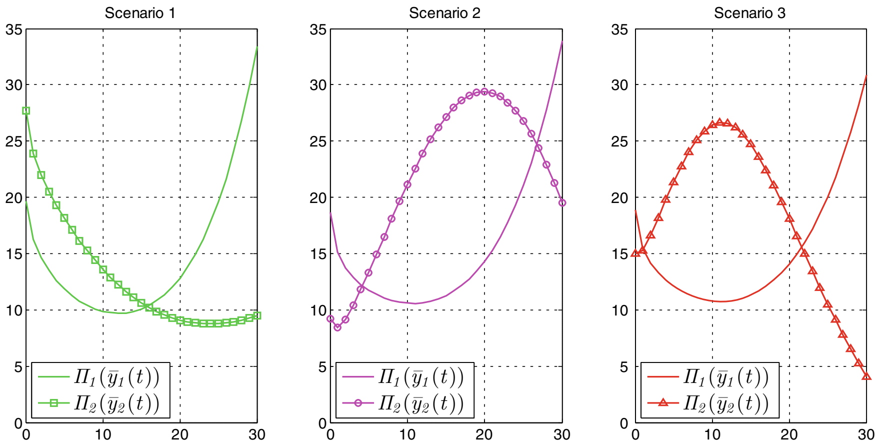


Fig. 5.18 Trajectories of maximum profit of a monopolist discriminating prices on two markets

time interval (0; 22), the price is higher on the second market and in the rest of the time horizon on the first market. All this can be stated on the basis of Fig. 5.15. Then it is confirmed in Fig. 5.16.

From Fig. 5.17, comparing it with Fig. 5.16, it can be noticed that the higher price of a product corresponds to the bigger supply of the product.[22] The monopolistic company wants to provide a bigger quantity of the product to a market on which a higher price can be set due to the lower absolute value price elasticity of demand.

The higher price of the product and the corresponding bigger supply on one of the markets than on the other market involve also the higher profit obtained from production and from sales of the product on a given market. It is reflected in Fig. 5.18. The higher profit is seen on this market where there is a higher optimal price and a bigger optimal supply.

5.4 Quantitative and Price Competition of Producers in a Duopoly

So far, in Sects. 5.1, 5.2 and 5.3, we have discussed rational decisions of producers acting in perfect competition and rational decisions of a monopolist when there is only one homogenous product under consideration. Let us now analyse another market structure, namely a duopoly. We will consider a duopoly in two distinct

[22] Time intervals in which the price on a given market is higher than on the other market do not have to coincide precisely with intervals in which the optimal supply is higher on a given market. This results from the fact that changes in values of functions: $\gamma(t)$, $a_i(t)$ and $b_i(t)$ affect the optimal price level and the optimal supply level with different strength.

cases—when duopolists manufacture one homogenous product and when each of them manufactures a product substitute for a product of her/his competitor.

Definition 5.12 A market structure is called a **duopoly** if on the market of a certain product (good or service) there are two producers,[23] each of them having an impact on the price of a product and on an output level and aiming at maximization of her/his own profit (there is no cartel collusion, they do not maximize the joint profit).

Definition 5.13 A market structure is called an **oligopoly** if on the market of a certain product (good or service) there are n producers ($n \geq 3$), each of them having an impact on the price of a product and on an output level and aiming at maximization of her/his own profit (there is no cartel collusion, they do not maximize the joint profit).

Among the duopoly and oligopoly models, one distinguishes **quantity competition** models (on quantity of a product) and **price competition models** (on price of a product).

In quantity competition models, which include **Cournot and Stackelberg duopoly and oligopoly models**, one assumes that producers manufacture a homogeneous (undifferentiated) product (good or service). In that case, they have to set the same price for the product. Thus, they cannot compete on the price of the product they manufacture, but they can compete with each other on output levels.

In price competition models, which include **Bertrand duopoly and oligopoly models**, one assumes that producers manufacture substitute (differentiated) products. In that case, they may set different prices for the products they manufacture. Thus, they can compete on the prices of the products they manufacture.

Due to the way we conduct the analysis in the whole book, we do not discuss the oligopoly models. We simply state that they are direct generalizations of duopoly models (the Cournot, Stackelberg and Bertrand ones) for cases where a number of producers acting on a market equal$n \geq 3$.

5.4.1 Cournot Duopoly Model and Its Equilibrium State

5.4.1.1 Static Approach

Let us apply the following set of assumptions:

[23] In Sect. 5.1, we consider a market of a product where two producers manufacture a product and determine an optimal supply levels to obtain maximum profits and where a function of demand for the product is exogenously determined. However, these producers have no influence on a product price. It is given by a market. Hence, these producers are considered as acting in the perfect competition. By Definition 5.12, the market discussed in Sect. 5.1 is not a case of duopoly.

(C1) Two producers ($i = 1, 2$) act on a market of a homogeneous (undifferentiated) product. Functions of their production total cost are as follows:

(5.218) $$\forall i = 1,2 \quad c_i^{tot}(y_i) = c_i^v(y_i) + c_i^f(y_i) = c_i y_i + d_i, \quad c_i, d_i > 0,$$

being the sum of variable cost functions:

(5.219) $$\forall i = 1,2 \quad c_i^v(y_i) = c_i y_i, \quad c_i > 0$$

and the fixed costs:

(5.220) $$\forall i = 1,2 \quad c_i^f(y_i) = d_i > 0.$$

Since the total cost functions are linear functions of output levels, we get that

(5.221) $$\forall i = 1,2 \quad \frac{\mathrm{d}c_i^{tot}(y_i)}{\mathrm{d}y_i} = \frac{\mathrm{d}c_i^v(y_i)}{\mathrm{d}y_i} = c_i > 0,$$

that is, the marginal total cost and the marginal variable cost for the i-th producer are equal and they are functions increasing in an output level.

(C2) A function of demand reported for a product by consumers, depending on its price set by producers, is as follows:

(5.222) $$y^d(p) = -ap + b, \quad a, b > 0,$$

where a denotes a measure of the consumers' reaction strength to a unit increase in the price of a product and b denotes a measure of a market capacity.

Since values of the demand function have to be non-negative, we get that

(5.223) $$p \in \left[0; \frac{b}{a}\right].$$

(C3) The total output by both producers matches the demand that consumers report by a given price of a product:

(5.224) $$y_1 + y_2 = y^d(p) = -ap + b, \quad a, b > 0.$$

(C4) The first producer wants to determine such an output level that guarantees the maximum profit for her/him taking an output level of the second producer as given:

(5.225) $$\Pi_1(y_1)|_{y_2=\text{const.}\geq 0} \to \max \qquad y_1 \geq 0.$$

(C5) The second producer wants to determine such an output level that guarantees the maximum profit for her/him taking an output level of the first producer as given:

(5.226) $$\Pi_2(y_2)|_{y_1=\text{const.}\geq 0} \to \max \qquad y_2 \geq 0.$$

A profit function of i-th producer ($i = 1, 2$) can be expressed as the difference between revenue from sales of a product and total cost of production:

(5.227) $$\forall i = 1, 2 \quad \Pi_i(y_i) = p(y)y_i - [c_i y_i + d_i] = [p(y) - c_i]y_i - d_i,$$

Substituting an inverse function of demand $p(y) = \frac{b-y}{a} = \alpha - \beta(y_1 + y_2)$, where $\alpha = \frac{b}{a}$, $\beta = \frac{1}{a}$, into Eq. (5.227), one obtains the profit functions of both producers as functions of their output levels:

- for the first producer:

(5.228) $$\begin{aligned}\Pi_1(y_1, y_2) &= [\alpha - \beta(y_1 + y_2)]y_1 - [c_1 y_1 + d_1] \\ &= [\alpha - c_1]y_1 - \beta y_1^2 - \beta y_1 y_2 - d_1.\end{aligned}$$

- for the second producer:

(5.229) $$\begin{aligned}\Pi_2(y_1, y_2) &= [\alpha - \beta(y_1 + y_2)]y_2 - [c_2 y_2 + d_2] \\ &= [\alpha - c_2]y_2 - \beta y_2^2 - \beta y_1 y_2 - d_2.\end{aligned}$$

When the output level of the second producer is taken as given, thus treated as a parameter, the necessary condition and the sufficient condition for the profit maximization problem of the first producer are the following[24]:

(5.230) $$\left.\frac{\partial \Pi_1(y_1, y_2)}{\partial y_1}\right|_{y_1=\bar{y}_1, y_2=\text{const.}\geq 0} = 0 \qquad \text{the necessary condition}$$

(5.231) $$\left.\frac{\partial^2 \Pi_1(y_1, y_2)}{\partial y_1^2}\right|_{y_1=\bar{y}_1,\ y_2=\text{const.}\geq 0} < 0 \qquad \text{the sufficient condition.}$$

[24] The profit function of the first (second) producer is a one-variable function when the supply of a product by the second (first) producer is set. In conditions (5.230)–(5.233), we use notions appropriate for the first- and second-order partial derivatives, but the necessary and sufficient conditions of the optimum existence refer actually to one-variable functions.

When the output level of the first producer is taken as given, thus treated as a parameter, the necessary condition and the sufficient condition for the profit maximization problem of the second producer are the following:

(5.232) $$\left.\frac{\partial \Pi_2(y_1, y_2)}{\partial y_2}\right|_{y_2=\bar{y}_2,\ y_1=\text{const.}\geq 0} = 0 \quad \text{the necessary condition,}$$

(5.233) $$\left.\frac{\partial^2 \Pi_2(y_1, y_2)}{\partial y_2^2}\right|_{y_2=\bar{y}_2,\ y_1=\text{const.}\geq 0} < 0 \quad \text{the sufficient condition.}$$

Deriving the necessary and sufficient conditions for the profit functions described by Eqs. (5.228)–(5.229), we get.

- for the first producer:

(5.234) $$\left.\frac{\partial \Pi_1(y_1, y_2)}{\partial y_1}\right|_{y_1=\bar{y}_1,\ y_2=\text{const.}\geq 0} = \alpha - c_1 - 2\beta\bar{y}_1 - \beta y_2 = 0,$$

(5.235) $$\left.\frac{\partial^2 \Pi_1(y_1, y_2)}{\partial y_1^2}\right|_{y_1=\bar{y}_1,\ y_2=\text{const.}\geq 0} = -2\beta < 0,$$

which means that for any (given) output level $y_2 \geq 0$ set by the second producer, the first producer obtains the maximum profit when $y_1 = \bar{y}_1$;
- for the second producer:

(5.236) $$\left.\frac{\partial \Pi_2(y_1, y_2)}{\partial y_2}\right|_{y_2=\bar{y}_2,\ y_1=\text{const.}\geq 0} = \alpha - c_2 - 2\beta\bar{y}_2 - \beta y_1 = 0$$

(5.237) $$\left.\frac{\partial^2 \Pi_2(y_1, y_2)}{\partial y_2^2}\right|_{y_2=\bar{y}_2,\ y_1=\text{const.}\geq 0} = -2\beta < 0,$$

which means that for any (given) output level $y_1 \geq 0$ set by the first producer, the second producer obtains the maximum profit when $y_2 = \bar{y}_2$.

From condition (5.234), it results that

(5.238) $$\bar{y}_1 = \frac{\alpha - c_1}{2\beta} - \frac{y_2}{2} \quad \boldsymbol{RL_1}.$$

This equation is called a **reaction line equation of the first producer**.

From condition (5.236), it results that

(5.239) $$\bar{y}_2 = \frac{\alpha - c_2}{2\beta} - \frac{y_1}{2} \quad \boldsymbol{RL_2}.$$

This equation is called a **reaction line equation of the second producer**.

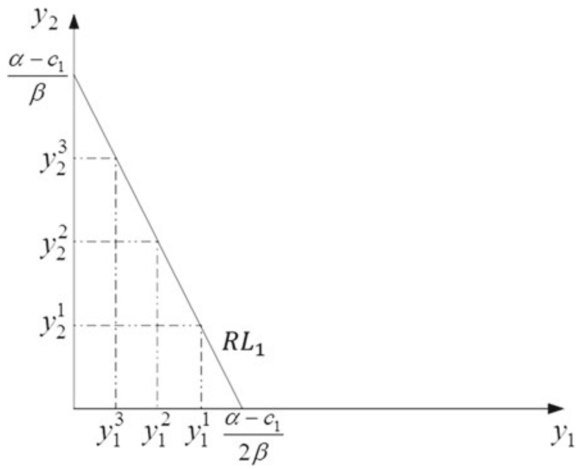


Fig. 5.19 Reaction line of the first producer

The respective reaction line equation describes the output level of the first (second) producer which, by the output level of the second (first) producer taken as given, guarantees the maximum profit for the first (second) producer (Figs. 5.19 and 5.20).

From Fig. 5.19 one can notice that, if the second producer supplied, respectively, $y_2^1, y_2^2, y_2^3 > 0$ units of the product on a market, then the first producer in order to maximize her/his profit should produce exactly $y_1^1, y_1^2, y_1^3 > 0$ units of the product. If the second producer supplied $y_2 = \frac{\alpha - c_1}{\beta}$ units of the product, then the first producer aiming at profit maximization should not manufacture this product. If the second producer did not supply the product on the market, then the first one would manufacture $\bar{y}_1 = \frac{\alpha - c_1}{2\beta}$ units of the product and would fully satisfy the demand reported for the product by consumers.

From Fig. 5.20 one can notice that, if the first producer supplied, respectively, $y_1^1, y_1^2, y_1^3 > 0$ units of the product on a market, then the second producer in order to maximize her/his profit should produce exactly $y_2^1, y_2^2, y_2^3 > 0$ units of the product. If the first producer supplied $y_1 = \frac{\alpha - c_2}{\beta}$ units of the product, then the second producer aiming at profit maximization should not manufacture this product. If the first producer did not supply the product, then the second one would manufacture $\bar{y}_2 = \frac{\alpha - c_2}{2\beta}$ units of the product and would fully satisfy the demand reported for the product by consumers.

From the reaction line equation RL_1, we can conclude that

$$y_2 = 0 \quad \Rightarrow \quad \bar{y}_1\big|_{RL_1} = \frac{\alpha - c_1}{2\beta}, \tag{5.240}$$

which means that, if the second producer was not present on the market, then the first producer would maximize the profit by supplying the product in a quantity

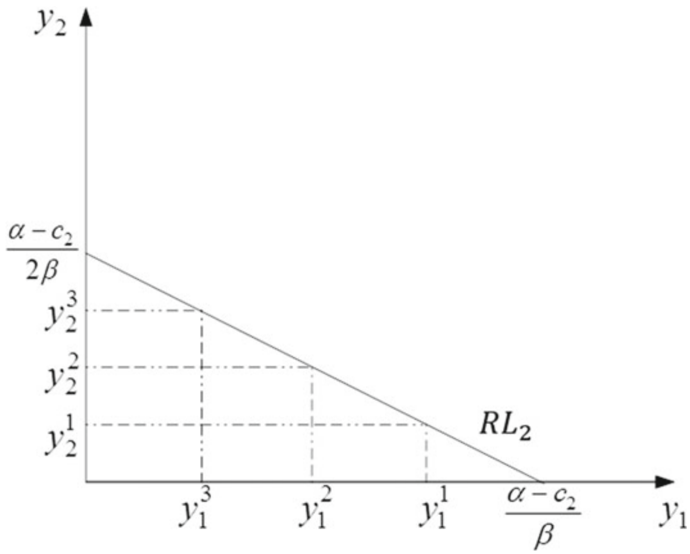


Fig. 5.20 Reaction line of the second producer

corresponding to the case of a pure monopoly. From the reaction line equation RL_1, it can also be stated that

(5.241) $$\bar{y}_1\big|_{RL_1} = 0 \quad \Leftrightarrow \quad \frac{\alpha - c_1}{2\beta} - \frac{y_2}{2} = 0 \quad \Leftrightarrow \quad y_2 = \frac{\alpha - c_1}{\beta},$$

which means that, if the second producer supplied $\frac{\alpha - c_1}{\beta}$ units of the product, then the first producer should exit the market since he/she would maximize the profit with zero output level. Let us notice that conditions (5.240) and (5.241) make economic sense when $c_1 < \alpha = \frac{b}{a}$. If the second producer rises her/his output level by one physical unit, then the first producer aiming at profit maximization has to reduce her/his own output level by half of the unit. This is derived from the reaction line equation as the following condition:

(5.242) $$\frac{\mathrm{d}\bar{y}_1}{\mathrm{d}y_2}\bigg|_{RL_1} = -\frac{1}{2}.$$

From the reaction line equation RL_1 we can conclude that

(5.243) $$y_1 = 0 \quad \Rightarrow \quad \bar{y}_2\big|_{RL_2} = \frac{\alpha - c_2}{2\beta},$$

which means that, if the first producer was not present on the market, then the second producer would maximize the profit by supplying the product in a quantity corresponding to the case of a pure monopoly. From the reaction line equation RL_2, it can be stated also that

(5.244) $$\bar{y}_2\big|_{RL_2} = 0 \quad \Leftrightarrow \quad \frac{\alpha - c_2}{2\beta} - \frac{y_1}{2} = 0 \quad \Leftrightarrow \quad y_1 = \frac{\alpha - c_2}{\beta},$$

which means that, if the first producer would supply $\frac{\alpha - c_1}{\beta}$ units of the product, then the second producer should exit the market since he/she would maximize the profit with zero output level. Let us notice that conditions (5.243) and (5.244) make economic sense when $c_2 < \alpha = \frac{b}{a}$. If the first producer rises her/his output level by one physical unit, then the second producer aiming at profit maximization has to reduce her/his own output level by half of the unit. This is derived from the reaction line equation as the following condition:

(5.245) $$\frac{d\bar{y}_2}{dy_1}\bigg|_{RL_2} = -\frac{1}{2}.$$

In order to find the equilibrium state in the Cournot duopoly model, one should solve the following system of equations:

(5.246) $$\alpha - c_1 - 2\beta\bar{y}_1 - \beta\bar{y}_2 = 0,$$

(5.247) $$\alpha - c_2 - 2\beta\bar{y}_2 - \beta\bar{y}_1 = 0.$$

From Eq. (5.246), we derive an expression:

(5.248) $$\bar{y}_2 = \frac{\alpha - c_1}{\beta} - 2\bar{y}_1,$$

which we substitute into expression (5.247) getting:

(5.249) $$\alpha - c_2 - 2\beta\left(\frac{\alpha - c_1}{\beta} - 2\bar{y}_1\right) - \beta\bar{y}_1 = 0,$$

and hence:

(5.250) $$\bar{y}_1 = \frac{\alpha + c_2 - 2c_1}{3\beta}$$

or

(5.251) $$\bar{y}_1 = \frac{b - a(2c_1 - c_2)}{3}.$$

Substituting (5.250) into (5.248), we get

(5.252) $$\bar{y}_2 = \frac{\alpha + c_1 - 2c_2}{3\beta}$$

or equivalently

$$\bar{y}_2 = \frac{b - a(2c_2 - c_1)}{3}. \tag{5.253}$$

In the Cournot duopoly model, the optimal output levels of the first and of the second producer in the equilibrium state are described by a vector:

$$\begin{aligned}\bar{\mathbf{y}}^{(C)} &= \left(\bar{y}_1^{(C)}, \bar{y}_2^{(C)}\right) = \left(\frac{\alpha + c_2 - 2c_1}{3\beta}, \frac{\alpha + c_1 - 2c_2}{3\beta}\right) \\ &= \left(\frac{b - a(2c_1 - c_2)}{3}, \frac{b - a(2c_2 - c_1)}{3}\right).\end{aligned} \tag{5.254}$$

Figure 5.21 presents a mechanism of reaching the equilibrium state in the Cournot duopoly model. It is not difficult to notice that each producer aiming at the maximization of her/his own profit and taking the output level set by the competitor as given will seek to have the output in her/his own firm on a level resulting from his/her reaction line. Accepting an output level different than the one resulting from his/her reaction line would be inconsistent with the profit maximization aim. As a consequence, both producers will accept as optimal these output levels which are indicated by the intersection of their reaction lines.

The equilibrium state exists, there is exactly one such state and it is globally stable. This means that, if the parameters of the profit function of each producer do not change, then as a result of rational behaviour of both producers the supply of the product is shared between both producers and equal to the demand reported

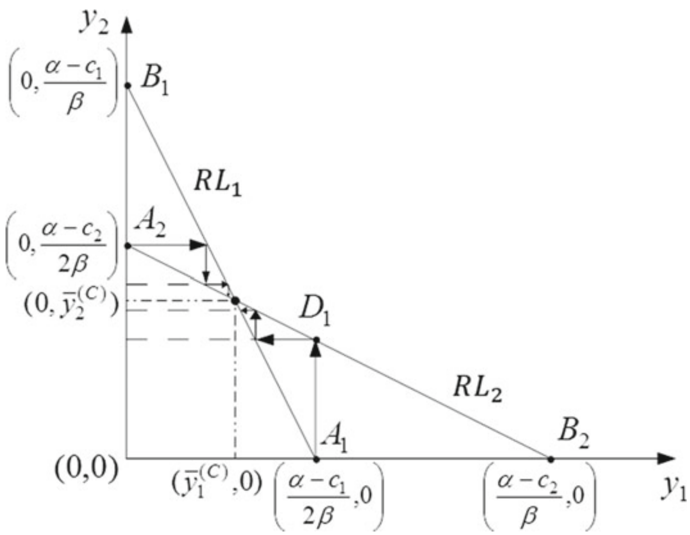


Fig. 5.21 Equilibrium state in Cournot duopoly model

for this product by consumers. These shares in the product supply provide the maximum profit for each producer.

On the basis of conditions (5.222), (5.250) and (5.252), one can determine the equilibrium price in the Cournot duopoly model:

$$\begin{aligned}\bar{p}^{(C)}\left(\bar{y}_1^{(C)}, \bar{y}_2^{(C)}\right) &= \alpha - \beta\left(\bar{y}_1^{(C)} + \bar{y}_2^{(C)}\right) \\ &= \frac{\alpha + c_1 + c_2}{3} = \frac{b + a(c_1 + c_2)}{3a},\end{aligned} \tag{5.255}$$

where: $\alpha = \frac{b}{a}$, $\beta = \frac{1}{a}$.

Let us analyse the sensitivity of the product optimal supply and of the equilibrium price to changes in values of the parameters of the Cournot duopoly model. We know that in the equilibrium state in the Cournot duopoly model the optimal supply is

$$\begin{aligned}\bar{\mathbf{y}}^{(C)} &= \left(y_1^{(C)}, y_2^{(C)}\right) = \left(\frac{\alpha + c_2 - 2c_1}{3\beta}, \frac{\alpha + c_1 - 2c_2}{3\beta}\right) \\ &= \left(\frac{b - a(2c_1 - c_2)}{3}, \frac{b - a(2c_2 - c_1)}{3}\right),\end{aligned} \tag{5.256}$$

which means that the supply of a product by each producer depends on the market capacity $b > 0$, the strength of consumers' reaction $a > 0$ to changes in a product price and on the marginal (variable or total) costs of production $c_1, c_2 > 0$.

Impact of market capacity on product supply by each producer

Let us determine partial derivatives of functions of each producer's product supply in the equilibrium state with respect to the market capacity:

$$\frac{\partial \bar{y}_1^{(C)}}{\partial b} = \frac{\partial \bar{y}_2^{(C)}}{\partial b} = \frac{1}{3}, \tag{5.257}$$

which means that, when the market capacity increases by one unit, the product supply by each producer increases by 1/3 of a physical unit. On this basis, we can state that in the Cournot duopoly model both producers have equal positions on a market.[25]

Impact of consumers' sensitivity to changes in a product price on product supply by each producer

$$\frac{\partial \bar{y}_1^{(C)}}{\partial a} = \frac{-2c_1 + c_2}{3}, \tag{5.258}$$

[25] One can check that the impact of the market capacity on the supply of a product by each producer in the Cournot duopoly model is weaker than in the case of a monopolistic market where there is only one producer.

(5.259) $$\frac{\partial \bar{y}_2^{(C)}}{\partial a} = \frac{-2c_2 + c_1}{3},$$

which means that how the consumers' sensitivity $a > 0$ to changes in a product price affects the product supply by each producer depends on the marginal total (variable) costs of production in both firms. Assuming that these marginal costs for both producers have similar values $c_1 \approx c_2$ (in the Cournot duopoly both producers have equal positions on a market), we can state that the stronger consumers' reaction to changes in a product price leads to a decrease in the product supply by each producer.

Impact of production marginal costs on product supply by each producer

(5.260) $$\frac{\partial \bar{y}_1^{(C)}}{\partial c_1} = -\frac{2}{3}a,$$

(5.261) $$\frac{\partial \bar{y}_2^{(C)}}{\partial c_1} = \frac{1}{3}a,$$

(5.262) $$\frac{\partial \bar{y}_1^{(C)}}{\partial c_2} = \frac{1}{3}a,$$

(5.263) $$\frac{\partial \bar{y}_2^{(C)}}{\partial c_2} = -\frac{2}{3}a.$$

Having:

(5.264) $$\frac{\partial \bar{y}_1^{(C)}}{\partial c_1} = \frac{\partial \bar{y}_2^{(C)}}{\partial c_2} = -\frac{2}{3}a \quad \text{and} \quad \frac{\partial \bar{y}_2^{(C)}}{\partial c_1} = \frac{\partial \bar{y}_1^{(C)}}{\partial c_2} = \frac{1}{3}a,$$

we can see that an increase in the marginal cost for the first (second) producer leads to a decrease in the product supply by the first (second) producer, which in absolute value is twice as high as the increase in the supply by the second (first) producer caused by an increase in the marginal cost for the first (second) producer.

Let us notice that, if the marginal costs for both producers were identical $c_1 = c_2 = c$, then from conditions (5.251) and (5.253), it results that

(5.265) $$\bar{y}_1^{(C)} = \bar{y}_2^{(C)} = \frac{b - ac}{3},$$

which means that,with the same marginal total (variable) costs of production both producers would share the market half and half.

Based on conditions (5.251) and (5.253), one can determine the total supply of a product in the equilibrium state in the Cournot duopoly model:

(5.266) $$\bar{y}^{(C)} = \bar{y}_1^{(C)} + \bar{y}_2^{(C)} = \frac{2\alpha - (c_1 + c_2)}{3\beta} = \frac{2b - a(c_1 + c_2)}{3},$$

which depends on the market capacity $b > 0$, the strength of consumers' reaction $a > 0$ to changes in a product price and on the marginal (variable or total) costs of production $c_1, c_2 > 0$.

Impact of market capacity on product total supply

(5.267)
$$\frac{\partial \bar{y}^{(C)}}{\partial b} = \frac{2}{3},$$

which means that, when the market capacity increases by one unit, the product total supply by both duopolists increases by 2/3 of a physical unit.[26]

Impact of consumers' sensitivity to changes in a product price on product total supply

(5.268)
$$\frac{\partial \bar{y}^{(C)}}{\partial a} = -\frac{c_1 + c_2}{3} < 0,$$

which means that the stronger the consumers' reaction to changes in a product price set by producers is, the lower the equilibrium total supply of a product is.

Impact of production marginal costs on product total supply

(5.269)
$$\frac{\partial \bar{y}^{(C)}}{\partial c_1} = \frac{\partial \bar{y}^{(C)}}{\partial c_2} = -\frac{1}{3}a < 0,$$

which means that a unit increase in the marginal total (variable) cost of production for any producer leads to an identical decrease in the product total supply in the Cournot duopoly model.

If the production marginal costs for both producers were identical $c_1 = c_2 = c$, then from condition (4.266), it results that the total supply would be equal[27]:

(5.270)
$$\bar{y}^{(C)} = \bar{y}_1^{(C)} + \bar{y}_2^{(C)} = \frac{2(\alpha - c)}{3\beta} = \frac{2(b - ac)}{3},$$

giving:

(5.271)
$$\frac{\partial \bar{y}^{(C)}}{\partial c} = -\frac{2}{3}a < 0,$$

which means that, with the same marginal costs for both producers, a unit increase in the marginal cost would mean the increase in the costs for both producers and

[26] One can check that the reaction of the total supply by duopolists in the Cournot model to a change in the market capacity is stronger than of the supply in a pure monopoly.

[27] One can check that the total supply of a product in the Cournot duopoly model is higher than the supply of a pure monopoly.

thus would lead to a decrease in the product total supply twice as strong as an increase in the cost just for one of the producers.

From condition (5.255), it is known that

$$\bar{p}^{(C)}\left(\bar{y}_1^{(C)}, \bar{y}_2^{(C)}\right) = \alpha - \beta\left(\bar{y}_1^{(C)} + \bar{y}_2^{(C)}\right) = \frac{\alpha + c_1 + c_2}{3} = \frac{b + a(c_1 + c_2)}{3a}, \tag{5.272}$$

which means that in the equilibrium state the price of a product set by both producers depends on the market capacity $b > 0$, the strength of consumers' reaction $a > 0$ to changes in a product price and on the marginal (variable or total) costs of production $c_1, c_2 > 0$.

Impact of market capacity on equilibrium price

$$\frac{\partial \bar{p}^{(C)}}{\partial b} = \frac{1}{3a} > 0, \tag{5.273}$$

which means that an increase in the market capacity leads to an increase in the price of a product.[28]

Impact of consumers' sensitivity to changes in a product price on equilibrium price

$$\frac{\partial \bar{p}^{(C)}}{\partial a} = -\frac{b}{3a^2} < 0, \tag{5.274}$$

which means that the stronger the consumers' reaction to changes in a product price is, the lower the equilibrium price set by both producers is.

Impact of production marginal costs on equilibrium price

$$\frac{\partial \bar{p}^{(C)}}{\partial c_1} = \frac{\partial \bar{p}^{(C)}}{\partial c_2} = \frac{1}{3} > 0, \tag{5.275}$$

which means that a unit increase in the marginal total (variable) cost of production for any producer leads to an identical increase in the equilibrium price in the Cournot duopoly model.

If the production marginal costs for both producers were identical $c_1 = c_2 = c$, then from condition (5.272), it results that the equilibrium price would be equal:

$$\bar{p}^{(C)} = \frac{b + 2ac}{3a}, \tag{5.276}$$

[28] One can check that the reaction of a product price set by duopolists in the Cournot model to an increase in the market capacity is weaker than of a product price set by a monopolist.

giving:

$$\frac{\partial \bar{p}^{(C)}}{\partial c} = \frac{2}{3} > 0, \tag{5.277}$$

which means that, with the same marginal costs for both producers, a unit increase in the marginal cost would mean the increase in the costs for both producers and thus would lead to an increase in the equilibrium price twice as strong as an increase in the cost just for one of the producers.

5.4.1.2 Dynamic Approach

In the analysis of the Cournot duopoly model presented in Sect. 5.4.1.1, we focus on the static approach, in particular on studying the equilibrium supply and the equilibrium price. Let us recall that in the Cournot duopoly model the equilibrium state exists, is only one and globally stable regardless of the values of the parameters a, b, c_1, c_2, about which it is enough to assume that they are all positive.[29] An optimal output level for a duopolist in the Cournot model, that is, the supply in the equilibrium state, for the first and for the second producers, respectively, equals:

$$\bar{y}_1^{(C)} = \frac{b - a(2c_1 - c_2)}{3}, \tag{5.278}$$

$$\bar{y}_2^{(C)} = \frac{b - a(2c_2 - c_1)}{3}. \tag{5.279}$$

We are interested in the equilibrium state with non-zero supplies by both producers: $\bar{y}_1 > 0$, $\bar{y}_2 > 0$. Therefore, we assume that values of the parameters satisfy the following inequalities:

$$2c_1 - c_2 < \frac{b}{a}, \tag{5.280}$$

$$2c_2 - c_1 < \frac{b}{a}. \tag{5.281}$$

The fulfilment of inequalities (5.280) and (5.281) is not necessary for the existence, uniqueness or global stability of the equilibrium state but is necessary to ensure positive equilibrium levels of the supply. The supply level in the equilibrium state cannot equal 0 because we deal with the case of a duopoly and zero supply of one of the producers would mean that he/she exits the market which turns into a monopoly.

[29] We make also additional assumptions about values of parameters to ensure that price levels and output levels are positive.

From the analysis conducted so far in Sect. 5.4.1.1, we know that a mechanism of reaching the equilibrium state is a sequence of iterations in determining the level of supply alternately by one producer and the other according to the given producer's reaction line. The successive stages of determining the supply levels can be identified with moments or periods in some time horizon whose end is indicated by the moment/period of reaching the state of equilibrium. If we want to interpret time as discrete, then iterations take place at equal intervals of time, for example, every one month. If time is treated as continuous, then subsequent iterations are interpreted as taking place at any consecutive moments, for example, the second iteration after a month, the third one after another 3 weeks, the fourth one after another 27 days, etc. In both cases, however, whether we interpret time as discrete or as continuous, with a given set of parameter values, the number of iterations is the same.

In addition to analysing the optimal values in the equilibrium state and the mechanism of reaching this state we are also interested how quickly this state is reached, that is, how many iterations are needed to determine the equilibrium supply levels and what determines the rate of convergence. From formulas (5.278)–(5.279), it can be seen that what distinguishes one producer from another, and at the same time determines the optimal output level by a given producer, are the marginal costs of production. We assume that both producers are rational, driven by profit maximization and, according to the assumption of the Cournot model, they have equal positions on a market. Besides the production marginal costs, they may also differ in the levels of the production fixed cost, but as we know, this cost does not affect producers' decisions regarding the output. Analysing equations of the reaction lines of both producers:

$$\bar{y}_1 = -\frac{1}{2}y_2 + \frac{b - ac_1}{2}, \tag{5.282}$$

$$\bar{y}_2 = -\frac{1}{2}y_1 + \frac{b - ac_2}{2}, \tag{5.283}$$

one can notice that the value that decides on the location of subsequent points on the given reaction line is the value of $b - ac_i$ $(i = 1, 2)$ and thus, in particular, the relationship between value of c_i and value of the quotient b/a.

Example 5.7 Two producers having equal positions act on a market of some homogeneous product. The demand for this product evolves according to a demand function:

$$y^d(p) = -ap + b, \qquad a, b > 0.$$

Production total costs for the first and for the second firm, respectively, are as follows:

$$c_1^{tot}(y_1) = c_1 y_1 + d_1,$$

$$c_2^{tot}(y_2) = c_2 y_2 + d_2,$$

where $d_1, d_2 \geq 0$ denote the fixed production costs and $c_1, c_2 > 0$ denote production marginal costs. The total output by both producers matches the demand for the product reported by consumers by a given price:

$$y_1 + y_2 = y^d(p).$$

Figures 5.22 and 5.23 present the reaction lines of duopolists, the equilibrium state and the mechanism of reaching this state, when the parameters of the demand function and the cost function have the following values: $a = 3, b = 18, c_1 = 2, c_2 = 1$. The optimal supply for the first producer is then $\bar{y}_1 = 3$ and for the second producer $\bar{y}_2 = 6$. These are the equilibrium supply levels.

Figure 5.22 illustrates the mechanism of reaching the equilibrium state when the first producer decides on the level of supply as first (Scenario 1), assuming the competitor's supply equals 0. The fact that the first producer makes the decision as first does not mean that he/she has an advantage over the second producer, but only allows us to assume the order of iterations, because producers' decisions do not have to be perfectly synchronized in time. We can see the order of making decisions by looking at the points in the reaction lines. The point indicating the first iteration, that is, point (6, 0), belongs to the reaction line of the first producer. Regardless of the starting point, thus regardless of what the first producer assumes

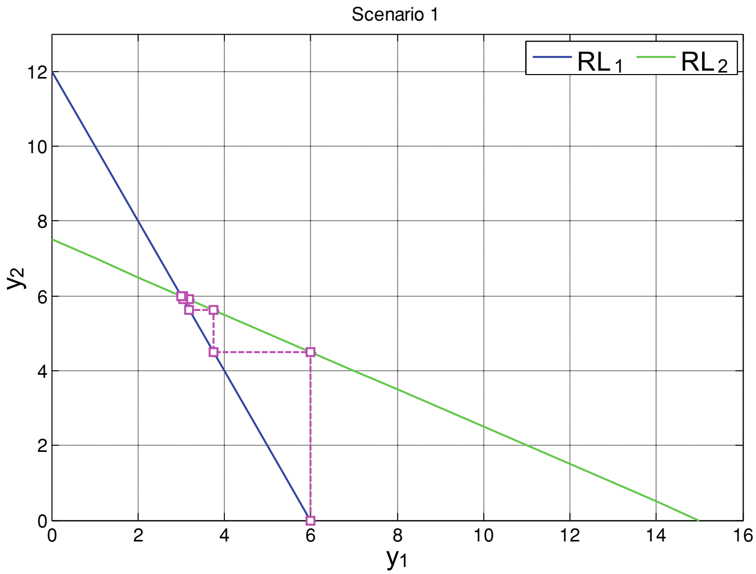


Fig. 5.22 Mechanism of reaching equilibrium state in Cournot duopoly model when first decision on supply is made by first producer

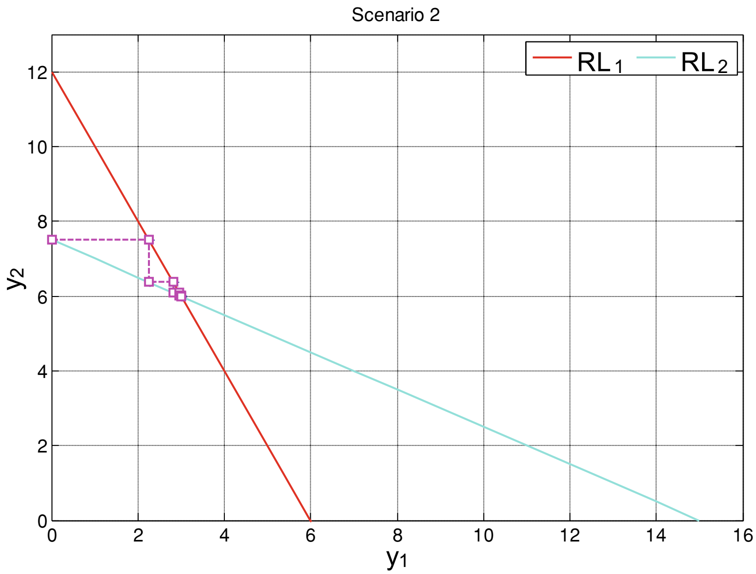


Fig. 5.23 Mechanism of reaching equilibrium state in Cournot duopoly model when first decision on supply is made by second producer

about the competitor's supply level, the equilibrium state is achieved after a certain number of iterations of the supply decisions.

The equilibrium state is also reached when the second producer makes the supply decision as first (Scenario 2), which is illustrated in Fig. 5.23. The point indicating the first iteration, that is, point (0, 8), belongs to the reaction line of the second producer. This time we assume that the second producer decides on the level of supply as first, assuming the competitor's supply equals 0. The equilibrium state is the same as before: $\bar{y}_1^{(C)} = 3$, $\bar{y}_2^{(C)} = 6$. The starting point could be as well some other point than (0, 8), but still belonging to the second producer reaction line and indicating what the second producer assumes about the competitor's supply level.

Tables 5.5 and 5.6 present trajectories of the equilibrium mechanism. We want to compare how fast output levels converge to the equilibrium levels when the difference between the production marginal costs for producers is smaller or bigger. The strength of consumers' reaction to changes in a product price and the market capacity are the same in both cases and amount to 3 and 18, respectively. From Table 5.5, one can see that, if $c_1 = 2$, $c_2 = 1$, the equilibrium state is reached in the 18th iteration and assuming an accuracy of two decimal places just right in the 12th iteration.

Table 5.6 presents the output level trajectories by an assumption that the marginal production costs are $c_1 = 3.4$, $c_2 = 1$. The equilibrium state is reached in the 19th iteration and assuming an accuracy of two decimal places just right

Table 5.5 Trajectories of output levels when $c_1 = 2$, $c_2 = 1$

Iteration number t	y_1	y_2	$\bar{y}_1^{(C)}$	$\bar{y}_2^{(C)}$
1	6.0000	0.0000		
2	6.0000	4.5000		
3	3.7500	4.5000		
4	3.7500	5.6250		
5	3.1875	5.6250		
6	3.1875	5.9063		
7	3.0469	5.9063		
8	3.0469	5.9766		
9	3.0117	5.9766		
10	3.0117	5.9941		
11	**3.0029**	**5.9941**		
12	3.0029	5.9985		
13	3.0007	5.9985		
14	3.0007	5.9996		
15	3.0002	5.9996		
16	3.0002	5.9999		
17	3.0000	5.9999		
18	3.0000	6.0000	3	6

in the 12th iteration. Therefore, there are no significant differences in the pace of convergence to the equilibrium state in both cases. Note, however, that for a given set of parameter values, assuming a greater difference in marginal costs between two producers is not possible due to condition $2c_1 - c_2 < \frac{b}{a}$.

Table 5.7 presents the output level trajectories by an assumption that the strength of consumers' reaction to changes in a product price is $a = 3$, and the market capacity is $b = 180$. Then we can consider the bigger difference in the production marginal costs for producers, for example, $c_1 = 30$, $c_2 = 1$. The equilibrium state is reached in the 22nd iteration. If an accuracy of two decimal places is assumed the equilibrium state is reached in the 16th iteration. However, taking $a = 3$, $b = 180$, $c_1 = 2$, $c_2 = 1$ or $c_1 = 1.1$, $c_2 = 1$, we obtain the equilibrium state with an accuracy of four decimal places also in the 22nd iteration, and with an accuracy of two places in the 15th iteration.[30] We see, then, that the differences in the pace of convergence between various considered cases with different sets of parameters' values are small.

[30] For these sets of parameters' values, we do not present tables with output levels' trajectories because they do not present significant differences in the rate of convergence to the equilibrium state.

Table 5.6 Trajectories of output levels when $c_1 = 3.4$, $c_2 = 1$

Iteration number t	y_1	y_2	$\bar{y}_1^{(C)}$	$\bar{y}_2^{(C)}$
1	3.9000	0.0000		
2	3.9000	5.5500		
3	1.1250	5.5500		
4	1.1250	6.9375		
5	0.4313	6.9375		
6	0.4313	7.2844		
7	0.2578	7.2844		
8	0.2578	7.3711		
9	0.2145	7.3711		
10	0.2145	7.3928		
11	0.2036	7.3928		
12	**0.2036**	**7.3982**		
13	0.2009	7.3982		
14	0.2009	7.3995		
15	0.2002	7.3995		
16	0.2002	7.3999		
17	0.2001	7.3999		
18	0.2001	7.4000		
19	0.2000	7.4000	0.2	7.4

From the trajectories presented in Tables 5.5, 5.6 and 5.7, it results that, with the linear function of demand and the linear functions of production costs, what has the greatest impact on the rate of convergence to the equilibrium state is the relationship between value of c_i $(i = 1, 2)$ and value of the quotient b/a.

Figures 5.24 and 5.25 present the mechanism of reaching the equilibrium state when we treat successive iterations of making a decision on supply levels as occurring sequentially in time. Figure 5.24 presents trajectories of the supply levels of duopolists in two scenarios, depending on which of the two producers decides on the output level as first. In both cases, the same equilibrium state is achieved: $\bar{y}_1^{(C)} = 3$, $\bar{y}_2^{(C)} = 6$.

Figure 5.25 presents trajectories of the total supply level and of the price set by a duopolist, in two scenarios considered. By a given set of parameters' values on a market where two producers with equal positions act, the total supply level is equal to 9 and the optimal price set by them is equal to 3.

Table 5.7 Trajectories of output levels when $c_1 = 30$, $c_2 = 1$

Iteration number t	y_1	y_2	$\bar{y}_1^{(C)}$	$\bar{y}_2^{(C)}$
1	45.0000	0.0000		
2	45.0000	66.0000		
3	12.0000	66.0000		
4	12.0000	82.5000		
5	3.7500	82.5000		
6	3.7500	86.6250		
7	1.6875	86.6250		
8	1.6875	87.6563		
9	1.1719	87.6563		
10	1.1719	87.9141		
11	1.0430	87.9141		
12	1.0430	87.9785		
13	1.0107	87.9785		
14	1.0107	87.9946		
15	1.0027	87.9946		
16	**1.0027**	**87.9987**		
17	1.0007	87.9987		
18	1.0007	87.9997		
19	1.0002	87.9997		
20	1.0002	87.9999		
21	1.0000	87.9999		
22	1.0000	88.0000	1	88

5.4.2 Stackelberg Duopoly Model and Its Equilibrium State

5.4.2.1 Static Approach

Let us apply the following set of assumptions:

(S1) Two producers ($i = 1, 2$) act on a market of a homogeneous (undifferentiated) product. The first producer is a **leader** and the second is a **follower**.

(S2) Functions of production total cost for producers are as follows:

(5.284) $$\forall i = 1, 2 \quad c_i^{tot}(y_i) = c_i^v(y_i) + c_i^f(y_i) = c_i y_i + d_i, \quad c_i, d_i > 0.$$

being the sum of variable cost functions:

(5.285) $$\forall i = 1, 2 \quad c_i^v(y_i) = c_i y_i, \quad c_i > 0$$

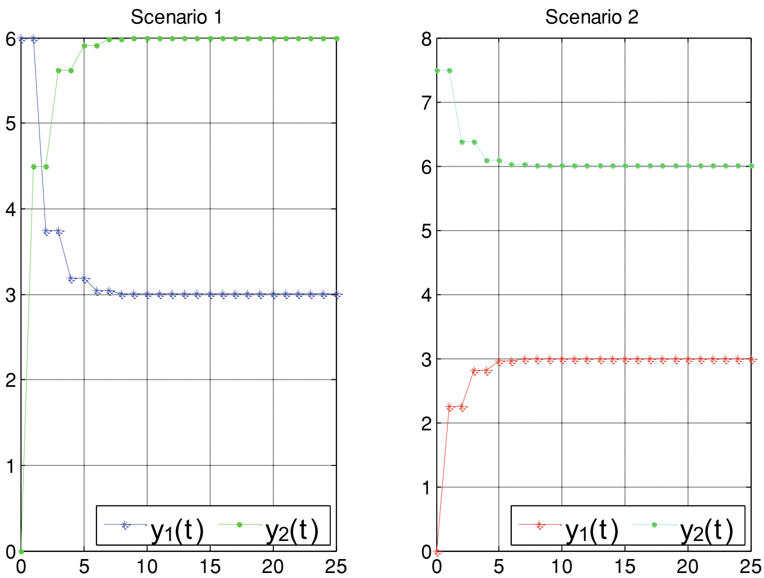


Fig. 5.24 Trajectories of output levels of duopolists in Cournot model

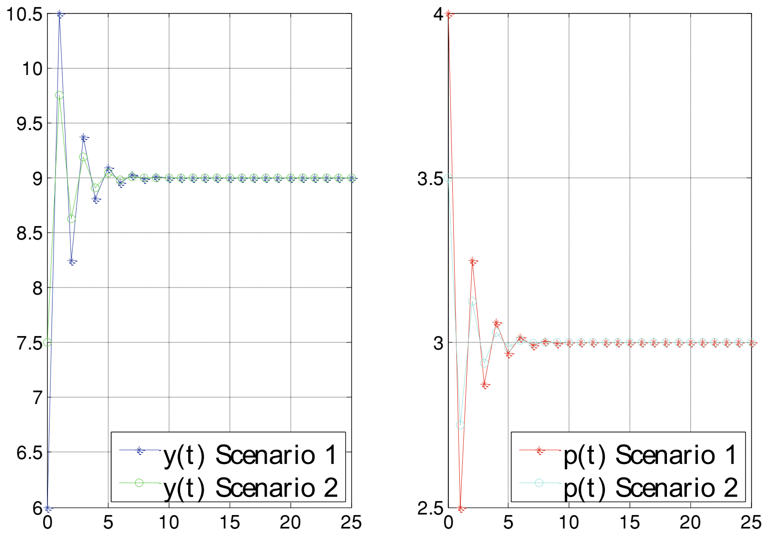


Fig. 5.25 Trajectories of total supply and of price set by duopolists in Cournot model

and the fixed costs:

(5.286) $$\forall i = 1, 2 \quad c_i^f(y_i) = d_i > 0.$$

Since the total cost functions are linear functions of output levels, we get that

(5.287) $$\forall i = 1, 2 \quad \frac{\mathrm{d}c_i^{tot}(y_i)}{\mathrm{d}y_i} = \frac{\mathrm{d}c_i^v(y_i)}{\mathrm{d}y_i} = c_i > 0,$$

that is the marginal total cost and the marginal variable cost for the i-th producer are equal and they are functions increasing in an output level.

(S3) A function of demand reported for a product by consumers, depending on its price set by producers, is as follows:

(5.288) $$y^d(p) = -ap + b \quad a, b > 0,$$

where a denotes a measure of the consumers' reaction strength to a unit increase in the price of a product and b denotes a measure of a market capacity.

Since values of the demand function have to be non-negative, we get that

(5.289) $$p \in \left[0; \frac{b}{a}\right].$$

(S4) The total output by both producers matches the demand that consumers report by a given price of a product:

(5.290) $$y_1 + y_2 = y^d(p) = -ap + b, \quad a, b > 0.$$

(S5) An inverse function of consumer demand for a product manufactured by producers has a form:

(5.291)
$$p(y_1, y_2) = \frac{b}{a} - \frac{1}{a}(y_1 + y_2) = \alpha - \beta(y_1 + y_2), \quad \alpha, \beta > 0, \ \alpha = \frac{b}{a}, \ \beta = \frac{1}{a}.$$

Let us notice that

(5.292) $$\frac{\partial p(y_1, y_2)}{\partial y_1} = \frac{\partial p(y_1, y_2)}{\partial y_2} = -\beta = -\frac{1}{a} < 0,$$

thus, no matter which producer increases the output by one physical unit it leads to a necessity to lower a product price by $\beta = \frac{1}{a}$ money units.

(S6) The first producer (the leader) wants to determine such an output level that guarantees the maximum profit for her/him:

(5.293) $$\Pi_1(y_1) \mapsto \max \qquad y_1 \geq 0.$$

A profit function of the first producer can be expressed as the difference between her/his revenue from sales of a product and total cost of production:

(5.294) $$\Pi_1(y_1) = p(y_1, y_2)y_1 - c_1y_1 - d_1.$$

Substituting the inverse function of demand (5.291) into Eq. (5.294), one obtains the profit function of the first producer as

(5.295) $$\begin{aligned}\Pi_1(y_1, y_2) =& [\alpha - \beta(y_1 + y_2)]y_1 - [c_1y_1 + d_1] \\ =& [\alpha - c_1]y_1 - \beta y_1^2 - \beta y_1 y_2 - d_1.\end{aligned}$$

(S7) The second producer (the follower) wants to determine such an output level that guarantees the maximum profit for her/him taking an output level of the first producer as given:

(5.296) $$\Pi_2(y_2)|_{y_1=\text{const.}\geq 0} \mapsto \max \qquad y_2 \geq 0.$$

A profit function of the second producer can be expressed as the difference between her/his revenue from sales of a product and total cost of production:

(5.297) $$\Pi_2(y_2) = p(y_1, y_2)y_2 - c_2y_2 - d_2.$$

Substituting the inverse function of demand (4.291) into Eq. (4.297), one obtains the profit function of the second producer as

(5.298) $$\begin{aligned}\Pi_2(y_1, y_2) =& [\alpha - \beta(y_1 + y_2)]y_2 - [c_2y_2 + d_2] \\ =& [\alpha - c_2]y_2 - \beta y_2^2 - \beta y_1 y_2 - d_2.\end{aligned}$$

Let us now derive optimal solutions to the profit maximization problems of both producers in a duopoly in the Stackelberg model.

The necessary condition and the sufficient condition for the profit maximization problem of the first producer are following:

(5.299) $$\left.\frac{\partial \Pi_1(y_1, y_2)}{\partial y_1}\right|_{y_1=\bar{y}_1} = 0 \qquad \text{the necessary condition,}$$

(5.300) $$\left.\frac{\partial^2 \Pi_1(y_1, y_2)}{\partial y_1^2}\right|_{y_1=\bar{y}_1} < 0 \qquad \text{the sufficient condition.}$$

When the output level of the first producer is taken as given, thus treated as a parameter, the necessary condition and the sufficient condition for the profit maximization problem of the second producer are following:

(5.301) $$\left.\frac{\partial \Pi_2(y_1, y_2)}{\partial y_2}\right|_{y_2=\bar{y}_2,\ y_1=\text{const.}\geq 0} = 0 \quad \text{the necessary condition,}$$

(5.302) $$\left.\frac{\partial^2 \Pi_2(y_1, y_2)}{\partial y_2^2}\right|_{y_2=\bar{y}_2,\ y_1=\text{const.}\geq 0} < 0 \quad \text{the sufficient condition.}$$

Deriving the necessary condition and the sufficient condition for the profit function of the second producer (the follower), we get

(5.303) $$\left.\frac{\partial \Pi_2(y_1, y_2)}{\partial y_2}\right|_{y_2=\bar{y}_2,\ y_1=\text{const.}\geq 0} = \alpha - c_2 - 2\beta\bar{y}_2 - \beta y_1 = 0$$

(5.304) $$\left.\frac{\partial^2 \Pi_2(y_1, y_2)}{\partial y_2^2}\right|_{y_2=\bar{y}_2,\ y_1=\text{const.}\geq 0} = -2\beta < 0,$$

which means that, for any (given) output level $y_1 \geq 0$ set by the first producer, the second producer obtains the maximum profit when $y_2 = \bar{y}_2$.

From condition (5.303), it results that

(5.305) $$\bar{y}_2 = \frac{\alpha - c_2}{2\beta} - \frac{y_1}{2} \quad \boldsymbol{RL_2}.$$

Equation (5.305) is called a **reaction line equation of the follower**. From the perspective of the follower, it describes her/his output level which, by the output level of the leader taken as given, guarantees the maximum profit for the follower. From the perspective of the leader, the equation describes a share of the market which, by a given output level of the leader, is to be captured by the follower.

Substituting expression (5.305) into the leader's profit function, one gets

(5.306)
$$\Pi_1(y_1, y_2) = [\alpha - c_1]y_1 - \beta y_1^2 - \beta y_1 y_2 - d_1 = \frac{\alpha - 2c_1 + c_2}{2} y_1 - \frac{\beta}{2} y_1^2 - d_1.$$

Then the necessary condition for the leader's profit function takes the form:

(5.307) $$\left.\frac{\partial \Pi_1(y_1, y_2)}{\partial y_1}\right|_{y_1=\bar{y}_1} = \frac{\alpha - 2c_1 + c_2}{2} - \beta\bar{y}_1 = 0,$$

and the sufficient condition for the leader's profit function is

(5.308) $$\left.\frac{\partial^2 \Pi_1(y_1, y_2)}{\partial y_1^2}\right|_{y_1=\bar{y}_1} = -\beta < 0.$$

From conditions (5.307) and (5.308), it results that for an output level:

(5.309) $$\bar{y}_1^{(S)} = \frac{\alpha - 2c_1 + c_2}{2\beta} = \frac{b - a(2c_1 - c_2)}{2}$$

the leader obtains the maximum profit.

Substituting expression (5.309) into Eq. (5.305):

(5.310) $$\bar{y}_2^{(S)} = \frac{\alpha - c_2}{2\beta} - \frac{\bar{y}_1^{(S)}}{2},$$

one derives the optimal output level of the follower:

(5.311) $$\bar{y}_2^{(S)} = \frac{\alpha - 3c_2 + 2c_1}{4\beta} = \frac{b - a(3c_2 - 2c_1)}{4}.$$

Let us notice that, in the Stackelberg duopoly model, there is only the reaction line of the follower (Fig. 5.26), who has to accept unconditionally the choice made by the leader. On the other hand, the leader, when deciding her/his optimal output level, does not have to take into account the decisions made by the follower. The product supply by the leader is determined by her/his profit function which depends on revenue from sales and on production total cost. The equilibrium state

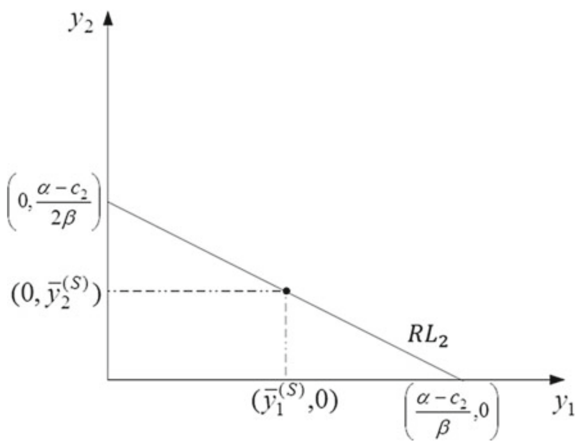


Fig. 5.26 Reaction line of follower and equilibrium state in Stackelberg duopoly model

in the Stackelberg duopoly model exists, there is exactly one such state and it is determined uniquely by the product supplies by the leader and by the follower. Deciding her/his output level the follower relies on the choice made by the leader as well as on the maximum profit that can be obtained when the leader has made a decision to manufacture $\bar{y}_1^{(S)}$ units of a product.

As a result, in the equilibrium state, the optimal supply of the product by the leader and by the follower in the Stackelberg duopoly model is given as

$$\bar{\mathbf{y}}^{(S)} = \left(\bar{y}_1^{(S)}, \bar{y}_2^{(S)}\right) = \left(\frac{\alpha + c_2 - 2c_1}{2\beta}, \frac{\alpha + 2c_1 - 3c_2}{4\beta}\right) = \left(\frac{b - a(2c_1 - c_2)}{2}, \frac{b - a(3c_2 - 2c_1)}{4}\right). \tag{5.312}$$

Then the total supply of the product equals:

$$\bar{y}^{(S)} = \bar{y}_1^{(S)} + \bar{y}_2^{(S)} = \frac{3\alpha - (2c_1 + c_2)}{4\beta} = \frac{3b - a(2c_1 + c_2)}{4}, \tag{5.313}$$

hence, the equilibrium price in the Stackelberg duopoly model is

$$\bar{p}^{(S)}\left(\bar{y}_1^{(S)}, \bar{y}_2^{(S)}\right) = \alpha - \beta\left(\bar{y}_1^{(S)} + \bar{y}_2^{(S)}\right) = \frac{\alpha + 2c_1 + c_2}{4} = \frac{b + a(2c_1 + c_2)}{4a}, \tag{5.314}$$

which means that the total supply of a product by both producers and the equilibrium price set by them depend on the market capacity $b > 0$, the strength of consumers' reaction $a > 0$ to changes in a product price and on the marginal (variable or total) costs of production $c_1, c_2 > 0$.

Let us analyse the sensitivity of the product optimal supply and of the equilibrium price to changes in values of the parameters of the Stackelberg duopoly model.

Impact of market capacity on product supply by each producer

Using expression (5.312), let us determine partial derivatives of functions of each producer's product supply in the equilibrium state with respect to the market capacity:

$$\frac{\partial \bar{y}_1^{(S)}}{\partial b} = \frac{1}{2} > \frac{1}{4} = \frac{\partial \bar{y}_2^{(S)}}{\partial b}, \tag{5.315}$$

which means that, when the market capacity increases by one unit, the product supply by the leader increases by 1/2 of a physical unit, while the product supply by the follower increases by 1/2 of a unit.

Impact of consumers' sensitivity to changes in a product price on product supply by each producer

(5.316)
$$\frac{\partial \bar{y}_1^{(S)}}{\partial a} = \frac{-2c_1 + c_2}{2},$$

(5.317)
$$\frac{\partial \bar{y}_2^{(S)}}{\partial a} = \frac{-3c_2 + 2c_1}{4},$$

which means that how the consumers' sensitivity $a > 0$ to changes in a product price affects the product supply by each producer depends on the marginal total (variable) costs of production in both firms. Let us consider three possible cases of reaction of the product supplies by both producers to an increase in the consumers' sensitivity to changes in a product price:

(1) if $c_1 > 0$ and $\frac{2}{3}c_1 < c_2 < 2c_1$, then the product supply by each producer (thus also the total supply) declines,
(2) if $c_1 > 0$ and $c_2 < \frac{2}{3}c_1$, then the product supply by the leader declines and the product supply by the follower increases,
(3) if $c_1 > 0$ and $c_2 > 2c_1$, then the product supply by the leader increases and the product supply by the follower declines.

Impact of production marginal costs on product supply by each producer

(5.318)
$$\frac{\partial \bar{y}_1^{(S)}}{\partial c_1} = -a < 0,$$

(5.319)
$$\frac{\partial \bar{y}_2^{(S)}}{\partial c_1} = \frac{1}{2}a > 0,$$

(5.320)
$$\frac{\partial \bar{y}_1^{(S)}}{\partial c_2} = \frac{1}{2}a > 0,$$

(5.321)
$$\frac{\partial \bar{y}_2^{(S)}}{\partial c_2} = -\frac{3}{4}a < 0,$$

which means that, when the marginal cost for the leader increases, the product supply by the leader declines and the product supply by the follower increases. In turn, when the marginal cost for the follower increases, the product supply by the leader increases and the product supply by the follower declines. However, we can notice in addition that reactions have different strengths in some cases. Reaction of the supply by each producer to an increase in the marginal cost of the competitor is equally strong. But the reaction of the supply by the leader to an increase in her/his marginal cost is stronger than the reaction of the supply by the follower to an increase in the follower's marginal cost.

The total supply of a product in the equilibrium state in the Stackelberg duopoly model equals:

$$\bar{y}^{(S)} = \bar{y}_1^{(S)} + \bar{y}_2^{(S)} = \frac{3\alpha - (2c_1 + c_2)}{4\beta} = \frac{3b - a(2c_1 + c_2)}{4}, \tag{5.322}$$

and also depends on the market capacity $b > 0$, the strength of consumers' reaction $a > 0$ to changes in a product price and on the marginal (variable or total) costs of production $c_1, c_2 > 0$.

Impact of market capacity on product total supply

$$\frac{\partial \bar{y}^{(S)}}{\partial b} = \frac{3}{4} > 0, \tag{5.323}$$

which means that, when the market capacity increases by one unit, the product total supply by both duopolists increases by 3/4 of a physical unit.

Impact of consumers' sensitivity to changes in a product price on product total supply

$$\frac{\partial \bar{y}^{(S)}}{\partial a} = -\frac{2c_1 + c_2}{4} < 0, \tag{5.324}$$

which means that the stronger the consumers' reaction to changes in a product price set by producers is, the lower the equilibrium total supply of a product is.

Impact of production marginal costs on product total supply

$$\frac{\partial \bar{y}^{(S)}}{\partial c_1} = -\frac{1}{2}a < 0, \tag{5.325}$$

$$\frac{\partial \bar{y}^{(S)}}{\partial c_2} = -\frac{1}{4}a < 0, \tag{5.326}$$

which means that a unit increase in the marginal total (variable) cost of production for any producer leads to a decrease in the product total supply. However, the reaction of the total supply to an increase in the leader's marginal cost is twice as strong as the reaction of the total supply to an increase in the follower's marginal cost.

If the production marginal costs for both producers were identical $c_1 = c_2 = c$, then from condition (5.322), it results that the total supply would be equal[31]:

$$\bar{y}^{(S)} = \bar{y}_1^{(S)} + \bar{y}_2^{(S)} = \frac{3(b - ac)}{4}, \tag{5.327}$$

[31] One can check that the total supply of a product in the Stackelberg duopoly model is higher than the supply of a pure monopoly.

giving:

$$\frac{\partial \bar{y}^{(S)}}{\partial c} = -\frac{3}{4}a > 0, \tag{5.328}$$

which means that, with the same marginal costs for both producers, a unit increase in the marginal cost would mean an increase in the costs for both producers and thus would lead to a decrease in the product total supply as strong as the sum of reactions to an increase in the costs for both the producers.

From condition (5.314), it is known that

$$\bar{p}^{(S)}\left(\bar{y}_1^{(S)}, \bar{y}_2^{(S)}\right) = \frac{\alpha + 2c_1 + c_2}{4} = \frac{b + a(2c_1 + c_2)}{4a}, \tag{5.329}$$

which means that, in the equilibrium state, the price of a product set by both producers depends on the market capacity $b > 0$, the strength of consumers' reaction $a > 0$ to changes in a product price and on the marginal (variable or total) costs of production $c_1, c_2 > 0$.

Impact of market capacity on equilibrium price

$$\frac{\partial \bar{p}^{(S)}}{\partial b} = \frac{1}{4a} > 0, \tag{5.330}$$

which means that an increase in the market capacity leads to an increase in the price of a product.

Impact of consumer sensitivity to changes in a product price on equilibrium price

$$\frac{\partial \bar{p}^{(S)}}{\partial a} = -\frac{b}{4a^2} < 0, \tag{5.331}$$

which means that the stronger the consumers' reaction to changes in a product price is, the lower the equilibrium price set by both producers is.

Impact of production marginal costs on equilibrium price

$$\frac{\partial \bar{p}^{(S)}}{\partial c_1} = \frac{1}{2} > 0, \tag{5.332}$$

$$\frac{\partial \bar{p}^{(S)}}{\partial c_2} = \frac{1}{4} > 0, \tag{5.333}$$

which means that a unit increase in the marginal total (variable) cost of production for any producer leads to an increase in the equilibrium price. However, this reaction is stronger when the leader's marginal cost increases.

If the production marginal costs for both producers were identical $c_1 = c_2 = c$, then from condition (5.329), it results that the equilibrium price would be equal:

(5.334) $$\bar{p}^{(S)} = \frac{b + 3ac}{4a}.$$

giving:

(5.335) $$\frac{\partial \bar{p}^{(S)}}{\partial c} = \frac{3}{4} > 0,$$

which means that, with the same marginal costs for both producers, a unit increase in the marginal cost would mean the increase in the costs for both producers and thus would lead to an increase in the equilibrium price as strong as the sum of reactions to an increase in the costs for both the producers.

5.4.2.2 Dynamic Approach

From the analysis of the Stackelberg duopoly model, presented in Sect. 5.4.2.1, we know that there is exactly one equilibrium state in this model, achieved immediately when the leader sets his/her level of the product supply to which the follower will adjust. The equilibrium state is determined uniquely, which means that the supply by each producer in the equilibrium state depends only on the assumed values of parameters in the demand function and in the production cost functions of both producers:

(5.336) $$\bar{y}_1^{(S)} = \frac{b - a(2c_1 - c_2)}{2},$$

(5.337) $$\bar{y}_2^{(S)} = \frac{b - a(3c_2 - 2c_1)}{4},$$

where

$$2c_1 - c_2 < \frac{b}{a}$$

and

$$3c_2 - 2c_1 < \frac{b}{a}$$

to ensure positive values of the optimal supplies.

A position of the leader means that a producer dictates her/his supply level to the follower who on this basis determines her/his optimal output level. The equilibrium state is thus achieved immediately and there is no change in producers' decisions on the supply if the value of any of the parameters a, b, c_1, c_2 does not change In the static approach, we assumed that these values are constant over time.

In the dynamic approach, we assume that they can be given as time-dependent functions: $a(t), b(t), c_1(t), c_2(t)$.

Let us assume that the demand function for a homogeneous product of duopolists has a form:

$$y^d(p(t)) = -a(t)p(t) + b(t), \qquad a(t), b(t) > 0, \tag{5.338}$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0, T]$, and T means time horizon. The production total cost of the leader and of the follower, respectively, are as follows:

$$c_1^{tot}(y_1(t)) = c_1(t)y_1(t) + d_1(t), \tag{5.339}$$

$$c_2^{tot}(y_2(t)) = c_2(t)y_2(t) + d_2(t), \tag{5.340}$$

where $d_1(t), d_2(t) \geq 0$ are the production fixed costs, and $c_1(t), c_2(t)$ are the production marginal costs.

By these assumptions in each period/at any moment of the considered time horizon, the supply of producers in the equilibrium state may have a different value:

$$\bar{y}_1^{(S)}(t) = \frac{b(t) - a(t)(2c_1(t) - c_2(t))}{2}, \tag{5.341}$$

$$\bar{y}_2^{(S)}(t) = \frac{b(t) - a(t)(3c_2(t) - 2c_1(t))}{4}. \tag{5.342}$$

From formulas (5.336)–(5.337) from the static approach, as well as from formulas (5.341)–(5.342) from the dynamic approach, we notice that the differences in the levels of supply between the leader and follower depend, among other things, on the position held on a market. Whereas these positions are established once in the whole considered time horizon, in the sense that one of the producers continues to be the leader and the other one continues to be the follower, differences in optimal supply levels between producers and changes taking place in these levels are determined by the time-variant production marginal costs for both producers.

The position held by a duopolist on a market affects the share he/she obtains in the market, that is, the share of her/his optimal supply in the total optimal supply of a product. This relationship can be seen on the basis of formulas (5.341)–(5.342). What also affects the shares of duopolists is the formation of their production marginal costs, especially while comparing the cost for one of the duopolists to the cost for the other. By equal marginal costs, the leader has 2/3 and the follower 1/3 of the market.[32] Beacuse marginal costs for the leader and the follower are

[32] Such distribution of market shares holds anytime when the production marginal costs for the leader and for the follower are the same. This results, besides the leader and the follower positions, from the adopted forms of the demand function and the functions of production total cost, which are assumed to be linear. With other forms of these functions, the distribution of market shares between the leader and the follower may be different.

different, their market shares take different values too. The leader has a bigger supply than the follower, and therefore, he/she holds a higher share in a market of a product, if:

(5.343) $$c_2(t) > \frac{6}{5}c_1(t) - \frac{1}{5} \cdot \frac{b(t)}{a(t)}, \qquad \forall t,$$

where $t = 0, 1, 2, \ldots, T$ or $t \in [0; T]$. The follower has a bigger supply than the leader and therefore a higher market share if the opposite inequality is satisfied:

(5.344) $$c_2(t) < \frac{6}{5}c_1(t) - \frac{1}{5} \cdot \frac{b(t)}{a(t)}.$$

From inequality (5.344), it results that the follower can have a higher market share than the leader only by a very low production marginal cost compared to the cost for the leader.

Example 5.8 Two producers act on a market for some homogeneous product. The first of them has a position of the leader and the other a position of the follower. The demand for this product evolves according to a demand function:

$$y^d(p(t)) = -3p(t) + 18, \qquad a(t), b(t) > 0.$$

The total output by both producers matches the demand for the product reported by consumers by a given price:

$$y_1(t) + y_2(t) = y^d(p(t)).$$

Production total costs for the leader and for the follower, respectively, are as follows:

$$c_1^{tot}(y_1(t)) = c_1(t)y_1 + 1,$$

$$c_2^{tot}(y_2(t)) = 2y_2(t) + 1,$$

where

$$c_1(t) = -\frac{2}{15}t + 4, \qquad \forall t \in [0; 30],$$

is a function of the leader's production marginal cost whose value depends on time.

In such a set of assumptions, only the marginal cost for the leader changes over time, and the following remains constant: the marginal cost for the follower, the strength of consumers' reaction to changes in a product price, the market capacity and the fixed production costs. Thanks to this set of assumptions we can focus on the impact of variability of the marginal costs' differences between the leader and the follower.

Trajectories of the production marginal costs are shown in Fig. 5.27. Initially, the cost for the leader is twice as high as for the follower. At moment $t = 15$, they are equal to each other. At the end of the time horizon, we make an extreme assumption that the marginal cost for the leader equals 0 to obtain the follower's supply equal to 0. We are interested in values of the leader's marginal cost by which the supply levels by the leader and by the follower are positive, that is, $c_1(t) \in (0; 4)$.

The follower has to adjust her/his supply to the output level that the leader decides to produce. Hence, the follower reacts according to the reaction line equation obtained from her/his profit maximization problem:

$$\bar{y}_2(t) = -\frac{1}{2}y_1(t) + 6.$$

Figure 5.28 presents the follower's reaction line and a trajectory of equilibrium states when the production marginal cost for the leader is time-variant. The highlighted point is the equilibrium state $\bar{y}^{(S)}(15) = (6, 3)$ established when the

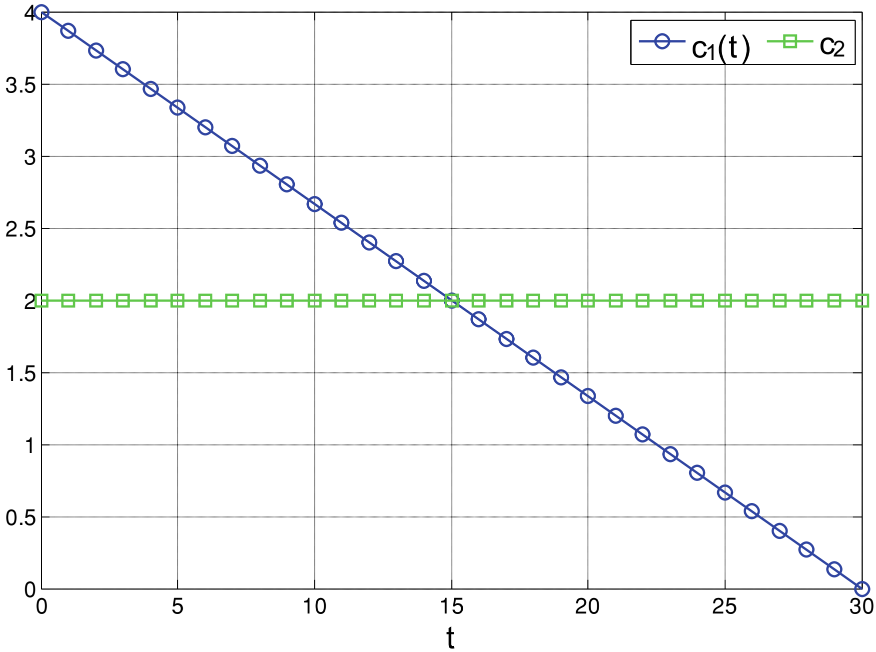


Fig. 5.27 Trajectories of marginal cost for a leader and a follower

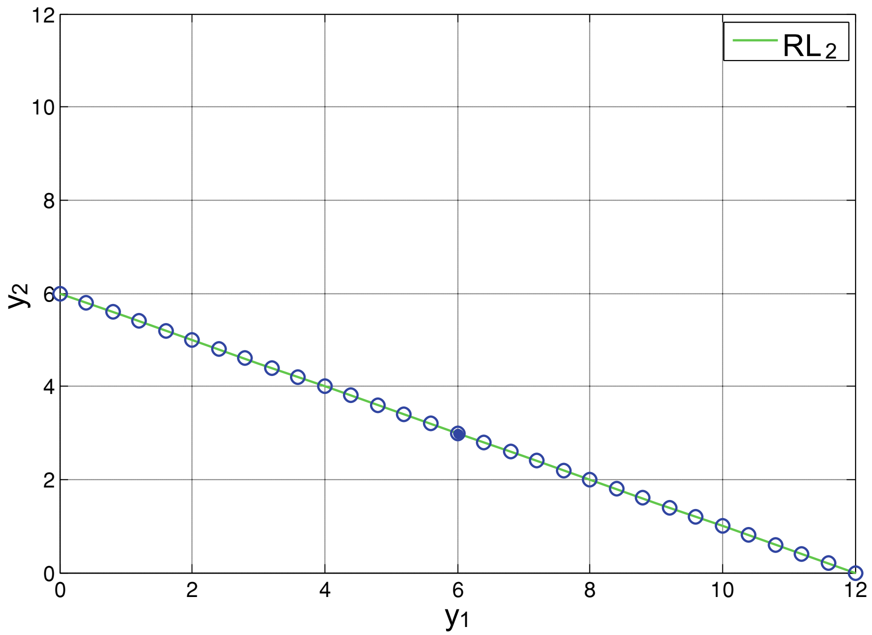


Fig. 5.28 Follower's reaction line and equilibrium state in Stackelberg duopoly model when leader's marginal cost varies in time

marginal costs for the leader and for the follower are the same and equal to 2. The distribution of market shares is then 2/3 for the leader and 1/3 for the follower. Equal shares are achieved at moment$t = 10$, when$\bar{y}^{(S)}(10) = (4, 4)$, by the production marginal costs: $c_1(10) \approx 2.67, c_2 = 2$. If the marginal cost for the leader exceeds 2.67 ($c_2 = 2$), then the follower has a higher market share; otherwise, the leader has a higher share. In the extreme case when $c_1(30) = 0$, the entire market belongs to the leader, but such a case is purely theoretical, because it means cost-free production for the leader. When the marginal cost for the follower is twice as high as for the leader: $c_1(22) \approx 1.07, c_2 = 2$, the leader achieves 85% of the market share, and the follower 15%.

Formation of the optimal supply over time, for both producers, is presented in Fig. 5.29. Their market shares are shown in Fig. 5.30. When the marginal cost for the leader decreases, her/his optimal supply level increases, while optimal supply level for the follower declines. The follower responds to each increase in the leader's supply reducing her/his own output level. Thanks to this, while taking the position of the follower, he/she can sell the output at a level set by himself/herself without fear that the leader will want to take this part of the market from him/her, because the follower has adjusted his/her output level to what the leader dictates.

Comparing Fig. 5.30 with Fig. 5.27, one can notice that a significant market share, over 60%, is achieved by the leader with his/her production marginal cost even slightly higher than for the follower: $c_1(13) \approx 2.27, c_2 = 2$. Reducing the

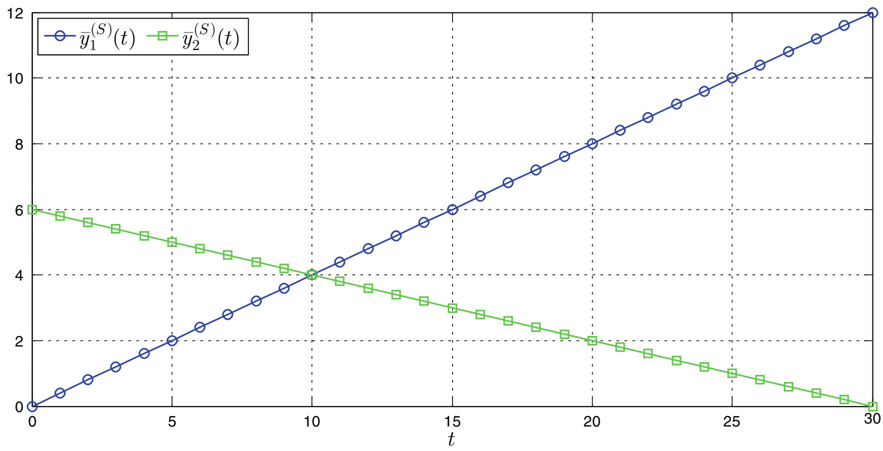


Fig. 5.29 Trajectories of optimal supplies in Stackelberg duopoly model when leader's marginal cost varies in time

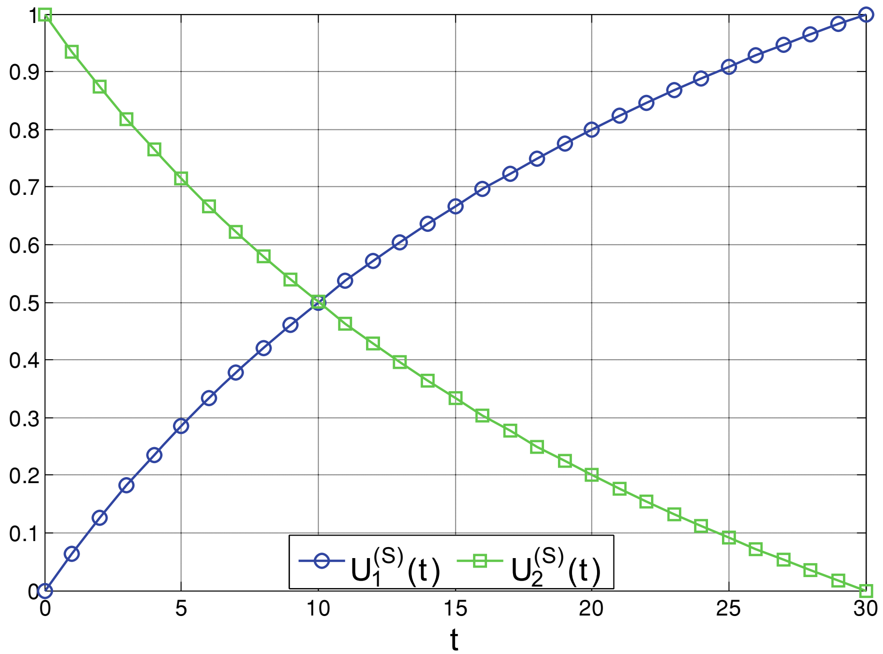


Fig. 5.30 Trajectories of market shares in Stackelberg duopoly model when leader's marginal cost varies in time

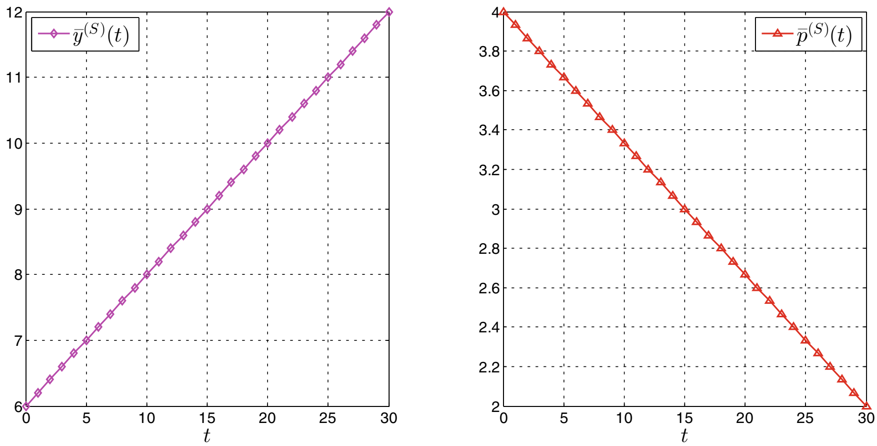


Fig. 5.31 Trajectories of optimal total supply and of optimal price in Stackelberg duopoly model when leader's marginal cost varies in time

leader's marginal cost to a level equal to or below $c_1(25) \approx 0.91$ allows her/him to achieve over 90% market share.

A decrease in the leader's marginal cost of production leads to an increase in the optimal total supply and a reduction in the optimal price of a product. The price of a product is determined by both producers on the basis, among others, of values of their production marginal costs, according to a formula:

$$p^{(S)}\left(\bar{y}^{(S)}(t)\right) = \frac{b(t) + a(t)(2c_1(t) + c_2(t))}{4a(t)},$$

similarly as in the static approach. A decrease in the production marginal cost for a given producer is profitable for her/him and unprofitable for her/his competitor (*ceteris paribus*). In order to sell a product in a quantity derived from the profit maximization problem, the producer has to accept a price level set together with the competitor. A reduction in the leader's marginal cost makes it possible to reduce the price of a product, leading to an increase in the demand for the product and in the optimal supply level which adjusts to the demand. This is reflected in Fig. 5.31.

5.4.3 Comparative Analysis of Cournot and Stackelberg Duopoly Models

Let us now compare results derived from models of a market of one homogenous product presented in the previous subchapter and sections: the pure monopoly model, the Cournot and the Stackelberg duopoly models.

In Tables 5.8a, 5.8b, 5.9a and 5.9b, we summarize information about the equilibrium price, the total supply and the individual supply be a given producer when a product price is the equilibrium price. The summary relates to a pure monopoly, a duopoly in the Cournot model and a duopoly in the Stackelberg model. We present also values of measures of reaction of the equilibrium total supply, the equilibrium individual supplies and the equilibrium price to changes in the market capacity$b$, the consumers' sensitivity a to changes in a product price and in the production marginal cost for each producer (c_1, c_2, c).

In these summaries, we distinguish two basic cases: when production marginal costs in a duopoly in the Cournot or in the Stackelberg model are different and

Table 5.8a Sensitivity analysis for total supply and equilibrium price in pure monopoly model, Cournot and Stackelberg duopoly models when marginal costs for two producers are different

Market characteristics	Pure monopoly[a]	Duopoly in Cournot model	Duopoly in stackelberg model
For total supply when $c_1 \neq c_2$	$\bar{y}^{(M)} = \frac{b-ac}{2}$	$\bar{y}^{(C)} = \frac{2b-a(c_1+c_2)}{3}$	$\bar{y}^{(S)} = \frac{3b-a(2c_1+c_2)}{4}$
$\frac{\partial \bar{y}^{(X)}}{\partial b}$ $X = M, C, S$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$
$\frac{\partial \bar{y}^{(X)}}{\partial a}$ $X = M, C, S$	$-\frac{1}{2}c$	$-\frac{c_1+c_2}{2}$	$-\frac{2c_1+c_2}{4}$
$\frac{\partial \bar{y}^{(X)}}{\partial c_1}$ $X = M, C, S$	$-\frac{1}{4}a$	$-\frac{1}{3}a$	$-\frac{1}{2}a$
$\frac{\partial \bar{y}^{(X)}}{\partial c_2}$ $X = M, C, S$	$-\frac{1}{4}a$	$-\frac{1}{3}a$	$-\frac{1}{4}a$
For equilibrium price when $c_1 \neq c_2$	$\bar{p}^{(M)} = \frac{b+ac}{2a}$	$\bar{p}^{(C)} = \frac{b+a(c_1+c_2)}{3a}$	$\bar{p}^{(S)} = \frac{b+a(2c_1+c_2)}{4a}$
$\frac{\partial \bar{p}^{(X)}}{\partial b}$ $X = M, C, S$	$\frac{1}{2a}$	$\frac{1}{3a}$	$\frac{1}{4a}$
$\frac{\partial \bar{p}^{(X)}}{\partial a}$ $X = M, C, S$	$-\frac{b}{2a^2}$	$-\frac{b}{3a^2}$	$-\frac{b}{4a^2}$
$\frac{\partial \bar{p}^{(X)}}{\partial c_1}$ $X = M, C, S$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{\partial \bar{p}^{(X)}}{\partial c_2}$ $X = M, C, S$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$

[a] For the pure monopoly model, we consider one producer and for the duopoly models - two producers of one homogenous product. Let us keep in mind that, to have the proper basis for comparison of results of the pure monopoly model with duopoly models, we assume that $c_1 + c_2 = 2c$, thus $c = \frac{1}{2}c_1 + \frac{1}{2}c_2$.

Table 5.8b Sensitivity analysis for total supply and equilibrium price in: pure monopoly model, Cournot and Stackelberg duopoly models when marginal costs for two producers are equal

Market characteristics	Pure monopoly[a]	Duopoly in cournot model	Duopoly in stackelberg model
For total supply when $c_1 = c_2 = c$	$\bar{y}^{(M)} = \frac{1}{2}(b - ac)$	$\bar{y}^{(C)} = \frac{2}{3}(b - ac)$	$\bar{y}^{(S)} = \frac{3}{4}(b - ac)$
$\frac{\partial \bar{y}^{(X)}}{\partial b}$ $X = M, C, S$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$
$\frac{\partial \bar{y}^{(X)}}{\partial a}$ $X = M, C, S$	$-\frac{1}{2}c$	$-\frac{2}{3}c$	$-\frac{3}{4}c$
$\frac{\partial \bar{y}^{(X)}}{\partial c}$ $X = M, C, S$	$-\frac{1}{2}a$	$-\frac{2}{3}a$	$-\frac{3}{4}a$
For equilibrium price when $c_1 = c_2 = c$	$\bar{p}^{(M)} = \frac{b+ac}{2a}$	$\bar{p}^{(C)} = \frac{b+2ac}{3a}$	$\bar{p}^{(S)} = \frac{b+3ac}{4a}$
$\frac{\partial \bar{p}^{(X)}}{\partial b}$ $X = M, C, S$	$\frac{1}{2a}$	$\frac{1}{3a}$	$\frac{1}{4a}$
$\frac{\partial \bar{p}^{(X)}}{\partial a}$ $X = M, C, S$	$-\frac{b}{2a^2}$	$-\frac{b}{3a^2}$	$-\frac{b}{4a^2}$
$\frac{\partial \bar{p}^{(X)}}{\partial c}$ $X = M, C, S$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$

[a] For the pure monopoly model we consider one producer and for the duopoly models we consider two producers of one homogenous product

when they are equal. In case of different costs to have the proper basis for comparison of results of the pure monopoly model with duopoly models, we assume that $c_1 + c_2 = 2c$, thus $c = \frac{1}{2}c_1 + \frac{1}{2}c_2$.

On the basis of measures given in Tables 5.8a and 5.8b, one can draw the following conclusions related to the total supply in the equilibrium state:

- the biggest equilibrium total supply is in the Stackelberg duopoly, the middle one in the Cournot duopoly and the smallest one in the pure monopoly;
- the biggest increase in the equilibrium total supply caused by an increase in the market capacity b ocurs in the Stackelberg duopoly, the middle one in the Cournot duopoly and the smallest one in the pure monopoly;
- the biggest decline of the equilibrium total supply caused by an increase in the consumers' sensitivity a to changes in a product price occurs in the Stackelberg duopoly, the middle one in the Cournot duopoly and the smallest one in the pure monopoly;
- the biggest decline of the equilibrium total supply caused by an increase in the first producer's marginal cost (Table 5.8a) occurs in the Stackelberg duopoly,

Table 5.9a Sensitivity analysis for individual supplies by each producer in pure monopoly model and C cournot and Stackelberg duopoly models when marginal costs for two producers are different

Market characteristics	Pure monopoly[a]	Duopoly in cournot model	Duopoly in Stackelberg model
For first producer's supply when $c_1 \neq c_2$	$\bar{y}^{(M)} = \frac{b-ac}{2}$	$\bar{y}_1^{(C)} = \frac{b-a(2c_1-c_2)}{3}$	$\bar{y}_1^{(S)} = \frac{b-a(2c_1-c_2)}{2}$
$\frac{\partial \bar{y}_1^{(X)}}{\partial b}$ $X = M, C, S$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{\partial \bar{y}_1^{(X)}}{\partial a}$ $X = M, C, S$	$-\frac{1}{2}c$	$\frac{-2c_1+c_2}{3}$	$\frac{-2c_1+c_2}{2}$
$\frac{\partial \bar{y}_1^{(X)}}{\partial c_1}$ $X = M, C, S$	$-\frac{1}{4}a$	$-\frac{2}{3}a$	$-a$
$\frac{\partial \bar{y}_1^{(X)}}{\partial c_2}$ $X = M, C, S$	$-\frac{1}{4}a$	$\frac{1}{3}a$	$\frac{1}{2}a$
For second producer's supply when $c_1 \neq c_2$	$\bar{y}^{(M)} = \frac{b-ac}{2}$	$\bar{y}_2^{(C)} = \frac{b-a(2c_2-c_1)}{3}$	$\bar{y}_2^{(S)} = \frac{b-a(3c_2-2c_1)}{4}$
$\frac{\partial \bar{y}_2^{(X)}}{\partial b}$ $X = M, C, S$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$\frac{\partial \bar{y}_2^{(X)}}{\partial a}$ $X = M, C, S$	$-\frac{c}{2}$	$\frac{-2c_2+c_1}{3}$	$\frac{-3c_2+2c_1}{4}$
$\frac{\partial \bar{y}_2^{(X)}}{\partial c_1}$ $X = M, C, S$	$-\frac{1}{4}a$	$\frac{1}{3}a$	$\frac{1}{2}a$
$\frac{\partial \bar{y}_2^{(X)}}{\partial c_2}$ $X = M, C, S$	$-\frac{1}{4}a$	$-\frac{2}{3}a$	$-\frac{3}{4}a$

[a] For the pure monopoly model we consider one producer and for the duopoly models we consider -two producers of one homogenous product. Let us keep in mind that, to have the proper basis for comparison of results of the pure monopoly model with duopoly models, we assume that $c_1+c_2 = 2c$, thus $c = \frac{1}{2}c_1 + \frac{1}{2}c_2$.

the middle one in the Cournot duopoly and the smallest one in the pure monopoly;

- a bigger decline of the equilibrium total supply caused by an increase in the second producer's marginal cost (Table 5.8a) occurs in the Cournot duopoly and a smaller one in the Stackelberg duopoly and in the pure monopoly;

Table 5.9b Sensitivity analysis for individual supplies by each producer in pure monopoly model and C cournot and Stackelberg duopoly models when marginal costs for two producers are equal

Market characteristics	Pure monopoly[a]	Duopoly in cournot model	Duopoly in Stackelberg model
For first producer's supply when $c_1 = c_2 = c$	$\bar{y}^{(M)} = \frac{1}{2}(b - ac)$	$\bar{y}_1^{(C)} = \frac{1}{3}(b - ac)$	$\bar{y}_1^{(S)} = \frac{1}{2}(b - ac)$
$\frac{\partial \bar{y}_1^{(X)}}{\partial b}$ $X = M, C, S$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{\partial \bar{y}_1^{(X)}}{\partial a}$ $X = M, C, S$	$-\frac{1}{2}c$	$-\frac{1}{3}c$	$-\frac{1}{2}c$
$\frac{\partial \bar{y}_1^{(X)}}{\partial c}$ $X = M, C, S$	$-\frac{1}{2}a$	$-\frac{1}{3}a$	$-\frac{1}{2}a$
For second producer's supply when $c_1 = c_2 = c$	$\bar{y}^{(M)} = \frac{b-ac}{2}$	$\bar{y}_2^{(C)} = \frac{1}{3}(b - ac)$	$\bar{y}_2^{(S)} = \frac{1}{4}(b - ac)$
$\frac{\partial \bar{y}_2^{(X)}}{\partial b}$ $X = M, C, S$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
$\frac{\partial \bar{y}_2^{(X)}}{\partial a}$ $X = M, C, S$	$-\frac{1}{2}c$	$-\frac{1}{3}c$	$-\frac{1}{4}c$
$\frac{\partial \bar{y}_2^{(X)}}{\partial c}$ $X = M, C, S$	$-\frac{1}{2}a$	$-\frac{1}{3}c$	$-\frac{1}{4}c$

[a] For the pure monopoly model we consider one producer and for the duopoly models we consider two producers of one homogenous product

- the biggest decline of the equilibrium total supply caused by an increase in both producers' marginal costs (Table 5.8b) occurs in the Stackelberg duopoly, the middle one in the Cournot duopoly and the smallest one in the pure monopoly.

On the basis of measures given in Tables 5.8a and 5.8b, one can draw the following conclusions related to the equilibrium price:

- the highest equilibrium price is in the pure monopoly, the middle one in the Cournot duopoly and the lowest one in the Stackelberg duopoly;
- the biggest increase in the equilibrium price caused by an increase in the market capacity b occurs in the pure monopoly, the middle one in the Cournot duopoly and the smallest one in the Stackelberg duopoly;
- the biggest decline of the equilibrium price caused by an increase in the consumers' sensitivity a to changes in a product price occurs in the pure monopoly,

the middle one in the Cournot duopoly and the smallest one in the Stackelberg duopoly;

- the highest rise of the equilibrium price caused by an increase in the first producer's marginal cost (Table 5.8a) occurs in the Stackelberg duopoly, the middle one in the Cournot duopoly and the lowest one in the pure monopoly;
- a higher rise of the equilibrium price caused by an increase in the second producer's marginal cost (Table 5.8a) occurs in the Cournot duopoly and a lower one in the Stackelberg duopoly and in the pure monopoly;
- the highest rise of the equilibrium price caused by an increase in both producers' marginal costs (Table 5.8b) occurs in the Stackelberg duopoly, the middle one in the Cournot duopoly and the smallest one in the pure monopoly.

On the basis of measures given in Table 5.9b, one can draw the following conclusions related to the equilibrium supplies by each producer when marginal costs for producers are different:

- the biggest equilibrium supply is by the monopolist and by the leader in the Stackelberg duopoly, the middle one is by any producer in the Cournot duopoly and the smallest one by the follower in the Stackelberg duopoly;
- the biggest increase in the individual supply caused by an increase in the market capacity b occurs for the monopolist and for the leader in the Stackelberg duopoly, the middle one for any producer in the Cournot duopoly and the smallest one for the follower in the Stackelberg duopoly;
- if $c_1 < \frac{1}{2}c_2$, then an increase in the consumers' sensitivity a to changes in a product price leads to an increase in the supply by the first producer in the Cournot duopoly and by the leader in the Stackelberg duopoly while the other producer's supply declines;
- if $c_2 < \frac{1}{2}c_1$ then an increase in the consumers' sensitivity a to changes in a product price leads to an increase in the supply by the second producer in the Cournot duopoly while the first producer's supply declines;
- if $c_2 < \frac{2}{3}c_1$, then an increase in the consumers' sensitivity a to changes in a product price leads to an increase in the supply by the follower in the Stackelberg duopoly while the first producer's supply declines;
- the biggest decline of the first producer's equilibrium supply caused by an increase in her/his marginal cost occurs in the Stackelberg duopoly, the middle one in the Cournot duopoly and the smallest one in the pure monopoly. An increase in the competitor's equilibrium supply is bigger in the Stackelberg model (the follower) than in the Cournot model;
- the biggest decline of the second producer's equilibrium supply caused by an increase in her/his marginal cost occurs in the Stackelberg duopoly, the middle one in the Cournot duopoly and the smallest one in the pure monopoly. An increase in the competitor's equilibrium supply is bigger in the Stackelberg model (the leader) than in the Cournot model.

On the basis of measures given in Table 5.9a, one can draw the following conclusions related to the equilibrium supplies by each producer when marginal costs for producers are equal:

- the biggest equilibrium supply is by the monopolist and by the leader in the Stackelberg duopoly, the middle one is by any producer in the Cournot duopoly and the smallest one by the follower in the Stackelberg duopoly;
- the biggest increase in the individual supply caused by an increase in the market capacity b occurs for the monopolist and for the leader in the Stackelberg duopoly, the middle one for any producer in the Cournot duopoly and the smallest one for the follower in the Stackelberg duopoly;
- the biggest decline of the equilibrium individual supply caused by an increase in the consumers' sensitivity a to changes in a product price occurs for the monopolist and for the leader in the Stackelberg duopoly, the middle one for any producer in the Cournot duopoly and the smallest one for the follower in the Stackelberg duopoly;
- the biggest decline of a given producer's equilibrium supply caused by an increase in her/his marginal cost occurs for the monopolist and for the leader in the Stackelberg duopoly, the middle one for any producer in the Cournot duopoly and the smallest one for the follower in the Stackelberg duopoly.

5.4.4 Bertrand Duopoly Model and Its Equilibrium State

5.4.4.1 Static Approach

Let us apply the following set of assumptions:

(B1) Two producers ($i = 1, 2$) act on a market of two heterogeneous (differentiated) substitute products. This means, among other things, that the demand reported for the first (second) product manufactured by the first (second) producer depends not only on its price but also on the price of the second (first) product manufactured by the second (first) producer. Contrary to the Cournot and Stackelberg duopoly models in the Bertrand duopoly model producers compete on prices of their products.

(B2) Functions of producers' production total cost are as follows[33]:

(5.345) $$\forall i = 1, 2 \quad c_i^{tot}(y_i) = c_i^v(y_i) + c_i^f(y_i) = c_i y_i + d_i, \quad c_i, d_i > 0,$$

being the sum of variable cost functions:

(5.346) $$\forall i = 1, 2 \quad c_i^v(y_i) = c_i y_i, \quad c_i > 0$$

[33] As previously, for the sake of simplicity, we assume that functions of production total costs are linear functions of output levels.

and the fixed costs:

(5.347) $$\forall i = 1, 2 \quad c_i^f(y_i) = d_i > 0.$$

Since the total cost functions are linear functions of output levels, we get that

(5.348) $$\forall i = 1, 2 \quad \frac{\mathrm{d}c_i^{tot}(y_i)}{\mathrm{d}y_i} = \frac{\mathrm{d}c_i^v(y_i)}{\mathrm{d}y_i} = c_i > 0,$$

that is the marginal total cost and the marginal variable cost for the i-th producer are equal and they are functions increasing in an output level.

(B3) A function of demand reported for a product of the first producer is as follows:

(5.349) $$y_1^d(p_1, p_2) = b_1 - a_1 p_1 + \gamma_1 p_2, \quad a_1, b_1, \gamma_1 > 0,$$

where:

(5.350) $$y_1^d(0, 0) = b_1,$$

hence, this parameter can be seen as a demand level for the first product when prices of both products equal 0. Yet, we are interested in cases when prices of both products are positive. We call this parameter a measure of **capacity of the first product's market**.

Parameter a_1 can be seen as an amount by which the demand for the first product approximately declines when its price is raised by one money unit and a price of the second product remains unchanged:

(5.351) $$\frac{\partial y_1^d}{\partial p_1} = -a_1$$

and we call it a measure of **sensitivity of the first product's consumers to changes in a price of the first product**.

Parameter γ_1 can be seen as an amount by which the demand for the first product approximately increases when a price of the second product is raised by one money unit and a price of the first product remains unchanged:

(5.352) $$\frac{\partial y_1^d}{\partial p_2} = \gamma_1,$$

and we call it a measure of **sensitivity of the first product's consumers to changes in a price of the second product**.

The change in a price of the first product on the demand for this product is generally presumed to have a stronger effect than the change in a price of the second product ($a_1 > \gamma_1$). It is also worth noticing that a rise in the price of the first product results in a decrease in the demand for this product, while a rise in the price of the second product leads to an increase in the demand for the first product. It stems from the fact that from the perspective of consumers these two products are substitutes for each other and they are ordinary goods.

(B4) A function of demand reported for a product of the first producer is as follows:

$$y_2^d(p_1, p_2) = b_2 - a_2 p_2 + \gamma_2 p_1, \quad a_2, b_2, \gamma_2 > 0, \tag{5.353}$$

where:

$$y_2^d(0, 0) = b_2, \tag{5.354}$$

hence, this parameter can be seen as a demand level for the second product when prices of both products equal 0. Yet, we are interested in cases when prices of both products are positive. We call this parameter a measure of **capacity of the second product's market**.

Parameter a_2 can be seen as an amount by which the demand for the second product approximately declines when its price is raised by one money unit and the price of the first product remains unchanged:

$$\frac{\partial y_2^d}{\partial p_2} = -a_2, \tag{5.355}$$

and we call it a measure of **sensitivity of the second product's consumers to changes in a price of the second product**.

Parameter γ_2 can be seen as an amount by which the demand for the second product approximately increases when the price of the first product is raised by one money unit and the price of the second product remains unchanged:

$$\frac{\partial y_2^d}{\partial p_1} = \gamma_2, \tag{5.356}$$

and we call it a measure of **sensitivity of the second product's consumers to changes in a price of the first product**.

Similarly as for the first product, one assumes that the change in the price of the second product on the demand for this product has a stronger effect than the change in the price of the first product ($a_2 > \gamma_2$). It is also worth noticing that a rise in the price of the second product results in a decrease in the demand for this product, while a rise in the price of the first product leads to an increase in the demand for the first product. It stems from the fact that from the perspective of consumers these two products are substitutes for each other and they are ordinary goods.

(B5) The supply of each product matches the demand that consumers report for this product:

(5.357) $$y_1^s(p_1, p_2) = y_1^d(p_1, p_2) = y_1 = b_1 - a_1 p_1 + \gamma_1 p_2,$$

(5.358) $$y_2^s(p_1, p_2) = y_2^d(p_1, p_2) = y_2 = b_2 - a_2 p_2 + \gamma_2 p_1.$$

(B6) The first producer wants to determine such a price level for her/his product that guarantees the maximum profit for her/him taking a price level of the second product as given:

(5.359) $$\Pi_1(p_1)|_{p_2=\text{const.}\geq 0} \mapsto \max \qquad p_1 \geq 0.$$

(B7) The second producer wants to determine such a price level for her/his product that guarantees the maximum profit for her/him taking a price level of the first product as given:

(5.360) $$\Pi_2(p_2)|_{p_1=\text{const.}\geq 0} \mapsto \max \qquad p_2 \geq 0.$$

From the set of assumptions presented above, it results that a function of revenue from sales:

- of the first product is expressed as

(5.361) $$\begin{aligned} r_1(p_1, p_2) &= p_1 y_1 = p_1(b_1 - a_1 p_1 + \gamma_1 p_2) \\ &= b_1 p_1 - a_1 p_1^2 + \gamma_1 p_1 p_2, \end{aligned}$$

- of the second product is expressed as

(5.362) $$\begin{aligned} r_2(p_1, p_2) &= p_2 y_2 = p_2(b_2 - a_2 p_2 + \gamma_2 p_1) \\ &= b_2 p_2 - a_2 p_2^2 + \gamma_2 p_1 p_2, \end{aligned}$$

From conditions (5.357)–(5.358), it results that a function of production total cost:

- for the first producer takes the form:

(5.363) $$\begin{aligned} c_1^{tot}(y_1) &= c_1 y_1 + d_1 = c_1(b_1 - a_1 p_1 + \gamma_1 p_2) + d_1 \\ &= c_1^{tot}(p_1, p_2), c_1, d_1 > 0, \end{aligned}$$

- for the second producer takes the form:

$$
\begin{aligned}
c_2^{tot}(y_2) &= c_2 y_2 + d_2 = c_2(b_2 - a_2 p_2 + \gamma_2 p_1) + d_2 \\
&= c_2^{tot}(p_1, p_2), c_2, d_2 > \ 0.c
\end{aligned} \tag{5.364}
$$

Hence, a profit function:

- for the first producer has a form:

$$
\begin{aligned}
\Pi_1(p_1, p_2) = &- (b_1 c_1 + d_1) + (b_1 + a_1 c_1) p_1 \\
&+ \gamma_1 p_2 (p_1 - c_1) - a_1 p_1^2,
\end{aligned} \tag{5.365}
$$

- for the second producer has a form:

$$
\begin{aligned}
\Pi_2(p_1, p_2) = &- (b_2 c_2 + d_2) + (b_2 + a_2 c_2) p_2 \\
&+ \gamma_2 p_1 (p_2 - c_2) - a_2 p_2^2.
\end{aligned} \tag{5.366}
$$

First producer
The necessary condition and the sufficient condition for the profit maximization problem of the first producer are the following:

$$
\left. \frac{\partial \Pi_1(p_1)}{\partial p_1} \right|_{p_1 = \bar{p}_1,\ p_2 = \text{const.} \geq 0} = 0 \quad \text{the necessary condition,} \tag{5.367}
$$

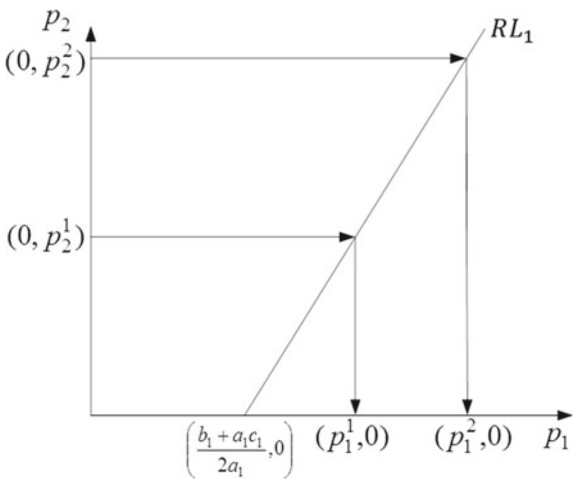
$$
\left. \frac{\partial^2 \Pi_1(p_1)}{\partial p_1^2} \right|_{p_1 = \bar{p}_1,\ p_2 = \text{const.} \geq 0} < 0 \quad \text{the sufficient condition.} \tag{5.368}
$$

Equation (5.365) gives

$$
\left. \frac{\partial \Pi_1(p_1)}{\partial p_1} \right|_{p = \bar{p}_1,\ p_2 = \text{const.} \geq 0} = b_1 + a_1 c_1 - 2 a_1 \bar{p}_1 + \gamma_1 p_2 = 0, \tag{5.369}
$$

$$
\left. \frac{\partial^2 \Pi_1(p_1)}{\partial p_1^2} \right|_{p_1 = \bar{p}_1,\ p_2 = \text{const.} \geq 0} = -2 a_1 < 0. \tag{5.370}
$$

Fig. 5.32 Reaction line of first producer



From conditions (5.369)–(5.370), it results that, by a given price $p_2 \geq 0$ of the second product, the first producer obtains the maximum profit when a price of her/his product is derived from an equation:

$$\bar{p}_1 = \frac{b_1 + a_1c_1}{2a_1} + \frac{\gamma_1}{2a_1}p_2 \quad \boldsymbol{RL_1}. \tag{5.371}$$

which is called a **line of the first producer's reaction** to changes in a price of the second product.

Figure 5.32 presents the way the first producer determines a price level of her/his product when the second producer has determined a price of the second product which is a substitute for the first product. Let us notice that, if the second producer set the price of the substitute product at a level of 0, p_2^1, p_2^2 money units, then the first producer aiming at profit maximization should set the price of her/his product, respectively, at the level of $\frac{b_1+a_1c_1}{2a_1}$, $p_1^1, p_1^2 > 0$ money units.

From Eq. (5.371), it results that

$$p_2 = 0 \quad \Rightarrow \quad \bar{p}_1 = \frac{b_1 + a_1c_1}{2a_1} > 0. \tag{5.372}$$

Condition (5.372) presents a price level of the first product that maximizes the profit of the first producer by zero price of the second product. Moreover, from Eq. (4.371), it follows that

$$\left.\frac{\mathrm{d}\bar{p}_1}{\mathrm{d}p_2}\right|_{RL_1} = \frac{\gamma_1}{2a_1} > 0, \tag{5.373}$$

which means that, if the second producer raises the price of the second product by one money unit, then the first producer aiming at profit maximization should raise the price of the first product by $\frac{\gamma_1}{2a_1}$ money units.

Second producer
The necessary condition and the sufficient condition for the profit maximization problem of the first producer are the following:

(5.374) $$\left.\frac{\partial \Pi_2(p_2)}{\partial p_2}\right|_{p_2=\bar{p}_2,\ p_1=\text{const.}\geq 0} = 0 \quad \text{the necessary condition,}$$

(5.375) $$\left.\frac{\partial^2 \Pi_2(p_2)}{\partial p_2^2}\right|_{p_2=\bar{p}_2,\ p_1=\text{const.}\geq 0} < 0 \quad \text{the sufficient condition.}$$

Equation (4.366) gives

(5.376) $$\left.\frac{\partial \Pi_2(p_2)}{\partial p_2}\right|_{p=\bar{p}_1,\ p_2=\text{const.}\geq 0} = b_2 + a_2c_2 - 2a_2\bar{p}_2 + \gamma_2 p_1 = 0,$$

(5.377) $$\left.\frac{\partial^2 \Pi_2(p_2)}{\partial p_2^2}\right|_{p_1=\bar{p}_1,\ p_2=\text{const.}\geq 0} = -2a_2 < 0.$$

From conditions (5.376)–(5.377), it results that, by a given price $p_1 \geq 0$ of the first product, the second producer obtains the maximum profit when a price of her/his product is derived from an equation:

(5.378) $$\bar{p}_2 = \frac{b_2 + a_2c_2}{2a_2} + \frac{\gamma_2}{2a_2} p_1 \quad \boldsymbol{RL_2},$$

which is called a **line of the second producer's reaction** to changes in a price of the first product.

Figure 5.33 presents the way the second producer determines a price level of her/his product when the first producer has determined a price of the second product which is a substitute for the second product. Let us notice that, if the first producer set the price of the substitute product at a level of 0, p_1^1, p_1^2 money units, then the second producer aiming at profit maximization should set the price of her/his product, respectively, at the level of $\frac{b_2+a_2c_2}{2a_2}$, p_2^1, $p_2^2 > 0$ money units.

From Eq. (5.378), it results that

(5.379) $$p_1 = 0 \quad \Rightarrow \quad \bar{p}_2 = \frac{b_2 + a_2c_2}{2a_2} > 0.$$

Condition (5.379) presents a price level of the second product that maximizes the profit of the second producer by zero price of the first product. Moreover, from Eq. (5.378), it follows that

(5.380) $$\left.\frac{\mathrm{d}\bar{p}_2}{\mathrm{d}p_1}\right|_{RL_2} = \frac{\gamma_2}{2a_2} > 0,$$

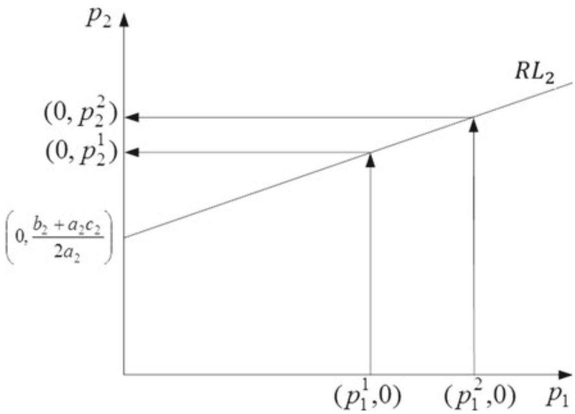


Fig. 5.33 Reaction line of second producer

which means that, if the first producer raises the price of the first product by one money unit, then the second producer aiming at profit maximization should raise the price of the second product by $\frac{\gamma_2}{2a_2}$ money units.

Figure 5.34 presents a mechanism of reaching the equilibrium state in the Bertrand duopoly model. It is not difficult to notice that each producer aiming at the maximization of her/his own profit and taking a price level of the other product set by the competitor as given will seek to have a price of her/his own product on a level resulting from his/her reaction line. Accepting a price level different than the one resulting from his/her reaction line would be inconsistent with the profit maximization aim. As a consequence, both producers will accept as optimal these price levels which are indicated by the intersection of their reaction lines.

The equilibrium state exists, there is exactly one such state and it is globally stable. This means that, if the parameters of the profit function of each producer do not change, then as a result of rational behaviour of both producers the optimal

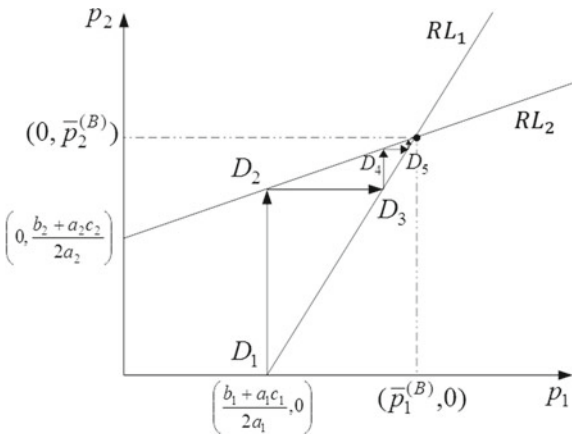


Fig. 5.34 Equilibrium state in Bertrand duopoly model

price levels of substitute products will be established such that each producer will achieve the maximum profit by these price levels.

In the Bertrand duopoly model, the set of equilibrium prices is determined as a solution to the system of Eqs. (5.369) and (5.376) written in a form:

(5.381) $$b_1 + a_1 c_1 = 2a_1 \bar{p}_1 - \gamma_1 \bar{p}_2,$$

(5.382) $$b_2 + a_2 c_2 = 2a_2 \bar{p}_1 - \gamma_2 \bar{p}_1.$$

The equilibrium price vector in the Bertrand duopoly model has a form:

(5.383)
$$\bar{\mathbf{p}}^{(B)} = \left(\bar{p}_1^{(B)}, \bar{p}_2^{(B)}\right) = \left(\frac{2a_2(b_1 + a_1 c_1) + \gamma_1(b_2 + a_2 c_2)}{4a_1 a_2 - \gamma_1 \gamma_2}, \frac{2a_1(b_2 + a_2 c_2) + \gamma_2(b_1 + a_1 c_1)}{4a_1 a_2 - \gamma_1 \gamma_2}\right).$$

The equilibrium price of each product depends on the capacities b_1, b_2 of markets of both products, the production marginal costs c_1, c_2 for both producers, the measures of consumers' sensitivities $a_1, a_2, \gamma_1, \gamma_2$to changes in prices of both products. From the assumption $a_i > \gamma_i, \ i = 1, 2$, it follows that a response of the demand for i-th product to a rise in the price of this product is negative and stronger than the positive response of the demand for i-th product to a rise in a price of the other product. Equation (5.383) ensures that prices of both products in the equilibrium state are positive.

Knowing the equilibrium prices $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2)$, determined by condition (5.383), on the basis of conditions (5.357) and (5.358), one can determine output levels of the first and of the second producers that guarantee maximum profits for both producers. One derives these levels from a system of equations:

(5.384) $$\bar{y}_1^s = b_1 - a_1 \bar{p}_1 + \gamma_1 \bar{p}_2,$$

(5.385) $$\bar{y}_1^s = b_2 - a_2 \bar{p}_2 + \gamma_2 \bar{p}_1,$$

whose solution is a vector of supplies of both products:

(5.386)
$$\bar{\mathbf{y}}^{(B)} = \left(\bar{y}_1^{(B)}, \bar{y}_2^{(B)}\right) = \left(\frac{a_1\left[2a_2(b_1 - a_1 c_1) + \gamma_1(b_2 - a_2 c_2 + \gamma_2 c_1)\right]}{4a_1 a_2 - \gamma_1 \gamma_2}, \frac{a_2\left[2a_1(b_2 - a_2 c_2) + \gamma_2(b_1 - a_1 c_1 + \gamma_1 c_2)\right]}{4a_1 a_2 - \gamma_1 \gamma_2}\right).$$

The output levels of both products at the equilibrium prices depend on the capacities b_1, b_2 of markets of both products, the production marginal costs c_1, c_2 for both producers, the consumers' sensitivities $a_1, a_2, \gamma_1, \gamma_2$to changes in prices of both products.

Let us analyse the sensitivity of the optimal prices of substitute products to changes in values of the parameters of the Bertrand duopoly model. In Tables 5.10a and 5.10b, there are given measures of reaction of the optimal prices of both products to changes in values of parameters of the demand functions and of the production total cost functions.

This sensitivity analysis involves determining the impact of change in a value of a single parameter on the optimal level of the price of i-th product (*ceteris paribus*). In the discussed Bertrand duopoly model, we assume that $a_i > \gamma_i,\ i =$

Table 5.10a Measures of response of first product's optimal price to changes in parameters' values

Characteristic	Value
$\frac{\partial \bar{p}_1^{(B)}}{\partial c_1}$	$\frac{2a_1a_2}{4a_1a_2-\gamma_1\gamma_2} > 0$
$\frac{\partial \bar{p}_1^{(B)}}{\partial c_2}$	$\frac{\gamma_1a_2}{4a_1a_2-\gamma_1\gamma_2} > 0$
$\frac{\partial \bar{p}_1^{(B)}}{\partial b_1}$	$\frac{2a_2}{4a_1a_2-\gamma_1\gamma_2} > 0$
$\frac{\partial \bar{p}_1^{(B)}}{\partial b_2}$	$\frac{\gamma_1}{4a_1a_2-\gamma_1\gamma_2} > 0$
$\frac{\partial \bar{p}_1^{(B)}}{\partial \gamma_1}$	$\frac{2a_2\gamma_2(b_1+a_1c_1)+4a_1a_2(b_2+a_2c_2)}{(4a_1a_2-\gamma_1\gamma_2)^2} > 0$
$\frac{\partial \bar{p}_1^{(B)}}{\partial \gamma_2}$	$\frac{2a_2\gamma_1(b_1+a_1c_1)+\gamma_1^2(b_2+a_2c_2)}{(4a_1a_2-\gamma_1\gamma_2)^2} > 0$
$\frac{\partial \bar{p}_1^{(B)}}{\partial a_1}$	$\frac{-2a_2[4b_1a_2+2\gamma_1(b_2+a_2c_2)+\gamma_1\gamma_2c_1]}{(4a_1a_2-\gamma_1\gamma_2)^2} < 0$
$\frac{\partial \bar{p}_1^{(B)}}{\partial a_2}$	$\frac{-\gamma_1\gamma_2[2(b_1+a_1c_1)+\gamma_1c_2]-4b_2a_1\gamma_1}{(4a_1a_2-\gamma_1\gamma_2)^2} < 0$

Table 5.10b Measures of response of first product's optimal price to changes in parameters' values

Characteristic	Value
$\frac{\partial \bar{p}_2^{(B)}}{\partial c_1}$	$\frac{\gamma_2a_1}{4a_1a_2-\gamma_1\gamma_2} > 0$
$\frac{\partial \bar{p}_2^{(B)}}{\partial c_2}$	$\frac{2a_1a_2}{4a_1a_2-\gamma_1\gamma_2} > 0$
$\frac{\partial \bar{p}_2^{(B)}}{\partial b_1}$	$\frac{\gamma_2}{4a_1a_2-\gamma_1\gamma_2} > 0$
$\frac{\partial \bar{p}_2^{(B)}}{\partial b_2}$	$\frac{2a_1}{4a_1a_2-\gamma_1\gamma_2} > 0$
$\frac{\partial \bar{p}_2^{(B)}}{\partial \gamma_1}$	$\frac{2a_1\gamma_2(b_2+a_2c_2)+\gamma_2^2(b_1+a_1c_1)}{(4a_1a_2-\gamma_1\gamma_2)^2} > 0$
$\frac{\partial \bar{p}_2^{(B)}}{\partial \gamma_2}$	$\frac{2a_1\gamma_1(b_2+a_2c_2)+4a_1a_2(b_1+a_1c_1)}{(4a_1a_2-\gamma_1\gamma_2)^2} > 0$
$\frac{\partial \bar{p}_2^{(B)}}{\partial a_1}$	$\frac{-\gamma_1\gamma_2[2(b_2+a_2c_2)+\gamma_2c_1]-4b_1a_2\gamma_2}{(4a_1a_2-\gamma_1\gamma_2)^2} < 0$
$\frac{\partial \bar{p}_2^{(B)}}{\partial a_2}$	$\frac{-2a_1[4b_2a_1+2\gamma_2(b_1+a_1c_1)+\gamma_1\gamma_2c_2]}{(4a_1a_2-\gamma_1\gamma_2)^2} < 0$

1, 2 which means that a response of the demand for i-th product to a rise in the price of this product is negative and stronger than the positive response of the demand for i-th product to a rise in the price of the other product.

On the basis of data presented in Table 5.10a, one can conclude that the first product's equilibrium price in the Bertrand model rises as a result of an increase (*ceteris paribus*) in the marginal costs c_1, c_2 for the first and for the second producer, the capacities b_1, b_2 of markets of both products, the sensitivity $\gamma_i (i = 1, 2)$ of i-th product's consumers to a rise in the price of the substitute j-th product. The first product's equilibrium price declines only as a result of an increase (*ceteris paribus*) in the sensitivity $a_i (i = 1, 2)$ of i-th product's consumers to a rise in the price of i-th product.

On the basis of data presented in Table 5.10b, one can conclude that the second product's equilibrium price in the Bertrand model rises as a result of an increase (*ceteris paribus*) in the marginal costs c_1, c_2 for the first and for the second producer, the capacities b_1, b_2 of markets of both products, the sensitivity $\gamma_i (i = 1, 2)$ of i-th product's consumers to a rise in the price of the substitute j-th product. The first product's equilibrium price declines only as a result of an increase (*ceteris paribus*) in the sensitivity $a_i (i = 1, 2)$ of i-th product's consumers to a rise in the price of i-th product.

Summary of the conclusions resulting from the sensitivity analysis of the optimal prices of substitute products in the Bertrand duopoly model shows their inherent symmetry. It depends much on the analytical forms of the demand functions assumed for both substitute products[34] and the production cost functions assumed for both producers.[35]

5.4.4.2 Dynamic Approach

In the analysis of the Bertrand duopoly model presented in Sect. 5.4.4.1, we focus on the static approach, in particular on studying the equilibrium prices and the equilibrium supplies. Let us recall that, in the Bertrand duopoly model, the equilibrium state exists, is only one and globally stable regardless of the values of the parameters$a_i, b_i, \gamma_i c_i$, $(i = 1, 2)$, about which it is enough to assume that they are all positive.[36]

The Bertrand duopoly model concerns price competition between two producers having equal positions on a market and manufacturing two substitute products. Hence, decisions of producers relate first of all to the setting of price levels for their products. The supply level results from the choice of the price which is determined on the basis of the profit maximization problem. The optimal price level of the duopolist's product in the Bertrand model, that is, the equilibrium

[34] Let us notice that forms of the demand functions are the same for both products, differing only in the indexation of parameters.

[35] Let us notice that forms of the production cost functions are the same for both producers, differing only in the indexation of parameters.

[36] We make also additional assumptions about values of parameters to ensure that price levels and output levels are positive.

price, for the first and for the second producers, respectively, equals:

(5.387) $$\bar{p}_1^{(B)} = \frac{2a_2(b_1 + a_1c_1) + \gamma_1(b_2 + a_2c_2)}{4a_1a_2 - \gamma_1\gamma_2},$$

(5.388) $$\bar{p}_2^{(B)} = \frac{2a_1(b_2 + a_2c_2) + \gamma_2(b_1 + a_1c_1)}{4a_1a_2 - \gamma_1\gamma_2}.$$

We are interested in the state of equilibrium that meets conditions: $\bar{p}_1 > 0$ and $\bar{p}_2 > 0$. Therefore, we assume that the values of the parameter satisfy an inequality:

(5.389) $$a_1a_2 > \frac{1}{4}\gamma_1\gamma_2.$$

Let us notice that having the demand for a given product assumed to respond stronger to a change in the price of this product than to a change in the price of the substitute product ($a_i > \gamma_i$, $i = 1, 2$) implies the inequality (5.389) is satisfied.

The fulfilment of inequality (5.389) is not necessary for the existence, uniqueness, or global stability of the equilibrium state but is necessary to ensure positive equilibrium price levels. The equilibrium price level cannot be equal to 0 because we deal with the case of a duopoly and zero price of some product would mean that its producer exits the market, which becomes a monopoly.

From the analysis conducted so far in Sect. 5.4.3.1, we know that a mechanism of reaching the equilibrium state is a sequence of iterations in determining the level of price alternately by one producer and the other according to the given producer's reaction line. The successive stages of determining the price levels can be identified with moments or periods in some time horizon whose end is indicated by the moment/period of reaching the state of equilibrium. If we want to interpret time as discrete, then iterations take place at equal intervals of time, for example, every 1 month. If time is treated as continuous, then subsequent iterations are interpreted as taking place at any consecutive moment, for example, the second iteration after a month, the third one after another 3 weeks, the fourth one after another 27 days, etc. In both cases, however, whether we interpret time as discrete or as continuous, with a given set of parameter values, the number of iterations is the same.

In addition to analysing the optimal values in the equilibrium state and the mechanism of reaching this state, we are also interested how quickly this state is reached, that is, how many iterations are needed to determine the equilibrium price levels and what determines the rate of convergence. From formulas (5.387)–(5.388), it can be seen that what distinguishes one producer from another and at the same time determines the optimal price level of a given product are the marginal costs of production, the market capacities, the consumers' sensitivity to changes in a price of a given product and the consumers' sensitivity to changes in a price of the substitute product. Two producers may therefore differ in as many as four dimensions related to the production side and to the demand side. Each of these aspects can influence the rate of convergence to the equilibrium state. Similarly as in the static approach equations of the reaction lines for both producers take the form:

$$\bar{p}_1 = \frac{\gamma_1}{2a_1} p_2 + \frac{b_1 + a_1 c_1}{2a_1}, \tag{5.390}$$

$$\bar{p}_2 = \frac{\gamma_2}{2a_2} p_1 + \frac{b_2 + a_2 c_2}{2a_2}. \tag{5.391}$$

According to Eq. (5.390) of the first producer's reaction line, he/she would set the highest price level if he/she assumed that its competitor's product price is zero, which means the competitor does not sell the product because a zero price would bring losses from production instead of profits. If that was the case, the first producer would have no competition and could set the price at the level that is chosen by a monopolist. However, there is a competitor on the market offering the substitute product, responding to a price level set by the first producer. The competitor, that is, the second producer, sets the price at the level resulting from the Eq. (4.391) of her/his reaction line. The first producer responds to this. After a certain number of iterations of price decisions, the equilibrium state is reached. It is defined by two equilibrium prices which ensure the maximum profit for each producer.

Example 5.9 Two producers having equal positions on a market offer two substitute products. The demand for these products evolves according to the following demand functions:

$$y_1^d(p_1, p_2) = -a_1 p_1 + \gamma_1 p_2 + b_1,$$

$$y_2^d(p_1, p_2) = -a_2 p_2 + \gamma_2 p_1 + b_2, \quad a_i, \gamma_i, b_i > 0, a_i > \gamma_i \quad i = 1, 2.$$

Production total costs for the first and for the second firms, respectively, are as follows:

$$c_1^{tot}(y_1) = c_1 y_1 + d_1,$$

$$c_2^{tot}(y_2) = c_2 y_2 + d_2,$$

where $d_1, d_2 \geq 0$ denote the fixed production costs and $c_1, c_2 > 0$ denote production marginal costs. An output level of each product matches the demand reported by consumers for this product by its given price:

$$y_i = y_i^d(p_1, p_2), \qquad i = 1, 2.$$

Figures 5.35 and 5.36 show the reaction lines of duopolists in the Bertrand model, the state of equilibrium and the mechanism of reaching the equilibrium when the parameters of the demand functions and the cost functions have the following values: $a_i = 3, b_i = 18, \gamma_1 = 1, c_i = 1, i = 1, 2$. The optimal price of each product is then 4.2. This is the equilibrium price.

Figure 5.35 illustrates the mechanism of reaching the equilibrium state when the first producer decides on the level of a price as first (Scenario 1), assuming the price of the competitor's product equals 0. The fact that the first producer makes the decision as first does not mean that he/she has an advantage over the second producer, but only allows us to assume the order of iterations because producers' decisions do not have to be perfectly synchronized in time. We can see the order of making decisions by looking at the points in the reaction lines. The point indicating the first iteration, that is point (3.5, 0), belongs to the reaction line of the first producer. Regardless of the starting point, thus regardless of what the first producer assumes about the competitor's price level, the equilibrium state is achieved after a certain number of iterations of the price decisions.

The equilibrium state is also reached when the second producer makes the price decision as first (Scenario 2), which is illustrated in Fig. 5.36. The point indicating the first iteration, that is, point (0, 3.5), belongs to the reaction line of the second producer. This time we assume that the second producer decides on the level of price as first, assuming a price of the competitor's product equals 0.

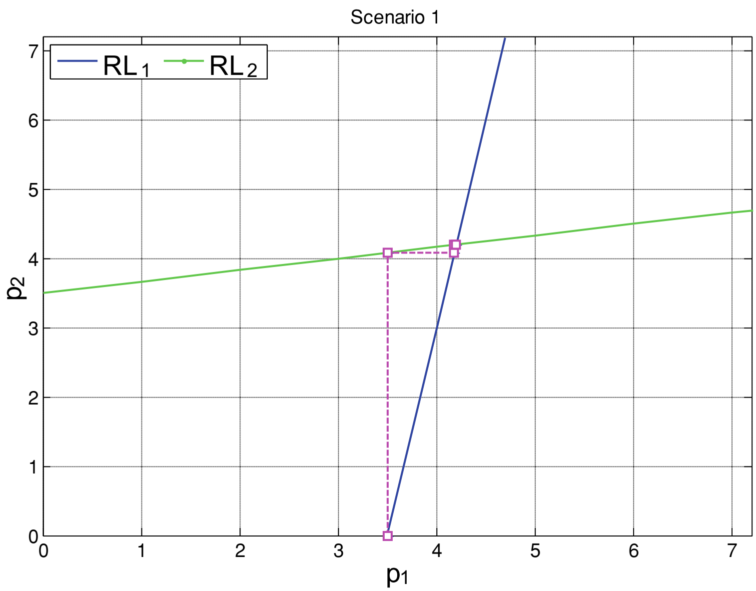


Fig. 5.35 Mechanism of reaching equilibrium state in Bertrand duopoly model when first decision on price is made by first producer

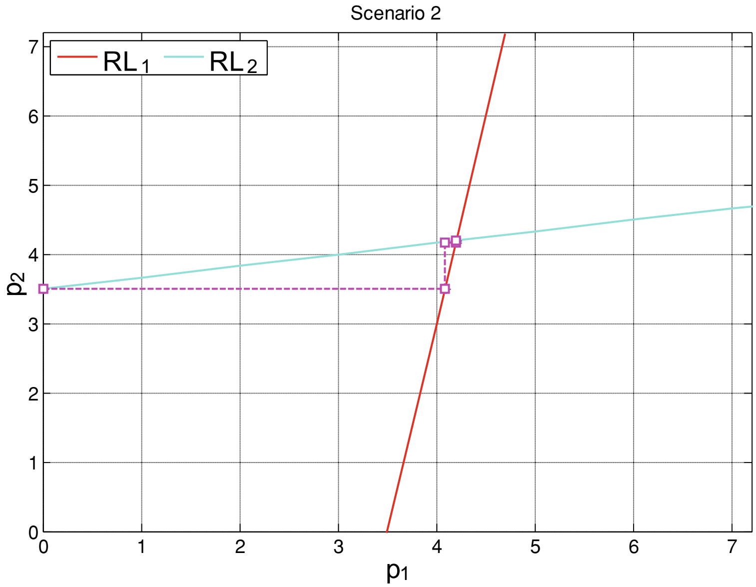


Fig. 5.36 Mechanism of reaching equilibrium state in Bertrand duopoly model when first decision on price is made by the second producer

The equilibrium state is the same as before: $\bar{p}_1^{(B)} = \bar{p}_2^{(B)} = 4.3$. The starting point could be as well some other point than $(0, 3.5)$, but still belonging to the second producer reaction line and indicating what the second producer assumes about the competitor's price level.

Table 5.11 presents a mechanism of reaching the equilibrium state when the demand for the substitute products of duopolists evolves the same way ($a_i = 3, b_i = 18, \gamma_i = 1, i = 1, 2$) and the producers do not differ from each other with the production marginal costs ($c_1 = c_2 = 1$). The equilibrium state is reached in the eighth iteration and assuming an accuracy of two decimal places just right in the fifth iteration. Let us treat this set of parameters' values as the baseline.

Table 5.12 presents trajectories of products' prices when assuming that consumers of a given product respond to changes in the price of this product: $a_1 = a_2 = 4$ stronger than in the baseline case. Values of the remaining parameters are the same as in the baseline set. The equilibrium state is reached in the eighth iteration and assuming an accuracy of two decimal places in the fourth iteration.

Table 5.13 presents trajectories of products' prices when assuming that consumers of a given product respond to changes in a price of the substitute product: $\gamma_1 = \gamma_2 = 2$ stronger than in the baseline case. Values of the remaining parameters are the same as in the baseline set. The equilibrium state is reached in the twelfth iteration and assuming an accuracy of two decimal places in the eighth iteration.

Table 5.14 presents trajectories of products' prices when assuming the market capacities for both products: $b_1 = b_2 = 19$ are bigger than in the baseline case.

Table 5.11 Trajectories of price levels when $a_i = 3,\ b_i = 18,\ \gamma_i = 1,\ c_i = 1$

Iteration number t	p_1	p_2	$\bar{p}_1^{(B)}$	$\bar{p}_2^{(B)}$
1	3.5000	0.0000		
2	3.5000	4.0833		
3	4.1806	4.0833		
4	4.1806	4.1968		
5	**4.1995**	**4.1968**		
6	4.1995	4.1999		
7	4.2000	4.1999		
8	4.2000	4.2000	4.2	4.2

Table 5.12 Trajectories of price levels when $a_1 = a_2 = 4$

Iteration number t	p_1	p_2	$\bar{p}_1^{(B)}$	$\bar{p}_2^{(B)}$
1	2.7500	0.0000		
2	2.7500	3.0938		
3	3.1367	3.0938		
4	**3.1367**	**3.1421**		
5	3.1428	3.1421		
6	3.1428	3.1428		
7	3.1429	3.1428		
8	3.1429	3.1429	3.1429	3.1429

Values of the remaining parameters are the same as in the baseline set. The equilibrium state is reached in the eighth iteration and assuming an accuracy of two decimal places in the fifth iteration.

Table 5.15 presents trajectories of products' prices when assuming the production marginal costs for both producers: $c_1 = c_2 = 2$ are higher than in the baseline case. Values of the remaining parameters are the same as in the baseline set. The equilibrium state is reached in the eighth iteration and assuming an accuracy of two decimal places in the fifth iteration.

We do not observe significant differences in the impacts of parameters' values for the rate of convergence to the equilibrium state. The biggest impact can be observed for values of parameters γ_1 and γ_2, that is parameters which describe how strong consumers of a product manufactured by a given producer react to changes in the price of the other product manufactured by her/his competitor.

Figures 5.37 and 5.38 present the mechanism of reaching the equilibrium state when we treat successive iterations of making decisions on price levels as occurring sequentially in time. The trajectories are presented by parameters' values

Table 5.13 Trajectories of price levels when $\gamma_1 = \gamma_2 = 2$

Iteration number t	p_1	p_2	$\bar{p}_1^{(B)}$	$\bar{p}_2^{(B)}$
1	3.5000	0.0000		
2	3.5000	4.6667		
3	5.0556	4.6667		
4	5.0556	5.1852		
5	5.2284	5.1852		
6	5.2284	5.2428		
7	5.2476	5.2428		
8	**5.2476**	**5.2492**		
9	5.2497	5.2492		
10	5.2497	5.2499		
11	5.2500	5.2499		
12	5.2500	5.2500	5.25	5.25

Table 5.14 Trajectories of price levels when $b_1 = b_2 = 19$

Iteration number t	p_1	p_2	$\bar{p}_1^{(B)}$	$\bar{p}_2^{(B)}$
1	3.6667	0.0000		
2	3.6667	4.2778		
3	4.3796	4.2778		
4	4.3796	4.3966		
5	**4.3994**	**4.3966**		
6	4.3994	4.3999		
7	4.4000	4.3999		
8	4.4000	4.4000	4.4	4.4

taken as in Table 5.13. Figure 5.37 presents trajectories of the price levels of duopolists in two scenarios, depending on which of the two producers decides about the price level as first. In both cases the same equilibrium state is achieved: $\bar{p}_1^{(B)} = \bar{p}_2^{(B)} = 5.25$.

Figure 5.38 presents trajectories of output levels. In both scenarios, the optimal output level is the same and also the same for both producers $\bar{y}_1^{(B)} = \bar{y}_2^{(B)} = 12.75$.

Table 5.15 Trajectories of price levels when $c_1 = c_2 = 2$

Iteration number t	p_1	p_2	$\bar{p}_1^{(B)}$	$\bar{p}_2^{(B)}$
1	4.0000	0.0000		
2	4.0000	4.6667		
3	4.7778	4.6667		
4	4.7778	4.7963		
5	**4.7994**	**4.7963**		
6	4.7994	4.7999		
7	4.8000	4.7999		
8	4.8000	4.8000	4.8	4.8

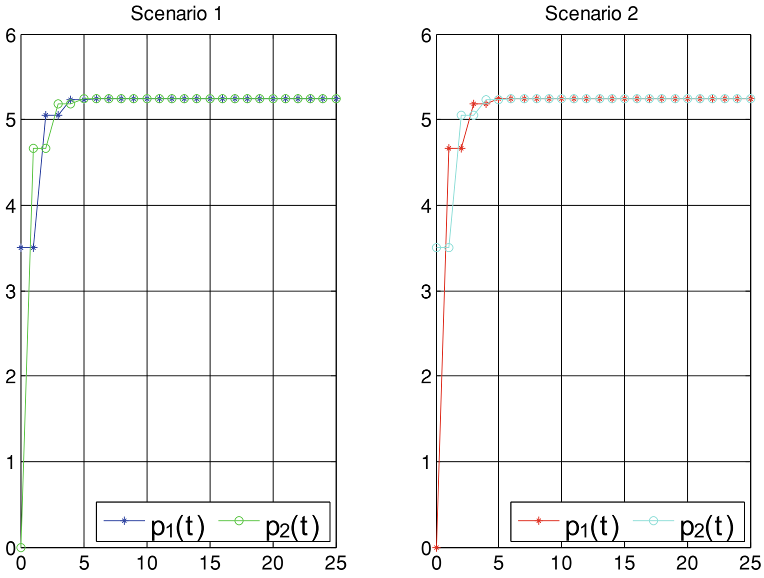


Fig. 5.37 Trajectories of price levels set by duopolists in Bertrand model

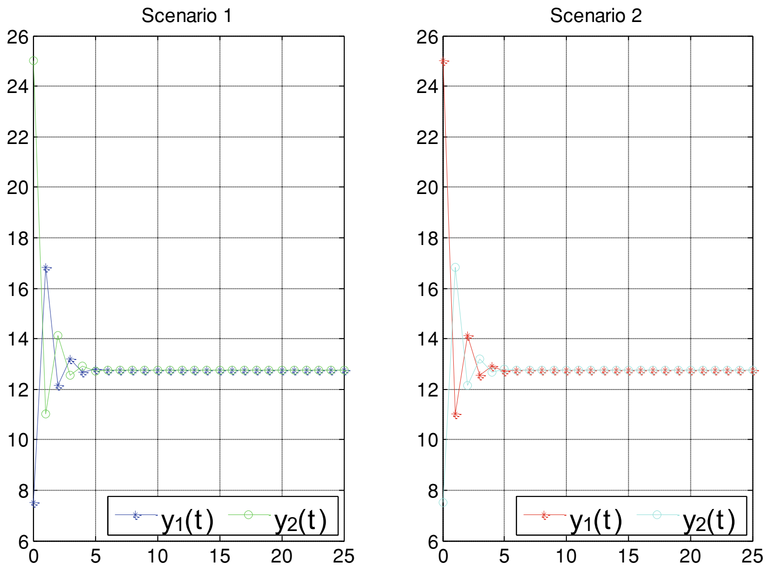


Fig. 5.38 Trajectories of output levels by duopolists in Bertrand model

5.5 Questions

1. What are a demand function and an inverse function of demand for one product?
2. What conditions need to be satisfied by exogenous linear: demand function and function of product supply to have a positive equilibrium price established on a market of this product?
3. How the optimal supplies by each of two producers and the optimal total supply react to changes in values of parameters of production cost functions and demand function when both producers act in perfect competition?
4. What are the relationships between a product price set by a monopolist and price elasticity of demand for this product?
5. How do the optimal supply and a product optimal price set by a monopolist react to changes in values of parameters of production cost function and demand function when there is an exogenously determined function of demand for product?
6. What is meant by discriminatory pricing practised by a monopolist for one product intended for two independent markets of the same product?
7. What are the conditions by which prices of one product supplied by a monopolist for two different markets are equal?
8. What is the difference between the perfect competition model presented in Sect. 5.1 and the Cournot duopoly model?

9. What are the conditions that should be satisfied by functions of demand for product in the Cournot, Stackelberg and Bertrand duopoly models?
10. How do firms acting on markets described by Cournot, Stackelberg and Bertrand duopoly models decide on their strategies of rational behaviour?
11. When do producers acting on a duopolistic market decide to compete on quantities of supplied product and when on prices?
12. What does it mean that two producers have equal positions on a duopolistic market and that one is a leader and the other is a follower?
13. What are the mechanisms of reaching an equilibrium state in the Cournot, Stackelberg and Bertrand duopoly models?
14. What conclusions can be drawn on the basis of comparative analysis of equilibrium states on a market of one product in case of pure monopoly, Cournot and Stackelberg duopoly models?

5.6 Exercises

E1. Determine an inverse function of demand and a function inverse to a given demand function: $y^d(p) = -ap^{\alpha} + b, \quad a, b > 0$. Draw graphs of these functions in the case when:

(a) $\alpha \in (0, 1)$,
(b) $\alpha > 1$.

E2. There is a market for a product with exogenously determined demand function and product supply function:

(a) $y^d(p) = -ap^2 + b,\ a, b > 0, \quad y^s(p) = cp^2 + d,\ c, d > 0,\ b > d,$
(b) $y^d(p) = -ap^{\frac{1}{2}} + b,\ a, b > 0, \quad y^s(p) = cp^{\frac{1}{2}} + d,\ c, d > 0,\ b > d.$

1. Determine the ranges of a product price, demand for the product, the product supply.
2. Draw graphs of the demand function and the supply function in space $\mathbb{R}^2_+$.
3. Determine an inverse function of demand and a function inverse to a given demand function. Draw their graphs in space $\mathbb{R}^2_+$.
4. Derive an equilibrium price by which $y^d(\bar{p}) = y^s(\bar{p})$, that is a price that equalizes the demand for a product and the product supply both expressed in the same physical units

E3. Two producers act in perfect competition on a market described in Sect. 4.1, supplying one homogenous product. A function of demand for the product is linear: $y^d(p) = -ap + b, \quad a, b > 0$, while functions of production total costs are nonlinear and of a form: $c^{tot}_j(y_j) = \gamma_j y_j^2 + \delta_j, \quad \gamma_j, \delta_j > 0,\ j = 1, 2.$

1. Determine the optimal supply of a product:
   (a) by the first producer,
   (b) by the second producer,
   (c) by both producers – the total supply.
2. Analyse sensitivity of the optimal supply by the first producer, by the second producer and the total supply by both producers to changes in values of parameters describing the market: a market capacity, strength of consumers' reaction to changes in a product price, production marginal cost of each producer.

E4. Three producers act in perfect competition on a market of one homogenous product. A function of demand for the product is linear $y^d(p) = -ap+b, \quad a, b > 0$, functions of production total costs are also linear: $c_j^{tot}(y_j) = \gamma_j y_j + \delta_j, \ \gamma_j, \delta_j > 0, \ j = 1, 2$.

1. Determine the optimal supply of a product by each of three producers.
2. Determine the optimal total supply by all three producers.
3. Analyse sensitivity of the optimal supply by each producer and the total supply by all three producers to changes in values of parameters describing the market: a market capacity, strength of consumers' reaction to changes in a product price, production marginal cost of each producer.
4. Generalize conclusions derived in points 1–3 to the case of r producers ($r \in \mathbb{N}, r \geq 3$).

E5. Consider two markets of one homogenous product. The first one is a perfect competition market described in Sect. 5.1, the second one is a monopolistic market presented in Sect. 5.2. Determine conditions by which the optimal supply of a product on the perfect competition market is

(a) bigger than,
(b) equal to,
(c) smaller than
the optimal supply of a product on the monopolistic market.

E6. Formulate and solve a problem of choice of the optimal supply and of the optimal price set by a monopolistic company considered in Example 5.3. Assume a nonlinear function of production total cost of a form: $c^{tot}(y) = \gamma y^2 + \delta, \quad \gamma, \delta > 0$. Analyse sensitivity of the optimal supply and of a product optimal price to changes in values of parameters describing the market: a market capacity, strength of consumers' reaction to changes in a product price, production marginal cost.

E7. Formulate and solve a problem of choice of the optimal supply and of the optimal price set by a monopolistic company considered in Example 5.4. Assume that the monopolist can supply her/his product to two different markets, thus regards discriminatory pricing. Assume a nonlinear function of production total cost:

$c^{tot}(y) = \gamma y^2 + \delta, \quad \gamma, \delta > 0$. Analyse how the optimal supply and a product optimal price react to changes in values of parameters describing the markets: market capacities, consumers' sensitivities to changes in a product price, production marginal cost.

E8. Consider the Cournot duopoly model when a production total cost function for i-th producer ($i = 1, 2$) is nonlinear and of a form: $c_i^{tot}(y) = c_i y_i^2 + d_i, \quad c_i, d_i > 0$. Determine optimal levels of: the product supply by each producer, the total supply by both producers, an equilibrium price. Analyse how these levels react to changes in values of parameters describing the market: a market capacity, strength of consumers' reaction to changes in a product price, production marginal costs.

E9. Consider the Stackelberg duopoly model when a production total cost function for i-th producer ($i = 1, 2$) is nonlinear and of a form: $c_i^{tot}(y) = c_i y_i^2 + d_i, \quad c_i, d_i > 0$. Determine the product optimal supply by: a leader, a follower, both producers and an equilibrium price. Analyse how these levels react to changes in values of parameters describing the market: a market capacity, strength of consumers' reaction to changes in a product price, production marginal costs.

E10. Compare conclusions drawn from solutions to exercises E6, E8 and E9 assuming they relate to a market of one homogenous market in cases of the: pure monopoly, Cournot duopoly, the Stackelberg duopoly.

E11. In Tables 5.16a and 5.16b there are given measures of reaction of the optimal supplies of two substitute products to changes in values of parameters of demand functions and of production total cost functions. The functions are linear and given in the same forms as in Sect. 5.4.4.1 where the Bertrand duopoly model is presented. Check if the given results are correct and on their basis formulate conclusions about the reactions.

E12. Two producers act on a market of two heterogeneous substitute products. The first producer (leader) can set an optimal price of her/his product on a level that guarantees her/him the maximum profit regardless a price level of a product supplied by the competitor. The second producer (follower) decides on a price level of her/his product depending on the other product's price set by the leader. There are given functions of:

(a) production total costs for both producers:

$$\forall i = 1, 2 \quad c_i^{tot}(y_i) = c_i^v(y_i) + c_i^f(y_i) = c_i y_i + d_i, \quad c_i, d_i > 0,$$

(b) demand for the first producer's (leader) product:

$$y_1^d(p_1, p_2) = b_1 - a_1 p_1, \quad a_1, b_1, > 0,$$

Table 5.16a Measures of response of first product's optimal supply to changes in parameters' values

Characteristic	Value
$\frac{\partial \bar{y}_1^{(B)}}{\partial c_1}$	$\frac{-a_1(2a_1a_2-\gamma_1\gamma_2)}{4a_1a_2-\gamma_1\gamma_2} < 0$
$\frac{\partial \bar{y}_1^{(B)}}{\partial c_2}$	$-\frac{a_1a_2\gamma_1}{4a_1a_2-\gamma_1\gamma_2} < 0$
$\frac{\partial \bar{y}_1^{(B)}}{\partial b_1}$	$\frac{2a_1a_2}{4a_1a_2-\gamma_1\gamma_2} > 0$
$\frac{\partial \bar{y}_1^{(B)}}{\partial b_2}$	$\frac{a_1\gamma_1}{4a_1a_2-\gamma_1\gamma_2} > 0$
$\frac{\partial \bar{y}_1^{(B)}}{\partial \gamma_1}$	$\frac{2a_1a_2[\gamma_2(b_1+a_1c_1)+2a_1(b_2-a_2c_2)]}{(4a_1a_2-\gamma_1\gamma_2)^2} > 0$
$\frac{\partial \bar{y}_1^{(B)}}{\partial \gamma_2}$	$\frac{a_1\gamma_1[2a_2(b_1+a_1c_1)+\gamma_1(b_2-a_2c_2)]}{(4a_1a_2-\gamma_1\gamma_2)^2} > 0$
$\frac{\partial \bar{y}_1^{(B)}}{\partial a_1}$	$-\frac{\gamma_1\gamma_2[2a_2(b_1-2a_1c_1)+\gamma_1(b_2-a_2c_2+\gamma_2c_1)]}{(4a_1a_2-\gamma_1\gamma_2)^2} < 0$
$\frac{\partial \bar{y}_1^{(B)}}{\partial a_2}$	$-\frac{a_1\gamma_1[\gamma_2(2b_1+2a_1c_1-\gamma_1c_2)+4a_1b_2]}{(4a_1a_2-\gamma_1\gamma_2)^2} < 0$

Table 5.16b Measures of response of second product's optimal supply to changes in parameters' values

Characteristic	Value
$\frac{\partial \bar{y}_2^{(B)}}{\partial c_1}$	$-\frac{a_1a_2\gamma_2}{4a_1a_2-\gamma_1\gamma_2} < 0$
$\frac{\partial \bar{y}_2^{(B)}}{\partial c_2}$	$\frac{-a_2(2a_1a_2-\gamma_1\gamma_2)}{4a_1a_2-\gamma_1\gamma_2} < 0$
$\frac{\partial \bar{y}_2^{(B)}}{\partial b_1}$	$\frac{a_2\gamma_2}{4a_1a_2-\gamma_1\gamma_2} > 0$
$\frac{\partial \bar{y}_2^{(B)}}{\partial b_2}$	$\frac{2a_1a_2}{4a_1a_2-\gamma_1\gamma_2} > 0$
$\frac{\partial \bar{y}_2^{(B)}}{\partial \gamma_1}$	$\frac{a_2\gamma_2[2a_1(b_2+a_2c_2)+\gamma_2(b_1-a_1c_1)]}{(4a_1a_2-\gamma_1\gamma_2)^2} > 0$
$\frac{\partial \bar{y}_2^{(B)}}{\partial \gamma_2}$	$\frac{2a_1a_2[\gamma_1(b_2+a_2c_2)+2a_2(b_1-a_1c_1)]}{(4a_1a_2-\gamma_1\gamma_2)^2} > 0$
$\frac{\partial \bar{y}_2^{(B)}}{\partial a_1}$	$-\frac{a_2\gamma_2[\gamma_1(2b_2+2a_2c_2-\gamma_2c_1)+4a_2b_1]}{(4a_1a_2-\gamma_1\gamma_2)^2} < 0$
$\frac{\partial \bar{y}_2^{(B)}}{\partial a_2}$	$-\frac{\gamma_1\gamma_2[2a_1(b_2-2a_2c_2)+\gamma_2(b_1-a_1c_1+\gamma_1c_2)]}{(4a_1a_2-\gamma_1\gamma_2)^2} < 0$

(c) demand for the second producer's (follower) product:

$$y_2^d(p_1, p_2) = b_2 - a_2p_2 + \gamma_2p_1, \quad a_2, b_2, \gamma_2 > 0.$$

The supply of each producer's product is assumed to match the demand reported for this product:

$$y_1^s(p_1, p_2) = y_1^d(p_1, p_2) = y_1 = \alpha_1 - \beta_1 p_1,$$

$$y_2^s(p_1, p_2) = y_2^d(p_1, p_2) = y_2. = \alpha_2 - \beta_2 p_2 + \gamma_2 p_1.$$

1. Formulate the profit maximization problem for:
   - the first producer: $\Pi_1(p_1) \mapsto \max \quad p_1 \geq 0$
   - the second producer: $\Pi_2(p_1, p_2)|_{p_1=\text{const.}\geq 0} \mapsto \max \quad p_1, p_2 \geq 0.$
2. Solve the profit maximization problems for both producers determining optimal levels of prices of both products and maximum profits of both producers.
3. Analyse how optimal prices of both products react to changes in values of parameters describing the market: market capacities, consumers' sensitivities to changes in prices of products, production marginal costs.
4. Draw reaction lines of both producers and illustrate an equilibrium state of this modified Bertrand duopoly model.
5. Determine the optimal supply of each product manufactured by a given producer.
6. State which of the strategies of setting an optimal price level of a product is more rational and more beneficial: the one of the leader or the one of the follower?

E13. Compare the original and the modified (E12) Bertrand duopoly models.[37] State if the leader position in the modified Bertrand duopoly model is more beneficial for the first producer than a market position equal to the one of the second producer when these two producers compete on prices.

E14. The demand for a product of a monopolistic company evolves according to a linear function of a form:

$$y^d(t) = -a(t)p(t) + b(t), \qquad a(t), b(t) > 0, \qquad \forall t = 0, 1, \ldots, 20.$$

A function of production total cost is given as

$$k^c\big(y^s(t)\big) = \gamma(t)y^s(t) + \delta(t), \qquad \gamma(t), \delta(t) > 0, \qquad \forall t = 0, 1, \ldots, 20.$$

1. Solve the profit maximization problem determining the optimal supply and an optimal price level by the following assumptions:

[37] The reference for the comparison can be the comparative analysis, presented in Sect. 5.4.3, of Cournot and Stackelberg duopoly models.

(a) $a(t) = -\frac{1}{t+1} + 2, \quad b(t) = 10, \quad \gamma(t) = 2,$
(b) $a(t) = 1, \quad b(t) = -0.025t^2 + 0.75t + 10, \quad \gamma(t) = 2,$
(c) $a(t) = 1, \quad b(t) = 10, \quad \gamma(t) = \frac{1}{t+1} + 1.$

2. Present trajectories of the optimal supply and of the optimal price.
3. Analyse how the optimal supply and the optimal price react to changes (*ceteris paribus*) in values of parameters describing the market: market capacity, consumers' sensitivity to changes in a product price, production marginal costs.

E15. Some monopolistic company considers discriminatory pricing for its product supplied to two different markets. The demand reported by consumers for i-th product ($i = 1, 2$) evolves according to a linear function of a form:

$$y_i^d(t) = -a_i(t)p_i(t) + b_i(t), \qquad a_i(t), b_i(t) > 0, \qquad i = 1, 2, \qquad \forall t \in [0; 20].$$

A function of production total cost is given as

$$c^{tot}\left(y^s(t)\right) = c(t)y^s(t) + d(t), \qquad c(t), d(t) > 0, \qquad t \in [0; 20],$$

where

$$c(t) = \frac{1}{t+1} + 1.$$

1. Solve the profit maximization problem determining the optimal supply and optimal levels of prices by the following assumptions:
   (a) $a_1(t) = 4 \cdot 0.98^t, \quad a_2(t) = -0.006t^2 + 0.1t + 4, \quad b_1(t) = b_2(t) = 15,$
   (b) $a_1(t) = a(t) = 4, \quad b_1(t) = 0.025t^2 - 0.5t + 15,$
   $b_2(t) = -0.025t^2 + 0.5t + 15,$
   (c) $a_1(t) = 4 \cdot 0.98^t, \quad a_2(t) = -0.006t^2 + 0.1t + 4,$
   $b_1(t) = 0.025t^2 - 0.5t + 15, \quad b_2(t) = -0.025t^2 + 0.5t + 15$
2. Present trajectories of the product optimal supplies intended for each of the markets and trajectories of the optimal prices of the product on each of the market.
3. Analyse how the optimal supplies and the optimal prices of the product supplied to both markets react to changes (*ceteris paribus*) in values of parameters describing the markets: market capacities, consumers' sensitivities to changes in a product price, production marginal cost. Analyse what the importance of differences in these values between two markets is.

E16. Two producers having equal positions act on a market of some homogeneous product. The demand for this product evolves according to a demand function:

$$y^d(p) = -ap + b, \qquad a, b > 0.$$

Production total costs for the first and for the second firm, respectively, are as follows:

$$c_1^{tot}(y_1) = c_1 y_1 + d_1, \qquad c_1, d_1 > 0,$$

$$c_2^{tot}(y_2) = c_2 y_2 + d_2, \qquad c_2, d_2 > 0.$$

The total output by both producers matches the demand for the product reported by consumers by a given price:

$$y_1 + y_2 = y^d(p).$$

1. Determine an equilibrium state in the Cournot duopoly model by the following assumptions:
   (a) $a = 2, \quad b = 20, \quad c_1 = 1, \quad c_2 = 1,$
   (b) $a = 2, \quad b = 20, \quad c_1 = 2, \quad c_2 = 1,$
   (c) $a = 2, \quad b = 200, \quad c_1 = 2, \quad c_2 = 1.$
2. Present a mechanism of reaching the equilibrium state when:
   (a) the first producer decides on the level of supply as first, assuming the competitor's supply equals 0,
   (b) the second producer decides on the level of supply as first, assuming the competitor's supply equals 0.

   State what the number of iterations in determining the levels of supply by each producer needed to reach the equilibrium state is.
3. Present trajectories of the optimal supplies by both producers.

E17. Two producers act on a market of some homogeneous product. The first of them has a position of the leader and the other a position of the follower. The demand for this product evolves according to a demand function:

$$y^d(p(t)) = -2p(t) + 20, \qquad a(t), b(t) > 0.$$

The total output by both producers matches the demand for the product reported by consumers by a given price:

$$y_1(t) + y_2(t) = y^d(p(t)).$$

Production total costs for the leader and for the follower, respectively, are as follows:

$$k_1^c(y_1(t)) = c_1(t)y_1 + 1,$$

$$k_2^c(y_2(t)) = 2.5y_2(t) + 1,$$

where

$$c_1(t) = -\frac{5}{16}t + 6\frac{1}{4}, \qquad \forall t \in [0; 20].$$

1. Determine an equation describing a line of reaction of the follower and an equilibrium state in the Stackelberg duopoly model when the production marginal cost for the leader varies in time. Present a graph of the follower's reaction line and the equilibrium states depending on changes in the marginal cost for the leader.
2. At which moment are marginal costs for the leader and for the follower equal? What are then their shares in the market of a product?
3. Present trajectories of the optimal supplies by the leader and by the follower.

E18. Two producers having equal positions on a market offer two substitute products. The demand for these products evolves according to the following demand functions:

$$y_1^d(p_1, p_2) = -a_1 p_1 + \gamma_1 p_2 + b_1,$$
$$y_2^d(p_1, p_2) = -a_2 p_2 + \gamma_2 p_1 + b_2, \qquad a_i, \gamma_i, b_i > 0, \qquad a_i > \gamma_i \quad i = 1, 2.$$

Production total costs for the first and for the second firm, respectively, are as follows:

$$c_1^{tot}(y_1) = c_1 y_1 + d_1, \qquad c_1, d_1 > 0,$$

$$c_2^{tot}(y_2) = c_2 y_2 + d_2, \qquad c_2, d_2 > 0.$$

An output level of each product matches the demand reported by consumers for this product by its given price:

$$y_i = y_i^d(p_1, p_2), \qquad i = 1, 2.$$

1. Determine an equilibrium state in the Cournot duopoly model by the following assumptions for $i = 1, 2$:
   (a) $a_i = 2, \quad b_i = 20, \quad \gamma_i = 1, \quad c_i = 1,$
   (b) $a_i = 3, \quad b_i = 20, \quad \gamma_i = 1, \quad c_i = 1,$
   (c) $a_i = 2, \quad b_i = 21, \quad \gamma_i = 1, \quad c_i = 1,$
   (d) $a_i = 2, \quad b_i = 20, \quad \gamma_i = 1.9, \quad c_i = 1,$
   (e) $a_i = 2, \quad b_i = 20, \quad \gamma_i = 1, \quad c_i = 2,$
   (f) $a_i = 3, \quad b_i = 21, \quad \gamma_i = 1.9, \quad c_i = 2.$
2. Present a mechanism of reaching the equilibrium state when:
   (a) the first producer decides on the level of a price as first, assuming a price of the competitor's product equals 0,

(b) the second producer decides on the level of a price as first, assuming a price of the competitor's product equals 0.

State what the number of iterations in determining the levels of a price by each producer needed to reach the equilibrium state is.

3. Present trajectories of the optimal prices of both producers' products.

6 Rationality of Choices Made by a Group of Producers and Consumers

In this chapter you will learn:

– what it means that the demand for a product and the supply of a product are described by functions determined exogenously or endogenously
– what is described in a Walrasian general equilibrium model with exogenous functions of demand and supply in a static approach and what in a dynamic approach
– what a net output space is and what its properties are
– what it means that in a general equilibrium Arrow-Debreu-McKenzie model functions of demand and supply are determined endogenously
– what is described in an Arrow-Debreu-McKenzie model in a static approach and what in a dynamic approach
– What the difference between discrete-time and continuous-time versions of a dynamic Arrow-Debreu-McKenzie model is.

In the first four chapters we considered situations of decision-making by individual consumers and producers or groups of them. In these analyses we accounted for the necessity of mutual adjustment of the demand and of the supply of products, consumer goods or production factors. Now we discuss a situation when a function of demand for one product or for two products and a function of supply of one product or of two products depend on prices of goods that are set by a market in conditions where global functions of demand for a product and global functions of supply result from rational decisions made independently by consumers and producers.

In Sects. 6.1 and 6.2 we present a static and a dynamic model of a market with exogenously determined functions of demand and of product supply. In Sects. 6.3 and 6.4 we discuss static and dynamic models of a market with endogenously determined functions of demand and of supply of products. We focus especially

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on vectors of Walrasian equilibrium prices and on a question of their asymptotic global stability. This chapter is a synthesis of the major conclusions and a specific culmination of the problems considered throughout book.

6.1 Static Market Model with Exogenous Functions of Supply and Demand

In our foregoing considerations we have concentrated on rational behaviour of individual consumers or producers. The rationality was identified with a situation in which every microeconomic agent is aware of the limitations occurring at a moment of making her/his decisions. Moreover, he/she is assumed to have the ability to formulate a decision-making criterion and ultimately to make a decision which, given the existing constraints, is assessed as the best, that is optimal. So far we have considered decision-making situations related mainly to determining the demand for a specific product or products by consumers, or the supply of a product or products by traders or producers. The key role in decision-making problems was played by prices of goods treated in the problems we considered as parameters or as decision variables. Identifying the optimal decision was connected to the concept of an equilibrium. The equilibrium was meant as a situation when a decision taken by a microeconomic agent is optimal.

Crucial problem we examined was the issue of balancing the supply and the demand for a specific product. The factor equalizing the demand and the supply of a given product was its price or prices of considered goods.

Let us first consider a market of a single good. Let us assume that a function of demand for the good is given, assumed to be decreasing in a price and for the sake of simplicity linear:

(6.1) $$y^d = -ap + b, \quad a, b > 0.$$

Parameter a can be seen as an amount by which the demand for the good approximately declines when its price is raised by one (notional) money unit and we call it a measure of **consumers' sensitivity to changes in a commodity price**. Parameter b can be seen as a demand level for the good when its price equals 0 and regarding the demand for the good we call it a measure of **market capacity**.

The demand function is decreasing in a price of the good since:

(6.2) $$\frac{\mathrm{d}y^d}{\mathrm{d}p} = -a < 0,$$

which means that when a price of the commodity is raised by one (notional) money unit then the demand for the good declines by a physical units.

Let us notice that the commodity price takes values in an interval:

(6.3) $$p \in \left[0; \frac{b}{a}\right],$$

and for the maximum price level $p = \frac{b}{a}$ the demand for the good equals zero. The demand for the good takes values in an interval:

$$y^d \in [0; b] \tag{6.4}$$

Let us also assume that the supply of the good is described by a linear supply function:

$$y^s = cp + d, \quad c, d > 0, \tag{6.5}$$

which is increasing in the price of the good:

$$\frac{\mathrm{d}y^s}{\mathrm{d}p} = c > 0. \tag{6.6}$$

Parameter c can be seen as an amount by which the supply of the good approximately increases when its price is raised by one (notional) money unit and we call it a measure of **reaction to changes in a commodity price**. Parameter d describes the supply of the good at zero price. We can interpret it as the minimum supply of a good and identify with a **level of stocks**.

Let us notice that for the commodity supply function (6.5) the price of the good may take values in an interval:

$$p \in [0; +\infty) \tag{6.7}$$

and the commodity supply in an interval:

$$y^s \in [d; +\infty). \tag{6.8}$$

Thus, the supply and the price of the good are constrained downwards and unconstrained upwards. So far we have considered situations in which optimal decisions made by microeconomic agents were constrained simultaneously by the demand and by the supply of a good or goods. In other words, the constraint on the demand is the supply, and the constraint on the supply of a good or goods is the demand. A natural question arises, namely, what a mechanism of demand and supply adjustment triggered by changes in prices of goods is.

Is there a price $\bar{p} > 0$ of a given good by which the demand and the supply of this good, both expressed in the same physical units, are equal? To find the answer to this question it is enough to solve an equation:

$$y^d = -ap + b = cp + d = y^s. \tag{6.9}$$

After simple transformations one gets a price level that equalizes the demand and the supply for a given good:

(6.10) $$\bar{p} = \frac{b - d}{a + c}.$$

Due to the assumed forms of the demand and the supply functions if $b > d$ then the equilibrium price $\bar{p} > 0$ exists and is defined uniquely. This means that the market capacity has to be greater than the level of stocks, otherwise a firm will decide to stop the production. We can also notice that the acceptable price level of a good is determined by condition (6.3) regarding the demand and by condition (6.7) regarding the supply of a good. This means that the equilibrium price, if it exists, belongs to interval $\left[0; \frac{b}{a}\right]$ which is an intersection of sets defined by conditions (6.3) and (6.7). It is not difficult to notice that the equilibrium price determined in condition (6.10) belongs to this condition.

Definition 6.1 A **function of excess demand** is mapping $z\colon \mathrm{int}\,\mathbb{R}_+ \to \mathbb{R}$ which assigns an excess demand for a good to any price $p > 0$ of this product:

(6.11) $$z(p) = y^d(p) - y^s(p).$$

Definition 6.2 A **Walrasian equilibrium price** is a level $\bar{p} > 0$ of a commodity price such that

(6.12) $$z(\bar{p}) = 0 \quad \Leftrightarrow \quad y^d(\bar{p}) = y^s(\bar{p}),$$

meaning that the demand and the supply of the good both expressed in the same physical units are equal.

We know that when $b > d$ then the Walrasian equilibrium price exists and is determined uniquely. Otherwise the equilibrium price does not exist.

Let us notice that the excess demand function defined in condition (6.11) is not homogeneous of degree zero:

(6.13) $$\neg\left[\forall p > 0, \forall \lambda > 0 \quad z(\lambda p) = \lambda^0 z(p) = z(p)\right],$$

since when a price of a given good changes then the excess demand for this good also changes. This property results from the assumption that the demand function and the commodity supply function are linear.

For the excess demand function (6.12) Walras's law is satisfied only for the equilibrium price:

(6.14) $$\exists \bar{p} > 0 \quad \bar{p}z(\bar{p}) = 0 \quad \Leftrightarrow \quad \bar{p}y^d(\bar{p}) = \bar{p}y^s(\bar{p}),$$

which means that a value of the global demand for a good is equal to the value of the global supply of this good.

However, for any positive price of a good which is not an equilibrium price Walras's law is not satisfied:

(6.15)
$$\forall p > 0, p \neq \bar{p} \quad pz(p) \neq 0 \quad \Leftrightarrow \quad py^d(p) \neq py^s(p) \quad \Leftrightarrow \quad y^d(p) \neq y^s(p).$$

Let us now consider a market of two goods and denote:

$i = 1, 2$—an index of goods,
$\mathbf{p} = (p_1, p_2) > (0, 0)$—a vector of prices of goods,
$\mathbf{y}^d(\mathbf{p}) = (y_1^d(\mathbf{p}), y_2^d(\mathbf{p})) \in \mathbb{R}_+^2$—a vector function of demand for two goods,
$\mathbf{y}^s(\mathbf{p}) = (y_1^s(\mathbf{p}), y_2^s(\mathbf{p})) \in \mathbb{R}_+^2$—a vector function of supply of two goods.

Let us assume that the function of demand for the first (second) good is decreasing in a price of this good and does not depend on a price of the second (first) good[1]:

(6.16)
$$y_1^d(\mathbf{p}) = -a_1 p_1 + b_1, \quad a_1, b_1 > 0,$$

(6.17)
$$y_2^d(\mathbf{p}) = -a_2 p_2 + b_2, \quad a_2, b_2 > 0.$$

From the system of Eqs. (6.16)–(6.17) it follows that

(6.18)
$$p_i \in \left[0; \frac{b_i}{a_i}\right], \quad i = 1, 2.$$

At the same time, by a given price interval for i-th good, a level of demand for this good belongs to an interval:

(6.19)
$$y_i^d(\mathbf{p}) \in [0; b_i], \quad i = 1, 2.$$

Let us also assume that the function of supply of the first (second) good is increasing in a price of this good and does not depend on a price of the second (first) good:

(6.20)
$$y_1^s(\mathbf{p}) = c_1 p_1 + d_1, \quad c_1, d_1 > 0,$$

(6.21)
$$y_2^s(\mathbf{p}) = c_2 p_2 + d_2, \quad c_2, d_2 > 0.$$

[1] This means that these two goods are independent of each other and therefore neither complementary nor substitute.

The system of Eqs. (6.20)–(6.21) shows that by positive values of parameters of the supply function of i-th good its price level should be non-negative. As a result the minimum supply of i-th good is

$$d_i \geq 0, \ i = 1, 2, \tag{6.22}$$

and a price level of i-th good satisfies a constraint resulting from the assumed form of the demand function for i-th good.

A function of excess demand for i-th good takes the form:

$$\begin{aligned} z_i(\mathbf{p}) &= y_i^d(\mathbf{p}) - y_i^s(\mathbf{p}) \\ &= -(a_i + c_i)p_i + b_i - d_i, \quad a_i, b_i, c_i, d_i > 0, \quad i = 1, 2. \end{aligned} \tag{6.23}$$

If for each good ($i = 1, 2$) it is satisfied that $b_i > d_i > 0$ then there exists a price vector of the Walrasian equilibrium:

$$\bar{\mathbf{p}} = \left(\frac{b_1 - d_1}{a_1 + c_1}, \frac{b_2 - d_2}{a_2 + c_2} \right) > (0, 0), \tag{6.24}$$

being a solution to a system of equations:

$$z_i(\mathbf{p}) = y_i^d(\mathbf{p}) - y_i^s(\mathbf{p}) = -(a_i + c_i)p_i + (b_i - d_i) = 0, \quad i = 1, 2, \tag{6.25}$$

by the assumption: $b_i > d_i > 0, \ a_i, c_i > 0 \quad i = 1, 2.$

Let us notice that the excess demand function:

- is not homogeneous of degree zero,
- satisfies Walras's law only for the Walrasian equilibrium price vector defined by the set of conditions (6.24).

In the examples discussed above we have shown that for a market of a single good, as well as for a market of two independent goods, there exists exactly one equilibrium price vector for which the supply and the demand for each of the goods, both expressed in the same physical units, are equal. We have also explained that Walras's law is satisfied only for the Walrasian equilibrium price vector.

Still there is a question of a mechanism that leads to a situation when starting from some initial price level of a good or from a system of goods' prices, one can generate the Walrasian equilibrium price system. In order to answer this question let us analyse the dynamic version of the market model of two goods.

6.2 Dynamic Market Model of Two Goods with Exogenous Functions of Supply and Demand

Let us take the following notation:

$i = 1, 2$—an index of goods,
$t = 0, 1, 2, ..., T$—time as a discrete variable,
$t \in [0; T]$—time as a continuous variable,
$T \to +\infty$—an upper limit of the considered time horizon,
$\mathrm{p}(t) = (p_1(t), p_2(t)) > (0, 0)$—a vector of prices of goods in period/at moment t,
$\mathbf{y}^d(\mathbf{p}(t)) = (y_1^d(\mathbf{p}(t)), y_2^d(\mathbf{p}(t))) \in \mathbb{R}_+^2$—a value of a vector function of demand for goods in period/at moment t,
$\mathbf{y}^s(\mathbf{p}(t)) = (y_1^s(\mathbf{p}(t)), y_2^s(\mathbf{p}(t))) \in \mathbb{R}_+^2$—a value of a vector function of supply of goods in period/at moment t.

Let us assume that the demand function for the first (second) good in period/at moment t is decreasing in a price of this good and does not depend on a price of the second (first) good[2]:

(6.26) $$y_1^d(\mathbf{p}(t)) = -a_1 p_1(t) + b_1, \quad a_1, b_1 > 0,$$

(6.27) $$y_2^d(\mathbf{p}(t)) = -a_2 p_2(t) + b_2, \quad a_2, b_2 > 0.$$

From the system of Eqs. (5.26)–(5.27) it results that in any period/at any moment t:

(6.28) $$p_i(t) \in \left[0; \frac{b_i}{a_i}\right], \ i = 1, 2.$$

At the same time, by a given interval of values for i-th good's price, the level of the demand for this good in any period/at any moment t belongs to an interval:

(6.29) $$y_i^d(\mathbf{p}(t)) \in [0; b_i], \ i = 1, 2.$$

Let us also assume that the function of the first (second) good's supply in period/at moment t is increasing in a price of this good and does not depend on a price of the second (first) good:

(6.30) $$y_1^s(\mathbf{p}(t)) = c_1 p_1(t) + d_1, \quad c_1, d_1 > 0,$$

[2] This means that these two goods are independent of each other and therefore neither complementary nor substitute.

(6.31) $$y_2^s(\mathbf{p}(t)) = c_2 p_2(t) + d_2, \quad c_2, d_2 > 0.$$

From the system of Eqs. (6.30)–(6.31) it results that when the function of the supply of i-th good has positive values of its parameters then i-th good's price level in any period/at any moment t should be non-negative. As a result the minimum supply of i-th good in any period/at any moment t equals:

(6.32) $$d_i \geq 0, \; i = 1, 2,$$

and the price level of i-th good in any period/at any moment t satisfies the constraint resulting from the assumed form of the demand function for i-th good.

The function of excess demand for i-th good in period/at moment t takes the form:

(6.33) $$\begin{aligned} &z_i(\mathbf{p}(t)) = y_i^d(\mathbf{p}(t)) - y_i^s(\mathbf{p}(t)) = -(a_i + c_i)p_i(t) + b_i - d_i, \\ &a_i, b_i, c_i, d_i > 0, \quad i = 1, 2. \end{aligned}$$

If for each good ($i = 1, 2$) it is satisfied that $b_i > d_i > 0$ then there exists the equilibrium price vector:

(6.34) $$\bar{\mathbf{p}} = \left(\frac{b_1 - d_1}{a_1 + c_1}, \frac{b_2 - d_2}{a_2 + c_2} \right) > (0, 0),$$

being a solution to a system of equations:

(6.35)
$$z_i(\mathbf{p}(t)) = y_i^d(\mathbf{p}(t)) - y_i^s(\mathbf{p}(t)) = -(a_i + c_i)p_i(t) + (b_i - d_i) = 0, \quad i = 1, 2,$$

by the assumption: $b_i > d_i > 0, a_i, c_i > 0 \quad i = 1, 2$.

Definition 6.3 A dynamic discrete-time model of a market of two goods with exogenous functions of the supply and of the demand is a system of difference equations:

(6.36) $$p_i(t + 1) = p_i(t) + \sigma_i z_i(\mathbf{p}(t)), \quad i = 1, 2,$$

with an initial condition:

(6.37) $$p_i(0) = \text{const.} > 0, \quad i = 1, 2,$$

where $\sigma_i > 0$ ($i = 1, 2$) denotes an arbitrarily determined measure of the sensitivity of a market of i-th good to the imbalance occurring there.

Let us notice that:

(6.38)
$$z_i(\mathbf{p}(t)) = 0 \quad \Leftrightarrow \quad y_i^d(\mathbf{p}(t)) = y_i^s(\mathbf{p}(t)) \quad \Rightarrow \quad p_i(t+1) = p_i(t), \quad i = 1, 2,$$

which means that on a market of i-th good or both goods there is no need to change prices because the price $p_i(t) > 0$ equalizes the supply and the demand for the i-th good both expressed in the same physical units. If such a situation concerns only one good, then we call it a **partial equilibrium**, and when it concerns both goods we call it a **general equilibrium** on markets of both goods.

$$\begin{aligned} &z_i(\mathbf{p}(t)) > 0 \Leftrightarrow y_i^d(\mathbf{p}(t)) > y_i^s(\mathbf{p}(t)) \\ &\Rightarrow p_i(t+1) > p_i(t), \quad i = 1, 2, \end{aligned} \tag{6.39}$$

which means that on a market of i-th good or both goods prices should be changed because the price $p_i(t) > 0$ does not equalize the supply and the demand for the i-th good both expressed in the same physical units. Because the demand exceeds the supply then the price of i-th good in period $t+1$ should be raised in comparison to its level in period t.

$$\begin{aligned} &z_i(\mathbf{p}(t)) < 0 \Leftrightarrow y_i^d(\mathbf{p}(t)) < y_i^s(\mathbf{p}(t)) \\ &\Rightarrow p_i(t+1) < p_i(t), \quad i = 1, 2, \end{aligned} \tag{6.40}$$

which means that on a market of i-th good or both goods prices should be changed because the price $p_i(t) > 0$ does not equalize the supply and the demand for the i-th good both expressed in the same physical units. Because the supply exceeds the demand then the price of i-th good in period $t+1$ should be reduced in comparison to its level in period t.

The system of Eqs. (6.36)–(6.37) uniquely defines a recursive rule for determining a price of i-th good in subsequent periods until equilibrium prices are reached on both markets. It is worth emphasizing that in subsequent periods the price of i-th good is determined uniquely. Nevertheless, we are interested only in such solutions to the system of difference Eqs. (6.36)–(6.37) in which prices of both goods are positive in any period.

Definition 6.4 A **feasible trajectory of prices** of i-th good in the dynamic discrete-time model of two goods is an infinite sequence of solutions to the difference equations system (6.36) with an initial condition $p_i(0) = \text{const.} > 0$ such that $\forall t = 0, 1, 2, \ldots \; p_i(t+1) > 0$.

The system of Eqs. (6.36)–(6.37) does not guarantee the existence of the equilibrium price vector $\bar{\mathbf{p}} > \mathbf{0}$. Additional conditions should be defined to ensure the existence and uniqueness of the equilibrium price vector, which is beyond the scope of this book. However, let us introduce one more concept that is related to the stability of the equilibrium state, if such a state exists.

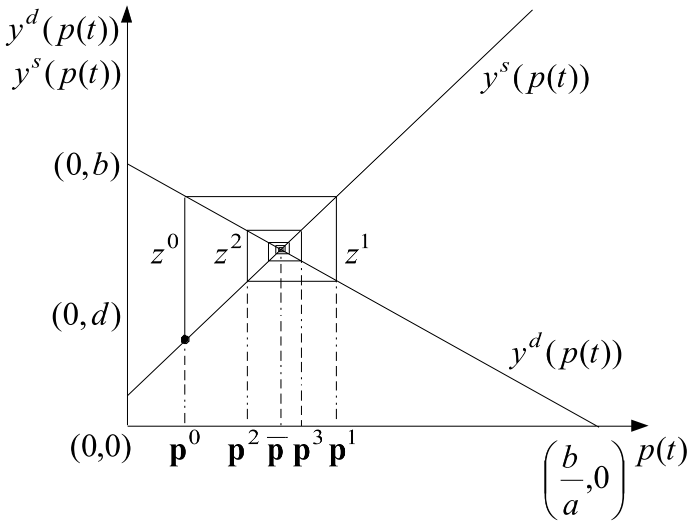

Fig. 6.1 Global asymptotic stability of Walrasian equilibrium price vector

Definition 6.5 A Walrasian equilibrium state $\bar{\mathbf{p}} > \mathbf{0}$in the dynamic discrete-time model of two goods is called **asymptotically globally stable** when a feasible trajectory $\mathbf{p}(t)$ of goods' prices satisfies a condition (Fig. 6.1):

(6.41) $$\lim_{t \to +\infty} \mathbf{p}(t+1) = \bar{\mathbf{p}}.$$

Definition 6.6 A dynamic continuous-time model of a market of two goods with exogenous functions of the supply and of the demand is a system of differential equations:

(6.42) $$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = \sigma_i z_i(\mathbf{p}(t)), i = 1, 2,$$

with an initial condition:

(6.43) $$p_i(0) = \text{const.} > 0, \quad i = 1, 2,$$

where $\sigma_i > 0$ ($i = 1, 2$) denotes an arbitrarily determined measure of the sensitivity of a market of i-th good to the imbalance occurring there.

Let us notice that

(6.44) $$z_i(\mathbf{p}(t)) = 0 \quad \Leftrightarrow \quad y_i^d(\mathbf{p}(t)) = y_i^s(\mathbf{p}(t)) \quad \Rightarrow \quad \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = 0, \; i = 1, 2,$$

which means that on a market of i-th good or both goods there is no need to change prices because the price $p_i(t) > 0$ equalizes the supply and the demand for the i-th good both expressed in the same physical units. If such a situation concerns only one good, then we call it a **partial equilibrium**, and when it concerns both goods we call it a **general equilibrium** on markets of both goods.

$$(6.45) \quad z_i(\mathbf{p}(t)) > 0 \quad \Leftrightarrow \quad y_i^d(\mathbf{p}(t)) > y_i^s(\mathbf{p}(t)) \quad \Rightarrow \quad \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} > 0, \; i = 1, 2,$$

which means that on a market of i-th good or both goods prices should be changed because the price $p_i(t) > 0$ does not equalize the supply and the demand for the i-th good both expressed in the same physical units. Because the demand exceeds the supply then the price of i-th good at moment $t + \Delta t$ should be raised in comparison to its level at moment t.

$$(6.46) \quad z_i(\mathbf{p}(t)) < 0 \quad \Leftrightarrow \quad y_i^d(\mathbf{p}(t)) < y_i^s(\mathbf{p}(t)) \quad \Rightarrow \quad \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} < 0, \; i = 1, 2,$$

which means that on a market of i-th good or both goods prices should be changed because the price $p_i(t) > 0$ does not equalize the supply and the demand for the i-th good both expressed in the same physical units. Because the supply exceeds the demand then the price of i-th good at moment $t + \Delta t$ should be reduced in comparison to its level at moment t.

The system of Eqs. (6.42)–(6.43) uniquely defines a recursive rule for determining a price of i-th good in subsequent moments until equilibrium prices are reached on both markets. It is worth emphasizing that in subsequent moments the price of i-th good is determined uniquely. Nevertheless, we are interested only in such solutions to the system of differentia Eqs. (6.42)–(6.43) in which prices of both goods are positive at any moment.

Definition 6.7 A **feasible trajectory of prices** of i-th good in the dynamic continuous-time model of two goods is an infinite sequence of solutions to the difference equations system (6.42) with an initial condition $p_i(0) = \text{const.} > 0$ such that $\forall t \in [0; +\infty) \, p_i(t + \Delta t) > 0$.

The system of Eqs. (6.42)–(6.43) does not guarantee the existence of the equilibrium price vector $\bar{\mathbf{p}} > \mathbf{0}$. Additional conditions should be defined to ensure the existence and uniqueness of the equilibrium price vector, which is beyond the scope of this book. However, let us also discuss the concept of the global asymptotic stability of the equilibrium state, if such a state exists.

Definition 6.8 A Walrasian equilibrium state$\bar{\mathbf{p}} > \mathbf{0}$ in the dynamic continuous-time model of two goods is called **asymptotically globally stable** when a feasible trajectory $\mathbf{p}(t)$ of goods' prices satisfies a condition:

$$\lim_{\substack{t\to+\infty\\ \Delta t\to 0}} \mathbf{p}(t+\Delta t) = \bar{\mathbf{p}}. \tag{6.47}$$

6.3 Static Arrow-Debreu-McKenzie Model[3]

Let us describe an economy in which we distinguish the part related to production and the part related to exchange (consumption).[4] Let us use the following notation:

$i = 1, 2$—an index of products which, depending on their intended use, may be consumer goods or production factors,
$k = 1, 2$—an index of (traders) consumers,
$j = 1, 2$—an index of producers,
$X^k \subseteq \mathbb{R}^2_+$—a goods space (a set of all bundles of goods available on a market with a metric $d_{NE}:\mathbb{R}^4_+ \to \mathbb{R}_+$ specified on this set, see Definition 1.2),
$Y^j \subseteq \mathbb{R}^2$—a production space of j-th producer in which the result of production is described by a vector of the net output.

Parameters

α_{kj}—a share of k-th consumer in the profit of j-th producer,
$\mathbf{a}^k = (a_{k1}, a_{k2}) \in \mathbb{R}^2_+$—an initial consumption bundle (initial endowment) of k-th consumer,

Variables

$\mathbf{p} = (p_1, p_2) \in \operatorname{int}\mathbb{R}^2_+$—a vector of prices of products (consumer goods or production factors).
$\mathbf{x}^k = (x_{k1}, x_{k2}) \in \mathbb{R}^2_+$—a vector of the demand for goods reported by k-th consumer,
$\mathbf{w}^j = (w_{j1}, w_{j2}) \in \mathbb{R}^2_+$—a vector of outputs in the production process of j-th producer,
$\mathbf{n}^j = (n_{j1}, n_{j2}) \in \mathbb{R}^2_+$—a vector of inputs of production factors in the production process of j-th producer,
$\mathbf{y}^j = \mathbf{w}^j - \mathbf{n}^j = (w_{j1} - n_{j1}, w_{j2} - n_{j2}) = (y_{j1}, y_{j2}) \in Y^j \subseteq \mathbb{R}^2$—a vector of the j-th producer's net output,
$\Pi_j(p_1, p_2)$—j-th producer's profit as a function of prices of products,

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[3] The Arrow-Debreu-McKenzie model should be interpreted as a general equilibrium model with endogenous functions of the supply and of the demand.
[4] A comprehensive extension of the issues discussed in this part of the chapter can be found in work (Panek, 2003).

$I^k(p_1, p_2) = \sum_{i=1}^{2} p_i a_{ki} + \sum_{j=1}^{2} \alpha_{kj} \Pi_j(p_1, p_2) > 0$—$k$-th consumer's income as a function of prices of goods,
$u^k : \mathbb{R}_+^2 \to \mathbb{R}$—an utility function of k-th consumer, describing her/his preferences (a relation of preferences),
$D^k(\mathbf{p}, I^k(\mathbf{p})) = \left\{(x_{k1}, x_{k2}) \in \mathbb{R}_+^2 \middle| p_1 x_{k1} + p_2 x_{k2} \le I^k(p_1, p_2)\right\} \subset X = \mathbb{R}_+^2$—a set of all consumption bundles of a value not exceeding k-th consumer's income.

Production

Each producer ($j = 1, 2$) wants to maximize the profit from sales of manufactured products. Each one is assumed to carry out a production process described by a vector of the net output:

$$\begin{aligned} \mathbf{y}^j &= \mathbf{w}^j - \mathbf{n}^j = \left(w_{j1} - n_{j1}, w_{j2} - n_{j2}\right) \\ &= \left(y_{j1}, y_{j2}\right) \in Y^j \subseteq \mathbb{R}^2, \ \ j = 1, 2, \end{aligned} \tag{6.48}$$

where:

$\mathbf{w}^j = (w_{j1}, w_{j2}) \in \mathbb{R}_+^2$—a vector of outputs in the production process of j-th producer,
$\mathbf{n}^j = (n_{j1}, n_{j2}) \in \mathbb{R}_+^2$—a vector of inputs of production factors in the production process of j-th producer,
$Y^j \subseteq \mathbb{R}^2$—a production space of j-th producer in which the result of production is described by a vector of the net output.

Let us notice that if the production process carried out by j-th producer is technologically feasible, that is:

$$\mathbf{z}^j = \left(\mathbf{n}^j, \mathbf{w}^j\right) \in Z \subseteq \mathbb{R}^4, \quad j = 1, 2, \tag{6.49}$$

then in this process, described in terms of the net output, the following situations are possible:

$$n_{ji} = w_{ji} = 0 \quad \Rightarrow \quad y_{ji} = 0, \tag{6.50}$$

i-th product is neither the input nor the output in the production process and its net output equals zero;

$$n_{ji} = w_{ji} > 0 \quad \Rightarrow \quad y_{ji} = 0, \tag{6.51}$$

i-th product is both the input and the output in the production process, but its net output equals zero;

(6.52) $$n_{ji} > 0, \quad w_{ji} = 0 \quad \Rightarrow \quad y_{ji} < 0,$$

i-th product is only the input in the production process, thus its net output in this production process is negative;

(6.53) $$n_{ji} = 0, \quad w_{ji} > 0 \quad \Rightarrow \quad y_{ji} > 0,$$

i-th product is only the output in the production process, therefore its net output in this production process is positive;

(6.54) $$n_{ji}, w_{ji} > 0 \quad \wedge \quad n_{ji} > w_{ji} \quad \Rightarrow \quad y_{ji} < 0,$$

i-th product is both the input and the output in the production process, and its net output is negative;

(6.55) $$n_{ji}, w_{ji} > 0 \quad \wedge \quad n_{ji} < w_{ji} \Rightarrow y_{ji} > 0,$$

i-th product is both the input and the output in the production process, and its net output is positive.

Definition 6.9 A **net output space** for j-th producer is a set of all net output vectors $\mathbf{y}^j = \mathbf{w}^j - \mathbf{n}^j = \left(y_{j1}, y_{j2}\right) \in Y^j \subseteq \mathbb{R}^2$ which are the results of technologically feasible production processes $\mathbf{z}^j = \left(\mathbf{n}^j, \boldsymbol{w}^j\right) \in Z \subseteq \mathbb{R}^4$, $j = 1, 2$ carried out by j-th producer. This set is defined with a norm $\|\mathbf{y}^j\|_{NE} = \max\limits_{i=1,2}\left|y_{ji}\right|$.

Assumption 6.1 About net output spaces we assume:

(1) $\mathbf{0} \in Y^j \subseteq \mathbb{R}^2$,
which means that if j-th producer aiming at profit maximization took losses then he/she could quit the production process in her/his company.

(2) $Y^j \cap \mathbb{R}^2_+ = \mathbf{0}$,
which means that if one of the coordinates of the vector $\mathbf{y}^j \in Y^j \subseteq \mathbb{R}^2$ is positive then the other has to be negative since it is impossible to have positive net output without taking any inputs.

(3) $Y^j \subseteq \mathbb{R}^2$ is a bounded and closed set (hence a compact set),
which means that we are interested in production processes in which inputs and outputs are finite. The fact that $Y^j \subseteq \mathbb{R}^2$ is a closed set can be interpreted so that a limit of any sequence of net output vectors belonging to Y^j belongs to this set too.

(4) $\forall \mathbf{y}^1, \mathbf{y}^2 \in Y^j \subseteq \mathbb{R}^2,\ \mathbf{y}^1 \neq \mathbf{y}^2,\ \forall \alpha, \beta > 0,\ \alpha + \beta = 1,\ \ \mathbf{z} = \alpha \mathbf{y}^1 + \beta \mathbf{y}^2 \in Y^j$ and

$$\exists \varepsilon > 0 \quad U_\varepsilon(\mathbf{z}) = \left\{ \mathbf{y} \in Y^j \middle| d_{NE}(\mathbf{y}, \mathbf{z}) < \varepsilon \right\} \subseteq Y^j,$$

which means that net output spaces of all producers are strictly convex sets in which production processes exhibit decreasing returns to scale.

By given prices $\mathbf{p} = (p_1, p_2) \in \operatorname{int} \mathbb{R}^2_+$ of products a profit maximization problem of j-th producer takes the form:

(6.56) $$p_1 y_{j1} + p_2 y_{j2} \to \max$$

(6.57) $$\mathbf{y}^j = \left(y_{j1}, y_{j2} \right) \in Y^j \subseteq \mathbb{R}^2.$$

The problem (6.56)–(6.57) is a nonlinear programming problem with a linear profit function and the compact strictly convex set $Y^j \subseteq \mathbb{R}^2$ of feasible solutions. Due to assumed properties of the j-th producer net output space this problem has a unique solution by a given vector $\mathbf{p} = (p_1, p_2) \in \operatorname{int} \mathbb{R}^2_+$ of prices of products that is the optimal net output vector $\bar{\mathbf{y}}^j = \left(\bar{y}_{j1}, \bar{y}_{j2} \right) \in Y^j \subseteq \mathbb{R}^2$ ensuring the maximum profit for j-the producer.

Definition 6.10 A **supply function** of j-th producer is a mapping $\bar{\mathbf{y}}^j : \mathbb{R}^2_+ \to \mathbb{R}^2$ which assigns an optimal vector of net output $\bar{\mathbf{y}}^j(\mathbf{p}) = \left(\bar{y}_{j1}(\mathbf{p}), \bar{y}_{j2}(\mathbf{p}) \right) \in Y^j \subseteq \mathbb{R}^2$ being the solution in the j-th producer's profit maximization problem (6.56)–(6.57) to any vector $\mathbf{p} = (p_1, p_2) \in \operatorname{int} \mathbb{R}^2_+$ of prices of products.

Theorem 6.1 If Assumption 6.1 is satisfied then $\forall \mathbf{p} > \mathbf{0}$ the supply function of j-th producer is:

(a) continuous and differentiable,
(b) homogeneous of degree zero:

(6.58) $$\forall \lambda > 0 \quad \bar{\mathbf{y}}^j(\lambda \mathbf{p}) = \lambda^0 \bar{\mathbf{y}}^j(\mathbf{p}) = \bar{\mathbf{y}}^j(\mathbf{p}),$$

which means that a proportional change in the prices of products does not lead to a change in the vector of the j-th producer's net output.

Definition 6.11 A **maximal profit function** of j-th producer is a mapping $\Pi^j : \operatorname{int} \mathbb{R}^2_+ \to \mathbb{R}$ which assigns the maximum profit $\Pi(\bar{\mathbf{y}}^j(\mathbf{p})) = p_1 \bar{y}_{j1}(\mathbf{p}) + p_2 \bar{y}_{j2}(\mathbf{p}) \in \mathbb{R}$ of j-th producer to any vector $\mathbf{p} = (p_1, p_2) \in \operatorname{int} \mathbb{R}^2_+$ of prices of products (Fig. 6.2).

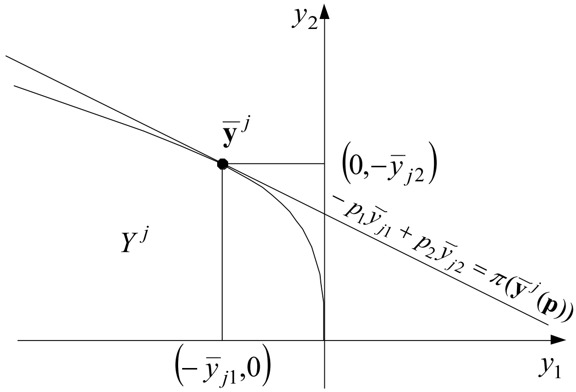

Fig. 6.2 Maximization problem of j-th producer

Theorem 6.2 If Assumption 6.1 is satisfied then $\forall \mathbf{p} > \mathbf{0}$ the profit function of j-th producer is.

(a) continuous and differentiable,
(b) positively homogeneous of first degree:

$$\forall\lambda > 0 \quad \Pi\left(\bar{\mathbf{y}}^j(\lambda\mathbf{p})\right) = \lambda^1\Pi\left(\bar{\mathbf{y}}^j(\mathbf{p})\right) = \lambda\Pi\left(\bar{\mathbf{y}}^j(\mathbf{p})\right), \tag{6.59}$$

which means that a proportional change in prices of products leads to the proportional change in the j-th producer's maximum profit.

Consumption

The k-th consumer wants purchase such a bundle of goods $\bar{\mathbf{x}}^k = (\bar{x}_{k1}, \bar{x}_{k2})$ whose value does not exceed the income of k-th consumer and whose utility is maximum and at the same time not less than utility of the initial bundle of goods $\mathbf{a}^k = (a_{k1}, a_{k2})$. By a given vector $\mathbf{p} = (p_1, p_2) \in \operatorname{int}\mathbb{R}^2_+$ of prices of goods the consumption utility maximization problem of k-th consumer can be written as

$$u^k(x_{k1}, x_{k2}) \mapsto \max, \tag{6.60}$$

$$p_1x_{k1} + p_2x_{k2} \leq I^k(p_1, p_2), \tag{6.61}$$

$$x_{k1}, x_{k2} \geq 0, \tag{6.62}$$

where $I^k(p_1, p_2) = p_1a_{k1} + p_2a_{k2} + \alpha_{k1}\Pi^1(\bar{\mathbf{y}}^1(p_1, p_2)) + \alpha_{k2}\Pi^2(\bar{\mathbf{y}}^2(p_1, p_2))$ denotes the income of k-th consumer.

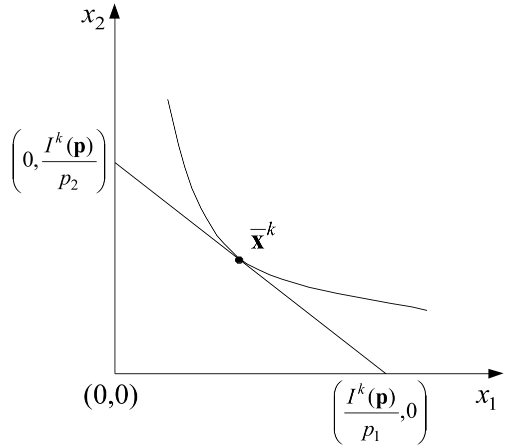

Fig. 6.3 Consumption utility maximization problem of k-th consumer

Assumption 6.2 The utility function of k-th consumer $u^k: \mathbb{R}^2_+ \to \mathbb{R}$ is increasing, differentiable, and strictly concave.

Then the problem (6.60)–(6.62) has exactly one optimal solution which belongs to the budget line and is of a form (Fig. 6.3):

$$\bar{\mathbf{x}}^k(\mathbf{p}) = \varphi^k\left(\mathbf{p}, I^k(\mathbf{p})\right) = \left(\alpha^k \frac{I^k(\mathbf{p})}{p_1}, \beta^k \frac{I^k(\mathbf{p})}{p_2}\right) > (0, 0),$$
$$\forall \alpha^k, \beta^k > 0, \quad \alpha^k + \beta^k = 1, \quad k = 1, 2. \tag{6.63}$$

Definition 6.12 A **demand function** of k-th consumer is a mapping $\varphi^k: \mathrm{int}\,\mathbb{R}^2_+ \to \mathrm{int}\,\mathbb{R}^2_+$ which assigns the optimal solution in the k-th consumer's consumption utility maximization problem (6.60)–(6.62) to any vector $\mathbf{p} = (p_1, p_2) \in \mathrm{int}\,\mathbb{R}^2_+$ of prices of goods:

$$\varphi^k(\mathbf{p}) = \varphi^k\left(\mathbf{p}, I^k(\mathbf{p})\right) = \left(\varphi_{k1}\left(\mathbf{p}, I^k(\mathbf{p})\right), \varphi_{k2}\left(\mathbf{p}, I^k(\mathbf{p})\right)\right)$$
$$= \bar{\mathbf{x}}^k(\mathbf{p}) = (\bar{x}_{k1}(\mathbf{p}), \bar{x}_{k2}(\mathbf{p})), \; k = 1, 2. \tag{6.64}$$

Theorem 6.3 If Assumption 6.2 is satisfied then $\forall \mathbf{p} > \mathbf{0}$ the demand function of k-th consumer is:

(a) continuous and differentiable,
(b) homogeneous of degree zero:

$$\forall \lambda > 0 \quad \bar{\mathbf{x}}^k(\lambda \mathbf{p}) = \lambda^0 \bar{\mathbf{x}}^k(\mathbf{p}) = \bar{\mathbf{x}}^k(\mathbf{p}), \tag{6.65}$$

which means that a proportional change in prices of goods does not lead to a change in the demand reported by k-th consumer.

Definition 6.13 An **indirect function of consumption utility** of k-th consumer is a mapping ν^k:$\operatorname{int}\mathbb{R}^2_+ \to \operatorname{int}\mathbb{R}_+$ which assigns the utility of the optimal bundle of goods being the solution in the k-th consumer's consumption utility maximization problem (5.60)–(5.62) to any vector $\mathbf{p} = (p_1, p_2) \in \operatorname{int}\mathbb{R}^2_+$ of prices of goods:

(6.66)
$$\nu^k(\mathbf{p}) = u^k(\bar{x}_{k1}(\mathbf{p}), \bar{x}_{k2}(\mathbf{p})), \quad k = 1, 2.$$

Theorem 6.4 If Assumption 6.2 is satisfied then $\forall \mathbf{p} > \mathbf{0}$ the indirect function of k-th consumer's utility is:

(a) continuous and differentiable,
(b) homogeneous of degree zero:

(6.67)
$$\forall \lambda > 0 \quad \nu^k(\lambda\mathbf{p}) = \lambda^0 \nu^k(\mathbf{p}) = \nu^k(\mathbf{p}),$$

which means that a proportional change in prices of goods does not lead to a change in the utility of the optimal bundle of k-th consumer.

General equilibrium

Definition 6.14 A **function of global demand** is an expression:

(6.68)
$$\bar{\mathbf{x}}(\mathbf{p}) = \bar{\mathbf{x}}^1(\mathbf{p}) + \bar{\mathbf{x}}^2(\mathbf{p}) = \begin{pmatrix} \bar{x}_{11}(\mathbf{p}) + \bar{x}_{21}(\mathbf{p}) \\ \bar{x}_{12}(\mathbf{p}) + \bar{x}_{22}(\mathbf{p}) \end{pmatrix},$$

describing the total demand of both consumers for each good.

Definition 6.15 A **function of global demand** is an expression:

(6.69)
$$\bar{\mathbf{y}}(\mathbf{p}) + \bar{\mathbf{a}} = \bar{\mathbf{y}}^1(\mathbf{p}) + \bar{\mathbf{y}}^2(\mathbf{p}) + \mathbf{a}^1 + \mathbf{a}^2 = \begin{pmatrix} \bar{y}_{11}(\mathbf{p}) + \bar{y}_{21}(\mathbf{p}) + a_{11} + a_{21} \\ \bar{y}_{12}(\mathbf{p}) + \bar{y}_{22}(\mathbf{p}) + a_{12} + a_{22} \end{pmatrix},$$

describing the total supply of each good.

Definition 6.16 A **function of excess demand** is a mapping $\mathbf{z}$:$\operatorname{int}\mathbb{R}^2_+ \to \mathbb{R}^2$ which assigns an excess demand to any vector $\mathbf{p} = (p_1, p_2) \in \operatorname{int}\mathbb{R}^2_+$ of prices. The function has a form:

(6.70)
$$\mathbf{z}(\mathbf{p}) = \bar{\mathbf{x}}(\mathbf{p}) - (\bar{\mathbf{y}}(\mathbf{p}) + \bar{\mathbf{a}})$$

or equivalently

$$\forall i = 1, 2 \quad z_i(\mathbf{p}) = \bar{x}_i(\mathbf{p}) - \left(\bar{y}_i(\mathbf{p}) + a_i\right). \tag{6.71}$$

Definition 6.17 A **partial equilibrium** on a market of i-th consumer good is a state when:

$$\begin{aligned} &\exists i \quad \exists\, \mathbf{p} > \mathbf{0} \quad z_i(\mathbf{p}) = \bar{x}_i(\mathbf{p}) - \left(\bar{y}_i(\mathbf{p}) + a_i\right) = 0 \\ &\Leftrightarrow \; \bar{x}_i(\mathbf{p}) = \bar{y}_i(\mathbf{p}) + a_i, \end{aligned} \tag{6.72}$$

meaning there exists a positive price vector such that the global demand for i-th good is equal to its global supply. We say then that on a market of i-th consumer good there is a partial equilibrium: the global demand for i-th good (expressed in physical units) equals the global supply of i-th good (expressed in the same physical units).

Definition 6.18 A **general equilibrium** (in the Walras sense) on a market of consumer goods is a state when:

$$\begin{aligned} &\forall i = 1, 2 \quad \exists\, \bar{\mathbf{p}} > \mathbf{0} \quad z_i(\bar{\mathbf{p}}) = \bar{x}_i(\bar{\mathbf{p}}) - \left(\bar{y}_i(\bar{\mathbf{p}}) + a_i\right) = 0 \\ &\Leftrightarrow \; \bar{x}_i(\bar{\mathbf{p}}) = \bar{y}_i(\bar{\mathbf{p}}) + a_i, \end{aligned} \tag{6.73}$$

meaning that there exists a positive price vector, called an **equilibrium (Walrasian) price vector** such that the global demand for each good is equal to its global supply. We then say that there is a general equilibrium on a market of consumer goods: the global demand for each good (expressed in physical units) equals its global supply (expressed in the same physical units).

Theorem 6.5 If Assumptions 6.1 and 6.2 are satisfied then $\forall \mathbf{p} > \mathbf{0}$ the excess demand function is:

(a) continuous and differentiable on $\operatorname{int} \mathbb{R}^2_+$,
(b) homogeneous of degree zero:

$$\forall i = 1, 2 \quad \forall \lambda > 0 \quad z_i(\lambda \mathbf{p}) = \lambda^0 z_i(\mathbf{p}) = z_i(\mathbf{p}), \tag{6.74}$$

which means that a proportional change in prices of all goods does not lead to a change in the excess demand for any good,
(c) satisfies Walras's law:

$$\begin{aligned} &\forall \mathbf{p} > \mathbf{0} \quad \sum_{i=1}^{2} p_i z_i(\mathbf{p}) = \sum_{i=1}^{2} p_i\left(\bar{x}_i(\mathbf{p}) - \left(\bar{y}_i(\mathbf{p}) + a_i\right)\right) = 0 \\ &\Leftrightarrow \quad \sum_{i=1}^{2} p_i \bar{x}_i(\mathbf{p}) = \sum_{i=1}^{2} p_i\left(\bar{y}_i(\mathbf{p}) + a_i\right), \end{aligned} \tag{6.75}$$

which means that for any price vector $\mathbf{p} = (p_1, p_2) > (0, 0)$ a value of the global demand for all goods is equal to a value of their global supply.

Note 6.1 The concept of the Walrasian equilibrium should be distinguished from Walras's law.

Note 6.2 The Walrasian equilibrium state described by the Walrasian equilibrium price vector may not exist, there may be exactly one such state or there may be more than one.

Note 6.3 The price vector of the Walrasian equilibrium (if it exists) is determined with an accuracy of a structure. Let us suppose that $\bar{\mathbf{p}} = \left(\bar{p}_1, \bar{p}_2\right) > \mathbf{0}$ is the Walrasian equilibrium price vector. Then we can present it in a form:

$$\bar{\mathbf{p}} = \left(\bar{p}_1, \bar{p}_2\right) = \bar{p}_1\left(1, \frac{\bar{p}_2}{\bar{p}_1}\right) = \lambda\left(1, \frac{\bar{p}_2}{\bar{p}_1}\right), \quad \text{where } \lambda = \bar{p}_1 > 0, \tag{6.76}$$

or

$$\bar{\mathbf{p}} = \left(\bar{p}_1, \bar{p}_2\right) = \bar{p}_2\left(\frac{\bar{p}_1}{\bar{p}_1}, 1\right) = \lambda\left(\frac{\bar{p}_1}{\bar{p}_1}, 1\right), \quad \text{where } \lambda = \bar{p}_2 > 0. \tag{6.77}$$

Theorem 6.6 If Assumptions 6.1 and 6.2 are satisfied then in the Arrow-Debreu-McKenzie model there exists at least one price vector of the Walrasian equilibrium, determined with an accuracy of a structure.

Note 6.4 The conditions, ensuring that in the Arrow-Debreu-McKenzie model exists exactly one Walrasian equilibrium price vector determined with an accuracy of a structure, are in the form of more complex assumptions. Therefore we will here not discuss them as part of the basic lecture.

Let us assume that there is given an initial endowment of two goods owned by two consumers described by a vector $\mathbf{a} = \left(\mathbf{a}^1, \mathbf{a}^2\right) = (a_{11}, a_{12}, a_{21}, a_{22}) \in \operatorname{int} \mathbb{R}^4_+$.

Definition 6.19 A vector $(\bar{\mathbf{x}}(\mathbf{p}), \bar{\mathbf{y}}(\mathbf{p}))$ is called an **allocation feasible** in the static Arrow-Debreu-McKenzie model when it meets a condition:

$$\begin{aligned} &\sum\nolimits_{k=1}^{2} \bar{\mathbf{x}}^k(\mathbf{p}) = \sum\nolimits_{j=1}^{2} \bar{\mathbf{y}}^j(\mathbf{p}) + \sum\nolimits_{k=1}^{2} \mathbf{a}^k \\ &\Leftrightarrow \begin{pmatrix} \bar{x}_{11}(\mathbf{p}) + \bar{x}_{21}(\mathbf{p}) \\ \bar{x}_{21}(\mathbf{p}) + \bar{x}_{22}(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \bar{y}_{11}(\mathbf{p}) + \bar{y}_{21}(\mathbf{p}) + a_{11} + a_{21} \\ \bar{y}_{12}(\mathbf{p}) + \bar{y}_{22}(\mathbf{p}) + a_{12} + a_{22} \end{pmatrix}, \end{aligned} \tag{6.78}$$

where:

$$\begin{aligned} \bar{\mathbf{x}}(\mathbf{p}) &= (\bar{x}_{11}(\mathbf{p}), \bar{x}_{12}(\mathbf{p}), \bar{x}_{21}(\mathbf{p}), \bar{x}_{22}(\mathbf{p})) \in \mathbb{R}^4_+, \\ \bar{\mathbf{y}}(\mathbf{p}) &= \left(\bar{y}_{11}(\mathbf{p}), \bar{y}_{12}(\mathbf{p}), \bar{y}_{21}(\mathbf{p}), \bar{y}_{22}(\mathbf{p})\right) \in \mathbb{R}^4. \end{aligned}$$

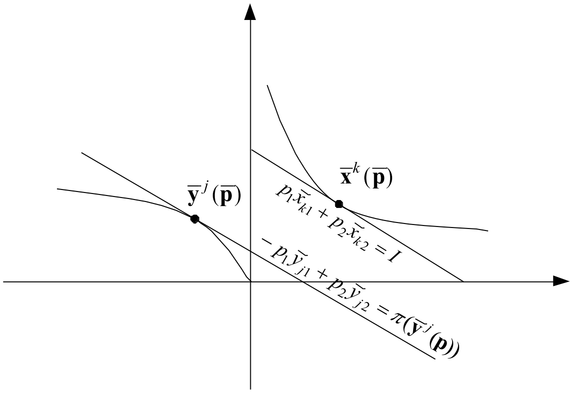

Fig. 6.4 General equilibrium in static Arrow-Debreu-McKenzie model

This means that $\forall \mathbf{p} > \mathbf{0}$ the global demand for i-th good is equal to the global supply of i-th good. It is worth emphasizing that the global supply and the global demand for i-th good are expressed in the same physical units (Fig. 6.4).

Definition 6.20 A **set of allocations feasible** in the static Arrow-Debreu-McKenzie model is a set:

(6.79)
$$F(\mathbf{a}) = \left\{ (\bar{\mathbf{x}}(\mathbf{p}), \bar{\mathbf{y}}(\mathbf{p})) \in \mathbb{R}^4_+ \times \mathbb{R}^4_+ | \sum\nolimits_{k=1}^{2} \bar{\mathbf{x}}^k(\mathbf{p}) = \sum\nolimits_{j=1}^{2} \bar{\mathbf{y}}^j(\mathbf{p}) + \sum\nolimits_{k=1}^{2} \mathbf{a}^k \right\}.$$

Definition 6.21 An allocation $(\bar{\mathbf{x}}(\mathbf{p}), \bar{\mathbf{y}}(\mathbf{p})) \in F(\mathbf{a})$ is called an **allocation accepted by traders** when it satisfies a condition:

$$u^k\left(\bar{\mathbf{x}}^k(\mathbf{p})\right) \geq u^k\left(\mathbf{a}^k\right) \quad \forall k = 1, 2. \tag{6.80}$$

Definition 6.22 A **set of allocations accepted by traders** is a set:

$$S(\mathbf{a}) = \left\{ (\bar{\mathbf{x}}(\mathbf{p}), \bar{\mathbf{y}}(\mathbf{p})) \in F(\mathbf{a}) | u^k\left(\bar{\mathbf{x}}^k(\mathbf{p})\right) \geq u^k\left(\mathbf{a}^k\right), k = 1, 2 \right\}. \tag{6.81}$$

Definition 6.23

An allocation $(\bar{\mathbf{x}}(\mathbf{p}), \bar{\mathbf{y}}(\mathbf{p})) \in S(\mathbf{a})$ accepted by traders is called a **Pareto optimal (efficient) allocation** if there is no other allocation accepted $(\mathbf{x}(\mathbf{p}), \mathbf{y}(\mathbf{p})) \in S(\mathbf{a})$ such that

$$\begin{aligned} &\forall k = 1, 2 \quad u^k\left(\mathbf{x}^k(\mathbf{p})\right) \geq u^k\left(\bar{\mathbf{x}}^k(\mathbf{p})\right), \\ &\exists k \quad u^k\left(\mathbf{x}^k(\mathbf{p})\right) > u^k\left(\bar{\mathbf{x}}^k(\mathbf{p})\right). \end{aligned} \tag{6.82}$$

Definition 6.24 A set $C(\mathbf{a})$ consisting of all allocations accepted by traders and Pareto optimal at the same time is called an **exchange core**.

Definition 6.25 A Pareto optimal allocation $(\bar{\mathbf{x}}(\bar{\mathbf{p}}), \bar{\mathbf{y}}(\bar{\mathbf{p}})) \in C(\mathbf{a})$ is called a **Walrasian equilibrium allocation** when the price vector $\bar{\mathbf{p}} = \lambda\left(1, \frac{\bar{p}_2}{\bar{p}_1}\right) > (0, 0),\ \lambda > 0$ is the Walrasian equilibrium price vector.

Definition 6.26 A set consisting of all Walrasian equilibrium allocations, that is a set:

(6.83) $$W(\mathbf{a}) = \{(\bar{\mathbf{x}}(\bar{\mathbf{p}}), \bar{\mathbf{y}}(\bar{\mathbf{p}})) \in C(\mathbf{a}) | \bar{\mathbf{x}}(\bar{\mathbf{p}}) = \bar{\mathbf{y}}(\bar{\mathbf{p}}) + \bar{\mathbf{a}}\} \in \mathbb{R}^4_+ \times \mathbb{R}^4_+,$$

is called a **set of Walrasian equilibrium allocations**.

Note 6.5 From Definitions 6.19–6.25 it follows that

(6.84) $$W(\mathbf{a}) \subseteq C(\mathbf{a}) \subseteq S(\mathbf{a}) \subseteq F(\mathbf{a}) \subset \mathbb{R}^4_+ \times \mathbb{R}^4_+$$

which means that each Walrasian equilibrium allocation is an allocation: Pareto optimal, accepted by traders and feasible.

Note 6.6 The reverse inclusion is not true, which means that not every feasible allocation is an accepted by traders, Pareto optimal or Walrasian equilibrium allocation.

6.4 Dynamic Arrow-Debreu-McKenzie Model

Let us describe an economy in which we distinguish the part related to production and the part related to exchange (consumption). Let us use the following notation:

$i = 1, 2$—an index of products which, depending on their intended use, may be consumer goods or production factors,
$k = 1, 2$—an index of (traders) consumers,
$j = 1, 2$—an index of producers,
$X^k(t) \subseteq \mathbb{R}^2_+$—a goods space (a set of all bundles of goods available on a market with a metric $d_{NE}:\mathbb{R}^4_+ \to \mathbb{R}_+$ specified on this set, see Definition 1.2) in period $t = 1, 2, ..., T$ or at moment $t \in [0; T]$,
$Y^j(t) \subseteq \mathbb{R}^2$—a production space of j-th producer in period $t = 1, 2, \ldots, T$ or at moment $t \in [0; T]$.

Parameters

α_{kj}—a share of k-th consumer in the profit of j-th producer,
$\mathbf{a}^k = (a_{k1}, a_{k2}) \in \mathbb{R}^2_+$—an initial consumption bundle (initial endowment) of k-th consumer,

Variables

$t = 1, 2, \ldots, T$—time as a discrete variable,
$t \in [0; T]$—time as a continuous variable,
T—time horizon, which can be finite or infinite,
$\mathbf{p}(t) = (p_1(t), p_2(t)) \in \operatorname{int}\mathbb{R}^2_+$—a vector of prices of products in period/at moment t,
$\mathbf{x}^k(t) = (x_{k1}(t), x_{k2}(t)) \in R^2_+$—a consumption bundle the k-th consumer wants to purchase in period/at moment t,
$\mathbf{w}^j(t) = (w_{j1}(t), w_{j2}(t)) \in \mathbb{R}^2_+$—a vector of outputs in the production process of j-th producer in period/at moment t,
$\mathbf{n}^j(t) = (n_{j1}(t), n_{j2}(t)) \in \mathbb{R}^2_+$—a vector of inputs of production factors in the production process of j-th producer in period/at moment t,
$\mathbf{y}^j(t) = \mathbf{w}^j(t) - \mathbf{n}^j(t) = \left(w_{j1}(t) - n_{j1}(t), w_{j2}(t) - n_{j2}(t)\right) = \left(y_{j1}(t), y_{j2}(t)\right) \in Y^j \subseteq \mathbb{R}^2$—a vector of the j-th producer's net output in period/at moment t,
$\Pi_j(p_1(t), p_2(t))$—j-th producer's profit as a function of prices of products in period/at moment t,
$I^k(p_1(t), p_2(t)) = \sum_{i=1}^2 p_i(t)a_{ki} + \sum_{j=1}^2 \alpha_{kj}\Pi_j(p_1(t), p_2(t)) > 0$—$k$-th consumer's income as a function of prices of goods in period/at moment$t$,
$u^k{:}\mathbb{R}^2_+ \to \mathbb{R}$—an utility function of k-th consumer, describing her/his preferences (a relation of preferences),
$D^k(\mathbf{p}(t), I^k(\mathbf{p}(t))) = \left\{(x_{k1}(t), x_{k2}(t)) \in \mathbb{R}^2_+ \middle| p_1(t)x_{k1}(t) + p_2(t)x_{k2}(t) \le I^k(p_1(t), p_2(t))\right\} \subset X = \mathbb{R}^2_+$—a set of all consumption bundles of a value not exceeding k-th consumer's income in period/at moment t.

Production

Each producer ($j = 1, 2$) in any period/at any moment t wants to maximize the profit from sales of manufactured products. Each one is assumed to carry out a production process described by a vector of the net output:

$$\begin{aligned}\mathbf{y}^j(t) &= \mathbf{w}^j(t) - \mathbf{n}^j(t) = \left(w_{j1}(t) - n_{j1}(t), w_{j2}(t) - n_{j2}(t)\right)\\ &= \left(y_{j1}(t), y_{j2}(t)\right) \in Y^j(t) \subseteq \mathbb{R}^2,\ \ j = 1, 2,\end{aligned} \tag{6.85}$$

where:

$\mathbf{w}^j(t) = (w_{j1}(t), w_{j2}(t)) \in \mathbb{R}^2_+$—a vector of outputs in the production process of j-th producer in period/at moment t,

$\mathbf{n}^j(t) = (n_{j1}(t), n_{j2}(t)) \in \mathbb{R}_+^2$—a vector of inputs of production factors in the production process of j-th producer in period/at moment t,
$Y^j(t) \subseteq \mathbb{R}^2$—a production space of j-th producer in period/at moment t in which a result of production is described by a vector of the net output.

Definition 6.27 A **net output space** for j-th producer in period/at moment t is a set of all net output vectors $\mathbf{y}^j(t) = \mathbf{w}^j(t) - \mathbf{n}^j(t) = \big(y_{j1}(t), y_{j2}(t)\big) \in Y^j(t) \subseteq \mathbb{R}^2$ which are the results of technologically feasible production processes $\mathbf{z}^j(t) = \big(\mathbf{n}^j(t), \boldsymbol{w}^j(t)\big) \in Z(t) \subseteq \mathbb{R}^4$, $j = 1, 2$ carried out by j-th producer. This set is defined with a norm $\|\mathbf{y}^j(t)\|_{NE} = \max_{i=1,2}\left|y_{ji}(t)\right|$ in period/at moment t.

Assumption 6.3 In any period/at any moment t Assumption 6.1 is satisfied.

By given prices $\mathbf{p}(t) = (p_1(t), p_2(t)) \in \operatorname{int}\mathbb{R}_+^2$ of products a profit maximization problem of j-th producer takes the form:

$$p_1(t)y_{j1}(t) + p_2(t)y_{j2}(t) \mapsto \max \tag{6.86}$$

$$\mathbf{y}^j(t) = \big(y_{j1}(t), y_{j2}(t)\big) \in Y^j \subseteq \mathbb{R}^2. \tag{6.87}$$

Problem (6.86)–(6.87) is a nonlinear programming problem with a linear profit function and the compact strictly convex set $Y^j(t) \subseteq \mathbb{R}^2$ of feasible solutions. Due to assumed properties of the j-th producer net output space this problem has a unique solution by a given vector $\mathbf{p}(t) = (p_1(t), p_2(t)) \in \operatorname{int}\mathbb{R}_+^2$ of prices of products that is the optimal net output vector $\bar{\mathbf{y}}^j(t) = \big(\bar{y}_{j1}(t), \bar{y}_{j2}(t)\big) \in Y^j(t) \subseteq \mathbb{R}^2$ ensuring in period/at moment t the maximum profit for j-the producer.

Definition 6.28 A **supply function** of j-th producer in period/at moment t is a mapping $\bar{\mathbf{y}}^j{:}\mathbb{R}_+^2 \to \mathbb{R}^2$ which assigns an optimal vector of net output $\bar{\mathbf{y}}^j(\mathbf{p}(t)) = \big(\bar{y}_{j1}(\mathbf{p}(t)), \bar{y}_{j2}(\mathbf{p}(t))\big) \in Y^j(t) \subseteq \mathbb{R}^2$ being the solution in the j-th producer's profit maximization problem (5.86)–(5.87) to any vector $\mathbf{p}(t) = (p_1(t), p_2(t)) \in \operatorname{int}\mathbb{R}_+^2$ of prices of products.

Consumption

The k-th consumer in period/at moment t wants to purchase such a bundle of goods $\bar{\mathbf{x}}^k(t) = (\bar{x}_{k1}(t), \bar{x}_{k2}(t))$ whose value does not exceed the income of k-th consumer and whose utility is maximum and at the same time not less than utility of the initial bundle of goods$\mathbf{a}^k = (a_{k1}, a_{k2})$. By a given vector $\mathbf{p}(t) = (p_1(t), p_2(t)) \in \operatorname{int}\mathbb{R}_+^2$ of prices of goods the consumption utility maximization problem of k-th consumer in period/at moment t can be written as

$$u^k(x_{k1}(t), x_{k2}(t)) \to \max \tag{6.88}$$

$$p_1(t)x_{k1}(t) + p_2(t)x_{k2}(t) \leq I^k(p_1(t), p_2(t)), \tag{6.89}$$

$$x_{k1}(t), x_{k2}(t) \geq 0, \tag{6.90}$$

where:

$$\begin{aligned} I^k(p_1(t), p_2(t)) =& p_1(t)a_{k1} + p_2(t)a_{k2} \\ &+ \alpha_{k1}\Pi^1\left(\bar{\mathbf{y}}^1(p_1(t), p_2(t))\right) + \alpha_{k2}\Pi^2\left(\bar{\mathbf{y}}^2(p_1(t), p_2(t))\right) \end{aligned} \tag{6.91}$$

denotes the income of k-th consumer in period/at moment t.

Assumption 6.4 The utility function of k-th consumer $u^k:\mathbb{R}^2_+ \to \mathbb{R}$ in any period/at any moment t is increasing, differentiable and strictly concave.

Then the problem (6.88)–(6.90) in period/at moment t has exactly one optimal solution which belongs to the budget line and is of a form:

$$\begin{aligned} &\bar{\mathbf{x}}^k(\mathbf{p}(t)) = \varphi^k\left(\mathbf{p}, I^k(\mathbf{p}(t))\right) = \left(\alpha^k \frac{I^k(\mathbf{p}(t))}{p_1(t)}, \beta^k \frac{I^k(\mathbf{p}(t))}{p_2(t)}\right) > (0,0), \\ &\forall \alpha^k, \beta^k > 0, \quad \alpha^k + \beta^k = 1, \quad k = 1, 2. \end{aligned} \tag{6.92}$$

Definition 6.29 A **demand function** of k-th consumer in period/at moment t is a mapping $\varphi^k:\operatorname{int}\mathbb{R}^2_+ \to \operatorname{int}\mathbb{R}^2_+$ which assigns the optimal solution in the k-th consumer's consumption utility maximization problem (5.88)–(5.90) to any vector $\mathbf{p}(t) = (p_1(t), p_2(t)) \in \operatorname{int}\mathbb{R}^2_+$ of prices of goods:

$$\varphi^k(\mathbf{p}(t)) = \varphi^k\left(\mathbf{p}(t), I^k(\mathbf{p}(t))\right) = \mathbf{x}^k(\mathbf{p}(t)), k = 1, 2. \tag{6.93}$$

Definition 6.30 An **indirect function of consumption utility** of k-th consumer in period/at moment t is a mapping $v^k:\operatorname{int}\mathbb{R}^2_+ \to \operatorname{int}\mathbb{R}_+$ which assigns the utility of the optimal bundle of goods being the solution in the k-th consumer's consumption utility maximization problem (6.88)–(6.90) to any vector $\mathbf{p}(t) = (p_1(t), p_2(t)) \in \operatorname{int}\mathbb{R}^2_+$ of prices of goods:

$$\nu^k(\mathbf{p}(t)) = u^k(\bar{x}_{k1}(\mathbf{p}(t)), \bar{x}_{k2}(\mathbf{p}(t))), \quad k = 1, 2. \tag{6.94}$$

General equilibrium

Definition 6.31 A **function of global demand** in period/at moment t is an expression:

(6.95) $$\bar{\mathbf{x}}(\mathbf{p}(t)) = \bar{\mathbf{x}}^{\mathbf{1}}(\mathbf{p}(t)) + \bar{\mathbf{x}}^{\mathbf{2}}(\mathbf{p}(t)) = \begin{pmatrix} \bar{x}_{11}(\mathbf{p}(t)) + \bar{x}_{21}(\mathbf{p}(t)) \\ \bar{x}_{12}(\mathbf{p}(t)) + \bar{x}_{22}(\mathbf{p}(t)) \end{pmatrix},$$

describing the total demand reported by both consumers for each good in period/at moment t.

Definition 6.32 A **function of global demand** in period/at moment t is an expression:

(6.96) $$\begin{aligned} &\bar{\mathbf{y}}(\mathbf{p}(t)) + \bar{\mathbf{a}} = \bar{\mathbf{y}}^{\mathbf{1}}(\mathbf{p}(t)) + \bar{\mathbf{y}}^{\mathbf{2}}(\mathbf{p}(t)) + \mathbf{a}^1 + \mathbf{a}^2 \\ &= \begin{pmatrix} \bar{y}_{11}(\mathbf{p}(t)) + \bar{y}_{21}(\mathbf{p}(t)) + a_{11} + a_{21} \\ \bar{y}_{12}(\mathbf{p}(t)) + \bar{y}_{22}(\mathbf{p}(t)) + a_{12} + a_{22} \end{pmatrix}, \end{aligned}$$

describing the total supply of each good in period/at moment t.

Definition 6.33 A **function of excess demand** in period/at moment t is a mapping $\mathbf{z}{:}\mathrm{int}\,\mathbb{R}^2_+ \rightarrow \mathbb{R}^2$ which assigns an excess demand to any vector $\mathbf{p}(t) = (p_1(t), p_2(t)) \in \mathrm{int}\,\mathbb{R}^2_+$ of prices. The function has a form:

(6.97) $$\mathbf{z}(\mathbf{p}(t)) = \bar{\mathbf{x}}(\mathbf{p}(t)) - (\bar{\mathbf{y}}(\mathbf{p}(t)) + \bar{\mathbf{a}})$$

or equivalently

(6.98) $$\forall i = 1, 2 \quad z_i(\mathbf{p}(t)) = \bar{x}_i(\mathbf{p}(t)) - \left(\bar{y}_i(\mathbf{p}(t)) + a_i\right).$$

Definition 6.34 A **partial equilibrium** on a market of i-th consumer good in period/at moment t is a state when:

(6.99) $$\begin{aligned} &\exists i \quad \exists\, \mathbf{p}(t) > \mathbf{0} \quad z_i(\mathbf{p}(t)) = \bar{x}_i(\mathbf{p}(t)) - \left(\bar{y}_i(\mathbf{p}(t)) + a_i\right) = 0 \\ &\Leftrightarrow \quad \bar{x}_i(\mathbf{p}(t)) = \bar{y}_i(\mathbf{p}(t)) + a_i, \end{aligned}$$

meaning there exists a positive price vector such that the global demand for i-th good is equal to its global supply. We say then that in period/at moment t on a market of i-th consumer good there is a partial equilibrium: the global demand for i-th good (expressed in physical units) equals the global supply of i-th good (expressed in the same physical units).

Definition 6.35 A **general equilibrium** (in the Walras sense) on a market of consumer goods in period/at moment t is a state when:

$$\begin{aligned} &\forall i = 1, 2 \quad \exists\, \bar{\mathbf{p}}(t) > \mathbf{0} \quad z_i(\bar{\mathbf{p}}(t)) = \bar{x}_i(\bar{\mathbf{p}}(t)) - \big(\bar{y}_i(\bar{\mathbf{p}}(t)) + a_i\big) = 0 \\ &\Leftrightarrow \quad \bar{x}_i(\bar{\mathbf{p}}(t)) = \bar{y}_i(\bar{\mathbf{p}}(t)) + a_i, \end{aligned} \tag{6.100}$$

meaning that there exists a positive price vector, called an **equilibrium (Walrasian) price vector** such that the global demand for each good is equal to its global supply in period/at moment t. We then say that in period/at moment t there is a general equilibrium on a market of consumer goods: the global demand for each good (expressed in physical units) equals its global supply (expressed in the same physical units).

Theorem 6.7 If Assumptions 6.3 and 6.4 are satisfied then in period/at moment t $\forall \mathbf{p}(t) > \mathbf{0}$ the excess demand function is:

1. continuous and differentiable on $\operatorname{int} \mathbb{R}^2_+$,
2. homogeneous of degree zero:

$$\forall i = 1, 2 \quad \forall \lambda > 0 \quad z_i(\lambda \mathbf{p}(t)) = \lambda^0 z_i(\mathbf{p}(t)) = z_i(\mathbf{p}(t)), \tag{6.101}$$

   Which means that a proportional change in prices of all goods occurring in period/at moment t does not lead to a change in the excess demand for any good,
3. satisfies Walras's law:

$$\begin{aligned} &\forall \mathbf{p}(t) > \mathbf{0} \quad \sum\nolimits_{i=1}^{2} p_i(t) z_i(\mathbf{p}(t)) = \sum\nolimits_{i=1}^{2} p_i\big(\bar{x}_i(\mathbf{p}(t)) - \big(\bar{y}_i(\mathbf{p}(t)) + a_i\big)\big) = 0 \\ &\Leftrightarrow \sum\nolimits_{i=1}^{2} p_i(t)\bar{x}_i(\mathbf{p}(t)) = \sum\nolimits_{i=1}^{2} p_i(t)\big(\bar{y}_i(\mathbf{p}(t)) + a_i\big), \end{aligned} \tag{6.102}$$

   which means that for any price vector $\mathbf{p}(t) = (p_1(t), p_2(t)) > (0, 0)$ a value of the global demand for all goods is equal to a value of their global supply.

Note 6.7 The concept of the Walrasian equilibrium should be distinguished from Walras's law.

Note 6.8 The Walrasian equilibrium state described by the Walrasian equilibrium price vector may not exist, there may be exactly one such state or there may be more than one.

Note 6.9 The price vector of the Walrasian equilibrium (if it exists) is determined with an accuracy of a structure. Let us suppose that $\bar{\mathbf{p}}(t) = \big(\bar{p}_1(t), \bar{p}_2(t)\big) > 0$ is the Walrasian equilibrium price vector. Then we can present it in a form:

$$\bar{\mathbf{p}}(t) = \big(\bar{p}_1(t), \bar{p}_2(t)\big) = \bar{p}_1(t)\left(1, \frac{\bar{p}_2(t)}{\bar{p}_1(t)}\right)$$

$$=\lambda\left(1, \frac{\bar{p}_2(t)}{\bar{p}_1(t)}\right), \quad \text{where } \lambda = \bar{p}_1(t) > 0, \tag{6.103}$$

or

$$\bar{\mathbf{p}}(t) = \big(\bar{p}_1(t), \bar{p}_2(t)\big) = \bar{p}_2(t)\left(\frac{\bar{p}_1(t)}{\bar{p}_1(t)}, 1\right)$$
$$=\lambda\left(\frac{\bar{p}_1(t)}{\bar{p}_1(t)}, 1\right), \quad \text{where } \lambda = \bar{p}_2(t) > 0. \tag{6.104}$$

Theorem 6.8 If Assumptions 6.3 and 6.4 are satisfied then in the Arrow-Debreu-McKenzie model there exists at least one price vector of the Walrasian equilibrium, determined with an accuracy of a structure.

Note 6.10 The conditions, ensuring that in the Arrow-Debreu-McKenzie model exists exactly one Walrasian equilibrium price vector determined with an accuracy of a structure, are in the form of more complex assumptions. Therefore we will here not provide them as part of the basic lecture.

Let us assume that there is given an initial endowment of two goods owned by two consumers described by a vector $\mathbf{a} = \big(\mathbf{a}^1, \mathbf{a}^2\big) = (a_{11}, a_{12}, a_{21}, a_{22}) \in \operatorname{int} \mathbb{R}^4_+$.

Definition 6.36 A vector $(\bar{\mathbf{x}}(\mathbf{p}(t)), \bar{\mathbf{y}}(\mathbf{p}(t)))$ is called an **allocation feasible** in period/at moment t in the dynamic Arrow-Debreu-McKenzie model when it meets a condition:

$$\sum_{k=1}^{2} \bar{\mathbf{x}}^k(\mathbf{p}(t)) = \sum_{j=1}^{2} \bar{\mathbf{y}}^j(\mathbf{p}(t)) + \sum_{k=1}^{2} \mathbf{a}^k$$
$$\Leftrightarrow \begin{pmatrix} \bar{x}_{11}(\mathbf{p}(t)) + \bar{x}_{21}(\mathbf{p}(t)) \\ \bar{x}_{21}(\mathbf{p}(t)) + \bar{x}_{22}(\mathbf{p}(t)) \end{pmatrix} = \begin{pmatrix} \bar{y}_{11}(\mathbf{p}(t)) + \bar{y}_{21}(\mathbf{p}(t)) + a_{11} + a_{21} \\ \bar{y}_{12}(\mathbf{p}(t)) + \bar{y}_{22}(\mathbf{p}(t)) + a_{12} + a_{22} \end{pmatrix}, \tag{6.105}$$

where:

$$\bar{\mathbf{x}}(\mathbf{p}(t)) = (\bar{x}_{11}(\mathbf{p}(t)), \bar{x}_{12}(\mathbf{p}(t)), \bar{x}_{21}(\mathbf{p}(t)), \bar{x}_{22}(\mathbf{p}(t))) \in \mathbb{R}^4_+,$$
$$\bar{\mathbf{y}}(\mathbf{p}(t)) = \big(\bar{y}_{11}(\mathbf{p}(t)), \bar{y}_{12}(\mathbf{p}(t)), \bar{y}_{21}(\mathbf{p}(t)), \bar{y}_{22}(\mathbf{p}(t))\big) \in \mathbb{R}^4.$$

This means that $\forall \mathbf{p}(t) > \mathbf{0}$ the global demand for i-th good is equal to the global supply of i-th good. It is worth emphasizing that the global supply and the global demand for i-th good are expressed in the same physical units.

Definition 6.37 A **set of allocations feasible** in the dynamic Arrow-Debreu-McKenzie model is a set:

$$F(\mathbf{a}) = \left\{(\bar{\mathbf{x}}(\mathbf{p}(t)), \bar{\mathbf{y}}(\mathbf{p}(t))) \in \mathbb{R}^4_+ \times \mathbb{R}^4_+ \mid \sum_{k=1}^{2} \bar{\mathbf{x}}^k(\mathbf{p}(t)) = \sum_{j=1}^{2} \bar{\mathbf{y}}^j(\mathbf{p}(t)) + \sum_{k=1}^{2} \mathbf{a}^k\right\}. \tag{6.106}$$

Definition 6.38 An allocation $(\bar{\mathbf{x}}(\mathbf{p}), \bar{\mathbf{y}}(\mathbf{p})) \in F(\mathbf{a})$ is called an **allocation accepted by traders** in period/at moment t when it satisfies a condition:

(6.107)
$$u^k\left(\bar{\mathbf{x}}^k(\mathbf{p}(t))\right) \geq u^k\left(\mathbf{a}^k\right) \quad \forall k = 1, 2.$$

Definition 6.39 A **set of allocations accepted by traders** in the dynamic Arrow-Debreu-McKenzie model is a set:

(6.108)
$$S(\mathbf{a}) = \left\{(\mathbf{x}(\mathbf{p}(t)), \bar{\mathbf{y}}(\mathbf{p}(t))) \in F(\mathbf{a}) | u^k\left(\mathbf{x}^k(\mathbf{p}(t))\right) \geq u^k\left(\mathbf{a}^k\right), k = 1, 2\right\}.$$

Definition 6.40 An allocation $(\bar{\mathbf{x}}(\mathbf{p}(t)), \bar{\mathbf{y}}(\mathbf{p}(t))) \in S(\mathbf{a})$ accepted by traders in period/at moment t is called a **Pareto optimal (efficient) allocation** if there is no other allocation accepted $(\mathbf{x}(\mathbf{p}(t)), \mathbf{y}(\mathbf{p}(t))) \in S(\mathbf{a})$ such that

(6.109)
$$\begin{aligned} &\forall k = 1, 2 \quad u^k\left(\mathbf{x}^k(\mathbf{p}(t))\right) \geq u^k\left(\bar{\mathbf{x}}^k(\mathbf{p}(t))\right), \\ &\exists k \quad u^k\left(\mathbf{x}^k(\mathbf{p}(t))\right) > u^k\left(\bar{\mathbf{x}}^k(\mathbf{p}(t))\right). \end{aligned}$$

Definition 6.41 A set $C(\mathbf{a})$ consisting of all allocations accepted by traders and Pareto optimal at the same time is called an **exchange core**.

Definition 6.42 A Pareto optimal allocation $(\bar{\mathbf{x}}(\bar{\mathbf{p}}(t)), \bar{\mathbf{y}}(\bar{\mathbf{p}}(t))) \in C(\mathbf{a})$ in the dynamic Arrow-Debreu-McKenzie model is called a **Walrasian equilibrium allocation** when the price vector $\bar{\mathbf{p}}(t) = \lambda\left(1, \frac{\bar{p}_2(t)}{\bar{p}_1(t)}\right) > (0, 0), \lambda > 0$ is the Walrasian equilibrium price vector.

Definition 6.43 A set consisting of all Walrasian equilibrium allocations, that is a set:

(6.110)
$$W(\mathbf{a}) = \left\{\left(\bar{\mathbf{x}}(\bar{\mathbf{p}}(t)), \bar{\mathbf{y}}(\bar{\mathbf{p}(t)})\right) \in C(\mathbf{a}) \middle| \bar{\mathbf{x}}(\bar{\mathbf{p}}(t)) = \bar{\mathbf{y}}(\bar{\mathbf{p}}(t)) + \bar{\mathbf{a}}\right\} \in \mathbb{R}^4_+ \times \mathbb{R}^4_+,$$

is called a **set of Walrasian equilibrium allocations** in period/at moment t.

Note 6.11 From Definitions 5.38–5.43 it follows that

(6.111)
$$W(\mathbf{a}) \subseteq C(\mathbf{a}) \subseteq S(\mathbf{a}) \subseteq F(\mathbf{a}) \subset \mathbb{R}^4_+ \times \mathbb{R}^4_+$$

which means that each Walrasian equilibrium allocation is an allocation: Pareto optimal, accepted by traders and feasible.

Note 6.12 The reverse inclusion is not true, which means that not every feasible allocation is an accepted by traders, Pareto optimal or Walrasian equilibrium allocation.

The main questions regarding a market described by the dynamic Arrow-Debreu-McKenzie model are:

- does a state of the Walrasian equilibrium exist on a consumer goods market?
- is there exactly one or at least one state of the Walrasian equilibrium?
- whether a state of the Walrasian equilibrium is asymptotically globally stable?

Definition 6.44 A **dynamic discrete-time Arrow-Debreu-McKenzie model** is a system of difference equations of a form:

(6.112) $$\forall i \quad p_i(t+1) = p_i(t) + \sigma_i z_i(p(t)),$$

with an initial condition:

(6.113) $$\forall i \quad p_i(0) = p_i^0 > 0,$$

(6.114) $$t = 0, 1, 2, \ldots$$

where $\sigma_i > 0$ denotes a measure of a broker 's sensitivity to an imbalance on i-th good's market, which for the sake of simplicity is assumed to be the same for markets of all goods: $\forall i \ \sigma_i = \sigma > 0$.

Condition (6.112) can be written in an equivalent form:

(6.115) $$\forall i \quad p_i(t+1) - p_i(t) = \sigma_i z_i(p(t)).$$

On the basis of conditions (6.112) and (6.115) it can be concluded that $\forall t = 0, 1, 2, \ldots$ and $\forall i = 1, 2$:

$$z_i(p(t)) > 0 \Rightarrow p_i(t+1) - p_i(t) > 0 \Rightarrow p_i(t+1) > p_i(t),$$
$$z_i(p(t)) < 0 \Rightarrow p_i(t+1) - p_i(t) < 0 \Rightarrow p_i(t+1) < p_i(t),$$
$$z_i(p(t)) = 0 \Rightarrow p_i(t+1) - p_i(t) = 0 \Rightarrow p_i(t+1) = p_i(t).$$

Equivalent conditions (6.112) and (6.115) lead to a simple recursive rule for determining prices of all products in subsequent periods of time. However, this rule does not ensure that the resulting price systems will make economic sense. We are not interested in situations where a price of any good is negative. Therefore our attention should be focused only on such solutions to systems of difference Eqs. (6.112) or (6.115), in which the vectors of consumer goods' prices determined on the basis of these solutions are positive: $\forall i \quad p_i(t+1) > 0$.

Definition 6.45 A **feasible price trajectory** in the dynamic discrete-time Arrow-Debreu-McKenzie model is an infinite sequence of solutions to the difference equations' system (6.112) with an initial condition $\mathbf{p}(0) = \mathbf{p}^0 > \mathbf{0}$ such that $\forall t = 0, 1, 2, \ldots \; \mathbf{p}(t+1) > \mathbf{0}$.

Assuming there exists a feasible price trajectory in the dynamic discrete-time Arrow-Debreu-McKenzie model, one is interested in the conditions of existence, uniqueness and stability of the Walrasian equilibrium state.

Definition 6.46 A Walrasian equilibrium state $\bar{\mathbf{p}} > \mathbf{0}$ is called **asymptotically globally stable** when a feasible trajectory of products' prices satisfies a condition:

$$\lim_{t \to +\infty} \mathbf{p}(t+1) = \bar{\mathbf{p}}. \tag{6.116}$$

Global stability means that any feasible trajectory of products' prices, starting from any initial price system $\mathbf{p}(0) = \mathbf{p}^0 > \mathbf{0}$, after reaching a state of the Walrasian equilibrium will remain in this state. The stability is also asymptotic one, because the state of the Walrasian equilibrium is a target state which, if exists, can be achieved in an infinite time horizon.

Definition 6.47 A **dynamic continuous-time Arrow-Debreu-McKenzie model** is a system of differential equations of a form:

$$\forall i \quad \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = \sigma_i z_i(\mathbf{p}(t)), \tag{6.117}$$

with an initial condition:

$$\forall i \quad p_i(0) = p_i^0 > 0, \tag{6.118}$$

$$t \in [0; +\infty). \tag{6.119}$$

where:

where $\sigma_i > 0$ denotes a measure of broker 's sensitivity to an imbalance on i-th good's market, which for the sake of simplicity is assumed to be the same for markets of all goods: $\forall i \; \sigma_i = \sigma > 0$.

On the basis of condition (6.117) it can be concluded that $\forall t \in [0; +\infty)$ and $\forall i = 1, 2$:

$$z_i(\mathbf{p}(t)) > 0 \Rightarrow \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} > 0 \Rightarrow p_i(t+1) > p_i(t),$$

$$z_i(\mathbf{p}(t)) < 0 \Rightarrow \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} < 0 \Rightarrow p_i(t+1) < p_i(t),$$
$$z_i(\mathbf{p}(t)) = 0 \Rightarrow \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = 0 \Rightarrow p_i(t+1) = p_i(t).$$

This simple recursive rule, described by conditions (6.117)–(6.118), shows how to determine prices of all goods in subsequent moments. However, it does not ensure that the resulting price systems will make economic sense. Therefore our attention should be focused only on such solutions, to systems of differential equations (6.117), in which the vectors of consumer goods' prices determined on the basis of these solutions are positive: $\forall i \quad p_i(t + \Delta t) > 0, \ \Delta t \to 0$.

Definition 6.48 A **feasible price trajectory** in the dynamic continuous-time Arrow-Debreu-McKenzie model is an infinite sequence of solutions to the differential equations system (6.117) with an initial condition $\mathbf{p}(0) = \mathbf{p}^0 > \mathbf{0}$ such that $\forall t \in [0; +\infty) \quad \mathbf{p}(t + \Delta t) > \mathbf{0}$.

Assuming that there exists a feasible price trajectory in the dynamic continuous-time Arrow-Debreu-McKenzie model, one is interested in conditions of existence, uniqueness and stability of the Walrasian equilibrium state.

Definition 6.49 A Walrasian equilibrium state$\bar{\mathbf{p}} > \mathbf{0}$ is called **asymptotically globally stable** when a feasible trajectory of goods' prices satisfies a condition:

$$\lim_{\substack{t \to +\infty \\ \Delta t \to 0}} \mathbf{p}(t + \Delta t) = \bar{\mathbf{p}}. \tag{6.120}$$

Global stability means that any feasible trajectory of products' prices, starting from any initial price system $\mathbf{p}(0) = \mathbf{p}^0 > \mathbf{0}$, after reaching a state of the Walrasian equilibrium will remain in this state. The stability is also asymptotic one, because the state of the Walrasian equilibrium is a target state which, if exists, can be achieved in an infinite time horizon.

6.5 Questions

1. Explain why an excess demand function, in the discussed model of a market of a single good with exogenously determined functions of supply and demand, is positively homogenous of degree 0 and satisfies Walras's law only for an equilibrium price.
2. Present analytical forms of functions of the demand and of the supply on a market of two goods so that these goods are
   (a) independent,
   (b) complementary,
   (c) substitute,

to each other.

3. Proceeding with the answer to question 2 state if each of these three cases: independency, substitutability, or complementarity makes economic sense in the considered model.
4. Justify that in the static Arrow-Debreu-McKenzie model vector functions of the demand and of the supply are determined endogenously.
5. What are the basic differences between the static Arrow-Debreu-McKenzie model and the static model of a market of two goods with exogenous functions of the supply and of the demand?
6. What is the difference between the Walrasian equilibrium state and Walras's law in the static Arrow-Debreu-McKenzie model?
7. What is the difference between the Walrasian equilibrium allocation and the Pareto optimal (efficient) allocation in the static Arrow-Debreu-McKenzie model?
8. How is a feasible trajectory of prices in the dynamic Arrow-Debreu-McKenzie model defined in its discrete-time or continuous-time version?
9. What is the significance of feasibility of a trajectory of goods' prices for its asymptotic convergence to the equilibrium state in the dynamic Arrow-Debreu-McKenzie model in the discrete-time or continuous-time version?
10. What does it mean that the Walrasian equilibrium price vector in the dynamic Arrow- Debreu-McKenzie model, in the discrete-time or continuous-time version, is asymptotically globally stable?

6.6 Exercises

E1. There is a market of a homogeneous product with exogenously determined demand function: $y^d(p) = -ap^\alpha + b, \quad a, b > 0$ and supply function: $y^s(p) = cp^\alpha + d, \quad c, d > 0$.

1. For a product price, the demand and the supply levels determine intervals of values resulting from analytical forms of the demand and supply functions.
2. Draw graphs of both functions in space $\mathbb{R}^2_+$ (in one figure). Determine by what values of parameters $b, d > 0$ there exists the equilibrium price and indicate this price in the figure, when:
   (a) $\alpha \in (0; 1)$,
   (b) $\alpha = 1$,
   (c) $\alpha > 1$.
3. Determine the equilibrium price.
4. Determine elasticities of the equilibrium price with regard to parameters of the supply and the demand functions that determine this price and give their economic interpretation.

E2. There is given a market of two products with exogenous demand functions:

$$\begin{aligned} y_1^d(\mathbf{p}) &= -\,a_1 p_1 + \gamma_1 p_2 + b_1, \\ y_2^d(\mathbf{p}) &= -\,a_2 p_2 + \gamma_2 p_1 + b_2, \quad a_i, b_i, \gamma_i > \ 0, \quad i = 1, 2 \end{aligned}$$

and exogenous supply functions:

$$\begin{aligned} y_1^s(\mathbf{p}) &= c_1 p_1 + \delta_1 p_2 + d_1, \\ y_2^s(\mathbf{p}) &= c_2 p_2 + \delta_2 p_1 + d_2, \quad c_i, d_i, \delta_i > \ 0, \quad i = 1, 2. \end{aligned}$$

1. For price of products, the demand and the supply levels determine intervals of values resulting from analytical forms of the demand and supply functions.
2. Check if the two products are:
   (a) independent,
   (b) complementary,
   (c) substitute,
   to each other.
3. Draw graphs of these functions in space $\mathbb{R}^3_+$ (in one figure). Determine by what values of parameters $b_i, d_i > 0$ $(i = 1, 2)$ there exists the equilibrium price vector and indicate this vector in the figure.
4. Determine the equilibrium price vector.
5. Determine elasticities of the equilibrium prices with regard to parameters of the supply and the demand functions that determine the prices and give economic interpretation of these elasticities.

E3. Present the model of a market of a single good from Exercise E1 as

(a) a dynamic discrete-time model,
(b) a dynamic continuous-time model.

Check whether in these models Walras's law is satisfied for an excess demand function. Selecting proper values for parameter $\sigma > 0$ in the dynamic discrete-time model determine a feasible price trajectory for 10 subsequent periods: $t = 1, 2, \ldots, 10$.

E4. Present the model of a market of two goods from Exercise E2 as

(a) a dynamic discrete-time model,
(b) a dynamic continuous-time model.

Check whether in these models Walras's law is satisfied for a vector function of the excess demand. Selecting proper values for parameter $\sigma > 0$ in the dynamic discrete-time model determine feasible price trajectories for 10 subsequent periods: $t = 1, 2, \ldots, 10$.

E5. An owner of a strawberry plantation hires one worker who has 24 units of time. The employee can allocate part of the time to work and part to rest. He/she owns 20% of shares in profits of the plantation. The only production factor in producing strawberries is the labour of the worker. The process of production of strawberries is described by a power production function of a form:

$$w_1 = f(n_2) = n_2^{0.5},$$

where w_1 denotes the output in the production process and n_2 denotes the input of labour. The plantation owner wants to maximize her/his profit from production and sales of strawberries for which their market price equals p_1 per unit (for example, one kilogramme, one crate etc.). The source of production cost is hiring the labour force for which its market price equals p_2 per one hour. The plantation owner, as a consumer, wants to maximize her/his utility $u^1(x_{11})$ described as an increasing function of consumed strawberries' quantity. Her/his income consists of 80% of shares in the profit of the plantation. The worker wants to maximize her/his utility described as an increasing function of consumed strawberries' quantity and of leisure:

$$u^2(x_{21}, x_{22}) = x_{21}^{0.5} x_{22}^{0.3}.$$

Income of the worker consists of 20% of shares in the profit of the plantation and a money value of 24 time units allocated partly to work and partly to leisure.

For the static Arrow-Debreu-McKenzie model:

1. Write a space of the net output of the strawberries' plantation. Note: according to the description of the exercise the input n_1 of strawberries equals 0. While labour is not produced, hence $w_2 = 0$.
2. Formulate and solve the profit maximization problem for the plantation owner.
3. Formulate and solve the consumption utility maximization problems for the plantation owner and for the worker.
4. Write a form of an excess demand function.
5. Determine a system of general equilibrium conditions and a Walrasian equilibrium price vector.
6. Determine components of a Walrasian equilibrium allocation.

E6. Consider a discrete-time version of the dynamic Arrow-Debreu-McKenzie model for the same data given as in Exercise E5. Initial prices are:

$$\mathbf{p}(0) = (10, 2).$$

Using formulas for the excess demand function and for a structure of the Walrasian equilibrium price vector derived in Exercise E5 for the static Arrow-Debreu-McKenzie model:

1. Determine trajectories of a price vector satisfying a system of equations of the dynamic discrete-time Arrow-Debreu-McKenzie model, taking a proportionality coefficient σ equal to $0.01, 0.05$ and 0.1. Calculate price ratios $\frac{p_2(t)}{p_1(t)}$ and compare them with the equilibrium price ratio $\frac{\bar{p}_2}{\bar{p}_1}$.
2. State which of trajectories determined in point 1 are feasible.
3. State if and when (in which period) a structure of prices stabilizes around the equilibrium price structure and whether it reaches this structure in time horizon $T = 30$.
4. Present graphs of the price trajectories in the state space.
5. Present graphs of the price trajectories as functions of time.

E7. Consider a continuous-time version of the dynamic Arrow-Debreu-McKenzie model for the same data given as in Exercise E5.

1. Determine trajectories of a price vector satisfying a system of equations of the dynamic continuous-time Arrow-Debreu-McKenzie model taking a proportionality coefficient σ equal to $0.01, 0.05, 0.1$ and determine whether these trajectories are feasible.
2. Determine if and when (at what moment) a structure of prices stabilizes around the equilibrium price structure.
3. Present graphs of price trajectories as functions of time.

7 Conclusion

In this chapter you will learn:

– what the basic ways to study microeconomic questions specified by economic agents, economic goods and social space of interaction are
– why neoclassical microeconomics is considered to be theory overly formalized in the mathematical sense
– what the symptoms of impasse in Walrasian general equilibrium theory in the 1970s that led to the changes in modern microeconomics were.

"For it is not enough to have a good mind, but the key is to apply it well"

Discourse on the Method

René Descartes

This handbook attempts to present elementary issues essentially related to the traditional approach to microeconomic research, which is closely related to the neoclassical mainstream economics.

The traditional approach to microeconomics, closely related to the works of Paul Samuelson and John R. Hicks, centres around four fundamental problems:

1. consumer theory—a study of the behaviour of households choosing consumer goods under budget constraints,
2. producer theory—studying the behaviour of enterprises that, while encountering technological limitations, are interested in maximizing profit or minimizing production costs,
3. the theory of the exchange of goods in markets where there is no competition or, accordingly, there is imperfect competition or perfect competition,

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4. the theory of economic efficiency, in which the concept of Pareto efficiency is used to assess the collective effectiveness of interactions occurring between economic agents and influenced by the exchange of goods.

In this approach, economic agents are assumed to operate rationally. This means that they have cognitive abilities and have enough information to define the criteria for evaluating various possible actions, as well as to identify internal and external constraints characteristic to a specific economic agent, affecting their choices. This way of understanding rationality is related to the idea of searching for the conditional optimum, which is related to the "*homo economicus*" paradigm. The subject of interest of microeconomics is primarily the study of the choices made by economic agents, i.e. the way they arbitrage among the possible options for various activities by comparing the benefits and losses incurred in achieving their goals or satisfying their interests. In this approach, one may observe an opposition between the opportunities (possible choices) and goals (benefits and expectations of economic agents). This can apply to a wide variety of options and goals. The neoclassical microeconomics considers the activities of enterprises, the aim of which is actually not to maximize the profit of their shareholders, but to maximize the utility of their owners' income. Another example would be "dynastic" households, whose aim is not to maximize the welfare of their members, but to maximize their own welfare while taking into account the well-being of numerous generations of descendants. Such approach may be regarded as conventional. It has the character of an even selection of, on the one hand, a set of goals and, on the other, the available means of their implementation by an abstract decision-making entity. This approach is not intended to describe the behaviour of specific "real" economic agents, but to provide a basis for predicting the overall consequences of such interactions. Traditional microeconomics can distinguish three main types of this kind of convention:

– "economic agents", or households and enterprises, which are considered as "black boxes", when in reality they are groups of individuals that may differ significantly in size (e.g. large multinational enterprises). But nothing should prevent us from "opening these black boxes" in order to conduct a microeconomic analysis in relation to the interior of the household or of the enterprise,
– "goods" that are of interest to economic agents. The concept of goods is defined with a certain degree of arbitrariness. They may vary in type and quality, may be produced conventionally or unconventionally (e.g. environmental pollution), legally or illegally (e.g. drugs),
– "social space of interaction", which in traditional microeconomics is identified with markets as spaces for transactions. However, one can also apply the microeconomic neoclassical analysis to all kinds of social transaction spaces, such as corporate internal markets, or formal and informal networks between economic agents.

The prevailing opinion about microeconomics in the neoclassical perspective is that it is strongly formal. Opinions of this type are partly related to the fact that the use of mathematical models is an important part of the neoclassical approach to economics, which, on the one hand, encounters a shortage of mathematical knowledge adequate to the complexity of the economic problems under consideration, and on the other hand, ignorance or reluctance to the use of formal language by economists acting as analysts of economic processes. Another distinguishing feature of the neoclassical approach is significant attachment to individual behaviour. This means that we have sufficient ability to relatively precisely define this type of action. The high importance attached to individual undertakings a priori leads to the rejection of the "hierarchical" vision of economic interactions and to its replacement by the vision of "horizontal" interactions. As a result, when interpreting economic facts and processes, the linear form of cause-effect relations is replaced with "co-causality" (relations of the feedback type).

The traditional approach to microeconomics is quite strongly related to Walras' theory of general equilibrium. However, due to some deadlock occurring in this theory felt particularly strongly in the 1970s, significant modifications were implemented thus giving rise to modern microeconomics. The first symptom of the changes taking place at that time was the expansion of research in microeconomics. One example is the search for relationships between microeconomics and macroeconomics, expressed in the creation of the theory of macroeconomics, firmly embedded in the theory of microeconomics. As a result of expanding the research area of microeconomics, new approaches and paradigms have emerged that are complementary and sometimes contradictory. At this point, it is worth presenting, as briefly as possible, two of them: the agency theory and the theory of contracts. In the case of agency theory, motivation and information play a key role. "Motivation" is understood here as the action of an economic agent (e.g. government, company director, …) inducing certain economic agents (citizens, company employees, …) to a strictly defined type of behaviour in conditions of limited access to full information necessary for effective operation. In other words, this is called information asymmetry. Information asymmetry may lead to negative selection or anti-selection when the result of a specific action is known only to one of at least two business agents implementing or intending to implement a joint action. This can lead to moral hazard (moral uncertainty) on the part of the economic agent who is fully knowledgeable about the specific action and its results. The asymmetry of access to information leads to an attempt to precisely define given activities in the form of an agreement accepted by all economic agents concerned, which should grant a certain information rent to economic agents having access to incomplete information. The theory of contracts extends this line of reasoning, by considering individual economic agents (consumers, producers, households, enterprises, …) in terms of binding contracts. An important aspect of these contracts is their incompleteness. This means that it is impossible to define all types of obligations between specific economic agents. The incompleteness of contracts is related to the notion of the right to make residual decisions. This concept relates to making decisions on matters not settled by the contract. The incompleteness of contracts impacts the limited rationality in the organization's behaviour in relation to unpredictable or non-verifiable activities.

The development of the theory of contracts contributed to the development of negotiation and renegotiation theories. In fact, the subject of interest of economic agents is not only to know the method and reasons for creating contracts but also to know the reasons for their positive or negative effects.

The aforementioned theories are related to the institutional stream of economics, treated by some economists as a sort of alternative for the overly formalized neoclassical trend. However, this way of understanding of the problem is not correct. The fact that some important real aspects of economic phenomena and processes can be identified within informal economic theories does not automatically imply that the explanations they provide are sufficient or efficient from the point of view of economic agents.

In these theories, we encounter the concept of uncertainty that accompanies the actions of economic agents. One of the effective tools for describing and solving such problems may be the theory of cooperative or non-cooperative games in the conditions of either full or incomplete access to information. Thus, there is a type of formal language that can be useful for describing and solving problems that involve imprecise definitions of economic phenomena and processes, or some other type of uncertainty that can be described in the language of probability or stochastics.

Among the interesting attempts to develop the scope of microeconomic analysis, it is worth mentioning those directions that are interdisciplinary in nature. They include economic law, which may attempt to create a microeconomic theory based on legislation and the legal system. Another example is the new political economy, which is closely related to political science. Finally, behavioural microeconomics may be an interesting alternative, resulting from stronger links between economics and psychology or sociology.

The discussed examples of evolution in contemporary microeconomics allow to draw one's attention to another aspect of economic education. When we start economic studies, we usually lack a more complete set of references for the acquired economic knowledge. In general, economic education takes place in an isolated area of economic knowledge. It is true that students have the opportunity to supplement their knowledge in the field of related social sciences (sociology, selected aspects of law, philosophy). Unfortunately, the didactic offer lacks approaches that would explicitly accept a specific relationship of the hierarchy of orders, creating an appropriate framework for microeconomic analyses. Accepting the idea of Pascal's hierarchy of orders, microeconomic analyses should take into account at least three levels of such hierarchy: ethical and moral order (superior order), legal and political order (intermediate order) and finally economic order (subordinate order). The idea of the descending hierarchy of orders formulated in this way is, in our opinion, the most interesting direction for the development of microeconomics in the future, as it would allow to analyze specific economic phenomena and processes in a broader context that would take into account the ethical, legal, political and social aspects of economic activities.

Going beyond the circle of traditional economic analyses, which only partially take into account their sociological, philosophical, psychological or legal aspects, does not necessarily imply deepening the false division that occurs in economic theory between quantitative (quantifiable) and qualitative (non-quantifiable) economics. On the contrary, the achievements of the so-called qualitative economy should go hand in hand with extending the scope of formalization of microeconomic problems, previously informal. An appropriate way out of this methodological deadlock might be to try to create a systems theory that takes into account the idea of a descending hierarchy of orders.

Mathematical Appendix

A.1 Elements of Logic and Theory of Sets

In the book we formulate statements to which one attributes a specified logical value: **truth** or **falsity**. Let us denote by p and q simple statements to which we attribute a value 0 if they are false or a value 1 if they are true.

A **negation** of a simple statement p is denoted by $\neg p$, which is to be read as "it is not that p".

Having simple statements one can form compound statements using logical connectives:

- a **conjunction** $p \wedge q$ of statements p and q is a true statement if and only if both statements p and q are true. We read it as: p and q.
- a **disjunction** $p \vee q$ of statements p and q is a true statement if and only if at least one of statements p or q is true. We read it as: p or q.
- an **implication** $p \Rightarrow q$ with an antecedent p and a consequent q is a false statement if and only if the antecedent p is true and the consequent q is false. We read it as: if p then q.
- an **equivalence** $p \Leftrightarrow q$ of statements p and q is a true statement if and only if both statements p and q have the same logical value. We read it as: p if and only if q.

According to De Morgan's laws:

- a negation of a disjunction is a conjunction of negations

(A.1) $$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q,$$

- negation of a conjunction is a disjunction of negations Table A.1

(A.2) $$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q.$$

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Table A.1 Logical values of simple and compound statements

p	q	$\neg p$	$\neg q$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
0	0	1	1	0	0	1	1	1	1	1	1
0	1	1	0	0	1	1	0	0	0	1	1
1	0	0	1	0	1	0	0	0	0	1	1
1	1	0	0	1	1	1	1	0	0	0	0

A concept of a **set** is so-called primary concept, that is something one does not define. A set consists of its elements which can be any mathematical objects of a selected type (for example: numbers, functions, derivatives, etc., also sets). A special case is a set $\emptyset$ which has no elements, called an **empty set**. To denote that a given mathematical object belongs or does not belong to a set one writes:

$x \in A$ — x belongs to set A,
$x \notin A$ — x does not belong to set A.

Two sets *A* and *B* are equal, which is written as $A = B$, if and only if every element of set A is an element of set B and every element of set B is an element of set A.

Set A is **contained in** set B (A is a **subset of** B), which is written as $A \subseteq B$, if and only if every element of set A is an element of set B. We also say that $A \subseteq B$ is an **inclusion** of A in B.

If $A \subseteq B \wedge A \neq B$ we say then that A is a **proper subset** of B.

Basic operations on sets.

- an **union of sets *A* and *B*** is a set:

(A.3) $$A \cup B = \{x | x \in A \vee x \in B\}$$

- an **intersection of sets *A* and *B*** (also a common part or a product) is a set:

(A.4) $$A \cap B = \{x | x \in A \wedge x \in B\}$$

- a **difference of sets *A* and *B*** is a set:

(A.5) $$A \backslash B = \{x | x \in A \wedge x \notin B\}$$

- a **Cartesian product of sets *A* and *B*** is a set:

(A.6) $$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

of all ordered pairs (a, b) in which the first element of a pair belongs to set A and the second element of a pair belongs to set B. If one considers Cartesian product $A \times A$ then it is called also the Cartesian product on A.[1]

[1] In the book we use a Cartesian product defined on a space of goods $X = \mathbb{R}^2_+$ (see Definition A.6).

Selected properties of sets

Definition A.1 An element $\mathbf{x} \in A \subseteq \mathbb{R}_+^2$ is called a limit of a sequence $\{\mathbf{x}^i\}_{i=1}^{+\infty}$ if $\lim_{i \to +\infty} d(\mathbf{x}^i, \mathbf{x}) \to 0$, which one writes as:

(A.7) $$\lim_{i \to +\infty} \mathbf{x}^i = \mathbf{x} \quad \text{or} \quad \mathbf{x}^i \to_{i \to +\infty} \mathbf{x}.$$

Definition A.2 Set $A \subseteq X = \mathbb{R}_+^2$ is called a **closed set** when:

(A.8) $$\forall \mathbf{x}^i \in A \quad \lim_{i \to +\infty} \mathbf{x}^i = \mathbf{x} \Rightarrow x \in A.$$

Definition A.3 Set $A \subseteq X = \mathbb{R}_+^2$ is called a **bounded set** when:

(A.9) $$\forall \mathbf{x}^1, \mathbf{x}^2 \in A \quad \exists N > 0 \quad d(\mathbf{x}^1, \mathbf{x}^2) < N.$$

Definition A.4 Set $A \subseteq X = \mathbb{R}_+^2$ is called a **compact set** when it is bounded and closed.

Definition A.5 Set $A \subseteq X = \mathbb{R}_+^2$ is called a **convex set** when:

(A.10) $$\forall \mathbf{x}^1, \mathbf{x}^2 \in A, \ \forall \alpha, \beta \geq 0, \ \alpha + \beta = 1 \quad \alpha \mathbf{x}^1 + \beta \mathbf{x}^2 \in A.$$

Definition A.6 A Cartesian product defined on a space of goods on $X = \mathbb{R}_+^2$ is a set:

(A.11) $$X \times X = \{(\mathbf{x}^1, \mathbf{x}^2) | \mathbf{x}^1 \in X, \mathbf{x}^1 \in X\},$$

of all ordered **pairs** of consumption bundles in which both bundles of goods (the first and the second one in a pair) belong to the space of goods.

Relation of preference as subset of Cartesian product and its properties

Definition A.7 A **relation of (weak) preference** is a set:

(A.12) $$P = \left\{\left(\mathbf{x}^1, \mathbf{x}^2\right) \in X \times X | \mathbf{x}^1 \succsim \mathbf{x}^2\right\},$$

of all ordered pairs of consumption bundles in which the first bundle is not worse (weakly preferred) than the second bundle.

Definition A.8 A **relation of strong preference** is a set:

(A.13) $$P_s = \{\left(\mathbf{x}^1, \mathbf{x}^2\right) \in X \times X | \mathbf{x}^1 \succ \mathbf{x}^2\},$$

of all ordered pairs of consumption bundles in which the first bundle is better (strongly preferred) than the second bundle.

Definition A.9 A **relation of consumer indifference** is a set:

(A.14) $$I = \{\left(\mathbf{x}^1, \mathbf{x}^2\right) \in X \times X | \mathbf{x}^1 \sim \mathbf{x}^2\},$$

of all ordered pairs of consumption bundles in which the first bundle is as good **(indifferent**) as the second bundle.

Definition A.10 A relation P of consumer (weak) preference is a relation of a **total preorder**, which means that it is complete and transitive:

(A.15) $$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathrm{X} = \mathbb{R}^2_+ \quad \mathrm{x}^1 \succsim \mathbf{x}^2 \vee \mathbf{x}^2 \succsim \mathbf{x}^1 \text{ (completeness)},$$

(A.16) $$\forall \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in X = \mathbb{R}^2_+ \quad \mathbf{x}^1 \succsim \mathbf{x}^2 \wedge \mathbf{x}^2 \succsim \mathbf{x}^3 \Rightarrow \mathbf{x}^1 \succsim \mathbf{x}^3 \text{(transitivity)}.$$

A.2 Linear Algebra

A n-dimensional vector $\mathbf{x}$ with coordinates being real numbers is an element/point of the n-dimensional[2] space $\mathbb{R}^n$ of real numbers. A notation:

[2] In the book we consider only one- or two-dimensional spaces of real numbers or real nonnegative numbers.

$\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ means that each coordinate of this vector is a real number,
$\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+$ means that each coordinate of this vector is a nonnegative real number,
$\mathbf{x} = (x_1, x_2) \in \operatorname{int}\mathbb{R}^2_+$ means that each coordinate of this vector is a positive real number.

Operations on vectors and on matrices

- addition:

(A.17) $$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2 \quad \mathbf{x}^1 + \mathbf{x}^2 = (x_{11} + x_{21}, x_{12} + x_{22}) \in \mathbb{R}^2$$

- subtraction:

(A.18) $$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2 \quad \mathbf{x}^1 - \mathbf{x}^2 = (x_{11} - x_{21}, x_{12} - x_{22}) \in \mathbb{R}^2$$

- multiplication by a number:

(A.19) $$\forall \lambda \in \mathbb{R}, \forall \mathbf{x} \in \mathbb{R}^2 \quad \lambda\mathbf{x} = (\lambda x_1, \lambda x_2) \in \mathbb{R}^2$$

- multiplication of vectors (scalar product, also inner or dot product)

(A.20) $$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2 \quad \mathbf{x}^1, \mathbf{x}^2 = \sum_{i=1}^{2} x_{1i}x_{2i} = x_{11}x_{21} + x_{12}x_{22}$$

Definition A.11 A **linear convex combination of vectors** $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n$ is a vector $\mathbf{x} \in \mathbb{R}^n$ such that:

(A.21) $$\mathbf{x} = \alpha\mathbf{x}^1 + \beta\mathbf{x}^2 \quad \text{where} \quad \alpha, \beta \geq 0, \ \alpha + \beta = 1.$$

Definition A.12 A **line segment with vectors** $\mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n$ as **endpoints** is a set of all linear convex combinations of vectors $\mathbf{x}^1, \mathbf{x}^2$:

(A.22) $$\left[\mathbf{x}^1, \mathbf{x}^2\right] = \left\{\alpha\mathbf{x}^1 + \beta\mathbf{x}^2 | \, \alpha, \beta \geq 0, \ \alpha + \beta = 1\right\} \subset \mathbb{R}^n.$$

Definition A.13 A **determinant of matrix**[3] $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is an expression:

(A.23) $$\det \mathbf{A} = |\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}.$$

[3] A value of a determinant is a number (scalar). A determinant of a matrix refers only to square matrices. In the case of matrices of higher orders (n≥3) there is a different, more complex definition of a determinant of a matrix.

A.3 Mathematical Analysis

Definition A.14 A **function** f **from** X **to** Y, denoted as $f: X \to Y$ is a mapping that to any element from set X assigns exactly one element from set Y. Set X is called a **domain of function** f (also set of arguments). Set Y is called a **codomain of function** f (also set of values).

Let us assume that $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$, where m, n are some natural numbers. If $n \geq 2$ then $f: X \to Y$ is called a **vector function**, which means its values are vectors. If $m = n = 1$ then $f: X \to Y$ is called a **one-variable scalar function**, which means its arguments are numbers and its values are also numbers (scalars). If $m = 2, n = 1$ then $f: X \to Y$ is called a two-variable scalar function, which means its arguments are vectors while its values are numbers (scalars).

Definition A.15 A **metric** is a mapping $d: \mathbb{R}^{2n} \to \mathbb{R}_+$ that satisfies all the three following conditions[4]:

1. $\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n \quad d(\mathbf{x}^1, \mathbf{x}^2) = 0 \Leftrightarrow \mathbf{x}^1 = \mathbf{x}^2$,
2. $\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n \quad d(\mathbf{x}^1, \mathbf{x}^2) = d(\mathbf{x}^2, \mathbf{x}^1)$,
3. $\forall \mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathbb{R}^n \quad d(\mathbf{x}^1, \mathbf{x}^3) \leq d(\mathbf{x}^1, \mathbf{x}^2) + d(\mathbf{x}^2, \mathbf{x}^3)$.

Examples of metrics

(a) Euclidean: $\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n \quad d_E(\mathbf{x}^1, \mathbf{x}^2) = \sqrt{(x_{11} - x_{21})^2 + \ldots + (x_{1n} - x_{2n})^2}$,
(b) Non-Euclidean: $\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^n \quad d_{NE}(\mathbf{x}^1, \mathbf{x}^2) = \max\{|x_{11} - x_{21}|, \ldots, |x_{1n} - x_{2n}|\}$.

Selected properties of function $f: \mathbb{R}_+^2 \to \mathbb{R}$.

Definition A.16 Function $f: \mathbb{R}_+^2 \to \mathbb{R}$ is called **continuous at point** $\mathbf{x} \in \mathbb{R}_+^2$ if for any sequence $\{\mathbf{x}^i\}_{i=1}^{+\infty} \subset \mathbb{R}_+^2$ it is satisfied that:

(A.24) $$\lim_{i \to +\infty} \mathbf{x}^i = \mathbf{x} \Rightarrow \lim_{i \to +\infty} f(\mathbf{x}^i) = f(\mathbf{x}).$$

Definition A.17 Function $f: \mathbb{R}_+^2 \to \mathbb{R}$ is called **continuous on space** $\mathbb{R}_+^2$ if it is continuous at every point of this space.

Definition A.18 Function $f: \mathbb{R}_+^2 \to \mathbb{R}$ is called **differentiable on space** $\mathbb{R}_+^2$ if its partial derivatives of 1st order:

(A.25) $$\frac{\partial f(x_1, x_2)}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1},$$

[4] In the book we use mainly assumptions that $n = 1$ or $n = 2$.

(A.26) $$\frac{\partial f(x_1, x_2)}{\partial x_2} = \lim_{\Delta x_2 \to 0} \frac{f(x_1, x_2 + \Delta x_2) - f(x_1, x_2)}{\Delta x_2},$$

are continuous on this space.

Formulas of selected elementary functions $f{:}\mathbb{R}_+^2 \to \mathbb{R}$.

(a) **constant function:** $y = f(\mathbf{x}) = a$,

(A.27) $$\frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{\partial f(x_1, x_2)}{\partial x_2} = 0,$$

(b) **linear function:** $y = f(\mathbf{x}) = a_1 x_1 + a_2 x_2$,

(A.28) $$\frac{\partial f(x_1, x_2)}{\partial x_1} = a_1, \quad \frac{\partial f(x_1, x_2)}{\partial x_2} = a_2,$$

(c) **power function:** $y = f(\mathbf{x}) = a x_1^{\alpha_1} x_2^{\alpha_2}$,

(A.29) $$\frac{\partial f(x_1, x_2)}{\partial x_1} = \alpha_1 a x_1^{\alpha_1 - 1} x_2^{\alpha_2}, \qquad \frac{\partial f(x_1, x_2)}{\partial x_2} = \alpha_2 a x_1^{\alpha_1} x_2^{\alpha_2 - 1},$$

(d) **logarithmic function:** $y = f(\boldsymbol{x}) = a_1 \ln x_1 + a_2 \ln x_2$,

(A.30) $$\frac{\partial f(x_1, x_2)}{\partial x_1} = \frac{a_1}{x_1}, \qquad \frac{\partial f(x_1, x_2)}{\partial x_2} = \frac{a_2}{x_2}.$$

Formulas for derivatives of sum, product and quotient of two functions of two variables

1. $f(\mathbf{x}) = g(\mathbf{x}) + h(\mathbf{x})$

(A.31) $$\frac{\partial f(\mathbf{x})}{\partial x_i} = \frac{\partial g(\mathbf{x})}{\partial x_i} + \frac{\partial h(\mathbf{x})}{\partial x_i}, i = 1, 2$$

2. $f(\mathbf{x}) = g(\mathbf{x}) \cdot h(\mathbf{x})$

(A.32) $$\frac{\partial f(\mathbf{x})}{\partial x_i} = \frac{\partial g(\mathbf{x})}{\partial x_i} h(\boldsymbol{x}) + \frac{\partial h(\mathbf{x})}{\partial x_i} g(\mathbf{x}), \ i = 1, 2$$

3. $f(\mathbf{x}) = \frac{g(\mathbf{x})}{h(\mathbf{x})}$

(A.33) $$\frac{\partial f(\boldsymbol{x})}{\partial x_i} = \frac{\frac{\partial g(\mathbf{x})}{\partial x_i} h(\boldsymbol{x}) - \frac{\partial h(\mathbf{x})}{\partial x_i} g(\mathbf{x})}{(h(\mathbf{x}))^2}, \quad i = 1, 2$$

Definition A.19 A function $f:\mathbb{R}_+^2 \rightarrow \mathbb{R}$ is called:

(a) **convex** on space $\mathbb{R}_+^2$ if

$$\begin{aligned}&\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}_+^2 \ \forall \alpha, \beta \geq 0 \ \alpha + \beta = 1\\ &f\left(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2\right) \geq \alpha f\left(\mathbf{x}^1\right) + \beta f\left(\mathbf{x}^2\right),\end{aligned} \tag{A.34}$$

(b) **concave** on space $\mathbb{R}_+^2$ if

$$\begin{aligned}&\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}_+^2 \ \forall \alpha, \beta \geq 0 \ \alpha + \beta = 1\\ &f\left(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2\right) \leq \alpha f\left(\mathbf{x}^1\right) + \beta f\left(\mathbf{x}^2\right),\end{aligned} \tag{A.35}$$

(c) **strictly convex** on space $\mathbb{R}_+^2$ if

$$\begin{aligned}&\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}_+^2, \mathbf{x}^1 \neq \mathbf{x}^2 \ \forall \alpha, \beta \geq 0 \alpha + \beta = 1\\ &f\left(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2\right) > \alpha f\left(\mathbf{x}^1\right) + \beta f\left(\mathbf{x}^2\right),\end{aligned} \tag{A.36}$$

(d) **strictly concave** on space $\mathbb{R}_+^2$ if

$$\begin{aligned}&\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}_+^2, \mathbf{x}^1 \neq \mathbf{x}^2 \ \forall \alpha, \beta \geq 0 \alpha + \beta = 1\\ &f\left(\alpha \mathbf{x}^1 + \beta \mathbf{x}^2\right) < \alpha f\left(\mathbf{x}^1\right) + \beta f\left(\mathbf{x}^2\right).\end{aligned} \tag{A.37}$$

Definition A.20 A function $f:\mathbb{R}_+^2 \rightarrow \mathbb{R}$ is called:

(a) **increasing** on space $\mathbb{R}_+^2$ if[5]

$$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}_+^2 \quad \mathbf{x}^1 \gneqq \mathbf{x}^2 \Rightarrow f\left(\mathbf{x}^1\right) > f\left(\mathbf{x}^2\right), \tag{A.38}$$

(b) **decreasing** on space $\mathbb{R}_+^2$ if

$$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}_+^2 \quad \mathbf{x}^1 \gneqq \mathbf{x}^2 \Rightarrow f\left(\mathbf{x}^1\right) < f\left(\mathbf{x}^2\right), \tag{A.39}$$

(c) **weakly increasing** on space $\mathbb{R}_+^2$ if

$$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}_+^2 \quad \mathbf{x}^1 \gneqq \mathbf{x}^2 \Rightarrow f\left(\mathbf{x}^1\right) \geq f\left(\mathbf{x}^2\right), \tag{A.40}$$

[5] An inequality $\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}_+^n \ \ \mathbf{x}^1 \gneqq \mathbf{x}^2$ means that at least one component x_i^1 of a vector $\mathbf{x}^1$ is bigger than the corresponding component x_i^2 of a vector $\mathbf{x}^2$ while the other components x_j^1 ($j = 1, 2, \ldots, n, \ j \neq i$, here $n = 2$) are bigger or equal to corresponding components x_j^2.

(d) **weakly decreasing** on space $\mathbb{R}^2_+$ if

(A.41) $$\forall \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}^2_+ \quad \mathbf{x}^1 \gneqq \mathbf{x}^2 \Rightarrow f\left(\mathbf{x}^1\right) \leq f\left(\mathbf{x}^2\right).$$

If a function $f{:}\mathbb{R}^2_+ \to \mathbb{R}$ is differentiable on its domain, then it is:

(a) increasing, when

(A.42) $$\frac{\partial u(\mathbf{x})}{\partial x_i} > 0, i = 1, 2,$$

(b) decreasing, when

(A.43) $$\frac{\partial u(\mathbf{x})}{\partial x_i} < 0, i = 1, 2,$$

(c) weakly increasing, when

(A.44) $$\frac{\partial u(\mathbf{x})}{\partial x_i} \geq 0, i = 1, 2,$$

(d) weakly decreasing, when

(A.45) $$\frac{\partial u(\mathbf{x})}{\partial x_i} \leq 0, i = 1, 2.$$

Unconditional optimization

In the whole book we consider searching for extremum points only for scalar, one-variable $\left(f : \mathbb{R}^{2+}_+ \to \mathbb{R}\right)$ or two-variable functions ($f{:}\mathbb{R}^2_+ \to \mathbb{R}$), which are assumed to be twice differentiable, thus with continuous derivatives of first and of second order. If they are additionally strictly concave then we search for maximum points. If they are strictly convex then we search for minimum points. We focus on extremum points $\bar{\mathbf{x}} > 0$.

In other words, we solve problems of unconditional optimization for a value of a function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ in an interior of its domain, that is in a set $\operatorname{int}\mathbb{R}^2_+$.

If a function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ is twice differentiable and strictly concave then we say that it reaches its **maximum** $f(\bar{\mathbf{x}})$ at point $\bar{\mathbf{x}} > 0$ if $\forall \mathbf{x} \in \operatorname{int}\mathbb{R}^2_+ \quad f(\bar{\mathbf{x}}) \geq f(\mathbf{x})$. Unconditional maximization problem for a function f has a form:

(A.46)
$$\bar{\mathbf{x}} = \arg \max_{\mathbf{x}\in \operatorname{int}\mathbb{R}^2_+} f(\mathbf{x}).$$

If a function $g\colon \mathbb{R}^2_+ \to \mathbb{R}$ is twice differentiable and strictly convex then we say that it reaches its **minimum** $g(\bar{\mathbf{x}})$ at point $\bar{\mathbf{x}} > 0$ if $\forall \mathbf{x} \in \operatorname{int}\mathbb{R}^2_+ \quad g(\bar{\mathbf{x}}) \leq g(\mathbf{x})$. Unconditional minimization problem for a function g has a form:

(A.47)
$$\bar{\mathbf{x}} = \arg \min_{\mathbf{x}\in \operatorname{int}\mathbb{R}^2_+} g(\mathbf{x}).$$

If function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ satisfies a condition:

(A.48)
$$\lim_{x_i \to 0^+} \frac{\partial f(\mathbf{x})}{\partial x_i} > 0 \wedge \lim_{x_i \to +\infty} \frac{\partial f(\mathbf{x})}{\partial x_i} < 0, \quad i = 1, 2,$$

then:

(A.49)
$$\exists_1\, \bar{\mathbf{x}} > 0 \quad \left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{x\,=\,\bar{\mathbf{x}}} = 0, \quad i = 1, 2.$$

When the function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ is striclty concave then (A.49) is a necessary and sufficient condition for the existence of a maximum of this function at point $\bar{\mathbf{x}} > 0$.

If function $g\colon \mathbb{R}^2_+ \to \mathbb{R}$ satisfies a condition:

(A.50)
$$\lim_{x_i \to 0^+} \frac{\partial g(\mathbf{x})}{\partial x_i} < 0 \wedge \lim_{x_i \to +\infty} \frac{\partial g(\mathbf{x})}{\partial x_i} > 0, \quad i = 1, 2,$$

then:

(A.51)
$$\exists_1\, \bar{\mathbf{x}} > 0 \quad \left.\frac{\partial g(\mathbf{x})}{\partial x_i}\right|_{x\,=\,\bar{\mathbf{x}}} = 0, \quad i = 1, 2.$$

When the function $g\colon \mathbb{R}^2_+ \to \mathbb{R}$ is striclty convex then (A.51) is a necessary and sufficient condition for the existence of a minimum of this function at point $\bar{\mathbf{x}} > 0$.

Definition A.21 Let a function $f : \mathbb{R}^2_+ \to \mathbb{R}$ be twice differentiable. Then a square and symmetric matrix of partial derivatives of second order for function f:

(A.52)
$$H(\mathbf{x}) = H(x_1, x_2) = \begin{bmatrix} \frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} & \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x_1,x_2)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} \end{bmatrix}$$

is called a Hessian (also Hessian matrix) of function f.

Definition A.22 A Hessian matrix $H(\mathbf{x})$ of a function $f : \mathbb{R}^2_+ \to \mathbb{R}$ is called a **negative definite** on its domain if $\forall \mathbf{x} \in \mathbb{R}^2_+$ a determinant of its principal submatrix of first order is negative:

(A.53)
$$\det H_1(\mathbf{x}) = \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} < 0,$$

while a determinant of its principal submatrix of 2nd order is positive:

(A.54)
$$\det H_2(\mathbf{x}) = \frac{\partial^2 f(\mathrm{x})}{\partial x_1^2} \cdot \frac{\partial^2 f(\mathrm{x})}{\partial x_2^2} - \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} \cdot \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1} > 0.$$

Definition A.23 A Hessian matrix $H(\mathbf{x})$ of a function $f : \mathbb{R}^2_+ \to \mathbb{R}$ is called a **negative definite** on its domain if $\forall \mathbf{x} \in \mathbb{R}^2_+$ determinants of all its principal submatrices are negative:

(A.55)
$$\det H_1(\mathbf{x}) = \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} > 0,$$

(A.56)
$$\det H_2(\mathbf{x}) = \frac{\partial^2 f(\mathrm{x})}{\partial x_1^2} \cdot \frac{\partial^2 f(\mathrm{x})}{\partial x_2^2} - \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} \cdot \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1} > 0.$$

Definition A.24 A Hessian matrix $H(\mathbf{x})$ of a function $f : \mathbb{R}^2_+ \to \mathbb{R}$ is called a **non-positive definite** on its domain if $\forall \mathbf{x} \in \mathbb{R}^2_+$ a determinant of its principal submatrix of first order is non-positive:

(A.57)
$$\det H_1(\mathbf{x}) = \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} \leq 0,$$

while a determinant of its principal submatrix of 2nd order is non-negative:

(A.58)
$$\det H_2(\mathbf{x}) = \frac{\partial^2 f(\mathrm{x})}{\partial x_1^2} \cdot \frac{\partial^2 f(\mathrm{x})}{\partial x_2^2} - \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} \cdot \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1} \geq 0.$$

Definition A.25 A Hessian matrix $H(\mathbf{x})$ of a function $f: \mathbb{R}_+^2 \to \mathbb{R}$ is called a **non-negative definite** on its domain if $\forall \mathbf{x} \in \mathbb{R}_+^2$ determinants of all its principal submatrices are non-negative:

$$\det H_1(\mathbf{x}) = \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} \geq 0, \quad \text{(A.59)}$$

$$\det H_2(\mathbf{x}) = \frac{\partial^2 f(\mathrm{x})}{\partial x_1^2} \cdot \frac{\partial^2 f(\mathrm{x})}{\partial x_2^2} - \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} \geq 0. \quad \text{(A.60)}$$

Theorem A.1 Let a function $f: \mathbb{R}_+^2 \to \mathbb{R}$ be twice differentiable. Then:

(i) function f is concave (convex) on its domain if and only if its Hessian is a non-positive (non-negative) definite matrix,
(ii) function f is strictly concave (strictly convex) on its domain if and only if its Hessian is a negative (positive) definite matrix.

Theorem A.2 If a function $f: \mathbb{R}_+ \to \mathbb{R}$ is twice differentiable, strictly concave and satisfies a condition: $\lim_{x \to 0^+} \frac{\mathrm{d}f(x)}{\mathrm{d}x} > 0 \wedge \lim_{x \to +\infty} \frac{\mathrm{d}f(x)}{\mathrm{d}x} < 0$, then there exists exactly one $\bar{x} > 0$ such that $\left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x = \bar{x}} = 0$ for which a function f reaches its maximum. A necessary and sufficient condition for existence of a stationary point $\bar{x}$ in which the function f reaches its maximum is $\left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x = \bar{x}} = 0$.

Theorem A.3 If a function $g: \mathbb{R}_+ \to \mathbb{R}$ is twice differentiable, strictly convex and satisfies a condition: $\lim_{x \to 0^+} \frac{\mathrm{d}g(x)}{\mathrm{d}x} < 0 \wedge \lim_{x \to +\infty} \frac{\mathrm{d}g(x)}{\mathrm{d}x} > 0$, then there exists exactly one $\bar{x} > 0$ such that $\left.\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right|_{x = \bar{x}} = 0$ for which a function f reaches its minimum. A necessary and sufficient condition for existence of a stationary point $\bar{x}$ in which the function f reaches its minimum is $\left.\frac{\mathrm{d}g(x)}{\mathrm{d}x}\right|_{x = \bar{x}} = 0$.

Theorem A.4 If a function $f: \mathbb{R}_+^2 \to \mathbb{R}$ is twice differentiable, strictly concave and satisfies a condition: $\lim_{x_i \to 0^+} \frac{\partial f(\mathbf{x})}{\partial x_i} > 0 \wedge \lim_{x_i \to +\infty} \frac{\partial f(\mathbf{x})}{\partial x_i} < 0, i = 1, 2$, then there exists exactly one $\bar{\mathbf{x}} > 0$ such that $\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x} = \bar{\mathbf{x}}} = 0, i = 1, 2$ for which a function f reaches its maximum. A necessary and sufficient condition for existence of a stationary point $\bar{\mathbf{x}}$ in which the function f reaches its maximum is $\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x} = \bar{\mathbf{x}}} = 0, i = 1, 2$.

Theorem A.5 If a function $g\colon \mathbb{R}^2_+ \to \mathbb{R}$ is twice differentiable, strictly convex and satisfies a condition: $\lim_{x_i \to 0^+} \frac{\partial f(\mathbf{x})}{\partial x_i} < 0 \wedge \lim_{x_i \to +\infty} \frac{\partial f(\mathbf{x})}{\partial x_i} > 0, i = 1, 2$, then there exists exactly one $\bar{\mathbf{x}} > 0$ such that $\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x} = \bar{\mathbf{x}}} = 0, i = 1, 2$ for which a function f reaches its minimum. A necessary and sufficient condition for the existence of a stationary point $\bar{\mathbf{x}}$ in which the function f reaches its minimum is $\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x} = \bar{\mathbf{x}}} = 0, i = 1, 2$.

Conditional optimization

In this book we consider many problems of conditional optimization that is all about searching for stationary points at which a function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ or $g\colon \mathbb{R}^2_+ \to \mathbb{R}$ reaches, respectively, its maximum or minimum in a nonempty, convex and compact set $B \subset X = \mathbb{R}^2_+$. Restricting the optimization problem to a subset B, thus introducing constraints in the optimization, one should apply concepts of local and global extrema.

If a strictly concave function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ satisfies a condition: $\lim_{x_i \to 0^+} \frac{\partial f(\mathbf{x})}{\partial x_i} > 0 \wedge \lim_{x_i \to +\infty} \frac{\partial f(\mathbf{x})}{\partial x_i} < 0, i = 1, 2$, then there exists exactly one $\bar{\mathbf{x}} > 0$ such that $\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x} = \bar{\mathbf{x}}} = 0, i = 1, 2$ for which a function f reaches its maximum. However, restricting the optimization problem to a subset B one should apply concepts of local and global maxima. A stationary point $\bar{\mathbf{x}} > 0$ at which a function f reaches its maximum in the whole domain is called a global maximum and denoted as $\bar{\mathbf{x}}^G$. It can be that $\bar{\mathbf{x}}^G \in B$ or $\bar{\mathbf{x}}^G \in X \backslash B$. When $\bar{\mathbf{x}}^G \in X \backslash B$ then a point $\bar{\mathbf{x}} \in B$ at which a function f reaches its maximum is called a **local maximum** of a function f in a set B and denoted as $\bar{\mathbf{x}}^L$.

If a strictly convex function $g\colon \mathbb{R}^2_+ \to \mathbb{R}$ satisfies a condition: $\lim_{x_i \to 0^+} \frac{\partial g(\mathbf{x})}{\partial x_i} < 0 \wedge \lim_{x_i \to +\infty} \frac{\partial g(\mathbf{x})}{\partial x_i} > 0, i = 1, 2$, then there exists exactly one $\bar{\mathbf{x}} > 0$ such that $\left.\frac{\partial g(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x} = \bar{\mathbf{x}}} = 0, i = 1, 2$ for which a function g reaches its minimum. However, restricting the optimization problem to a subset B one should apply concepts of local and global minima. A stationary point $\bar{\mathbf{x}} > 0$ at which a function g reaches its minimum in the whole domain is called a global minimum and denoted as $\bar{\mathbf{x}}^G$. It can be that $\bar{\mathbf{x}}^G \in B$ or $\bar{\mathbf{x}}^G \in X \backslash B$. In a case when $\bar{\mathbf{x}}^G \in X \backslash B$ then a point $\bar{\mathbf{x}} \in B$ at which a function g reaches its minimum is called a **local minimum** of a function g in a set B and denoted[6] as $\bar{\mathbf{x}}^L$.

[6] Here there is no need to distinguish denotation of stationary points between maxima and minima (local or global ones) because once we have formulated an optimization problem we know whether we search for a maximum or for a minimum.

Conditional maximization problem for a function f has a form:

(A.61) $$\bar{\mathbf{x}} = \arg \max_{\mathbf{x} \in B \subset \operatorname{int} \mathbb{R}^2_+} f(\mathbf{x}),$$

and unconditional minimization problem for a function g has a form:

(A.62) $$\bar{\mathbf{x}} = \arg \min_{\mathbf{x} \in B \subset \operatorname{int} \mathbb{R}^2_+} g(\mathbf{x}).$$

In the book we consider conditional optimization problems when a set B has a form:

(a) $B = \left\{\mathbf{x} \in \mathbb{R}^2_+ | p_1x_1 + p_2x_2 \leq I,\ x_1 \leq b_1,\ x_2 \leq b_2\right\}$,
(b) $B = \left\{\mathbf{x} \in \mathbb{R}^2_+ | p_1x_1 + p_2x_2 \leq I\right\}$,
(c) $B = \left\{\mathbf{x} \in \mathbb{R}^2_+ | \ x_1 \leq b_1,\ x_2 \leq b_2\right\}$,
(d) $B = \left\{\mathbf{x} \in \mathbb{R}^2_+ | h(\boldsymbol{x}) = y^o\right\}$,

where: $b_1, b_2, I, y^0 = \text{const.} > 0$.

If we are interested only in positive stationary points: $\bar{\mathbf{x}} > 0$ in the conditional optimization problem and B is described by constraints determining upper limit values for a vector $\mathbf{x} \in B \subset \mathbb{R}^2_+$ then two cases are possible. When $\bar{\mathbf{x}}^G \notin B$ then $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L \leq \mathbf{b}$. We say then that constraints on variables are binding. When $\bar{\mathbf{x}}^G \in B$ then $\bar{\mathbf{x}} = \bar{\mathbf{x}}^G \leq \mathbf{b}$. We say then that constraints on variables are not binding. In this second case, the conditional optimization problem reduces in fact to the unconditional optimization.

It is worth noticing that conditional optimization problems (A.61) and (A.62) are all about determining optimal solutions to a nonlinear programming problem for strictly concave (strictly convex) objective function in a nonempty, convex and compact set $B = \left\{\mathbf{x} \in \mathbb{R}^2_+ : \mathbf{x} \leq \mathbf{b}\right\} \subset X = \mathbb{R}^2_+$.

Let us formulate a maximization problem for values of a function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ in a nonempty, convex and compact set[7] $B \subset X = \mathbb{R}^2_+$:

(A.63) $$f(x_1, x_2) \mapsto \max$$

(A.64) $$x_1 \leq b_1,$$

(A.65) $$x_2 \leq b_2,$$

(A.66) $$h(x_1, x_2) \leq b_3,$$

[7] The set B is described by conditions (A.64)–(A.67).

(A.67) $$x_1, x_2 \geq 0.$$

where:

$f\colon \mathbb{R}^2_+ \to \mathbb{R}$ — twice differentiable and strictly concave,
$h\colon \mathbb{R}^2_+ \to \mathbb{R}$ — twice differentiable and concave,
$b_1, b_2, b_2 > 0$ — parameters.

Problem (A.63)–(A.67) can be expressed using a Lagrange function

(A.68) $$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \lambda_1(b_1 - x_1) + \lambda_2(b_2 - x_2) + \lambda_3(b_3 - h(x_1, x_2)).$$

Theorem A.6 (the Kuhn-Tucker theorem) Let a function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ be twice differentiable and strictly concave and a function $h\colon \mathbb{R}^2_+ \to \mathbb{R}$ twice differentiable and concave on a nonempty, convex and compact set $B \subset X = \mathbb{R}^2_+$. Then $\bar{\mathbf{x}} \geq 0$ is an optimal solution to problem (A.63)–(A.67) if and only if there exists a pair $\left(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}}\right) \geq 0$ that satisfies a set of conditions:

(A.69) $$\left.\frac{\partial L\left(\mathbf{x}, \bar{\boldsymbol{\lambda}}\right)}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} \leq 0, \; i = 1, 2$$

(A.70) $$\sum_{i=1}^{2} \bar{x}_i \left.\frac{\partial L\left(\mathbf{x}, \bar{\boldsymbol{\lambda}}\right)}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = 0,$$

(A.71) $$\left.\frac{\partial L(\bar{\mathbf{x}}, \boldsymbol{\lambda})}{\partial \lambda_j}\right|_{\boldsymbol{\lambda}=\bar{\boldsymbol{\lambda}}} \geq 0, \; j = 1, 2, 3,$$

(A.72) $$\sum_{j=1}^{3} \lambda_j \left.\frac{\partial L(\bar{\mathbf{x}}, \boldsymbol{\lambda})}{\partial \lambda_i}\right|_{\boldsymbol{\lambda}=\bar{\boldsymbol{\lambda}}} = 0,$$

where:

(A.73) $$\left.\frac{\partial L\left(\mathbf{x}, \bar{\boldsymbol{\lambda}}\right)}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} - \bar{\lambda}_i - \bar{\lambda}_3 \left.\frac{\partial h(x_1, x_2)}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}}, \quad i = 1, 2,$$

(A.74) $$\left.\frac{\partial L(\bar{\mathbf{x}}, \boldsymbol{\lambda})}{\partial \lambda_i}\right|_{\boldsymbol{\lambda}=\bar{\boldsymbol{\lambda}}} = b_i - \bar{x}_i, \quad i = 1, 2,$$

(A.75) $$\left.\frac{\partial L(\bar{\mathbf{x}}, \boldsymbol{\lambda})}{\partial \lambda_3}\right|_{\boldsymbol{\lambda}=\bar{\boldsymbol{\lambda}}} = b_3 - h(\bar{x}_1, \bar{x}_2).$$

Conditions (A.69) and (A.71) take forms:

(A.76)
$$\bar{x}_1\left(\frac{\partial f(\mathbf{x})}{\partial x_1}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}} - \bar{\lambda}_1 - \bar{\lambda}_3\frac{\partial h(x_1,x_2)}{\partial x_1}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}}\right) + \bar{x}_2\left(\frac{\partial f(\mathbf{x})}{\partial x_2}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}} - \bar{\lambda}_2 - \bar{\lambda}_3\frac{\partial h(x_1,x_2)}{\partial x_2}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}}\right) = 0$$

(A.77)
$$\bar{\lambda}_1(b_1 - \bar{x}_1) + \bar{\lambda}_2(b_2 - \bar{x}_2) + \bar{\lambda}_3(b_3 - h(\bar{x}_1, \bar{x}_2)) = 0,$$

where: $\bar{\lambda}_j = \frac{\partial f(\mathbf{x})}{\partial b_j}\Big|_{\mathbf{x}=\bar{\mathbf{x}}} \geq 0,\ \ j = 1,2,3$ means an optimal Lagrange multiplier which determines by how much the maximum value of a function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ approximately increases when a value of parameter b_j increases by one notional unit.

If $\bar{\lambda}_j > 0$ $(j = 1,2,3)$ then the j-th constraint is binding. When $\bar{\lambda}_j = 0$ then the j-th constraint is not binding.

Let us assume that we are interested only in a positive optimal solution $\bar{\mathbf{x}} > 0$ to problem (A.63)–(A.67). Then condition (A.75) is satisfied if and only if:

(A.78)
$$\frac{\partial f(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}} - \bar{\lambda}_i - \bar{\lambda}_3\frac{\partial h(x_1,x_2)}{\partial x_i}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}} = 0,\ i = 1,2.$$

If additionally $\forall j = 1,2,3\ \ \bar{\lambda}_j = 0$ then condition (A.75) takes the form:

(A.79)
$$\frac{\partial f(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\bar{\mathbf{x}}} = 0,\ i = 1,2.$$

This takes place when no constraint is binding, which means that:

(A.80)
$$\bar{\mathbf{x}} = \bar{\mathbf{x}}^G \leq \mathbf{b}.$$

When none of the constraints describing a set B is binding then an optimal solution $\bar{\mathbf{x}} = \bar{\mathbf{x}}^G$ to problem (A.63)–(A.67) is identical to a global maximum that a strictly concave function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ reaches in a space $X = \mathbb{R}^2_+$. Then the conditional maximization problem is the same as the unconditional maximization problem.

This gives us a useful conclusion. If we want to determine the optimal solution to problem (A.63)–(A.67) we should find a global maximum of a function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ in its domain $X = \mathbb{R}^2_+$. If $\bar{\mathbf{x}}^G \in B$ then the optimal solution to problem (A.63)–(A.67) is $\bar{\mathbf{x}} = \bar{\mathbf{x}}^G \leq \mathbf{b}$.

When $\forall j = 1,2,3\ \bar{\lambda}_j > 0$ then each constraint is binding and condition (A.77) is satisfied in the initial form. In the same time from Condition (A.76) we get that:

(A.81)
$$\bar{x}_i = b_i, \quad i = 1,2$$

and

$$h(\bar{x}_1, \bar{x}_2) = b_3. \tag{A.82}$$

In this case an optimal solution to problem (A.63)–(A.67) is a vector $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L$ such that $\bar{\mathbf{x}}^G \gneqq \bar{\mathbf{x}}^L$. Then a stationary point $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L$ is called a local maximum of a function $f: \mathbb{R}^2_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}^2_+$.

In any of the remaining six cases, when at least one optimal Lagrange multiplier is non-zero, we proceed in a similar way. As a result, on the basis of conditions (A.75) and (A.76), one gets an optimal solution $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L$ such that $\bar{\mathbf{x}}^G \gneqq \bar{\mathbf{x}}^L$. A stationary point $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L$ is then called a local maximum of a function $f: \mathbb{R}^2_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}^2_+$.

If in the conditional optimization problem one concerns a strictly convex function $g: \mathbb{R}^2_+ \to \mathbb{R}$ and searches for a stationary point at which this function reaches its minimum then one can proceed in a similar way as for a strictly concave function and searching a maximum. The minimization problem can be expressed in an equivalent form when one substitutes a strictly convex function $g: \mathbb{R}^2_+ \to \mathbb{R}$ with such a strictly concave function $f: \mathbb{R}^2_+ \to \mathbb{R}$ that $\forall \mathbf{x} \subset \mathbb{R}^2_+ \quad g(\mathbf{x}) = -f(\mathbf{x})$ and searches for a stationary point in a set B at which a function f reaches its maximum.

The Kuhn-Tucker theorem refers to a strictly concave function, while the comments concern a positive optimal solution $\bar{\mathbf{x}} > 0$ in a set B. It is not difficult to formulate analogical comments in cases when an optimal solution is non-negative $\bar{\mathbf{x}} \geq 0$ or when a function $f: \mathbb{R}^2_+ \to \mathbb{R}$ is just concave (not strictly) and a function $g: \mathbb{R}^2_+ \to \mathbb{R}$ is just convex (not strictly).

Let us present conclusions resulting from the Kuhn-Tucker theorem for two special cases of problem (A.63)–(A.67).

Case 1 Let a function $f: \mathbb{R}^2_+ \to \mathbb{R}$ be strictly concave and twice differentiable. Let us assume that $h(x_1, x_2) = p_1 x_1 + p_2 x_2, \quad b_1 = b_2 = +\infty, \quad b_3 = I > 0$. Then problem (A.63)–(A.67) can be expressed in a form:

$$f(x_1, x_2) \mapsto \max \tag{A.83}$$

$$p_1 x_1 + p_2 x_2 \leq I, \tag{A.84}$$

$$x_1, x_2 \geq 0, \tag{A.85}$$

to which the following Lagrange function corresponds:

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \lambda(I - p_1 x_1 - p_2 x_2). \tag{A.86}$$

If we assume additionally that a function $f : \mathbb{R}^2_+ \mapsto \mathbb{R}$ satisfies a condition:

$$\lim_{x_i \to 0^+} \frac{\partial f(\mathbf{x})}{\partial x_i} > 0 \wedge \lim_{x_i \to +\infty} \frac{\partial f(\mathbf{x})}{\partial x_i} < 0, \quad i = 1, 2, \tag{A.87}$$

then the Kuhn-Tucker theorem can be formulated as follows.

Theorem A.7 (the Kuhn-Tucker theorem) Let a function $f : \mathbb{R}^2_+ \to \mathbb{R}$ be twice differentiable and strictly concave and a function $h : \mathbb{R}^2_+ \to \mathbb{R}$ twice differentiable and concave on a nonempty, convex and compact set $B = \left\{ \mathbf{x} \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \le I \right\} \subset X = \mathbb{R}^2_+$. Then a vector $\bar{\mathbf{x}} \ge 0$ is an optimal solution to problem (A.83)–(A.85) if and only if there exists a pair $(\bar{\mathbf{x}}, \bar{\lambda}) \ge 0$ that satisfies a set of conditions:

$$\left. \frac{\partial L(\mathbf{x}, \bar{\lambda})}{\partial x_i} \right|_{\mathbf{x} = \bar{\mathbf{x}}} \le 0, \; i = 1, 2, \tag{A.88}$$

$$\sum_{i=1}^{2} \bar{x}_i \left. \frac{\partial L(\mathbf{x}, \bar{\lambda})}{\partial x_i} \right|_{\mathbf{x} = \bar{\mathbf{x}}} = 0, \tag{A.89}$$

$$\left. \frac{\partial L(\bar{\mathbf{x}}, \lambda)}{\partial \lambda} \right|_{\lambda = \bar{\lambda}} \ge 0, \tag{A.90}$$

$$\bar{\lambda} \left. \frac{\partial L(\bar{\mathbf{x}}, \lambda)}{\partial \lambda} \right|_{\lambda = \bar{\lambda}} = 0, \tag{A.91}$$

where:

$$\left. \frac{\partial L(\mathbf{x}, \bar{\lambda})}{\partial x_i} \right|_{\mathbf{x} = \bar{\mathbf{x}}} = \left. \frac{\partial f(\mathbf{x})}{\partial x_i} \right|_{\mathbf{x} = \bar{\mathbf{x}}} - \bar{\lambda} p_i, \quad i = 1, 2, \tag{A.92}$$

$$\left. \frac{\partial L(\bar{\mathbf{x}}, \lambda)}{\partial \lambda} \right|_{\lambda = \bar{\lambda}} = I - p_1 \bar{x}_1 - p_2 \bar{x}_2. \tag{A.93}$$

Conditions (A.89) and (A.91) take forms:

$$\bar{x}_1 \left(\left. \frac{\partial f(\mathbf{x})}{\partial x_1} \right|_{\mathbf{x} = \bar{\mathbf{x}}} - \bar{\lambda} p_1 \right) + \bar{x}_2 \left(\left. \frac{\partial f(\mathbf{x})}{\partial x_2} \right|_{\mathbf{x} = \bar{\mathbf{x}}} - \bar{\lambda} p_2 \right) = 0, \tag{A.94}$$

$$\bar{\lambda}(I - p_1 \bar{x}_1 - p_2 \bar{x}_2) = 0, \tag{A.95}$$

where: $\bar{\lambda} = \left. \frac{\partial f(\mathbf{x})}{\partial I} \right|_{\mathbf{x} = \bar{\mathbf{x}}} \ge 0$ means an optimal Lagrange multiplier, which determines by how much the maximum value of a function $f : \mathbb{R}^2_+ \to \mathbb{R}$ approximately increases when a value of parameter I increases by one notional unit.

If $\bar{\lambda} > 0$ then constraint (A.84) is binding. When $\bar{\lambda} = 0$ then the constraint is not binding.

Let us assume that we are interested only in a positive optimal solution $\bar{\mathbf{x}} > 0$ to problem (A.83)–(A.85) then condition (A.94) is satisfied if and only if:

(A.96) $$\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} - \bar{\lambda} p_i = 0, \quad i = 1, 2.$$

If additionally $\bar{\lambda} = 0$ then condition (A.96) takes the form:

(A.97) $$\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = 0, \; i = 1, 2.$$

This takes place when constraint (A.84) is not binding, which means that:

(A.98) $$\bar{\mathbf{x}} = \bar{\mathbf{x}}^G.$$

When the constraint describing a set B is not binding then an optimal solution $\bar{\mathbf{x}} = \bar{\mathbf{x}}^G$ to problem (A.83)–(A.85) is identical to a global maximum that a strictly concave function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ reaches in a space $X = \mathbb{R}^2_+$. Then the conditional maximization problem is the same as the unconditional maximization problem.

When $\bar{\lambda} > 0$ then the constraint is binding and condition (A.94) is satisfied in the initial form. At the same time from condition (A.95) we get that:

(A.99) $$p_1\bar{x}_1 + p_2\bar{x}_2 = I.$$

In this case an optimal solution to problem (A.83)–(A.85) is a vector $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L$ such that $\bar{\mathbf{x}}^G \gneqq \bar{\mathbf{x}}^L$. Then a stationary point $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L$ is called a local maximum of a function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}^2_+$.

Let us consider a case when a function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ is strictly concave, twice differentiable and increasing in its whole domain. This means that we do not assume condition (A.87). Let us assume that we are interested only in a positive optimal solution $\bar{\mathbf{x}} > 0$ of problem (A.83)–(A.85). In such a case a function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ has no global maximum[8] $\bar{\mathbf{x}}^G > 0$. But since there is the binding constraint (A.84) the function has an optimal solution $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L$ which can be derived from a system of equations:

(A.100) $$\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \bar{\lambda} p_i, \; i = 1, 2,$$

(A.101) $$p_1\bar{x}_1 + p_2\bar{x}_2 = I.$$

[8] Strictly speaking, there does not exist a finite optimal solution to problem (A.82)–(A.84), which would be equivalent to the global maximum.

where: $\bar{\lambda} > 0$ means an optimal Lagrange multiplier, which determines by how much the maximum value of a function $f: \mathbb{R}_+^2 \to \mathbb{R}$ approximately increases when a value of parameter $I > 0$ increases by one notional unit.

Case 2 Let a function $f: \mathbb{R}_+^2 \to \mathbb{R}$ be increasing, strictly concave and twice differentiable. Let us assume that $h(x_1, x_2) = p_1 x_1 + p_2 x_2, \quad b_1 = b_2 > 0, \quad b_3 = +\infty$. Then problem (A.63)–(A.67) can be expressed in a form:

$$f(x_1, x_2) \mapsto \max \tag{A.102}$$

$$x_1 \leq b_1, \tag{A.103}$$

$$x_2 \leq b_2, \tag{A.104}$$

$$x_1, x_2 \geq 0, \tag{A.105}$$

to which the following Lagrange function corresponds:

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \lambda_1 (b_1 - x_1) + \lambda_2 (b_2 - x_2) \tag{A.106}$$

If we assume additionally that a function $f: \mathbb{R}_+^2 \to \mathbb{R}$ satisfies condition (A.87) then the Kuhn-Tucker theorem can be formulated as follows.

Theorem A.8 (the Kuhn-Tucker theorem) Let a function $f: \mathbb{R}_+^2 \to \mathbb{R}$ be twice differentiable and strictly concave on a nonempty, convex and compact set $B = \{\mathbf{x} \in \mathbb{R}_+^2 | \, x_1 \leq b_1, x_2 \leq b_2\} \subset X = \mathbb{R}_+^2$. Then a vector $\bar{\mathbf{x}} \geq 0$ is an optimal solution to problem (A.102)–(A.105) if and only if there exists a pair $(\bar{\mathbf{x}}, \bar{\boldsymbol{\lambda}}) \geq 0$ that satisfies a set of conditions:

$$\left. \frac{\partial L(\mathbf{x}, \bar{\boldsymbol{\lambda}})}{\partial x_i} \right|_{\mathbf{x}=\bar{\mathbf{x}}} \leq 0, \; i = 1, 2, \tag{A.107}$$

$$\sum\nolimits_{i=1}^{2} \bar{x}_i \left. \frac{\partial L(\mathbf{x}, \bar{\boldsymbol{\lambda}})}{\partial x_i} \right|_{\mathbf{x}=\bar{\mathbf{x}}} = 0, \tag{A.108}$$

$$\left. \frac{\partial L(\bar{\mathbf{x}}, \boldsymbol{\lambda})}{\partial \lambda_j} \right|_{\boldsymbol{\lambda}=\bar{\boldsymbol{\lambda}}} \geq 0, \; j = 1, 2, \tag{A.109}$$

$$\bar{\lambda}_1 \left. \frac{\partial L(\bar{\mathbf{x}}, \boldsymbol{\lambda})}{\partial \lambda_1} \right|_{\boldsymbol{\lambda}=\bar{\boldsymbol{\lambda}}} + \bar{\lambda}_2 \left. \frac{\partial L(\bar{\mathbf{x}}, \boldsymbol{\lambda})}{\partial \lambda_2} \right|_{\boldsymbol{\lambda}=\bar{\boldsymbol{\lambda}}} = 0, \tag{A.110}$$

where:

(A.111) $$\left.\frac{\partial L\left(\mathbf{x}, \bar{\boldsymbol{\lambda}}\right)}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} - \bar{\lambda}_i, \quad i = 1, 2,$$

(A.112) $$\left.\frac{\partial L(\bar{\mathbf{x}}, \lambda)}{\partial \lambda_i}\right|_{\boldsymbol{\lambda}=\bar{\boldsymbol{\lambda}}} = b_i - \bar{x}_i, i = 1, 2.$$

Conditions (A.108) and (A.110) take forms:

(A.113) $$\bar{x}_1\left(\left.\frac{\partial f(\mathbf{x})}{\partial x_1}\right|_{\mathbf{x}=\bar{\mathbf{x}}} - \bar{\lambda}_1\right) + \bar{x}_2\left(\left.\frac{\partial f(\mathbf{x})}{\partial x_2}\right|_{\mathbf{x}=\bar{\mathbf{x}}} - \bar{\lambda}_2\right) = 0,$$

(A.114) $$\bar{\lambda}_1(b_1 - \bar{x}_1) + \bar{\lambda}_2(b_2 - \bar{x}_2) = 0,$$

where: $\bar{\lambda}_i = \left.\frac{\partial f(\mathbf{x})}{\partial b_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} \geq 0, i = 1, 2$ means an optimal Lagrange multiplier which determines by how much the maximum value of a function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ approximately increases when a value of parameter b_i increases by one notional unit.

If $\bar{\lambda}_i > 0$ $(i = 1, 2)$ then the i-th constraint is binding. When $\bar{\lambda}_i = 0$ then the i-th constraint is not binding.

Let us assume that we are interested only in a positive optimal solution $\bar{\mathbf{x}} > 0$ to problem (A.102)–(A.105) then condition (A.113) is satisfied if and only if:

(A.115) $$\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} - \bar{\lambda}_i = 0, \ i = 1, 2.$$

If additionally $\forall i = 1, 2 \quad \bar{\lambda}_i = 0$ then condition (A.115) takes the form:

(A.116) $$\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = 0, \ i = 1, 2.$$

This takes place when no constraint on resources is binding, which means that:

(A.117) $$\bar{\mathbf{x}} = \bar{\mathbf{x}}^G.$$

When none of the constraints describing a set B is binding then an optimal solution $\bar{\mathbf{x}} = \bar{\mathbf{x}}^G$ to problem (A.102)–(A.105) is identical to a global maximum that a strictly concave function $f\colon \mathbb{R}^2_+ \to \mathbb{R}$ reaches in a space $X = \mathbb{R}^2_+$. Then the conditional maximization problem is the same as the unconditional maximization problem.

In the case when $\forall i = 1, 2 \ \bar{\lambda}_i > 0$ then each constraint is binding and condition (A.113) is satisfied in the initial form. At the same time from condition (A.113), we get that:

(A.118) $$\bar{x}_i = b_i, \quad i = 1, 2$$

In this case, an optimal solution to problem (A.102)–(A.105) is a vector $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L$ such that $\bar{\mathbf{x}}^G \gneqq \bar{\mathbf{x}}^L$. Then a stationary point $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L$ is called a local maximum of a function $f: \mathbb{R}^2_+ \to \mathbb{R}$ in a set $B \subset X = \mathbb{R}^2_+$.

Let us consider a case when a function $f: \mathbb{R}^2_+ \to \mathbb{R}$ is strictly concave, twice differentiable and increasing in its whole domain. This means that we do not assume condition (A.87). If we are interested only in a positive optimal solution $\bar{\mathbf{x}} > 0$ of problem (A.102)–(A.105). In such a case a function $f: \mathbb{R}^2_+ \to \mathbb{R}$ has no global maximum[9] $\bar{\mathbf{x}}^G > 0$. But since there are the binding constraints (A.103) and (A.104), the function has an optimal solution $\bar{\mathbf{x}} = \bar{\mathbf{x}}^L$ which can be derived from a system of equations:

(A.119) $$b_1 = \bar{x}_1,$$

(A.120) $$b_2 = \bar{x}_2.$$

From condition (A.115), one gets in addition the following:

(A.121) $$\left.\frac{\partial f(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\bar{\mathbf{x}}} = \bar{\lambda}_i, \; i = 1, 2,$$

where: $\bar{\lambda}_i > 0, i = 1, 2$ means an optimal Lagrange multiplier, which determines by how much the maximum value of a function $f: \mathbb{R}^2_+ \to \mathbb{R}$ approximately increases when a value of parameter $b_i > 0$ increases by one notional unit.

A4 Difference and Differential Equations

Definition A.26 A **first increment (difference) of variable** $y(t)$ is an expression of a form[10]:

(A.122) $$\Delta y(t) = y(t+1) - y(t).$$

Definition A.27 A ***n*-th order increment of variable** $y(t)$ is the first increment of $(n-1)$-th order increment:

(A.123) $$\Delta^n y(t) = \Delta\left(\Delta^{n-1} y(t)\right).$$

[9] Strictly speaking, there does not exist a finite optimal solution to problem (A.102)–(A.105), which would be equivalent to the global maximum.

[10] In the book we exploit difference and differential equations when a variable that means time is treated as an independent variable. When time is meant as a discrete variable then one exploits difference equations or sets of difference equations. When time is meant as a continuous variable then one exploits differential equations or sets of differential equations. In Definitions A.26–A.32 time variable can be replaced by any variable x that is discrete (Definitions A.26–A.30) or continuous (Definitions A.31 and A.32).

Definition A.28 A ***n*-th order difference equation** is an expression of a form:

(A.124) $$F\left(\Delta^n y(t), \Delta^{n-1} y(t), ..., \Delta y(t), y(t), t\right) = 0.$$

Definition A.29 A ***n*-th order linear difference equation** is an expression of a form:

(A.125) $$y(t+n) + a_1(t)y(t+n-1) + ... + a_n(t)y(t) = g(t),$$

where: $a_i(t),\ i = 1, 2, ..., n,\ g(t),\ a_n(t) \neq 0$ mean any sequence of numbers.

Definition A.30 A **first-order linear difference equation** is an expression of a form:

(A.126) $$y(t+1) + ay(t) = c.$$

Definition A.31 An **ordinary differential equation of *n*-th order** is an expression of a form:

(A.127) $$F\left(t, y, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}, ..., \frac{\mathrm{d}^n y}{\mathrm{d}t^n}\right) = 0.$$

Definition A.32 An **ordinary linear differential equation of first order** is an expression of a form:

(A.128) $$\frac{\mathrm{d}y}{\mathrm{d}t} + p(t)y = q(t).$$

Note A.1 An ordinary linear differential equation of first order is called homogenous when $q(t) \equiv 0$. Otherwise is called **heterogeneous**.

Definition A.33 Any function $y(t)$ that transforms an ordinary differential equation of n-th order into an identity is called a **solution to this differential equation**.

Transition from difference equations to differential equations

There is given a difference equation of a form:

(A.129) $$x(t+1) - x(t) = g(x(t)).$$

Let us present its approximation for any $\Delta t \in [0; 1]$:

(A.130) $$x(t+\Delta t) - x(t) \cong \Delta t g(x(t)).$$

If $\Delta t = 0$ then one gets an identity. When $\Delta t = 1$ then (A.130) is equivalent to (A.129).

When $\Delta t \in (0; 1)$ then (A.130) is a sufficiently good linear approximation of (A.129) if: (A.131)

$$\forall x \in [x(t); x(t+1)], g(x) \cong g(x(t)) \tag{A.131}$$

.

Dividing (A.130) on both sides by Δt and assuming $\Delta t \to 0$ one gets:

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t} \cong g(x(t)). \tag{A.132}$$

Differential Eq. (A.132) is a counterpart of difference equation (A.129) in a case when t and $t+1$ are "small".

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Glossary

Allocation – a result of distribution of consumer goods among all traders (consumers). The distribution refers to the demand for consumer goods as well as to the supply of goods. In a simple model of exchange and in the Arrow-Hurwicz model, the distribution of the supply of consumer goods is determined by endowment of consumer goods in consumptions bundles that traders come to a market with. The distribution of the demand for consumer goods is determined implicitly by relations of preference of consumers and in the Arrow-Hurwicz model explicitly by Marshallian demand functions, each of them corresponding to a given consumer.

Allocation accepted by traders – any allocation feasible with regard to an initial allocation such that its consumption bundles have their utility not smaller than the utility of bundles that traders come to a market with.

Allocation blocked by traders – any allocation feasible with regard to an initial allocation such that there exists some other feasible allocation (with regard to an initial allocation) that is better for at least one of traders and not worse for the other traders.

Allocation feasible with regard to an initial allocation – any allocation describing the demand of each trader for each consumer good such that its resulting total demand reported by all consumers for each good equals the supply of this good.

Allocation, initial allocation – distribution of the supply of all consumer goods among all traders who come to a market with consumptions bundles in order to exchange them for optimal bundles.

Allocation optimal (efficient) in the Pareto sense – any allocation feasible with regard to initial allocation such that there does not exist any other feasible allocation (with regard to an initial allocation) that would be better for at least one of traders and not worse for the other traders.

Allocation, Walrasian equilibrium allocation – a Pareto optimal (efficient) allocation that is established on a market of consumer goods by Walrasian equilibrium prices.

Arrow-Debreu-McKenzie model – an example of a general equilibrium model. It is a model in which one distinguishes in the economy the part related to production and the part related to exchange (consumption). It can be presented in a

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static or dynamic version (with discrete or continuous time). Within this framework, one determines conditions for the existence, uniqueness and asymptotic global stability of a Walrasian equilibrium state defined with an accuracy of a structure.

Arrow-Hurwicz model – an example of a market equilibrium model. It is a model in which one describes only this part of the economy that is related to exchange (consumption). It can be presented in a static or dynamic version (with discrete or continuous time) . Within this framework, one determines conditions for the existence, uniqueness and asymptotic global stability of a Walrasian equilibrium state defined with an accuracy of a structure.

Average fixed cost of production – (unit fixed cost)—the production fixed cost per output unit.

Average revenue (turnover) from sales of product – the revenue (turnover) from sales of a product per output unit.

Average total cost of production (unit total cost) – production total cost per output unit.

Average variable cost of production (unit variable cost) – production variable cost per output unit.

Break-even points (profitability thresholds) – levels of inputs of particular production factors or of the product supply indicating by which levels of these variables a firm having production activity has zero profit. For inputs (or for the supply) in intervals indicated by the break-even points a firm generates non-negative profits, outside these intervals the firm incurs financial losses.

Budget line (budget constraint) – a set of all consumption bundles whose money value, by given prices of consumer goods, is equal to an income of a given consumer.

Budget set – a set of all consumption bundles whose money value, by given prices of consumer goods, is not greater than an income of a given consumer.

Capital, human capital – knowledge and skills of people that result from cumulating effects of schooling, competencies acquired at work and more generally of education. In some production processes, especially when studied from a point of view of macroeconomics, treated as one of production factors.

Capital, physical capital – tangible assets used in production processes, treated as a production factor.

Ceteris paribus – an expression meaning that when one determines an effect of a change in some single variable on a value of some function there is an assumption that values of other variables remain unchanged. This assumption serves analyzing the influence of a separated single factor.

Competition, imperfect competition – a situation on a market of a given product when at least one of the assumptions describing the perfect competition is not satisfied.

Competition, perfect competition – a situation on a market of a given product described by a set of the four following assumptions[11]:

– **market atomization:** a number of agents (consumers/producers) reporting the demand for a given product (treated as a consumer good/production factor) or reporting the supply (producers) of this product is big enough for each of them not to have decisive impact on a level of a product price or on conditions of exchange of the product. As a consequence each producer takes the price of the product as given by the market and treats it as a parameter adjusting the output level by her/his firm.

– **homogeneity of a product:** products manufactured by firms are all undifferentiated,

– **transparency:** each economic agent has perfect knowledge about a supply level and about a price of the product,

– **liquidity:** there are no barriers hindering entry to or exit from the market of the product since any decision of such type does not involve additional costs.

Competition, price competition – a situation on a market of two (duopoly), a few (oligopoly) or many (monopolistic competition[12]) heterogeneous products when producers manufacture and sell products presented by them and perceived by consumers as substitutive. Then the producers compete on price levels trying each one to maximize her/his profit. An example of a model describing the price competition is the duopoly Bertrand model.

Competition, quantity competition – a situation on a market of one homogenous product when two (duopoly), a few (oligopoly) or many (oligopoly) producers manufacture and sell the same product and compete on supply levels trying each one to reach the biggest share of the market at the same time to maximize her/his profit. Examples of models describing the quantity competition are the Cournot and the Stackelberg duopoly models.

Complementary consumers goods (complements) – such goods that a given consumer wants to have them always in some proportion of quantities. As a consequence a utility level of a consumption bundle increases (decreases) only when quantities of all goods increase (decrease) in a specific proportion. Treating goods as complements is generally subjective and describes the preferences of a consumer in a certain way determined by a weakly increasing utility function. From a point of view of demand reaction, two consumer goods are called

[11] In the considerations about the perfect competition presented in the book of crucial importance is the fact whether a firm has or does not have influence on a price of a product manufactured by the firm.

[12] In the book we do not study a market structure called monopolistic competition (not to confuse with a monopoly nor with a monopolist) since this is the most complex market structure. Analyzing such concepts as price competition, quantity competition, profit maximization or production cost minimization is much more clear in less complex models, as those presented in the book. Monopolistic competition can be seen as a market structure intermediate between an oligopoly and the perfect competition.

complementary when a rise in a price of one of these goods results in a decrease in the demand for the other good.

Complementary consumer goods, perfect complements – specific complementary goods such that a proportion in which a consumer wants to have them is always the same, regardless of quantities of goods in a consumption bundle. Preferences towards perfect complements are described by a Koopmans-Leontief utility function.

Complementary production factors – such production factors that in a given production process a producer needs to use them always in some proportion of inputs. As a consequence an output level increases (decreases) only when inputs of all production factors increase (decrease) in a specific proportion. In a production process, an output level depending on inputs of such production factors is described by a weakly increasing utility function. When the proportion of the usage of such production factors is always the same, regardless of inputs of these factors, then they are called perfectly complementary.

Consumer Giffen good – an inferior good for which a rise in a price of this good results in an increase in the demand for this good.

Consumer goods' space – a set of all bundles of goods available on a market along with a non-Euclidean metric specified on this set.

Consumer inferior good – a good for which a rise (decline) in a consumer's income results in a decrease (increase) in the demand for this good.

Consumer independent goods – such goods that when there is a rise in a price of one of these goods it does not affect the demand for the other good.

Consumer normal good – a good for which a rise (decline) in a consumer's income results in an increase (decrease) in the demand for this good.

Consumer ordinary good – a good for which a rise in a price of this good results in a decrease in the demand for this good.

Consumer Veblen good – a normal good for which an increase in a price of this good results in an increase in the demand for this good.

Consumption bundle – one of primary concepts in the theory of consumer choice, defined as a vector in which i-th coordinate describes non-negative quantity of i-th consumer good expressed in physical units. These quantities of goods are owned by a given consumer or considered to be purchased by her/him, depending on the description of a question.

Contract curve – a set of Pareto-optimal allocations having given some initial allocation. It is presented graphically in the Edgeworth box.

Cost, opportunity cost – when it comes to purchasing products (consumer goods or production factors) it means the loss of value or benefit that would be incurred by purchasing some products relative to purchasing other goods. In general, this type of cost can be related to any economic activity in comparison to some other activity, not only to purchasing products.

Cost, transaction cost – cost incurred during negotiating and implementing a trade agreement.

Difference equation – an equation that determines dependency of an unknown function on its differences (increments) of various orders. If it relates to differences up to n-th order then it is called n-th order difference equation. It is a discrete counterpart of a differential equation.

Differential equation – an equation that determines dependency of an unknown function on its derivatives of various orders. If it relates to derivatives up to n-th order then it is called n-th order differential equation. It is a continuous counterpart of a difference equation.

Discriminatory pricing – pricing policy that serves maximizing total profit from production and sales of a homogenous product on different markets at different prices.

Duopoly model, Bertrand model – an example of a model describing a market of two substitute products on which two producers with equal market positions act. It is an example of description of price competition between two producers who, by given exogenous demand functions, aim at maximizing profits. When setting an optimal price for her/his product each producer takes into account a price of the substitute product manufactured by her/his competitor. By given values of parameters of the functions of demand for the substitute products and of parameters of production total cost functions, there exists exactly one equilibrium price vector by which each producer maximizes her/his profits.

Duopoly model, Cournot model – an example of a model describing a market of one homogenous product on which two producers with equal market positions act. It is an example of description of quantity competition between two producers who, by a given exogenous demand function, aim at maximizing profits. When determining the optimal supply of her/his product, each producer takes into account the supply of the product manufactured by her/his competitor. By given values of parameters of the function of demand for the homogenous product and of parameters of production total cost functions, there exists exactly one equilibrium supply vector by which each producer maximizes her/his profits. As a consequence, there exists an equilibrium price that equalizes total supply of the product with global demand for the product.

Duopoly model, Stackelberg model – an example of a model describing a market of one homogenous product on which two producers with unequal market positions act. One of them, called a leader, has the dominant position. The other, called a follower, adjusts her/his decisions on the supply of the product to decisions made by the leader. The leader, determining her/his optimal level of the product supply, does not take into account decisions of the follower but aiming at profit maximization takes into account a demand function and her/his own profit function. The model is an example of quantity competition between two producers who, by the given exogenous demand function, aim at maximizing profits. By given values of parameters of the function of demand for the homogenous product and of parameters of production total cost functions there exists exactly one equilibrium supply vector by which each producer maximizes her/his profits. As a consequence, there exists an equilibrium price which equalizes total supply of the product with global demand for the product. Here the optimal product supply by the leader is a solution to an unconditional leader's profit maximization problem, while the optimal product supply by the follower is a solution to a conditional follower's profit maximization problem.

Economic agent – a single microeconomic decision-maker in a model of some aspect of the economy, for example: consumer, trader, producer.

Economics – a subdivision of knowledge (a discipline) in the branch of social science in which one discusses and studies rational ways of the resource usage by an individual, groups of individuals or a society. It examines how these economic agents use the resources to manufacture goods and services and distribute them among groups and individuals who form a given society in a given time horizon.

Edgeworth box – a geometric illustration used in a case of two traders and two consumer goods in the simple model of exchange and in the Arrow-Hurwicz model to present a set of all allocations feasible with regard to an initial allocation. The Edgeworth box is created by overlapping two coordinate systems, each of which is associated with the first or the second trader.

Effect, income effect – one of the effects described in the Slutsky equation. It consists in determining how the demand for a given good is affected by relatively smaller purchasing power of a consumer caused by an uncompensated increase in a price of this good or in a price of the other good. If its sign is negative the good is normal, if positive the good is inferior.

Effect, price effect – one of the effects described in the Slutsky equation. It can relate to one good or to a relationship between two goods. If it relates to one good then it consists in determining how a change in a price of a given good affects the demand for this good (*ceteris paribus* — the price of the other good and a consumer's income remain unchanged). Then if its sign is negative the good is ordinary, if positive the good is a Giffen good. If it relates to two goods then it consists in determining how a change in a price of one good affects the demand for the other good (*ceteris paribus* — the price of the other good and a consumer's income remain unchanged). Then if its sign is negative the goods are mutual complements, if positive they are mutual substitutes.

Effect, substitution effect – one of the effects described in the Slutsky equation. It can relate to one good or to a relationship between two goods. If it relates to one good then its sign is always negative and consists in determining how a change in a price of a given good affects the compensated demand for this good. If it relates to two goods then it consists in determining how a change in a price of a given good affects the compensated demand for the other good. An increase in a price of one good results in a decrease in utility of an optimal consumption bundle. But this utility's decrease is compensated by a hypothetical increase in a consumer's income such that a new optimal consumption bundle has the same utility level as the bundle before the increase in the price of the good.

Elasticity of function – a characteristic of any function that describes by approximately what % a value of the function changes (increases, decreases or remains unchanged) when a value of one of its arguments increases by 1% and values of the other arguments do not change.

Elasticity, price elasticity of demand for good with respect to its price (simple price elasticity) – a characteristic of a function of demand for a given good that describes by approximately what % the demand for this good changes (increases, decreases or remains unchanged) when a price of this good increases by 1% and prices of the other goods and a consumer's income do not change.[13]

Elasticity, price elasticity of demand for good with respect to price of other good (cross or mixed price elasticity) – a characteristic of a function of demand for a given good that describes by approximately what % the demand for this good changes (increases, decreases or remains unchanged) when a price of some other good increases by 1% and prices of remaining goods and a consumer's income do not change.

Elasticity, price elasticity of demand for good with respect to consumer's income (income elasticity) – a characteristic of a function of demand for a given good that describes by approximately what % the demand for this good changes (increases, decreases or remains unchanged) when a consumer's income of increases by 1% and prices of consumer goods not change.

Elasticity of production with respect to production factor – a characteristic of a production function that describes by approximately what % an output level changes (increases, decreases or remains unchanged) when an input level of this production factor increases by 1% and inputs of the other production factors do not change.

Elasticity of production with respect to scale of inputs – a characteristic of a production function that describes by approximately what % an output level changes (increases, decreases or remains unchanged) when an input level of each production factor increases by 1%. If the production function is positively homogenous of order $\theta>0$ then a value of this elasticity equals θ.

Elasticity of substitution of one consumer good by the other good in a consumption bundle with a given utility level – a relative measure of substitution that describes by approximately what % one should raise quantity of the other good when quantity of the given good has been reduced by 1%, in order to keep the given utility of a consumption bundle unchanged.

Elasticity of substitution of one production factor by the other production factor in a vector of inputs by which the output can be produced at a given level – a relative measure of substitution that describes by approximately what % one should raise quantity of the other production factor when an input of the given production factor has been reduced by 1%, in order to keep the given level of output unchanged.

Equilibrium price – a level of a price by which the supply and the reported demand, both expressed in the same physical units, are equal to each other.

[13] Interpretations of price elasticities and income elasticity of demand for a given good refer to the cases when the demand for goods is described by the Marshallian demand function.

Exchange core – a set defined in the Arrow-Hurwicz model or in the Arrow-Debreu-McKenzie model as consisting of all allocations accepted by all traders and Pareto optimal at the same time

Exogenous function of supply of product – arbitrarily determined (not as a result of derivations of a model under consideration) functional dependency of the supply of a product on its price. Most of the time one assumes that the supply of a product is a function increasing in a price of the product. Sometimes such an assumption is referred to as the law of supply.[14]

Exogenous function of demand for product – arbitrarily determined (not as a result of derivations of a model under consideration) functional dependency of the demand for a product on its price. Most of the time one assumes that the demand of a product is a function decreasing in a price of the product. Sometimes such an assumption is referred to as the law of demand.

Follower on market of product – a producer (or a firm owned by her/him) who acting on a market of a given product in conditions described by the Stackelberg duopoly (or oligopoly) model (homogenous product, quantity competition) makes decisions optimal for her/him depending on decisions made by a leader of the market (a producer with a dominant position). There can be one or a few followers.

Function – a mapping that to any element from a domain (set of arguments) assigns exactly one element from a codomain (set of values).

Function of compensated demand (Hicksian demand function) – a function that describes dependency of the consumer's demand for consumer goods on prices of these goods and on a utility level of a consumption bundle. It is derived as an optimal solution to a consumer's expenditure minimization problem.

Function of conditional demand for production factors – a function that describes dependency of the producer's demand for production factors on prices of these factors and on an output level. It is derived as an optimal solution to a cost minimization problem when producing the output at a fixed level.

Function of consumer demand (Marshallian demand function) – a function that describes dependency of the consumer's demand for consumer goods on prices of these goods and on the consumer's income. It is derived as an optimal solution to a consumption utility maximization problem.

Function of consumer's expenditure – a function that describes how the minimum expenditure, which a consumer incurs to purchase a consumption bundle with a given utility level, depends on prices of consumer goods and on the utility level. It results from a consumer's expenditure minimization problem.

Function of demand for production factors – a function that describes dependency of the producer's demand for production factors on prices of these factors

[14] Treating this assumption as a general economic law is an exaggeration. Let us notice that a number of conditions has to be implied to have this assumption satisfied.

and on a price of product manufactured by the producer. It is derived as an optimal solution to a profit maximization problem with regard to inputs of production factors.

Function of excess demand – a difference between a global supply function and a global demand function.

Function of firm's profit – a difference between a function of revenue from sales of product and a function of production total cost. It describes dependency of the profit a firm can obtain on production factors' inputs or on output level.

Function of firm's maximal profit – a function that describes how the maximum profit, which a firm can obtain, depends on prices of production factors, a price of a product manufactured by the firms and on the fixed cost of production. It results from a profit maximization problem with regard to inputs of production factors or with regard to output level.

Function of global supply – a sum of supply functions of all producers.

Function of global demand – a sum of demand functions of all consumers.

Function homogenous of order zero – a function such that when values of all its arguments increase proportionally then a value of the function does not change.

Function of marginal cost – a function that describes how a production total (or variable) cost[15] level increases when an input of some production factor or when an output level increases by one notional unit.

Function of marginal profit – a difference between a function of marginal revenue from sales of a product and a function of marginal production total cost. All these functions are considered as functions of production factors inputs or of an output level. It describes how a profit level changes when an input of some production factor or when an output level increases by one notional unit.

Function of marginal revenue from sales – a function that describes how the revenue from sales of a product increases when an input of some production factor or when an output level increases by one notional unit.

Function positively homogenous of order θ>0 — a function such that when values of all its arguments increase proportionally then a value of the function increases: proportionally ($\theta = 0$), less than proportionally ($\theta < 0$), more than proportionally ($\theta > 0$).

Function of production fixed cost – a constant function that describes cost related to a production process that does not depend explicitly on inputs of production factors nor on the output level.

Function of production total cost – a function that describes dependency of cost of production on production factors' inputs or on output level. It is a sum of functions of variable cost and of fixed cost.

[15] A function of marginal total cost and a function of marginal variable cost are identical since a function of the fixed cost is constant. A function of marginal fixed cost equals 0 and does not matter in the marginal total cost.

Function of production variable cost – a function that describes cost related to a production process that depends explicitly on inputs of production factors or on the output level.

Function of product supply – a function that describes dependency of the product supply by a producer on prices of production factors and on a price of product manufactured by the producer. It is derived as an optimal solution to a profit maximization problem with regard to output level.

Function of revenue from sales of product – a function that describes dependency of the revenue (turnover) from sales of a product on production factors' inputs or on output level. It is one of basic financial characteristics of a production process. In the mathematical sense it is the output level of a manufactured product multiplied by its price.

Gossen's first law – a property saying that a marginal utility of any good decreases with an increase of quantity of this good in a consumption bundle. It is satisfied in case of any increasing, differentiable and strictly concave utility function.

Gossen's second law – a property saying that when a consumption bundle is optimal (as an optimal solution to a consumption utility maximization problem or to a consumer's expenditure minimization problem) then a marginal rate of substitution of one good by the other good in this bundle is constant and equal to the ratio of prices of these two goods. The law is satisfied in case of any increasing, differentiable and strictly concave utility function.

Graph of function – a set of all vectors in which all coordinates (one or more, depending on whether a function has one or more variables), except the last one, are arguments of a given function and the last coordinate is a value of this function corresponding to these arguments. A geometric illustration of this set is presented as a set of points in the Cartesian coordinate system.

Growth rate of function – a characteristic of any function that describes by approximately what % a value of the function changes (increases, decreases or remains unchanged) when a value of one of its arguments increases by one notional unit and values of the other arguments do not change.

Growth speed of function – a characteristic of any function that describes by how much approximately a value of the function changes (increases, decreases or remains unchanged) when a value of one of its arguments increases by one notional unit and values of the other arguments do not change. For a differentiable utility function the growth speed is called a marginal utility of a good. For a differentiable production function the growth speed is called a marginal productivity (marginal effectiveness) of a production factor.

Heterogeneous products – differentiated products that are offered on a market, presented by producers and perceived by consumers as substitutive. In fact they do not have to differ much in physical features to be regarded as heterogeneous. Other qualities, like a brand, the popularity, commercials, make them seen as heterogeneous.

Holism – research approach opposed to methodological individualism. It relies on a belief that society is not a simple sum of individuals who form it and that properties of a given society influence a lot behaviour and activities of the individuals.

Homogeneous product – undifferentiated products that are offered on a market, presented by producers and perceived by consumers as the same one product. In fact they may differ a little in physical features but still be regarded by consumers as homogeneous.

Homo oeconomicus – man as an individual who manages her/his resources, behaves rationally and makes decisions rational from a point of view of economic criteria.

Homo sapiens – man as a reasoning and intelligent being.

Homo socialis – man as a social being.

Imperfect competition firm – a firm acting on a market of one homogenous product or on a market of substitute differentiated products under prevailing imperfect competition conditions. What distinguishes the imperfect competition firm from the perfect competition one is that the former has some market power when deciding on a price of its product, independently of or dependently on competing firms.

Income, nominal income – any income of a consumer for which purchasing power is not determined.

Income, real income – any income of a consumer for which purchasing power is determined.

Indifference curve – a set of all consumption bundles whose utility is the same from a point of view of a given consumer' preferences.

Indirect function of consumption utility – a function that describes maximum utility of a consumption bundle that a consumer can have by given prices of consumer goods and a given consumer's income. It is derived as an optimal solution to a consumption utility maximization problem.

Inverse function – a function inverse to a certain function is a mathematical concept described in Definition 5.3 or 5.8. In this concept, it does not matter what the interpretation of variables is since the choice of the variable is arbitrary. Graphs of functions inverse to each other are symmetric with respect to the 45^{o} line.

Inverse function of consumer demand – an identity transformation of a demand function. From a point of view of a producer, it describes dependency of a product price on a supply level. It is used in a profit function of a producer who has market power to set a price of her/his product. When formulating the profit function it allows to take into account properties of the demand for the product described by the demand function, such as a market capacity or sensitivity of consumers to changes in the product price.

Isoquant of production – a set of all vectors of production factor' inputs by which the output can be produced at a fixed given level.

Kuhn-Tucker theorem – a theorem determining necessary and sufficient conditions for the existence of an optimal solution to a nonlinear mathematical programming problem. It is applicable when searching for a conditional optimum (maximum, minimum) of a given function (called objective function).

Lagrange function – a function that allows to formulate any problem of mathematical programming (linear or nonlinear one). It is useful in searching for conditional extrema for objective functions in conditional optimization problems. It is formed as a sum of an objective function and of constraints (conditions) multiplied by so-called Lagrange multipliers.

Lagrange multiplier – a variable that is used in a Lagrange function to introduce a constraint determined in a conditional optimization problem. In a Lagrange function there are as many multipliers as there are the constraints. An optimal multiplier describes by how much an optimal value of an objective function changes when a value used to formulate a constraint increases by one notional unit.

Leader on market of product – a producer (or a firm owned by her/him) who, acting on a market of a given product in conditions described by the Stackelberg duopoly (or oligopoly) model (homogenous product, quantity competition), makes decisions optimal for her/him not depending on decisions made by other producers who are called followers. There can be one or a few leaders.

Line of producer's reaction – a concept occurring in the Cournot, Stackelberg and Bertrand duopoly models. It describes a set of optimal levels of the product supply (Cournot and Stackelberg duopolies) or of optimal levels of a product price (Bertrand duopoly) for a given producer when he/she takes the supply level or the price level set by the other producer as given. In the Stackelberg duopoly model the line of reaction is determined only for the follower since the leader does not make her/his decisions on the supply or on the price as dependent on decisions made by the follower.

Macroeconomics – a branch of economics in which one discusses and studies the behaviour of the collectivity of consumers and producers and to describe it uses aggregate volumes related to the entire economy to know, understand and explain economic phenomena and processes taking place in the economy treated as a whole.

Marginal demand for good with respect to consumer's income – a measure of reaction of the demand for a given good that describes by approximately how many physical units the demand for this good changes (increases, decreases or remains unchanged) when a consumer's income ($j \neq I$ *or* $j=1$) increases by one notional money unit and prices of consumers goods do not change.

Marginal demand for i-th good with respect to price of j-th good – a measure of reaction of the demand for *i*-th good that describes by approximately how many physical units the demand for this good changes (increases, decreases or remains unchanged) when a price of *j*-th good ($j \neq i$ *or* $j=i$) increases by one notional money unit and prices of the other goods and a consumer's income do not change.

Marginal productivity of production factor (growth speed of production) – one of the characteristics of a production function. It describes by approximately how many physical units an output level changes (increases, decreases or remains unchanged) when an input of a given production factor increases by one notional unit and inputs of the other production factors in the vector of inputs do not change.

Marginal rate of substitution of one good by another good – an absolute measure of substitutability of two goods that describes by approximately how many units one should raise a quantity of the other good in a consumption bundle when a quantity of the given good has been reduced by one unit, in order to keep the consumption bundle utility unchanged in comparison to the bundle before the changes.

Marginal rate of substitution of one production factor by other factor – an absolute measure of substitutability of two production factors that describes by approximately how many units one should raise an input of the other production factor in a vector of inputs when a quantity of the given factor has been reduced by one unit, in order to keep the output level unchanged in comparison to the vector of inputs before the changes.

Marginal utility of consumer good – one of the characteristics of a utility function. It describes by approximately how many units a utility of a consumption bundle changes when a quantity of a given good increases by one notional unit and quantity of the other goods in the bundle does not change.

Marginal utility of income – a measure that describes by approximately how many units a utility of an optimal consumption bundle (a value of an indirect utility function) changes (usually increases) when a consumer's income increases by one notional money unit and prices of consumer goods do not change. It is equal to an optimal Lagrange multiplier as well as to a marginal utility of a money unit for the purchase of any of the goods.

Marginal value of function – from the point of view of microeconomics it is a characteristic of any function that describes by how much approximately a value of the function changes (increases, decreases or remains unchanged) when a value of one of its arguments increases by 1, and values of the other arguments do not change. In the mathematical sense it is a first-order derivative of a function (partial derivative in case of a function of two or more variables).

Market – a group of buyers and sellers of a certain product (good or service).

Market capacity – the maximum demand that consumers can report on a market for a given product at a product price equal to zero.

Market economy – an economy in which distribution of resources among various applications is made as a result of decentralized decisions of economic agents (producers, firms, consumers, households, government) who interact with each other on markets of products (goods and services).

Market failure – inability of the market mechanism to distribute resources among various applications.

Mesoeconomics – a branch of economics in which one discusses and studies the processes of management of resources by economic agents who act within distinct economic sectors or activities.

Methodological individualism – research approach consisting in a belief that in order to understand social reality one has to concentrate on an individual, not on society as a whole. It is that because society is an outcome of activities of individuals who undergo various transformations being a result of these activities. This approach is typical for traditional microeconomics and in general for the neoclassical school.

Metric, Euclidean metric – an example of a metric, a mapping satisfying specified properties and determined by a specified formula.[16] In the theory of a consumer choice it is used as a measure of distance between two consumption bundles. This metric is applicable when quantities of all consumer goods in consumption bundles are expressed in the same physical units.

Metric, non-Euclidean metric – an example of a metric, a mapping satisfying specified properties and determined by a specified formula.[17] In the theory of a consumer choice, it is used as a measure of distance between two consumption bundles. To determine it one does not need to use the same physical units for quantities of consumer goods in consumption bundles. Its formula does not cause any interpretation issues.

Microeconomics – a branch of economics in which one describes and analyzes markets as well as behaviour and activities of individual economic agents such as consumers (households) or producers (firms).

Model of market with exogenous demand and supply functions – a simplified version of the Arrow-Debreu-McKenzie model. A function of demand for a product and a function of the product supply are determined here exogenously, that is they do not result from equations of the model.

Model of imperfect competition market – description of a market of a certain product or a bundle of products, in which at least one of the assumptions of perfect competition market is not satisfied.

Model of perfect competition market – description of a market of a certain product or a bundle of products consisting of four assumptions: market atomization, homogeneity of a product (or a bundle of products), transparency and liquidity.

Monopolist – an owner of a firm who is the only one supplier of a certain product on a market.

Monopoly – a market on which there is only one firm supplying a certain product on this market. Very often this name is also identified with this only one firm.

Monopoly, institutional monopoly – a case of a monopoly in which a monopolist is the only producer who holds an exclusive license for the manufacturing and selling a certain product.

Monopoly, technological monopoly – a case of amonopoly in which a monopolist is the only producer who owns technology and infrastructure needed to manufacture and sell a certain product.

Oligopoly – an example of the imperfect competition. A market on which at least three producers act. At the same time, it does not satisfy the assumptions of the perfect competition model. It can be a market of one homogenous product or a few substitute differentiated products.

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[16] These properties and the formula can be found in the Mathematical appendix.

[17] These properties and the formula can be found in the Mathematical appendix.

Optimal consumption bundle – a bundle of consumer goods which, by a given relation of consumer's preference, is not worse than any other bundle from a set of all consumption bundles available on a market. There can be exactly one or many optimal bundles (infinitely many assuming perfect divisibility of physical units of goods). In particular, optimal consumption bundles are optimal solutions to consumption utility maximization problems or to consumer's expenditure minimization problems. If certain conditions are satisfied, optimal bundles defined in these two ways can be the same.

Optimality conditions in optimization problems[18] – conditions on basis of which one searches for optimal solution of a given optimization problem. They can be one of three types:

– **necessary** - if the condition is a consequent (statement q) in an implication: $p \Rightarrow q$,

– **sufficient** - if the condition is an antecedent (statement p) in an implication: $p \Rightarrow q$,

– **necessary and sufficient in the same time** - if the condition is used in an equivalence: $p \Leftrightarrow q$.

Optimization, conditional optimization – searching for such a stationary point at which a certain function, called an objective function, reaches its optimal (maximal or minimal) value given there are additional conditions on values of decisive variables described by a system of equations or inequalities. In the mathematical sense we search only the specific subset of the domain of the function. When such a point is found then it can be a local optimum, different than a global optimum, if the constraining conditions are binding and or it can be a global optimum if the constraining conditions are not binding.

Optimization, unconditional optimization – searching for such a stationary point at which a certain function, called an objective function, reaches its optimal (maximal or minimal) value. It is called unconditional because there is no additional condition on decisive variables besides the reaching the optimal value. In the mathematical sense we search the whole domain of the function.

Optimum – an argument for which a certain function, called an objective function, has the largest (maximum) or the smallest (minimum) value. It can be determined without additional conditions (unconditional optimum) or with some additional conditions (conditional optimum) described by a system of equations or inequalities.

Optimum, economic optimum – the best choice for an economic agent (consumer, trader, producer) from a point of view of a given choice criterion (optimality criterion). For example, a consumer wants to know what quantities of goods give her/him the highest utility by a given income or what quantities

[18] Not to be confused with optimality criteria, which can be for example the highest utility of a consumption bundle, the highest profit from production and sales, the lowest expenditure incurred on a consumption bundle, the lowest total production cost.

of goods guarantee the lowest expenditure by a given fixed utility level. A producer wants to know what inputs of production factors or what supply level allow her/him to reach the highest profit or what inputs of production factor guarantee the lowest production cost by a given fixed output level.

Pareto frontier – a set of Pareto-optimal allocations determined with respect to a given initial allocation. When the Edgeworth box is used a Pareto frontier is also called a contract curve.

Perfect competition firm – a firm acting on a market of one homogenous product under prevailing perfect competition conditions. What distinguishes the perfect competition firm from the imperfect competition one is that the former has no market power to set a price of its product, thus takes the price given by the market.

Problem of consumer's expenditure minimization – a problem in which one searches for such a consumption bundle by which a consumer incurs the minimum cost of purchasing the bundle with a given fixed utility level.

Problem of consumption utility maximization – a problem in which one searches for such a consumption bundle by which a consumer has the maximum utility and whose value does not exceed a consumer's income.

Problem of cost minimization when producing output at fixed level – a problem in which one searches for such a vector of production factor' inputs by which a producer incurs the minimum cost of producing an output at a given fixed level.

Problem of firm profit maximization with regard to inputs of production factors – a problem in which one searches for such a vector of production factors' inputs by which a firm obtains the maximum profit.

Problem of firm profit maximization with regard to output level – a problem in which one searches for such an output level by which a firm obtains the maximum profit.

Production explicit cost – expenditures incurred by a firm in order to purchase or hire production factors.

Production factors – all service and material resources used by a firm in a production process to manufacture a given product. Among the production factors one specifies: physical capital, labour, land, human capital, social capital, intellectual capital, informational capital.

Production fixed cost – cost related to a production process that does not depend on inputs of production factors nor on the output level.

Production function – a formal description of a set of technologically effective production processes. It assigns maximum quantity of a product that can be produced when using any given inputs of production factors

Production function, Cobb–Douglas production function – a power production function, positively homogenous of first order. It is an only case of a power production function being an example of a neoclassical production function that is increasing, having zero value for zero inputs and satisfying Inada's limit properties.

Production implicit cost (imputed/implied/notional cost) – not financial cost related to what a firm needs to decide in order to use already owned or hired production factors.

Production process – a process of transforming a bundle of inputs (of production factors) into a bundle of outputs (of products).

Production process, technologically feasible process – such a production process in which it is possible to obtain any (not necessarily maximum) quantity of a product using given inputs of production factors.

Production process, technologically effective process – such a technologically feasible production process in which it is possible to obtain a maximum quantity of a product using given inputs of production factors.

Production space – a set of all technologically feasible production processes with a non-Euclidean norm defined on this set.

Production total cost – a sum of variable and of the fixed production cost.

Production variable cost – cost related to a production process that depends on inputs of production factors or on the output level.

Profit – a difference between revenue (turnover) from sales of a product and total cost of its production.

Profit, accounting profit – economic profit with added implicit costs. The addition of the implicit costs results from the fact that when the accounting profit is determined only explicit costs are taken into account.

Profit, economic profit – a difference between revenue (turnover) from sales of a product and total cost of its production, in which one accounts for explicit and implicit costs.

Relation of consumer indifference – a subset of a Cartesian product on the consumer goods' space consisting of all ordered pairs of consumption bundles in which the first bundle is as good (indifferent) as the second bundle. This means also that the utility of the first bundle in the pair equals the utility of the second bundle.

Relation of consumer strong preference – a subset of a Cartesian product on the consumer goods' space consisting of all ordered pairs of consumption bundles in which the first bundle is better (strongly preferred) than the second bundle. This means also that the utility of the first bundle in the pair is bigger than the utility of the second bundle.

Relation of consumer (weak) preference – a subset of a Cartesian product on the consumer goods' space consisting of all ordered pairs of consumption bundles in which the first bundle is not worse (weakly preferred) than the second bundle. This means also that the utility of the first bundle in the pair is not less than the utility of the second bundle.

Resources of production factors – specified quantities of production factors owned by a firm. If the resources owned by a firm do (not) constrain a possibility to make optimal decisions then we call them (un)constrained. In such a case we say that the constraints on resources of production factors are (not) binding.

Returns to scale – a change in an output level caused by a simultaneous and proportional increase (decrease) in inputs of all production factors. One distinguishes:

– **constant returns to scale (proportional revenues)** - when inputs of all production factors are increased/decreased proportionally then the output level increases/decreases proportionally,

– **decreasing returns to scale (decreasing revenues)** - when inputs of all production factors are increased/decreased proportionally then the output level increases/decreases less than proportionally,

– **increasing returns to scale (increasing revenues)** - when inputs of all production factors are increased/decreased proportionally then the output level increases/decreases more than proportionally.

Revenue (turnover) from sales of product – the amount of income for a firm generated by selling the product that this firm manufactures.

Sensitivity analysis (comparative statistics) – analysis of the impact of single variables (values of parameters) on an optimal value of a selected variable. Most of the time it is conducted in terms of *ceteris paribus* using partial derivatives of a function that describes the selected variable and depends on strictly determined parameters of a model under consideration.

Simple model of exchange – a simplified version of the Arrow-Hurwicz model in which one does not take into account any financial characteristics of consumer goods' exchange, that is prices of consumer goods and incomes of traders. The transaction of exchange, if it is made, results in an allocation, which is Pareto optimal and at the same time accepted by all traders. Determination of the allocation depends on an initial allocation and preferences of traders described by utility functions, by an assumption that each trader's behaviour is rational.

Slutsky equation – an equation showing that a change in the demand for i-th good caused by a change in a price of j-th good (price effect) is a sum of a substitution effect and of an income effect, each caused by the change in j-th good's price. The equation is derived by specific assumptions about the utility level and a consumer's income level that result in equal values of the Marshallian demand function and the Hicksian demand (compensated demand) function.

Stationary point – an argument for which a value of a given function's first derivative (or values of first-order partial derivatives in the case of a function of two or more variables) equals zero. It is also said that the stationary point satisfies a first-order condition. Stationary points are points at which a given function can have its local extrema, but it does not have to since the first order condition is necessary but not sufficient.

Strategy of firm – behaviour of a firm aiming at reaching an optimal goal by given constraining conditions. When there are no constraints on resources of production factors, thus a firm can obtain any quantities of these factors, then we call a strategy long-term. When resources of production factors are constrained, thus a firm can obtain only limited quantities of these factors, then we call the strategy short -term.[19]

Substitution – a possibility of replacing a given quantity of one product by a specific quantity of some other product. It refers to consumer goods and then is considered from a point of view of a consumer who wants to have a certain utility of her/his consumption bundle. Or it refers to production factors and then is considered from a point of view of a producer who wants to obtain a certain output level in her/his firm.

Substitute consumer goods – such goods that a given consumer considers replacing a given quantity of one good by a specific quantity of the other good as bringing no significant change in her/his utility level from a consumption bundle in comparison to the bundle before the changes. From a point of view of demand reaction, two consumer goods are called substitutes when a rise in a price of one of these goods results in an increase in the demand for the other good.

Substitute consumer goods, perfect substitutes – specific substitute goods such that a proportion in which a consumer regards replacing one good by the other is always the same, regardless of quantities of goods in a consumption bundle. Preferences towards perfect substitutes are described by a linear utility function.

Substitute production factors – such production factors that in a given production process a producer considers replacing a given input of one production factor by a specific input of the other factor as bringing no significant change in an output level in comparison to the inputs before the changes. When the proportion of the substitution of such production factors is always the same, regardless of inputs of these factors, then they are called perfectly substitute.

Path of price expansion of demand – a sequence of optimal solutions to consumption utility maximization problems, each of which corresponds to a change in a price of one of goods, as compared to an initial level of this price, with the price of the other good unchanged and a consumer's income unchanged.

Path of income expansion of demand – a sequence of optimal solutions to consumption utility maximization problems, each of which corresponds to a change in a consumer's income as compared to an initial level of income, with the prices of both goods unchanged.

Utility function – a function that assigns some number to any consumption bundle in such a way that describes a consumer's relation of (weak) preference.

Walrasian general equilibrium state – a state determined by a set of prices constituting an equilibrium price vector by which the global supply of each good equals the global demand for this good, both expressed in the same physical units. The Walrasian equilibrium price vector, if it exists, is determined with an accuracy of a structure, that is with an accuracy of a multiplication by a positive

[19] This way of reasoning (distinction between long-term and short-term of the firm activity) is not completely correct from the methodological point of view. It would be, if time, considered as continuous or discrete variable, was present explicitly among variables describing the firm activity.

number. The structure of this vector is a ratio of a price of each good and a price of some chosen good which is then called a *numéraire* good.

Walrasian partial equilibrium state – a state determined by a set of prices by which the global supply of good equals the global demand for this good, both expressed in the same physical units. This concepts refers to a single market, where only one good is considered. Thus, it is possible that by given prices of goods there is a few partial equilibrium states for a few markets, but in the same time there is no general equilibrium state by these prices.

Walras's law – a property of the Arrow-Hurwicz model and of the Arrow-Debreu-McKenzie model saying that for any positive prices of goods a money value of the global (reported by all consumers on a market) total (for all goods) demand for goods is equal to the money value of their global (by all suppliers) total (of all goods) supply.

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