SYNCHRONIZATION AND ANTI-SYNCHRONIZATION OF CHAOTIC SYSTEMS USING ADAPTIVE INTEGRAL SLIDING MODE

By

Umar Farooq

MEE143016

A thesis submitted to the

Electrical Engineering Department

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

Faculty of Engineering

Capital University of Science and Technology

Islamabad

December 2016

Dedication

This thesis is dedicated to my Parents, Teachers and Friends

For their endless love, support and encouragement

CAPITAL UNIVERSITY OF SCIENCE & TECHNOLOGY
ISLAMABAD

CERTIFICATE OF APPROVAL

Synchronization and Anti-synchronization of Chaotic systems Using Adaptive Integral Sliding Mode

by

Umar Farooq

MEE143016

THESIS EXAMINING COMMITTEE

S No	Examiner	Name	Organization
(a)	External Examiner	Dr. Nisar Ahmed	GIKI, Swabi
(b)	Internal Examiner	Dr. Aamer Iqbal Bhatti	CUST, Islamabad
(c)	Supervisor	Dr. Fazal ur Rehman	CUST, Islamabad

Dr. Fazal ur Rehman
Thesis Supervisor
January, 2017

Dr. Noor Muhammad Khan
Head of Department
Department of Electrical Engineering
Dated : January, 2017

Dr. Imtiaz Ahmad Taj
Dean
Faculty of Engineering
Dated : January, 2017

ACKNOWLEDGMENT

First and foremost I would like to thank Allah almighty, who gave me the courage and support to continue my graduate studies with the research work. The determination granted by Allah helped me to tolerate the hard time to produce this thesis.

I cannot forget the ideal man of the world and most respectable personality for whom Allah created the whole universe, Prophet Mohammed (Peace Be Upon Him).

I would like to thank to my supervisor, Dr. Fazal-ur-Rehman, with whom it has been an honor to work. He allowed me tremendous freedom in choosing my area of research and gave me outstanding guidance all along the way. His "outside of the box" thinking has taught me to do the same and for that I am grateful to him.

I am highly grateful to my loving parents and family who kept me motivated, guided and focused throughout my Master. Their help in various regards contributed in keeping my moral high. I extend my heartiest thanks to my colleagues in control system research group, especially Mr. Ibrahim Shah, Mr. Waseem Abbasi and Mr. Abdullah. They always guided me whenever I was in need of that.

DECLARATION

It is declared that this is an original piece of my own work, except where otherwise acknowledged in text and references. This work has not been submitted in any form for another degree or diploma at any university or other institution for tertiary education and shall not be submitted by me in future for obtaining any degree from this or any other University or Institution.

Umar Farooq

MEE143016

December 2016

ABSTRACT

This thesis presents new control strategy based on Sliding Mode and Adaptive Integral Sliding Mode for the synchronization and anti-synchronization of chaotic systems. Two cases are considered: (i) systems with known parameters, and (ii) systems with unknown parameters. In case (i) the synchronization and anti-synchronization are achieved through sliding mode control, while in case (ii) the adaptive integral sliding mode control is used. To employ the adaptive integral sliding mode control, the error system is transformed into a special structure containing nominal part and some unknown terms. The unknown terms are computed adaptively. Then the error system is stabilized using integral sliding mode control. The stabilizing controller for the error system is constructed which consists of the nominal control plus some compensator control. The compensator controller and the adapted laws are derived on the basis of Lyapunov stability theory. Three numerical examples, (i) Lorenz system (ii) hyper-chaotic Lorenz-Stenflo system and (iii) hyper-chaotic memristor oscillator systems are shown to illustrate and validate the synchronization schemes presented in this thesis.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

1.1 Introduction

All physical systems are nonlinear by nature. In order to attain the better understanding about the dynamical behavior of the different nonlinear systems, an interesting and important phenomenon is to investigate the synchronization between these dynamical systems. Synchronization, observed as naturally occurring process, has significant impact in diverse areas of engineering, sciences and even in the social life. Synchronization of nonlinear systems is an attractive area among the researchers of different disciplines due to its numerous applications in the fields of engineering and technology. Noteworthy efforts by researchers have been devoted to investigate the problem of synchronization of nonlinear systems. To address the problem of synchronization of nonlinear systems demands the investigation of different dynamical parameters associated with nonlinear systems such as, unknown dynamics which have strong influence on synchronization. This dynamical property of nonlinear urge to be investigated along with nonlinear systems, because their impact on the performance of nonlinear systems cannot be ignored. In different nonlinear systems, different parameters can be source of instability and degrade the closed-loop performance of nonlinear systems. Figure 1.1 demonstrates the basic model for synchronization of nonlinear master-slave systems through an appropriate controller. Convergence of error is assured by selection of a suitable control signal *u(t)*.

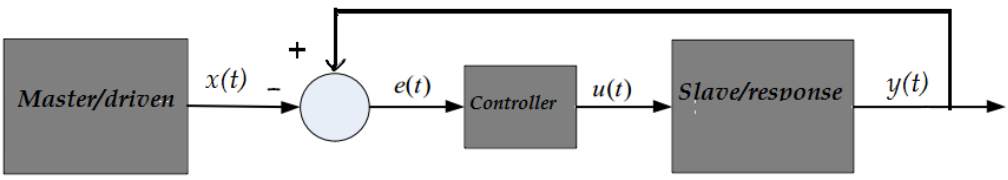

Figure 1.1: Block diagram of synchronization using controller.

This work describes a scheme that applies master/slave chaotic synchronization systems. A sliding plane is chosen to design a sliding mode controller to ensure robustness. In the presence of system uncertainty, the slave chaotic system is then synchronized with the master. The Simulation results indicate that the synchronization error state asymptotically converges to the origin of the phase plane, implying that the master/slave chaotic system synchronization is achieved using Sliding Mode Control for known parameters while the Integral Sliding Mode Control for unknown parameters is in operation.

1.1.1 Overview

Synchronization and Anti-synchronization of chaotic systems is the rudimentary determination of this research work. We need to bring error system to region from any initial condition. The techniques used are Sliding Mode Control and Adaptive Integral Sliding mode Control. Appropriate Hurwitz sliding surface and a Lyapunov function are selected for the stabilizing controller. Adaptive laws strictly obey Lyapunov function of stability analysis. There are three systems uses in the thesis and these results are verified through simulation studies using MATLAB.

1.1.2 Motivation

The history of the attempts made for inventing, building and designing systems, having the capability to control the models which have parameters, whose values are unknown is long and rich. The chaotic systems have been a topic of interest for the researchers over the last three/four decades. It is hardly possible to avoid contact with chaotic systems. Such problems arise in our daily life. Some of these problems are simple to solve but there are control problems with more complications. Synchronization of nonlinear systems contains diverse area of application in almost every field of life. It is quite difficult to discuss all the application areas in this short section, however some active research areas and applied examples of the synchronization are described.

1.2 Problem statement

The purpose of this study is to develop appropriate synchronization schemes for two nonlinear systems working according to master-slave principal.

- That addresses Complete Synchronization and anti-synchronization of two identical nonlinear chaotic systems.

1.3 Application of Research

As we are dealing with the chaotic systems, there are many examples of these systems in our daily life. The chaotic systems are applicable in our daily life, like the worldwide weather, the heart and brain. We have begun to understand that the tools of chaotic theory can be applied on the way to understanding, manipulation, and control of a variety of systems, with numerous of the practical application coming after 1990. Chaotic system is applicable in actual-world as epileptic seizure, heart fibrillation, neural process, chemical reactions, climate, industrial control processes, and many more.

1.4 Structure of the Thesis

Chapter 2: Literature review

This chapter will give us a review the literature published about the synchronization of chaotic systems. On the basis of literature review, a decision is taken about the proposed control strategy to use for selected example of synchronization systems.

Chapter 3: Proposed algorithm

This chapter contains the proposed algorithm synchronization and anti-synchronization of chaotic systems. Adaptive Integral Sliding Mode developed to investigate the problem of synchronization and anti-synchronization of nonlinear systems under the known and unknown parameters, using lyapunov function to check stability.

Chapter 4: Numerical examples

This chapter present synchronization of two identicle chaotic systems with known and unknown parameters, the proposed control technique, Sliding Mode Control for known parameters, while Adaptive Integral Sliding Mode Control for unknown parameters.

Chapter 5: Conclusion and future work

A brief conclusion of thesis is outlined in this Chapter. Moreover, some future research proposals are suggested for the researchers interested to work in the area of synchronization and anti-synchronization of nonlinear systems.

Chapter 2

LITRATURE REVIEW

2.1 Introduction

This chapter presents the literature review of synchronization and anti-synchronization of chaotic systems and sliding mode control. This study will help to design a new control strategy for synchronization and anti-synchronization of chaotic systems.

2.2 Chaos

Chaos theory describes the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems. A dynamical system is called chaotic if it satisfies the three properties: boundedness, infinite recurrence and sensitive dependence on initial conditions [1], which popularly known as the butterfly effect.

Chaos is characterized by the way a dynamical system which does not repeat itself, even though the system is governed by deterministic equations [2]. In the same way that time and the frequency are used to identify chaotic signals, phase-plane and correlation are used to identify the attractor and randomness of the chaotic system. The attractor is a region of the state space from which there are no exit paths. That is, points that get close enough to an attractor remain close even if they are slightly disturbed. Attractors contain of single state called an equilibrium state, or a cycle of states called a limit cycle [2]. For chaotic systems, the attractor does not settle to one of these but explores all of the state space around the attractor for all time without ever repeating.

Chaotic systems have been invoked as details for, or as causal significantly to explanation of, actual-world behaviors. Several examples are epileptic seizure, heart fibrillation, neural process, chemical reactions, climate, industrial control processes and so far forms of message encryption. Aside from irregular performance of actual-world systems, chaos is as well invoked to make clear features such as the real trajectories exhibited in a specified state space or the sojourn times of trajectories in exacting regions of state space [4, 7]. The nature of scientific details in the literature on chaos is carefully under-discussed to put it gently.

Figure 2.1 show the trajectory of the Lorenz attractor in the phase space, depicting the stretching and folding properties [3], which can be seen when plotting the states of the system against each other.

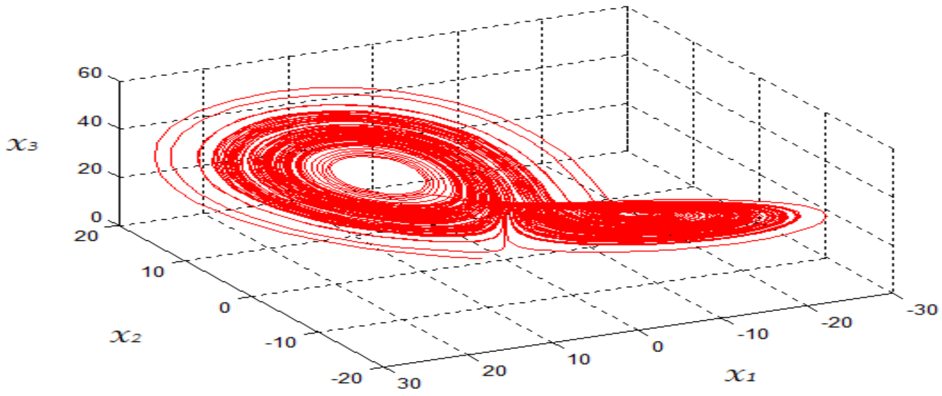

Figure 2.1: The phase portrait of x_1, x_2, x_3

A significant development in chaos theory occurred when Lorenz discovered a chaotic system of a weather model [2]. Subsequently, Rössler discovered a chaotic system in 1976 [13]. Chaos theory has applications in several fields of science and engineering such as oscillators, dynamos, Tokamak systems, chemical reactions, neural networks, neurology, biology, electrical circuit's cryptosystems, memristors random bit generator etc.

Some of real life phenomena exhibit linear behavior, whereas others seem to be nonlinear. Swirling smoke from cigarettes, a waving flag in wind, randomly dribbling water through faucet, behavior of petrol flow inside piping, a chart of human vascularisation, biological populations all accommodate a sort of chaotic order [1].

A mathematical model of chaos was first suggested by Lorenz, meteorology, in 1963 [5]. After Lorenz up to 2005 a variety of popular chaotic systems were introduced by Rikitake [4, 6, 7], R¨ossler [8], Shimizu-Morioka [9], HindmarshRose [10], Chua [11], Rucklidge [12], Sprott [13], Chen [14], L¨u [15] and Liu [16]. In parallel with the developments, chaos and chaotic systems have been used in many scientific disciplines such as engineering, computing, communication, medicine, biology, management-finance and consumer electronics [17]. Accord-ingly, many novel chaotic and hyper-chaotic systems displaying different dynamical behaviors have appeared in literature [18, 19, 26].

A hyperchaotic attractor is typically defined as chaotic behavior with at least two positive Lyapunov exponents. Combined with one null exponent along the flow and one negative exponent to ensure the boundness of the solution, the minimal dimension for a hyperchaotic system is 4.

Recently, there has been great interest in chaotic research on hyperchaotic systems and their applications in secure communications, data encryption, etc. The first four-dimensional hyperchaotic system was discovered by O.E. Rössler in 1976. This figure is taken from [19].

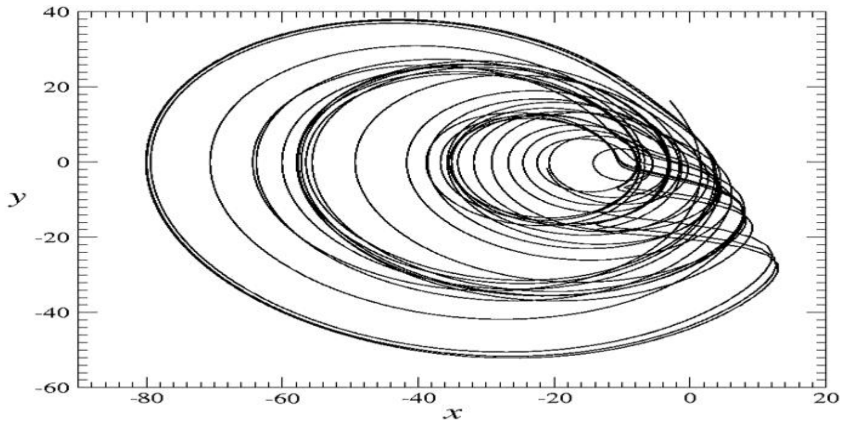

Figure 2.2: First Rossler hyperchaos.

Recently, the generation of hyper chaos and the hyperchaotic circuit realization have attracted researchers' increasing attention. The hyperchaotic system has at least two positive Lyapunov exponents, indicating that its dynamics are expanded in more than one direction simultaneously. For the autonomous continuous system, the dimension of a hyperchaotic attractor must be at least four, however, for a chaotic attractor, three-dimension is enough and it has just a single positive Lyapunov exponent. Therefore, compared with ordinary chaotic system, hyperchaotic system has more complicated and richer dynamics so as to be better used in many chaos-needed fields.

This thesis presents control strategy for chaotic control system. The core determination of this work is to introduce a new control technique for chaotic systems A sliding mode control technique is introduced for known, while adaptive integral sliding mode control technique with unknown parameters of chaotic system.

2.3 Synchronization

Synchronization is based on the concept of closeness of the frequencies between different periodic oscillations generated by two systems that one is a master or drive system and the other one is a slave or response system. While chaos synchronization is unlikely to achieve, even if chaotic subsystems are little different with almost the same initial states, their outputs tend to diverge from each other dramatically as time evolves.

Synchronization processes are ubiquitous in our lives, which play a very important role in many different contexts, such as synchronous communication, signal synchronization (for example, synchronization between video and audio signals), firefly bioluminescence. Synchronization occurs ubiquitously in natural and synthetic systems, such as neural systems, biological systems, social systems, communication systems, and the Internet, geostationary satellite, synchronous motor, database synchronization and Synchronization is a typical collective behavior in nature and technology, such as the synchronous swing of clocks, the generation of harmonic oscillator, and the flocking phenomenon [10].

In the last two decades, there has been considerable interest devoted to the synchronization of chaotic and hyperchaotic systems. In their seminal paper in 1990, Pecora and Carroll [11] introduced a method to synchronize two identical chaotic systems and showed that it was possible for some chaotic systems to be completely synchronized. Subsequently, chaos synchronization has been applied in a wide variety of fields including physics [12], chemistry [13], ecology [14], secure communications [15-16], cardiology [17], robotics [18], complex dynamical networks etc.

In the past twenty years, various types of synchronization have been proposed and investigated, e.g., complete synchronization [7, 8], lag synchronization [9–10], anticipated synchronization [12], phase synchronization [13], project synchronization [17], generalized synchronization [19], etc. As a special case of generalized synchronization, anti-synchronization is achieved when the sum of the state variables of master and slave system converges to zero asymptotically. It has been experimentally and numerically verified that the coupled chaotic systems can achieve anti-synchronization [21–25]. Recently, some control methods have been utilized to anti-synchronize two identical or non-identical chaotic systems and derive theoretically some sufficient anti-synchronization conditions, e.g., observer control [28], active control [29], back-stepping control [36], adaptive control [29,30], sliding mode control [41], nonlinear control [37], $H\infty$ control [39], etc.

In this thesis, a new control scheme based on the adaptive integral sliding mode control for the chaotic synchronization of two identical chaotic systems is used. The sliding mode control method is often used because of its inherent advantages of easy realization, fast response and good transient performance as well as its insensitivity to parameter uncertainties and external disturbances.

Dutch researcher Christian Huygens was probably the first scientist who observed and described the synchronization phenomena in seventeenth century. In 1658, Christian Huygens investigated the synchronization between two weekly coupled pendulum clocks [8]. Despite the study of the first synchronization phenomena, the actual work on synchronization of nonlinear systems was started late in 1920. After few years in 1927, Balthasar Vander Pol extended the efforts of W. H. Reck and J. H. Vincent by obtaining the theoretical and practical results for synchronization [8]. Modern nonlinear dynamics revived in 1990s, when different new dynamical properties of nonlinear systems were explored and innovative work of numerical methods were recognized for controllability and stability analysis of the dynamical nonlinear systems. Peccora and Carrol [9] gives the idea of synchronization of nonlinear (chaotic) systems, by investigating the properties of two nonlinear systems and described that two nonlinear systems can be synchronized by linking them with a common signal. After the inspirational work of Peccora and Carrol, on synchronization of dynamical systems, this problem attracted a great number of researchers from different fields of engineering and sciences. Considerable research work has been carried out to investigate the synchronization phenomena in different nonlinear systems and different control strategies have been developed [10].

Research work on synchronization of nonlinear systems is briefly revisited as follows. Since after the pioneer work on synchronization of two identical nonlinear systems, namely, response and drive systems [9], the problem of synchronization of nonlinear systems has been extensively studied in both theoretical and practical systems. The

Study of synchronization is evolved with the dynamical parameters of nonlinear systems such as unknown parameters etc.

There are some main types of synchronization:

(1) Complete Synchronization (2) Generalized Synchronization (3) Phase Synchronization (4) Lag Synchronization (5) Projective synchronization (6) Anticipatory Synchronization.

This is illustrated in table 2.1:

(1)	**Complete synchronization**: When driven and response meet to be exactly same. $\lim_{t \to \infty} \|Y(t) - X(t)\| = 0$.
(2)	**Generalized synchronization**: Synchronization between the states of two systems by a functional relation is defined as generalized synchronization. $\lim_{t \to \infty} \|Y(t) - FX(t)\| = 0$ .where F is constant.
(3)	**Phase synchronization**: When their phase difference remains bounded and amplitudes remain uncorrelated. $\|\varphi_1(t) - \varphi_2(t)\| = 0.$ Where, $\varphi_1(t)$ and $\varphi_2(t)$ indicate the phases of two coupled oscillators.
(4)	**Lag synchronization**: when dynamics is described by delay differential equations. One of the oscillators follows of other. $\|X_1(t) - X_1(t+\tau)\| = 0,$ Where τ is delay.
(5)	**Projective synchronization**: The state of master $X(t)$ and response system $Y(t)$ synchronize with respect to scaling factor α. $Y(t) = \alpha X(t)$.
(6)	**Anticipatory Synchronization:** Anticipatory synchronization is defined as the states of the drive system anticipate the states of the master system with a time delay $\tau > 0$.

2.3.1 Complete Synchronization

The trajectories of the driven and the response systems converge to be accurately the same. This is the first and the simplest form of synchronization [8, 20]. This occurs in coupled somehow the same systems and well referred as a conventional synchronization or an identical synchronization.

Chaotic systems are dynamical systems that defy synchronization, due to their essential feature of displaying high sensitivity to initial conditions [9]. As a result, two identical chaotic systems starting at nearly the same initial points in phase space develop onto trajectories which become uncorrelated in the course of the time. Nevertheless, it has been shown that it is possible to synchronize these kind of systems, to make them evolving on the same chaotic trajectory [3, 8]. When one deals with coupled identical systems, synchronization appears as the equality of the state variables while evolving in time. We refer to this type of synchronization as complete synchronization (CS).

Two continuous-time chaotic systems:

$$\dot{x}(t) = F(x(t)) \tag{2.1}$$

and

$$\dot{y}(t) = H(y(t)) \tag{2.2}$$

are said to be complete synchronization if:

$$\lim_{t \to \infty} \| y(t) - x(t) \| = 0 \tag{2.3}$$

2.4 Sliding Mode Basics

2.4.1 Sliding Mode Control

Sliding mode control (SMC) is a linear as well as nonlinear control system design technique with inherent robustness properties beside parametric changes, turbulence and perturbation etc. Unlike the classical, recent and healthy control systems design method. This method is based on the on-off sort of control, where the controller switches the control path depending upon the value of predefined polynomial which is a purpose of the system position and is term as sliding surface. Due to the nature of the controller it is considered as an alternating type of control technique and the controller formation is relatively simple to design and execute. As well being a control systems technique, it is also used for the irritation estimation and rejection. SMC [8] is a variable structure control systems design technique. The very basic notion of sliding mode control was given in [10].

There are two fundamental points of interest of the sliding mode control. First is that dynamic behavior of the structure might be customized by the specific selection of sliding function. Furthermore, the closed loop reaction develops in to absolutely unresponsive to some specific uncertainties. The concept of optimal sliding mode control design was discussed in [11]. The very important control strategies for sliding mode control of indeterminate systems were briefly discussed in [12, 13]. The basic principle of SMC is outlined in the following important references [14, 15].

2.4.2 Integral Sliding Mode Control

Integral Sliding Mode Control (ISMC) was initially proposed to impose a sliding mode from the start of the system response, which means a controller based on ISMC ideas can give satisfied to matched uncertainties all through the entire system response [16]. In this section, Integral Sliding Mode (ISM) controllers is explain, and the particular features linked with ISMC design are discussed. It is assumed that state information is presented for the controller design. As established in this chapter, when using sliding mode based technique, the system state trajectories are insensitive to matched uncertainties while in the sliding mode [16].

In Integral Sliding Mode Control (ISMC), it is assumed that there exists a nominal plant, for which an appropriately designed state feedback controller has already been designed to make sure asymptotic stability of the closed-loop system, and to satisfy predefined performance specifications. A discontinuous controller is 'added' to the nominal state feedback controller to make sure the nominal performance is preserved, and the system is not sensitive to outer disturbances [17]. This design philosophy provides the opportunity to retro-fit an ISMC to the existing baseline controller to compensate for the matched uncertainties and external disturbances all through the system response.

2.4.3 Sliding Surface Design

This section explores variable structure control (VSC) as a speedy swapped feedback control causing in sliding mode [14, 16]. According to a rule, the gain in every feedback track switches between two values that depend on the value of the state at every point. The reason for transferring control function is to make the system state trail onto a pre-indicated plane in the state space and to keep up system state path on this surface for consequent time. The surface is known as a switching surface. The feedback track has gained one when the plant states route is "above" the surface and different gains if the path is "beneath" the surface. This surface characterizes the principle for proper switching. This surface is similarly named the sliding surface.

The uncertainties and disturbances are always present in practical system and in such cases, discontinuous control ensures robustness. Figure 2.2 shows the reaching phase (RP), sliding mode (SM) and sliding surface (SS).

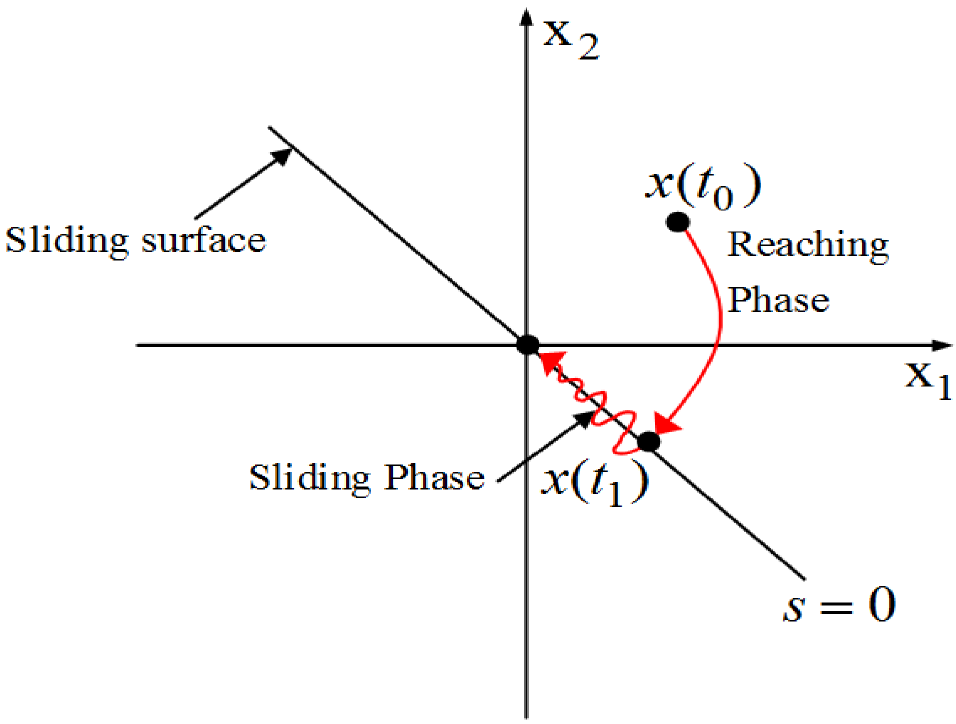

Figure 2.3: The Sliding Phase, Reaching Phase and Sliding Surface.

2.4.4 Chattering Phenomenon

This part is about an indication of the chattering phenomenon. In sliding mode scheme the control signal exhibits high frequency oscillation called chattering. Such chattering has much negative effect in real world applications. This phenomenon may lead to large unwanted oscillations that degrade performance of the system. In order to avoid chattering effect, various solutions of this problem have been proposed in [17], [18] i.e. the boundary layer design. A new design scheme based on estimation of sliding variable was presented [19]. The method based on the describing function approach was developed for chattering analysis of the system in the presence of the un-modeled dynamics. Another way to reduce chattering effect is by means of Second Order Sliding Mode (SOSM) and the High Order Sliding Mode (HOSM) control techniques. Figure 2.3 shows the chattering effect.

In above section main property and drawback of chattering have been discussed. To reduce the chattering effect some approaches were proposed. The interesting and the most recent method for the removal of the chattering is the second or higher order sliding mode control theory.

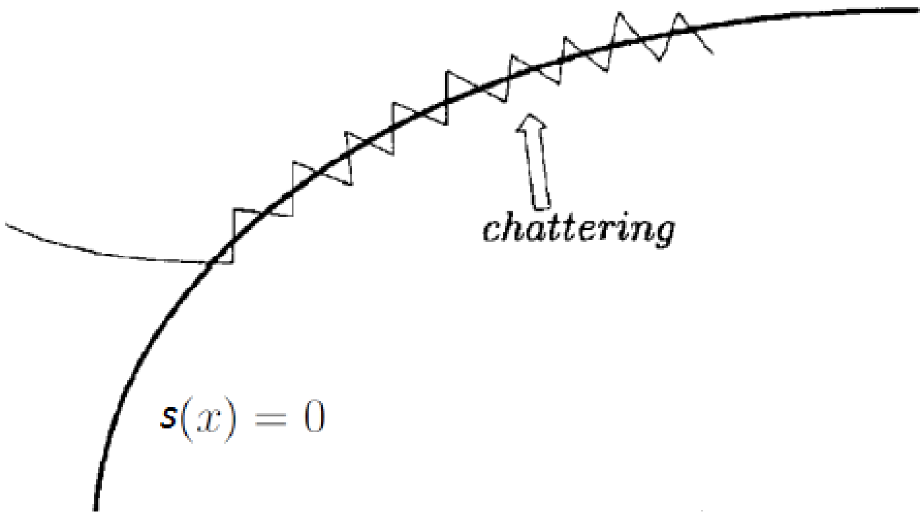

Figure 2.4: The Chattering Effect.

Chapter 3

A new control algorithm for complete synchronization and anti-synchronization based on sliding mode

3.1 Introduction

In this chapter a new control technique is presented to achieve Complete Synchronization (CS) and anti-synchronization between two identical chaotic systems. Two cases are considered, (i) with known parameters (ii) With unknown parameters. Case (i) is solved with sliding mode control while case (ii) solved using adaptive integral sliding mode control.

3.2 Problem formulation

Consider the two chaotic systems:

$$\dot{w} = l(w) + L(w)\theta \tag{3.1}$$

$$\dot{s} = d(s) + D(s)\vartheta + u \tag{3.2}$$

Where $w = (w_1, w_2, ..., w_n)^T \in R^n$ and $(s_1, s_2, ... s_n)^T \in R^n$ are state vectors of the drive system (3.1) and response system (3.2) respectively. $\theta \in \Re^p$ and $\vartheta \in \Re^q$ are real vectors of known parameters. $L(w) \in R^{n \times p}$ and $D(s) \in R^{n \times q}$ are matrices. $l(w) \in R^n$ and $d(s) \in R^m$ are vectors of nonlinear functions, and $u(w,s) \in R^m$ is the real control vector.

The error is defined as:

$$e = s - qw \tag{3.3}$$

Where $q = 1$, for synchronization and $q = -1$, for anti-synchronization.

Then error dynamics is:

$$\dot{e} = \dot{s} - q\dot{w} = d(s) + D(s)\theta + u - q\{l(w) + L(w)\vartheta\} \tag{3.4}$$

The control objective in complete synchronization is to design u, such that error system (3.4) becomes asymptotically stable.

Case 1: synchronization and anti-synchronization with known parameter:

Consider the following system:

$$\dot{x} = l(w) + L(w)\theta$$

$$\dot{y} = d(s) + D(s)\vartheta + u$$

Define the error as:

$$e = s - qw \tag{3.5}$$

Where $e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \in R^n$. By taking the derivative of equation (3.5) with respect time, the dynamical error system is obtained as:

$$\dot{e} = \dot{w} - q\dot{s} = d(s) + D(s)\vartheta + u - q\{l(w) + L(w)\theta\} \tag{3.6}$$

By choosing

$$u = -d(s) - D(s)\vartheta + q\{l(w) + L(w)\theta\} + ee$$

Where, $ee = \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ e_n \\ v \end{bmatrix}$

Here v is the new input, and then system (3.6) becomes:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \vdots &= \vdots \\ \dot{e}_n &= v \end{aligned} \tag{3.7}$$

Define the Hurwitz sliding surface system (3.7) as:

$$\sigma = e_1 + \sum_{i=2}^{n-1} c_i e_i + e_n$$
$$\dot{\sigma} = \dot{e}_1 + \sum_{i=2}^{n-1} c_i \dot{e}_i + \dot{e}_n \qquad (3.8)$$
$$\dot{\sigma} = e_2 + \sum_{i=2}^{n-1} c_i e_{i+1} + v$$

By choosing $v = -e_2 - \sum_{i=2}^{n-1} c_i e_{i+1} - k\sigma$, we have $\dot{\sigma} = -k\,\sigma$. Therefore the system (3.7) is asymptotically stable.

From this we conclude that, $\sigma \to 0$, therefore $(e_1, e_2 \ldots e_n) \to 0$.

Case 2: synchronization and anti-synchronization with unknown parameter:

Let $\hat{\theta}$, $\hat{\vartheta}$ be estimate of θ, ϑ respectively, $\tilde{\vartheta} = \vartheta - \vartheta$ and $\tilde{\theta} = \theta - \hat{\theta}$ be error in estimating θ, ϑ .

Then equation (3.1) and (3.2) can be written as:

$$\dot{w} = l(w) + L(w)\hat{\theta} + L(w)\tilde{\theta} \qquad (3.9)$$

$$\dot{s} = d(s) + D(s)\hat{\vartheta} + G(s)\tilde{\vartheta} + u \qquad (3.10)$$

Define the error as:

$$e = w - qs \qquad (3.11)$$

By taking the derivative of equation (3.11) with respect time, we have:

$$\dot{e} = \dot{w} - q\dot{s}$$
$$\dot{e} = d(s) + D(s)\hat{\vartheta} + D(s)\tilde{\vartheta} + u - q\{l(w) + L(w)\hat{\theta} + L(w)\tilde{\theta}\} \qquad (3.12)$$

By choosing

$$u = -d(s) - D(s)\hat{\vartheta} + q\{l(w) + L(w)\hat{\theta}\} + ee$$

Then equation (3.12) becomes

$$\dot{e} = ee + D(s)\tilde{\vartheta} - q\,L(w)\tilde{\theta}$$

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_n \end{bmatrix} = \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ v \end{bmatrix} + D(s)\tilde{\vartheta} - q\,L(w)\tilde{\theta} \tag{3.13}$$

To employ the integral sliding mode control, choose the nominal system for (3.13) as:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \vdots &= \vdots \\ \dot{e}_n &= v_0 \end{aligned} \tag{3.14}$$

Define the Hurwitz sliding surface system (3.14) as:

$$\begin{aligned} \sigma_0 &= e_1 + \sum_{i=2}^{n-1} c_i e_i + e_n \\ \dot{\sigma}_0 &= \dot{e}_1 + \sum_{i=2}^{n-1} c_i \dot{e}_i + \dot{e}_n \\ \dot{\sigma}_0 &= e_2 + \sum_{i=2}^{n-1} c_i e_{i+1} + v_0 \end{aligned} \tag{3.15}$$

By choosing $v_0 = -e_2 - \sum_{i=2}^{n-1} c_i e_{i+1} - k\sigma_0$, we have $\dot{\sigma}_0 = -k\,\sigma_0$. Therefore the system (3.14) is asymptotically stable.

Now choose the sliding surface for the system (3.13) as: $\sigma = \sigma_0 + z$

Where, z is some integral term computed later, to avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$. Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later.

Where, $C = [1\, c_1\, c_2\, \ldots c_{n-1}\, 1]$ is chosen in such a way that σ become Hurwitz polynomial.

Then,

$$
\begin{aligned}
\sigma &= C\,e + z \\
\dot{\sigma} &= C\,\dot{e} + \dot{z} \\
\dot{\sigma} &= C\,[ee + D(s)\tilde{\vartheta} - qL(w)\tilde{\theta}] + \dot{z} \\
&= e_2 + \sum_{i=2}^{n-1} c_i e_i + v_0 + v_s + CD(s)\tilde{\vartheta} - qCL(w)\tilde{\theta} + \dot{z}
\end{aligned}
$$

By choosing a Lyapunov function: $V = \frac{1}{2}\sigma^2 + \frac{1}{2}\tilde{\theta}^T\tilde{\theta} + \frac{1}{2}\tilde{\vartheta}^T\tilde{\vartheta}$ design the adaptive laws for $\tilde{\theta}, \hat{\theta}, \tilde{\vartheta}, \hat{\vartheta}$ and compute v_s such that $\dot{V} < 0$.

Consider a Lyapunov function $V = \frac{1}{2}\sigma^2 + \frac{1}{2}\tilde{\theta}^T\tilde{\theta} + \frac{1}{2}\tilde{\vartheta}^T\tilde{\vartheta}$. Then $\dot{V} < 0$ if the adaptive laws for $\tilde{\theta}, \hat{\theta}, \tilde{\vartheta}, \hat{\vartheta}$ and the value of v_s are chosen as:

$$
\begin{aligned}
&\dot{z} = -e_2 - \sum_{i=1}^{n-1} c_i e_{i+1} - v_0, \quad v_s = -k\sigma \\
&\dot{\tilde{\theta}} = \sigma\, q\, L^T(w) C^T - k_1\tilde{\theta} \\
&\dot{\tilde{\vartheta}} = -\sigma\, D^T(s) C^T - k_2\tilde{\vartheta} \qquad \text{where}, k, k_1, k_2 > 0
\end{aligned}
\tag{3.16}
$$

Proof:

$$
\begin{aligned}
\dot{V} &= \sigma\dot{\sigma} + \tilde{\theta}^T\dot{\tilde{\theta}} + \tilde{\vartheta}^T\dot{\tilde{\vartheta}} \\
&= \sigma\{e_2 + \sum_{i=1}^{n-1} c_i e_{i+1} + v_0 + v_s + D(s)\tilde{\vartheta} - q\,L(w)\tilde{\theta} + \dot{z}\} + \tilde{\theta}^T\dot{\tilde{\theta}} + \tilde{\vartheta}^T\dot{\tilde{\vartheta}} \\
&= \sigma[e_2 + \sum_{i=1}^{n-1} c_i e_{i+1} + v_0 + v_s + \dot{z}] + \tilde{\theta}^T\{\dot{\tilde{\theta}} - q\,\sigma\,L^T(w)C^T\} + \tilde{\vartheta}^T\{\dot{\tilde{\vartheta}} + \sigma\,D^T(s)C^T\}
\end{aligned}
$$

By using

$$
\begin{aligned}
&\dot{z} = -e_2 - \sum_{i=1}^{n-1} c_i e_{i+1} - v_0, \quad v_s = -k\sigma \\
&\dot{\tilde{\theta}} = \sigma\, q\, L^T(w) C^T - k_1\tilde{\theta} \\
&\dot{\tilde{\vartheta}} = -\sigma\, D^T(s) C^T - k_2\tilde{\vartheta}, \text{where}, k, k_1, k_2 > 0
\end{aligned}
$$

We have

$$
\dot{V} = -k\sigma^2 - k_1\tilde{\theta}^T\tilde{\theta} - k_2\tilde{\vartheta}^T\tilde{\vartheta}
$$

From this we conclude that $\sigma, \tilde{\theta}, \tilde{\vartheta} \to 0$, since $\sigma \to 0$, therefore $(e_1, e_2 \ldots e_n) \to 0$.

Chapter 4

Numerical Examples

Introduction

In this chapter, three different numerical examples are considered to verify the proposed control strategy.

4.1 Numerical Example 1: (Lorenz system)

The following example is taken from [24], where synchronization for this system is obtained by using finite-time controller, while we used sliding mode control for known parameter and integral sliding mode control technique for unknown parameter to achieve synchronization of chaotic system. We compare our result with given results in [24], and our error result approaching to zero faster as compare to result presented in [24].

Consider the Lorenz system [24] as a master system

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= cx_1 - x_2 - x_1x_2 \\ \dot{x}_3 &= x_1x_2 - bx_3 \end{aligned} \tag{4.1}$$

and the slave system

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + u_1 \\ \dot{y}_2 &= cy_1 - y_2 - y_1y_2 + u_2 \\ \dot{y}_3 &= y_1y_2 - by_3 + u_3 \end{aligned} \tag{4.2}$$

When system parameters are chosen as: $a = 10,\ b = 8/3,\ c = 28,$ then this system shows chaotic behavior with initial conditions: $x(0) = [1,2,0]^T$

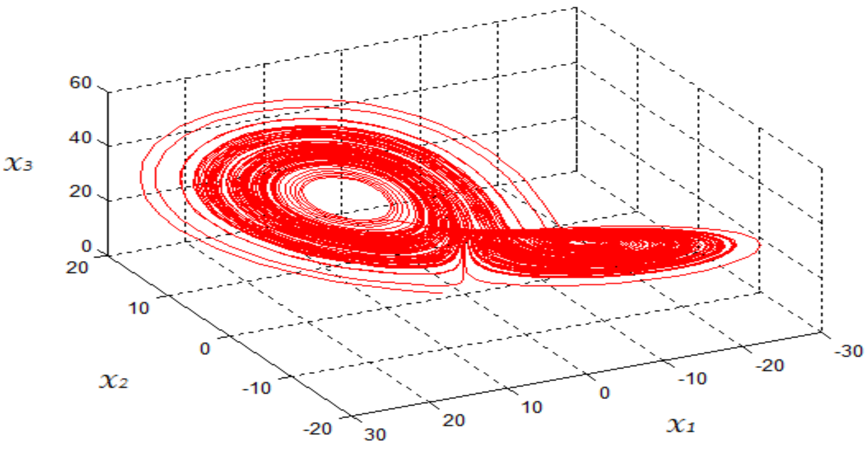

Figure 4.1: The phase portrait of x, y, z

Case 1: synchronization and anti-synchronization with known parameters:

The error signals are defined as:

$$e_1 = y_1 - q\,x_1, e_2 = y_2 - q\,x_2, e_3 = y_3 - q\,x_3 \qquad (4.3)$$

Where $q = 1$, for synchronization and $q = -1$, for anti-synchronization.

Then the error dynamics becomes:

$$\begin{aligned} e_1 &= \dot{y}_1 - q\ \dot{x}_1 = a(y_2 - y_1) - q\ a(x_2 - x_1) + u_1 \\ e_2 &= \dot{y}_2 - q\ \dot{x}_2 = cy_1 - y_2 - y_1 y_3 - q\ (cx_1 - x_2 - x_1 x_3) + u_2 \\ e_3 &= \dot{y}_3 - q\ \dot{x}_3 = y_1 y_2 - by_3 - q\ (x_1 x_2 - bx_3) + u_3 \end{aligned} \qquad (4.4)$$

By choosing

$$\begin{aligned} u_1 &= -a(y_2 - y_1) + q\ a(x_2 - x_1) + e_2 \\ u_2 &= -(cy_1 - y_2 - y_1 y_3) + q\ cx_1 - x_2 - x_1 x_3 + e_3 \\ u_3 &= -(y_1 y_2 - by_3) + q\ x_1 x_2 - bx_3 + v \end{aligned}$$

Where v is the new input, which can be written as:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= v \end{aligned} \qquad (4.5)$$

Define the Hurwitz sliding surface for nominal system (4.4) as:

$$\sigma_0 = (1 + \frac{d}{dt})^2 e_1 = e_1 + 2e_2 + e_3$$

Then,

$\dot{\sigma}_0 = \dot{e}_1 + 2\dot{e}_2 + \dot{e}_3 = e_2 + 2e_3 + v$

By choosing $v = -e_2 - 2e_3 - k\sigma_0$, $k > 0$, we have $\dot{\sigma}_0 = -k\sigma_0$. Therefore the error system (4.4) is asymptotically stable.

In simulations, the initial conditions are chosen as given in [2], $x(0) = [1,2,0]^T$, $y(0) = [0,1,2]^T$. The values of parameters are: $a = 10,\ b = 8/3,\ c = 28$.

Consider a Lyapunov function: $V = 0.5(\sigma^2{}_0\ )$

Then,

$\dot{V} = \sigma_0\dot{\sigma}_0 = \sigma_0(-k\sigma_0) = -k\sigma_0{}^2$

From this we conclude that $\sigma_0 \to 0$, since σ_0 are Hurwitz therefore $e_i \to 0, i = 1,\ldots,3$, therefore the systems (4.5) are asymptotically stable.

Simulation results:

For synchronization we set q=1 in eq. (4.3):

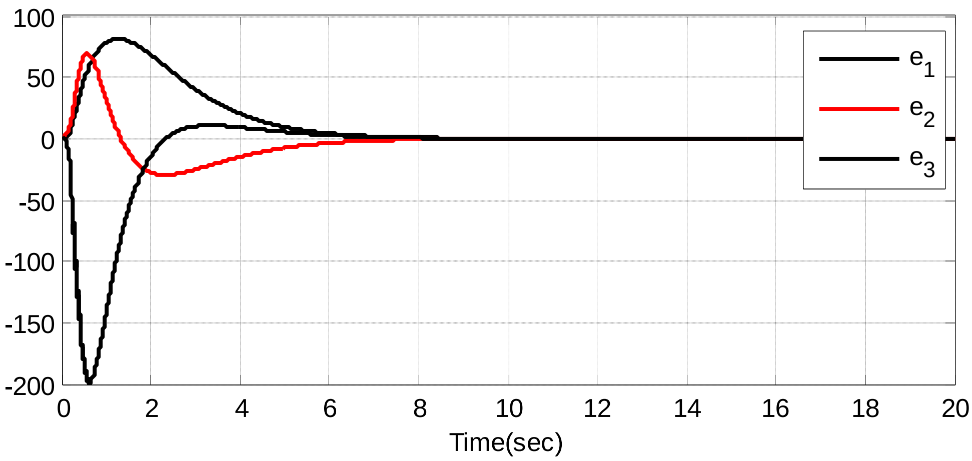

Figure 4.2: Time history of error variables for known parameters.

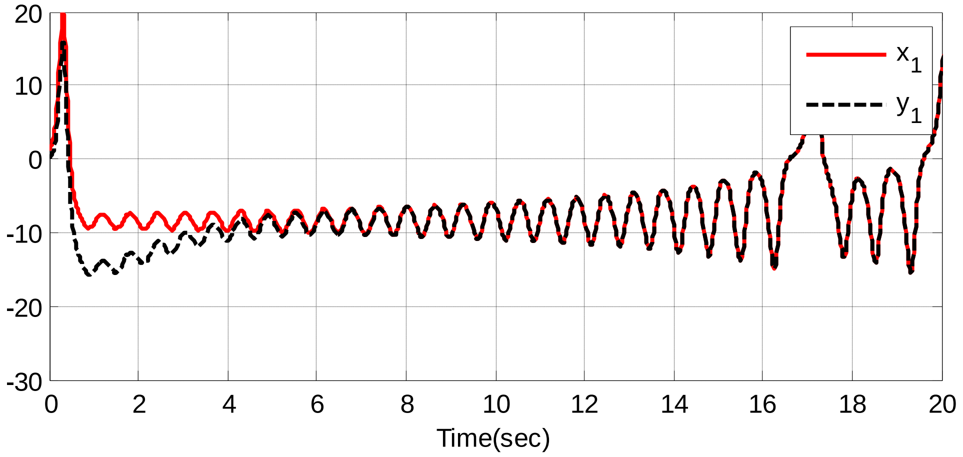

Figure 4.3: Synchronization between x_1 and y_1 for known parameters.

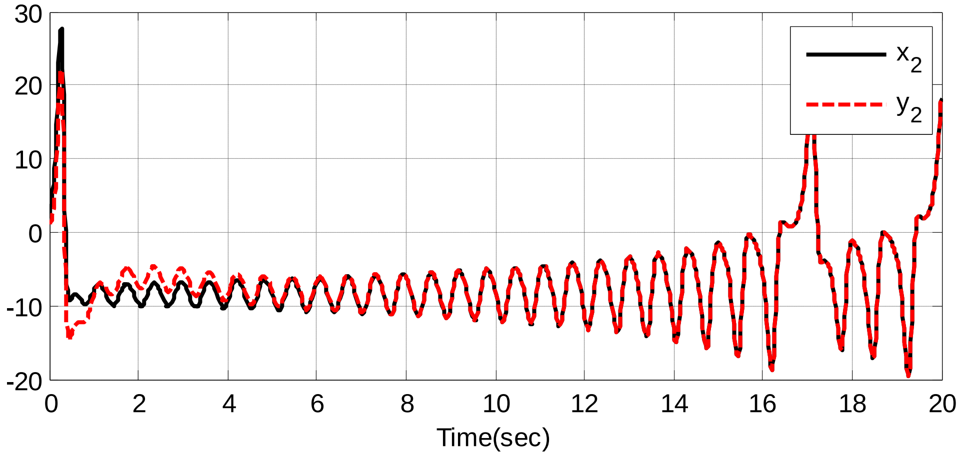

Figure 4.4: Synchronization between x_2 and y_2 for known parameters.

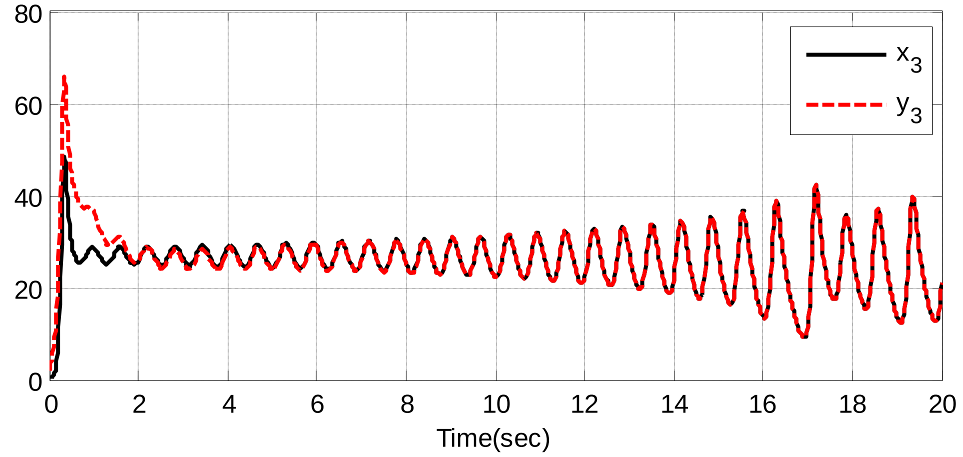

Figure 4.5: Synchronization between x_3 and y_3 for known parameters.

For anti-synchronization we set q= -1 in eq. (4.15):

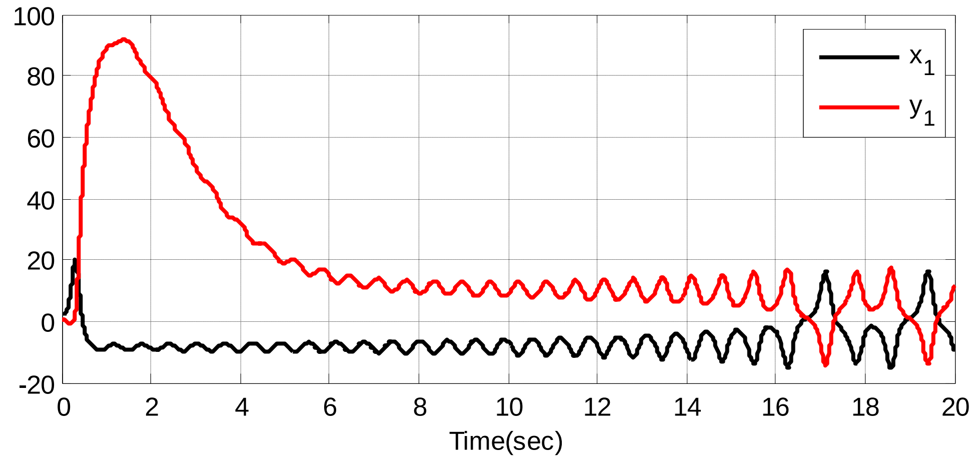

Figure 4.6: Anti-synchronization between x_1 and y_1 for known parameters.

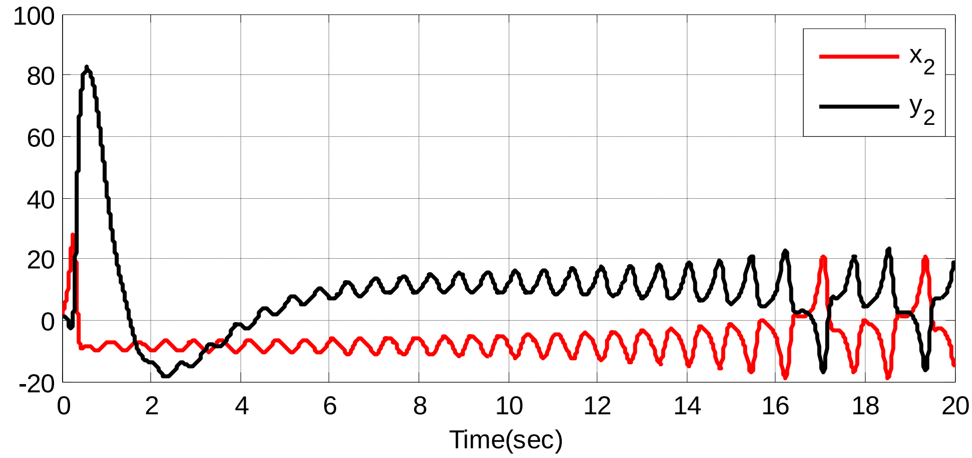

Figure 4.7: Anti-synchronization between x_2 and y_2 for known parameters.

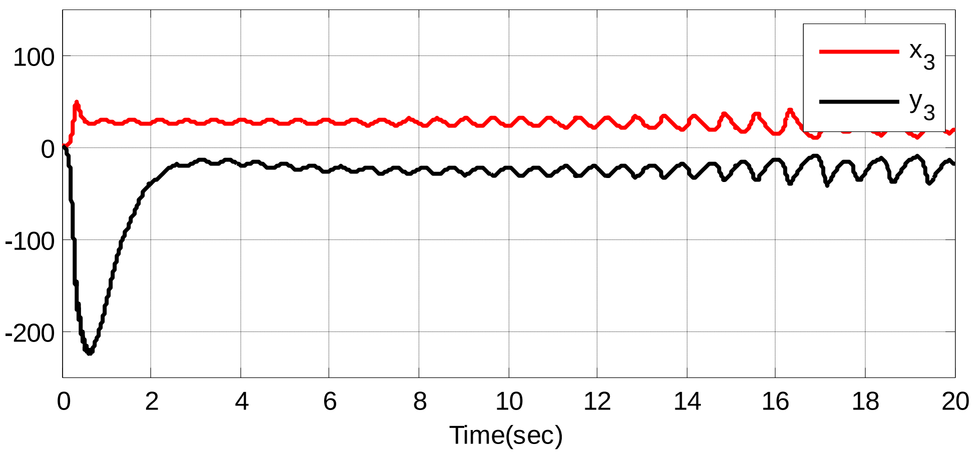

Figure 4.8: Anti-synchronization between x_3 and y_3 for known parameters.

Case 2: synchronization and anti-synchronization with unknown parameters

Let $\hat{a}, \hat{b}, \hat{c},$ be estimated of $a, b, c,$ and let $\tilde{a} = a - \hat{a}, \tilde{b} = b - \hat{b}, \tilde{c} = c - \hat{c}$ be errors.

Thus system (4.1) and (4.2) can be written as:

$$\begin{aligned} \dot{x}_1 &= \hat{a}(x_2 - x_1) + \tilde{a}(x_2 - x_1) \\ \dot{x}_2 &= \hat{c}x_1 + \tilde{c}x_1 - x_2 - x_1x_3 \\ \dot{x}_3 &= x_1x_2 - \hat{b}x_3 - \tilde{b}x_3 \end{aligned} \tag{4.6}$$

$$\begin{aligned} \dot{y}_1 &= \hat{a}(y_2 - y_1) + \tilde{a}(y_2 - y_1) + u_1 \\ \dot{y}_2 &= \hat{c}y_1 + \tilde{c}y_1 - y_2 - y_1y_3 + u_2 \\ \dot{y}_3 &= y_1y_2 - \hat{b}y_3 - \tilde{b}y_3 + u_3 \end{aligned} \tag{4.7}$$

The error signals are defined as:

$$e_1 = y_1 - q\,x_1, e_2 = y_2 - q\,x_2, e_3 = y_3 - q\,x_3 \tag{4.8}$$

Then the dynamics of the error system becomes:

$$\begin{aligned} &\dot{e}_1 = \dot{y}_1 - q\,\dot{x}_1, \dot{e}_2 = \dot{y}_2 - q\,\dot{x}_2, \dot{e}_3 = \dot{y}_3 - q\,\dot{x}_3 \\ &\dot{e}_1 = \dot{y}_1 - q\,\dot{x}_1 = (\hat{a}(y_2 - y_1) + \tilde{a}(y_2 - y_1)) + u_1 - q\,(\hat{a}(x_2 - x_1) + \tilde{a}(x_2 - x_1)) \\ &\dot{e}_2 = \dot{y}_2 - q\,\dot{x}_2 = (\hat{c}y_1 + \tilde{c}y_1 - y_2 - y_1y_3) + u_2 - q\,(\hat{c}x_1 + \tilde{c}x_1 - x_2 - x_1x_3) \\ &\dot{e}_3 = \dot{y}_3 - q\,\dot{x}_3 = (y_1y_2 - \hat{b}y_3 - \tilde{b}y_3) + u_3 - q\,(x_1x_2 - \hat{b}x_3 - \tilde{b}x_3) \end{aligned} \tag{4.9}$$

By choosing

$$\begin{aligned} u_1 &= -\hat{a}(y_2 - y_1) + q\,\hat{a}(x_2 - x_1) + e_2 \\ u_2 &= -(\hat{c}y_{11} - y_2 - y_1y_3) + q\,(\hat{c}x_1 - x_2 - x_1x_3) + e_3 \\ u_3 &= -(y_1y_2 - \hat{b}y_3) + q\,(x_1x_2 - \hat{b}x_3) + v \end{aligned}$$

Where v is the new input, the system (4.9) can be written as:

$$\begin{aligned} \dot{e}_1 &= \tilde{a}(y_2 - y_1) - q\,\tilde{a}(x_2 - x_1) + e_2 \\ \dot{e}_2 &= \tilde{c}y_1 - q\,\tilde{c}x_1 + e_3 \\ \dot{e}_3 &= -\tilde{b}y_3 + q\,\tilde{b}x_3 + v \end{aligned} \tag{4.10}$$

To employ the integral sliding mode control, choose the nominal system for (4.10) as:

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= v_0 \end{aligned} \tag{4.11}$$

Define the Hurwitz sliding surface for nominal system (4.11) as: $\sigma_0 = (1+\frac{d}{dt})^2 e_1 = e_1 + 2e_2 + e_3$

Then

$$\dot{\sigma}_0 = \dot{e}_1 + 2\dot{e}_2 + \dot{e}_3 = e_2 + 2e_3 + v$$

By choosing $v = -e_2 - 2e_3 - k\,\sigma_0,\ k > 0$, we have $\dot{\sigma}_0 = -k\,\sigma_0$. Therefore the nominal system (4.9) is asymptotically stable.

Now choose the sliding surface for the system (4.8) as: $\sigma = \sigma_0 + z = e_1 + 2e_2 + e_3 + z$

Where, z is some integral term computed later. To avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$.

Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later.

Then,

$$\begin{aligned}\dot{\sigma} &= \dot{e}_1 + 2\dot{e}_2 + \dot{e}_3 + \dot{z} \\ &= \tilde{a}(y_2 - y_1) - q\tilde{a}(x_2 - x_1) + e_2 + 2\tilde{c}y_1 - q2\tilde{c}x_1 \\ &+ 2e_3 - \tilde{b}y_3 + q\tilde{b}x_3 + v_0 + v_s + \dot{z}\end{aligned}$$

By choosing a Lyapunov function: $V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2)$, design the adaptive laws for $\tilde{a}\ ,\hat{a}\ ,\tilde{b}\ ,\hat{b}\ ,\tilde{c},\hat{c}$ and compute v_s such that $\dot{V} < 0$.

Consider a Lyapunov function: $V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2)$. Then $\dot{V} < 0$ if the adaptive laws for $\tilde{a}\ ,\hat{a}\ ,\tilde{b}\ ,\hat{b}\ ,\tilde{c},\hat{c}$ and the value of v_s are chosen as:

$$\begin{aligned}&\dot{z} = -e_2 - 2e_3 - k\sigma_0 k > 0 - v_0\,, \qquad v_s = -k\sigma \\ &\dot{\tilde{a}} = -\sigma(y_2 - y_1) + \sigma q(x_2 - x_1) - k\,\tilde{a},\ \dot{\hat{a}} = -\dot{\tilde{a}} \\ &\dot{\tilde{b}} = \sigma\, y_3 - \sigma q\, x_3 - k\,\tilde{b},\ \ \dot{\hat{b}} = -\dot{\tilde{b}} \\ &\dot{\tilde{c}} = -2\,\sigma\, y_1 + 2\,\sigma q\, x_1 - k\,\tilde{c} \quad\ \dot{\hat{c}} = -\dot{\tilde{c}}\end{aligned} \tag{4.12}$$

Proof:

Since

$$
\begin{aligned}
\dot{V} &= \sigma\dot{\sigma} + \tilde{a}\ \dot{\tilde{a}}\ + \tilde{b}\ \dot{\tilde{b}} + \tilde{c}\ \dot{\tilde{c}} \\
&= \sigma(\tilde{a}(y_2 - y_1) - q\tilde{a}(x_2 - x_1) + e_2 + 2\tilde{c}y_1 - 2\tilde{c}x_1 \\
&+ 2e_3 - \tilde{b}y_3 + q\tilde{b}x_3 + v_0 + v_s + \dot{z}) + \tilde{a}\ \dot{\tilde{a}}\ + \tilde{b}\ \dot{\tilde{b}} + \tilde{c}\ \dot{\tilde{c}}\ + \tilde{d}\ \dot{\tilde{d}} \\
&= \sigma(e_2 + 2e_3 + v_0 + v_s + \dot{z}) + \tilde{a}\ (\dot{\tilde{a}}\ + \sigma(y_2 - y_1) - \sigma(x_2 - x_1)\ ) \\
&+ \tilde{b}\ (\dot{\tilde{b}} + \sigma y_4 - \sigma x_4) + \tilde{c}\ (\dot{\tilde{c}}\ + 2\sigma y_1 - 2\sigma x_1\ )
\end{aligned}
$$

By using

$$
\begin{aligned}
&\dot{z} = -e_2 - 2e_3 - k\sigma_0 k > 0 - v_0\,, \qquad v_s = -k\sigma \\
&\dot{\tilde{a}} = -\sigma(y_2 - y_1) + q\sigma(x_2 - x_1) - k\,\tilde{a},\ \dot{\hat{a}} = -\dot{\tilde{a}} \\
&\dot{\tilde{b}} = \sigma\, y_3 - q\sigma\, x_3 - k\,\tilde{b},\ \ \dot{\hat{b}} = -\dot{\tilde{b}} \\
&\dot{\tilde{c}} = -\,2\,\sigma\, y_1 + q2\,\sigma\, x_1 - k\,\tilde{c} \quad\ \dot{\hat{c}} = -\dot{\tilde{c}}
\end{aligned}
$$

We have

$$
\dot{V} = -k\sigma^2 - k_1\tilde{a}_1^2 - k_2\tilde{b}^{\,2} - k_3\tilde{c}^{\,2}\,.
$$

From this we conclude that $\sigma, \tilde{a}, \tilde{b}, \tilde{c} \rightarrow 0$. Since $\sigma \rightarrow 0$, therefore $e = (e_1, e_2, e_3) \rightarrow 0$.

In simulations, the initial conditions are chosen as: $x(0) = [1,2,0]^T$, $y(0) = [0,1,2]^T$. The true values of the unknown parameters are chosen as: $\alpha = 10,\ b = 8/3,\ c = 28.$

Simulation result:

For synchronization we set q=1 in eq. (4.8):

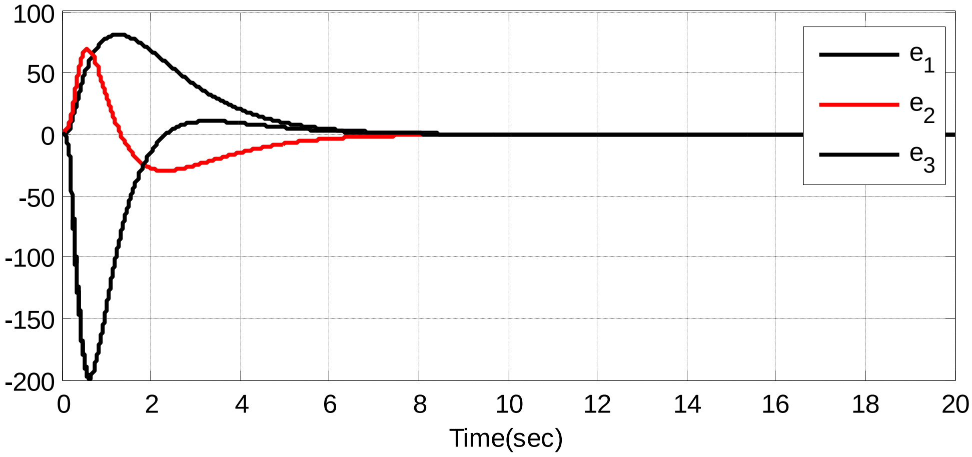

Figure 4.9: Time history of error variable of unknown parameters.

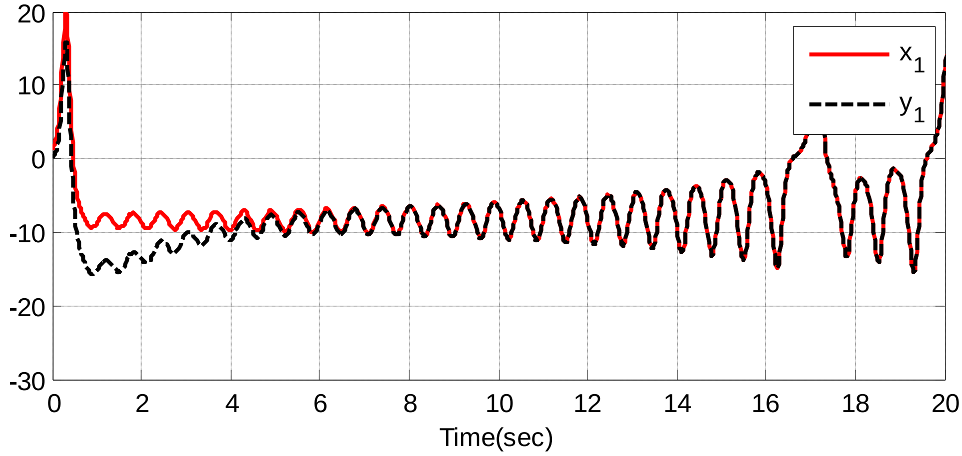

Figure 4.10: Synchronization between x_1 and y_1 for unknown parameters.

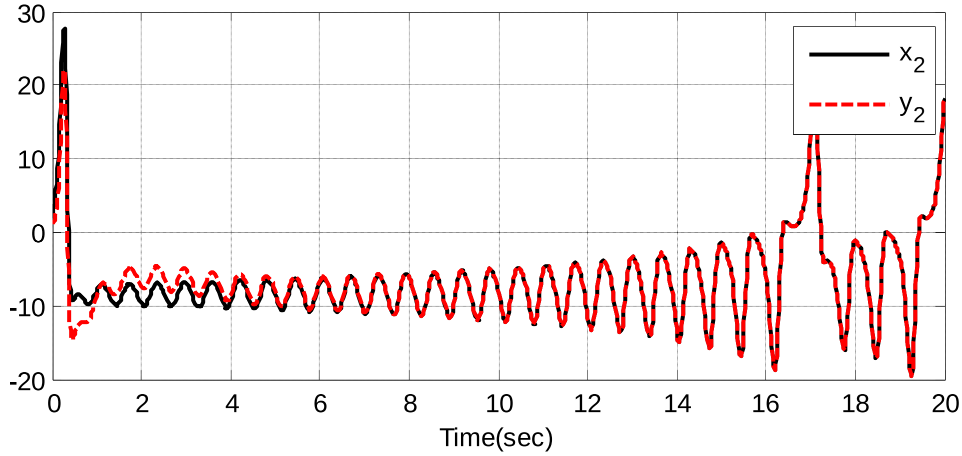

Figure 4.11: Synchronization between x_2 and y_2 for unknown parameters.

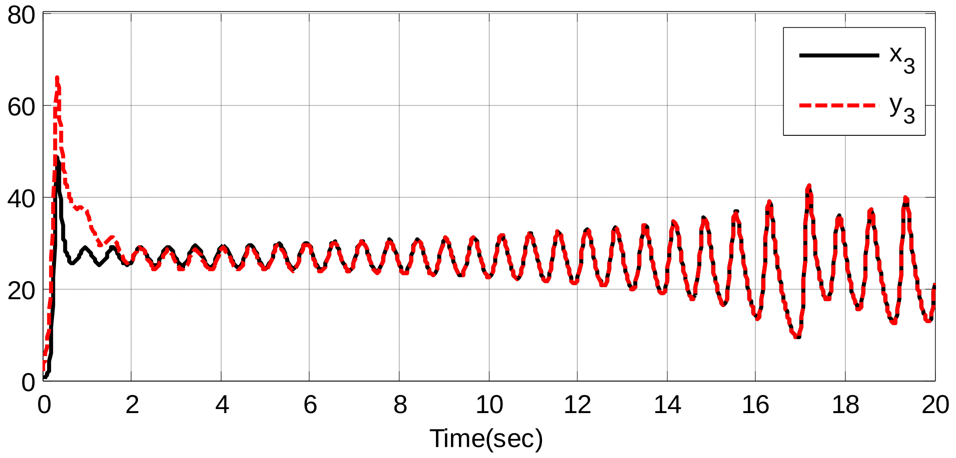

Figure 4.12: Synchronization between x_3 and y_3 for unknown parameters.

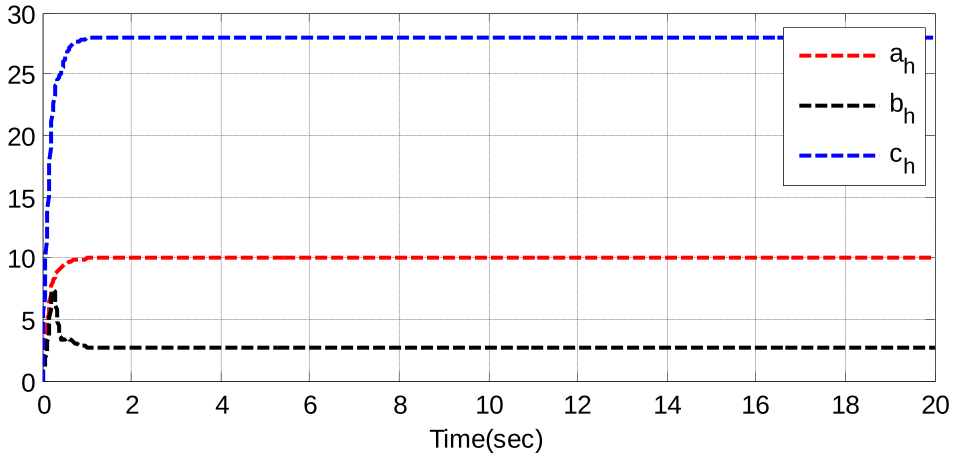

Figure 4.13: Estimation of Unknown parameters a_h, b_h and c_h for synchronization.

For anti-synchronization we set q= -1 in eq. (4.20):

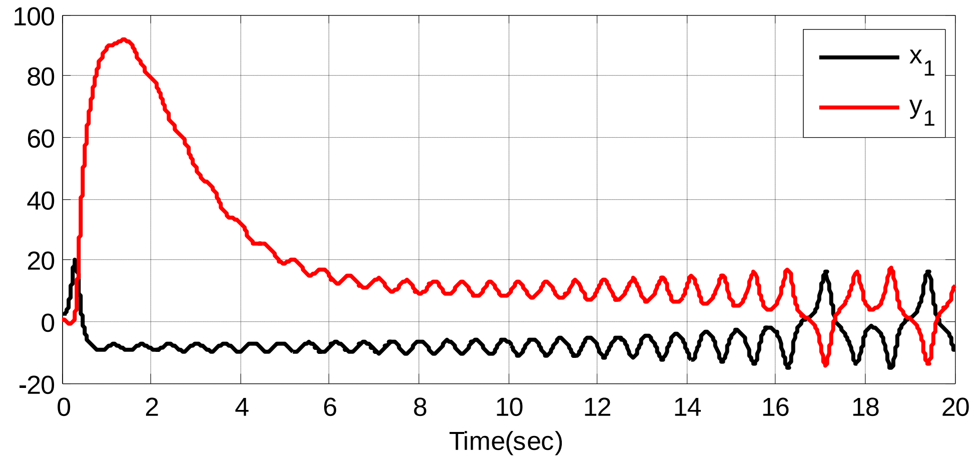

Figure 4.14: Anti-synchronization between x_1 and y_1 for unknown parameters.

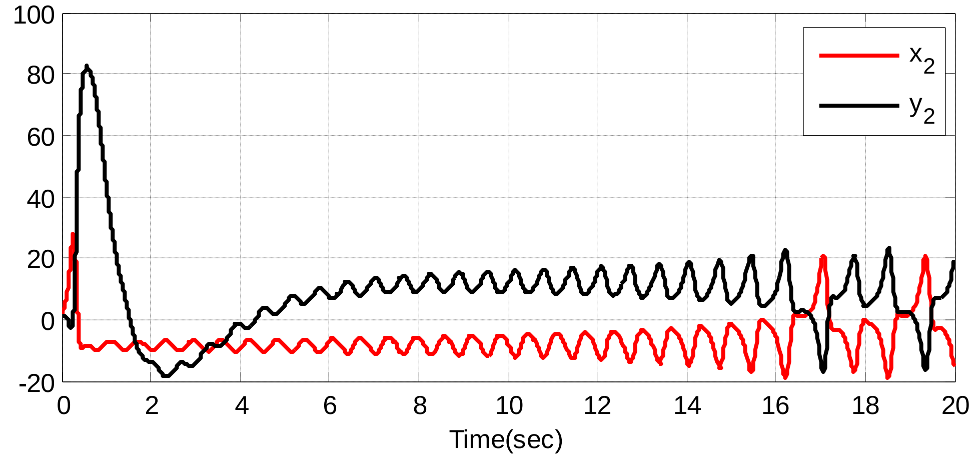

Figure 4.15: Anti-synchronization between x_2 and y_2 for unknown parameters.

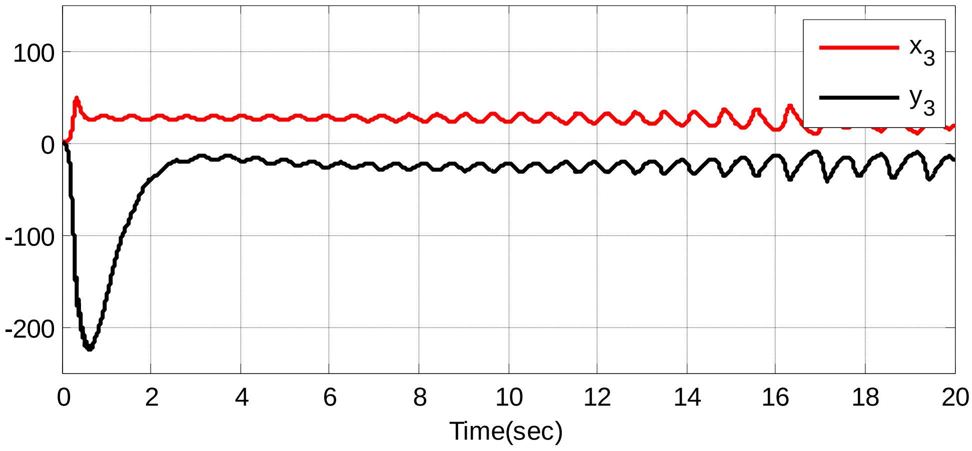

Figure 4.16: Anti-synchronization between x_3 and y_3 for unknown parameters.

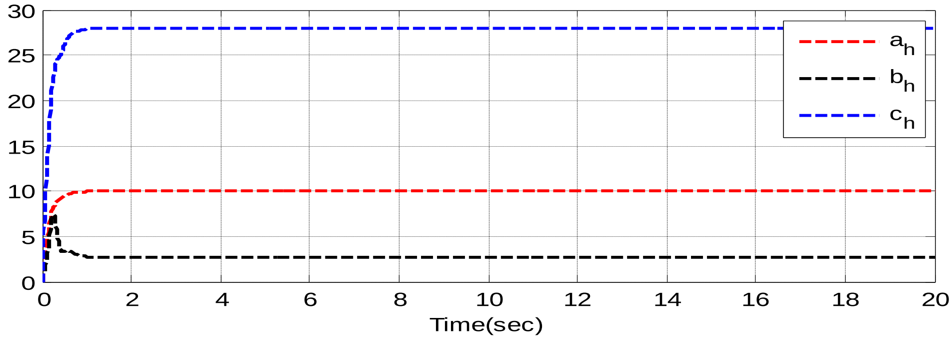

Figure 4.17: Estimation of Unknown parameters a_h, b_h, c_h for Anti-synchronization.

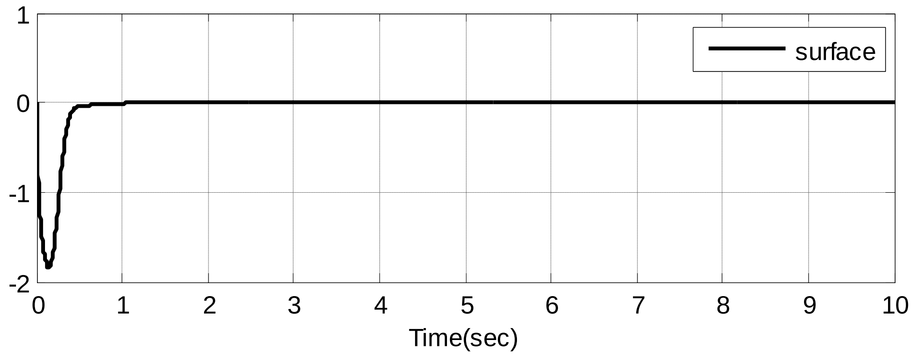

Figure 4.18: Time history of surface.

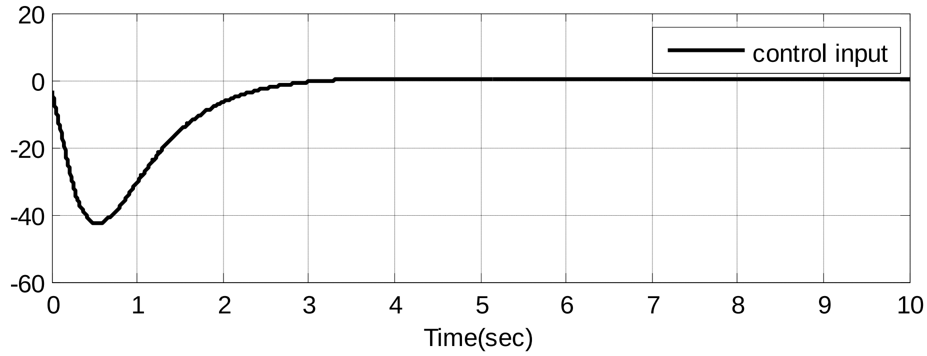

Figure 4.19: Time history of control input.

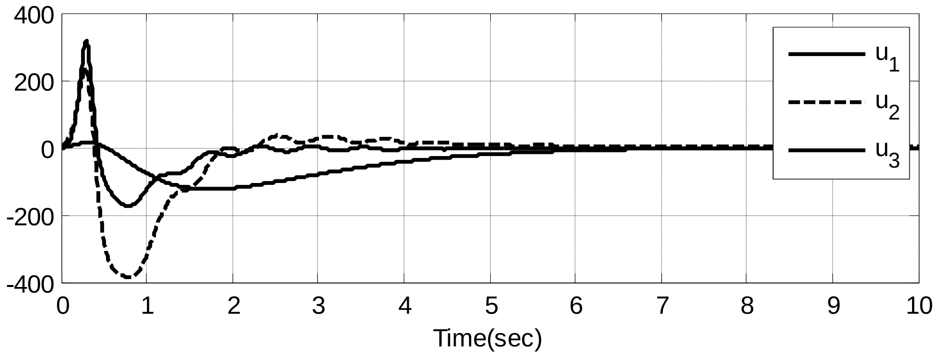

Figure 4.20: Time history of adaptive controllers.

4.2 Numerical Example 2: (Lorenz-Stenflo system)

The following example is taken from [25], based on hyper-chaotic Lorenz-Stenflo system. where synchronization for this system is obtained using Linear-feedback control technique system, while we used sliding mode control for known parameter and adaptive integral sliding mode control technique for unknown parameter to achieve synchronization of chaotic system. we compare our result with given results in [25], and our error result approaching to zero faster as compare to result presented in [25].

Consider the Lorenz-Stenflo system [25] as a master system

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + bx_4 \\
\dot{x}_2 &= x_1(c - x_3) - x_2 \\
\dot{x}_3 &= x_1 x_2 - dx_3 \\
\dot{x}_4 &= -x_1 - ax_4
\end{aligned} \tag{4.13}$$

and the slave system

$$\begin{aligned}
\dot{y}_1 &= a(y_2 - y_1) + by_4 + u_1 \\
\dot{y}_2 &= y_1(c - y_3) - y_2 + u_2 \\
\dot{y}_3 &= y_1 y_2 - dy_3 + u_3 \\
\dot{y}_4 &= -y_1 - ay_4 + u_4
\end{aligned} \tag{4.14}$$

When system parameters are chosen as: $a = 1, b = 1.5, c = 26, d = 0.7$ and $\sigma = 0.3$, then this system shows chaotic behavior with initial conditions: $x(0) = [1,2,3,4]^T$.

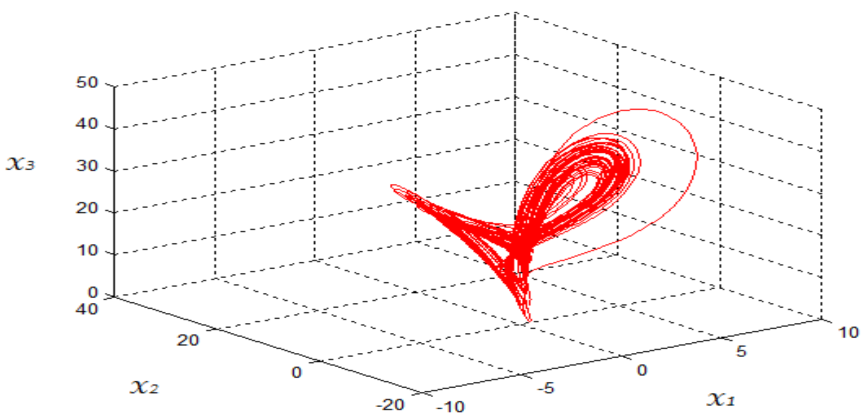

Figure 4.21: The phase portrait of x_1, x_2, x_3.

Case 1: synchronization and anti-synchronization with known parameters:

The error signals are defined as:

$$e_1 = y_1 - qx_1, e_2 = y_2 - qx_2, e_3 = y_3 - qx_3,\ e_4 = y_4 - qx_4 \tag{4.15}$$

Where $q = 1$, for synchronization and $q = -1$, for anti-synchronization.

Then the error dynamics becomes:

$$\begin{aligned}
\dot{e}_1 &= \dot{y}_1 - q\dot{x}_1 = a(y_2 - y_1) + by_4 - qa(x_2 - x_1) + bx + u_1 \\
\dot{e}_2 &= \dot{y}_2 - q\dot{x}_2 = y_1(c - y_3) - y_2 - qx_1(c - x_3) - x_2 + u_2 \\
\dot{e}_3 &= \dot{y}_3 - q\dot{x}_3 = y_1 y_2 - dy_3 - qx_1 x_2 - dx_3 + u_3 \\
\dot{e}_4 &= \dot{y}_4 - q\dot{x}_4 = -y_1 - ay_4 - q(-x_1 - ax_4) + u_4
\end{aligned} \tag{4.16}$$

By choosing

$$\begin{aligned}
u_1 &= -(a(y_2 - y_1) + by_4) + q(a(x_2 - x_1) + bx) + \mathrm{e}_2 \\
u_2 &= -(y_1(c - y_3) - y_2) + q(x_1(c - x_3) - x_2) + e_3 \\
u_3 &= -(y_1 y_2 - dy_3) + q(x_1 x_2 - dx_3) + e_4 \\
u_4 &= -(-y_1 - ay_4) + q((-x_1 - ax_4)) + v
\end{aligned}$$

Where v is the new input, which can be written as:

$$\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
\dot{e}_3 &= e_4 \\
\dot{e}_4 &= v
\end{aligned} \tag{4.17}$$

Define the Hurwitz sliding surface for system (4.16) as: $\sigma_0 = (1 + \frac{d}{dt})^3 e_1 = e_1 + 3e_2 + 3e_3 + e_4$

Then,

$$\dot{\sigma}_0 = \dot{e}_1 + 3\dot{e}_2 + 3\dot{e}_3 + \dot{e}_3 + \dot{e}_4 = e_2 + 3e_3 + 3e_4 + e_4 + v$$

By choosing $v = -e_2 - 3e_3 - 3e_4 - k\sigma_0, k > 0$, we have $\dot{\sigma}_0 = -k\,\sigma_0$. Therefore the nominal system (4.16) is asymptotically stable.

In simulations, the initial conditions are chosen as: $x(0) = [1,2,3,4]^T$, $y(0) = [5,6,7,8]^T$. The true value of the known parameters is: $a = 1, b = 1.5, c = 26, d = 0.7$ and $\sigma = 0.3$.

Consider a Lyapunov function: $V = 0.5(\sigma^2{}_0\ )$

Then

$$\dot{V} = \sigma_0\dot{\sigma}_0 = \sigma_0(-k\sigma_0) = -k\sigma_0{}^2$$

From this we conclude that $\sigma_0 \to 0$, since σ_0 are Hurwitz therefore $e_i \to 0, i = 1,\ldots,4$, therefore the systems (4.17) are asymptotically stable.

Simulation results:

For synchronization we set q=1 in eq. (4.15):

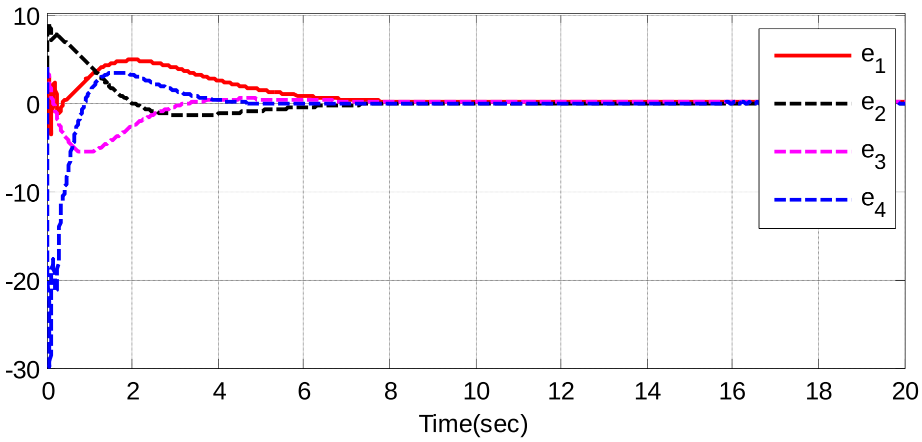

Figure 4.22: Time history of error variables for known parameters.

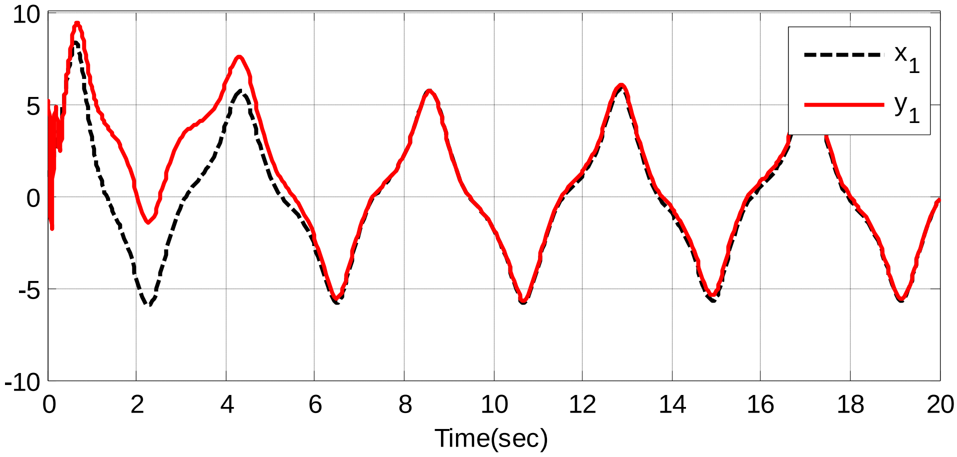

Figure 4.23: Synchronization between x_1 and y_1 for known parameters.

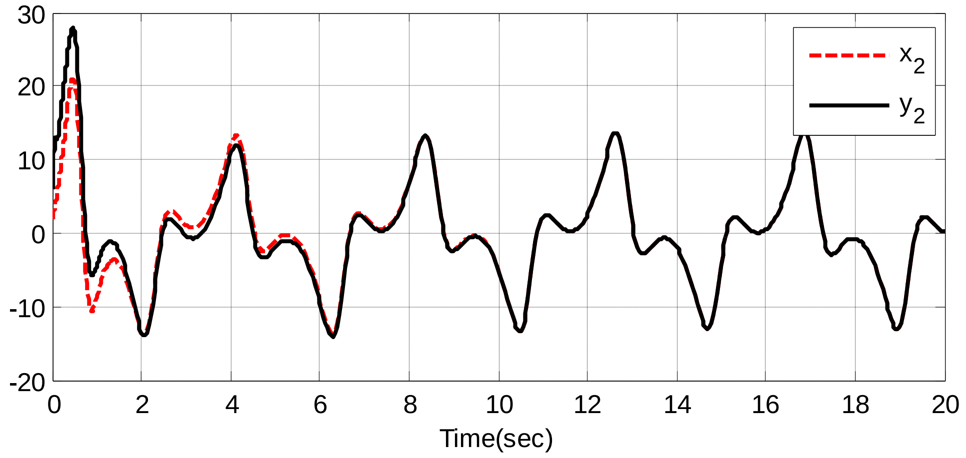

Figure 4.24: Synchronization between x_2 and y_2 of known parameters.

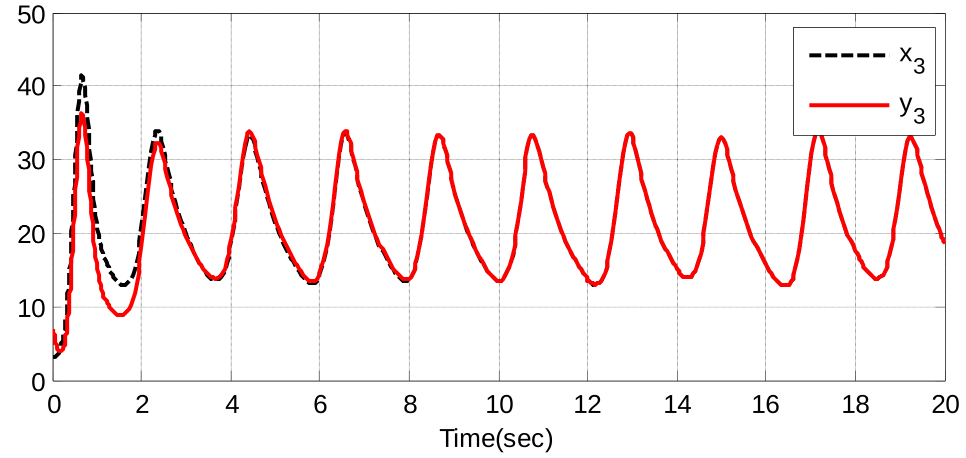

Figure 4.25: Synchronization between x_3 and y_3 known parameters.

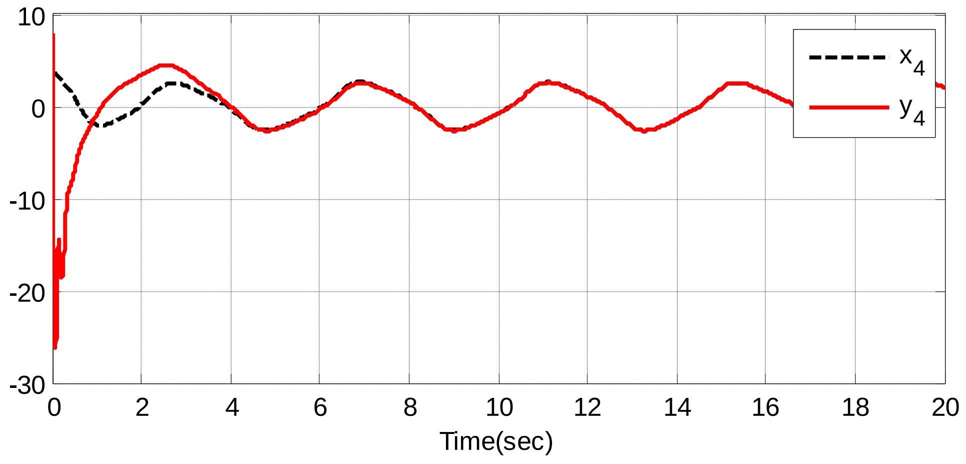

Figure 4.26: Synchronization between x_4 and y_4 of known parameters.

For anti-synchronization we set q= -1 in eq. (4.15):

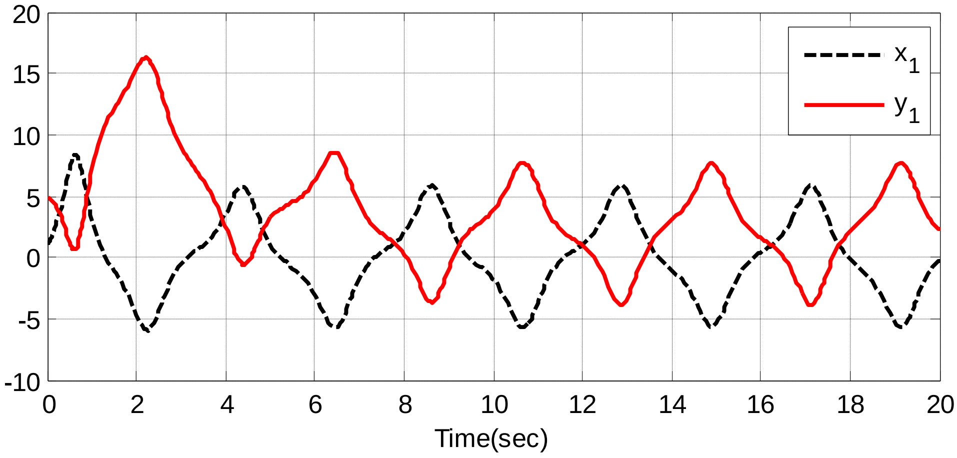

Figure 4.27: Anti-synchronization between x_1 and y_1 of known parameters.

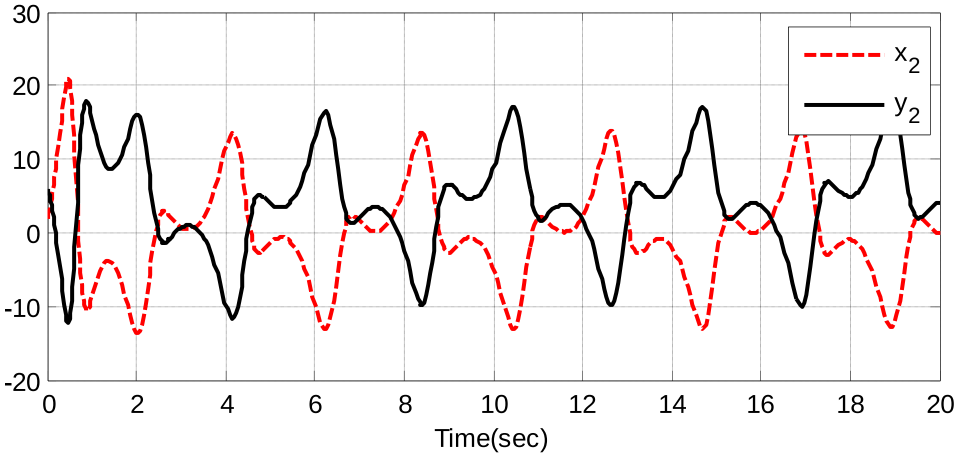

Figure 4.28: Anti-synchronization between x_2 and y_2 of known parameters.

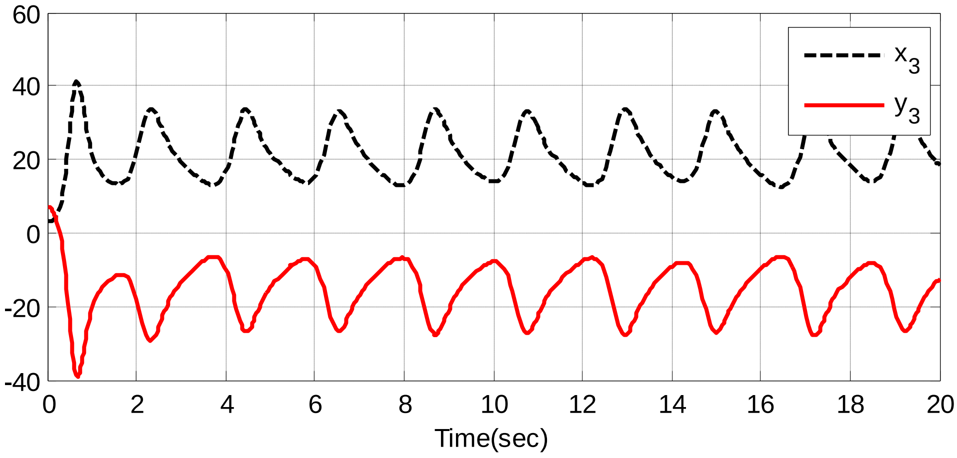

Figure 4.29: Anti-synchronization between x_3 and y_3 of known parameters.

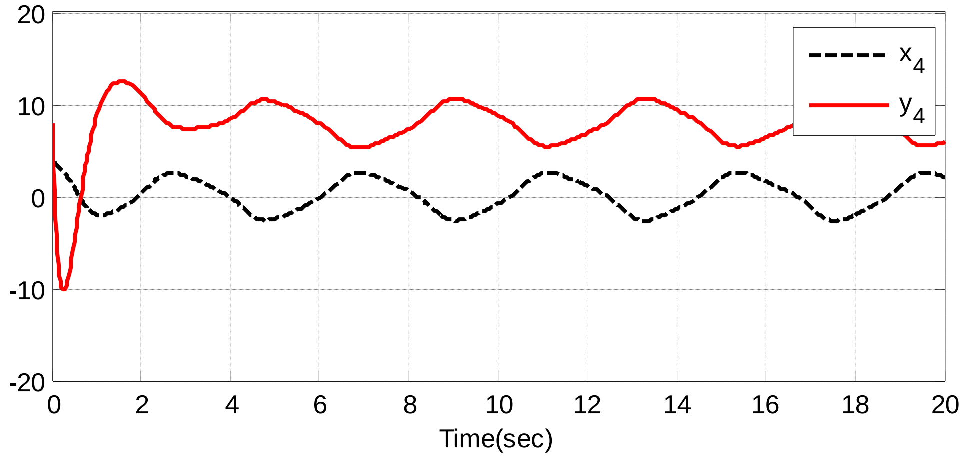

Figure 4.30: Anti-synchronization between x_4 and y_4 of known parameters.

Case 2: synchronization and anti-synchronization with unknown parameters

Let $\hat{a},\hat{b},\hat{c},\hat{d}$ be estimated of a,b,c,d and let $\tilde{a}=a-\hat{a},\ \tilde{b}=b-\hat{b},\ \tilde{c}=c-\hat{c},\ \tilde{d}=d-\hat{d}$ be error.

Thus system (4.13) and (4.14) can be written as:

$$\begin{aligned}
\dot{x}_1 &= \hat{a}(x_2-x_1)+\hat{b}x_4+\tilde{a}(x_2-x_1)+\tilde{b}x_4 \\
\dot{x}_2 &= x_1\hat{c}+x_1\tilde{c}-x_1x_3-x_1 \\
\dot{x}_3 &= x_1x_2-\hat{d}x_3-\tilde{d}x_3 \\
\dot{x}_4 &= -x_1-\hat{a}x_4-\tilde{a}x_4
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
\dot{y}_1 &= \hat{a}(y_2-y_1)+\hat{b}y_4+\tilde{a}(y_2-y_1)+\tilde{b}y_4 \\
\dot{y}_2 &= y_1\hat{c}+y_1\tilde{c}-y_1y_3-y_1 \\
\dot{y}_3 &= y_1y_2-\hat{d}y_3-\tilde{d}y_3 \\
\dot{y}_4 &= -y_1-\hat{a}y_4-\tilde{a}y_4
\end{aligned} \tag{4.19}$$

The error signals are defined as:

$$e_1=y_1-qx_1, e_2=y_2-qx_2, e_3=y_3-qx_3, e_4=y_4-qx_4 \tag{4.20}$$

Then the dynamics of the error system becomes:

$$\dot{e}_1=\dot{y}_1-q\dot{x}_1, \dot{e}_2=\dot{y}_2-q\dot{x}_2, \dot{e}_3=\dot{y}_3-q\dot{x}_3, \dot{e}_4=\dot{y}_4-q\dot{x}_4$$

$$
\begin{aligned}
\dot{e}_1 &= \dot{y}_1 - q\dot{x}_1 = (\hat{a}(y_2 - y_1) + \hat{b}y_4 + \tilde{a}(y_2 - y_1) + \tilde{b}y_4) \\
&\quad - q(\hat{a}(x_2 - x_1) + \hat{b}x_4 + \tilde{a}(x_2 - x_1) + \tilde{b}x_4) + u_1 \\
\dot{e}_2 &= \dot{y}_2 - q\dot{x}_2 = (y_1\hat{c} + y_1\tilde{c} - y_1y_3 - y_1) - q(x_1\hat{c} + x_1\tilde{c} - x_1x_3 - x_1) + u_2 \\
\dot{e}_3 &= \dot{y}_3 - q\dot{x}_3 = (y_1y_2 - \hat{d}y_3 - \tilde{d}y_3) - q(x_1x_2 - \hat{d}x_3 - \tilde{d}x_3) + u_3 \\
\dot{e}_4 &= \dot{y}_4 - q\dot{x}_4 = (-y_1 - \hat{a}y_4 - \tilde{a}y_4) - q(-x_1 - \hat{a}x_4 - \tilde{a}x_4) + u_4
\end{aligned}
\tag{4.21}
$$

By choosing

$$
\begin{aligned}
u_1 &= -(\hat{a}(y_2 - y_1) + \hat{b}y_4) + q(\hat{a}(x_2 - x_1) + \hat{b}x_4) + e_2 \\
u_2 &= -(y_1\hat{c} - y_1y_3 - y_1) + q(x_1\hat{c} - x_1x_3 - x_1) + e_3 \\
u_3 &= -(y_1y_2 - \hat{d}y_3) + q(x_1x_2 - \hat{d}x_3) + e_4 \\
u_4 &= -(-y_1 - \hat{a}y_4) + q(-x_1 - \hat{a}x_4) + v
\end{aligned}
$$

Where v is the new input, the system (4.21) can be written as:

$$
\begin{aligned}
\dot{e}_1 &= (\tilde{a}(y_2 - y_1) + \tilde{b}y_4) - q(\tilde{a}(x_2 - x_1) + \tilde{b}x_4) + e_2 \\
\dot{e}_2 &= y_1\tilde{c} - qx_1\tilde{c} + e_3 \\
\dot{e}_3 &= -\tilde{d}y_3 + q\tilde{d}x_3 + e_4 \\
\dot{e}_4 &= -\tilde{a}y_4 + q\tilde{a}x_4 + v
\end{aligned}
\tag{4.22}
$$

To employ the integral sliding mode control, choose the nominal system for (4.22) as:

$$
\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
\dot{e}_3 &= e_4 \\
\dot{e}_4 &= v_0
\end{aligned}
\tag{4.23}
$$

Define the Hurwitz sliding surface for nominal system (4.23) as:

$$
\sigma_0 = (1 + \frac{d}{dt})^3 e_{1r} = e_1 + 3e_2 + 3e_3 + e_4
$$

Then,

$$
\dot{\sigma}_0 = \dot{e}_1 + 3\dot{e}_2 + 3\dot{e}_3 + \dot{e}_4 = e_2 + 3e_3 + 3e_4 + e_5 + v
$$

By choosing $v = -e_2 - 3e_3 - 3e_4 - k\sigma_0,\ k > 0$, we have $\dot{\sigma}_0 = -k\sigma_0$. Therefore the nominal system (4.23) is asymptotically stable.

Now choose the sliding surface for the system (4.22) as: $\sigma = \sigma_0 + z = e_1 + 3e_2 + 3e_3 + e_4 + z$

where, z is some integral term computed later. To avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$.

Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later. Then,

$$\begin{aligned}
\dot{\sigma} &= \dot{e}_1 + 3\dot{e}_2 + 3\dot{e}_3 + \dot{e}_4 + \dot{z} \\
&= \tilde{a}(y_2 - y_1) + \tilde{b}y_4 - q\tilde{a}(x_2 - x_1) - \tilde{b}x_4 + e_2 + 3y_1\tilde{c} - q3x_1\tilde{c} \\
&+ 3e_3 - 3\tilde{d}y_3 + q3\tilde{d}x_3 + 3e_4 - \tilde{a}y_4 + \tilde{a}x_4 + v_0 + v_s + \dot{z}
\end{aligned}$$

By choosing a Lyapunov function: $V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2)$, design the adaptive laws for $\tilde{a}\ ,\hat{a}\ ,,\tilde{b}\ ,\hat{b}\ ,\tilde{c},\hat{c},\tilde{d},\hat{d}$ and compute v_s such that $\dot{V} < 0$.

Consider a Lyapunov function: $V = \frac{1}{2}\sigma^2 + \frac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2)$. Then $\dot{V} < 0$ if the adaptive laws for $\tilde{a}\ ,\hat{a}\ ,,\tilde{b}\ ,\hat{b}\ ,\tilde{c},\hat{c},\tilde{d},\hat{d}$ and the value of v_s are chosen as:

$$\begin{aligned}
&\dot{z} = -e_2 - 3e_3 - 3e_4 - k\sigma_0 k > 0 - v_0\,, \qquad v_s = -k\sigma \\
&\dot{\tilde{a}} = -\sigma(y_2 - y_1) + q\sigma(x_2 - x_1) + \sigma\, y_4 - q\sigma\, x_4 - k\,\tilde{a},\ \dot{\hat{a}} = -\dot{\tilde{a}} \\
&\dot{\tilde{b}} = -\sigma\, y_4 + q\sigma\, x_4 - k\,\tilde{b},\ \ \dot{\hat{b}} = -\dot{\tilde{b}} \\
&\dot{\tilde{c}} = -3\sigma\, y_1 + 3q\,\sigma\, x_1 - k\,\tilde{c} \quad\ \dot{\hat{c}} = -\dot{\tilde{c}} \\
&\dot{\tilde{d}} = 3\sigma\, y_3 - 3q\sigma\, x_3 - k\,\tilde{d} \qquad \dot{\hat{d}} = -\dot{\tilde{d}}
\end{aligned} \tag{4.24}$$

Proof:

Since

$$\begin{aligned}
\dot{V} &= \sigma\dot{\sigma} + \tilde{a}\ \dot{\tilde{a}}\ + \tilde{b}\ \dot{\tilde{b}} + \tilde{c}\ \dot{\tilde{c}}\ + \tilde{d}\ \dot{\tilde{d}} \\
&= \sigma(\tilde{a}(y_2 - y_1) + \tilde{b}y_4 - q\tilde{a}(x_2 - x_1) - \tilde{b}x_4 + e_2 + 3y_1\tilde{c} - q3x_1\tilde{c} \\
&+ 3e_3 - 3\tilde{d}y_3 + 3q\tilde{d}x_3 + 3e_4 - \tilde{a}y_4 + q\tilde{a}x_4 + v_0 + v_s + \dot{z}) + \tilde{a}\ \dot{\tilde{a}}\ + \tilde{b}\ \dot{\tilde{b}} + \tilde{c}\ \dot{\tilde{c}}\ + \tilde{d}\ \dot{\tilde{d}} \\
&= \sigma(\dot{e}_1 + 3\dot{e}_2 + 3\dot{e}_3 + \dot{e}_4 + \dot{z}) + \tilde{a}\ (\dot{\tilde{a}}\ + \sigma(y_2 - y_1) - \sigma(x_2 - x_1) - \sigma y_4 + \sigma x_4) \\
&+ \tilde{b}\ (\dot{\tilde{b}} + \sigma y_4 - \sigma x_4) + \tilde{c}\ (\dot{\tilde{c}}\ + \sigma 3y_1 - \sigma 3x_1)\ + \tilde{d}\ (\dot{\tilde{d}}\ - \sigma 3y_3 + \sigma 3x_3)
\end{aligned}$$

By using

$\dot{z} = -e_2 - 3e_3 - 3e_4 - k\sigma_0 k > 0 - v_0$, $v_s = -k\sigma$

$\dot{\tilde{a}} = -\sigma(y_2 - y_1) + q\sigma(x_2 - x_1) + \sigma y_4 - q\sigma x_4 - k\tilde{a}$, $\dot{\hat{a}} = -\dot{\tilde{a}}$

$\dot{\tilde{b}} = -\sigma y_4 + q\sigma x_4 - k\tilde{b}$, $\dot{\hat{b}} = -\dot{\tilde{b}}$

$\dot{\tilde{c}} = -3\sigma y_1 + q3\sigma x_1 - k\tilde{c}$ $\dot{\hat{c}} = -\dot{\tilde{c}}$

$\dot{\tilde{d}} = 3\sigma y_3 - q3\sigma x_3 - k\tilde{d}$ $\dot{\hat{d}} = -\dot{\tilde{d}}$

We have

$\dot{V} = -k\sigma^2 - k_1\tilde{a}_1^2 - k_2\tilde{b}^2 - k_3\tilde{c}^2 - k_4\tilde{d}^2$.

From this we conclude that $\sigma, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \to 0$. Since $\sigma \to 0$, therefore $e = (e_1, e_2, e_3, e_4) \to 0$.

In simulations, the initial conditions are chosen as: $x(0) = [1,2,3,4]^T$, $y(0) = [5,6,7,8]^T$. The true value of the unknown parameters are chosen as: $a = 1, b = 1.5, c = 26, d = 0.7$.

Simulation result:

For synchronization we set q=1 in eq. (4.20)

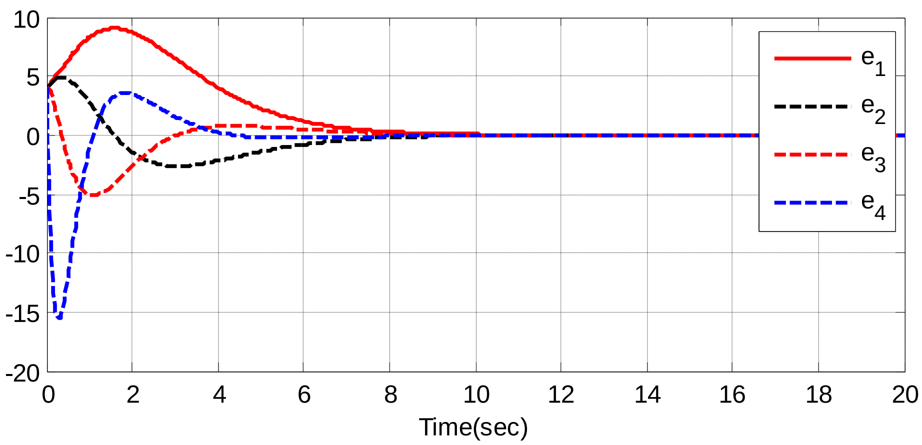

Figure 4.31: Time history of error variables of unknown parameters

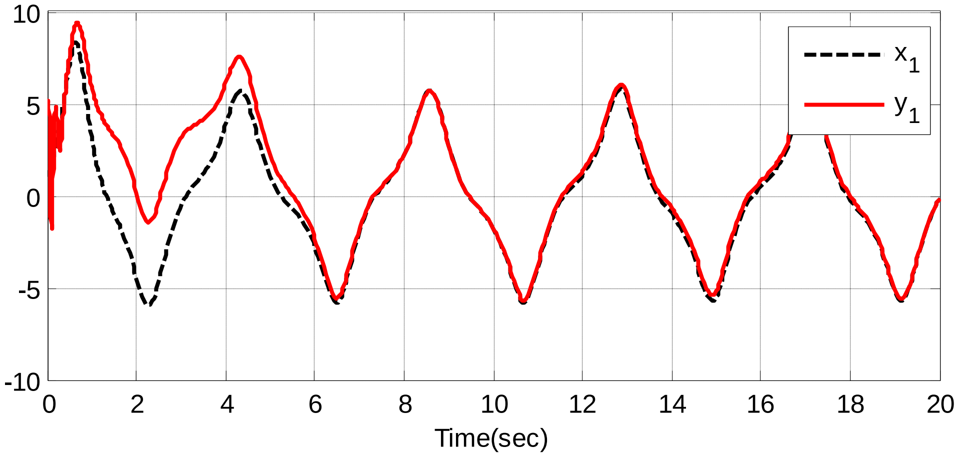

Figure 4.32: Synchronization between x_1 and y_1 for unknown parameters.

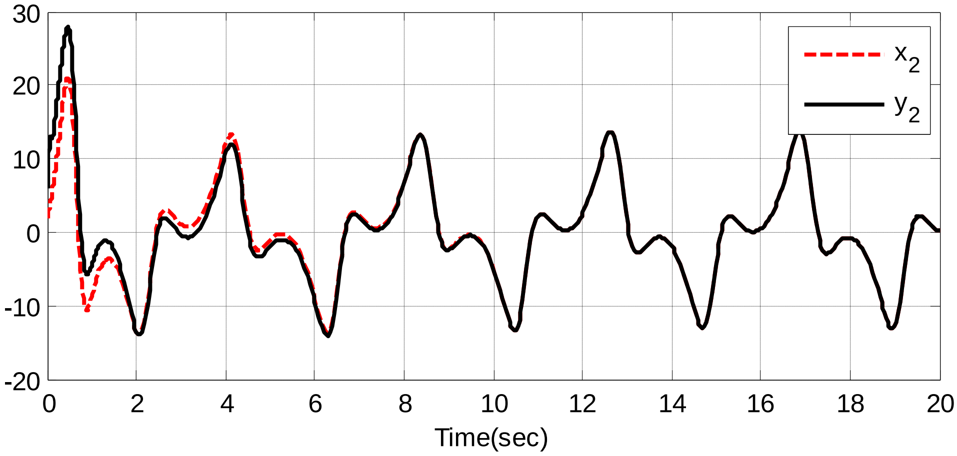

Figure 4.33: Synchronization between x_2 and y_2 for unknown parameters.

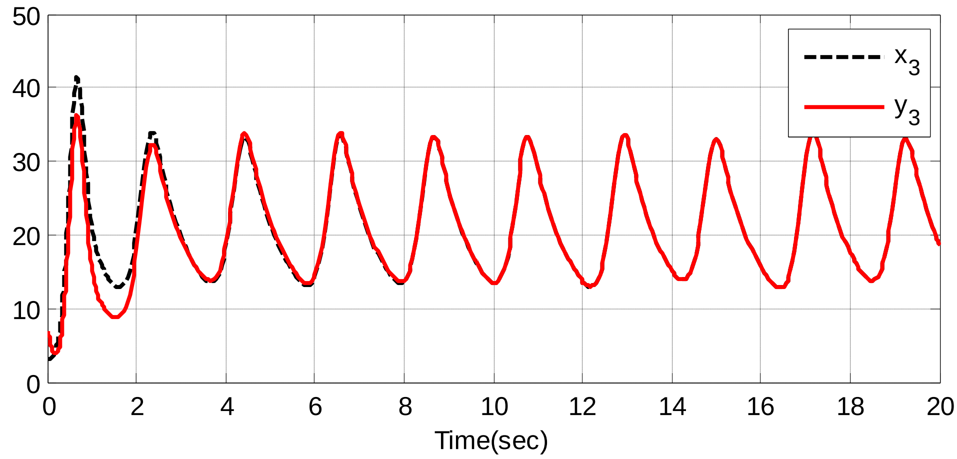

Figure 4.34: Synchronization between x_3 and y_3 for unknown parameters.

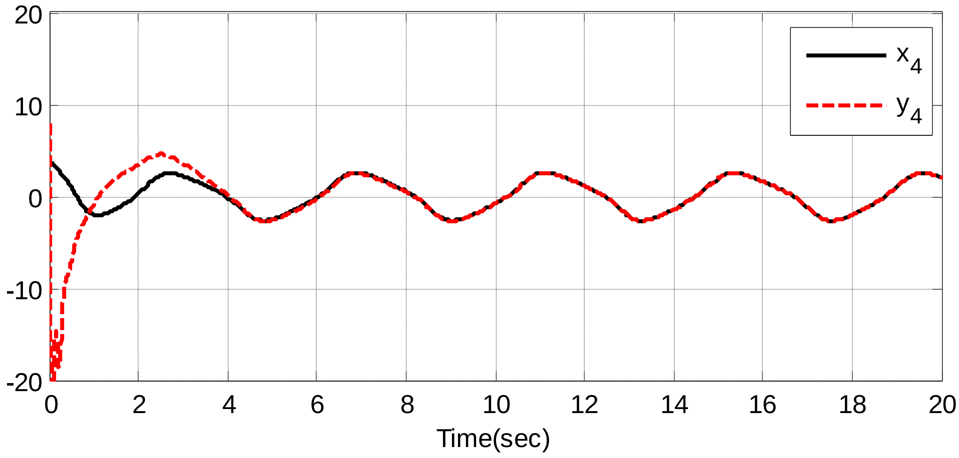

Figure 4.35: Synchronization between x_4 and y_4 for unknown parameters.

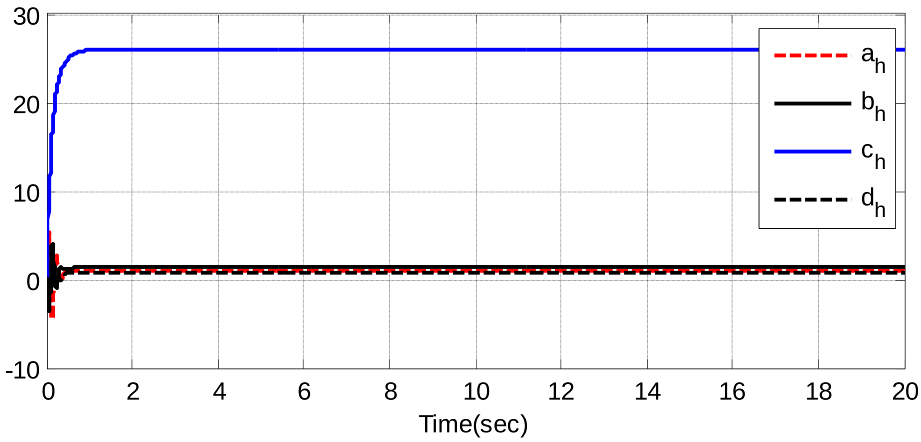

Figure 4.36: Estimation of unknown parameters a_h, b_h, c_h and d_h for synchronization.

For anti-synchronization we set q= -1 in eq. (4.20):

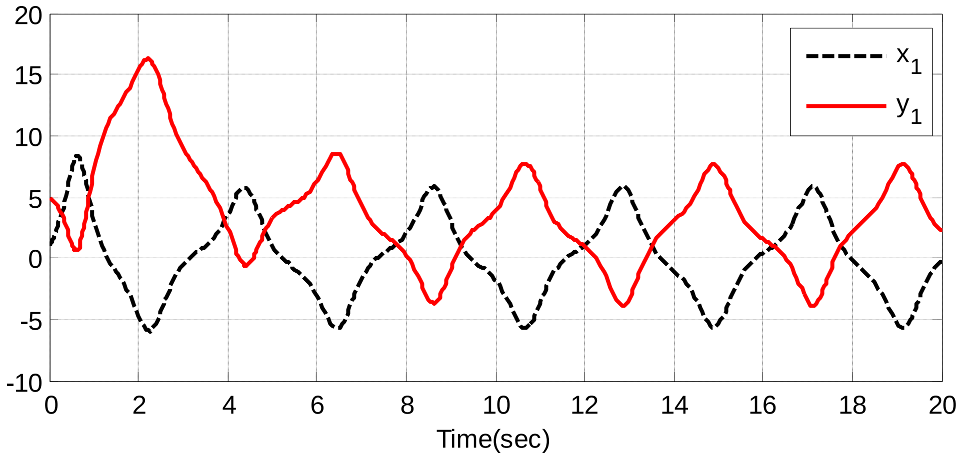

Figure 4.37: Anti-synchronization between x_1 and y_1 for unknown parameters.

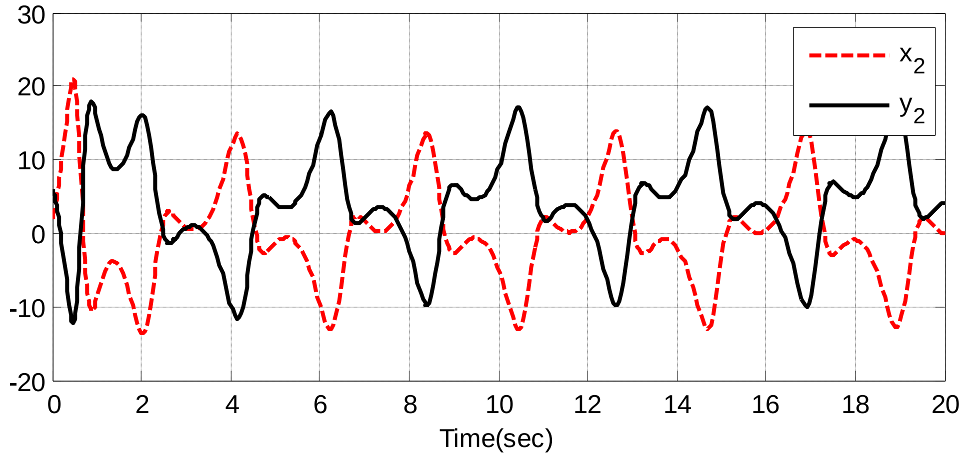

Figure 4.38: Anti-synchronization between x_2 and y_2 for unknown parameters.

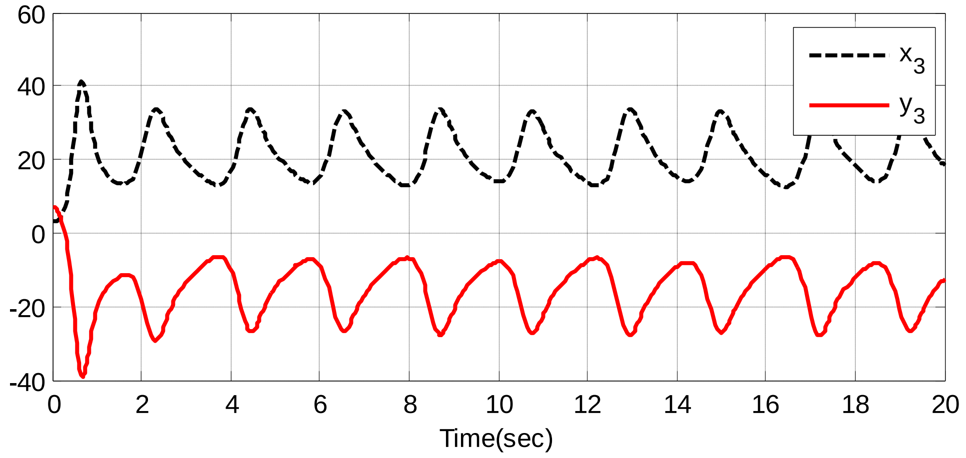

Figure 4.39: Anti-synchronization between x_3 and y_3 for unknown parameters.

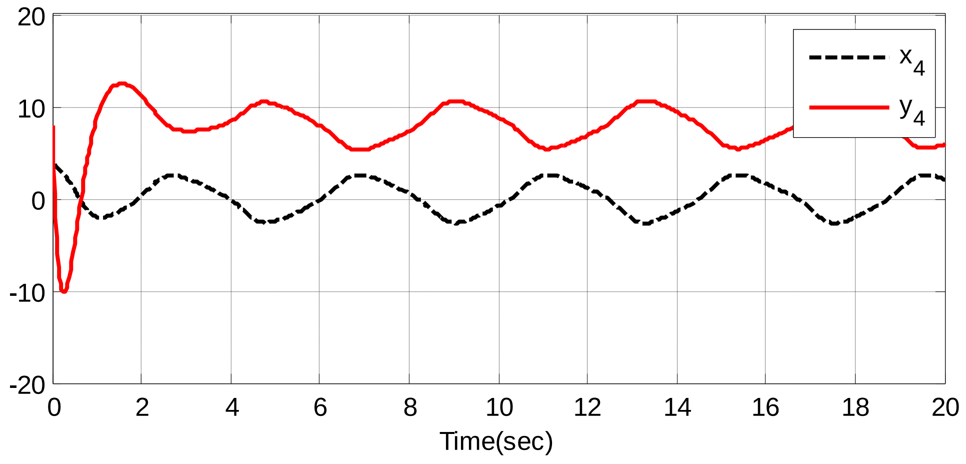

Figure 4.40: Anti-synchronization between x_4 and y_4 for unknown parameters.

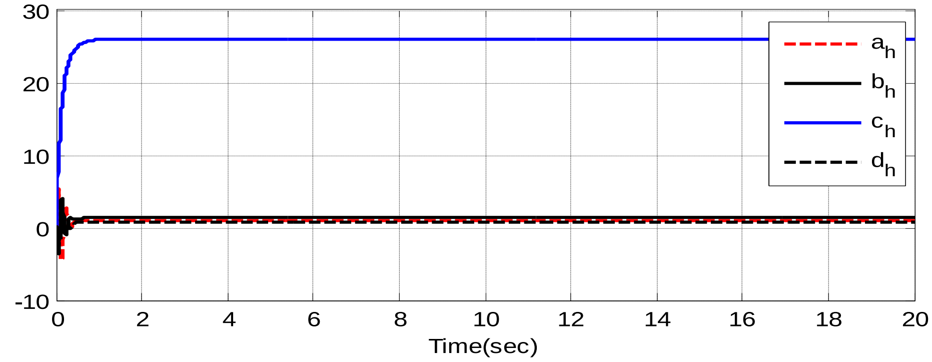

Figure 4.41:Estimation of Unknown parameters a_h, b_h, c_h, d_h for Antisynchronization.

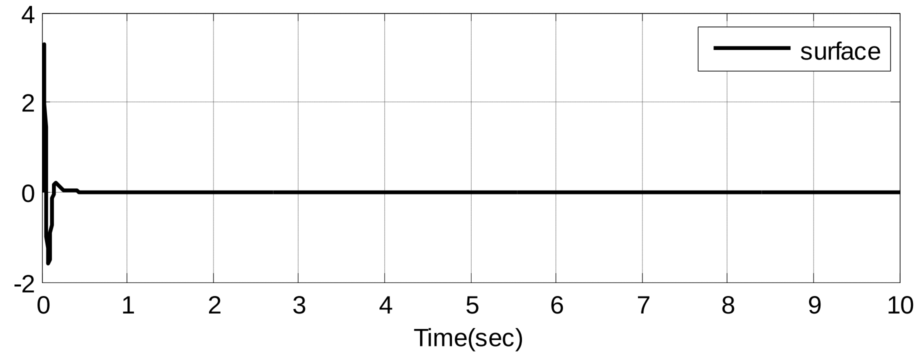

Figure 4.42: Time history of surface.

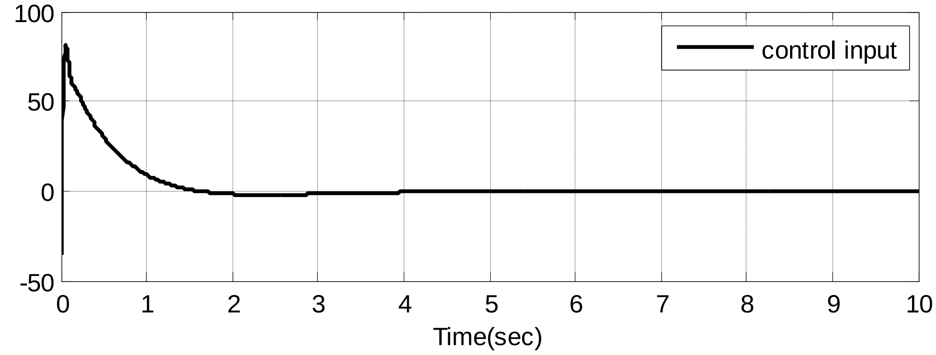

Figure 4.43: Time history of input controller.

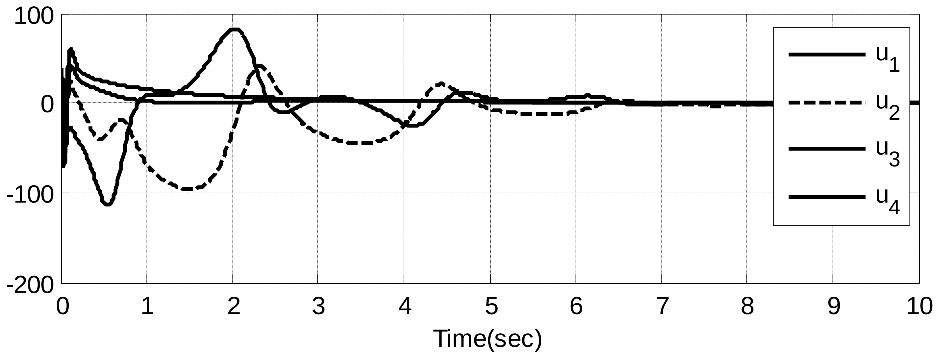

Figure 4.44: Time history of adaptive controllers.

4.3 NUMERICAL EXAMPLE 3: (Memristor system)

The following example is taken from [23], where synchronization for this system is obtained by using sufficient asymptotic stability condition based on Lyapunov theory, while we used sliding mode control for known parameter and integral sliding mode control technique for unknown parameter to achieve complete synchronization (CS). We compare our result with given results in [23], and our error result approaching to zero faster as compare to result presented in [23].

The drive system is taken as the following system:

$$\begin{aligned}
\dot{x}_1 &= \alpha_1 x_3 - \alpha_2 x_1 - \alpha_3 x_1 x_5^2 \\
\dot{x}_2 &= -\alpha_4 x_3 + \alpha_4 x_4 \\
\dot{x}_3 &= \alpha_5 x_2 - \alpha_5 x_1 - \alpha_6 x_3 \\
\dot{x}_4 &= -\alpha_7 x_2 \\
\dot{x}_5 &= x_1
\end{aligned} \tag{4.25}$$

and the following system is considered as the response system:

$$\begin{aligned}
\dot{y}_1 &= \alpha_1 y_3 - \alpha_2 y_1 - \alpha_3 y_1 y_5^2 + u_1 \\
\dot{y}_2 &= -\alpha_4 y_3 + \alpha_4 y_4 + u_2 \\
\dot{y}_3 &= \alpha_5 y_2 - \alpha_5 y_1 - \alpha_6 y_3 + u_3 \\
\dot{y}_4 &= -\alpha_7 y_2 + u_4 \\
\dot{y}_5 &= y_1
\end{aligned} \tag{4.26}$$

The true values of the parameters are taken as: $\alpha_1 = 9, \alpha_2 = -10.8, \alpha_3 = 10.8, \alpha_4 = 1$, $\alpha_5 = 30, \alpha_6 = 30, \alpha_7 = 15$. In simulations, the initial conditions are chosen as: $(x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)) = (1, 2, -2, -3, 1)$

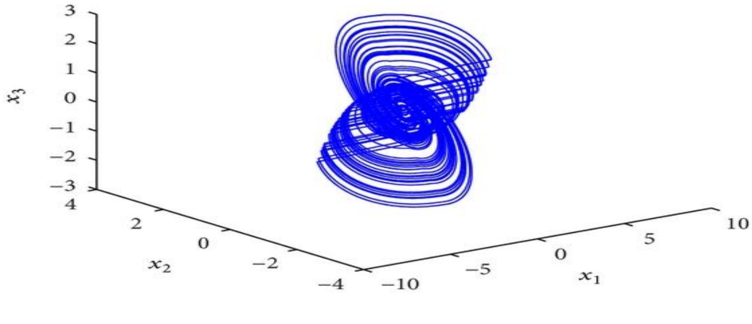

Figure 4.45: The phase portrait of x_1, x_2, x_3

Case 1: Synchronization and anti-synchronization with known parameters:

The error signals are defined as:

$$e_1 = y_1 - qx_1, e_2 = y_2 - qx_2, e_3 = y_3 - qx_3,\; e_4 = y_4 - qx_4,\; e_5 = y_5 - qx_5 \qquad (4.27)$$

Where $q = 1$, for synchronization and $q = -1$, for anti-synchronization.

The dynamics of the error system is:

$$\begin{aligned}
\dot{e}_1 &= \dot{y}_1 - q\dot{x}_1 = \alpha_1 y_3 - \alpha_2 y_1 - \alpha_3 y_1 y_5^2 + u_1 - q\{\alpha_1 x_3 - \alpha_2 x_1 - \alpha_3 x_1 x_5^2\} \\
\dot{e}_2 &= \dot{y}_2 - q\dot{x}_2 = -\alpha_4 y_3 + \alpha_4 y_4 + u_2 - q\{-\alpha_4 x_3 + \alpha_4 x_4\} \\
\dot{e}_3 &= \dot{y}_3 - q\dot{x}_3 = \alpha_5 y_2 - \alpha_5 y_1 - \alpha_6 y_3 + u_3 - q\{\alpha_5 x_2 - \alpha_5 x_1 - \alpha_6 x_3\} \\
\dot{e}_4 &= \dot{y}_4 - q\dot{x}_4 = -\alpha_7 y_2 + u_4 - q\{-\alpha_7 x_2\} \\
\dot{e}_5 &= \dot{y}_5 - q\dot{x}_5 = y_1 - qx_1 = e_1
\end{aligned} \qquad (4.28)$$

By choosing

$$\begin{aligned}
u_1 &= -\alpha_1 y_3 + \alpha_2 y_1 + \alpha_3 y_1 y_5^2 + q\{\alpha_1 x_3 - \alpha_2 x_1 - \alpha_3 x_1 x_5^2\} + e_2 \\
u_2 &= \alpha_4 y_3 - \alpha_4 y_4 + q\{-\alpha_4 x_3 + \alpha_4 x_4\} + e_3 \\
u_3 &= -\alpha_5 y_2 + \alpha_5 y_1 + \alpha_6 y_3 + q\{\alpha_5 x_2 - \alpha_5 x_1 - \alpha_6 x_3\} + e_4 \\
u_4 &= \alpha_7 y_2 + q\{-\alpha_7 x_2\} + v
\end{aligned},$$

where v is the new input, the system (4.28) can be written as:

$$\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
\dot{e}_3 &= e_4 \\
\dot{e}_4 &= e_5 \\
\dot{e}_5 &= v
\end{aligned} \qquad (4.29)$$

Define the Hurwitz sliding surface for system (17) as: $\sigma_0 = (1 + D)^4 e_1$

Then

$$\dot{\sigma}_0 = \dot{e}_1 + 4\dot{e}_2 + 6\dot{e}_3 + 4\dot{e}_4 + \dot{e}_5 = e_2 + 4e_3 + 6e_4 + 4e_5 + v$$

By choosing $v = -e_2 - 4e_3 - 6e_4 - 4e_5 - k\sigma, k > 0$, we have $\dot{\sigma}_0 = -k\sigma_0$. Therefore the system (4.29) is asymptotically stable.

In simulations, the initial conditions are chosen as: $(x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)) = (1, 2, -2, -3, 1)$

$(y_1(0), y_2(0), y_3(0), y_4(0), y_5(0)) = (2,1,0.5,-2,-1)$ and the true values of the parameters are taken as: $\alpha_1 = 9, \alpha_2 = -10.8, \alpha_3 = 10.8, \alpha_4 = 1$, $\alpha_5 = 30, \alpha_6 = 30, \alpha_7 = 15$.

The numerical results are considered as two cases:

(i). **Synchronization**: Choosing the scaling parameter $q = 1$, the synchronization of the drive system (4.25) and response system (4.26) is achieved as indicated by the convergence of the error state variables to zero.

(ii). **Anti-synchronization**: Choosing the scaling parameter $q = -1$, the anti-synchronization of the drive system (4.25) and response system (4.26) is achieved as indicated by the convergence of the error state variables to zero.

Simulation results:

For synchronization we set q=1 in eq. (4.27)

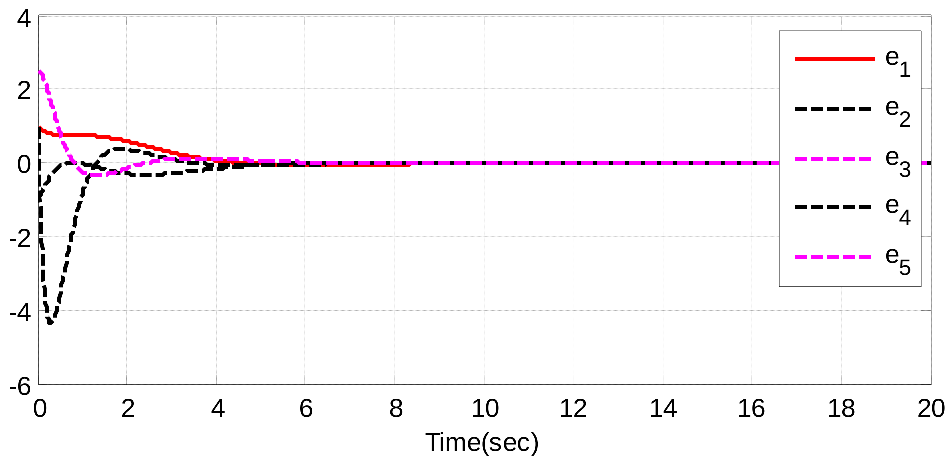

Figure 4.46: Time history of error variables for known parameters.

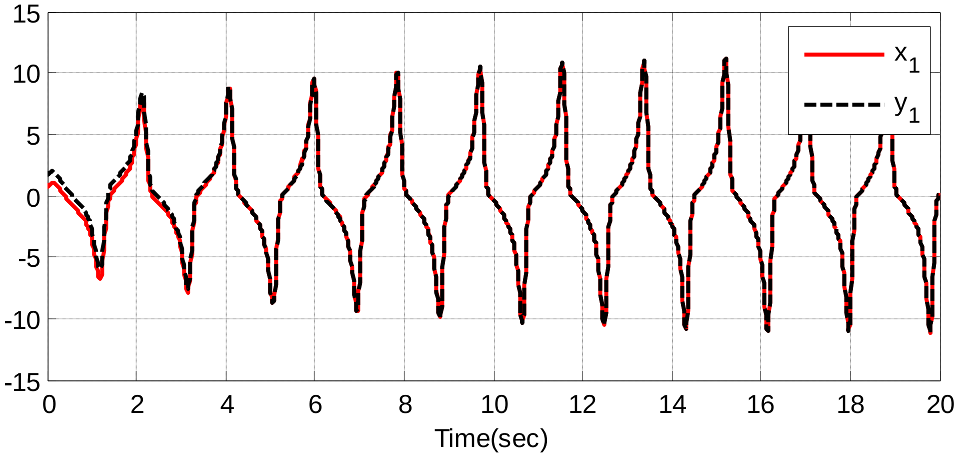

Figure 4.47: Synchronization between x_1 and y_1 of known parameters.

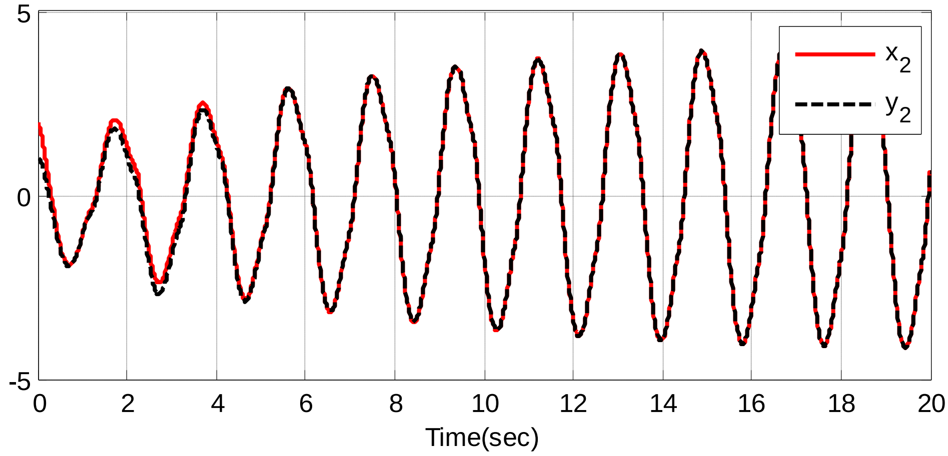

Figure 4.48: Synchronization between x_1 and y_1 of known parameters.

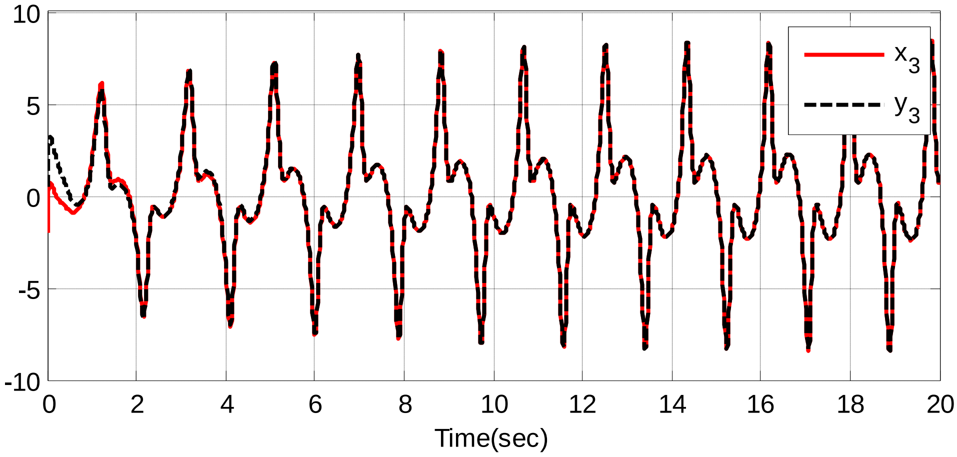

Figure 4.49: Synchronization between x_3 and y_3 of known parameters.

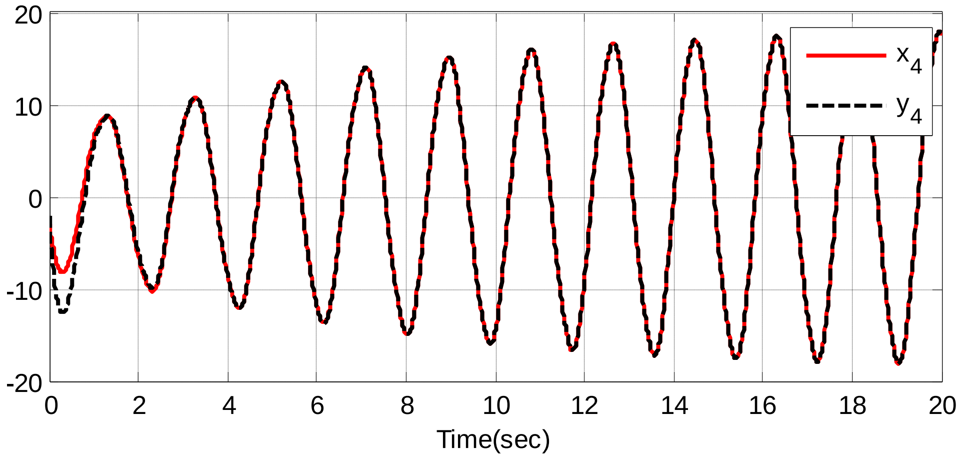

Figure 4.50: Synchronization between x_4 and y_4 of known parameters.

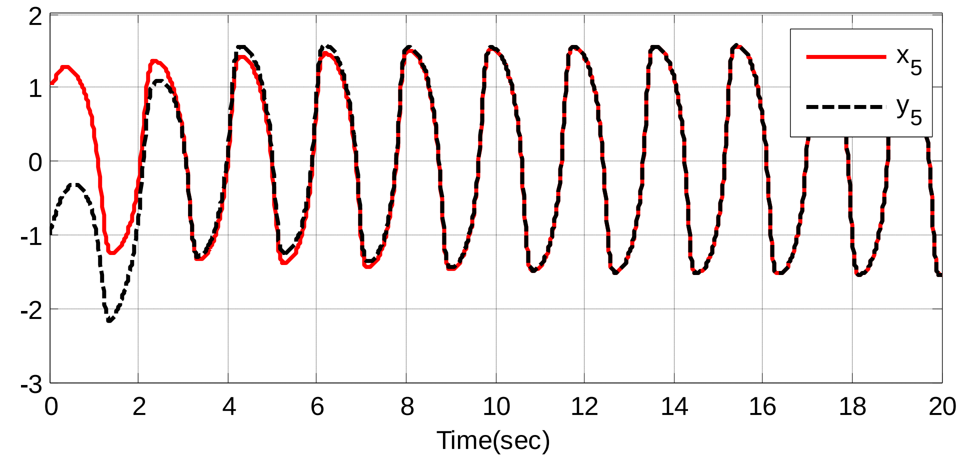

Figure 4.51: Synchronization between x_5 and y_5 of known parameters.

For anti-synchronization we set q= -1 in eq. (4.27):

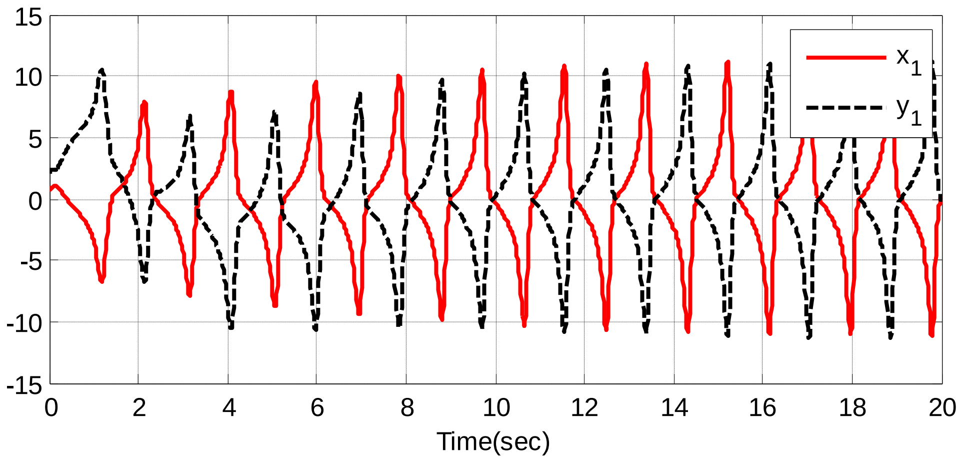

Figure 4.52: Anti-synchronization between x_1 and y_1 of known parameters.

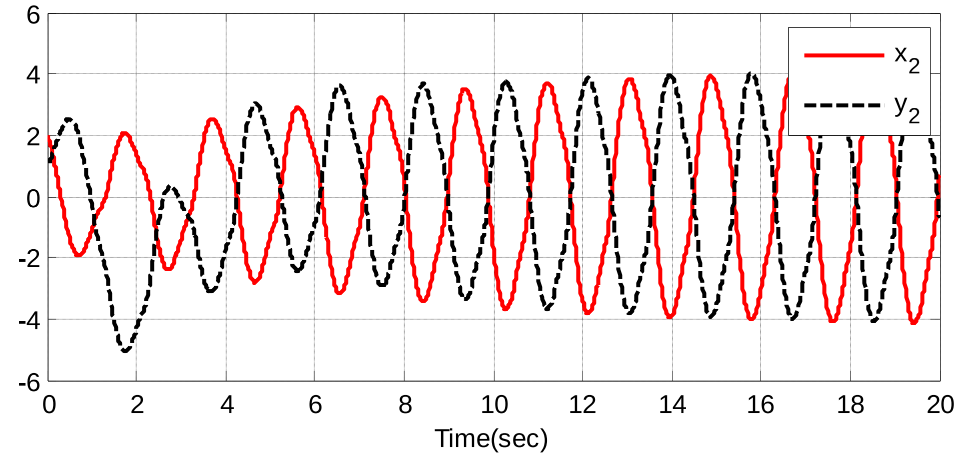

Figure 4.53: Anti-synchronization between x_2 and y_2 of known parameters.

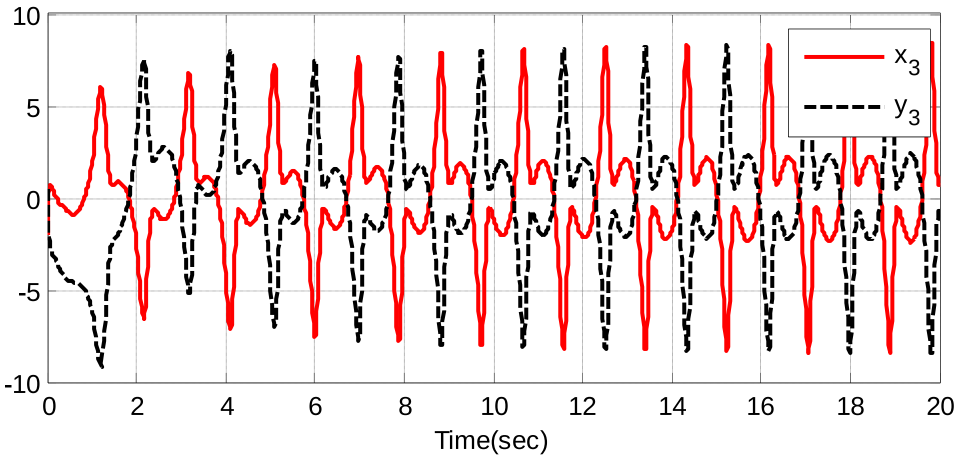

Figure 4.54: Anti-synchronization between x_3 and y_3 of known parameters.

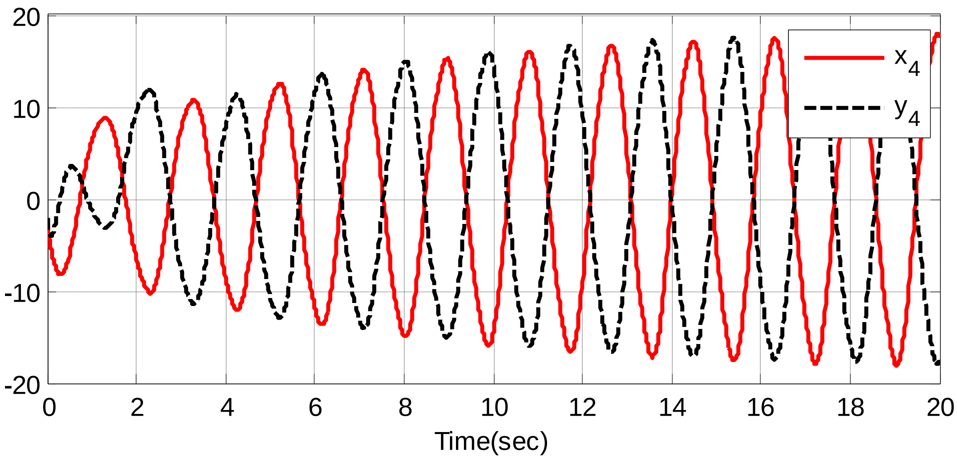

Figure 4.55: Anti-synchronization between x_4 and y_4 of known parameters.

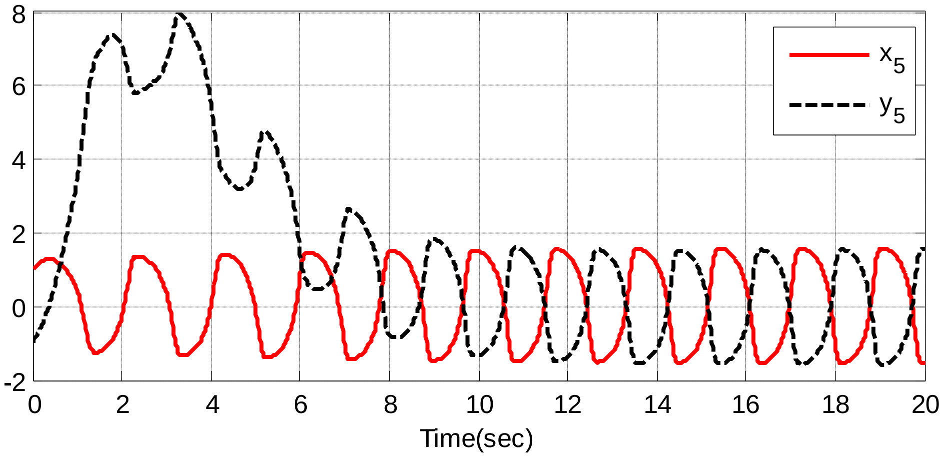

Figure 4.56: Anti-synchronization between x_5 and y_5 of known parameters.

Case 2: synchronization and anti-synchronization with unknown parameters.

Let $\hat{\alpha}_i, i = 1,\ldots,7$ be the estimates of $\alpha_i, i = 1,\ldots,7$ and $\tilde{\alpha}_i = \alpha_i - \hat{\alpha}_i, i = 1,\ldots,7$ be the error in estimation of $\alpha_i, i = 1,\ldots,7$ respectively. Therefore the systems (4.25) and (4.26) can be written as:

$$
\begin{aligned}
\dot{x}_1 &= \hat{\alpha}_1 x_3 + \tilde{\alpha}_1 x_3 - \hat{\alpha}_2 x_1 - \tilde{\alpha}_2 x_1 - \hat{\alpha}_3 x_1 x_5^2 - \tilde{\alpha}_3 x_1 x_5^2 \\
\dot{x}_2 &= -\hat{\alpha}_4 x_3 - \tilde{\alpha}_4 x_3 + \hat{\alpha}_4 x_4 + \tilde{\alpha}_4 x_4 \\
\dot{x}_3 &= \hat{\alpha}_5 x_2 + \tilde{\alpha}_5 x_2 - \hat{\alpha}_5 x_1 - \tilde{\alpha}_5 x_1 - \hat{\alpha}_6 x_3 - \tilde{\alpha}_6 x_3 \\
\dot{x}_4 &= -\hat{\alpha}_7 x_2 - \tilde{\alpha}_7 x_2 \\
\dot{x}_5 &= x_1
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
\dot{y}_1 &= \hat{\alpha}_1 y_3 + \tilde{\alpha}_1 y_3 - \hat{\alpha}_2 y_1 - \tilde{\alpha}_2 y_1 - \hat{\alpha}_3 y_1 y_5^2 - \tilde{\alpha}_3 y_1 y_5^2 + u_1 \\
\dot{y}_2 &= -\hat{\alpha}_4 y_3 - \tilde{\alpha}_4 y_3 + \hat{\alpha}_4 y_4 + \tilde{\alpha}_4 y_4 + u_2 \\
\dot{y}_3 &= \hat{\alpha}_5 y_2 + \tilde{\alpha}_5 y_2 - \hat{\alpha}_5 y_1 - \tilde{\alpha}_5 y_1 - \hat{\alpha}_6 y_3 - \tilde{\alpha}_6 y_3 + u_3 \\
\dot{y}_4 &= -\hat{\alpha}_7 y_2 - \tilde{\alpha}_7 y_2 + u_4 \\
\dot{y}_5 &= y_1
\end{aligned}
\tag{3.31}
$$

Then the dynamics of the error system becomes:

$$
\begin{aligned}
\dot{e}_1 &= \dot{y}_1 - q\dot{x}_1 = \hat{\alpha}_1 y_3 + \tilde{\alpha}_1 y_3 - \hat{\alpha}_2 y_1 - \tilde{\alpha}_2 y_1 - \hat{\alpha}_3 y_1 y_5^2 - \tilde{\alpha}_3 y_1 y_5^2 + u_1 \\
&\quad - q\{\hat{\alpha}_1 x_3 + \tilde{\alpha}_1 x_3 - \hat{\alpha}_2 x_1 - \tilde{\alpha}_2 x_1 - \hat{\alpha}_3 x_1 x_5^2 - \tilde{\alpha}_3 x_1 x_5^2\} \\
\dot{e}_2 &= \dot{y}_2 - q\dot{x}_2 = -\hat{\alpha}_4 y_3 - \tilde{\alpha}_4 y_3 + \hat{\alpha}_4 y_4 + \tilde{\alpha}_4 y_4 + u_2 \\
&\quad - q\{-\hat{\alpha}_4 x_3 - \tilde{\alpha}_4 x_3 + \hat{\alpha}_4 x_4 + \tilde{\alpha}_4 x_4\} \\
\dot{e}_3 &= \dot{y}_3 - q\dot{x}_3 = \hat{\alpha}_5 y_2 + \tilde{\alpha}_5 y_2 - \hat{\alpha}_5 y_1 - \tilde{\alpha}_5 y_1 - \hat{\alpha}_6 y_3 - \tilde{\alpha}_6 y_3 + u_3 \\
&\quad - q\{\hat{\alpha}_5 x_2 + \tilde{\alpha}_5 x_2 - \hat{\alpha}_5 x_1 - \tilde{\alpha}_5 x_1 - \hat{\alpha}_6 x_3 - \tilde{\alpha}_6 x_3\} \\
\dot{e}_4 &= \dot{y}_4 - q\dot{x}_4 = -\hat{\alpha}_7 y_2 - \tilde{\alpha}_7 y_2 + u_4 - q\{-\hat{\alpha}_7 x_2 - \tilde{\alpha}_7 x_2\} \\
\dot{e}_5 &= \dot{y}_5 - q\dot{x}_5 = y_1 - qx_1 = e_1
\end{aligned}
\tag{4.32}
$$

By choosing

$$
\begin{aligned}
u_1 &= -\hat{\alpha}_1 y_3 + \hat{\alpha}_2 y_1 + \hat{\alpha}_3 y_1 y_5^2 + q\{\hat{\alpha}_1 x_3 - \hat{\alpha}_2 x_1 - \hat{\alpha}_3 x_1 x_5^2\} + e_2 \\
u_2 &= \hat{\alpha}_4 y_3 - \hat{\alpha}_4 y_4 + q\{-\hat{\alpha}_4 x_3 + \hat{\alpha}_4 x_4\} + e_3 \\
u_3 &= u_3 = -\hat{\alpha}_5 y_2 + \hat{\alpha}_5 y_1 + \hat{\alpha}_6 y_3 + q\{\hat{\alpha}_5 x_2 - \hat{\alpha}_5 x_1 - \hat{\alpha}_6 x_3\} + e_4 \\
u_4 &= \hat{\alpha}_7 y_2 + q\{-\hat{\alpha}_7 x_2\} + v
\end{aligned}
$$

where v is the new input, the system (4.32) can be written as:

$$
\begin{aligned}
\dot{e}_5 &= e_1 \\
\dot{e}_1 &= e_2 + \tilde{\alpha}_1 y_3 - \tilde{\alpha}_2 y_1 - \tilde{\alpha}_3 y_1 y_5^2 - q\tilde{\alpha}_1 x_3 + q\tilde{\alpha}_2 x_1 + q\tilde{\alpha}_3 x_1 x_5^2 \\
\dot{e}_2 &= e_3 - \tilde{\alpha}_4 y_3 + \tilde{\alpha}_4 y_4 + q\tilde{\alpha}_4 x_3 - q\tilde{\alpha}_4 x_4 \\
\dot{e}_3 &= e_4 + \tilde{\alpha}_5 y_2 - \tilde{\alpha}_5 y_1 - \tilde{\alpha}_6 y_3 - q\tilde{\alpha}_5 x_2 + q\tilde{\alpha}_5 x_1 + q\tilde{\alpha}_6 x_3 \\
\dot{e}_4 &= v - \tilde{\alpha}_7 y_2 + q\tilde{\alpha}_7 x_2
\end{aligned}
\tag{4.33}
$$

Choose the nominal system for (4.33) as:

$$
\begin{aligned}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
\dot{e}_3 &= e_4 \\
\dot{e}_4 &= e_5 \\
\dot{e}_5 &= v_0
\end{aligned}
\tag{4.34}
$$

Define the sliding surface for nominal system (4.34) as: $\sigma_0 = (1 + D)^4 e_1$

Then

$$
\dot{\sigma}_0 = \dot{e}_1 + 4\dot{e}_2 + 6\dot{e}_3 + 4\dot{e}_4 + \dot{e}_5 = e_2 + 4e_3 + 6e_4 + 4e_5 + v
$$

By choosing $v = -e_2 - 4e_3 - 6e_4 - 4e_5 - k\sigma, k > 0$, we have $\dot{\sigma}_0 = -k\sigma_0$. Therefore the system (4.34) is asymptotically stable.

Now choose the sliding surface for system (4.33) as: $\sigma = \sigma_0 + z = e_1 + 2e_2 + e_3 + z$

where, z is some integral term computed later. To avoid the reaching phase, choose $z(0)$ such that $\sigma(0) = 0$. Choose $v = v_0 + v_s$ where, v_0 is the nominal input and v_s is compensator term computed later. Then

$$
\begin{aligned}
\dot{\sigma} &= \dot{e}_5 + 4\dot{e}_1 + 6\dot{e}_2 + 4\dot{e}_3 + \dot{e}_4 + \dot{z} \\
&= e_1 + 4e_2 + 4\tilde{\alpha}_1 y_3 - 4\tilde{\alpha}_2 y_1 - 4\tilde{\alpha}_3 y_1 y_5^2 - 4q\tilde{\alpha}_1 x_3 + 4q\tilde{\alpha}_2 x_1 + 4q\tilde{\alpha}_3 x_1 x_5^2 \\
&+ 6e_3 - 6\tilde{\alpha}_4 y_3 + 6\tilde{\alpha}_4 y_4 + 6q\tilde{\alpha}_4 x_3 - 6q\tilde{\alpha}_4 x_4 \\
&+ 4e_4 + 4\tilde{\alpha}_5 y_2 - 4\tilde{\alpha}_5 y_1 - 4\tilde{\alpha}_6 y_3 - 4q\tilde{\alpha}_5 x_2 + 4q\tilde{\alpha}_5 x_1 + 4q\tilde{\alpha}_6 x_3 \\
&+ v_0 + v_s - \tilde{\alpha}_7 y_2 + q\tilde{\alpha}_7 x_2 + \dot{z}
\end{aligned}
\tag{4.35}
$$

By choosing a Lyapunov function $V = \frac{1}{2}\sigma^2 + \frac{1}{2}\sum_{i=1}^{7}\tilde{\alpha}_i^2$, design the adaptive laws for $\tilde{\alpha}_i, \hat{\alpha}_i, i = 1,\ldots,7$ and compute v_s such that $\dot{V} < 0$.

Consider a Lyapunov function $V = \frac{1}{2}\sigma^2 + \frac{1}{2}\sum_{i=1}^{7}\tilde{\alpha}_i^2$. Then $\dot{V} < 0$ if the adaptive laws for $\tilde{\alpha}_i, \hat{\alpha}_i, i = 1,\ldots,7$ and the value of v_s are chosen as:

$$
\begin{aligned}
&\dot{z} = -e_1 - 4e_2 - 6e_3 - 4e_4 - v_0, \quad v_s = -k\,s \\
&\dot{\tilde{\alpha}}_1 = -4\sigma\, y_3 + 4q\sigma\, x_3 - k_1\tilde{\alpha}_1 \;\&\; \dot{\hat{\alpha}}_1 = -\dot{\tilde{\alpha}}_1 \\
&\dot{\tilde{\alpha}}_2 = 4\sigma\, y_1 - 4q\sigma\, x_1 - k_2\tilde{\alpha}_2 \;\&\; \dot{\hat{\alpha}}_2 = -\dot{\tilde{\alpha}}_2 \\
&\dot{\tilde{\alpha}}_3 = 4\sigma\, y_1 y_5^2 - 4q\sigma\, x_1 x_5^2 - k_3\tilde{\alpha}_3 \;\&\; \dot{\hat{\alpha}}_3 = -\dot{\tilde{\alpha}}_3 \\
&\dot{\tilde{\alpha}}_4 = 6\sigma y_3 - 6\sigma y_4 - 6q\sigma x_3 + 6q\sigma x_4 - k_4\tilde{\alpha}_4 \;\&\; \dot{\hat{\alpha}}_4 = -\dot{\tilde{\alpha}}_4 \\
&\dot{\tilde{\alpha}}_5 = -4\sigma y_2 + 4\sigma y_1 + 4\sigma q x_2 - 4\sigma q x_1 - k_5\tilde{\alpha}_5 \;\&\; \dot{\hat{\alpha}}_5 = -\dot{\tilde{\alpha}}_5 \\
&\dot{\tilde{\alpha}}_6 = 4\sigma y_3 - 4\sigma q x_3 - k_6\tilde{\alpha}_6 \;\&\; \dot{\hat{\alpha}}_6 = -\dot{\tilde{\alpha}}_6 \\
&\dot{\tilde{\alpha}}_7 = \sigma\, y_2 - \sigma q\, x_2 - k_7\tilde{\alpha}_7 \;\&\; \dot{\hat{\alpha}}_7 = -\dot{\tilde{\alpha}}_7,\; k, k_i > 0, i = 1,\ldots,7
\end{aligned}
\tag{4.36}
$$

Proof:

Since

$$\begin{aligned}
\dot{V} &= \sigma\dot{\sigma} + \tilde{\alpha}_1\dot{\tilde{\alpha}}_1 + \tilde{\alpha}_2\dot{\tilde{\alpha}}_2 + \tilde{\alpha}_3\dot{\tilde{\alpha}}_3 + \tilde{\alpha}_4\dot{\tilde{\alpha}}_4 + \tilde{\alpha}_5\dot{\tilde{\alpha}}_5 + \tilde{\alpha}_6\dot{\tilde{\alpha}}_6 + \tilde{\alpha}_7\dot{\tilde{\alpha}}_7 \\
&= \sigma\{e_1 + 4e_2 + 4\tilde{\alpha}_1 y_3 - 4\tilde{\alpha}_2 y_1 - 4\tilde{\alpha}_3 y_1 y_5^2 - 4q\tilde{\alpha}_1 x_3 + 4q\tilde{\alpha}_2 x_1 + 4q\tilde{\alpha}_3 x_1 x_5^2 \\
&+ 6e_3 - 6\tilde{\alpha}_4 y_3 + 6\tilde{\alpha}_4 y_4 + 6q\tilde{\alpha}_4 x_3 - 6q\tilde{\alpha}_4 x_4 \\
&+ 4e_4 + 4\tilde{\alpha}_5 y_2 - 4\tilde{\alpha}_5 y_1 - 4\tilde{\alpha}_6 y_3 - 4q\tilde{\alpha}_5 x_2 + 4q\tilde{\alpha}_5 x_1 + 4q\tilde{\alpha}_6 x_3 \\
&+ v_0 + v_s - \tilde{\alpha}_7 y_2 + \tilde{\alpha}_7 x_2 + \dot{z}\} + \tilde{\alpha}_1\dot{\tilde{\alpha}}_1 + \tilde{\alpha}_2\dot{\tilde{\alpha}}_2 + \tilde{\alpha}_3\dot{\tilde{\alpha}}_3 + \tilde{\alpha}_4\dot{\tilde{\alpha}}_4 + \tilde{\alpha}_5\dot{\tilde{\alpha}}_5 + \tilde{\alpha}_6\dot{\tilde{\alpha}}_6 + \tilde{\alpha}_7\dot{\tilde{\alpha}}_7 \\
&= \sigma\{e_1 + 4e_2 + 6e_3 + 4e_4 + v_0 + v_s + \dot{z}\}\} \\
&+ \tilde{\alpha}_1\{\dot{\tilde{\alpha}}_1 + 4\sigma y_3 - 4q\sigma x_3\} + \tilde{\alpha}_2\{\dot{\tilde{\alpha}}_2 - 4\sigma y_1 + 4q\sigma x_1\} + \tilde{\alpha}_3\{\dot{\tilde{\alpha}}_3 - 4\sigma y_1 y_5^2 + 4q\sigma x_1 x_5^2\} \\
&+ \tilde{\alpha}_4\{\dot{\tilde{\alpha}}_4 + 6\sigma y_4 - 6\sigma y_3 + 6q\sigma x_3 - 6q\sigma x_4\} \\
&+ \tilde{\alpha}_5\{\dot{\tilde{\alpha}}_5 + 4\sigma y_2 - 4\sigma y_1 - 4q\sigma x_2 + 4q\sigma x_1\} + \tilde{\alpha}_6\{\dot{\tilde{\alpha}}_6 - 4\sigma y_3 + 4q\sigma x_3\} + \tilde{\alpha}_7\{\dot{\tilde{\alpha}}_7 - \sigma y_2 + \sigma q x_2\}
\end{aligned}$$

By using

$$\begin{aligned}
&\dot{z} = -e_1 - 4e_2 - 6e_3 - 4e_4 - v_0, \quad v_s = -k\,s \\
&\dot{\tilde{\alpha}}_1 = -4\sigma y_3 + 4q\sigma x_3 - k_1\tilde{\alpha}_1 \;\&\; \dot{\hat{\alpha}}_1 = -\dot{\tilde{\alpha}}_1 \\
&\dot{\tilde{\alpha}}_2 = 4\sigma y_1 - 4q\sigma x_1 - k_2\tilde{\alpha}_2 \;\&\; \dot{\hat{\alpha}}_2 = -\dot{\tilde{\alpha}}_2 \\
&\dot{\tilde{\alpha}}_3 = 4\sigma y_1 y_5^2 - 4q\sigma x_1 x_5^2 - k_3\tilde{\alpha}_3 \;\&\; \dot{\hat{\alpha}}_3 = -\dot{\tilde{\alpha}}_3 \\
&\dot{\tilde{\alpha}}_4 = 6\sigma y_3 - 6\sigma y_4 - 6q\sigma x_3 + 6q\sigma x_4 - k_4\tilde{\alpha}_4 \;\&\; \dot{\hat{\alpha}}_4 = -\dot{\tilde{\alpha}}_4 \\
&\dot{\tilde{\alpha}}_5 = -4\sigma y_2 + 4\sigma y_1 + 4\sigma q x_2 - 4\sigma q x_1 - k_5\tilde{\alpha}_5 \;\&\; \dot{\hat{\alpha}}_5 = -\dot{\tilde{\alpha}}_5 \\
&\dot{\tilde{\alpha}}_6 = 4\sigma y_3 - 4\sigma q x_3 - k_6\tilde{\alpha}_6 \;\&\; \dot{\hat{\alpha}}_6 = -\dot{\tilde{\alpha}}_6 \\
&\dot{\tilde{\alpha}}_7 = \sigma y_2 - \sigma q x_2 - k_7\tilde{\alpha}_7 \;\&\; \dot{\hat{\alpha}}_7 = -\dot{\tilde{\alpha}}_7,\; k, k_i > 0, i = 1,\ldots,7
\end{aligned}$$

We have

$$\dot{V} = -k\sigma^2 - k_1\tilde{\alpha}_1^2 - k_2\tilde{\alpha}_2^2 - k_3\tilde{\alpha}_3^2 - k_4\tilde{\alpha}_4^2 - k_5\tilde{\alpha}_5^2 - k_6\tilde{\alpha}_6^2 - k_7\tilde{\alpha}_7^2 < 0$$

From this we conclude that $\sigma, \tilde{\alpha}_i, i = 1,\ldots,7 \to 0$. Since $\sigma \to 0$, therefore $e_1, e_2, e_3, e_4, e_5 \to 0$.

In simulations, the initial conditions are chosen as: $(x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)) = (1, 2, -2, -3, 1)$, $(y_1(0), y_2(0), y_3(0), y_4(0), y_5(0)) = (2,1,0.5,-2,-1)$ and the true values of the parameters are taken as: $\alpha_1 = 9, \alpha_2 = -10.8, \alpha_3 = 10.8, \alpha_4 = 1$, $\alpha_5 = 30, \alpha_6 = 30, \alpha_7 = 15$.

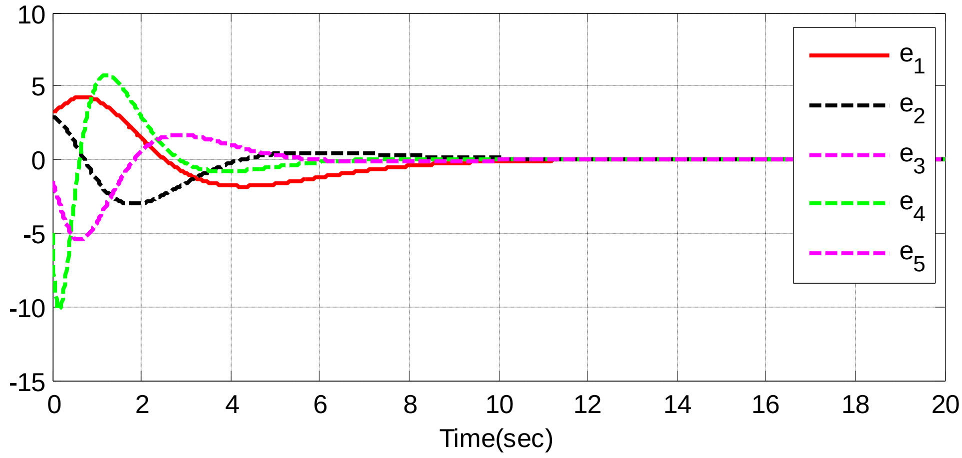

Figure 4.57: Time history of error variables for unknown parameters.

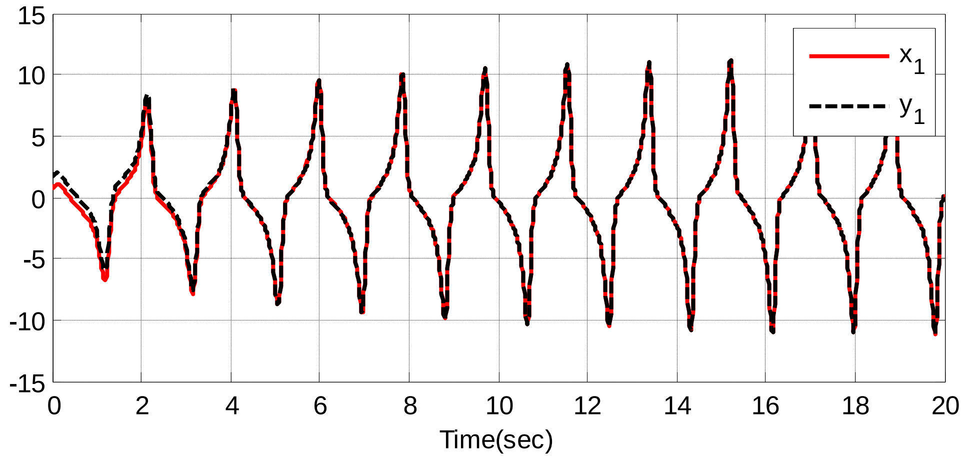

Figure 4.58: Synchronization between x_1 and y_1 for unknown parameters.

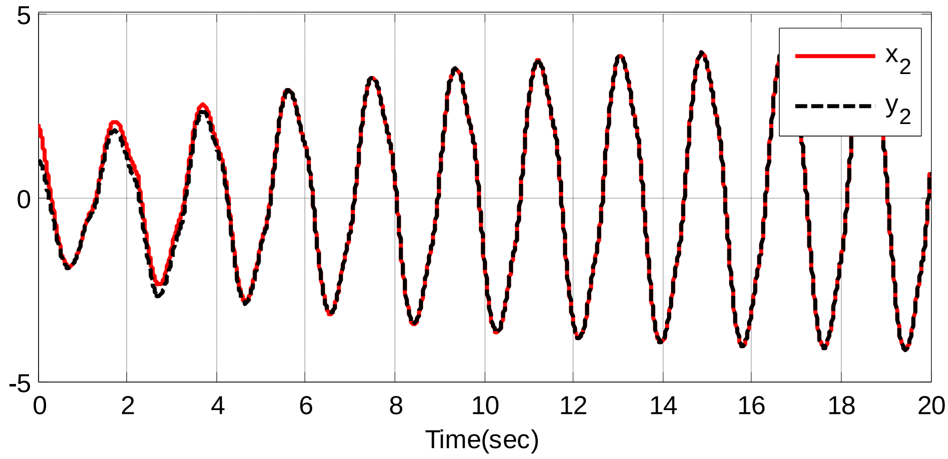

Figure 4.59: Synchronization between x_2 and y_2 for unknown parameters.

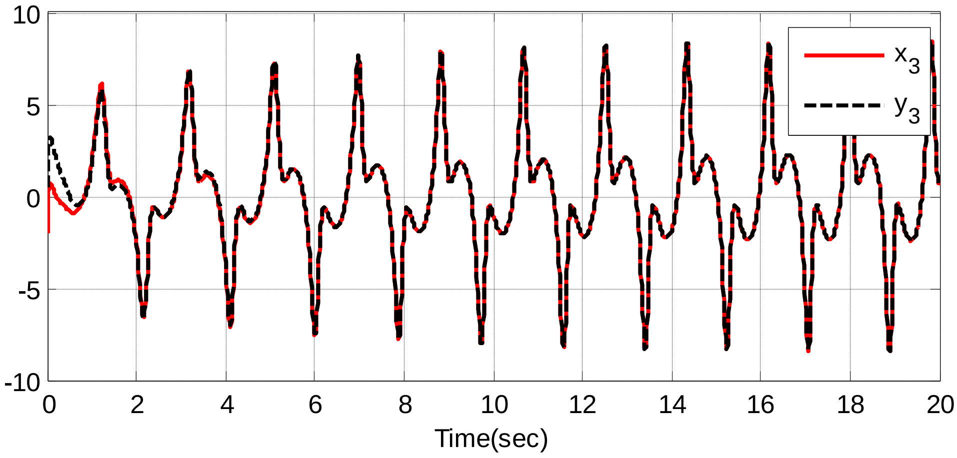

Figure 4.60: Synchronization between x_3 and y_3 for unknown parameters.

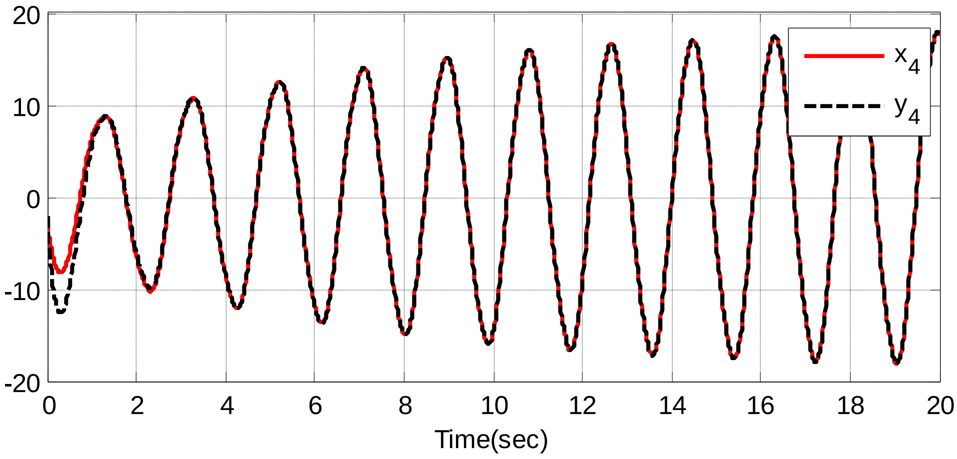

Figure 4.61: Synchronization between x_4 and y_4 for unknown parameters.

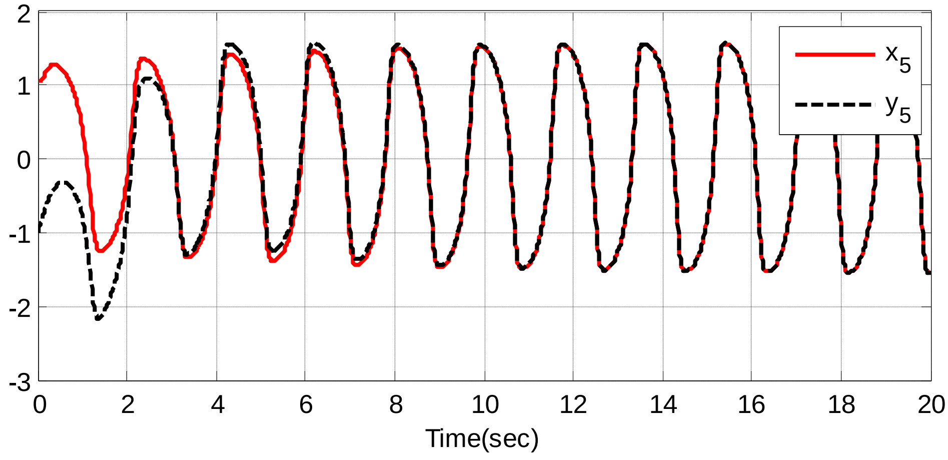

Figure 4.62: Synchronization between x_5 and y_5 for unknown parameters.

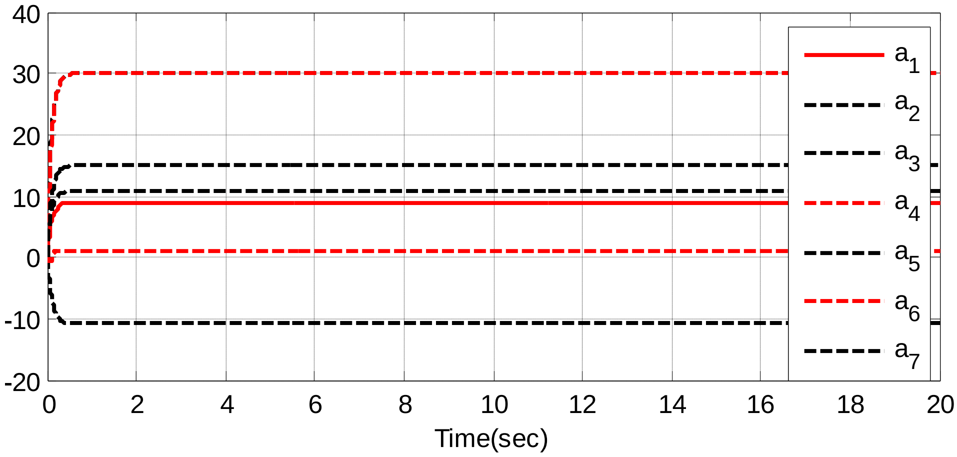

Figure4.63:Estimation unknown parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ synchronization.

For anti-synchronization we set q=-1 in eq. (4.33)

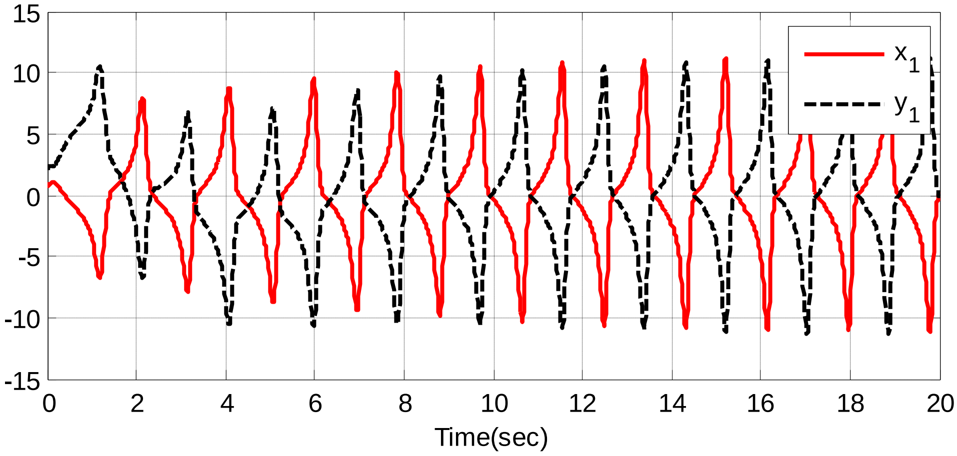

Figure 4.64: Anti-synchronization between x_1 and y_1 for unknown parameters.

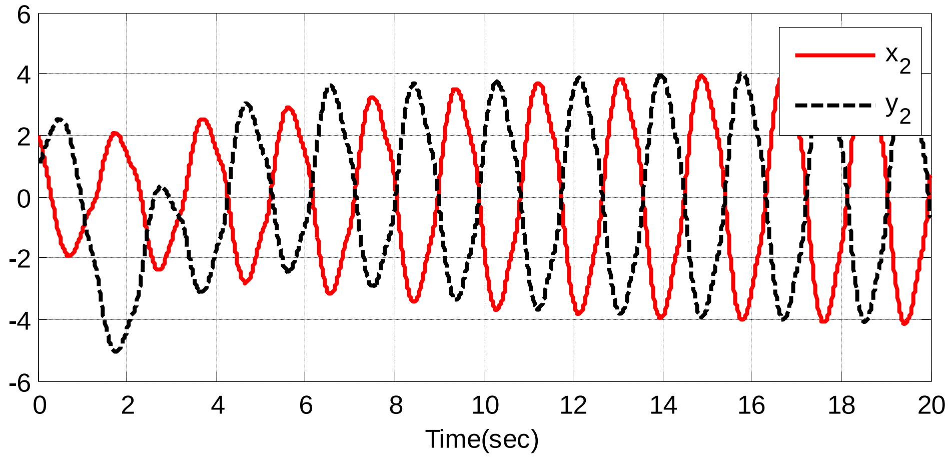

Figure 4.65: Anti-synchronization between x_2 and y_2 for unknown parameters.

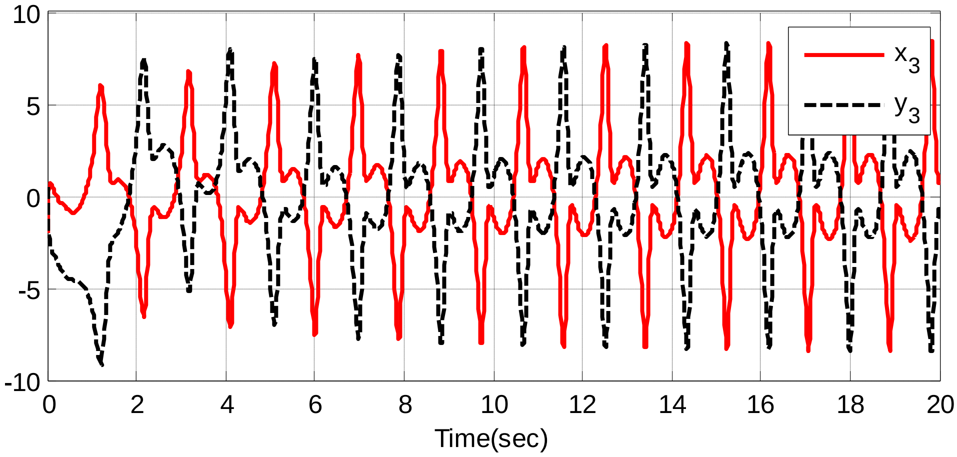

Figure 4.66: Anti-synchronization between x_3 and y_3 for unknown parameters.

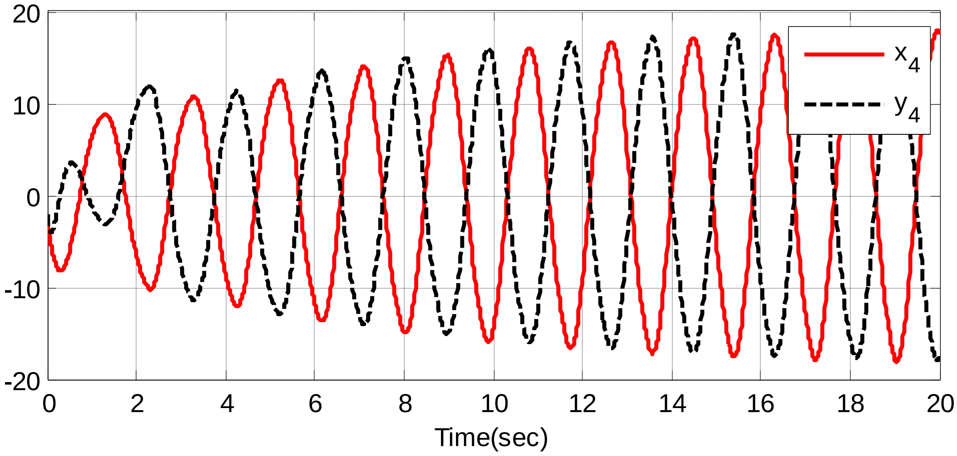

Figure 4.67: Anti-synchronization between x_4 and y_4 for unknown parameters.

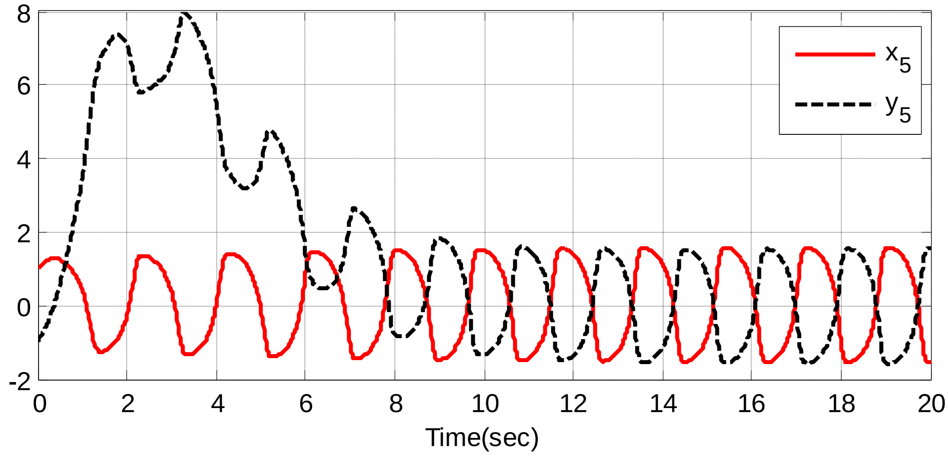

Figure 4.68: Anti-synchronization between x_5 and y_5 for unknown parameters.

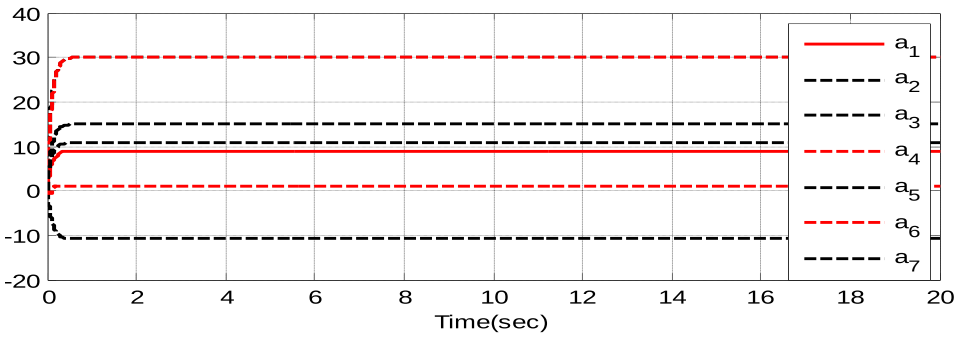

Figure 4.69:Estimation of Unknown $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ of Anti-synchronization.

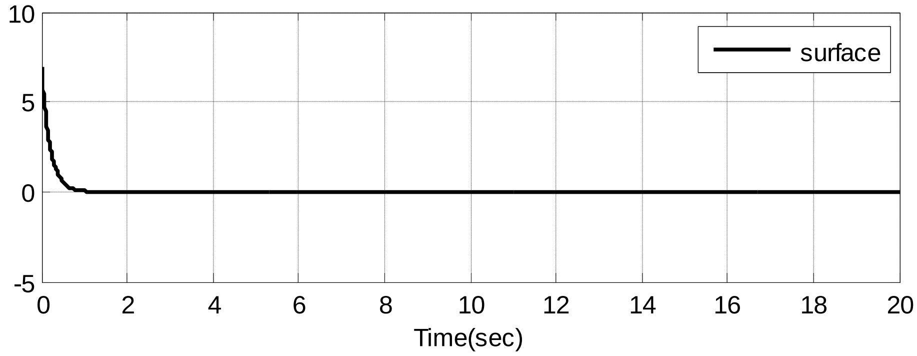

Figure 4.70: Time history of surface.

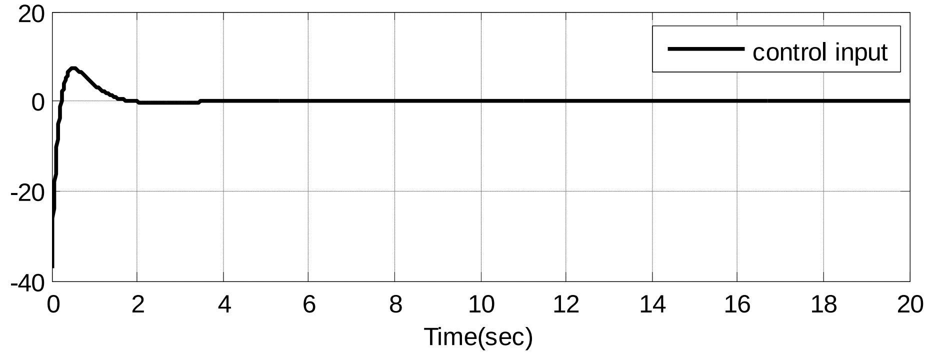

Figure 4.71: Time history of control input.

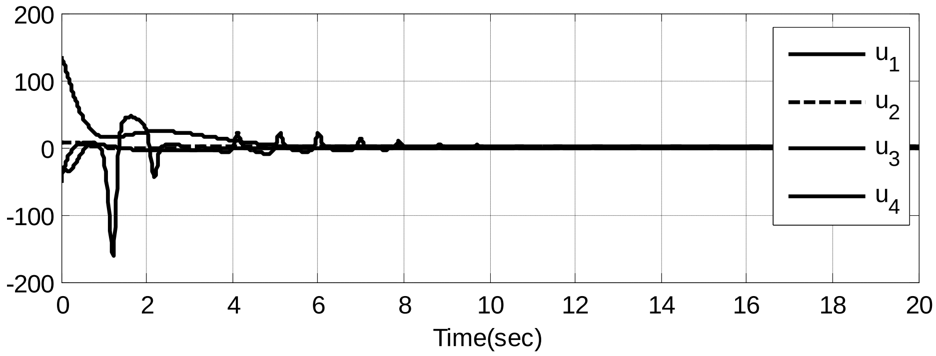

Figure 4.72: Time history of adaptive controllers.

Chapter 5

CONCLUSION AND FUTURE WORK

5.1 Introduction

The work done by human being can never be complete. Taking into account this reality, this chapter is aimed to explain the results and conclusion of this research thesis.

5.2 Conclusion

This thesis presents the synchronization and anti-synchronization scheme between two identical chaotic systems. Two cases are considered (i) systems with known parameters, and (ii) systems with unknown parameters. In case (i) the synchronization and anti-synchronization are achieved through sliding mode control, while in case (ii) the adaptive integral sliding mode control is used. To employ the adaptive integral sliding mode control, the error system is transformed into a special structure containing nominal part and some unknown terms. The unknown terms are computed adaptively. Then the error system is stabilized using integral sliding mode control. The stabilizing controller for the error system is constructed which consists of the nominal control plus some compensator control. To avoid the chattering phenomenon, smooth continuous compensator control is used instead of traditional discontinuous control. The compensator controller and the adapted law are derived in such a way that the time derivative of a Lyapunov function becomes strictly negative. Numerical simulations are shown to illustrate and validate the synchronization schemes presented in this thesis.

5.3 Future Work

After the successful completion of this thesis, some recommendations are made which must be considered in future work. The proposed control algorithm is also applicable to other chaotic systems. After going through some hard efforts to produce this research work, there are some areas which need future attention. The obvious future work is to implement the proposed control algorithm and control strategy on the practical chaotic system and compare the simulated result with the real simulation of chaotic systems.

REFERENCES

[1] Broer, Henk, and Floris Takens. *Dynamical systems and chaos*. Vol. 172. Springer Science & Business Media, 2010.

[2] Berliner, L. Mark. "Statistics, probability and chaos." *Statistical Science*(1992): 69-90.

[3] Haroun, Mohamed. *Secure communications based on chaotic systems*. Diss. University of Victoria, 2015.

[4] Oestreicher, Christian. "A history of chaos theory." *Dialogues Clin Neurosci*9.3 (2007): 279-89.

[5] Ives, Crystal. "Human beings as chaotic systems." *Life Science Tehcnology*(2004).

[6] Mondal, Sanjoy, and Chitralekha Mahanta. "Adaptive integral higher order sliding mode controller for uncertain systems." *Journal of Control Theory and Applications* 11.1 (2013): 61-68.

[7] Boccaletti, Stefano, et al. "The synchronization of chaotic systems." *Physics reports* 366.1 (2002): 1-101.

[8] Li, Xian-Feng, et al. "Complete (anti-) synchronization of chaotic systems with fully uncertain parameters by adaptive control." *Nonlinear Dynamics*63.1-2 (2011): 263-275.

[9] Pikovsky, Arkady, Michael Rosenblum, and Jürgen Kurths. *Synchronization: a universal concept in nonlinear sciences*. Vol. 12. Cambridge university press, 2003.

[10] Balmforth, Neil, et al. "Master-slave synchronization and the Lorenz equations." *Chaos: An Interdisciplinary Journal of Nonlinear Science* 7.3 (1997): 392-394.

[11] Yau, Her-Terng. "Design of adaptive sliding mode controller for chaos synchronization with uncertainties." *Chaos, Solitons & Fractals* 22.2 (2004): 341-347.

[12] Zhang, Hao, Xi-Kui Ma, and Wei-Zeng Liu. "Synchronization of chaotic systems with parametric uncertainty using active sliding mode control." *Chaos, Solitons & Fractals* 21.5 (2004): 1249-1257.

[13] Mondal, Amit, Mitul Islam, and Nurul Islam. "Robust antisynchronization of chaos using sliding mode control strategy." *Pramana* 84.1 (2015): 47-67.

[14] Perruquetti W., &Barbot, J. P. *Sliding mode control in engineering*. CRC Press.2002

[15] Bartolini Giorgio, et al. "Modern sliding mode control theory." *Lecture Notes in Control and Information Sciences* 375.7: 23-49. 2008

[16] Edwards Christopher, and Sarah Spurgeon. *Sliding mode control: theory and applications*. CRC Press, 1998

[17] Ghamati, Mina, and Saeed Balochian. "Design of adaptive sliding mode control for synchronization Genesio–Tesi chaotic system." *Chaos, Solitons & Fractals* 75 (2015): 111-117.

[18] Tseng M. L., & Chen, M. S. *Chattering reduction of sliding mode control by low-pass filtering the control signal*. *Asian Journal of control*, *12*(3), 392-398. 2010.

[19] Kharel, Rupak. *Design and Implementation of Secure Chaotic Communication Systems*. Diss. Northumbria University, 2011.

[20] Yang, Tao. "A survey of chaotic secure communication systems." *International Journal of Computational Cognition* 2.2 (2004): 81-130.

[21] Lu, Junan, Xiaoqun Wu, and Jinhu Lü. "Synchronization of a unified chaotic system and the application in secure communication." *Physics Letters A*305.6 (2002): 365-370.

[22] Martinez-Guerra, Rafael, and Wen Yu. "Chaotic synchronization and secure communication via sliding-mode observer." *International Journal of Bifurcation and Chaos* 18.01 (2008): 235-243.

[23] Jin-E, Zhang. "Combination-combination hyperchaos synchronization of complex memristor oscillator system." *Mathematical Problems in Engineering* 2014 (2014).

[24] Ren, Tao, Zhi-liang Zhu, and Hai Yu. "Design of finite-time synchronization controller and its application to security communication system." *Appl. Math*8.1 (2014): 387-391.

[25] Li, Zhanguo, and Zheng Zhang. "Secure communication scheme with hyper-chaotic Lorenz-Stenflo system." *Electronics, Communications and Control (ICECC), 2011 International Conference on*. IEEE, 2011.

[26] Fridman L., & Levant, A. Higher order sliding modes as a natural phenomenon in control theory.In *Robust Control via variable structure and Lyapunov techniques* (pp. 107-133).Springer Berlin Heidelberg. 1996

[27] Ablay G., &Aldemir, T. Observer based sliding mode control of uncertain chaotic systems. In *Electrical and Electronics Engineering, 2009.ELECO 2009.International Conference on* (pp. II-307).IEEE.November, 2009

[28] Jawaada, Wafaa, Mohd Salmi Md Noorani, and M. Mossa Al-Sawalha. "Anti-synchronization of chaotic systems via adaptive sliding mode control." *Chinese Physics Letters* 29.12 (2012): 120505.

[29] Rabiei, Zahra, G. Reza Bidari, and Naser Pariz. "Synchronization Between Two Different Chaotic SystemsUsing Adaptive Sliding Mode Control." *International Journal of Information and Electronics Engineering* 3.1 (2013): 83.

[30] Yau, Her-Terng. "Design of adaptive sliding mode controller for chaos synchronization with uncertainties." *Chaos, Solitons & Fractals* 22.2 (2004): 341-347.

[31] Jang, Ming-Jyi, Chieh-Li Chen, and Cha'o-Kuang Chen. "Sliding mode control of chaos in the cubic Chua's circuit system." *International Journal of Bifurcation and Chaos* 12.06 (2002): 1437-1449.

[32] Zhao, Zhan-Shan, Jing Zhang, and Lian-Kun Sun. "Sliding Mode Control in Finite Time Stabilization for Synchronization of Chaotic Systems." *ISRN Applied Mathematics* 2013 (2013).

[33] Dadras, Sara, and Hamid Reza Momeni. "Adaptive sliding mode control of chaotic dynamical systems with application to synchronization." *Mathematics and Computers in Simulation* 80.12 (2010): 2245-2257.

[34] Yan, Jun-Juh, Meei-Ling Hung, and Teh-Lu Liao. "Adaptive sliding mode control for synchronization of chaotic gyros with fully unknown parameters." *Journal of Sound and Vibration* 298.1 (2006): 298-306.

[35] He, Qie, Ling Wang, and Bo Liu. "Parameter estimation for chaotic systems by particle swarm optimization." *Chaos, Solitons & Fractals* 34.2 (2007): 654-661.

[36] Wang, C., and S. S. Ge. "Adaptive synchronization of uncertain chaotic systems via backstepping design." *Chaos, Solitons & Fractals* 12.7 (2001): 1199-1206.

[37] Li, Damei, Jun-an Lu, and Xiaoqun Wu. "Linearly coupled synchronization of the unified chaotic systems and the Lorenz systems." *Chaos, Solitons & Fractals* 23.1 (2005): 79-85.

[38] Pecora, Louis M., and Thomas L. Carroll. "Synchronization in chaotic systems." *Physical review letters* 64.8 (1990): 821.

[39] Tanaka, Kazuo, Takayuki Ikeda, and Hua O. Wang. "Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: quadratic stabilizability, H_∞ control theory, and linear matrix inequalities." *IEEE Transactions on Fuzzy systems* 4.1 (1996): 1-13.

[40] Xiao-run, L. I., Z. H. A. O. Liao-ying, and Z. H. A. O. Guang-zhou. "Sliding mode control for synchronization of chaotic systems with structure or parameters mismatching." *Journal of Zhejiang University Science A* 6.6 (2005): 571-576.

[41] Hou, Yi-You, Ben-Yi Liau, and Hsin-Chieh Chen. "Synchronization of unified chaotic systems using sliding mode controller." *Mathematical Problems in Engineering* 2012 (2012).